ESTIMATION OF PARAMETERS OF GAUSSIAN RANDOM VARIABLES USING ROBUST DIFFERENTIAL GEOMETRIC TECHNIQUES

A Thesis

by

SUDHA YELLAPANTULA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2009

Major Subject: Electrical Engineering

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Approved by:

Chair of Committee,	Don R. Halverson
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ABSTRACT

Estimation of Parameters of Gaussian Random Variables

Using Robust Differential Geometric Techniques.

(May 2009)

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Most signal processing systems today need to estimate parameters of the underlying probability distribution, however quantifying the robustness of this system has always been difficult. This thesis attempts to quantify the performance and robustness of the Maximum Likelihood Estimator (MLE), and a robust estimator, which is a Huber-type censored form of the MLE. This is possible using differential geometric concepts of slope. We compare the performance and robustness of the robust estimator, and its behaviour as compared to the MLE. Various nominal values of the parameters are assumed, and the performance and robustness plots are plotted. The results showed that the robustness was high for high values of censoring and was lower as the censoring value decreased. This choice of the censoring value was simplified since there was an optimum value found for every set of parameters. This study helps in future studies which require quantifying robustness for different kinds of estimators.

To my husband, Sudhakar Mahajanam

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I would like to thank my mother, Y. Sushma Murty and my father, Y.G.K. Murty, for their selfless sacrifices and unceasing support. They are like solid rock, I know I can always count on them. Also, my mother-in-law M.V. Lakshmi, who has encouraged me like her own daughter. I would also like to extend my thanks to my dear little sister Suchitra, who always supports me no matter what. Also, to my brother-in-law, Neelachal for being there when needed.

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CHAPTER I

INTRODUCTION

A. Background to Estimation Theory

Modern estimation theory is the foundation of many signal processing systems designed to extract information and estimate the values of a group of parameters. Some examples of signal processing applications include Communications, Control, Radar, Sonar, Speech, Image Analysis, Biomedicine and Seismology [1]. All these systems have the common problem of estimating the values of some parameters based on continuous waveforms. Estimation theory can be broadly classified into parameter estimation and estimation of random variables; we focus on parameter estimation and its corresponding robustness.

1. Parameter Estimation

In parameter estimation, the underlying distribution is determined a priori. In other words, it is assumed that the distribution of variables being assessed belong to known parametrized families of probability distributions. The data can be modeled as random variables ($\overrightarrow{X}$), and the underlying probability density function (pdf) is parametrized by the unknown parameter. The estimator can be thought of as a rule that assigns a value for each realization of X. Given an N-point data set, {X[0], X[1],. . . ,X[n]}, parameter estimation is the process of defining an estimator which best estimates the underlying unknown parameter [2], [3].

$$\hat{\theta} = g\left(X\left[0\right], X\left[1\right], \ldots, X\left[n\right]\right)$$

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where $g(\cdot)$ is the function to be determined by the estimator. When the assumptions are correct, an optimal (in some well-defined sense, such as minimum variance) estimator can often be found. However a significant problem with parameter estimation is that the estimator performance is highly sensitive to changes in the underlying model. In certain cases, if the assumptions are violated even slightly, the estimator may completely break.

2. Robust Estimation

Robust Estimation provides estimators that emulate classical estimation methods, but they are not unduly affected by outliers or other minor departures from the assumed model. They are more 'robust' than parametric estimators and have better performance than the non-parametric estimators. The definition given by Huber states that robustness signifies insensitivity to small deviations from the assumptions. Some of the desirable properties of a robust estimator are that the estimator should have optimal or nearly optimal performance at the assumed model. Secondly, the estimator should be robust in that small deviations from the model assumption should not cause a breakdown. The robustness theory developed by Huber [4] is largely qualitative and difficult to quantify. The new methods developed by Halverson, et.al [5], [6], [7], [8], [9], [10] use novel differential geometric techniques which are quantitative and intuitive in nature. This work focuses on robust estimation of gaussian parameters.

B. Background to Differential Geometry

1. Introduction to Differential Geometry Terminology

Some intuitive understanding of differential geometry terms are explained below, to help understand the equations that follow [11] :

1. Euclidean space: An n-dimensional space with notions of distance and angle that obey the Euclidean relationships is called an n-dimensional Euclidean space. An essential property of a Euclidean space is its flatness.

2. Non-Euclidean space: While there is essentially only one Euclidean space of each dimension, there are many non-Euclidean spaces of each dimension. Non-Euclidean spaces can be constructed by systematically deforming Euclidean spaces. Two types of non-euclidean spaces are: a. Hyperbolic geometry, b. Elliptical geometry.

3. Consider two straight lines indefinitely extended in a two-dimensional plane that are both perpendicular to a third line. In Euclidean geometry the lines remain at a constant distance from each other, and are known as parallels. In hyperbolic geometry they curve away from each other, increasing in distance as one moves further from the points of intersection with the common perpendicular; these lines are often called ultraparallels. In elliptic geometry the lines curve towards each other and eventually intersect.

4. Reimmanian geometry: Riemannian geometry deals with a broad range of geometries whose metric properties vary from point to point, as well as two standard types of Non-Euclidean geometry, spherical geometry and hyperbolic geometry, as well as Euclidean geometry itself. Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds, smooth manifolds with a Riemannian metric, i.e. with an inner product on the tangent space at each point which varies

smoothly from point to point. This gives in particular local notions of angle, length of curves, surface area, and volume. From those some other global quantities can be derived by integrating local contributions.

5. Manifold: A manifold is a mathematical space in which every point has a neighborhood which resembles Euclidean space, but in which the global structure may be more complicated. In a one-dimensional manifold (or one-manifold), every point has a neighborhood that looks like a segment of a line. In a two-manifold, every point has a neighborhood that looks like a disk.

6. Differential Manifold: A differentiable manifold is a type of manifold that is locally similar enough to Euclidean space to allow one to do calculus. A differentiable manifold can be described using mathematical maps, called coordinate charts, collected in a mathematical atlas. One may then apply ideas from calculus while working within the individual charts, since these lie in Euclidean spaces to which the usual rules of calculus apply. The notion of a differentiable manifold refines that of a manifold by requiring the transitions i.e., the function which changes the coordinate systems between charts to be differentiable.

7. Tensor: A tensor is an object which extends the notion of scalar, vector, and matrix. Tensors allow one to express physical laws in a form that applies to any coordinate system: an association of a different (mathematical) tensor with each point of a geometric space, varying continuously with position.

2. Formula Used for Robustness Analysis

The formula used for Robustness Analysis is derived by Vishal Varma [12], [13] and is stated below as follows:

Given a performance function $\underline{P} = h(\cdot)$, where $\underline{P}$ can be modelled as a Reimmanian surface, with a reimannian metric g, and h is a function on it, then the maximum directional derivative is given by

$$(D_{\vec{X}} h) = \sqrt{\nabla h G^{-1} \nabla h^T} \tag{1.1}$$

where G is a covariant tensor matrix for the given Reimannian metric which is defined by the curve; while ∇ represents the gradient of the function.

CHAPTER II

THEORY: ROBUST ESTIMATION

A. Maximum Likelihood Estimation of Gaussian Random Variables

1. Introduction

The maximum likelihood estimation is a popular way of obtaining practical estimators. MLE has the asymptotic properties of being unbiased, acheiving the Cramier Rao Lower Bound (CRLB) and having a gaussian pdf. Consider a set of data points which have a guassian distribution with mean μ_0 and variance θ_0. Let the estimates of the mean and variance be denoted by $\hat{\mu}$ and $\hat{\theta}$. The likelihood function can be written as:

$$f(\vec{X};\hat{\mu},\hat{\theta}) = \left(\frac{1}{\sqrt{2\pi\hat{\theta}}}\right)^n e^{\left(-\frac{(x_1-\hat{\mu})^2}{2\hat{\theta}} - \frac{(x_2-\hat{\mu})^2}{2\hat{\theta}} - \dots \frac{(x_n-\hat{\mu})^2}{2\hat{\theta}}\right)} \tag{2.1}$$

The maximum likelihood estimates can be found by differentiating the log-likelihood function with respect to each of the parameters and setting them to zero.

2. Estimation of Mean

The best estimate of the mean μ using maximum likelihood estimation is obtained by taking the partial derivative with respect to $\hat{\mu}$ of the log-likelihood function and setting it to zero.

$$f_{X_1X_2X_3\dots X_n}(x_1x_2\dots x_n) = \left(\frac{1}{\sqrt{2\pi\hat{\theta}}}\right)^n e^{(-\frac{(x_1-\hat{\mu})^2}{2\hat{\theta}} - \frac{(x_2-\hat{\mu})^2}{2\hat{\theta}} - \dots \frac{(x_n-\hat{\mu})^2}{2\hat{\theta}})} \tag{2.2}$$

Taking the natural logarithm on both sides and taking the partial derivative with respect to $\hat{\mu}$, we obtain the following:

$$\frac{\partial}{\partial\hat{\mu}} ln(f) = \frac{\partial}{\partial\hat{\mu}}\left[-\frac{(X_1-\hat{\mu})^2}{2\hat{\theta}} - \frac{(X_2-\hat{\mu})^2}{2\hat{\theta}} - \dots - \frac{(X_n-\hat{\mu})^2}{2\hat{\theta}}\right] \tag{2.3}$$

$$= \frac{2(X_1 - \hat{\mu})}{2\hat{\theta}} + \frac{2(X_2 - \hat{\mu})}{2\hat{\theta}} + ... + \frac{2(X_n - \hat{\mu})}{2\hat{\theta}} \tag{2.4}$$

$$\frac{\partial}{\partial \hat{\mu}} ln(f) = 0 \Rightarrow \frac{2(X_1 - \hat{\mu})}{2\hat{\theta}} + \frac{2(X_2 - \hat{\mu})}{2\hat{\theta}} + ... + \frac{2(X_n - \hat{\mu})}{2\hat{\theta}} = 0 \tag{2.5}$$

$$\Rightarrow X_1 + X_2 + ... + X_n - n\hat{\mu} = 0 \tag{2.6}$$

$$\hat{\mu} = \frac{X_1 + X_2 + ... + X_n}{n} \tag{2.7}$$

3. Estimation of Variance

Expression to calculate the best estimate for the variance θ is obtained by taking the partial derivative with respect to $\hat{\theta}$ of the log-likelihood function, and setting it to zero.

$$\frac{\partial}{\partial \hat{\theta}} ln(f) = 0 \Rightarrow -\frac{n}{2\hat{\theta}} + \frac{1}{2\hat{\theta}^2} \left[(X_1 - \hat{\mu})^2 + (X_2 - \hat{\mu})^2 + ... + (X_n - \hat{\mu})^2 \right] = 0 \tag{2.8}$$

$$\hat{\theta} = \frac{1}{n} \sum_i (X_i - \hat{\mu})^2 \tag{2.9}$$

The expression for $\hat{\theta}$ can be further simplified by substituting the expression for $\hat{\mu}$ as follows:

$$\hat{\theta} = \frac{1}{n} \sum_i \left(X_i - \frac{1}{n} \sum_j X_j \right)^2 \tag{2.10}$$

$$= \frac{1}{n} \sum_i \left[X_i^2 + \frac{1}{n^2} \left(\sum_j X_j \right)^2 - \left(\frac{2X_i}{n} \right) \sum_j X_j \right] \tag{2.11}$$

$$\theta = \frac{1}{n} \sum_i X_i^2 - \frac{1}{n^2} (\sum_i X_i)^2 \tag{2.12}$$

4. Expression for the Mean Square Error

Mean square error is defined as :

$$err = E\left[(\theta - \hat{\theta})^2 + (\mu - \hat{\mu})^2 \right] \tag{2.13}$$

where $\hat{\theta}$ and $\hat{\mu}$ are the best estimates of the mean and the variance, while θ and μ are the true values. Substituting the values for $\hat{\theta}$ and $\hat{\mu}$, we obtain:

$$Error = E\left[(\frac{1}{n}\sum_i X_i^2 - \frac{1}{n^2}(\sum_i X_i)^2 - \theta)^2 + (\frac{1}{n}\sum_i X_i - \mu)^2\right] \tag{2.14}$$

$$\begin{aligned} Error = \theta^2 + \mu^2 + \frac{1}{n^2}E\left[\left(\sum_i X_i^2\right)^2\right] + \frac{1}{n^4}E\left[\left(\sum_i X_i\right)^4\right] + \\ + \left(\frac{2\theta}{n^2} + \frac{1}{n^2}\right)E\left[\left(\sum_i X_i\right)^2\right] - \frac{2\theta}{n}E\left[\sum_i X_i^2\right] - \\ - \frac{2\mu}{n}E\left[\sum_i X_i\right] - \frac{2}{n^3}E\left[\left(\sum_i X_i^2\right)\left(\sum_j X_j\right)^2\right] \end{aligned} \tag{2.15}$$

The above expression can be simplified by using the fact that the samples are independant and identically distributed (iid). A general expression can be found in terms of n. This can be done by expanding each of the terms, and combining like terms. The following relations hold for all i:

$$E\left[X_i\right] = \mu \tag{2.16}$$

$$E\left[X_i^2\right] = \theta + \mu^2 \tag{2.17}$$

$$E\left[X_i^3\right] = 3\mu\theta + \mu^3 \tag{2.18}$$

$$E\left[X_i^4\right] = 3\theta^2 + 6\theta\mu^2 + \mu^4 \tag{2.19}$$

Since these relations hold for all i, the expression for the error can be simplified by combining like terms. The expectation for each of the terms in the expression for the error is given below:

$$E\left[\left(\sum_i X_i^2\right)^2\right] = E\left[nX_1^4 + 2\binom{n}{2}X_1^2X_2^2\right] \tag{2.20}$$

$$\text{for } n = 2 \qquad E\left[\left(\sum_i X_i\right)^4\right] = E\left[nX_1^4 + 6\binom{n}{2}X_1^2X_2^2 + 8\binom{n}{2}X_1^3X_2\right] \tag{2.21}$$

for $n = 3$ $$E\left[\left(\sum_i X_i\right)^4\right] = E\left[nX_1^4 + 6\binom{n}{2}X_1^2X_2^2 + 12n\binom{n-1}{2}X_1^2X_2X_3 + \right.$$
$$\left. +8\binom{n}{2}X_1^3X_2\right] \tag{2.22}$$

for $n \geq 4$ $$E\left[\left(\sum_i X_i\right)^4\right] = E\left[nX_1^4 + 6\binom{n}{2}X_1^2X_2^2 + 12n\binom{n-1}{2}X_1^2X_2X_3\right.$$
$$\left. +4!\binom{n}{4}X_1X_2X_3X_4 + 8\binom{n}{2}X_1^3X_2\right] \tag{2.23}$$

$$E\left[\left(\sum_i X_i\right)^2\right] = E\left[nX_1^2 + 2\binom{n}{2}X_1X_2\right] \tag{2.24}$$
$$E\left[\sum_i X_i^2\right] = E\left[nX_1^2\right] \tag{2.25}$$
$$E\left[\sum_i X_i\right] = E\left[nX_1\right] \tag{2.26}$$

for $n = 2$ $$E\left[\left(\sum_i X_i^2\right)\left(\sum_j X_j\right)^2\right] = E\left[nX_1^4 + 2\binom{n}{2}X_1^2X_2^2 + \right.$$
$$\left. +4\binom{n}{2}X_1^3X_2\right] \tag{2.27}$$

for $n \geq 3$ $$E\left[\left(\sum_i X_i^2\right)\left(\sum_j X_j\right)^2\right] = E\left[nX_1^4 + 2\binom{n}{2}X_1^2X_2^2 + \right.$$
$$\left. +4\binom{n}{2}X_1^3X_2 + 2n\binom{n-1}{2}X_1^2X_2X_3\right] \tag{2.28}$$

Substituting the values of the expectation for each of the terms, the general expression for error is for $n \geq 4$:

$$\begin{aligned} ERROR = \theta^2 + \mu^2 + \left(\frac{1}{n} + \frac{1}{n^3} - \frac{2}{n^2}\right) E\left[X_1^4\right] \\ + \left(\frac{2\theta+1}{n} - 2\theta\right) E\left[X_1^2\right] \\ + \left(\frac{(n)!}{(n-2)!}\left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}\right)\right)\left(E\left[X_1^2\right]\right)^2 \\ + \left(\left(\frac{2\theta+1}{n^2}\right)\frac{(n)!}{(n-2)!}\right)\left(E\left[X_1\right]\right)^2 \\ + \left(\left(-\frac{2}{n^2} + \frac{6}{n^3}\right)\frac{(n-1)!}{(n-3)!}\right) E\left[X_1^2\right]\left(E\left[X_1\right]\right)^2 \\ + \left(\left(\frac{4}{n^4} - \frac{4}{n^3}\right)\frac{(n)!}{(n-2)!}\right) E\left[X_1^3\right] E\left[X_1\right] \\ + \frac{(n)!}{n^4(n-4)!}\left(E\left[X_1\right]\right)^4 \\ + (-2\mu) E\left[X_1\right] \end{aligned} \tag{2.29}$$

Substituting the values of the expectation for each of the terms, the general expression for error, in terms of the coefficients is given below:

$$ERROR = \frac{\theta}{n^2}(2\theta n - \theta + n) \qquad \text{for } n \geq 4 \tag{2.30}$$

B. Robust Estimator g(x)

1. Introduction

A robust estimator can be found by a Huber-type censoring, where the Maximum likelihood estimator is censored. The new estimator $g(X)$ is equal to the ML estimator over a range 'L', after which, the estimator is truncated to a constant value (also

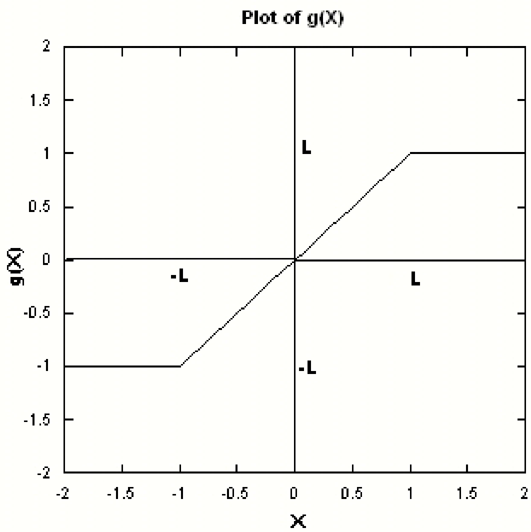

Fig. 1. Limiting function g(X)

equal to L). Mathematically, the function can be represented as:

$$g(X) = \left\{ \begin{array}{cc} -L, & -\infty < X < -L \\ X, & -L < X < L \\ L, & L < X < \infty \end{array} \right\} \tag{2.31}$$

For example, the robust estimator for X is truncated to $-L$ and L as can be seen in Figure 1.

2. Estimation of Mean, Variance and Mean Square Error

Replacing X with g(X), the mean, variance and error expressions are as given below :

$$\hat{\mu} = \frac{g\left(X_1\right) + g\left(X_2\right) ... + g\left(X_n\right)}{n} \tag{2.32}$$

$$\hat{\theta} = \frac{1}{n} \sum_i g^2\left(X_i\right) - \frac{1}{n^2} \left(\sum_i g\left(X_i\right)\right)^2 \tag{2.33}$$

$$Error = E\left[\left(\frac{1}{n}\sum_i g^2\left(X_i\right) - \frac{1}{n^2}\left(\sum_i g\left(X_i\right)\right)^2 - \theta\right)^2 + \left(\frac{1}{n}\sum_i g\left(X_i\right) - \mu\right)^2\right] \quad (2.34)$$

$$Error = \theta^2 + \mu^2 + \frac{1}{n^2}E\left[\left(\sum_i g^2\left(X_i\right)\right)^2\right] + \frac{1}{n^4}E\left[\left(\sum_i g\left(X_i\right)\right)^4\right] +$$
$$\left(\frac{2\theta}{n^2} + \frac{1}{n^2}\right)E\left[\left(\sum_i g\left(X_i\right)\right)^2\right] - \frac{2\theta}{n}E\left[\sum_i g^2\left(X_i\right)\right] - \frac{2\mu}{n}E\left[\sum_i g\left(X_i\right)\right]$$
$$-\frac{2}{n^3}E\left[\left(\sum_i g^2\left(X_i\right)\right)\left(\sum_j g\left(X_j\right)\right)^2\right] \quad (2.35)$$

Since the samples are i.i.d. , the following relations hold:

$$E\left[g\left(X_1\right)\right] = E\left[g\left(X_2\right)\right] = ... = E\left[g\left(X_n\right)\right]$$

and similarly other like terms can be combined. This leads to the general expression for the error, in terms of the expectations of functions of $g(X)$.

$$\begin{aligned} err = \theta^2 + \mu^2 + \left(\frac{1}{n} + \frac{1}{n^3} - \frac{2}{n^2}\right)E\left[g^4\left(X_1\right)\right] \\ + \left(\frac{2\theta+1}{n} - 2\theta\right)E\left[g^2\left(X_1\right)\right] \\ + \left(\frac{(n)!}{(n-2)!}\left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4}\right)\right)E\left[g^2\left(X_1\right)g^2\left(X_2\right)\right] \\ + \left(\left(\frac{2\theta+1}{n^2}\right)\frac{(n)!}{(n-2)!}\right)E\left[g\left(X_1\right)g\left(X_2\right)\right] \\ + \left(\left(-\frac{2}{n^2} + \frac{6}{n^3}\right)\frac{(n-1)!}{(n-3)!}\right)E\left[g^2\left(X_1\right)g\left(X_2\right)g\left(X_3\right)\right] \\ + \left(\left(\frac{4}{n^4} - \frac{4}{n^3}\right)\frac{(n)!}{(n-2)!}\right)E\left[g^3\left(X_1\right)g\left(X_2\right)\right] \\ + \frac{1}{n^4}\frac{(n)!}{(n-4)!}E\left[g\left(X_1\right)g\left(X_2\right)g\left(X_3\right)g\left(X_4\right)\right] \\ + (-2\mu)E\left[g\left(X_1\right)\right] \end{aligned} \quad (2.36)$$

Each of the 8 expectations can then be broken down into 3 integrals. The integrals are evaluated in Matlab, and the final answer to each of the expectation expressions is given in terms of the erf function, which is defined in Matlab as twice the integral of the gaussian distribution function with zero mean and variance = 0.5.

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{2.37}$$

Hence, $erf(\infty) = 1$ and $erf(-\infty) = -1$.

$$f(X) = \frac{1}{\sqrt{2\pi\theta}} e^{\frac{-(X-\mu)^2}{2\theta}} \tag{2.38}$$

$$\begin{aligned} E\left[g\left(X_1\right)\right] &= \int_{-\infty}^{-L} g\left(X_1\right) f(X_1) dX_1 + \int_{-L}^{L} g\left(X_1\right) f(X_1) dX_1 + \int_{L}^{\infty} g\left(X_1\right) f(X_1) dX_1 \\ &= \int_{-\infty}^{-L} \left(-L\right) f(X_1) dX_1 + \int_{-L}^{L} X_1 f(X_1) dX_1 + \int_{L}^{\infty} L f(X_1) dX_1 \\ &= \left(\frac{\mu}{2} + \frac{L}{2}\right) erf\left(\frac{L+\mu}{\sqrt{2\theta}}\right) + \left(\frac{\mu}{2} - \frac{L}{2}\right) erf\left(\frac{L-\mu}{\sqrt{2\theta}}\right) + \frac{\theta}{\sqrt{2\pi\theta}} \left[e^{\frac{-(L+\mu)^2}{2\theta}} - e^{\frac{-(L-\mu)^2}{2\theta}}\right] \end{aligned} \tag{2.39}$$

$$\begin{aligned} E\left[g^2\left(X_1\right)\right] &= \int_{-\infty}^{-L} g^2\left(X_1\right) f(X_1) dX_1 + \int_{-L}^{L} g^2\left(X_1\right) f(X_1) dX_1 + \int_{L}^{\infty} g^2\left(X_1\right) f(X_1) dX_1 \\ &= \int_{-\infty}^{-L} L^2 f(X_1) dX_1 + \int_{-L}^{L} X_1^2 f(X_1) dX_1 + \int_{L}^{\infty} L^2 f(X_1) dX_1 \\ &= L^2 + \left(\frac{\mu^2+\theta}{2} - \frac{L^2}{2}\right)\left[erf\left(\frac{L+\mu}{\sqrt{2\theta}}\right) + erf\left(\frac{L-\mu}{\sqrt{2\theta}}\right)\right] - \\ &\quad -\sqrt{\frac{\theta}{2\pi}}\left[\left(L-\mu\right) e^{\frac{-(L+\mu)^2}{2\theta}} + \left(L+\mu\right) e^{\frac{-(L-\mu)^2}{2\theta}}\right] \end{aligned} \tag{2.40}$$

$$E\left[g^3\left(X_1\right)\right] = \int_{-\infty}^{-L} g^3\left(X_1\right) f(X_1) dX_1 + \int_{-L}^{L} g^3\left(X_1\right) f(X_1) dX_1 + \int_{L}^{\infty} g^3\left(X_1\right) f(X_1) dX_1$$

$$E\left[g^3\left(X_1\right)\right] = \left(\frac{3\mu\theta + \mu^3}{2}\right)\left(I_1 + I_2\right) + \frac{L^3}{2}\left(I_1 - I_2\right) +$$

$$+\sqrt{\frac{\theta}{2\pi}}\left[\left(L^2 - \mu L + \mu^2 + 2\theta\right) E_1 + \left(L^2 + -\mu L + \mu^2 + 2\theta\right) E_2\right] \tag{2.41}$$

$$E\left[g^4\left(X_1\right)\right] = \int_{-\infty}^{-L} \left(-L^3\right) f(X_1) dX_1 + \int_{-L}^{L} X_1^3 f(X_1) dX_1 + \int_{L}^{\infty} L^3 f(X_1) dX_1$$

$$= L^4 + \left(\frac{\mu^4 + 6\mu^2\theta + 3\theta^2}{2} - \frac{L^4}{2}\right)\left[erf\left(\frac{L+\mu}{\sqrt{2\theta}}\right) + erf\left(\frac{L-\mu}{\sqrt{2\theta}}\right)\right]$$

$$-\sqrt{\frac{\theta}{2\pi}}\left[\left(L^3 - \mu L^2 + \mu^2 L - \mu^2 - 5\theta^2 + 3\theta L\right) e^{\frac{-(L+\mu)^2}{2\theta}} +\right.$$

$$\left. + \left(L^3 + \mu L^2 + \mu^2 L + \mu^2 + 5\theta^2 + 3\theta L\right) e^{\frac{-(L-\mu)^2}{2\theta}}\right] \tag{2.42}$$

To simplify notation, let the following integrals be replaced by alphabets as given below:

$$I_1 = erf\left(\frac{L+\mu}{\sqrt{2\theta}}\right) \tag{2.43}$$

$$I_2 = erf\left(\frac{L-\mu}{\sqrt{2\theta}}\right) \tag{2.44}$$

$$E_1 = e^{\frac{-(L+\mu)^2}{2\theta}} \tag{2.45}$$

$$E_2 = e^{\frac{-(L-\mu)^2}{2\theta}} \tag{2.46}$$

The expectations of functions of the new estimator can now be simplified as follows:

$$E\left[g\left(X_1\right)\right] = \left(\frac{\mu}{2} + \frac{L}{2}\right) I_1 + \left(\frac{\mu}{2} - \frac{L}{2}\right) I_2 + \sqrt{\frac{\theta}{2\pi}}\left(E_1 + E_2\right) \tag{2.47}$$

$$E\left[g^2\left(X_1\right)\right] = L^2 + \left(\frac{\mu^2+\theta}{2} - \frac{L^2}{2}\right)\left(I_1+I_2\right) - \sqrt{\frac{\theta}{2\pi}}\left[\left(L-\mu\right)E_1 + \left(L+\mu\right)E_2\right] \tag{2.48}$$

$$\begin{aligned} E\left[g^3\left(X_1\right)\right] &= \left(\frac{3\mu\theta+\mu^3}{2}\right)\left(I_1+I_2\right) + \frac{L^3}{2}\left(I_1-I_2\right) + \\ &+\sqrt{\frac{\theta}{2\pi}}\left[\left(L^2-\mu L+\mu^2+2\theta\right)E_1 + \left(L^2-\mu L+\mu^2+2\theta\right)E_2\right] \end{aligned} \tag{2.49}$$

$$\begin{aligned} E\left[g^4\left(X_1\right)\right] &= L^4 + \left(\frac{\mu^4+6\mu^2\theta+3\theta^2}{2} - \frac{L^4}{2}\right)\left(I_1+I_2\right) \\ &-\sqrt{\frac{\theta}{2\pi}}\left[\left(L^3-\mu L^2+\mu^2 L-\mu^2-5\theta^2+3\theta L\right)E_1+\right. \\ &\left.+\left(L^3+\mu L^2+\mu^2 L+\mu^2+5\theta^2+3\theta L\right)E_2\right] \end{aligned} \tag{2.50}$$

The expression for error can be simplified as:

$$\begin{aligned} ERROR = \theta^2+\mu^2 &+ \left(\frac{1}{n}+\frac{1}{n^3}-\frac{2}{n^2}\right)E\left[g^4\left(X_1\right)\right] \\ &+\left(\frac{2\theta+1}{n}-2\theta\right)E\left[g^2\left(X_1\right)\right] \\ &+\left(\frac{(n)!}{(n-2)!}\left(\frac{1}{n^2}-\frac{2}{n^3}+\frac{3}{n^4}\right)\right)\left(E\left[g^2\left(X_1\right)\right]\right)^2 \\ &+\left(\left(\frac{2\theta+1}{n^2}\right)\frac{(n)!}{(n-2)!}\right)\left(E\left[g\left(X_1\right)\right]\right)^2 \\ &+\left(\left(-\frac{2}{n^2}+\frac{6}{n^3}\right)\frac{(n-1)!}{(n-3)!}\right)E\left[g^2\left(X_1\right)\right]\left(E\left[g\left(X_1\right)\right]\right)^2 \\ &+\left(\left(\frac{4}{n^4}-\frac{4}{n^3}\right)\frac{(n)!}{(n-2)!}\right)E\left[g^3\left(X_1\right)\right]E\left[g\left(X_1\right)\right] \\ &+\frac{1}{n^4}\frac{(n)!}{(n-4)!}\left(E\left[g\left(X_1\right)\right]\right)^4 \\ &+\left(-2\mu\right)E\left[g\left(X_1\right)\right] \end{aligned} \tag{2.51}$$

C. Error, Performance, Slope, Robustness

The mean square error for both the MLE and the Robust Estimator is a function of mean (μ) and (θ).

$$MeanSquareError = f(\mu; \theta) \tag{2.52}$$

In most practical problems, we have a constant signal to noise ratio. An approximation of signal-to-noise ratio (defined k) is

$$k = \frac{\mu^2}{\theta} \tag{2.53}$$

This thesis attempts to find the performance and robustness of the estimators on this constant signal-to-noise ratio manifold.
From (2.53), we can write

$$\mu = \sqrt{k\theta} \tag{2.54}$$

Also, we know that the maximum slope occurs as given by equation (1.1).

$$SLOPE^2 = \left(\frac{\partial P}{\partial \theta}\right)\left(\frac{1}{g_{11}}\right)\left(\frac{\partial P}{\partial \theta}\right)^T \tag{2.55}$$

where

$$g_{11} = 1 + \left(\frac{\partial \mu}{\partial \theta}\right)^2 \tag{2.56}$$

$$g_{11} = \frac{k + 4\theta}{4\theta} \tag{2.57}$$

and P is the performance surface of the estimator. Since $\frac{\partial P}{\partial \theta}$ is a scalar value, of one dimension, hence, the SLOPE value can be simplified as:

$$SLOPE^2 = \left(\frac{\partial P}{\partial \theta}\right)^2\left(\frac{1}{g_{11}}\right) \tag{2.58}$$

For the robust estimator, the error is given by (2.51). The slope is found in a similar manner, using (2.58). The slope is calculated using simulations in Matlab.

D. Cost Function J

The Mean Square Error and the Slope have been calculated in the previous sections, for the Robust Estimator. The question now arises, can the user decide how much emphasis he/she wants on the robustness and how much on the performance? It is possible that at the onset, maximum emphasis could be placed on robustness. Later, the emphasis can be shifted towards performance. This kind of quantitative robustness can be acheived by a composite cost function J. Performance, P is defined as the inverse of the mean square error. Also, robustness is defined as the inverse of the slope.

$$P = \frac{1}{\text{ERROR}} \tag{2.59}$$

$$R = \frac{1}{\text{SLOPE}} \tag{2.60}$$

The composite function J can be defined as

$$J = rR + (1-r)\,P \qquad 0 \le r \le 1 \tag{2.61}$$

A value of r=1, puts 100% emphasis on robustness, while r=0, puts 100% emphasis on perfofrmance.

1. Error and Slope for MLE

The general expression for Error is given by (2.30). This error is plotted (for $n = 4, 5, 6, \ldots, 20$). As can be seen in Figure 2, the error at any given variance is the highest for the least number of samples ($n = 4$) and decreases as the number of samples increases. This is consistent with our intuitive understanding as well. The error in this case was found to be independent of the signal to noise ratio, k as well. In Figure 3, the slopes for $k = 0.1$ are plotted against variance for various number of samples. We can see that for any given variance, the slope is the lowest for $n = 4$ and the highest for $n = 20$. This is again what would be expected because as the number of samples increase, the error decreases, but the robustness of our measurements decreases (hence the slope increases). Figures 4 and 5 show how the slope varies with the variance for different values of k and all have the same general trend as in Figure 3. Figure 6 shows a 3D plot of error which varies with both mean and variance, while Figure 7 is the performance curve, the inverse of error. As the variance increases, the error increases (and the performance decreases) while the robustness increases as shown in Figure 9 and the slope decreases, as shown in Figure 8.

2. Plots of the Robust Estimator, versus L

Figure 10 shows the error for the robust estimator for $n = 4$, $k = 10$ and three different variances. Error is plotted against the censoring height L. With high censoring (low values of L) we have really high error, which asymptotes to the MLE value of error as the censoring becomes negligible (high values of L). In this case too, the error is the highest for the highest variance. Similarly, Figures 11 and 12 show the error plots for different values of n and k. Both the plots show the same behavior as explained in Figure 10. Figures 13, 14 and 15 are the performance plots and they are the inverse plots of Figures 10, 11 and 12 respectively. We notice that there exists an optimum value of L, where the performance is higher than the MLE. Figures 16, 17 and 18 are the plots of the slope versus L. It can be seen that the slope is the lowest for very high censoring and it asymptotes to the MLE value as L tends to infinity. There exists a value of L for which the slope is the highest (most non-robust). Figures 19, 20 and 21 are the inverse of the previous 3 plots. We need to find an optimum L which maximises both performance and robustness for a given set of parameters.

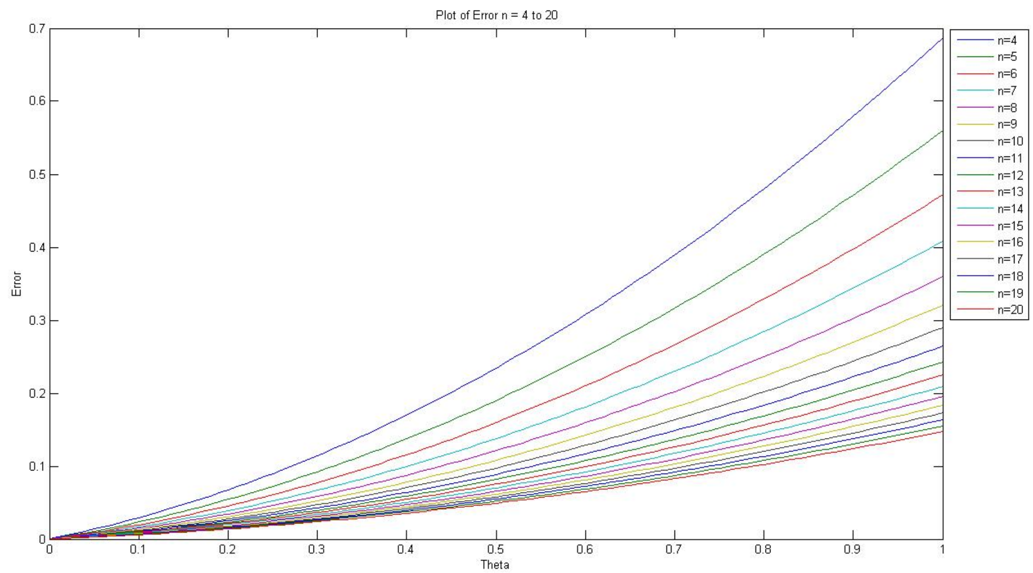

Fig. 2. Error vs variance, for the MLE,(is independent of k), for all n

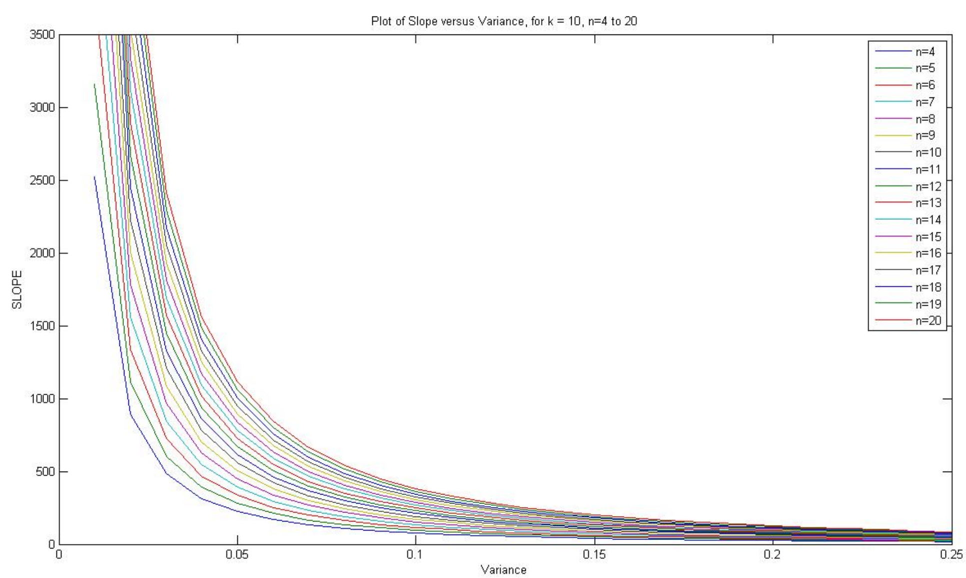

Fig. 3. Slope versus variance, for all n, k = 0.1

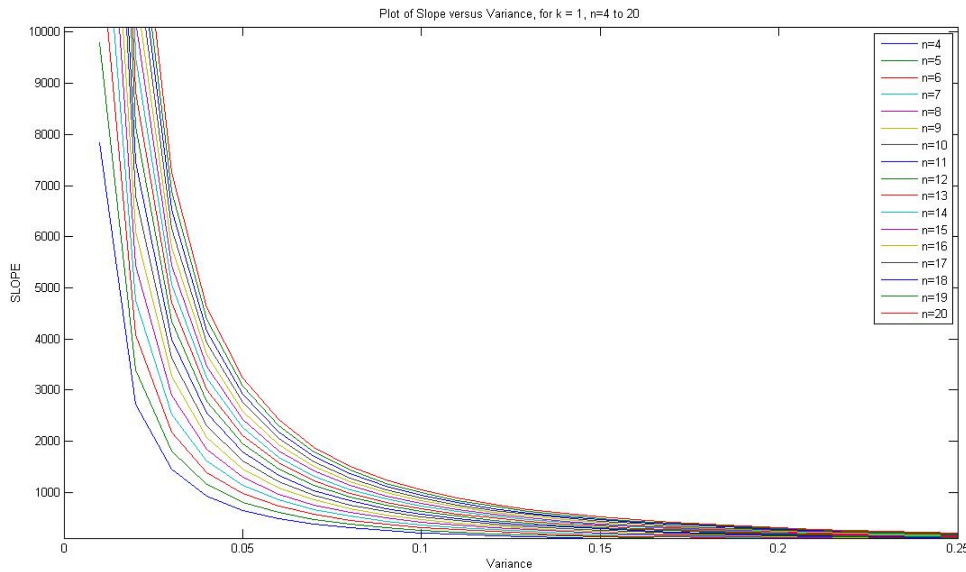

Fig. 4. Slope versus variance, for all n, k = 1

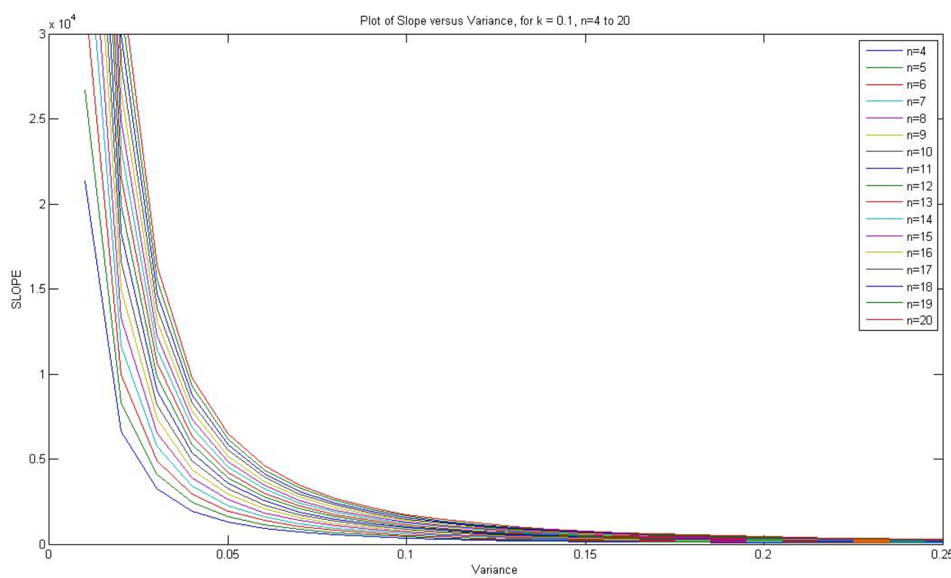

Fig. 5. Slope versus variance, for all n, k = 10

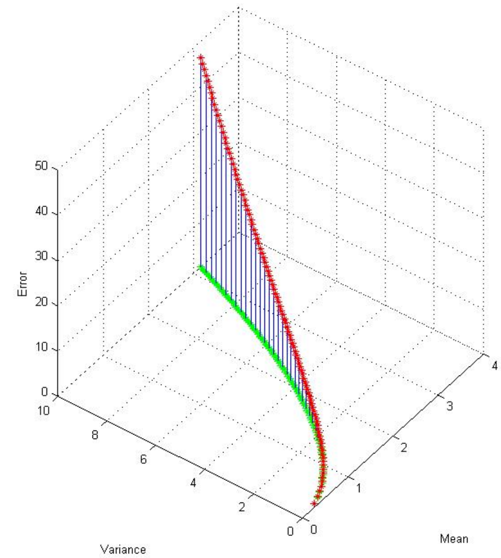

Fig. 6. Error (n=5), over the constant k manifold

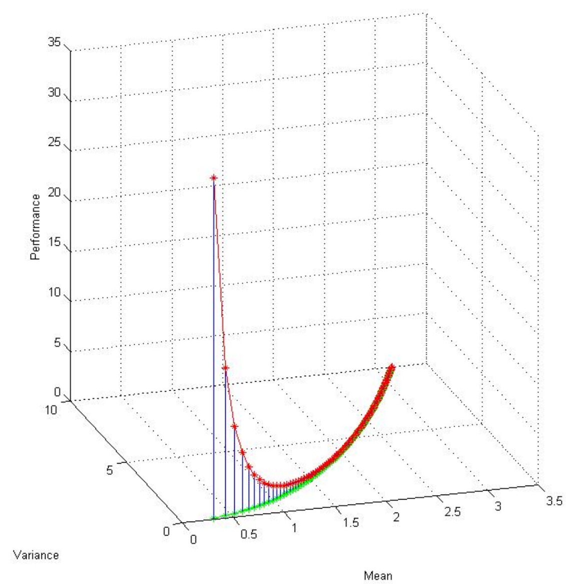

Fig. 7. Performance (n=5), over the constant k manifold

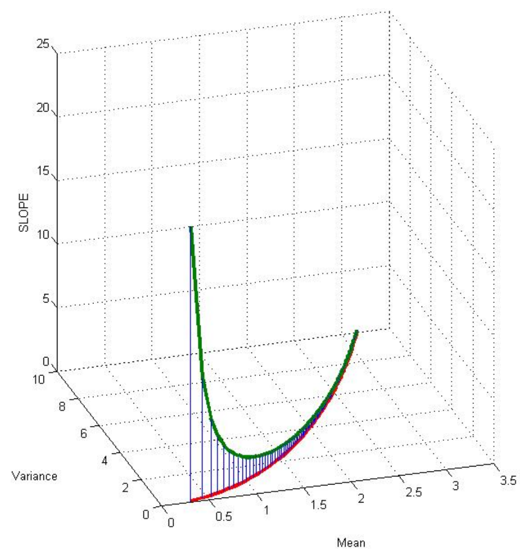

Fig. 8. Slope (n=5), over the constant k manifold

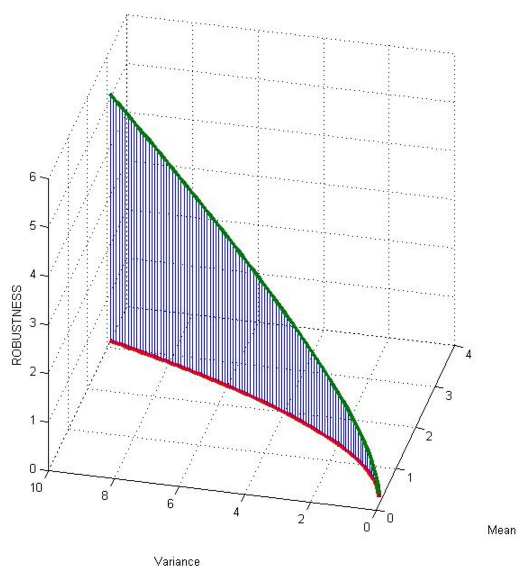

Fig. 9. Robustness (n=5), over the constant k manifold

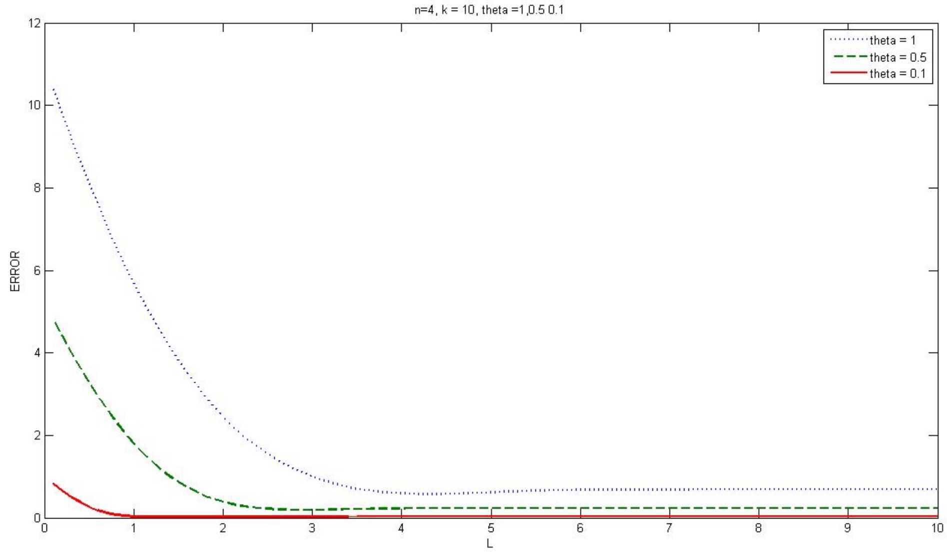

Fig. 10. Error versus L, n=4, k=10, theta=0.1, 0.5, 1

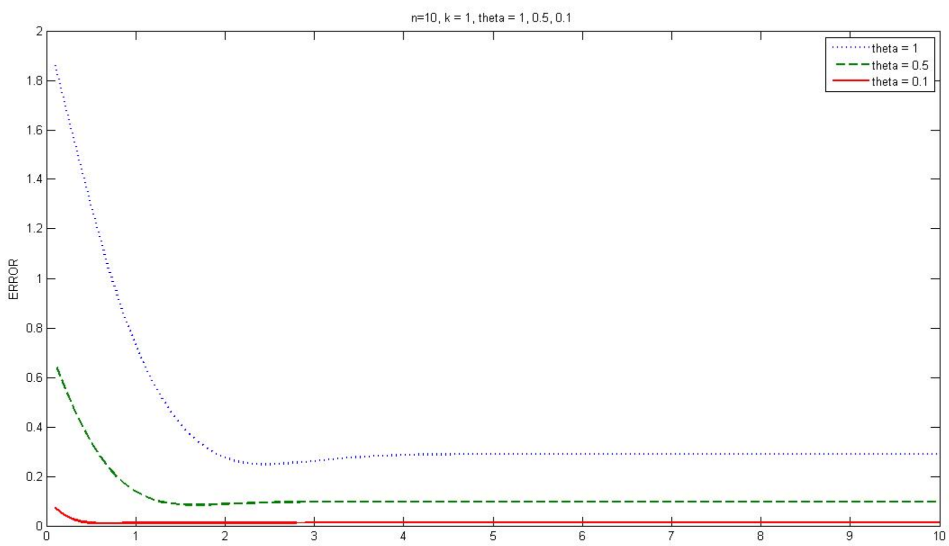

Fig. 11. Error versus L, n=10, k=1, theta=0.1, 0.5, 1

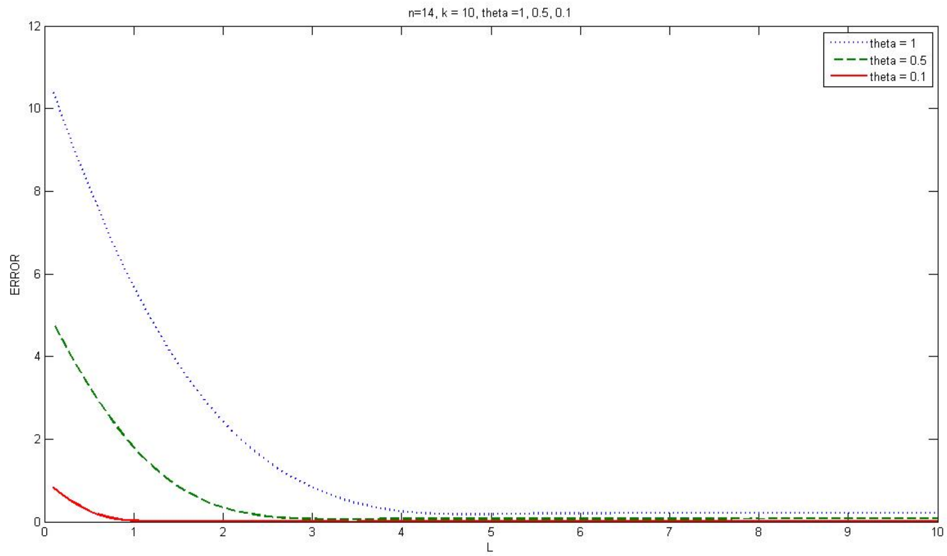

Fig. 12. Error versus L, n=14, k=10, theta=0.1, 0.5, 1

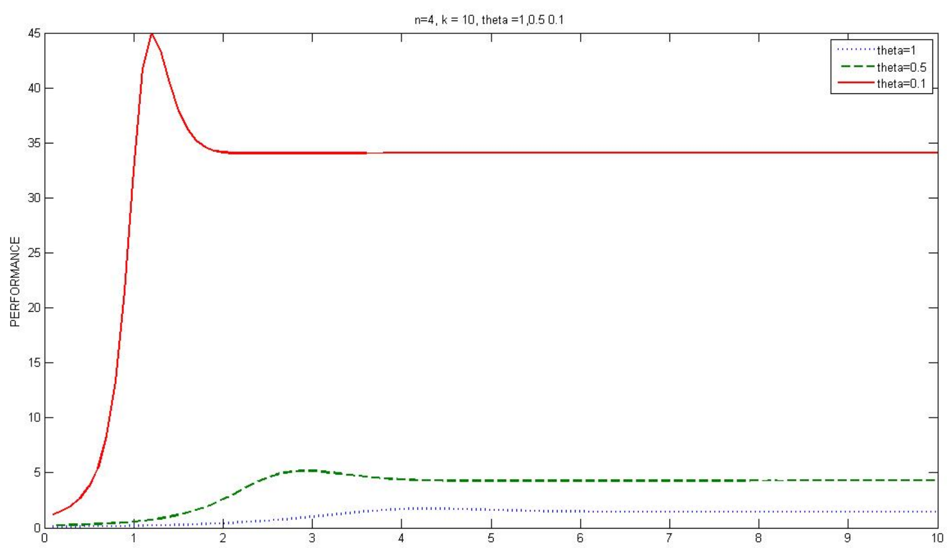

Fig. 13. Performance versus L, n=4, k=10, theta=0.1, 0.5, 1

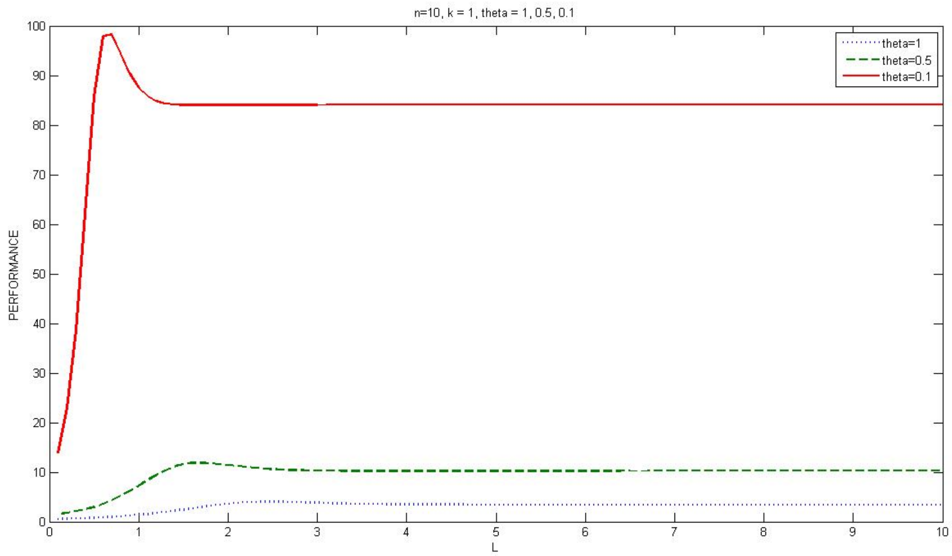

Fig. 14. Performance versus L, n=10, k=1, theta=0.1, 0.5, 1

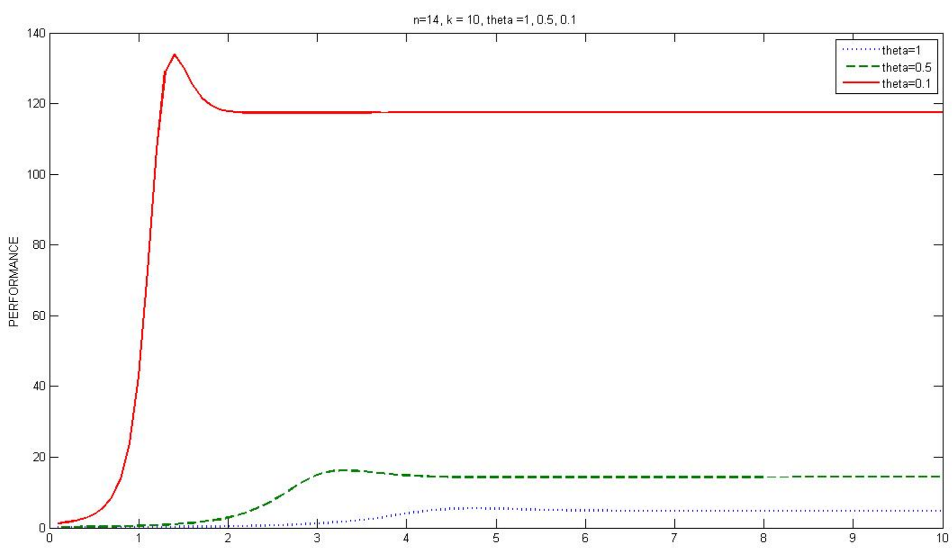

Fig. 15. Performance versus L, n=14, k=10, theta=0.1, 0.5, 1

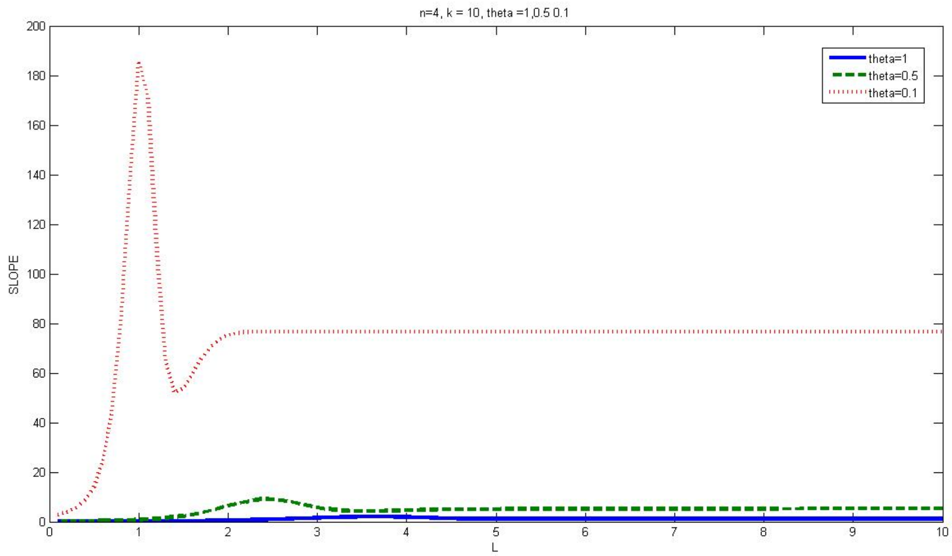

Fig. 16. Slope versus L, n=4, k=10, theta=0.1, 0.5, 1

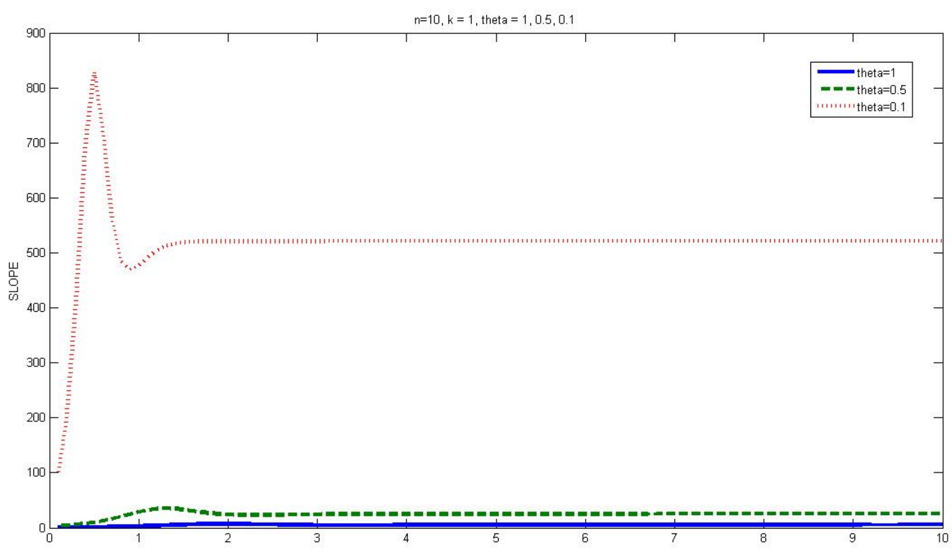

Fig. 17. Slope versus L, n=4, k=10, theta=0.1, 0.5, 1

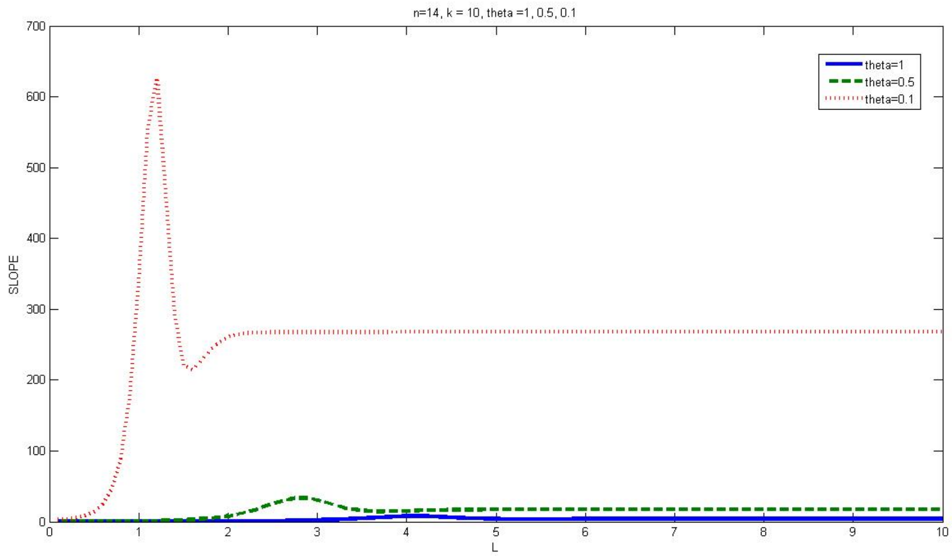

Fig. 18. Slope versus L, n=4, k=10, theta=0.1, 0.5, 1

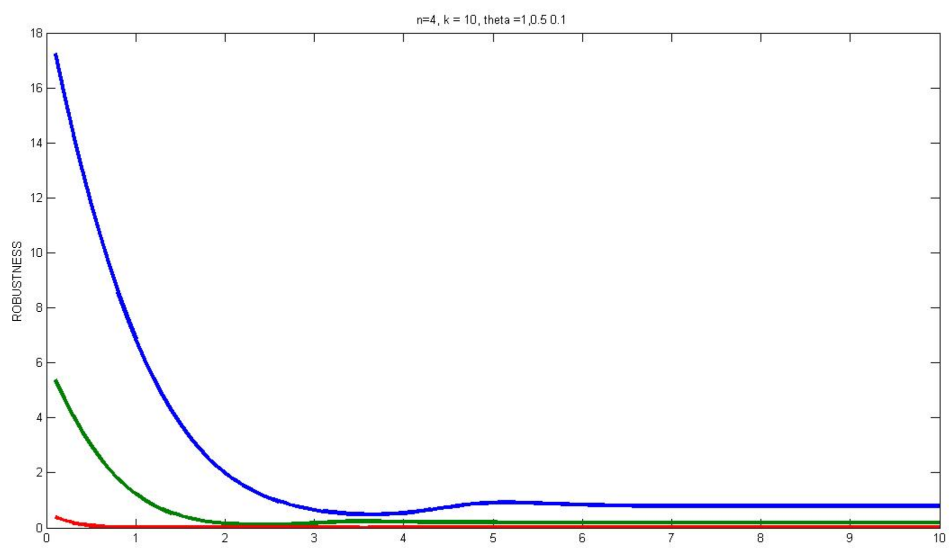

Fig. 19. Robustness versus L, n=4, k=10, theta=0.1, 0.5, 1

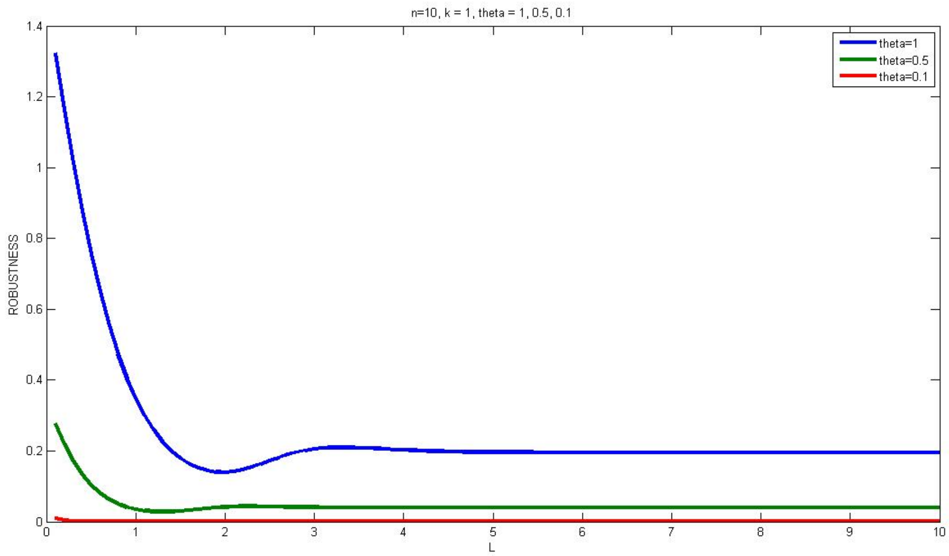

Fig. 20. Robustness versus L, n=10, k=1, theta=0.1, 0.5, 1

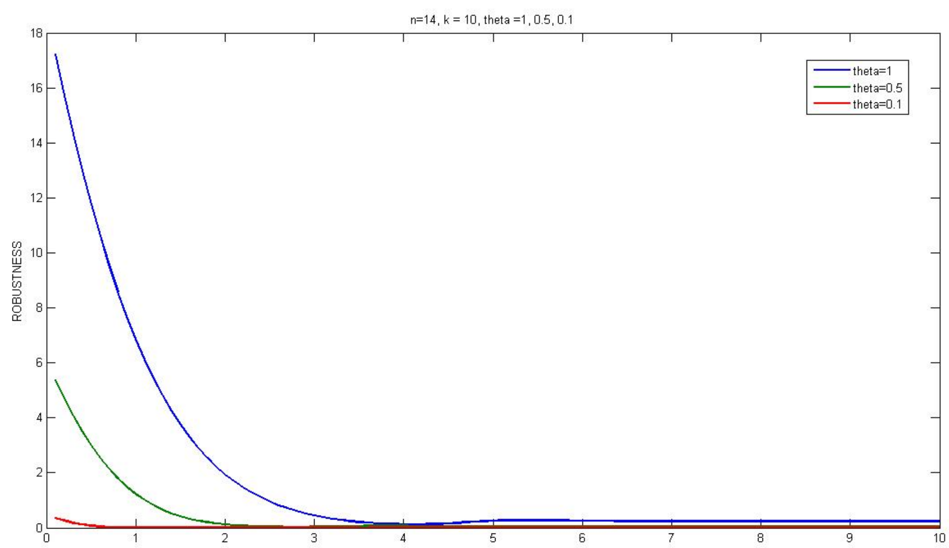

Fig. 21. Robustness versus L, n=14, k=10, theta=0.1, 0.5, 1

CHAPTER III

RESULTS

The plots of normalized performance and normalized robustness are plotted for different values of n, k, θ and L. The error is calculated using (2.51). Given a value of n, k, θ, perfomance and robustness are calculated for all L. Let $\vec{P}$ and $\vec{R}$ denote the values of performance and robustness for different L values.

$$P(normalized) = \frac{\vec{P} - min|\vec{P}|}{max|\vec{P}| - min|\vec{P}|} \tag{3.1}$$

$$R(normalized) = \frac{\vec{R} - min|\vec{R}|}{max|\vec{R}| - min|\vec{R}|} \tag{3.2}$$

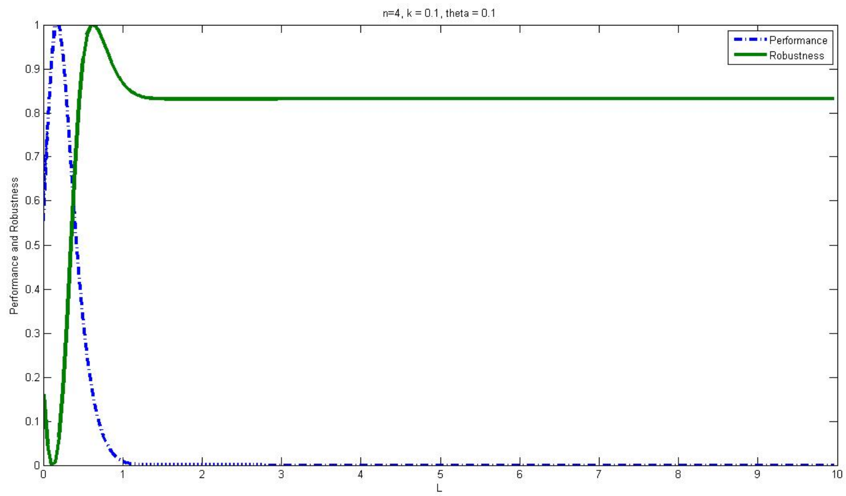

Fig. 22. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=0.1; $\theta = 0.1$

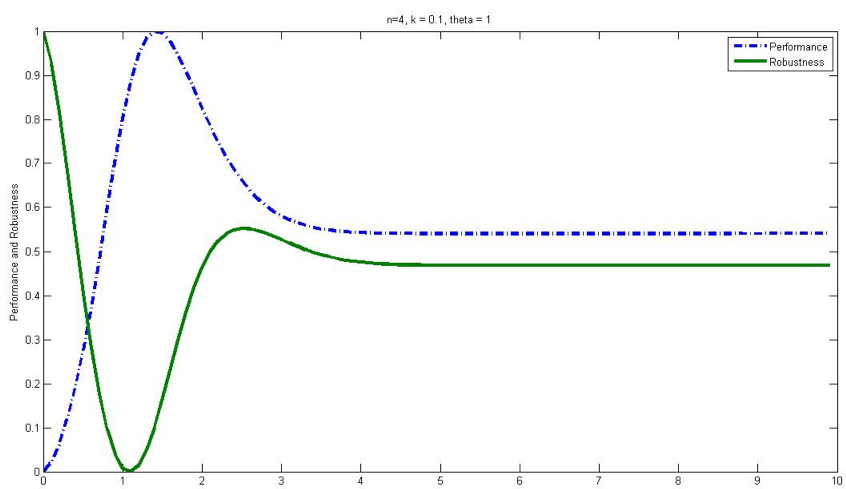

Fig. 23. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=0.1; $\theta = 1$

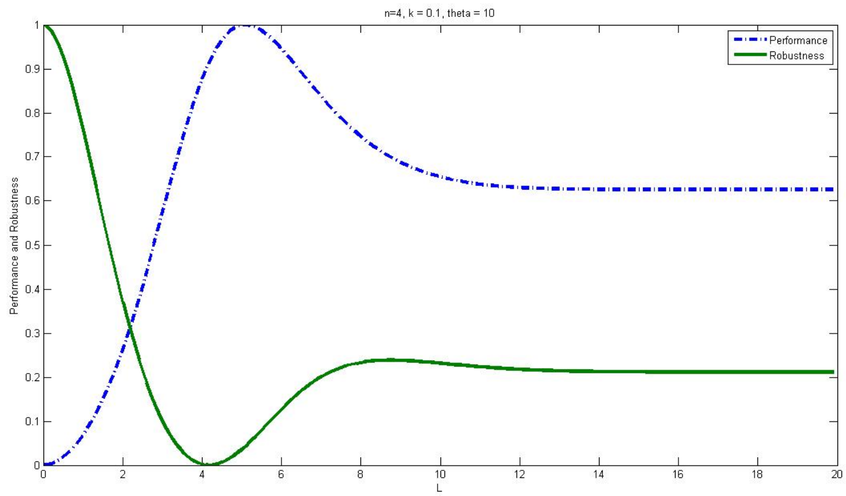

Fig. 24. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=0.1; $\theta = 10$

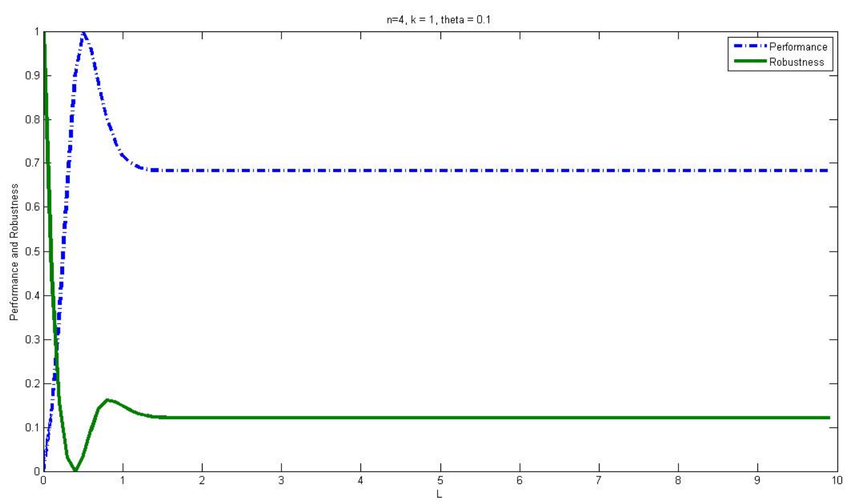

Fig. 25. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=1; $\theta = 0.1$

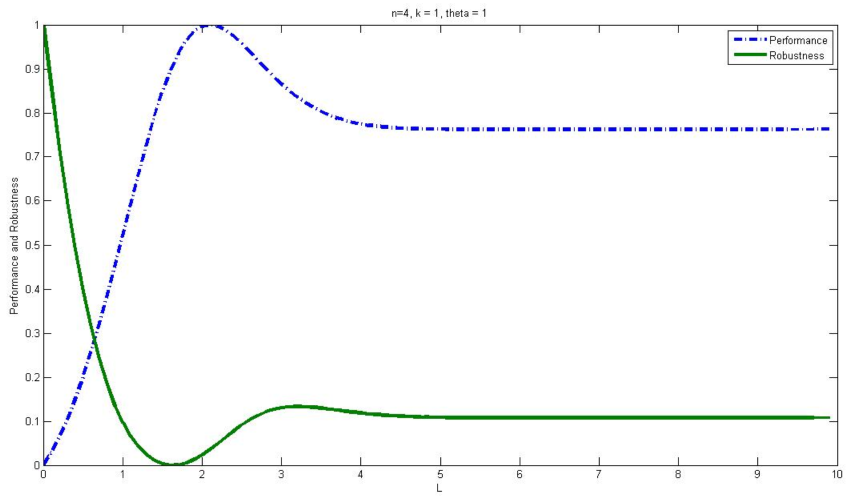

Fig. 26. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=1; $\theta = 1$

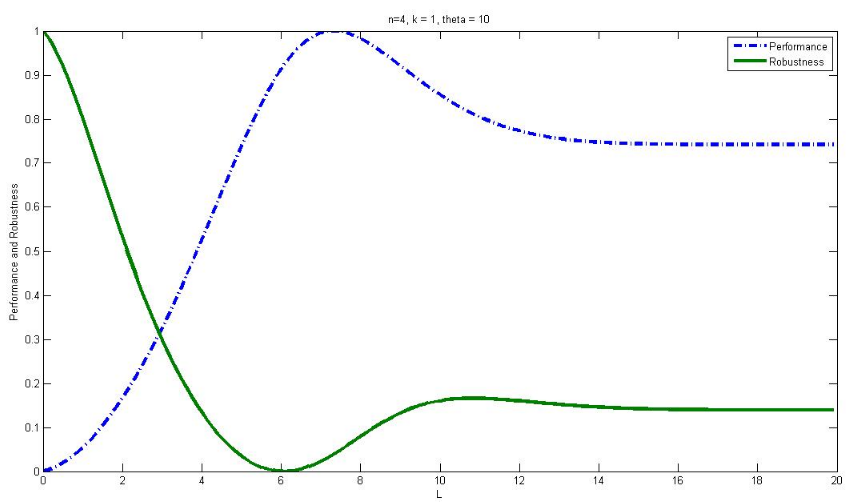

Fig. 27. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=1; $\theta = 10$

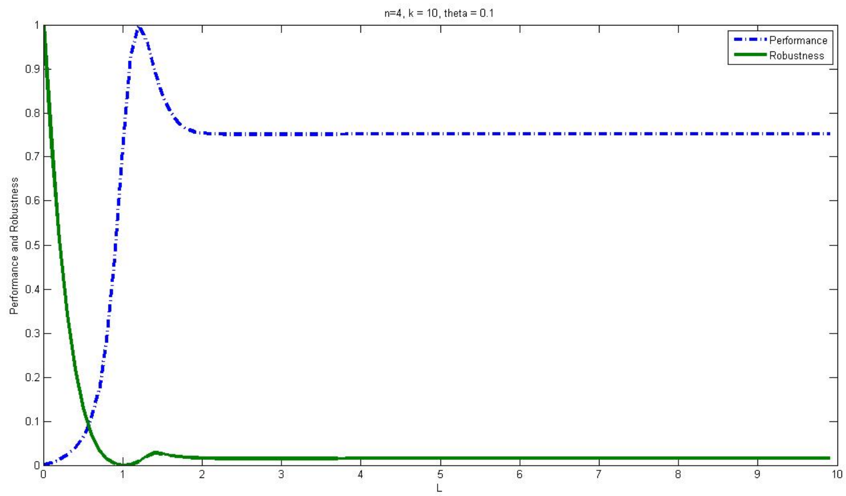

Fig. 28. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=10; $\theta = 0.1$

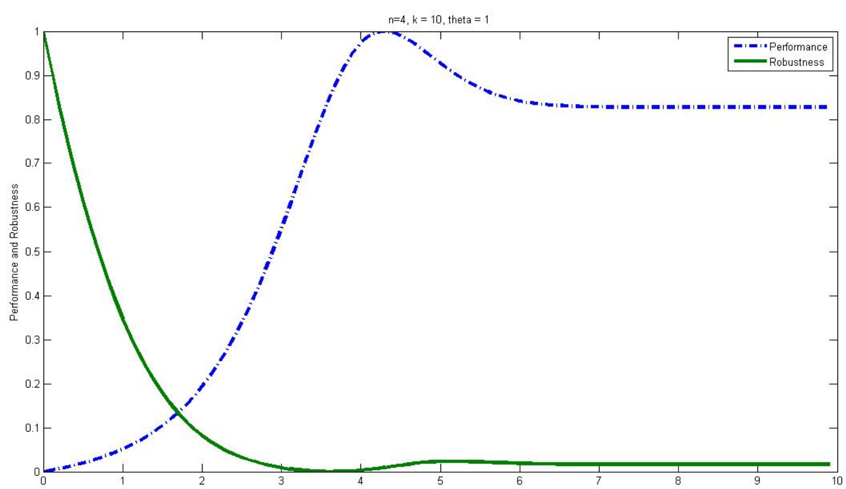

Fig. 29. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=10; $\theta = 1$

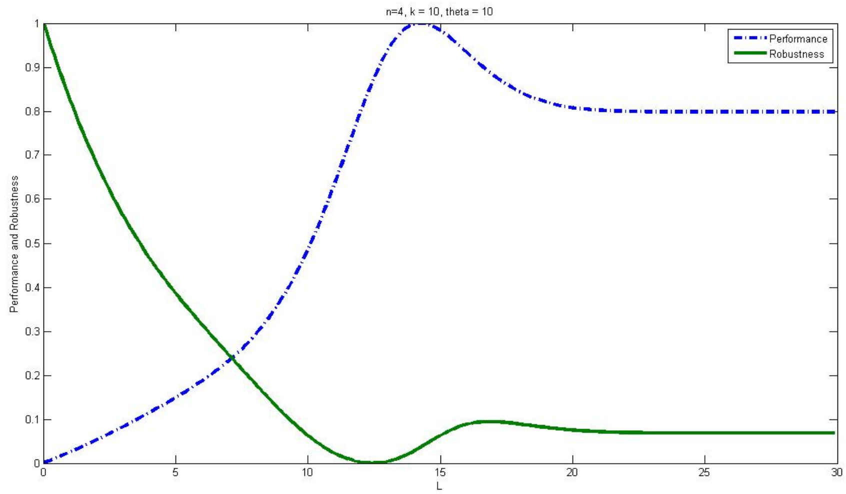

Fig. 30. Plot of Robustness and Performance versus censoring value 'L' for n=4; k=10; $\theta = 10$

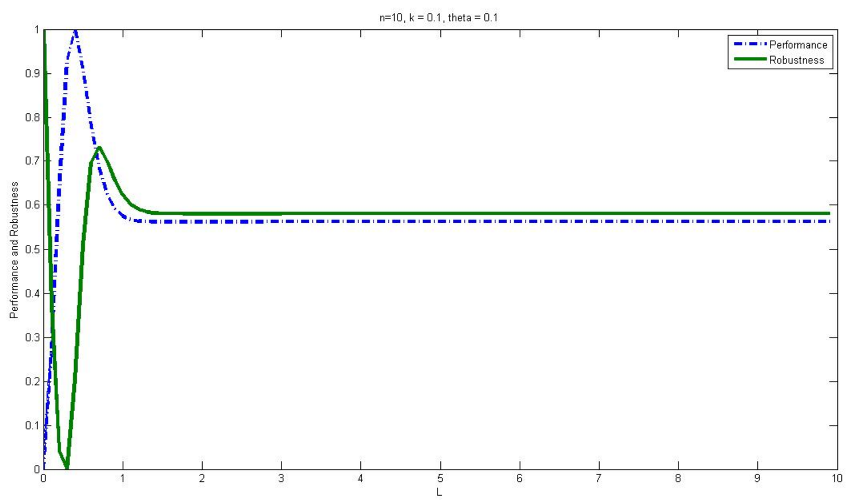

Fig. 31. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=0.1; $\theta = 0.1$

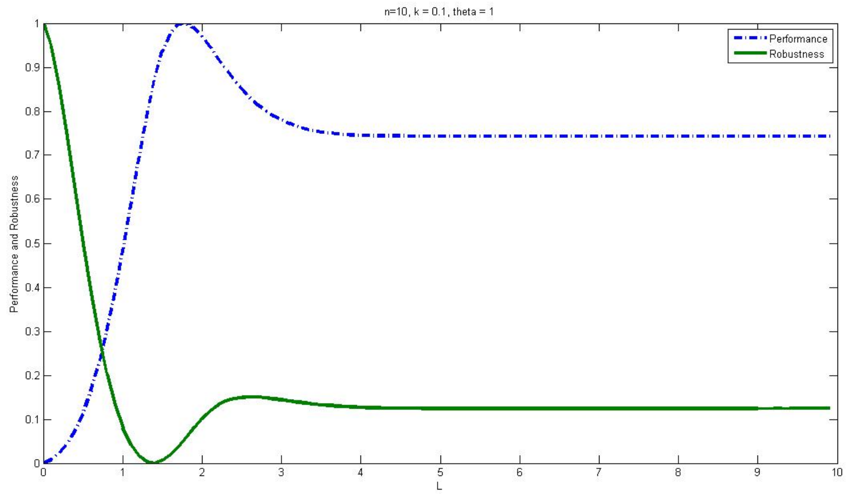

Fig. 32. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=0.1; $\theta = 1$

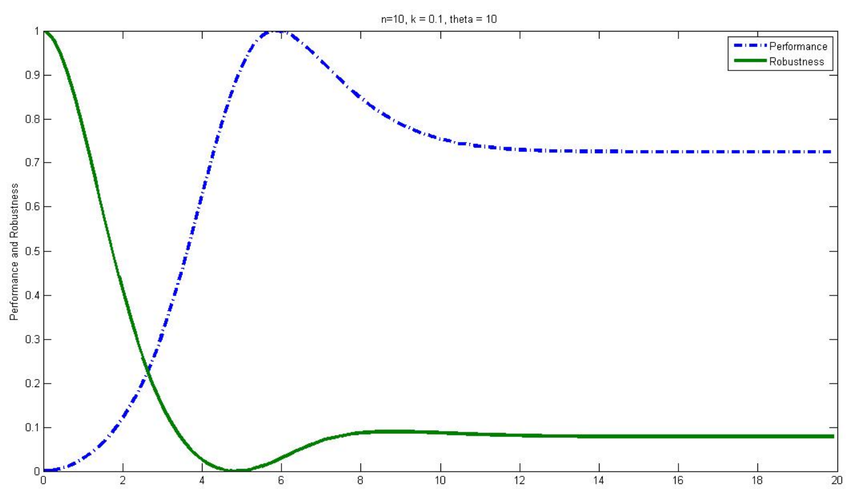

Fig. 33. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=0.1; $\theta = 10$

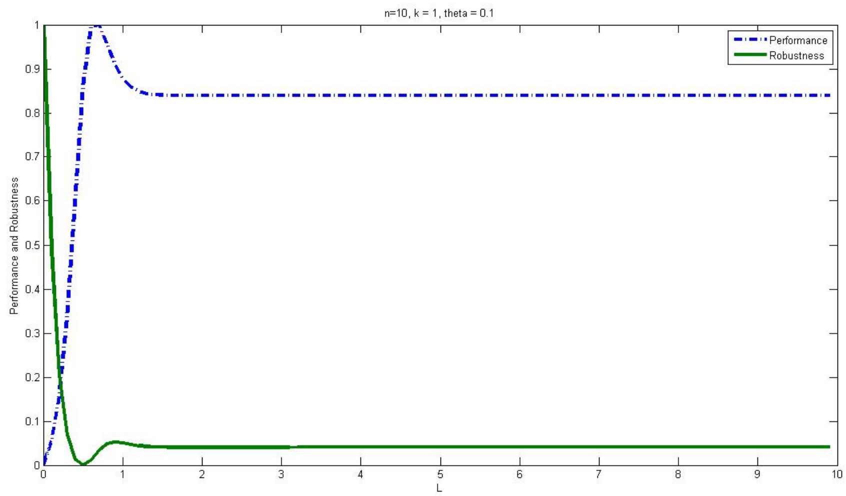

Fig. 34. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=1; $\theta = 0.1$

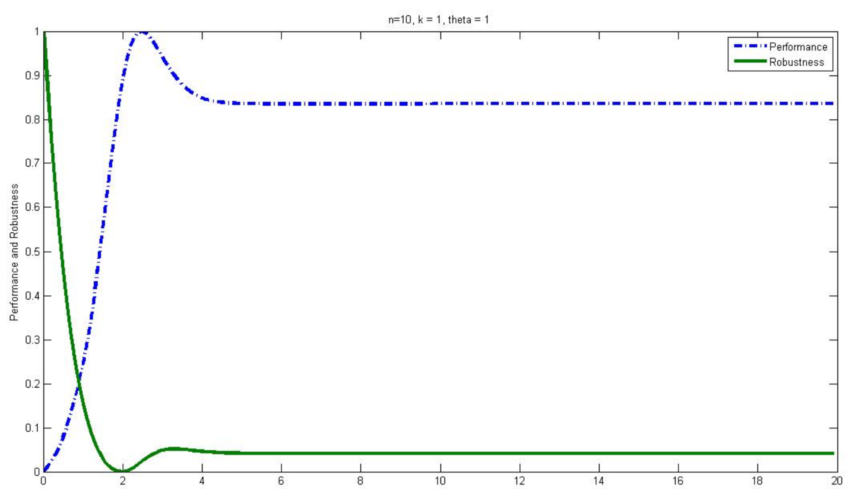

Fig. 35. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=1; $\theta = 1$

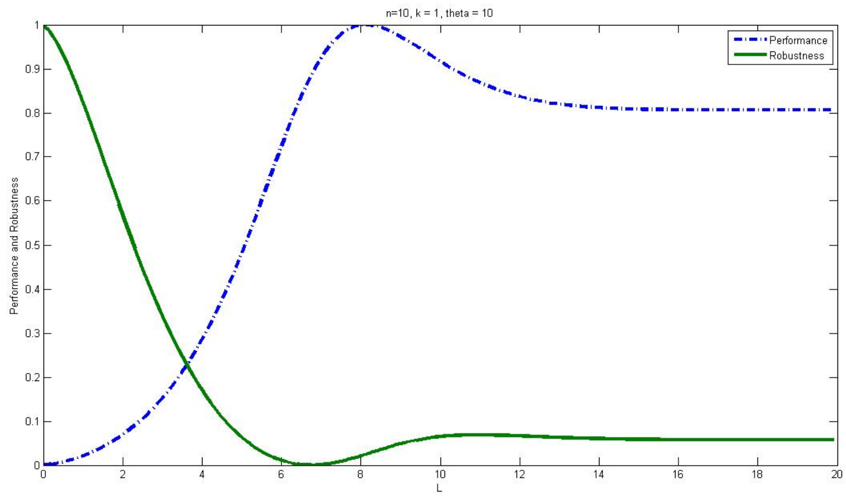

Fig. 36. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=1; $\theta = 10$

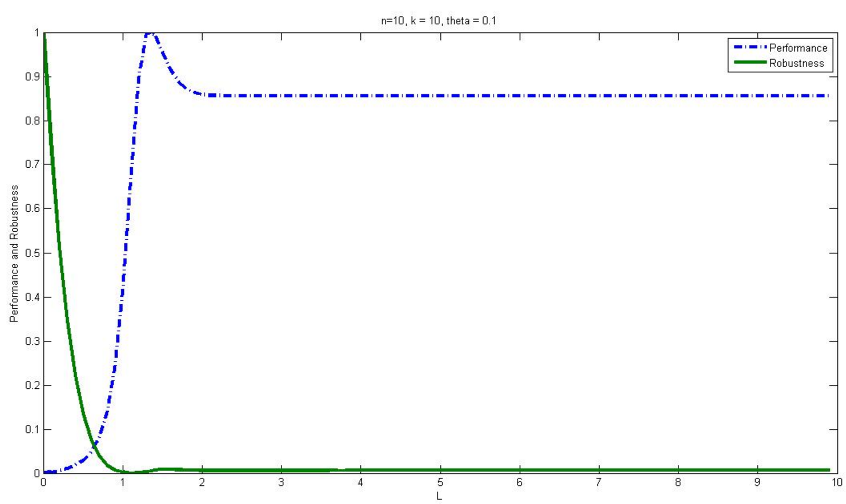

Fig. 37. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=10; $\theta = 0.1$

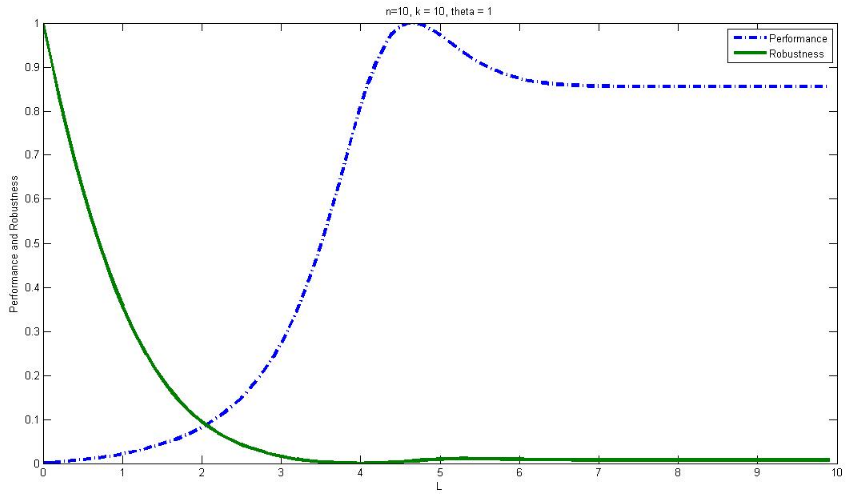

Fig. 38. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=10; $\theta = 1$

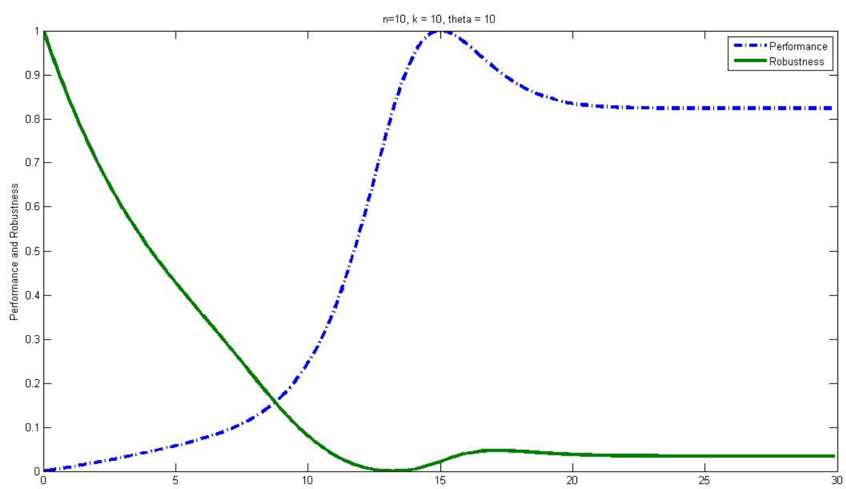

Fig. 39. Plot of Robustness and Performance versus censoring value 'L' for n=10; k=10; $\theta = 10$

We can conclude by observing the plots of normalized robustness and normalized performance as shown in Figures 22- 39 for different nominal values of θ,k that, as expected, the robustness is very high for high censoring, and is low as the censoring value tends to infinity (no censoring). The choice of emphasis between performance and robustness is actually greatly simplified, since there is an L value for every set of paramters, below which we have higher robustness and above which we have higher performance. The choice of L which maximises the cost function for different values of 'r' is given in the Tables I- XVIII which follow.

Table I. Choice of L for r values, $n = 4, k = 0.1, \theta = 0.1$

$n = 4, k = 0.1, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	0.151
0.1	0.201
0.2	0.201
0.3	0.201
0.4	0.201
0.5	0.501
0.6	0.551
0.7	0.551
0.8	0.601
0.9	0.601
1	0.601

Table II. Choice of L for r values, $n = 4, k = 0.1, \theta = 10$

$n = 4, k = 0.1, \theta = 10$	
r(emphasis)	L Value (maximum)
0	1.401
0.1	1.501
0.2	1.501
0.3	1.601
0.4	1.801
0.5	2.001
0.6	2.201
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table III. Choice of L for r values, $n = 4, k = 0.1, \theta = 1$

$n = 4, k = 0.1, \theta = 1$	
r(emphasis)	L Value (maximum)
0	5.101
0.1	5.101
0.2	5.201
0.3	5.301
0.4	5.501
0.5	5.901
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table IV. Choice of L for r values, $n = 4, k = 1, \theta = 0.1$

$n = 4, k = 1, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	0.501
0.1	0.501
0.2	0.501
0.3	0.501
0.4	0.501
0.5	0.601
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table V. Choice of L for r values, $n = 4, k = 1, \theta = 1$

$n = 4, k = 1, \theta = 1$	
r(emphasis)	L Value (maximum)
0	2.101
0.1	2.101
0.2	2.101
0.3	2.201
0.4	2.301
0.5	2.401
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table VI. Choice of L for r values, $n = 4, k = 1, \theta = 10$

$n = 4, k = 1, \theta = 10$	
r(emphasis)	L Value (maximum)
0	7.301
0.1	7.401
0.2	7.501
0.3	7.601
0.4	7.901
0.5	8.301
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table VII. Choice of L for r values, $n = 4, k = 10, \theta = 0.1$

$n = 4, k = 10, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	1.201
0.1	1.201
0.2	1.201
0.3	1.201
0.4	1.201
0.5	1.201
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table VIII. Choice of L for r values, $n = 4, k = 10, \theta = 1$

$n = 4, k = 10, \theta = 1$	
r(emphasis)	L Value (maximum)
0	4.301
0.1	4.301
0.2	4.301
0.3	4.301
0.4	4.301
0.5	4.301
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table IX. Choice of L for r values, $n = 4, k = 10, \theta = 10$

$n = 4, k = 10, \theta = 10$	
r(emphasis)	L Value (maximum)
0	14.201
0.1	14.301
0.2	14.401
0.3	14.501
0.4	14.601
0.5	14.901
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table X. Choice of L for r values, $n = 10, k = 0.1, \theta = 0.1$

$n = 10, k = 0.1, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	0.401
0.1	0.401
0.2	0.401
0.3	0.501
0.4	0.501
0.5	0.601
0.6	0.601
0.7	0.601
0.8	0.001
0.9	0.001
1	0.001

Table XI. Choice of L for r values, $n = 10, k = 0.1, \theta = 1$

$n = 10, k = 0.1, \theta = 1$	
r(emphasis)	L Value (maximum)
0	1.801
0.1	1.801
0.2	1.801
0.3	1.801
0.4	1.901
0.5	2.001
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XII. Choice of L for r values, $n = 10, k = 0.1, \theta = 10$

$n = 10, k = 0.1, \theta = 10$	
r(emphasis)	L Value (maximum)
0	5.901
0.1	5.901
0.2	5.901
0.3	6.001
0.4	6.001
0.5	6.201
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XIII. Choice of L for r values, $n = 10, k = 1, \theta = 0.1$

$n = 10, k = 1, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	0.701
0.1	0.701
0.2	0.701
0.3	0.701
0.4	0.701
0.5	0.701
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XIV. Choice of L for r values, $n = 10, k = 1, \theta = 1$

$n = 10, k = 1, \theta = 1$	
r(emphasis)	L Value (maximum)
0	2.501
0.1	2.501
0.2	2.501
0.3	2.501
0.4	2.501
0.5	2.601
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XV. Choice of L for r values, $n = 10, k = 1, \theta = 10$

$n = 10, k = 1, \theta = 10$	
r(emphasis)	L Value (maximum)
0	8.101
0.1	8.201
0.2	8.201
0.3	8.301
0.4	8.401
0.5	8.501
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XVI. Choice of L for r values, $n = 10, k = 10, \theta = 0.1$

$n = 10, k = 10, \theta = 0.1$	
r(emphasis)	L Value (maximum)
0	1.301
0.1	1.301
0.2	1.301
0.3	1.301
0.4	1.301
0.5	1.301
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XVII. Choice of L for r values, $n = 10, k = 10, \theta = 1$

$n = 10, k = 10, \theta = 1$	
r(emphasis)	L Value (maximum)
0	4.601
0.1	4.601
0.2	4.701
0.3	4.701
0.4	4.701
0.5	4.701
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

Table XVIII. Choice of L for r values, $n = 10, k = 10, \theta = 10$

$n = 10, k = 10, \theta = 10$	
r(emphasis)	L Value (maximum)
0	15.001
0.1	15.001
0.2	15.101
0.3	15.101
0.4	15.201
0.5	15.301
0.6	0.001
0.7	0.001
0.8	0.001
0.9	0.001
1	0.001

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VITA

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