

CENTER MANIFOLD ANALYSIS OF DELAYED LIÉNARD EQUATION AND  
ITS APPLICATIONS

A Thesis

by

SIMING ZHAO

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2009

Major Subject: Aerospace Engineering

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## ABSTRACT

Center Manifold Analysis of Delayed Liénard Equation and Its Applications.

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Liénard Equations serve as the elegant models for oscillating circuits. Motivated by this fact, this thesis addresses the stability property of a class of delayed Liénard equations. It shows the existence of the Hopf bifurcation around the steady state. It has both practical and theoretical importance in determining the criticality of the Hopf bifurcation. For such purpose, center manifold analysis on the bifurcation line is required. This thesis uses operator differential equation formulation to reduce the infinite dimensional delayed Liénard equation onto a two-dimensional manifold on the critical bifurcation line. Based on the reduced two-dimensional system, the so called Poincaré-Lyapunov constant is analytically determined, which determines the criticality of the Hopf bifurcation. Numerics based on a Matlab bifurcation toolbox (DDE-Biftool) and Matlab solver (DDE-23) are given to compare with the theoretical calculation. Two examples are given to illustrate the method.

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## CHAPTER I

## INTRODUCTION

## A. Motivation and Literature Review

Starting from the early days of nonlinear dynamics, there has been a great deal of research in nonlinear oscillators. In particular, during the development of radio and vacuum tubes, Liénard equations were intensely studied since they served as elegant models of oscillating circuits. These equations are second-order differential equations of the form [1]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad (1.1)$$

it is a generalization of the Van der Pol and unforced Duffing equation. It can be interpreted mechanically as the equation of motion for a unit mass subjecting to a damping force  $-f(x)\dot{x}$  and a restoring force  $-g(x)$ . With appropriate hypotheses on  $f(x)$  and  $g(x)$ , this equation admits an unique stable limit cycle. See [2] for a proof.

This thesis considers a delayed version of the above Liénard equation, which assumes the form

$$\ddot{x}(t) + f(x(t))\dot{x} + g(x(t - \tau)) = 0, \quad (1.2)$$

where  $f, g \in C^4$ ,  $f(0) = K > 0$ ,  $g(0) = 0$ ,  $g'(0) = 1$ , and  $\tau > 0$  is a finite time delay.

Equation of type (1.2) is called delay differential equation (DDE). At each time instant, the behavior of the system not only depends on the current states, but also on the past ones. A good exposition of delay equations can be found in [3]. Several classical methods developed to study ordinary differential equations can be well applied to DDEs, these include the method of multiple scales [4], the Lindstedt-

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Poincaré method [5, 6], harmonic balance [7] and the averaging method [8]. Among all the available methods, center manifold theory [9] is one of the most rigorous mathematical tools to study local bifurcations of delay differential equations [10].

Luk [11] and Zhang [12] derived necessary and sufficient conditions for the boundedness of all solutions by using Lyapunov-functional approach. Zhang [13] and Omari [14] showed the existence of periodic orbits for this equation. Metzner [15] proved the existence of periodic orbits in a Duffing-type equation with delay and studied the effects of delay on the solvability of the problem. Acosta and Lizana [16] and Xu and Lu [17] investigated the linear stability of equation (1.2). Xu and Lu [17] performed a center manifold based Hopf bifurcation analysis. Campbell *et al.* [18] also used center manifold analysis of single and double Hopf bifurcations of a similar second order delay-differential-equation. Colonius [19] studied optimal periodic control of (1.2).

## B. Problem Statement

The goal of this thesis is to investigate the bifurcation structure of the steady state of equation (1.2), i.e. how time delay and the nonlinearities affect system dynamics. This is an important problem of both application and theoretical merit.

In this thesis, center manifold reduction is used to project this infinite dimensional system onto a two dimensional local manifold which describes the local dynamics of the the system. Xu and Lu [17] first proposed this problem. But in their analysis, the curvature of the center manifold, caused by the quadratic terms, was not accounted. It will be considered in this thesis.

The main challenge of this thesis is the analytical determination of the criticality of the Hopf bifurcation. It requires operator formulation and large symbolic



calculations.

### C. Outline of the Thesis

This thesis is organized as follows. Mathematical preliminaries required are reviewed in Chapter II. Main results are presented in Chapter III. Chapter IV contains numerical simulations. Chapter V shows two examples to illustrate this method. Lastly, Chapter VI concludes the thesis.

## CHAPTER II

### MATHEMATICAL PRELIMINARIES

Center manifold analysis of delay differential equations requires mathematical tools such as advanced calculus and functional analysis. The work here is an extension and application of well established theory on delay differential equations ([10]). Main mathematical techniques that will be used throughout this thesis are the topics of this chapter.

#### A. Delay Differential Equation

In most engineering applications, system is modeled by ordinary differential equation which assumes future behavior of the system is uniquely determined by the present states and independent of the past. In delay differential equation (DDE), the past exerts its influence on the future in a significant manner. In this section, some basics on DDE will be discussed. The contents followed Hale [10] and Niculescu [20].

##### 1. Definition, Existence and Uniqueness

This thesis will adapt definitions from [10]. Let  $C([a, b], R^n)$  be the Banach space of continuous functions mapping the interval  $[a, b]$  into  $R^n$ . If  $t_0 \in R, d \geq 0$  and  $x \in C([t_0 - \tau, t_0 + d], R^n)$ , then for any  $t \in [t_0, t_0 + d]$ , let  $\mathbf{x}_t \in C$  be defined by  $\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta), -\tau \leq \theta \leq 0$ . The following relation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}_t), \tag{2.1}$$

is called functional differential equation. Equations of this type are very general and it includes delay differential equation (DDE).

A function  $\mathbf{x}$  is said to be a solution of (2.1) if there exists  $t_0 \in R, d \geq 0$

such that  $\mathbf{x} \in C([t_0 - \tau, t_0 + A], R^n)$  and  $\mathbf{x}$  satisfies (2.1) for  $t \in (t_0, t_0 + d)$ . In this case  $\mathbf{x}$  is defined as a solution of (2.1) on  $[t_0 - \tau, t_0 + d)$  with initial condition  $\mathbf{x}(t_0 + \delta) = \phi(\delta), \delta \in [-\tau, 0]$ .

If the function  $\mathbf{f}(t, \mathbf{x})$  is continuous and it satisfies local Lipschitz condition in terms of  $\mathbf{x}$ , then solution  $\mathbf{x}$  defined above exists and is unique. The solution is also continuous dependent on the initial data  $\phi(\delta), \delta \in [-\tau, 0]$ .

## 2. Step Method

*Step method* was first proposed by Bellman and Cooke [3]. Almost all existing DDE numerical solvers are based on this method. More discussion and further references can be found in Cryer [21]. Consider the following DDE

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{x}(t - \tau)), \quad (2.2)$$

with initial data  $\mathbf{x}(t_0 + \delta) = \phi(\delta), \delta \in [-\tau, 0]$ .

In order to propagate the equation, the 'minimum' amount of initial data necessary for the existence of a solution is the initial function defined in the interval  $[t_0 - \tau, t_0]$ . From (2.2), it can be observed that the solution  $\mathbf{x}(t_0, \phi)$  on the interval  $[t_0, t_0 + \tau]$  is defined by the solution of

$$\dot{\mathbf{x}}_1 = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}_1(t - \tau)) = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}(t - \tau)), \quad \forall t \in [t_0, t_0 + \tau) \quad (2.3)$$

since the initial data  $\mathbf{x}(t - \tau), t \in [t_0, t_0 + \tau)$  is given, solution  $\mathbf{x}(t_0, \phi)$  on the interval  $[t_0, t_0 + \tau)$  can be solved as an ordinary differential equation with initial condition  $\mathbf{x}_1(t_0) = \mathbf{x}(t_0) = \phi(0)$ .

Apply same procedure on the next interval  $[t_0 + \tau, t_0 + 2\tau]$ , the solution  $\mathbf{x}(t_0, \phi)$

on  $[t_0 + \tau]$  is defined by the solution of

$$\dot{\mathbf{x}}_2 = \mathbf{f}(t, \mathbf{x}_2, \mathbf{x}_2(t - \tau)) = \mathbf{f}(t, \mathbf{x}_2, \mathbf{x}_1(t - \tau)), \quad \forall t \in [t_0 + \tau, t_0 + 2\tau) \quad (2.4)$$

where  $\mathbf{x}_1(t - \tau) = \mathbf{x}_1(\delta)$ ,  $\delta \in [t_0, t_0 + \tau)$ .

By iteration, this procedure can be continued, the solution of DDE (2.2) can be solved up to any finite time.

### 3. Stability Property of DDE

Let  $C_H = \psi \in C : |\psi| < H$ . Consider system (2.1) with  $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$  which satisfies all conditions for existence and uniqueness of the solutions. Solution  $\mathbf{x} = \mathbf{0}$  is a trivial solution of the system, it's stability is defined as

**Definition 1.** [10]

*The solution  $\mathbf{x} = \mathbf{0}$  of (2.1) is called stable at  $t_0, t_0 \geq 0$  if*

1. *There exists a  $b = b(t_0) > 0$  such that  $\psi \in C_b = \psi \in C : |\psi| < b$  implies the solution  $\mathbf{x}(t_0, \psi)$  of (2.1) exists for  $t \geq t_0$  and  $\mathbf{x}_t(t_0, \psi)$  is in  $C_H = \psi \in C : |\psi| < H$  for  $t \geq t_0$ .*
2. *For every  $\epsilon > 0$ , there exists a  $\delta = \delta(t_0, \epsilon) > 0$  such that  $\psi \in C_\delta = \psi \in C : |\psi| < \delta$  implies the solution  $\mathbf{x}(t_0, \psi)$  satisfies  $\mathbf{x}_t(t_0, \psi) \in C_\epsilon = \psi \in C : |\psi| < \epsilon$  for  $t \geq t_0$ .*

*The solution  $\mathbf{x} = \mathbf{0}$  of (2.1) is called asymptotically stable at  $t_0, t_0 \geq 0$  if it is stable and there exists an  $H_0 = H_0(t_0)$  such that  $\psi \in C_{H_0}$  implies the solution  $\mathbf{x}(t_0, \psi)$  satisfies*

$$\lim_{t \rightarrow \infty} |\mathbf{x}_t(t_0, \psi)| = 0.$$

*The solution  $\mathbf{x} = \mathbf{0}$  of (2.1) is called unstable at  $t_0$  if it is not stable at  $t_0$ .*

Similar like ODE, Lyapunov stability analysis can also be applied into DDE. If

there exists a continuous function  $V : R^+ \times C_H \rightarrow R$ , define

$$\dot{V}(t, \psi) = \limsup_{h \rightarrow 0} \frac{1}{h} [V(t+h, \mathbf{x}_{t+h}(t, \psi)) - V(t, \psi)], \quad (2.5)$$

where  $\mathbf{x}_{t+h}(t, \psi)$  is the solution of (2.1). Then the following theorem can be applied to determine Lyapunov stability.

**Theorem 1.** [10]

*Suppose  $\mathbf{f}$  takes closed bounded sets of  $R^+ \times C_H$  into closed bounded sets of  $R^n$ . Suppose  $u(s)$ ,  $v(s)$  and  $w(s)$  are continuous functions for  $s \in [0, H)$ . Both  $u(s)$  and  $v(s)$  are positive and nondecreasing for  $s \neq 0$ ,  $u(0) = v(0) = 0$ ,  $w(s)$  is nonnegative and nondecreasing. If there exists a continuous function  $V : R^+ \times C_H \rightarrow R$  such that*

$$u(|\psi(0)|) \leq V(t, \psi) \leq v(|\psi|),$$

$$\dot{V}(t, \psi) \leq -w(|\psi(0)|),$$

*then solution  $\mathbf{x} = \mathbf{0}$  of (2.1) is uniformly stable. If, in addition,  $w(s) > 0$  for  $s > 0$  while  $w(s)$  is nondecreasing, then the solution  $\mathbf{x} = \mathbf{0}$  of (2.1) is uniformly asymptotically stable.*

Local linear stability of the solution  $\mathbf{x} = \mathbf{0}$  is determined by the characteristic equation of the linearized equation around  $\mathbf{x} = \mathbf{0}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}(t - \tau), \quad (2.6)$$

the following defines the characteristic equation of (2.1) around  $\mathbf{x} = \mathbf{0}$

$$\mathcal{F}(\lambda) = \det(\lambda I - A - B e^{-\lambda\tau}). \quad (2.7)$$

This is a transcendental equation with infinite number of zeros. One of common ways in analyzing this equation is to substitute  $\lambda = i\omega$  into it, which corresponds to

stability boundary of the system. Later in the thesis, the procedure will be further elaborated. For more discussions on the distribution of zeros of such characteristic equation, see Bellman and Cooke [3], Stépán [22] and MacDonald [23].

## B. Center Manifold Theory

Center manifold theory is one of the most powerful tools to study local behaviors of a dynamical system. It forms the foundation for bifurcation theory. Here in this section, center manifold theory for ODE will be briefly introduced. It systematically followed Kuznetsov [24].

First, consider the following linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad (2.8)$$

where  $\mathbf{A}$  is a  $n \times n$  matrix. The system has invariant subspaces  $E^s$ ,  $E^c$  and  $E^u$  which are spanned by the generalized eigenvectors which in turn corresponds to eigenvalues having negative, zero and positive real parts. Solutions of this system will decay to zero/neither grow nor decay/become unbounded if initial conditions are started from  $E^s/E^c/E^u$ .

If the linearized version of a nonlinear system around the fixed point has zero eigenvalues, this equilibrium is called non-hyperbolic fixed point. The main idea for center manifold theory is to find an invariant manifold passing through this fixed point to which the system could be restricted in order to study local behaviors of the fixed point. Consider the following nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad (2.9)$$

where  $\mathbf{f}$  is sufficiently smooth,  $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ . Let the eigenvalues of the Jacobian ma-

trix evaluated at the fixed point  $\mathbf{x}_0 = \mathbf{0}$  be  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Assume that there are  $n_-/n_0/n_+$  eigenvalues with negative/zero/positive real parts, they span linear eigenspaces  $E^s, E^c$  and  $E^u$  correspondingly. Rewriting (2.9) in an *eigenbasis* formed by all eigenvectors of its Jacobian matrix

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{B}\mathbf{u} + \mathbf{g}(\mathbf{u}, \mathbf{v}), \\ \dot{\mathbf{v}} &= \mathbf{C}\mathbf{v} + \mathbf{h}(\mathbf{u}, \mathbf{v}),\end{aligned}\tag{2.10}$$

where  $\mathbf{u} \in \mathbb{R}^{n_0}$ ,  $\mathbf{v} \in \mathbb{R}^{n_+ + n_-}$ , matrix  $\mathbf{B}$  has all its eigenvalues on the imaginary axis. A center manifold  $W^c$  of the system (2.10) can be locally represented as the graph of a smooth function (Figure 1)

$$W^c = \{(\mathbf{u}, \mathbf{v}) : \mathbf{v} = \mathbf{V}(\mathbf{u}, \mathbf{v})\},$$

where  $\mathbf{V} : \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_+ + n_-}$ . Since  $W^c$  is tangent at the fixed point,  $\mathbf{V} = \mathcal{O}(\|\mathbf{u}\|^2)$ .

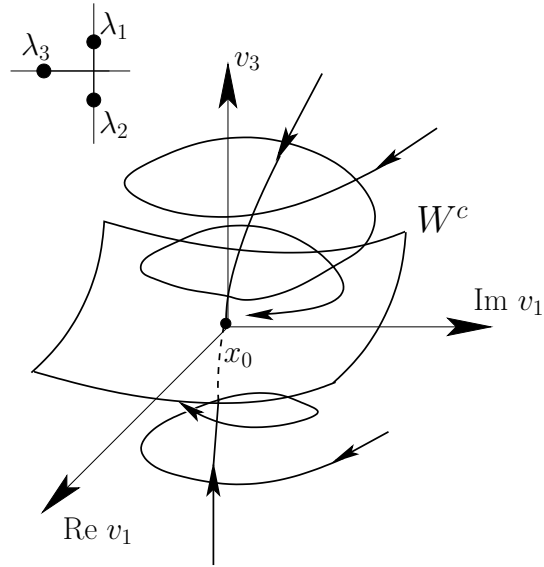


Fig. 1. A 2-D center manifold as the graph of a function  $\mathbf{v} = \mathbf{V}(\mathbf{u})$ .

The following theorem tells the existence and property of the center manifold. It is beautifully introduced and proved by Carr [25].

**Theorem 2.** *There exists a smooth  $n_0$ -dimensional invariant manifold  $\mathbf{V}^c$  of (2.10) which is tangent to  $E^c$  at  $t = 0$ . The dynamics of (2.10) restricted to the center manifold in the neighborhood of the fixed point is given by*

$$\dot{\mathbf{u}} = \mathbf{B}\mathbf{u} + \mathbf{g}(\mathbf{u}, \mathbf{V}(\mathbf{u})), \quad (2.11)$$

*the above system is topologically the same with (2.10).*

The obvious question now is how to compute the center manifold. Differentiating  $\mathbf{v} = \mathbf{V}(\mathbf{u})$  with respect to time implies  $\dot{\mathbf{v}} = D\mathbf{V}\dot{\mathbf{u}}$ . Substituting (2.10) into the above differentiated form gives a condition each center manifold needs to satisfy

$$\mathcal{N}(\mathbf{V}(\mathbf{u})) \equiv D\mathbf{V}[\mathbf{B}\mathbf{u} + \mathbf{g}(\mathbf{u}, \mathbf{V}(\mathbf{u}))] - \mathbf{C}\mathbf{v} - \mathbf{h}(\mathbf{u}, \mathbf{V}(\mathbf{u})) = 0, \quad (2.12)$$

the above equation is a quasilinear partial differential equation which is a necessary condition for  $\mathbf{V}(\mathbf{u})$  to be an invariant manifold.

This equation is very difficult to solve. But the following theorem gives a method for computing an approximate solution of (2.12), the proof is also in Carr [25]

**Theorem 3.** *Let  $\phi : \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_+ + n_-}$  be a mapping with  $\phi(\mathbf{0}) = D\phi(\mathbf{0}) = \mathbf{0}$  such that  $\mathcal{N}(\phi(\mathbf{u})) = \mathbf{O}(|\mathbf{u}|^q)$  as  $\mathbf{u} \rightarrow \mathbf{0}$  for some  $q > 1$ . Then*

$$|\mathbf{V}(\mathbf{u}) - \phi(\mathbf{u})| = \mathbf{O}(|\mathbf{u}|^q), \quad \mathbf{u} \rightarrow \mathbf{0}. \quad (2.13)$$

Theorem 2 allows the center manifold to be computed to any desired degree of accuracy by solving (2.12) to the same degree of accuracy. For this reason, power series expansions will work out nicely. In this thesis, second order power series expansion will be used to approximate the center manifold.



## CHAPTER III

MAIN RESULTS<sup>1</sup>

The main results of this thesis are presented in this chapter. Followed by linear stability analysis, operator differential form is formulated in order to project the infinite dimensional system onto a two dimensional manifold. The so called Poincaré-Lyapunov constant is calculated based on the projected equation.

## A. Model Description and Linear Stability Analysis

The equation considered here assumes the form

$$\ddot{x}(t) + f(x(t))\dot{x} + g(x(t - \tau)) = 0, \quad (3.1)$$

where  $f, g \in C^4$ ,  $f(0) = K > 0$ ,  $g(0) = 0$ ,  $g'(0) = 1$ , and  $\tau > 0$  is a finite time delay. Obviously,  $x \equiv 0$  is the null solution of the equation.

Equation (3.1) can be rewritten in the following state space form

$$\begin{aligned} \dot{x}(t) &= y(t) - S(x(t)), \\ \dot{y}(t) &= -g(x(t - \tau)), \end{aligned} \quad (3.2)$$

where  $S(x) = \int_0^x f(\delta)d\delta$ .

Expanding (3.2) in the neighborhood of the null solution up to third order yields

$$\begin{aligned} \dot{x}(t) &= y(t) - Kx(t) + ax^2(t) + bx^3(t), \\ \dot{y}(t) &= -x(t - \tau) + cx^2(t - \tau) + dx^3(t - \tau), \end{aligned} \quad (3.3)$$

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<sup>1</sup>Reprinted with permission from S.M. Zhao and T. Kalmár-Nagy, Center manifold analysis of the delayed Liénard equation, Delay Differential Equations-Recent Advances and New Directions, pp. 203219, Balachandran, B., Kalmár-Nagy, T., Gilsinn, D.E. (Eds.), copyright[2009] by Springer.

where  $a = -\frac{1}{2}f'(0)$ ,  $b = -\frac{1}{6}f''(0)$ ,  $c = -\frac{1}{2}g''(0)$ , and  $d = -\frac{1}{6}g'''(0)$ . By defining a new vector  $\mathbf{z} = (x, y)^T$ , the above system can be written in vector form

$$\dot{\mathbf{z}}(t) = \mathbf{L}\mathbf{z}(t) + \mathbf{R}\mathbf{z}(t - \tau) + \mathbf{f}(\mathbf{z}), \quad (3.4)$$

where

$$\mathbf{L} = \begin{pmatrix} -K & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{f}(\mathbf{z}) = \begin{pmatrix} ax^2 + bx^3 \\ cx^2(t - \tau) + dx^3(t - \tau) \end{pmatrix}. \quad (3.5)$$

The above third order Taylor expanded form (3.3) up to is topological equivalent to the original nonlinear system [26]. The center manifold analysis in this thesis is based on this expanded form (3.3).

Note that state space form (3.2) is not unique. The reason why (3.2) is used here, is that the nonlinearity  $\mathbf{f}(\mathbf{z})$  only contains the first component of the state vector  $\mathbf{z}$ , which can simplify further calculations.

Neglecting the higher order terms yields the following linear system which will be used to carry out linear stability analysis

$$\dot{\mathbf{z}}(t) = \mathbf{L}\mathbf{z}(t) + \mathbf{R}\mathbf{z}(t - \tau), \quad (3.6)$$

whose characteristic equation reads

$$\lambda^2 + K\lambda + e^{-\lambda\tau} = 0. \quad (3.7)$$

On the stability boundary, the characteristic equation (3.7) has a pair of pure imaginary roots. To find out such roots, substituting  $\lambda = i\omega$ ,  $\omega > 0$  into (3.7) and

separating the real and imaginary parts

$$\begin{aligned}\omega^2 &= \cos \omega\tau, \\ K\omega &= \sin \omega\tau.\end{aligned}\tag{3.8}$$

Using simple trigonometry, (3.8) can be reduced to

$$\begin{aligned}K &= \sqrt{\frac{1 - \omega^4}{\omega^2}}, \\ \tau &= \frac{1}{\omega}(2n\pi + \arctan \frac{K}{\omega}),\end{aligned}\tag{3.9}$$

where  $0 < \omega < 1$  and  $n$  is a nonnegative integer.

Note that (3.9) describes infinitely many curves in the  $(\tau, K)$  plane, but the the first branch ( $n = 0$ ) (shown in Figure 2) actually separates the stable and unstable regions (this is a consequence of a theorem by Hale [10]).

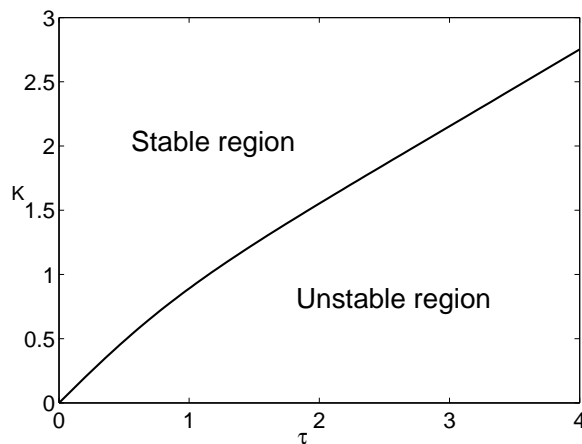


Fig. 2. Linear stability boundary of the delayed Liénard equation.

For a fixed time delay  $\tau$ , the critical value of the bifurcation parameter  $K$  (i.e. on the stability boundary) will be denoted by  $k$ . A necessary condition for the

existence of periodic orbits is that by changing bifurcation parameter  $K$ , the critical characteristic roots cross the imaginary axis with nonzero velocity, i.e.  $d\lambda/dK|_{K=k} \neq 0$ . Taking the first derivative with respect to  $K$  in (3.7) and using (3.9) gives

$$\gamma = \operatorname{Re} \frac{d\lambda}{dK} \Big|_{K=k} = \frac{\omega^2(2 + k\tau)}{(k - \tau\omega^2)^2 + (2\omega + k\tau\omega)^2} > 0, \quad (3.10)$$

which means that the root crossing velocity is positive. This velocity  $\gamma$  will later be used to estimate the vibration amplitude.

In order to prove the existence of the Hopf bifurcation, criticality of such bifurcation needs to be addressed, which requires normal form of the Hopf bifurcation on the local manifold around the origin. It is a well established procedure from Hale [10]. The following sections will closely follow a friendly-explained version by Kalmár-Nagy *et al.* [27].

## B. Operator Differential Equation Formulation

Generally delay differential equations can be expressed as abstract evolution equations on the Banach space  $\mathcal{H}$  of continuously differentiable functions  $\mu : [-\tau, 0] \rightarrow \mathbb{R}^2$

$$\dot{\mathbf{z}}_t = \mathcal{D}\mathbf{z}_t + \mathcal{F}(\mathbf{z}_t), \quad (3.11)$$

where the shift of time  $\mathbf{z}_t(\varphi) \in \mathcal{H}$  is defined as

$$\mathbf{z}_t(\varphi) = \mathbf{z}(t + \varphi), \quad \varphi \in [-\tau, 0]. \quad (3.12)$$

The linear operator  $\mathcal{D}$  at the critical bifurcation parameter assumes the form

$$\mathcal{D}\mathbf{u}(\varphi) = \begin{cases} \frac{d}{d\varphi}\mathbf{u}(\varphi), & \varphi \in [-\tau, 0) \\ \mathbf{L}\mathbf{u}(0) + \mathbf{R}\mathbf{u}(-\tau), & \varphi = 0 \end{cases},$$

while the nonlinear operator is written as

$$\mathcal{F}(\mathbf{u})(\varphi) = \begin{cases} \mathbf{0}, & \varphi \in [-\tau, 0) \\ \mathbf{f}(\mathbf{u}), & \varphi = 0 \end{cases},$$

$$\mathbf{f}(\mathbf{u}(\varphi)) = \begin{pmatrix} au_1^2(0) + bu_1^3(0) \\ cu_1^2(-\tau) + du_1^3(-\tau) \end{pmatrix}.$$

In order to calculate the center manifold, the adjoint operator needs to be defined on the adjoint space  $\mathcal{H}^*$  of continuously differentiable functions  $\theta : [0, \tau] \rightarrow \mathbb{R}^2$ , which assumes the form

$$\mathcal{D}^*\mathbf{u}(\theta) = \begin{cases} -\frac{d}{d\varphi}\mathbf{u}(\varphi), & \varphi \in (0, \tau] \\ \mathbf{L}^*\mathbf{u}(0) + \mathbf{R}^*\mathbf{u}(\tau), & \varphi = 0 \end{cases}$$

with respect to the bilinear form  $(\cdot, \cdot) : \mathcal{H}^* \times \mathcal{H} \rightarrow \mathbb{R}$

$$(v, \mu) = v^*(0)\mu(0) + \int_{-\tau}^0 v^*(\delta + \tau)\mathbf{R}\mu(\delta)d\delta. \quad (3.13)$$

Since the critical eigenvalues of the linear operator  $\mathcal{D}$  just coincide with the critical characteristic roots of the characteristic function  $D(\lambda, K)$ , the Hopf bifurcation can be studied on the two-dimensional center manifold embedded in the infinite dimensional phase space. The center subspace is spanned by the real and imaginary parts of the complex eigenfunction  $\mathbf{p}(\varphi)$  of  $\mathcal{D}$  corresponding to the critical characteristic roots  $i\omega$ . The complex eigenfunction  $\mathbf{p}(\varphi)$  and the eigenfunctions  $\mathbf{q}(\theta)$  of the adjoint operator  $\mathcal{D}^*$  can be found from

$$\begin{aligned} \mathcal{D}\mathbf{p}(\varphi) &= i\omega\mathbf{p}(\varphi), \\ \mathcal{D}^*\mathbf{q}(\theta) &= -i\omega\mathbf{q}(\theta). \end{aligned} \quad (3.14)$$

The general solutions to (3.14) are of the form

$$\begin{aligned}\mathbf{p}(\varphi) &= \mathbf{p}_1(\varphi) + i\mathbf{p}_2(\varphi) = \mathbf{c}e^{i\omega\varphi}, \\ \mathbf{q}(\theta) &= \mathbf{q}_1(\varphi) + i\mathbf{q}_2(\varphi) = \mathbf{d}e^{i\omega\theta},\end{aligned}\tag{3.15}$$

the constants  $\mathbf{p}$  and  $\mathbf{q}$  are found by using the boundary conditions embedded in the operator equations (3.14)

$$\begin{aligned}(i\omega\mathbf{I} - \mathbf{L} - e^{-i\omega\tau}\mathbf{R})\mathbf{c} &= \mathbf{0}, \\ (-i\omega\mathbf{I} - \mathbf{L}^T - e^{i\omega\tau}\mathbf{R}^T)\mathbf{d} &= \mathbf{0}.\end{aligned}$$

The vectors  $\mathbf{p}$  and  $\mathbf{q}$  should not be aligned, i.e. the bilinear form of  $\mathbf{q}$  and  $\mathbf{p}$  should be nonzero

$$(\mathbf{q}, \mathbf{p}) = \beta \neq 0,\tag{3.16}$$

the constant  $\beta$  can be freely chosen. From the bilinear form defined by (3.13), the following equation can be achieved

$$\begin{aligned}(\mathbf{q}, \mathbf{p}) &= \mathbf{q}^*(0)\mathbf{p}(0) + \int_{-\tau}^0 \mathbf{q}^*(\xi + \tau)\mathbf{R}\mathbf{p}(\xi)d\xi \\ &= \mathbf{d}^*\mathbf{c} + \mathbf{d}^*\mathbf{R}\mathbf{c}e^{-i\omega\tau} \int_{-\tau}^0 d\xi \\ &= \mathbf{d}^*(\mathbf{I} + \tau e^{-i\omega\tau}\mathbf{R})\mathbf{c}.\end{aligned}\tag{3.17}$$

To summarize, the vectors  $\mathbf{c}$  and  $\mathbf{d}$  can be found from the following equations

$$(i\omega\mathbf{I} - \mathbf{L} - e^{-i\omega\tau}\mathbf{R})\mathbf{c} = \mathbf{0},\tag{3.18}$$

$$(i\omega\mathbf{I} + \mathbf{L}^T + e^{i\omega\tau}\mathbf{R}^T)\mathbf{d} = \mathbf{0},\tag{3.19}$$

$$\mathbf{d}^*(\mathbf{I} + \tau e^{-i\omega\tau}\mathbf{R})\mathbf{c} = 2.\tag{3.20}$$

There are 4 complex unknowns for the above three complex equations. (3.18)

and (3.19) result in (note that  $c_1$  and  $d_2$  are complex variables)

$$\mathbf{c} = \begin{pmatrix} 1 \\ k + i\omega \end{pmatrix} c_1, \quad \mathbf{d} = \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} d_2.$$

The following equation can be obtained from (3.20)

$$c_1 d_2^* [k - \omega^2 \tau + i(2\omega + k\omega\tau)] = 2. \quad (3.21)$$

Fixing 1 unknown (choosing  $c_1 = 1$ ) yields

$$d_2^* = \frac{2}{k - \omega^2 \tau + i(2\omega + k\omega\tau)}. \quad (3.22)$$

After separating the real and imaginary part,  $\mathbf{c}$  and  $\mathbf{d}$  have the following form

$$\begin{aligned} \mathbf{c}_1 = \operatorname{Re} \mathbf{c} &= \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}, \\ \mathbf{c}_2 = \operatorname{Im} \mathbf{c} &= \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \\ \mathbf{d}_1 = \operatorname{Re} \mathbf{d} &= \begin{pmatrix} d_{11} \\ d_{12} \end{pmatrix} = \Omega \begin{pmatrix} \omega^2(1 + \frac{1}{2}k\tau) \\ \frac{1}{2}(k - \omega^2\tau) \end{pmatrix}, \\ \mathbf{d}_2 = \operatorname{Im} \mathbf{d} &= \begin{pmatrix} d_{21} \\ d_{22} \end{pmatrix} = \Omega \begin{pmatrix} -\frac{\omega}{2}(k - \omega^2\tau) \\ \omega(1 + \frac{1}{2}k\tau) \end{pmatrix}, \end{aligned}$$

where  $\Omega = \frac{4\gamma}{\omega^2(2+k\tau)}$ .

Decomposing the solution  $\mathbf{z}_t(\varphi)$  into two components  $y_1(t)$  and  $y_2(t)$  lying in the center subspace and the infinite-dimensional component  $\mathbf{w}$  transverse to the center

subspace

$$\begin{aligned}\mathbf{z}_t(\varphi) &= y_1(t)\mathbf{p}_1(\varphi) + y_2(t)\mathbf{p}_2(\varphi) + \mathbf{w}(t)(\varphi), \\ y_1(t) &= (\mathbf{q}_1, \mathbf{z}_t) |_{\varphi=0}, \quad y_2(t) = (\mathbf{q}_2, \mathbf{z}_t) |_{\varphi=0}.\end{aligned}$$

These new coordinates above transform the operator differential equation (3.11) into

$$\dot{y}_1 = \omega y_2 + \mathbf{q}_1^T(0)\mathbf{F}, \quad (3.23)$$

$$\dot{y}_2 = -\omega y_1 + \mathbf{q}_2^T(0)\mathbf{F}, \quad (3.24)$$

$$\dot{\mathbf{w}} = \mathcal{D}\mathbf{w} + \mathcal{F}(\mathbf{z}_t) - \mathbf{q}_1^T(0)\mathbf{F}\mathbf{p}_1 - \mathbf{q}_2^T(0)\mathbf{F}\mathbf{p}_2. \quad (3.25)$$

Note that the nonlinear operator in (3.25) should be written as

$$\mathcal{F}(y_1\mathbf{p}_1 + y_2\mathbf{p}_2 + \mathbf{w}) = \begin{cases} \mathbf{0}, & \varphi \in [-\tau, 0) \\ \mathbf{F}, & \varphi = 0 \end{cases},$$

where  $\mathbf{F} = (f_1, f_2)^T$  and  $f_1$  and  $f_2$  are given as (neglecting terms higher than third order)

$$f_1 = a(w_1(0) + y_1)^2 + b(w_1(0) + y_1)^3 = a(y_1^2 + 2y_1w_1(0)) + by_1^3, \quad (3.26)$$

$$\begin{aligned}f_2 &= c(w_1(-\tau) + \cos \omega\tau y_1 - \sin \omega\tau y_2)^2 + d(w_1(-\tau) + \cos \omega\tau y_1 - \sin \omega\tau y_2)^3 \\ &= c(\omega^4 y_1^2 + k^2 \omega^2 y_2^2 - 2k\omega^3 y_1 y_2 + 2\omega^2 w_1(-\tau) y_1 - 2k\omega w_1(-\tau) y_2) \\ &\quad + d(\omega^6 y_1^3 - k^3 \omega^3 y_2^3 - 3k\omega^5 y_1^2 y_2 + 3k^2 \omega^4 y_1 y_2^2).\end{aligned} \quad (3.27)$$

In the next section, dynamics of  $y_1$  and  $y_2$  in the center manifold will be derived by approximating  $\mathbf{w}(y_1, y_2)(\varphi)$  by quadratic terms (higher order terms of  $\mathbf{w}$  are not relevant for local Hopf bifurcation analysis).



### C. Center Manifold Reduction

The center manifold is tangent to  $y_1 - y_2$  plane at the origin, and is locally invariant and attractive to the flow of (3.11). Notice that when  $a = c = 0$  (symmetric nonlinearities), the center manifold coincides with the center sub-space which is spanned by the eigenfunctions calculated earlier. Since the nonlinearities considered here are not always symmetric, center manifold can be a nonlinear surface. Consider the following quadratic form

$$\mathbf{w}(y_1, y_2)(\varphi) = \frac{1}{2}(\mathbf{h}_1(\varphi)y_1^2 + 2\mathbf{h}_2(\varphi)y_1y_2 + \mathbf{h}_3(\varphi)y_2^2). \quad (3.28)$$

The time derivative of  $\varphi$  can be expressed by differentiating the right-hand side of the above equation via substituting (3.23) and (3.24)

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{h}_1y_1\dot{y}_1 + \mathbf{h}_2y_2\dot{y}_1 + \mathbf{h}_2y_1\dot{y}_2 + \mathbf{h}_3y_2\dot{y}_2 \\ &= \dot{y}_1(\mathbf{h}_1y_1 + \mathbf{h}_2y_2) + \dot{y}_2(\mathbf{h}_2y_1 + \mathbf{h}_3y_2) \\ &= -\omega\mathbf{h}_2y_1^2 + \omega(\mathbf{h}_1 - \mathbf{h}_3)y_1y_2 + \omega\mathbf{h}_2y_2^2 + O(y^3). \end{aligned} \quad (3.29)$$

Comparing the coefficients of  $y_1^2$ ,  $y_1y_2$ ,  $y_2^2$  with another form of  $\dot{\mathbf{w}}$  from (3.25) yields the following boundary value problem

$$\begin{aligned} \frac{1}{2}\dot{\mathbf{h}}_1 &= -\omega\mathbf{h}_2 + \mathbf{m}f_{111} + \mathbf{n}f_{211}, \\ \dot{\mathbf{h}}_2 &= \omega\mathbf{h}_1 - \omega\mathbf{h}_3 + \mathbf{m}f_{112} + \mathbf{n}f_{212}, \\ \frac{1}{2}\dot{\mathbf{h}}_3 &= \omega\mathbf{h}_2 + \mathbf{m}f_{122} + \mathbf{n}f_{222}, \end{aligned} \quad (3.30)$$

$$\begin{aligned} \frac{1}{2}(\mathbf{Lh}_1(0) + \mathbf{Rh}_1(-\tau)) &= -\omega\mathbf{h}_2(0) + \mathbf{m}(0)f_{111} + \mathbf{n}(0)f_{211} - \mathbf{s}_1, \\ \mathbf{Lh}_2(0) + \mathbf{Rh}_2(-\tau) &= \omega\mathbf{h}_1(0) - \omega\mathbf{h}_3(0) + \mathbf{m}(0)f_{112} + \mathbf{n}(0)f_{212} - \mathbf{s}_2, \\ \frac{1}{2}(\mathbf{Lh}_3(0) + \mathbf{Rh}_3(-\tau)) &= \omega\mathbf{h}_2(0) + \mathbf{m}(0)f_{122} + \mathbf{n}(0)f_{222} - \mathbf{s}_3, \end{aligned} \quad (3.31)$$

with

$$\mathbf{m}(\varphi) = d_{11}\mathbf{p}_1(\varphi) + d_{21}\mathbf{p}_2(\varphi),$$

$$\mathbf{n}(\varphi) = d_{12}\mathbf{p}_1(\varphi) + d_{22}\mathbf{p}_2(\varphi).$$

The coefficients of the quadratic terms can be extracted from (3.26) and (3.27)

as

$$f_{111} = a, \quad f_{211} = c\omega^4,$$

$$f_{112} = 0, \quad f_{212} = -2ck\omega^3,$$

$$f_{122} = 0, \quad f_{222} = ck^2\omega^2.$$

Introducing the following notations

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix}, \quad \mathbf{C}_{6 \times 6} = \omega \begin{pmatrix} \mathbf{0} & -2\mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & 2\mathbf{I} & \mathbf{0} \end{pmatrix},$$

$$\mathbf{s} = \begin{pmatrix} 2\mathbf{S}_0\mathbf{s}_1 \\ \mathbf{S}_0\mathbf{s}_2 \\ 2\mathbf{S}_0\mathbf{s}_3 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 2\mathbf{N}_0\mathbf{s}_1 \\ \mathbf{N}_0\mathbf{s}_2 \\ 2\mathbf{N}_0\mathbf{s}_3 \end{pmatrix},$$

$$\mathbf{s}_1 = \begin{pmatrix} f_{111} \\ f_{211} \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} f_{112} \\ f_{212} \end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix} f_{122} \\ f_{222} \end{pmatrix},$$

$$\mathbf{S}_0 = \begin{pmatrix} d_{11} & d_{12} \\ kd_{11} + \omega d_{21} & kd_{12} + \omega d_{22} \end{pmatrix},$$

$$\mathbf{N}_0 = \begin{pmatrix} d_{21} & d_{22} \\ kd_{21} - \omega d_{11} & kd_{22} - \omega d_{12} \end{pmatrix},$$

Equation (3.30) can be written as the following inhomogeneous differential equation

$$\frac{d}{d\varphi}\mathbf{h}(\varphi) = \mathbf{C}\mathbf{h} + \mathbf{s}\cos\omega\varphi + \mathbf{n}\sin\omega\varphi, \quad (3.32)$$

the above linear ODE has the general solution form

$$\mathbf{h}(\varphi) = e^{\mathbf{C}\varphi}\mathbf{K} + \mathbf{M}\cos\omega\varphi + \mathbf{N}\sin\omega\varphi. \quad (3.33)$$

Substituting this solution form into (3.32) yields the following equations which solve for  $\mathbf{M}$ ,  $\mathbf{N}$  and  $\mathbf{K}$

$$\begin{pmatrix} \mathbf{C}_{6\times 6} & -\omega\mathbf{I}_{6\times 6} \\ \omega\mathbf{I}_{6\times 6} & \mathbf{C}_{6\times 6} \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} = - \begin{pmatrix} \mathbf{s} \\ \mathbf{n} \end{pmatrix}, \quad (3.34)$$

$$\mathbf{P}\mathbf{h}(0) + \mathbf{Q}\mathbf{h}(-\tau) = \mathbf{s} - \mathbf{r}, \quad (3.35)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} \end{pmatrix} - \mathbf{C}_{6\times 6},$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 2\mathbf{s}_1 \\ \mathbf{s}_2 \\ 2\mathbf{s}_3 \end{pmatrix}.$$

The expressions for  $\mathbf{w}_1(0)$  and  $\mathbf{w}_1(-\tau)$  are given by

$$\begin{aligned} \mathbf{w}_1(0) &= \frac{1}{2}((M_1 + K_1)y_1^2 + 2(M_3 + K_3)y_1y_2 + (M_5 + K_5)y_2^2) \\ &= h_{110}y_1^2 + h_{210}y_1y_2 + h_{310}y_2^2, \end{aligned} \quad (3.36)$$

$$\begin{aligned}
\mathbf{w}_1(-\tau) &= \frac{1}{2}((e^{-\mathbf{C}\tau}\mathbf{K} |_1 + M_1 \cos \omega\tau - N_1 \sin \omega\tau)y_1^2 \\
&\quad + 2(e^{-\mathbf{C}\tau}\mathbf{K} |_3 + M_3 \cos \omega\tau - N_3 \sin \omega\tau)y_1y_2 \\
&\quad + (e^{-\mathbf{C}\tau}\mathbf{K} |_5 + M_5 \cos \omega\tau - N_5 \sin \omega\tau)y_2^2) \\
&= h_{11\tau}y_1^2 + h_{21\tau}y_1y_2 + h_{31\tau}y_2^2.
\end{aligned} \tag{3.37}$$

From (3.36) and (3.37), the first, third and fifth component of  $\mathbf{M}$ ,  $\mathbf{N}$ ,  $\mathbf{K}$ ,  $e^{-\mathbf{C}\tau}\mathbf{K}$  can be calculated

$$\begin{aligned}
\begin{pmatrix} M_1 \\ M_3 \\ M_5 \end{pmatrix} &= -\frac{2}{3\omega} \begin{pmatrix} ad_{21} + c\omega^2(-2d_{12}k\omega + d_{22}(2k^2 + \omega^2)) \\ -ad_{11} + c\omega^2(d_{22}k\omega + d_{12}(k^2 - \omega^2)) \\ 2ad_{21} + c\omega^2(2d_{12}k\omega + d_{22}(k^2 + 2\omega^2)) \end{pmatrix}, \\
\begin{pmatrix} N_1 \\ N_3 \\ N_5 \end{pmatrix} &= \frac{2}{3\omega} \begin{pmatrix} ad_{11} + c\omega^2(2d_{22}k\omega + d_{12}(2k^2 + \omega^2)) \\ ad_{21} + c\omega^2(d_{12}k\omega + d_{22}(-k^2 + \omega^2)) \\ 2ad_{11} + c\omega^2(-2d_{22}k\omega + d_{12}(k^2 + 2\omega^2)) \end{pmatrix}, \\
\begin{pmatrix} K_1 \\ K_3 \\ K_5 \end{pmatrix} &= \zeta \begin{pmatrix} \omega(-2ak(\omega^2 - 1) + 3c\omega^2(4 + k^4 - 2\omega^2 - \omega^4)) \\ a(1 - 4\omega^2 - 2k^2\omega^2) + ck(1 + 2\omega^4) \\ \omega(2ak(-1 + \omega^2) + c(2k^2 + 8\omega^2 - 2\omega^4)) \end{pmatrix}, \\
&= \begin{pmatrix} e^{-\mathbf{C}\tau}\mathbf{K} |_1 \\ e^{-\mathbf{C}\tau}\mathbf{K} |_3 \\ e^{-\mathbf{C}\tau}\mathbf{K} |_5 \end{pmatrix} = \\
&\zeta \begin{pmatrix} \omega(-2ak(1 + 2\omega^4) + c\omega^2(9 + 2k^2 + 2k^4 + (2k^2 - 6)\omega^2 + 8k^2\omega^4)) \\ a(1 - 4\omega^6) + ck\omega^2(-1 - 5k^2 + (7 + 4k^4)\omega^2 - 4k^2\omega^4) \\ \omega(2ak(1 + 2\omega^4) + c(3k^2 - 2 + (8 - 8k^2)\omega^2 + 8k^4\omega^4)) \end{pmatrix},
\end{aligned}$$

where  $\zeta = \frac{2\omega}{5+12\omega^4-8\omega^6}$ .

#### D. Hopf Bifurcation Analysis

In order to restrict a third-order approximation of system (3.23) and (3.24) to the two-dimensional center manifold calculated in the previous section, the dynamics of  $y_1$  and  $y_2$  is assumed to have following norm form

$$\begin{aligned}\dot{y}_1 &= \omega y_2 + a_{20}y_1^2 + a_{11}y_1y_2 + a_{02}y_2^2 + a_{30}y_1^3 + a_{21}y_1^2y_2 + a_{12}y_1y_2^2 + a_{03}y_2^3, \\ \dot{y}_2 &= -\omega y_1 + b_{20}y_1^2 + b_{11}y_1y_2 + b_{02}y_2^2 + b_{30}y_1^3 + b_{21}y_1^2y_2 + b_{12}y_1y_2^2 + b_{03}y_2^3.\end{aligned}\quad (3.38)$$

Using the 10 out of these 14 coefficients  $a_{jk}, b_{jk}$ , the so called Poincaré-Lyapunov constant  $\Delta$  can be calculated ([9])

$$\begin{aligned}\Delta &= \frac{1}{8\omega}((a_{20} + a_{02})(-a_{11} + b_{20} - b_{02}) + (b_{20} + b_{02})(a_{20} - a_{02} + b_{11})) \\ &\quad + \frac{1}{8}(3a_{30} + a_{12} + b_{21} + 3b_{03}).\end{aligned}\quad (3.39)$$

Based on the center manifold calculation, the 10 coefficients are as follows

$$\begin{aligned}a_{20} &= ad_{11} + c\omega^4d_{12}, \\ b_{20} &= ad_{21} + c\omega^4d_{22}, \\ a_{11} &= -2ck\omega^3d_{12}, \\ b_{11} &= -2ck\omega^3d_{22}, \\ a_{02} &= ck^2\omega^2d_{12}, \\ b_{02} &= ck^2\omega^2d_{22}, \\ a_{30} &= (2ah_{110} + b)d_{11} + (2c\omega^2h_{11\tau} + d\omega^6)d_{12}, \\ a_{12} &= 2ah_{310}d_{11} + (2c\omega^2h_{31\tau} - 2ck\omega h_{21\tau} + 3dk^2\omega^4)d_{12}, \\ b_{21} &= 2ah_{210}d_{21} + (2c\omega^2h_{21\tau} - 2ck\omega h_{11\tau} - 3dk\omega^5)d_{22}, \\ b_{03} &= -(2ck\omega h_{31\tau} + dk^3\omega^3)d_{22}.\end{aligned}$$

Substituting all these coefficients into (3.39) yields (tedious simplification)

$$\Delta = l_1 d_{12} + l_2 d_{22}, \quad (3.40)$$

where

$$\begin{aligned} l_1 &= \frac{3}{8}d\omega^2 + \frac{a^2}{4}\omega\zeta(1 + 4\omega^2 - 2\omega^4) - \frac{ac}{2}k\omega\zeta(1 + \omega^2 + \omega^4) \\ &\quad + \frac{c^2}{4}\omega\zeta\left(\frac{11}{2} + k^2 + 2\omega^2 + 12\omega^4 - 12\omega^6\right), \\ l_2 &= \frac{3}{8}b\omega - \frac{3}{8}dk\omega + \frac{a^2}{2}k\omega^2\zeta(1 - \omega^2) + \frac{ac}{4}\zeta\left(\frac{7}{2} + \omega^2 + 10\omega^4 - 10\omega^6\right) \\ &\quad + \frac{c^2}{4}k\zeta\left(-\frac{11}{2} + \omega^2 - 12\omega^4 + 12\omega^6\right), \end{aligned} \quad (3.41)$$

where  $\zeta = \frac{2\omega}{5+12\omega^4-8\omega^6}$ .

This is the main result in this thesis. All the calculations are prepared for this constant whose sign determines the criticality of Hopf bifurcation (negative/positive sign of  $\Delta$  implies super-critical/sub-critical Hopf bifurcation).

## CHAPTER IV

## SIMULATION RESULTS

In order to validate the above center manifold calculation, continuation-based DDE-Biftool [28, 29] and Matlab numerical solver DDE-23 are used. By defining  $\alpha = \frac{\gamma}{\Delta}$ , the vibration amplitude in the neighborhood of the bifurcation point is estimated as

$$r = \sqrt{\alpha(K - k)}. \quad (4.1)$$

The delayed Liénard equation

$$\begin{aligned} \dot{x}(t) &= y(t) - Kx(t) + ax^2(t) + bx^3(t), \\ \dot{y}(t) &= -x(t - \tau) + cx^2(t - \tau) + dx^3(t - \tau), \end{aligned} \quad (4.2)$$

with three different sets of parameters (see Table I) was solved by continuation (DDE-Biftool) and numerical integration (DDE-23). For  $\tau = 1, 2, 3, 4$ , Table II shows the value of  $k$  at the bifurcation point, the critical frequency  $\omega$  and the percent error  $\varepsilon = 100|\frac{\alpha - \alpha_{num}}{\alpha}|$ . The numerical approximation  $\alpha_{num}$  has been obtained from the DDE-Biftool results (using amplitudes corresponding to values of the bifurcation parameter  $K$  such that  $|K - k| \leq 0.0005k$ ) by least-squares fit.

Figures 3, 4 and 5 show the amplitude estimate of (4.2) for the parameters from Table I. The solid line, dots, and rectangles show the analytical amplitude estimate based on (4.1), the DDE-Biftool results, and numerical results by DDE-23, respectively. The DDE-23 results are obtained by combining numerical integration, estimation of amplitude decay/growth, and bisection to locate periodic orbits.

Finally, by randomly choosing  $a, b, c, d$  from  $[-10, 10]$  and  $\tau$  from  $[0, 5]$ , 1000 DDE-Biftool simulations were performed and the amplitude results were compared with the analytical ones. It shows that  $\alpha$  agrees very well with  $\alpha_{num}$  (approximation

Table I. Sets of parameter values of three different delayed Liénard equations.

	a	b	c	d
I	1	-2	5	-2
II	-3	-2	1	-4
III	-3	-1	4	3

Table II. Critical bifurcation parameter  $k$ , critical frequency  $\omega$  and error  $\varepsilon$  evaluated at different time delays  $\tau$ .

$\tau$	1	2	3	4
$k$	0.891	1.552	2.154	2.753
$\omega$	0.824	0.601	0.454	0.360
$\varepsilon_I$	0.13%	0.10%	0.15%	0.11%
$\varepsilon_{II}$	0.91%	0.94%	0.30%	0.25%
$\varepsilon_{III}$	0.29%	0.15%	0.09%	0.15%



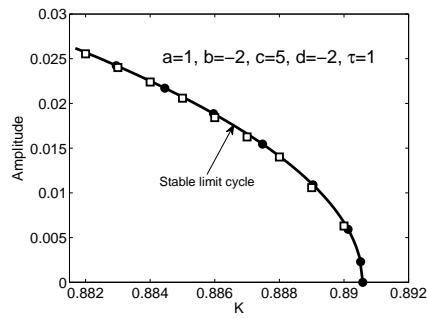
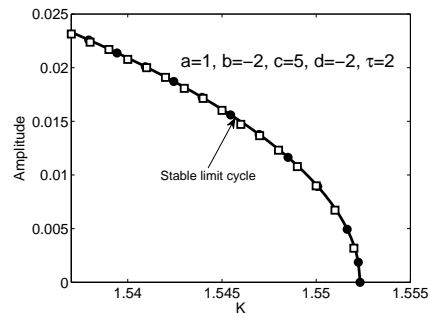
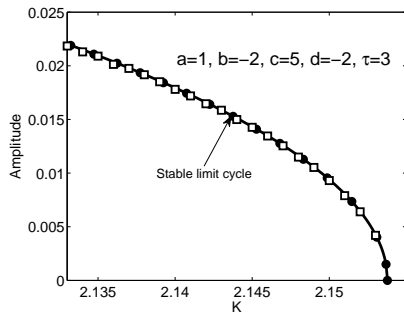
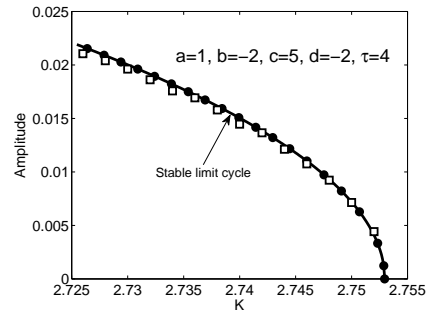
(a)  $\tau = 1$ (b)  $\tau = 2$ (c)  $\tau = 3$ (d)  $\tau = 4$ 

Fig. 3. Amplitude estimate for  $K$  for case I of Table I. The solid line is the estimated vibration amplitude based on (4.1), the dots and rectangles are results obtained from DDE-Biftool and DDE-23, respectively. The bifurcation is sub-critical.

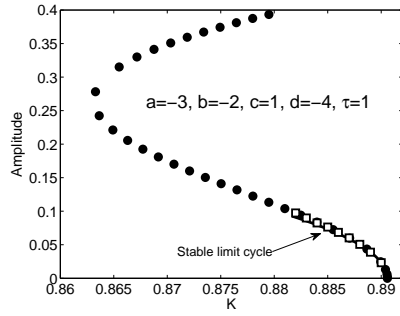
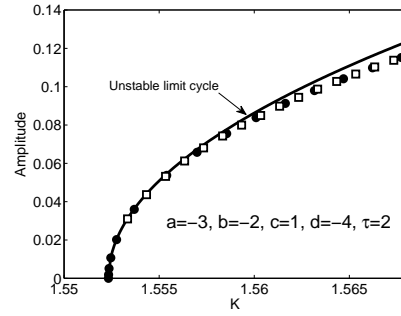
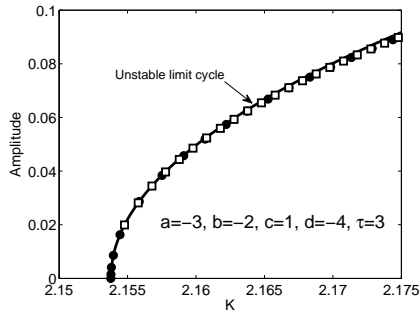
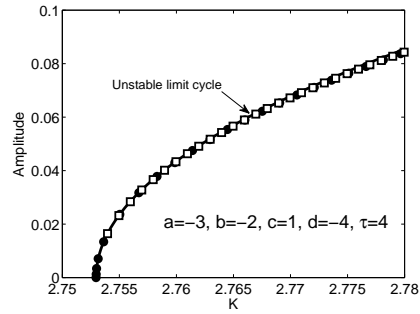
(a)  $\tau = 1$ (b)  $\tau = 2$ (c)  $\tau = 3$ (d)  $\tau = 4$ 

Fig. 4. Amplitude estimate for  $K$  for case II of Table I. The solid line is the estimated vibration amplitude based on (4.1), the dots and rectangles are results obtained from DDE-Biftool and DDE-23, respectively.

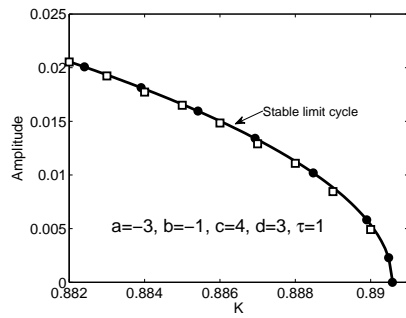
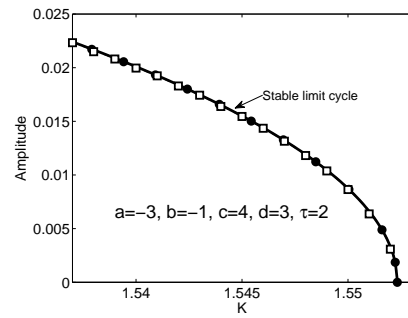
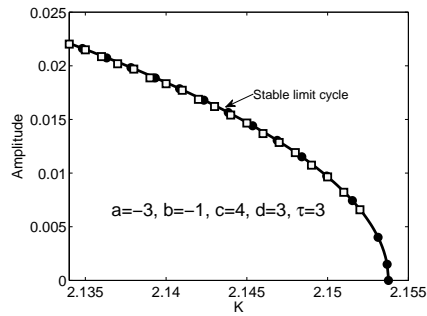
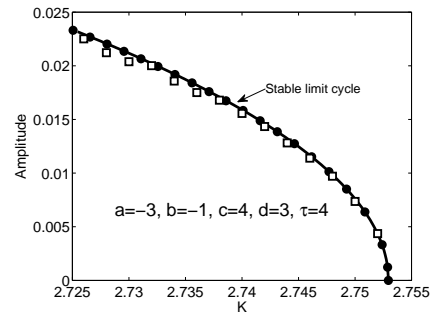
(a)  $\tau = 1$ (b)  $\tau = 2$ (c)  $\tau = 3$ (d)  $\tau = 4$ 

Fig. 5. Amplitude estimate for  $K$  for case III of Table I. The solid line is the estimated vibration amplitude based on (4.1), the dots and rectangles are results obtained from DDE-Biftool and DDE-23, respectively. The bifurcation is super-critical.

is again based on amplitudes within 0.05% of the critical value  $k$ ): the mean error is 0.8%, with a small variance of  $5.43 \times 10^{-4}$ .

## CHAPTER V

## APPLICATION OF CENTER MANIFOLD ANALYSIS

In this chapter, two examples of Liénard equation will be presented. The first one is the so-called Sunflower Equation which describes the helical movement of the tip of sunflower. The second one deals with free vibration of a nonlinear damped mass-spring system with delayed force.

## A. Hopf Bifurcation in the Sunflower Equation

Israelson and Johnson [30] proposed the following equation

$$y'' + \frac{A}{\epsilon}y' + \frac{B}{\epsilon}\sin y(\tilde{t} - \epsilon) = 0, \quad (5.1)$$

to explain the helical movement of the tip of sunflower. The upper part of the stem performs a rotating movement.  $y(\tilde{t})$  is the angle of the plant with respect to the vertical line, the delay factor  $\epsilon$  corresponds to a geotropic reaction time in the effect due to accumulation of the growth hormone alternatively on both side of the plant. The parameters  $A$  and  $B$  can be obtained experimentally.

Somolinos [31] proved the existence of periodic solutions for (5.1), this result covers both small amplitude limit cycle generated by Hopf bifurcation and large amplitude limit cycles. Casal and Freedman [5] computed a perturbation expansion based on the Lindstedt-Poincaré method. MacDonald [7] performed first order harmonic balance of (5.1).

Introducing a time scaling  $t \rightarrow \sqrt{\frac{B}{\epsilon}}\tilde{t}$  and expanding (5.1) about the null solution up to third order

$$\ddot{x} + \frac{A}{\tau}\dot{x} + x(t - \tau) - \frac{1}{6}x^3(t - \tau) = 0, \quad (5.2)$$

where  $\tau = \sqrt{B\epsilon}$ ,  $x(t) = y(\sqrt{\frac{\epsilon}{B}}\tilde{t})$ .

This equation is in the same form as (3.1) with  $K = \frac{A}{\tau}$ ,  $a = b = c = 0$ , and  $d = \frac{1}{6}$ , therefore previous results can be directly applied. The characteristic equation has the following form

$$\lambda^2 + \frac{A}{\tau}\lambda + e^{-\lambda\tau} = 0. \quad (5.3)$$

On the stability boundary

$$\begin{aligned} \omega^2 &= \cos \omega\tau, \\ \frac{A}{\tau}\omega &= \sin \omega\tau. \end{aligned} \quad (5.4)$$

The parametric curve  $(\tau(\omega), A_{cr}(\omega))$  describes the stability boundary (see Figure 6).

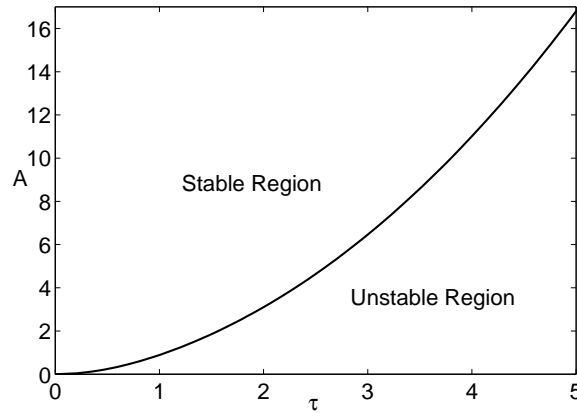


Fig. 6. Linear stability boundary of the Sunflower equation.

Now substituting  $\frac{A_{cr}}{\tau} = k$  into the Poincaré-Lyapunov constant formula (3.40)

$$\Delta = -\frac{\Omega}{32\tau}(A_{cr}\omega^2 + \tau^2) < 0, \quad (5.5)$$

where  $\Omega = \frac{4\tau^2}{(A_{cr}-\tau^2\omega^2)^2+(2\tau\omega+A_{cr}\tau\omega)^2}$ . In order to obtain the amplitude estimate, rooting

crossing velocity on the stability curve is required

$$\gamma = \operatorname{Re} \frac{d\lambda}{dA} \Big|_{A=A_{cr}} = -\frac{\omega^2}{4\tau}(2 + A_{cr})\Omega < 0. \quad (5.6)$$

It can be concluded that the Hopf bifurcation of the Sunflower equation is always super-critical. This conclusion is in full agreement with the earlier studies referred above. From (4.1), (5.5) and (5.6), the amplitude of the limit cycle can be estimated as

$$r = 2\omega \sqrt{\frac{-2(2 + A_{cr})(A - A_{cr})}{A_{cr}\omega^2 + \tau^2}}. \quad (5.7)$$

Figure 7a shows the amplitude estimate for  $B = 4$  and  $\epsilon = 1$ . The solid line denotes the plot of the analytical result based on (5.7), while the dots and triangles correspond to the numerical results of DDE-Biftool based on the original equation (5.1) and the Taylor expanded one (5.2). Figure 7b shows the  $x - \dot{x}$  plot corresponding to point C ( $A = 3, B = 4, \epsilon = 1$ ) in Figure 7a. The initial function is chosen as  $x(t) = 0.1, \quad t \in (-1, 0]$ .

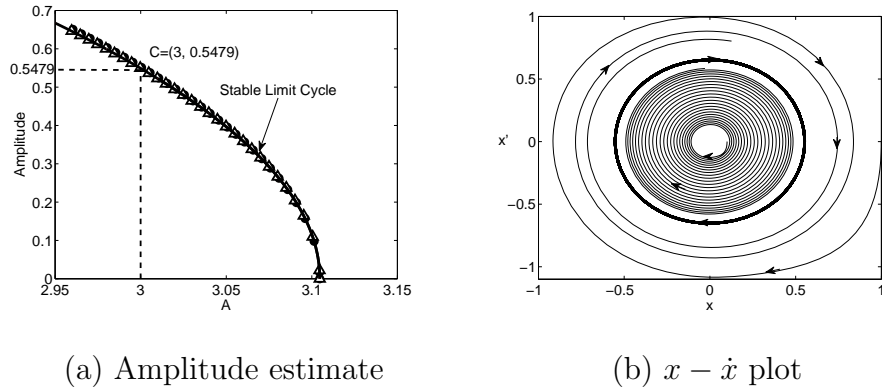


Fig. 7. Simulation results of the sunflower equation 5.1 and 7.

## B. Self-Excitation of a Delayed Liénard Oscillator

Free vibration of a nonlinear damped mass-spring system with delayed force can be modeled as

$$\ddot{x} + K\dot{x} + x(t - \tau) + \beta x^3(t - \tau) = 0, \quad (5.8)$$

where  $K > 0$  is the positive damping force and time delay  $\tau$  is caused by the time-dependence of the deformation of the spring element when the spring is no longer an idea elastic body. For detailed discussion of this problem, readers are referred to [32].

In [33], the resonance of this positive damped equation is analyzed by applying averaging method. Das and Chatterjee [34] studied Hopf bifurcation of the same equation by using multiple scales. Wang *et al.* [35] considered it through energy analysis and averaging technique.

This equation has the same form as (3.1) with  $a = b = c = 0, d = -\beta$ , therefore the previous results can be directly applied, substituting all these coefficients into the Poincaré-Lyapunov constant formula (3.40)

$$\Delta = \frac{3\gamma\beta(k\omega^2 + \tau)}{4\omega^2(2 + k\tau)}. \quad (5.9)$$

From (5.9) it can be interpreted that the direction of the Hopf bifurcation is determined by the sign of  $\beta$ , the bifurcation is sub-critical for  $\beta > 0$  and super-critical for  $\beta < 0$ , this result agrees well with the ones in the literature. Figure 8 shows the amplitude estimate of (5.8) when  $\beta > 0$  and  $\beta < 0$ .



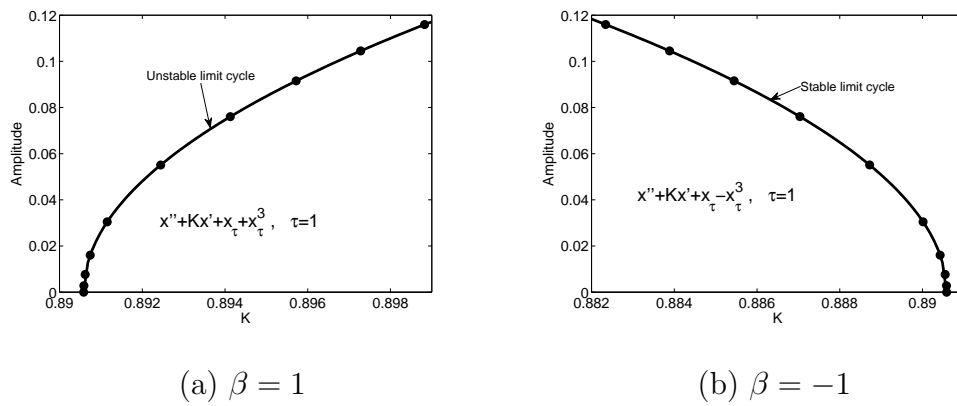


Fig. 8. Amplitude estimate for  $K$  when fixing time delay  $\tau = 1$ , the solid line denotes the estimated vibration amplitude by using our pre-calculated Poincaré-Lyapunov constant, the dot points are obtained from DDE-Biftool.

## CHAPTER VI

## CONCLUSION

This thesis has shown the existence of the Hopf bifurcation around the null solution of a class of delayed Liénard equation using center manifold analysis. Based on a projected two-dimensional manifold, a closed-form criterion for the criticality of the Hopf bifurcation is derived. The amplitude estimate for the bifurcating limit cycle was obtained by using the calculated root crossing velocity ( $\gamma$ ) and Poincaré-Lyapunov constant ( $\Delta$ ). The analytical results agree well with the numerical ones obtained from DDE-Biftool and DDE-23. Finally, two examples have been discussed to illustrate the method.

## REFERENCES

- [1] A. Liénard, “Etude des oscillations entretenues,” *Rev. Gén. Electr.*, vol. 23, pp. 901–946, 1928.
- [2] L. Perko, *Differential Equations and Dynamical Systems*, New York: Springer, 2001.
- [3] R. Bellman and K.L. Cooke, *Differential-Difference Equations*, New York: Academic Press, 1963.
- [4] H.Y. Hu, E.H. Dowell, and L.N. Virgin, “Resonances of a harmonically forced Duffing oscillator with time delay state feedback,” *Nonlinear Dyn.*, vol. 15, no. 4, pp. 311–327, 1998.
- [5] A. Casal and M. Freedman, “A Poincaré-Lindstedt approach to bifurcation problems for differential-delay equations,” *IEEE Trans. Autom. Control*, vol. 25, no. 5, pp. 967–973, 1980.
- [6] H.C. Morris, “A perturbative approach to periodic solutions of delay-differential equations,” *IMA J. Appl. Math.*, vol. 18, no. 1, pp. 15–24, 2001.
- [7] N. MacDonald, “Harmonic balance in delay-differential equations,” *J. Sound Vib.*, vol. 186, no. 4, pp. 649–656, 1995.
- [8] B. Lehman and S. Weibel, “Moving averages for periodic delay differential and difference equations,” *Lect. Notes Contr. Inform. Sci.*, vol. 228, pp. 158–183, 1998.
- [9] B.D. Hassard, N.D. Kazarinoff, and Y.H. Wan, *Theory and Applications of Hopf Bifurcation*, London U.K.: Cambridge University Press, 1981, vol. 41.

- [10] J.K. Hale, *Theory of Functional Differential Equations*, Applied Math. Sciences, New York : Springer-Verlag, 1977.
- [11] W. Luk, “Some results concerning the boundedness of solutions of Liénard equations with delay,” *SIAM J. Appl. Math.*, vol. 30, no. 4, pp. 768–774, 1976.
- [12] B. Zhang, “On the retarded Liénard equation,” *Proc. Amer. Math. Soc.*, vol. 115, no. 3, pp. 779–785, 1992.
- [13] B. Zhang, “Periodic solutions of the retarded Liénard equation,” *Ann. Mat. Pura Appl.*, vol. 172, no. 1, pp. 25–42, 1997.
- [14] P. Omari and F. Zanolin, “Periodic solutions of Liénard equations,” *Rend. Semin. Mat. Univ. Padova*, vol. 72, pp. 203–230, 1984.
- [15] G. Metzen, “Existence of periodic solutions of second order differential equations with delay,” *Proc. Amer. Math. Soc.*, vol. 103, no. 3, pp. 765–772, 1988.
- [16] A. Acosta and M. Lizana, “Hopf bifurcation for the equation  $x''(t)+f(x(t))x'(t)+g(x(t-r))=0$ ,” *Divulgaciones Matemáticas*, vol. 13, pp. 35–43, 2005.
- [17] J. Xu and Q.S. Lu, “Hopf bifurcation of time-delay Liénard equations,” *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 9, no. 5, pp. 939–951, 1999.
- [18] S.A. Campbell, J. Bélair, T. Ohira, and J. Milton, “Limit cycles, tori, and complex dynamics in a second-order differential equation with delayed negative feedback,” *J. Dynam. Differential Equations*, vol. 7, no. 1, pp. 213–236, 1995.
- [19] F. Colonius, “Optimal periodic control of retarded Liénard equation,” *Distributed parameter systems, in Proc. 2nd Int. Conf.*, pp. 77–91, Voraú, Austria, 1984.

- [20] S.I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Berlin: Springer, 2001.
- [21] C.W. Cryer, *Numerical Methods for Functional Differential Equations*, New York: Academic Press, 1972.
- [22] G. Stépán, “Modelling nonlinear regenerative effects in metal cutting,” *Phil. Trans. R. Soc. A*, pp. 739–757, 2001.
- [23] N. MacDonald, *Biological Delay Systems: Linear Stability Theory*, London U.K.: Cambridge University Press, 1989.
- [24] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, New York: Springer, 2004.
- [25] J. Carr, *Applications of Center Manifold Theory*, Applied Math. Sciences, New York : Springer-Verlag, 1981.
- [26] O. Arino, M.L. Hbid, and E.A. Dads, *Delay Differential Equations and Applications*, Dordrecht, The Netherlands: Springer, 2006.
- [27] T. Kalmár-Nagy, G. Stépán, and F.C. Moon, “Subcritical hopf bifurcation in the delay equation model for machine tool vibrations,” *Nonlinear Dyn.*, vol. 26, no. 2, pp. 121–142, 2001.
- [28] K. Engelborghs, T. Luzyanina, and D. Roose, “Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL,” *ACM Trans. Math. Software*, vol. 28, no. 1, pp. 1–21, 2002.
- [29] K. Engelborghs, T. Luzyanina, and G. Samaey, “DDE-BIFTOOL v. 2.00: a matlab package for bifurcation analysis of delay differential equations,” *Tech-*

*nical Report TW-330*, Department of Computer Science, K.U.Leuven, Leuven, Belgium, 2001.

- [30] D. Israelson and A. Johnson, “Theory of circumnutations in helianthus annuus,” *Physiol. Plant*, vol. 20, pp. 957–976, 1967.
- [31] A.S. Somolinos, “Periodic solutions of the sunflower equation:  $x'' + (a/r)x' + (b/r) \sin x(t-r) = 0$ ,” *Q. Appl. Math.*, vol. 35, pp. 465–478, 1978.
- [32] I.U.N. Rabinov, *Elements of Hereditary Solid Mechanics*, Moscow: Mir Publishers, 1980.
- [33] Y. Yoshitake, J. Inoue, and A. Sueoka, “Vibrations of a forced self-excited system with time lag,” *Bull. JSME*, vol. 26, no. 211, pp. 1943–1951.
- [34] S.L. Das and A. Chatterjee, “Multiple scales without center manifold reductions for delay differential equations near Hopf bifurcations,” *Nonlinear Dyn.*, vol. 30, no. 4, pp. 323–335, 2002.
- [35] Z.H. Wang and H.Y. Hu, “An energy analysis of the local dynamics of a delayed oscillator near a Hopf bifurcation,” *Nonlinear Dyn.*, vol. 46, no. 1, pp. 149–159, 2006.

## VITA

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