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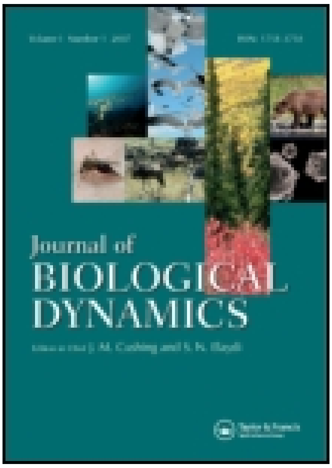

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Journal of Biological Dynamics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tjbd20

Finite difference approximations for a size-structured population model with distributed states in the recruitment

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Published online: 03 Jun 2014.

To cite this article: Azmy S. Ackleh, József Z. Farkas, Xinyu Li & Baoling Ma (2015) Finite difference approximations for a size-structured population model with distributed states in the recruitment, Journal of Biological Dynamics, 9:sup1, 2-31, DOI: 10.1080/17513758.2014.923117

To link to this article: http://dx.doi.org/10.1080/17513758.2014.923117

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Journal of Biological Dynamics, 2015
Vol. 9, Suppl. 1, 2–31, http://dx.doi.org/10.1080/17513758.2014.923117

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Finite difference approximations for a size-structured population model with distributed states in the recruitment

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(*Received 22 January 2014; accepted 6 May 2014*)

We consider a size-structured population model where individuals may be recruited into the population at different sizes. First- and second-order finite difference schemes are developed to approximate the solution of the model. The convergence of the approximations to a unique weak solution is proved. We then show that as the distribution of the new recruits become concentrated at the smallest size, the weak solution of the distributed states-at-birth model converges to the weak solution of the classical Gurtin–McCamy-type size-structured model in the weak* topology. Numerical simulations are provided to demonstrate the achievement of the desired accuracy of the two methods for smooth solutions as well as the superior performance of the second-order method in resolving solution-discontinuities. Finally, we provide an example where supercritical Hopf-bifurcation occurs in the limiting single state-at-birth model and we apply the second-order numerical scheme to show that such bifurcation also occurs in the distributed model.

Keywords: continuous structured population models; distributed states-at-birth; finite difference approximations; convergence theory; existence and uniqueness of solutions

1. Introduction

Continuous structured population models are frequently used to study fundamental questions of population dynamics, see e.g. [1,2,5–11,17,22]. These models assume that individuals are distinguished from one another by characteristics such as body length, height, weight, maturity level, and age, etc. These characteristics are often referred to as size in general. In the classical one-dimensional case, size-structured models are formulated in terms of a nonlocal hyperbolic partial differential equation (PDE) describing the dynamics of the density $u(x,t)$ together with an initial value $u_0(x)$ and a boundary condition at $x = x_0$. Here x is the structuring variable size. The boundary condition describes the inflow of newborns in the population. In most of these models, it is assumed that all the newborns have the same size x_0. It is clear in the case when x represents age and $x_0 = 0$. However, this assumption is not appropriate for many phenomena. For example, newborns of human beings and other mammals can have different body lengths and weights at

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birth. In cell populations, where large enough cells with different sizes will divide into two new daughter cells through mitosis and cytokinesis, there is no fixed size for the newly divided daughter cell when joining the population. Another example comes from modelling fragmentation and coagulation in systems of reacting polymers where aggregates of different sizes coalesce to form larger clusters or break apart into smaller ones [3,14,23]. In all of these examples, the recruitment cannot be accurately modelled by simply imposing one boundary condition at the x_0.

Population models with distributed states-at-birth thus were introduced and studied for example in [2,7,11,22]. Calsina and Saldana [7] considered a very general size-structured model where individuals may be recruited into the population at different sizes. The recruitment of new individuals is demonstrated in the PDE and modelled by a Lipschitz operator. They studied well-posedness of the model and established global existence and uniqueness of solutions utilizing results from the theory of nonlinear evolution equations. Tucker and Zimmerman [22] studied an age–size-structured population model which assumes that size-at-birth is distributed. They proved the existence of unique solutions to the model using a contraction mapping argument. The local asymptotic stability of equilibria is also discussed using results from the theory of strongly continuous semigroups of bounded linear operators. Distributed recruitment terms also appear in structured population models dealing with cell division [17] and in modelling reacting polymers by means of fragmentation models [16].

In this paper we consider the following nonlinear Gurtin–MacCamy-type model with a distributed recruitment term (see, e.g. [11]) and will refer to it as the distributed size-structured model (DSSM):

$$\begin{aligned}
&\frac{\partial}{\partial t}p(s,t)+\frac{\partial}{\partial s}(\gamma(s,Q(t))p(s,t))=-\mu(s,Q(t))p(s,t)\\
&\quad+\int_0^1\beta(s,y,Q(t))p(y,t)\,\mathrm{d}y,\quad s\in(0,1),t\in(0,T),\\
&\gamma(0,Q(t))p(0,t)=0,\quad t\in[0,T],\\
&p(s,0)=p^0(s),\quad s\in[0,1].
\end{aligned}\tag{1}$$

Here, $p(s,t)$ denotes the density of individuals of size s at time t. Therefore, $Q(t)=\int_0^1 p(s,t)\,\mathrm{d}s$ provides the total population at time t. The functions γ and μ represent the individual growth and mortality rate, respectively. It is assumed that individuals may be recruited into the population at different sizes with $\beta(s,y,Q)$ being the rate at which one individual of size y gives birth to an individual of size s when the total population is Q. There is no-inflow of individuals through the boundary $s=0$ since $p(0,t)=0$ for all $t\geq 0$.

Farkas *et al.* [11] analysed the asymptotic behaviour of solutions of model (1) using positive perturbation arguments and results from the spectral theory of positive semigroups. In [2], the question of the existence of non-trivial steady states is studied based on the reformulation of the problem (1) as an integral equation. However, to our knowledge, numerical schemes for computing approximate solutions of the distributed-rate model (1) have not been developed. Thus, in this paper we focus on the development of finite difference schemes to approximate the solution of model (1). Efficient schemes are essential for solving optimal control problems or parameter estimation problems governed by model (1) as such problems require solving the model numerous times before an optimal solution is obtained.

Furthermore, we establish a connection between the model (1) and the following classical size-structured model (CSSM) which will be referred to as CSSM for abbreviation:

$$\frac{\partial}{\partial t}p(s,t)+\frac{\partial}{\partial s}(\gamma(s,Q(t))p(s,t))=-\mu(s,Q(t))p(s,t),\quad s\in(0,1),\ t\in(0,T),$$

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$$\gamma(0, Q(t))p(0, t) = \int_0^1 \tilde{\beta}(y, Q(t))p(y, t)\,\mathrm{d}y, \quad t \in [0, T], \tag{2}$$
$$p(s, 0) = p^0(s), \quad s \in [0, 1].$$

Here $\tilde{\beta}$ is the fertility rate of individuals of size y at population level Q and the rest of the functions and parameters have similar interpretations as in model (1). We show that as the distribution of the new recruits become concentrated at the smallest size, the weak solution of Equation (1) converge in the weak* topology to the weak solution of Equation (2). To our knowledge, this is the first theoretical result that connects the two models.

This paper is organized as follows. Assumptions and notation are introduced in Section 2. In Section 3, we present a first-order explicit upwind (FOEU) scheme for solving the DSSM and prove its convergence to a unique weak solution with bounded total variation (TV). In Section 4, we present a second-order explicit finite difference scheme and prove its convergence. In Section 5, we establish the connection between DSSM and CSSM. Section 6 is devoted to numerical simulations and to the construction of a simple example in which supercritical Hopf-bifurcation occurs. We give concluding remarks in Section 7.

2. Assumptions and notation

Let $\mathbb{D}_1 = [0, 1] \times [0, \infty)$ and $\mathbb{D}_2 = [0, 1] \times [0, 1] \times [0, \infty)$. Let c be a sufficiently large positive constant. Throughout the paper we impose the following regularity conditions on the functions involved in the DSSM.

(H1) $\gamma(s, Q)$ is continuously differentiable with respect to s and Q, $\gamma_s(s, Q)$ and $\gamma_Q(s, Q)$ are Lipschitz continuous in s with Lipschitz constant c, uniformly in Q. Moreover, $0 < \gamma(s, Q) \leq c$ for $s \in [0, 1)$ and $\gamma(1, Q) = 0$.

(H2) $0 \leq \mu(s, Q) \leq c$, μ is Lipschitz continuous in s and Q with Lipschitz constant c.

(H3) $0 \leq \beta(s, y, Q) \leq c$, $\beta(s, y, Q)$ is Lipschitz continuous in Q with Lipschitz constant c, uniformly in s and y. Moreover, for every partition $\{s_i\}_{i=1}^N$ of $[0, 1]$, we have

$$\sup_{(y,Q)\in[0,1]\times[0,\infty)} \sum_{i=1}^{N} |\beta(s_i, y, Q) - \beta(s_{i-1}, y, Q)| \leq c.$$

(H4) $p^0 \in \mathrm{BV}([0, 1])$, where BV stands for the space of functions with bounded TV, and $p^0(s) \geq 0$.

Now we give the definition of a weak solution to the DSSM as follows.

Definition 2.1 *A function* $p \in \mathrm{BV}([0, 1] \times [0, T])$ *is called a weak solution of the DSSM model (1) if it satisfies*:

$$\begin{aligned} &\int_0^1 p(s, t)\phi(s, t)\,\mathrm{d}s - \int_0^1 p^0(s)\phi(s, 0)\,\mathrm{d}s \\ &\quad = \int_0^t \int_0^1 p(s, \tau)[\phi_\tau(s, \tau) + \gamma(s, Q(\tau))\phi_s(s, \tau) - \mu(s, Q(\tau))\phi(s, \tau)]\,\mathrm{d}s\,\mathrm{d}\tau] \\ &\qquad + \int_0^t \int_0^1 \int_0^1 \beta(s, y, Q(\tau))p(y, \tau)\phi(s, \tau)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\tau \end{aligned} \tag{3}$$

for every test function $\phi \in C^1([0, 1] \times [0, T])$ *and* $t \in [0, T]$.

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Suppose that the intervals $[0,1]$ and $[0,T]$ are divided into N and L subintervals, respectively. The following notation will be used throughout the paper: $\Delta s = 1/N$ and $\Delta t = T/L$. The discrete mesh points are given by $s_i = i\Delta s$, $t_k = k\Delta t$ for $i = 0,1,\ldots,N$, $k = 0,1,\ldots,L$. For ease of notation, we take a uniform mesh with constant sizes Δs and Δt. More general nonuniform meshes can be similarly considered. We shall denote by p_i^k and Q^k the finite difference approximation of $p(s_i,t_k)$ and $Q(t_k)$, respectively. We also let

$$\gamma_i^k = \gamma(s_i,Q^k), \quad \mu_i^k = \mu(s_i,Q^k), \quad \beta_{i,j}^k = \beta(s_i,y_j,Q^k).$$

We define the ℓ^1 and ℓ^∞ norms and the TV seminorm of the grid functions p^k by

$$\|p^k\|_1 = \sum_{i=1}^{N} |p_i^k|\Delta s, \quad \|p^k\|_\infty = \max_{0\le i\le N} |p_i^k|, \quad \mathrm{TV}(p^k) = \sum_{i=0}^{N-1} |p_{i+1}^k - p_i^k|,$$

and the finite difference operators by

$$\Delta_+ p_i^k = p_{i+1}^k - p_i^k, \quad 0 \leqslant i \leqslant N-1, \quad \Delta_- p_i^k = p_i^k - p_{i-1}^k, \quad 1 \leqslant i \leqslant N.$$

Throughout the discussion, we impose the following Courant–Friedrichs–Lewy condition concerning Δs and Δt:

(H5) $c(3\Delta t/2\Delta s) + c\Delta t \leqslant 1$.

3. A first-order upwind scheme

We first discretize model (1) using the following FOEU scheme:

$$\begin{aligned}
&\frac{p_i^{k+1} - p_i^k}{\Delta t} + \frac{\gamma_i^k p_i^k - \gamma_{i-1}^k p_{i-1}^k}{\Delta s} = -\mu_i^k p_i^k + \sum_{j=1}^{N} \beta_{i,j}^k p_j^k \Delta s, \quad 1 \le i \le N,\ 0 \le k \le L-1,\\
&\gamma_0^k p_0^k = 0, \quad 0 \le k \le L,\\
&p_i^0 = p^0(s_i), \quad 0 \le i \le N,
\end{aligned} \tag{4}$$

where the total population Q^k is discretized by a right-hand sum $Q^k = \sum_{i=1}^N p_i^k \Delta s$.

We can equivalently write the first part of Equation (4) as follows:

$$\begin{aligned}
p_i^{k+1} = \frac{\Delta t}{\Delta s}\gamma_{i-1}^k p_{i-1}^k + \left(1 - \frac{\Delta t}{\Delta s}\gamma_i^k - \mu_i^k \Delta t\right) p_i^k + \left(\sum_{j=1}^{N} \beta_{i,j}^k p_j^k \Delta s\right)\Delta t,&\\
1 \le i \le N,\ 0 \le k \le L-1.&
\end{aligned} \tag{5}$$

The boundary condition $\gamma(0,Q(t))p(0,t) = 0$ and assumption (H1) imply that $p_0^k = 0$ for $k \ge 0$. One can easily see that under assumptions (H1)–(H5), $p_i^{k+1} \ge 0$, for $i = 1,2,\ldots,N$ and $k = 0,1,\ldots,L-1$. Therefore, the scheme (4) has a unique nonnegative solution.

3.1. *Estimates for the first-order finite difference approximations*

In this section we use techniques similar to [4,21]. We begin by establishing an ℓ_1 bound on the approximations.

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LEMMA 3.1 *The following estimate holds*:

$$\|p^k\|_1 \le (1+c\Delta t)^k\|p^0\|_1 \le (1+c\Delta t)^L\|p^0\|_1 \le \exp(cT)\|p^0\|_1 \equiv M_1, \quad k=0,1,\dots,L.$$

Proof Multiplying Equation (5) by Δs and summing over $i=1,2,\dots,N$, we have

$$\sum_{i=1}^{N} p_i^{k+1}\Delta s = \sum_{i=1}^{N} p_i^k\Delta s - \Delta t\sum_{i=1}^{N}(\gamma_i^k p_i^k - \gamma_{i-1}^k p_{i-1}^k) - \sum_{i=1}^{N} p_i^k\mu_i^k\Delta s\Delta t + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\beta_{i,j}^k p_j^k\Delta s\right)\Delta s\Delta t.$$

Therefore by assumptions (H1)–(H3) and the second part of Equation (4)

$$\begin{aligned}\|p^{k+1}\|_1 &\le \|p^k\|_1 - \Delta t(\gamma_N^k p_N^k - \gamma_0^k p_0^k) + c\|p^k\|_1\Delta t\\ &= (1+c\Delta t)\|p^k\|_1,\end{aligned}$$

which then implies the estimate. ■

Note that $Q^k = \sum_{i=1}^{k} p_i^k\Delta s = \|p^k\|_1 \le M_1$. We now define $\mathbb{D}_3 = [0,1]\times[0,M_1]$.

LEMMA 3.2 *The following estimate holds*:

$$\|p^k\|_\infty \le (1+2c\Delta t)^k\|p^0\|_\infty \le (1+2c\Delta t)^L\|p^0\|_\infty \le \exp(2cT)\|p^0\|_\infty, \quad k=0,1,\dots,L.$$

Proof Since $p_0^k = 0$ for $k \ge 0$, $\|p^{k+1}\|_\infty$ is obtained at p_i^{k+1} for some $1 \le i \le N$. From Equation (5) and assumptions (H1), (H3) and (H5), we have

$$\begin{aligned}\|p^{k+1}\|_\infty &\le \frac{\Delta t}{\Delta s}\gamma_{i-1}^k\|p^k\|_\infty + \left(1 - \frac{\Delta t}{\Delta s}\gamma_i^k - \mu_i^k\Delta t\right)\|p^k\|_\infty + c\|p^k\|_\infty\Delta t\\ &\le \|p^k\|_\infty + \sup_{\mathbb{D}_3}|\gamma_s|\|p^k\|_\infty\Delta t + c\|p^k\|_\infty\Delta t\\ &\le (1+2c\Delta t)\|p^k\|_\infty.\end{aligned}$$

■

LEMMA 3.3 *There exists a positive constant* M_2 *such that* $\mathrm{TV}(p^k) \le M_2$, $k=0,1,\dots,L$.

Proof From the first part of Equation (4), we have

$$\begin{aligned}p_{i+1}^{k+1} - p_i^{k+1} &= (p_{i+1}^k - p_i^k) - \frac{\Delta t}{\Delta s}[(\gamma_{i+1}^k p_{i+1}^k - \gamma_i^k p_i^k) - (\gamma_i^k p_i^k - \gamma_{i-1}^k p_{i-1}^k)]\\ &\quad - \Delta t(\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k) + \sum_{j=1}^{N}(\beta_{i+1,j}^k - \beta_{i,j}^k)p_j^k\Delta s\Delta t.\end{aligned}$$

Simple calculations yield

$$\begin{aligned}(\gamma_{i+1}^k p_{i+1}^k - \gamma_i^k p_i^k) - (\gamma_i^k p_i^k - \gamma_{i-1}^k p_{i-1}^k) &= \gamma_{i+1}^k(p_{i+1}^k - p_i^k) + (\gamma_{i+1}^k - \gamma_i^k)p_i^k - \gamma_i^k(p_i^k - p_{i-1}^k)\\ &\quad - (\gamma_i^k - \gamma_{i-1}^k)p_i^k\\ &= \gamma_{i+1}^k(p_{i+1}^k - p_i^k) - \gamma_i^k(p_i^k - p_{i-1}^k) + (\gamma_i^k - \gamma_{i-1}^k)\\ &\quad (p_i^k - p_{i-1}^k) + [(\gamma_{i+1}^k - \gamma_i^k) - (\gamma_i^k - \gamma_{i-1}^k)]p_i^k.\end{aligned}$$

Therefore, for $1 \le i \le N-1$,

$$p_{i+1}^{k+1} - p_i^{k+1} = \left(1 - \frac{\Delta t}{\Delta s}\gamma_{i+1}^k\right)(p_{i+1}^k - p_i^k) + \frac{\Delta t}{\Delta s}\gamma_i^k(p_i^k - p_{i-1}^k) - \frac{\Delta t}{\Delta s}(\gamma_i^k - \gamma_{i-1}^k)(p_i^k - p_{i-1}^k)$$
$$- \frac{\Delta t}{\Delta s}[(\gamma_{i+1}^k - \gamma_i^k) - (\gamma_i^k - \gamma_{i-1}^k)]p_i^k - \Delta t(\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k)$$
$$+ \sum_{j=1}^{N}(\beta_{i+1,j}^k - \beta_{i,j}^k)p_j^k \Delta s \Delta t. \tag{6}$$

Summing Equation (6) over $i = 0, 1, \dots, N-1$ and applying assumptions (H1) and (H5) we arrive at

$$\mathrm{TV}(p^{k+1}) = |p_1^{k+1} - p_0^{k+1}| + \sum_{i=1}^{N-1}|p_{i+1}^{k+1} - p_i^{k+1}|$$
$$= p_1^{k+1} + \sum_{i=1}^{N-1}|p_{i+1}^k - p_i^k| - \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}(\gamma_{i+1}^k|p_{i+1}^k - p_i^k| - \gamma_i^k|p_i^k - p_{i-1}^k|)$$
$$+ \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}|\gamma_i^k - \gamma_{i-1}^k||p_i^k - p_{i-1}^k| + \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}|(\gamma_{i+1}^k - \gamma_i^k) - (\gamma_i^k - \gamma_{i-1}^k)|p_i^k$$
$$+ \Delta t\sum_{i=1}^{N-1}|\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k| + \sum_{i=1}^{N-1}\sum_{j=1}^{N}|\beta_{i+1,j}^k - \beta_{i,j}^k|p_j^k \Delta s \Delta t. \tag{7}$$

By Equation (5) and assumptions (H1)–(H3),

$$p_1^{k+1} = \frac{\Delta t}{\Delta s}\gamma_0^k p_0^k + \left(1 - \frac{\Delta t}{\Delta s}\gamma_1^k - \mu_1^k \Delta t\right)p_1^k + \left(\sum_{j=1}^{N}\beta_{1,j}^k p_j^k \Delta s\right)\Delta t$$
$$\le p_1^k - \frac{\Delta t}{\Delta s}\gamma_1^k p_1^k + c\|p^k\|_1 \Delta t. \tag{8}$$

It can be seen from assumption (H1) that

$$\sum_{i=1}^{N-1}\left(\gamma_{i+1}^k|p_{i+1}^k - p_i^k| - \gamma_i^k|p_i^k - p_{i-1}^k|\right) = \gamma_N^k|p_N^k - p_{N-1}^k| - \gamma_1^k|p_1^k - p_0^k| = -\gamma_1^k p_1^k \tag{9}$$

and

$$\frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}|(\gamma_{i+1}^k - \gamma_i^k) - (\gamma_i^k - \gamma_{i-1}^k)|p_i^k = \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}|\gamma_s(\hat{s}_{i+1}, Q^k) - \gamma_s(\hat{s}_i, Q^k)|p_i^k \Delta s$$
$$\le \Delta t\sum_{i=1}^{N-1}2cp_i^k \Delta s \le 2c\|p^k\|_1 \Delta t, \tag{10}$$

where $\hat{s}_i \in [s_{i-1}, s_i]$ and $\hat{s}_{i+1} \in [s_i, s_{i+1}]$.

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By assumption (H2),

$$\sum_{i=1}^{N-1} |\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k| \Delta t \leq \Delta t \sum_{i=1}^{N-1} |\mu_{i+1}^k - \mu_i^k| p_{i+1}^k + \sum_{i=1}^{N-1} \sup_{\mathbb{D}_3} \mu |p_{i+1}^k - p_i^k| \Delta t$$
$$\leq c\|p^k\|_1 \Delta t + c \sum_{i=1}^{N-1} |p_{i+1}^k - p_i^k| \Delta t. \tag{11}$$

By assumption (H3),

$$\sum_{i=1}^{N-1} \sum_{j=1}^{N} |\beta_{i+1,j}^k - \beta_{i,j}^k| p_j^k \Delta s \Delta t = \sum_{j=1}^{N} \left(\sum_{i=1}^{N-1} |\beta_{i+1,j}^k - \beta_{i,j}^k| \right) p_j^k \Delta s \Delta t = c\|p^k\|_1 \Delta t. \tag{12}$$

A combination of Equations (7)–(12) then yields

$$\mathrm{TV}(p^{k+1}) \leq p_1^k - \frac{\Delta t}{\Delta s} \gamma_1^k p_1^k + c\|p^k\|_1 \Delta t + \sum_{i=1}^{N-1} |p_{i+1}^k - p_i^k| + \frac{\Delta t}{\Delta s} \gamma_1^k p_1^k$$
$$+ \frac{\Delta t}{\Delta s} |\gamma_s| \Delta s \sum_{i=1}^{N-1} |p_i^k - p_{i-1}^k| + c\|p^k\|_1 \Delta t + c\|p^k\|_1 \Delta t$$
$$+ c \sum_{i=1}^{N-1} |p_{i+1}^k - p_i^k| \Delta t + 2c\|p^k\|_1 \Delta t. \tag{13}$$

Therefore, from assumption (H1), Lemmas 3.1 and 3.2, there exist positive constants c_1 and c_2 such that

$$\mathrm{TV}(p^{k+1}) \leq (1 + c_1 \Delta t)\, \mathrm{TV}(p^k) + c_2 \Delta t,$$

which leads to the desired result. ■

LEMMA 3.4 *There exists a positive constant M_3 such that for any $q_1 > q_2 > 0$ the following estimate holds*:

$$\sum_{i=1}^{N} \left| \frac{p_i^{q_1} - p_i^{q_2}}{\Delta t} \right| \Delta s \leq M_3 (q_1 - q_2).$$

Proof By Equation (5) and assumptions (H1)–(H3), we have

$$\sum_{i=1}^{N} \left| \frac{p_i^{k+1} - p_i^k}{\Delta t} \right| \Delta s = \sum_{i=1}^{N} \left| \gamma_{i-1}^k p_{i-1}^k - \gamma_i^k p_i^k - \mu_i^k p_i^k \Delta s + \sum_{j=1}^{N} \beta_{i,j}^k p_j^k \Delta s \Delta s \right|$$
$$\leq \sum_{i=1}^{N} |\gamma_i^k - \gamma_{i-1}^k| p_{i-1}^k + \sum_{i=1}^{N} \gamma_i^k |p_i^k - p_{i-1}^k| + \sum_{i=1}^{N} \mu_i^k p_i^k \Delta s$$
$$+ \sum_{i=1}^{N} \sum_{i=j}^{N} \beta_{i,j}^k p_j^k \Delta s \Delta s$$
$$\leq c\, \mathrm{TV}(p^k) + 3c\|p^k\|_1.$$

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Thus, by Lemmas 3.1 and 3.3 there exists a positive constant M_3 such that

$$\sum_{i=1}^{N}\left|\frac{p_i^{k+1}-p_i^k}{\Delta t}\right|\Delta s \le M_3.$$

Therefore,

$$\sum_{i=1}^{N}\left|\frac{p_i^{q_1}-p_i^{q_2}}{\Delta t}\right|\Delta s \le \sum_{i=1}^{N}\sum_{k=q_2}^{q_1}\left|\frac{p_i^{k+1}-p_i^k}{\Delta t}\right|\Delta s \le M_3(q_1-q_2).$$

■

3.2. *Convergence of the first-order finite difference approximations to the unique weak solution*

Following similar notation as in [21] we define a set of functions $\{P_{\Delta s,\Delta t}\}$ by $\{P_{\Delta s,\Delta t}(s,t)\} = p_i^k$ for $s \in [s_{i-1}, s_i), t \in [t_{k-1}, t_k), i = 1, 2, \dots, N$, and $k = 1, 2, \dots, L$. Then by Lemmas 3.1–3.4, the set of functions $\{P_{\Delta s,\Delta t}\}$ is compact in the topology of $\mathcal{L}^1((0,1)\times(0,T))$. Hence, following the proof of Lemma 16.7 on p. 276 in [21] we obtain the following result.

Theorem 3.5 *There exists a subsequence of functions* $\{P_{\Delta s_r,\Delta t_r}\} \subset \{P_{\Delta s,\Delta t}\}$ *which converges to a function* $p \in \mathrm{BV}([0,1]\times[0,T])$ *in the sense that for all* $t > 0$,

$$\int_0^1 |P_{\Delta s_r,\Delta t_r} - p(s,t)|\,\mathrm{d}s \longrightarrow 0,$$

$$\int_0^T\int_0^1 |P_{\Delta s_r,\Delta t_r} - p(s,t)|\,\mathrm{d}s\,\mathrm{d}t \longrightarrow 0$$

as $r \to \infty$ (*i.e.* $\Delta a_r, \Delta s_r, \Delta t_r \to 0$). *Furthermore, there exists a constant* M_4 *depending on* $\|p^0\|_{\mathrm{BV}([0,1]\times[0,T])}$ *such that the limit function satisfies*

$$\|p\|_{\mathrm{BV}([0,1]\times[0,T])} \le M_4.$$

We show in the next theorem that the limit function $p(s,t)$ constructed by the finite difference scheme is a weak solution of the DSSM model (1).

Theorem 3.6 *The limit function* $p(s,t)$ *defined in Theorem 3.5 is a weak solution of problem (1). Moreover, it satisfies*

$$\|p\|_{L^\infty((0,1)\times(0,T))} \le \exp(2cT)\|p^0\|_\infty.$$

Proof The fact that $p(s,t)$ is a weak solution with bounded TV follows from Lemma 3.1–3.4 and Lemma 16.9 on p. 280 of [21]. The bound on $\|p\|_{L^\infty((0,1)\times(0,T))}$ is obtained by taking the limit in the bounds of the difference approximation in Lemma 3.2. ■

The following theorem guarantees the continuous dependence of the solution p_i^k of Equation (4) with respect to the initial condition p_i^0.

Theorem 3.7 *Let* $\{p_i^k\}$ *and* $\{\hat{p}_i^k\}$ *be solutions of Equation (4) corresponding to the initial conditions* $\{p_i^0\}$ *and* $\{\hat{p}_i^0\}$, *respectively. Then there exists a positive constant* δ *such that*

$$\|p^{k+1} - \hat{p}^{k+1}\|_1 \le (1+\delta t)\|p^k - \hat{p}^k\|_1 \quad \textit{for all } k \ge 0.$$

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Proof Let $u_i^k = p_i^k - \hat{p}_i^k$ for $i = 0, 1, \dots, N$ and $k = 0, 1, \dots, L$. Then by Equation (5) u_i^k satisfies

$$\begin{aligned} u_i^{k+1} &= \frac{\Delta t}{\Delta s}(\gamma_{i-1}^k p_{i-1}^k - \hat{\gamma}_{i-1}^k \hat{p}_{i-1}^k) + (p_i^k - \hat{p}_i^k) - \frac{\Delta t}{\Delta s}(\gamma_i^k p_i^k - \hat{\gamma}_i^k \hat{p}_i^k) \\ &\quad - \Delta t(\mu_i^k p_i^k - \hat{\mu}_i^k \hat{p}_i^k) + \sum_{j=1}^N (\beta_{i,j}^k p_j^k - \hat{\beta}_{i,j}^k \hat{p}_j^k)\Delta s \Delta t, \quad 1 \le i \le N,\ 0 \le k \le L-1, \\ u_0^{k+1} &= p_0^{k+1} - \hat{p}_0^{k+1} = 0, \quad 0 \le k \le L-1. \end{aligned} \tag{14}$$

Here $\hat{Q}^k = \sum_{i=1}^N \hat{p}_i^k$, $\hat{\gamma}_i^k = \gamma(s_i, t_k, \hat{Q}^k)$ and similar notation are used for $\hat{\mu}_i^k$ and $\hat{\beta}_{i,j}^k$. Using the first part of Equation (14) and assumption (H5), we obtain

$$\begin{aligned} |u_i^{k+1}| &\le \left(1 - \frac{\Delta t}{\Delta s}\gamma_i^k - \Delta t \mu_i^k\right)|u_i^k| + \frac{\Delta t}{\Delta s}\gamma_{i-1}^k |u_{i-1}^k| + \Delta t|(\gamma_{i-1}^k - \hat{\gamma}_{i-1}^k)\hat{p}_{i-1}^k - (\gamma_i^k - \hat{\gamma}_i^k)\hat{p}_i^k| \\ &\quad + \Delta t|\mu_i^k - \hat{\mu}_i^k|\hat{p}_i^k + \sum_{j=1}^N \beta_{i,j}^k |\mu_i^k| \Delta s \Delta t + \sum_{j=1}^N |\beta_{i,j}^k - \hat{\beta}_{i,j}^k|\hat{p}_j^k \Delta s \Delta t \\ &\le \left[1 - \mu_i^k \Delta t + \left(\sum_{j=1}^N \beta_{i,j}^k \Delta s\right)\Delta t\right]|u_i^k| - \frac{\Delta t}{\Delta s}(\gamma_i^k |u_i^k| - \gamma_{i-1}^k |u_{i-1}^k|) \\ &\quad + \frac{\Delta t}{\Delta s}|(\gamma_{i-1}^k - \hat{\gamma}_{i-1}^k)\hat{p}_{i-1}^k - (\gamma_i^k - \hat{\gamma}_i^k)\hat{p}_i^k| + |\mu_i^k - \hat{\mu}_i^k|\hat{p}_i^k \Delta t + \sum_{j=1}^N |\beta_{i,j}^k - \hat{\beta}_{i,j}^k|\hat{p}_j^k \Delta s \Delta t. \end{aligned}$$

Multiplying the above inequality by Δs and summing over $i = 1, 2, \dots, N$, we have

$$\begin{aligned} \sum_{i=1}^N |u_i^{k+1}|\Delta s &\le \sum_{i=1}^N \left[1 - \Delta t \mu_i^k + \left(\sum_{j=1}^N \beta_{i,j}^k \Delta s\right)\Delta t\right]|u_i^k|\Delta s \\ &\quad - \Delta t \sum_{i=1}^N (\gamma_i^k |u_i^k| - \gamma_{i-1}^k |u_{i-1}^k|) + \Delta t \sum_{i=1}^N |(\gamma_{i-1}^k - \hat{\gamma}_{i-1}^k)\hat{p}_{i-1}^k - (\gamma_i^k - \hat{\gamma}_i^k)\hat{p}_i^k| \\ &\quad + \Delta t \sum_{i=1}^N |\mu_i^k - \hat{\mu}_i^k|\hat{p}_i^k \Delta s + \Delta t \sum_{i=1}^N \sum_{j=1}^N |\beta_{i,j}^k - \hat{\beta}_{i,j}^k|\hat{p}_j^k \Delta s \Delta s. \end{aligned} \tag{15}$$

Here by assumptions (H2) and (H3)

$$\sum_{i=1}^N \left[1 - \mu_i^k \Delta t + \left(\sum_{j=1}^N \beta_{i,j}^k \Delta s\right)\Delta t\right]|u_i^k|\Delta s \le \sum_{i=1}^N (1 + c\Delta t)|u_i^k|\Delta s = (1 + c\Delta t)\|u^k\|_1. \tag{16}$$

By assumption (H1) and the second part of Equation (14), one obtains

$$\sum_{i=1}^N (\gamma_i^k |u_i^k| - \gamma_{i-1}^k |u_{i-1}^k|) = (\gamma_N^k |\mu_N^k| - \gamma_0^k |u_0^k|) = \gamma_0^k |u_0^k| = 0. \tag{17}$$

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By assumption (H1),

$$
\begin{aligned}
&\sum_{i=1}^{N} |(\gamma_{i-1}^k - \hat{\gamma}_{i-1}^k)\hat{p}_{i-1}^k - (\gamma_i^k - \hat{\gamma}_i^k)\hat{p}_i^k| \\
&\quad \le \sum_{i=1}^{N} |\gamma_{i-1}^k - \hat{\gamma}_{i-1}^k||\hat{p}_i^k - \hat{p}_{i-1}^k| + \sum_{i=1}^{N} |(\gamma_i^k - \hat{\gamma}_i^k) - (\gamma_{i-1}^k \quad - \hat{\gamma}_{i-1}^k)|\hat{p}_i^k \\
&\quad \le \sum_{i=1}^{N} |\gamma_Q(s_{i-1}, \bar{Q}^k)||Q^k - \hat{Q}^k||\hat{p}_i^k - \hat{p}_{i-1}^k| \\
&\qquad + \sum_{i=1}^{N} |\gamma_Q(s_i, \bar{Q}^k)(Q^k - \hat{Q}^k) - \gamma_Q(s_{i-1}, \bar{Q}^k)(Q^k - \hat{Q}^k)|\hat{p}_i^k \\
&\quad \le |Q^k - \hat{Q}^k| \sup_{\mathbb{D}_3} |\gamma_Q| \,\mathrm{TV}(\hat{p}^k) + |Q^k - \hat{Q}^k| \sum_{i=1}^{N} |\gamma_Q(s_i, \bar{Q}^k) - \gamma_Q(s_{i-1}, \bar{Q}^k)|\hat{p}_i^k \\
&\quad \le |Q^k - \hat{Q}^k| \left[\sup_{\mathbb{D}_3} |\gamma_Q| \,\mathrm{TV}(\hat{p}^k) + c \sum_{i=1}^{N} \hat{p}_i^k \Delta s \right] \\
&\quad = |Q^k - \hat{Q}^k| [\sup_{\mathbb{D}_3} |\gamma_Q| \,\mathrm{TV}(\hat{p}^k) + c\|\hat{p}^k\|_1],
\end{aligned}
\tag{18}
$$

where $\bar{Q}^k$ is between Q^k and $\hat{Q}^k$.

By assumption (H2),

$$
\sum_{i=1}^{N} |\mu_i^k - \hat{\mu}_i^k|\hat{p}_i^k \Delta s = \sum_{i=1}^{N} c|(Q^k - \hat{Q}^k)|\hat{p}_i^k \Delta s \le c|Q^k - \hat{Q}^k|\|\hat{p}^k\|_1. \tag{19}
$$

From assumption (H3) we obtain

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} |\beta_{i,j}^k - \hat{\beta}_{i,j}^k|\hat{p}_j^k \Delta s \Delta s \Delta t &\le \Delta t \sum_{i=1}^{N}\sum_{j=1}^{N} c|Q^k - \hat{Q}^k|\hat{p}_j^k \Delta s \Delta s \\
&\le c|Q^k - \hat{Q}^k|\Delta t \sum_{i=1}^{N}\sum_{j=1}^{N} \hat{p}_j^k \Delta s \Delta s \\
&\le c|Q^k - \hat{Q}^k| \left(\sum_{j=1}^{N} \hat{p}_j^k \Delta s \right) \left(\sum_{i=1}^{N} \Delta s \right) \Delta t \\
&= c\|\hat{p}^k\|_1 |Q^k - \hat{Q}^k| \Delta t.
\end{aligned}
\tag{20}
$$

A combination of Equations (15)–(20) and assumptions (H1)–(H3) implies that there exists a positive constant $\tilde{M}$ such that

$$
\|u^{k+1}\|_1 \le (1 + c\Delta t)\|u^k\|_1 + \tilde{M}|Q^k - \hat{Q}^k|\Delta t.
$$

Note that

$$
|Q^k - \hat{Q}^k| = |\sum_{i=1}^{N} (p_i^k - \hat{p}_i^k)\Delta s| \le \sum_{i=1}^{N} |p_i^k - \hat{p}_i^k|\Delta s \le \sum_{i=1}^{N} |u_i^k|\Delta s = \|u^k\|_1.
$$

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Therefore,

$$\|u^{k+1}\|_1 \le (1 + c\Delta t + \tilde{M}\Delta t)\|u^k\|_1.$$

Let $\delta = c + \tilde{M}$ and we obtain the result. ■

In the next theorem we prove that the BV solution defined in Theorem 3.7 is unique using a technique similar to that in [4].

THEOREM 3.8 *Suppose that p and $\hat{p}$ are bounded variation weak solutions of problem (1) corresponding to initial conditions $\{p^0\}$ and $\{\hat{p}^0\}$, respectively. Then there exists a positive constant ρ such that*

$$\|p(\cdot,t) - \hat{p}(\cdot,t)\|_1 \le \rho\|p(\cdot,0) - \hat{p}(\cdot,0)\|_1.$$

Proof Assume that Q is a given Lipschitz continuous function and consider the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial}{\partial t}p(s,t) + \frac{\partial}{\partial s}(\gamma(s,Q(t))p(s,t)) = -\mu(s,Q(t))p(s,t) + \int_0^1 \beta(s,y,Q(t))p(y,t)\,\mathrm{d}y,\\
&\quad s \in (0,1],\ t \in (0,T], \qquad\qquad (21)\\
&\gamma(0,Q(t))p(0,t) = 0, \quad t \in [0,T],\\
&p(s,0) = p^0(s), \quad s \in [0,1].
\end{aligned}$$

Since Equation (21) is a linear problem with local boundary conditions, it has a unique weak solution. Actually, a weak solution can be defined as a limit of the finite difference approximation with the given numbers $Q^k = Q(t_k)$ and the uniqueness can be established by using similar techniques as in [24]. In addition, as in the proof of Theorem 3.7, we can show that if p_i^k and $\hat{p}_i^k$ are solutions of the difference scheme (4) corresponding to the given functions Q^k and $\hat{Q}^k$, respectively, then there exist positive constants c_1 and c_2 such that

$$\|u^{k+1}\|_1 \le (1 + c_1\Delta t)\|u^k\|_1 + c_2|Q^k - \hat{Q}^k|\Delta t, \qquad (22)$$

with $u^k = p^k - \hat{p}^k$.

Equation (22) leads to

$$\|u^k\|_1 \le (1 + c_1\Delta t)^k\|u^0\|_1 + c_2\Delta t\sum_{r=0}^{k-1}(1 + c_1\Delta t)^r|Q^{k-r-1} - \hat{Q}^{k-r-1}|.$$

Hence

$$\|u^k\|_1 \le (1 + c_1\Delta t)^k\left(\|u^0\|_1 + c_2\Delta t\sum_{r=0}^{k-1}|Q^{k-r-1} - \hat{Q}^{k-r-1}|\right). \qquad (23)$$

Now from Theorem 3.5 one can take the limit in Equation (23) to obtain

$$\|u(\cdot,t)\|_1 \le \mathrm{e}^{c_1T}\left(\|u^0\|_1 + c_2\int_0^t |Q(l) - \hat{Q}(l)|\,\mathrm{d}l\right), \qquad (24)$$

where $u(\cdot,t) = p(\cdot,t) - \hat{p}(\cdot,t)$ and $p(\cdot,t)$ are the unique solutions of problem (21) with any set of given functions $Q(t)$ and $\hat{Q}(t)$. We then apply the estimate given in Equation (24) for the

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corresponding solutions of Equation (21) with two specific functions $Q(t)$ and $\hat{Q}(t)$ which are constructed using the limits obtained in Theorem 3.6 as follows:

$$Q(t) = \int_0^1 p(s,t)\,\mathrm{d}s, \quad \hat{Q}(t) = \int_0^1 \hat{p}(s,t)\,\mathrm{d}s.$$

Thus, we have

$$\begin{aligned} |Q(t) - \hat{Q}(t)| &= \left| \int_0^1 p(s,t)\,\mathrm{d}s - \int_0^1 \hat{p}(s,t)\,\mathrm{d}s \right| \\ &\le \int_0^1 |p(s,t) - \hat{p}(s,t)|\,\mathrm{d}s \\ &= \int_0^1 |u(s,t)|\,\mathrm{d}s = \|u(\cdot,t)\|_1. \end{aligned}$$

Therefore,

$$\int_0^t |Q(t) - \hat{Q}(t)|\,\mathrm{d}l \le \int_0^t \|u(\cdot,t)\|_1\,\mathrm{d}l.$$

Thus,

$$\|u(\cdot,t)\|_1 \le \mathrm{e}^{c_1 T}\left(\|u^0\|_1 + c_2 \int_0^t \|u(\cdot,t)\|_1\,\mathrm{d}l \right).$$

Using Gronwall's inequality we have

$$\|u(\cdot,t)\|_1 \le \mathrm{e}^{(c_1 T + c_2 T\,\mathrm{e}^{c_1 T})}\|u^0\|_1.$$

The result follows by letting $\rho = \mathrm{e}^{(c_1 T + c_2 T\,\mathrm{e}^{c_1 T})}$. ■

4. A second-order finite difference scheme

To achieve an accurate approximation the first-order upwind scheme we discussed in the previous section would require many grid points and thus is time-consuming. In this section we develop the following second-order finite difference scheme for the DSSM based on minmod MUSCL schemes [15,20] (SOEM).

$$\begin{gathered} \frac{p_i^{k+1} - p_i^k}{\Delta t} + \frac{\hat{f}_{i+1/2}^k - \hat{f}_{i-1/2}^k}{\Delta s} = -\mu_i^k p_i^k + \sum_{j=0}^{N}{}^{\bigstar} \beta_{i,j}^k p_j^k \Delta s, \quad i = 1, 2, \ldots, N,\ k = 0, 1, \ldots, L-1, \\ \gamma_0^k p_0^k = 0, \quad k = 0, 1, \ldots, L, \end{gathered} \tag{25}$$

with the initial condition $p_i^0 = p^0(s_i)$. Here Q^k is discretized using a second-order Trapezoidal rule.

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That is,

$$Q^k = \sum_{i=0}^{N}{}^{\star}\, p_i^k \Delta s = \frac{1}{2} p_0^k \Delta s + \sum_{i=1}^{N-1} p_i^k \Delta s + \frac{1}{2} p_N^k \Delta s.$$

Similarly,

$$\sum_{j=0}^{N}{}^{\star}\, \beta_{i,j}^k p_j^k \Delta s = \frac{1}{2} \beta_{i,0}^k p_0^k \Delta s + \sum_{j=1}^{N-1} \beta_{i,j}^k p_j^k \Delta s + \frac{1}{2} \beta_{i,N}^k p_N^k \Delta s.$$

The finite difference scheme (25) can be rewritten as

$$p_i^{k+1} = p_i^k - \frac{\Delta t}{\Delta s}(\hat{f}_{i+1/2}^k - \hat{f}_{i-1/2}^k) - \mu_i^k p_i^k \Delta t + \left(\sum_{j=0}^{N}{}^{\star}\, \beta_{i,j}^k p_j^k \Delta s \right) \Delta t, \quad i = 1, 2, \ldots, N. \tag{26}$$

Here the limiter is defined as

$$\hat{f}_{i+1/2}^k = \begin{cases} \gamma_i^k p_i^k + \frac{1}{2}(\gamma_{i+1}^k - \gamma_i^k) p_i^k + \frac{1}{2} \gamma_i^k mm(\Delta_+ p_i^k, \Delta_- p_i^k), & i = 2, \ldots, N-2, \\ \gamma_i^k p_i^k, & i = 0, 1, N-1, N. \end{cases} \tag{27}$$

The minmod function *mm* is defined by

$$mm(a,b) = \frac{\operatorname{sign}(a) + \operatorname{sign}(b)}{2} \min(|a|, |b|).$$

Therefore,

$$0 \leqslant \frac{mm(a,b)}{a} \leqslant 1 \quad \text{and} \quad 0 \leqslant \frac{mm(a,b)}{b} \leqslant 1 \quad \forall a, b \neq 0.$$

As in [20] we define B_i^k and D_i^k by

$$B_i^k = \begin{cases} \dfrac{1}{2} \left(\gamma_{i+1}^k + \gamma_i^k + \gamma_i^k \dfrac{mm(\Delta_+ p_i^k, \Delta_- p_i^k)}{\Delta_- p_i^k} - \gamma_{i-1}^k \dfrac{mm(\Delta_- p_i^k, \Delta_- p_{i-1}^k)}{\Delta_- p_i^k} \right), & i = 3, \ldots, N-2, \\ \dfrac{1}{2} \left(\gamma_{i+1}^k + \gamma_i^k + \gamma_i^k \dfrac{mm(\Delta_+ p_i^k, \Delta_- p_i^k)}{\Delta_- p_i^k} \right), & i = 2, \\ \dfrac{1}{2} \left(2\gamma_i^k - \gamma_{i-1}^k \dfrac{mm(\Delta_- p_i^k, \Delta_- p_{i-1}^k)}{\Delta_- p_i^k} \right), & i = N-1, \\ \gamma_i^k, & i = 1, N, \end{cases}$$

$$D_i^k = \begin{cases} \dfrac{1}{2}(\Delta_+ \gamma_i^k + \Delta_- \gamma_i^k), & i = 3, \ldots, N-2, \\ \dfrac{1}{2} \Delta_+ \gamma_i^k + \Delta_- \gamma_i^k, & i = 2, \\ \dfrac{1}{2} \Delta_- \gamma_i^k, & i = N-1, \\ \Delta_- \gamma_i^k, & i = 1, N. \end{cases}$$

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Note that

$$2(B_i^k - D_i^k) = \begin{cases} \gamma_i^k \left(1 + \dfrac{mm(\Delta_+ p_i^k, \Delta_- p_i^k)}{\Delta_- p_i^k}\right) + \gamma_{i-1}^k \left(1 - \dfrac{mm(\Delta_- p_i^k, \Delta_- p_{i-1}^k)}{\Delta_- p_i^k}\right), & i = 3, \ldots, N-2, \\ 2\gamma_{i-1}^k + \gamma_i^k \dfrac{mm(\Delta_+ p_i^k, \Delta_- p_i^k)}{\Delta_- p_i^k}, & i = 2, \\ \gamma_i^k + \gamma_{i-1}^k \left(1 - \dfrac{mm(\Delta_- p_i^k, \Delta_- p_{i-1}^k)}{\Delta_- p_i^k}\right), & i = N-1, \\ 2\gamma_{i-1}^k, & i = 1, N. \end{cases}$$

One can easily see from assumption (H1) that

$$|B_i^k| \leqslant \frac{3}{2} \sup_{\mathbb{D}_1} |\gamma| \leqslant \frac{3}{2} c, \quad B_i^k - D_i^k \geq 0. \tag{28}$$

The finite difference scheme (25) can then be written in a more compact way as follows:

$$\begin{aligned} p_i^{k+1} = {} & \left(1 - \frac{\Delta t}{\Delta s} B_i^k - \mu_i^k \Delta t\right) p_i^k + \frac{\Delta t}{\Delta s} (B_i^k - D_i^k) p_{i-1}^k \\ & + \left(\sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{i,j}^k p_j^k \Delta s \right) \Delta t \quad \text{for } i = 1, 2, \ldots, N. \end{aligned} \tag{29}$$

4.1. *Estimates of the second-order finite difference approximations*

From a biological point of view, it is very important that our scheme preserves non-negativity of solutions. We will first show this property in the following lemma.

Lemma 4.1 *The finite difference scheme (25) has a unique nonnegative solution.*

Proof From assumption (H4), we have $p_i^0 \geq 0$ for $i = 0, 1, \ldots, N$. Also, by the second part of Equation (25) and assumption (H1), $p_0^k = 0$ for $k \geq 0$. Moreover, by assumptions (H1)–(H3) and (H5), one observes that

$$1 - \frac{\Delta t}{\Delta s} B_i^k - \mu_i^k \Delta t \geq 1 - \frac{\Delta t}{\Delta s} \frac{3}{2} \sup_{\mathbb{D}_1} |\gamma| - \sup_{\mathbb{D}_1} |\mu| \Delta t \geq 1 - \frac{\Delta t}{\Delta s} \frac{3}{2} c - c\Delta t \geq 0. \tag{30}$$

Therefore, by induction it follows that $p_i^k \geq 0$ for $i = 1, 2, \ldots, N$, $k \geq 1$, and thus the system has a unique nonnegative solution. ■

The next lemma shows that the numerical approximations are bounded in ℓ^1 norm.

Lemma 4.2 *For some positive constant M_5, the following estimate holds*:

$$\|p^k\|_1 \leq \exp(cT) \|p^0\|_1 \equiv M_5 \quad \textit{for } k = 0, 1, \ldots, L. \tag{31}$$

Proof Multiplying the first part of (26) by Δs and summing over $i = 1, 2, \ldots, N$, we have

$$\|p^{k+1}\|_1 = \sum_{i=1}^{N} p_i^k \Delta s - \sum_{i=1}^{N} (\hat{f}_{i+1/2}^k - \hat{f}_{i-1/2}^k)\Delta t - \sum_{i=1}^{N} \mu_i^k p_i^k \Delta t \Delta s + \sum_{i=1}^{N} \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^k p_j^k \Delta s \right) \Delta s \Delta t$$

$$= \|p^k\|_1 - (\gamma_N^k p_N^k - \gamma_0^k p_0^k)\Delta t - \sum_{i=1}^{N} \mu_i^k p_i^k \Delta s \Delta t + \sum_{i=1}^{N} \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^k p_j^k \Delta s \right) \Delta s \Delta t.$$

Therefore, by assumptions (H1)–(H3) one can see that

$$\begin{aligned} \|p^{k+1}\|_1 &\le \|p^k\|_1 + \sum_{i=1}^{N} \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^k \Delta s p_j^k \right) \Delta s \Delta t \\ &\le \|p^k\|_1 + c \sum_{i=1}^{N} \|p^k\|_1 \Delta s \Delta t \\ &\le (1 + c\Delta t)\|p^k\|_1, \end{aligned}$$

which implies the estimate. ■

Note that

$$Q^k = \sum_{i=0}^{N}{}^{\star} p_i^k \Delta s = \sum_{i=1}^{N} p_i^k \Delta s - \frac{1}{2} p_N^k \Delta s \leqslant \sum_{i=1}^{N} p_i^k \Delta s = \|p^k\|_1 \le M_5.$$

We now define $\mathbb{D}_4 = [0,1] \times [0, M_5]$.
The following lemma establishes l^∞ bounds of the numerical approximations.

Lemma 4.3 *There exists a positive constant M_6 such that*

$$\|p^k\|_\infty \leqslant M_6 \quad \textit{for } k = 0, 1, \ldots, L.$$

Proof If $\|p^k\|_\infty$ is obtained at the left boundary, then $\|p^k\|_\infty = p_0^k = 0$ for $k \ge 0$. Otherwise, assume that $p_i^{k+1} = \|p^{k+1}\|_\infty$, for some $1 \leqslant i \leqslant N$. From Equation (29), assumptions (H1)–(H3) and (H5) we have

$$\begin{aligned} \|p^{k+1}\|_\infty &\leqslant \left(1 - \frac{\Delta t}{\Delta s} B_i^k - \mu_i^k \Delta t\right) \|p^k\|_\infty + \frac{\Delta t}{\Delta s}(B_i^k - D_i^k)\|p^k\|_\infty + \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^k \Delta s \right) \|p^k\|_\infty \Delta t \\ &\leqslant (1 + c\Delta t)\|p^k\|_\infty - \frac{\Delta t}{\Delta s} D_i^k \|p^k\|_\infty. \end{aligned}$$

By assumption (H1), $|\gamma_i^k - \gamma_{i-1}^k| = |\gamma_s(\hat{s}_i, Q^k)|\Delta s \leqslant c\Delta s$ and thus $-D_i^k \le \frac{3}{2} c \Delta s$.
Therefore,

$$\|p^{k+1}\|_\infty \leqslant (1 + \tfrac{5}{2} c \Delta t)^{k+1} \|p^0\|_\infty.$$

The result then follows easily from the above inequality. ■

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In the next lemma we show that the approximations p_i^k are of bounded TV.

Lemma 4.4 *There exists a constant M_7 such that*

$$\mathrm{TV}(p^k) \leqslant M_7 \quad \textit{for } k = 0, 1, \dots, L.$$

Proof From Equation (29), we have

$$\begin{aligned} p_{i+1}^{k+1} - p_i^{k+1} &= \left(1 - \frac{\Delta t}{\Delta s} B_{i+1}^k\right)(p_{i+1}^k - p_i^k) + \frac{\Delta t}{\Delta s}(B_i^k - D_i^k)(p_i^k - p_{i-1}^k) - \frac{\Delta t}{\Delta s}(D_{i+1}^k - D_i^k)p_i^k \\ &\quad - \Delta t(\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k) + \left(\sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{i+1,j}^k p_j^k \Delta s\right)\Delta t - \left(\sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{i,j}^k p_j^k \Delta s\right)\Delta t, \end{aligned}$$

for $i = 1, \dots, N-1$.

Therefore,

$$\begin{aligned} \mathrm{TV}(p^{k+1}) &= |p_1^{k+1} - p_0^{k+1}| + \sum_{i=1}^{N-1} |p_{i+1}^{k+1} - p_i^{k+1}| \\ &\leqslant |p_1^{k+1} - p_0^{k+1}| + \sum_{i=1}^{N-1} \left| \left(1 - \frac{\Delta t}{\Delta s} B_{i+1}^k\right)(p_{i+1}^k - p_i^k) + \frac{\Delta t}{\Delta s}(B_i^k - D_i^k)(p_i^k - p_{i-1}^k) \right| \\ &\quad + \sum_{i=1}^{N-1} |D_{i+1}^k - D_i^k| p_i^k \frac{\Delta t}{\Delta s} + \sum_{i=1}^{N-1} |\mu_{i+1}^k p_{i+1}^k - \mu_i^k p_i^k| \Delta t \\ &\quad + \sum_{i=1}^{N-1} \left| \sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{i+1,j}^k p_j^k \Delta s - \sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{i,j}^k p_j^k \Delta s \right| \Delta t \\ &= |p_1^{k+1} - p_0^{k+1}| + I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{32}$$

We now estimate the bound of $\mathrm{TV}(p^k)$ term by term.

$$\begin{aligned} |p_1^{k+1} - p_0^{k+1}| &= \left(1 - \frac{\Delta t}{\Delta s} B_1^k - \mu_1^k \Delta t\right) p_1^k + \frac{\Delta t}{\Delta s}(B_1^k - D_1^k) p_0^k + \left(\sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{1,j}^k p_j^k \Delta s\right)\Delta t \\ &= \left(1 - \frac{\Delta t}{\Delta s}\gamma_1^k - \mu_1^k \Delta t\right) p_1^k + \left(\sideset{}{^{\star}}\sum_{j=0}^{N} \beta_{1,j}^k p_j^k \Delta s\right)\Delta t. \end{aligned} \tag{33}$$

By assumptions (H1) and (H5),

$$\begin{aligned} I_1 &\leqslant \sum_{i=1}^{N-1} \left(1 - \frac{\Delta t}{\Delta s} B_{i+1}^k\right)|p_{i+1}^k - p_i^k| + \frac{\Delta t}{\Delta s}(B_i^k - D_i^k)|p_i^k - p_{i-1}^k| \\ &\leqslant \sum_{i=1}^{N-1} |p_{i+1}^k - p_i^k| - \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}(B_{i+1}^k |p_{i+1}^k - p_i^k| - B_i^k |p_i^k - p_{i-1}^k|) - \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1} D_i^k |p_i^k - p_{i-1}^k| \\ &\leqslant \sum_{i=1}^{N-1} |p_{i+1}^k - p_i^k| - \frac{\Delta t}{\Delta s}(B_N^k |p_N^k - p_{N-1}^k| - B_1^k |p_1^k - p_0^k|) - \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1} D_i^k |p_i^k - p_{i-1}^k| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{N-1}|p_{i+1}^k - p_i^k| + \frac{\Delta t}{\Delta s}\gamma_1^k p_1^k + \frac{\Delta t}{\Delta s}\sum_{i=1}^{N-1}\frac{3}{2}\sup_{\mathbb{D}_4}|\gamma_i^k - \gamma_{i-1}^k|\ \ |p_i^k - p_{i-1}^k| \\
&\leqslant \sum_{i=1}^{N-1}|p_{i+1}^k - p_i^k| + \frac{\Delta t}{\Delta s}\gamma_1^k p_1^k + \frac{3}{2}c\,\mathrm{TV}(p^k)\Delta t. \qquad (34)
\end{aligned}
$$

By assumption (H1),

$$
\begin{aligned}
I_2 &= \sum_{i=3}^{N-3}|D_{i+1}^k - D_i^k|p_i^k\frac{\Delta t}{\Delta s} + \sum_{i=1,2,N-2,N-1}|D_{i+1}^k - D_i^k|p_i^k\frac{\Delta t}{\Delta s} \\
&\leqslant \sum_{i=3}^{N-3}|D_{i+1}^k - D_i^k|p_i^k\frac{\Delta t}{\Delta s} + 8\sup_{\mathbb{D}_4}|D_i^k|p_i^k\frac{\Delta t}{\Delta s} \\
&\leqslant \sum_{i=3}^{N-3}|D_{i+1}^k - D_i^k|p_i^k\frac{\Delta t}{\Delta s} + 12c\parallel p^k\parallel_\infty \Delta t. \qquad (35)
\end{aligned}
$$

From assumption (H1) we have

$$
\begin{aligned}
|D_{i+1}^k - D_i^k| &= \tfrac{1}{2}|(\Delta_+\gamma_{i+1}^k + \Delta_-\gamma_{i+1}^k) - (\Delta_+\gamma_i^k + \Delta_-\gamma_i^k)| \\
&= \tfrac{1}{2}|(\gamma_{i+2}^k - \gamma_{i+1}^k) - (\gamma_i^k - \gamma_{i-1}^k)| \\
&= \tfrac{1}{2}|\gamma_s(\hat{s}_{i+2}, Q^k)\Delta s - \gamma_s(\hat{s}_i, Q^k)\Delta s| \\
&\le \tfrac{1}{2}c|\hat{s}_{i+2} - \hat{s}_i|\Delta s = c(\Delta s)^2, \qquad (36)
\end{aligned}
$$

where $\hat{s}_i \in [s_{i-1}, s_i]$ and $\hat{s}_{i+2} \in [s_{i+1}, s_{i+2}]$ for $i = 3, 4, \dots, N-3$.

Therefore by combining Equations (35) and (36) we obtain

$$
I_2 \leqslant \sum_{i=3}^{N-3} c(\Delta s)^2 p_i^k \frac{\Delta t}{\Delta s} + 12c\|p^k\|_\infty \Delta t \leqslant c\|p^k\|_1\Delta t + 12c\|p^k\|_\infty\Delta t. \qquad (37)
$$

We have from assumption (H2) that

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{N-1}|(\mu_{i+1}^k - \mu_i^k)p_{i+1}^k + \mu_i^k(p_{i+1}^k - p_i^k)|\Delta t \\
&\le c\Delta s\sum_{i=1}^{N-1}p_{i+1}^k\Delta t + \sup_{\mathbb{D}_4}\mu\sum_{i=1}^{N-1}|p_{i+1}^k - p_i^k|\Delta t \\
&\le c\|p^k\|_1\Delta t + c\,\mathrm{TV}(p^k)\Delta t. \qquad (38)
\end{aligned}
$$

By assumption (H3),

$$
I_4 \le \sum_{j=0}^{N}{}^{\star}\left(\sum_{i=1}^{N-1}|\beta_{i+1,j}^k - \beta_{i,j}^k|\right)p_j^k\Delta s\Delta t \le c\sum_{j=0}^{N}{}^{\star}\ p_j^k\Delta s\Delta t \le c\|p^k\|_1\Delta t. \qquad (39)
$$

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A combination of Equations (32)–(39) then leads to

$$\begin{aligned}\mathrm{TV}(p^{k+1}) &= \left(1-\frac{\Delta t}{\Delta s}\gamma_1^k-\mu_1^k\Delta t\right)p_1^k+\left(\sideset{}{^{\star}}\sum_{j=0}^{N}\beta_{1,j}^k p_j^k\Delta s\right)\Delta t+\sum_{i=1}^{N-1}|p_{i+1}^k-p_i^k|+\frac{\Delta t}{\Delta s}\gamma_1^k p_1^k\\ &\quad+\frac{3}{2}c\Delta t\,\mathrm{TV}(p^k)+c\|p^k\|_1\Delta t+12c|p^k\|_\infty\Delta t+c\|p^k\|_1\Delta t\\ &\quad+c\,\mathrm{TV}(p^k)\Delta t+c\|p^k\|_1\Delta t\\ &\le(1+c_1\Delta t)\mathrm{TV}(p^k)+c_2\Delta t, \end{aligned}\tag{40}$$

where $c_1=\frac{5}{2}c$ and $c_2=4cM_5+12cM_6$. The result then follows. ■

Next we will show that the finite difference approximations are ℓ_1 Lipschitz continuous in t.

Lemma 4.5 *There exists a positive constant M_8 such that for any $m>n>0$ the following estimates hold*:

$$\sum_{i=1}^{N}\left|\frac{p_i^m-p_i^n}{\Delta t}\right|\Delta s\le M_8(m-n).$$

Proof From Equation (29) and assumptions (H1)–(H3), we have

$$\begin{aligned}\sum_{i=1}^{N}\left|\frac{p_i^{k+1}-p_i^k}{\Delta t}\right|\Delta s &= \sum_{i=1}^{N}|-B_i^kp_i^k-\mu_i^kp_i^k\Delta s+B_i^kp_{i-1}^k-D_i^kp_{i-1}^k+\sideset{}{^{\star}}\sum_{j=0}^{N}\beta_{i,j}^kp_j^k\Delta s\Delta s|\\ &\le\frac{3}{2}\sup_{\mathbb{D}_4}\gamma\sum_{i=1}^{N}|p_i^k-p_{i-1}^k|+\sup_{\mathbb{D}_4}\mu\|p^k\|_1+\sum_{i=1}^{N}\frac{3}{2}\sup_{\mathbb{D}_4}|\gamma_{i+1}^k-\gamma_i^k|p_{i-1}^k\\ &\quad+\sum_{i=1}^{N}\sideset{}{^{\star}}\sum_{j=0}^{N}\sup_{\mathbb{D}_4}|\beta|p_j^k\Delta s\Delta s\\ &\le\frac{3}{2}c\,\mathrm{TV}(p^k)+c\|p^k\|_1+\frac{3}{2}c\|p^k\|_1+c\|p^k\|_1. \end{aligned}\tag{41}$$

Thus by Lemmas 4.2 and 4.4 there exists a positive constant M_8 such that

$$\sum_{i=1}^{N}\left|\frac{p_i^{k+1}-p_i^k}{\Delta t}\right|\Delta s\le M_8.$$

Therefore,

$$\sum_{i=1}^{N}\left|\frac{p_i^m-p_i^n}{\Delta t}\right|\Delta s\le\sum_{i=1}^{N}\sum_{k=n}^{m-1}\left|\frac{p_i^{k+1}-p_i^k}{\Delta t}\right|\Delta s\le M_8(m-n).$$

■

4.2. *Convergence of the second-order finite difference approximations*

We again follow similar notation as in [21] and define a set of functions $\{P_{\Delta s,\Delta t}\}$ by $\{P_{\Delta s,\Delta t}(s,t)\}=p_i^k$ for $s\in[s_{i-1},s_i)$, $t\in[t_{k-1},t_k)$, $i=1,2,\dots,N$ and $k=1,2,\dots,L$. Then by Lemmas 4.2–4.5,

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the set of functions $\{P_{\Delta s,\Delta t}\}$ is compact in the topology of $\mathcal{L}^1((0,1)\times(0,T))$. Hence following the proof of Lemma 16.7 on p. 276 in [21] we obtain the following result.

THEOREM 4.6 *There exists a subsequence of functions* $\{P_{\Delta s_r,\Delta t_r}\} \subset \{P_{\Delta s,\Delta t}\}$ *which converges to a function* $p \in \mathrm{BV}([0,1]\times[0,T])$ *in the sense that for all* $t > 0$,

$$\int_0^1 |P_{\Delta s_r,\Delta t_r} - p(s,t)|\,\mathrm{d}s \longrightarrow 0,$$

$$\int_0^T \int_0^1 |P_{\Delta s_r,\Delta t_r} - p(s,t)|\,\mathrm{d}s\,\mathrm{d}t \longrightarrow 0$$

as $r \to \infty$ (*i.e.* $\Delta a_r, \Delta s_r, \Delta t_r \to 0$). *Furthermore, there exist constants* M_9 *depending on* $\|p^0\|_{\mathrm{BV}([0,1]\times[0,T])}$ *such that the limit function satisfies*

$$\|p\|_{\mathrm{BV}([0,1]\times[0,T])} \le M_9.$$

We show in the next theorem that the limit function $p(s,t)$ constructed by the finite difference scheme is a weak solution to problem (1).

THEOREM 4.7 *The limit function* $p(s,t)$ *defined in Theorem 4.6 is a weak solution of the DSSM. Moreover, it satisfies*

$$\|p\|_{L^\infty((0,1)\times(0,T))} \le \exp(\tfrac{5}{2}cT)\|p^0\|_\infty.$$

Proof Let $\phi \in C^1([0,1]\times[0,T])$ and denote the value of $\phi(s_i,t_k)$ by ϕ_i^k.

Multiplying Equation (26) by ϕ_i^{k+1} and rearranging some terms we have

$$\begin{aligned} p_i^{k+1}\phi_i^{k+1} - p_i^k\phi_i^k &= p_i^k(\phi_i^{k+1} - \phi_i^k) + \frac{\Delta t}{\Delta s}[\hat{f}_{i-1/2}^k(\phi_i^{k+1} - \phi_{i-1}^{k+1}) + (\hat{f}_{i-1/2}^k\phi_{i-1}^{k+1} - \hat{f}_{i+1/2}^k\phi_i^{k+1})] \\ &\quad - \mu_i^k p_i^k \phi_i^{k+1}\Delta t + \sum_{j=0}^{N}{}^{\star}\, \beta_{i,j}^k p_j^k \phi_i^{k+1}\Delta s\Delta t. \end{aligned} \tag{42}$$

Multiplying the above equation by Δs, summing over $i = 1, 2, \ldots, N$, $k = 0, 1, \ldots, L-1$, and applying $p_0^k = 0$ and $\gamma_N^k = 0$ we obtain,

$$\begin{aligned} \sum_{i=1}^{N}(p_i^L\phi_i^L - p_i^0\phi_i^0)\Delta s &= \sum_{k=0}^{L-1}\sum_{i=1}^{N} p_i^k \frac{\phi_i^{k+1} - \phi_i^k}{\Delta t}\Delta s\Delta t \\ &\quad + \sum_{k=0}^{L-1}\sum_{i=0}^{N-1} \hat{f}_{i+1/2}^k \frac{\phi_{i+1}^{k+1} - \phi_i^{k+1}}{\Delta s}\Delta s\Delta t - \sum_{k=0}^{L-1}\sum_{i=1}^{N} \mu_i^k p_i^k \phi_i^{k+1}\Delta s\Delta t \\ &\quad + \sum_{k=1}^{L-1}\sum_{i=1}^{N}\sum_{j=1}^{N}{}^{\star}\, \beta_{i,j}^k p_j^k \phi_i^{k+1}\Delta s\Delta t\Delta s. \end{aligned} \tag{43}$$

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Note that by Equation (27), one has

$$\begin{aligned}\sum_{k=0}^{L-1}\sum_{i=0}^{N-1}\hat{f}_{i+1/2}^{k}\frac{\phi_{i+1}^{k+1}-\phi_{i}^{k+1}}{\Delta s}\Delta s\Delta t = \sum_{k=0}^{L-1}\Bigg[&\gamma_0^k p_0^k + \gamma_1^k p_1^k + \gamma_{N-1}^k p_{N-1}^k \\ &+ \sum_{i\in J_1}\frac{\gamma_i^k+\gamma_{i+1}^k}{2}p_i^k + \sum_{i\in J_2}\frac{\gamma_{i+1}^k p_i^k + \gamma_i^k p_{i+1}^k}{2} \\ &+ \sum_{i\in J_3}\frac{\gamma_{i+1}^k p_i^k + 2\gamma_i^k p_i^k - \gamma_i^k p_{i-1}^k}{2}\Bigg]\frac{\phi_{i+1}^{k+1}-\phi_i^{k+1}}{\Delta s}\Delta s\Delta t,\end{aligned} \tag{44}$$

where $J_1=\{2\le i\le N-2 : \operatorname{sign}(\Delta_+p_i^k)\operatorname{sign}(\Delta_-p_i^k)=-1,\quad \text{or}\quad \operatorname{sign}(\Delta_+p_i^k)\operatorname{sign}(\Delta_-p_i^k)=0\}$, $J_2=\{2\le i\le N-2 : \Delta_-p_i^k\ge\Delta_+p_i^k>0,\quad\text{or}\quad \Delta_-p_i^k\le\Delta_+p_i^k<0\}$, $J_3=\{2\le i\le N-2 : \Delta_+p_i^k>\Delta_-p_i^k>0,\quad\text{or}\quad \Delta_+p_i^k<\Delta_-p_i^k<0\}$. One can easily check that $J_1\cup J_2\cup J_3=\{2,3,\dots,N-3,N-2\}$. Now we could rewrite Equation (43) as

$$\begin{aligned}\sum_{i=1}^{N}(p_i^L\phi_i^L-p_i^0\phi_i^0)\Delta s = \sum_{k=0}^{L-1}\sum_{i=1}^{N}p_i^k\frac{\phi_i^{k+1}-\phi_i^k}{\Delta t}\Delta s\Delta t + \sum_{k=0}^{L-1}\Bigg[&\gamma_0^k p_0^k + \gamma_1^k p_1^k + \gamma_{N-1}^k p_{N-1}^k \\ + \sum_{i\in J_1}\frac{\gamma_i^k+\gamma_{i+1}^k}{2}p_i^k + \sum_{i\in J_2}\frac{\gamma_{i+1}^k p_i^k+\gamma_i^k p_{i+1}^k}{2}& \\ + \sum_{i\in J_3}\frac{\gamma_{i+1}^k p_i^k + 2\gamma_i^k p_i^k - \gamma_i^k p_{i-1}^k}{2}\Bigg]\frac{\phi_{i+1}^{k+1}-\phi_i^{k+1}}{\Delta s}\Delta s\Delta t& \\ - \sum_{k=0}^{L-1}\sum_{i=1}^{N}\mu_i^k p_i^k\phi_i^{k+1}\Delta s\Delta t + \sum_{k=1}^{L-1}\sum_{i=1}^{N}\sum_{j=1}^{N}{}^{\star}\,\beta_{i,j}^k p_j^k\phi_i^{k+1}\Delta s\Delta t\Delta s.&\end{aligned} \tag{45}$$

Since p_i^k is piecewise constant and ϕ is smooth, and the integrals are limits of step functions, we have

$$\begin{aligned}&\int_0^1 P_{\Delta s,\Delta t}(s,t)\phi(s,t)\,\mathrm{d}s + \delta_1 - \int_0^1 P_{\Delta s,\Delta t}(s,0)\phi(s,0)\,\mathrm{d}s + \delta_2 \\ &= \int_0^t\int_0^1 P_{\Delta s,\Delta t}(s,\tau)\phi_\tau(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau + \delta_3 + \int_0^t\Bigg\{\int_0^{\Delta s}\gamma(s,Q(\tau))P_{\Delta s,\Delta t}(s,\tau)\phi_s(s,\tau)\,\mathrm{d}s \\ &\quad + \int_{1-\Delta s}^1\gamma(s,Q(\tau))P_{\Delta s,\Delta t}(s,\tau)\phi_s(s,\tau)\,\mathrm{d}s + \int_{J_1}\gamma(s,Q(\tau))P_{\Delta s,\Delta t}(s,\tau)\phi_s(s,\tau)\,\mathrm{d}s \\ &\quad + \int_{J_2}\gamma(s,Q(\tau))P_{\Delta s,\Delta t}(s,\tau)\phi_s(s,\tau)\,\mathrm{d}s + \int_{J_3}\gamma(s,Q(\tau))P_{\Delta s,\Delta t}(s,\tau)\phi_s(s,\tau)\,\mathrm{d}s\Bigg\}\,\mathrm{d}\tau + \delta_4 \\ &\quad - \int_0^t\int_0^1 P_{\Delta s,\Delta t}(s,\tau)\mu(s,Q(\tau))\phi(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau + \delta_5 \\ &\quad + \int_0^t\int_0^1\phi(s,\tau)\int_0^1 P_{\Delta s,\Delta t}(s,\tau)\beta(s,y,Q(\tau))\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\tau + \delta_6.\end{aligned} \tag{46}$$

$\delta_i\to 0, i=1,2,\dots,6$, as $\Delta s,\Delta t\to 0$ and by the choice of the initial values $\int_0^1 P_{\Delta s,\Delta t}(s,0)\phi(s,0)\,\mathrm{d}s\to\int_0^1 p^0(s)\phi(s,0)\,\mathrm{d}s$ as $\Delta s\to 0$. By Theorem 4.6 $\int_0^1|P_{\Delta s,\Delta t}-p(s,t)|\,\mathrm{d}s\to 0$ and

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$\int_0^t \int_0^1 |P_{\Delta s,\Delta t} - p(s,t)|\, \mathrm{d}s\, \mathrm{d}t \to 0$ as $\Delta s, \Delta t \to 0$. Combining the above fact and Equation (46) and following a similar argument used in the proof of Lemma (16.9) on p. 280 of [21], we can show that the limit of the difference approximations in Theorem 4.6 is a weak solution to problem (1). The bound on $\|p\|_{L^\infty}$ is obtained by taking the limit in the bounds of the difference approximation in Lemma 4.3. ∎

Remark The scheme (25) is of second order in space (except at the boundary) but only first order in time. To obtain second-order accuracy in time we will use the following TV diminishing Runge–Kutta time discretization: Let

$$\begin{aligned} p_i^{(1)} &= p_i^k - \frac{\Delta t}{\Delta s}(\hat{f}_{i+1/2}^k - \hat{f}_{i-1/2}^k) - \mu_i^k p_i^k \Delta t + \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^k p_j^k \Delta s \right) \Delta t, \quad i = 1,2,\ldots,N, \\ p_0^{(1)} &= 0, \\ Q^{(1)} &= \sum_{i=0}^{N}{}^{\star} p_i^{(1)} \Delta s \end{aligned} \tag{47}$$

and compute

$$\begin{aligned} p_i^{k+1} &= \frac{1}{2}\left(p_i^k + p_i^{(1)} - \frac{\Delta t}{\Delta s}(\hat{f}_{i+1/2}^{(1)} - \hat{f}_{i-1/2}^{(1)}) - \mu_i^{(1)} p_i^{(1)} \Delta t + \left(\sum_{j=0}^{N}{}^{\star} \beta_{i,j}^{(1)} p_j^{(1)} \Delta s \right) \Delta t \right), \\ &\quad i = 1,2,\ldots,N, \\ p_0^{k+1} &= 0, \\ Q^{k+1} &= \sum_{i=0}^{N}{}^{\star} p_i^{k+1} \Delta s, \end{aligned} \tag{48}$$

where in Equations (47)–(48) we use the notation $\mu_i^{(1)} = \mu(s_i, Q^{(1)})$ and $\beta_{i,j}^{(1)} = \beta(s_i, y_j, Q^{(1)})$. As for $\hat{f}_{i+1/2}^{(1)}$, it is the same as $\hat{f}_{i+1/2}^{k+1}$ with $p_i^{(1)}$ replacing p_i^{k+1} and $\gamma(s_i, Q^{(1)})$ replacing $\gamma_i^{k+1} = \gamma(s_i, Q^{k+1})$. This gives a second-order scheme in both space and time which shares the same stability and convergence properties as scheme (25) (see [12]).

5. Weak* connection between CSSM and DSSM

The aim of this section is to establish a relationship between solutions of the single state-at-birth model CSSM and the distributed states-at-birth model DSSM. In particular, we show that if the distribution of new recruits in the DSSM becomes concentrated at the left-boundary ($s = 0$), then solutions of the DSSM converge to solutions of the CSSM in the weak* topology. To this end, we have the following theorem:

THEOREM 5.1 *Let* $\{\beta_n(s,y,Q)\}_{n\geq 1}$ *be a sequence of reproductive functions of DSSM. Assume* $\beta_n(s,y,Q) = \beta_{1,n}(s)\beta_2(y,Q)$ *such that*

(A) $\beta_2 \in C^1([0,1] \times [0,\infty))$ *and* $0 \leq \beta_2(y,Q) \leq c$.
(B) $\beta_{1,n} \in C^1([0,1])$ *and* $\int_0^1 \beta_{1,n}(s)\, \mathrm{d}s = 1$ *for each* $n \geq 1$.
(C) *For every test function* $\xi \in C[0,1]$, $\int_0^1 \beta_{1,n}(s)\xi(s)\, \mathrm{d}s \to \xi(0)$, *as* $n \to \infty$.

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Then the weak solution p_n *of DSSM (1) corresponding to* β_n *converges to the weak solution,* $\hat{p}$*, of CSSM (2) in the weak* topology, i.e. as* $n \to \infty$, $\int_0^1 p_n(s,t)\eta(s)\,\mathrm{d}s \to \int_0^1 \hat{p}(s,t)\eta(s)\,\mathrm{d}s$ *for every* $\eta \in C[0,1]$.

Proof It can be seen that β_n satisfies assumption (H3). Thus for each β_n, there exists a weak solution p_n of DSSM (1) which satisfies Equation (3). We denote the total population by $Q_n(t) = \int_0^1 p_n(s,t)\,\mathrm{d}s$. Now, let $\phi \equiv 1$ in (3) and apply property (B) we get

$$\int_0^1 p_n(s,t)\,\mathrm{d}s - \int_0^1 p^0(s)\,\mathrm{d}s = -\int_0^t\int_0^1 p_n(s,\tau)\mu(s,Q_n(\tau))\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t\int_0^1 \beta_2(y,Q_n(\tau))p_n(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{49}$$

Since $\mu \geq 0$ and $p_n \geq 0$, By (A) one has

$$\|p_n(\cdot,t)\|_1 \leq \|p^0\|_1 + c\int_0^t \|p_n(\cdot,\tau)\|_1\,\mathrm{d}\tau. \tag{50}$$

Using Gronwall's inequality, we have

$$\|p_n(\cdot,t)\|_1 \leq \exp(ct)\|p^0\|_1 \leq \exp(cT)\|p^0\|_1. \tag{51}$$

Combining Equation (51) and assumption (H4) one can easily see that the solutions p_n of DSSM are bounded in L^1 norm uniformly in n. Thus, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ that converges in the weak* topology to $\hat{p}$ as $n_i \to \infty$. More specifically, for every $\eta \in C[0,1]$, $\int_0^1 p_{n_i}(s,t)\eta(s)\,\mathrm{d}s \to \int_0^1 \hat{p}(s,t)\eta(s)\,\mathrm{d}s$ as $n_i \to \infty$. Letting $\eta \equiv 1$, we get $Q_{n_i} \to \hat{Q}$ as $n_i \to \infty$. Since by assumption (H1), $\gamma(s,Q)$ is continuously differentiable with respect to s and Q, $\gamma(s,Q_{n_i}) \to \gamma(s,\hat{Q})$, as $n_i \to \infty$. Similarly, applying assumptions (H2) and (A) one gets $\mu(s,Q_{n_i}) \to \mu(s,\hat{Q})$ and $\beta_2(s,Q_{n_i}) \to \beta_2(s,\hat{Q})$ as $n_i \to \infty$.

Now letting $n_i \to \infty$ in Equation (3) and applying (C) we obtain

$$\begin{aligned}\int_0^1 \hat{p}(s,t)\phi(s,t)\,\mathrm{d}s - \int_0^1 p^0(s)\phi(s,0)\,\mathrm{d}s \\ = \int_0^t\int_0^1 \hat{p}(s,\tau)[\phi_\tau(s,\tau) + \gamma(s,\hat{Q}(\tau))\phi_s(s,\tau) - \mu(s,\hat{Q}(\tau))\phi(s,\tau)]\,\mathrm{d}s\,\mathrm{d}\tau \\ + \int_0^t \phi(0,\tau)\int_0^1 \beta_2(y,\hat{Q}(\tau))\hat{p}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau,\end{aligned} \tag{52}$$

for any $\phi \in C^1([0,1]\times[0,T])$. Therefore $\hat{p}$ satisfies Equation (1.2) in [4] and thus is a weak solution of the CSSM with the initial condition $p^0(s)$ and reproduction function $\beta_2(s,Q)$. Since the weak solution of CSSM is unique [4] we get that $p_n \to \hat{p}$ the unique weak solution of CSSM. ■

6. Numerical simulations and examples

In this section we present several numerical simulations to demonstrate the performance of the FOEU scheme (4) and the second-order explicit scheme (25) developed in the previous sections. To better demonstrate their capability in solving the DSSM we compare the schemes with another second-order explicit upwind (SOEU) method [19], which is given by

$$\begin{aligned}
&\frac{p_i^{k+1}-p_i^k}{\Delta t}+\frac{\gamma_i^k p_i^k}{\Delta s}=-\mu_i^k p_i^k+\sum_{j=0}^{N}{}^{\star}\;\beta_{i,j}^k p_j^k\,\Delta s, \quad i=1,\\
&\frac{p_i^{k+1}-p_i^k}{\Delta t}+\frac{3\gamma_i^k p_i^k-4\gamma_{i-1}^k p_{i-1}^k}{\Delta s}=-\mu_i^k p_i^k+\sum_{j=0}^{N}{}^{\star}\;\beta_{i,j}^k p_j^k\,\Delta s, \quad i=2,\\
&\frac{p_i^{k+1}-p_i^k}{\Delta t}+\frac{3\gamma_i^k p_i^k-4\gamma_{i-1}^k p_{i-1}^k+\gamma_{i-2}^k p_{i-2}^k}{\Delta s}=-\mu_i^k p_i^k+\sum_{j=0}^{N}{}^{\star}\;\beta_{i,j}^k p_j^k\,\Delta s,\\
&\quad 3\le i\le N,\ 0\le k\le L-1,\\
&\gamma_0^k p_0^k=0, \quad 0\le k\le L,\\
&p_i^0=p^0(s_i), \quad 0\le i\le N.
\end{aligned} \tag{53}$$

Here Q^k is discretized by the same Trapezoidal rule as used in scheme (25). We also utilize scheme (25) to investigate the connection between the two population models: DSSM and CSSM. At last we apply the numerical scheme (25) to show supercritical Hopf-bifurcation in a distributed states-at-birth model. Throughout this section we use uniformly spaced grid points for both s and t.

6.1. *Validation of the numerical methods against an exact solution*

This example is merely designed to test the order of accuracy of the schemes for smooth solutions and thus may be biologically irrelevant. To this end we choose the parameter values such that the resulting model is nonlinear and would yield an exact solution. Let the initial condition be $p^0(s)=s$. The rest of the parameter values are chosen to be the following:

$$\begin{aligned}
T&=8.0,\\
\beta(s,y,Q)&=1+4sQ,\\
\gamma(s,Q)&=\frac{(1-s)}{2},\\
\mu(s,Q)&=2Q.
\end{aligned}$$

With this choice of model ingredients it can be easily verified that $p(s,t)=s\,\mathrm{e}^t$ is an exact solution of the DSSM. Given the exact solution, we can show numerically the order of accuracy of the schemes by means of an error table. We ran seven simulations for each scheme with step sizes being halved with each successive simulation. Then we calculated the corresponding L^1 norm of the error in each simulation for all schemes. In the initial simulation we let $N=10$ and $L=40$. Based on these consecutive L_1 errors we calculated the orders of accuracy, and listed the results in Table 1. This table indicates that the designed order of accuracy is obtained by all three schemes for this smooth solution of the model.

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Table 1. L^1 errors and orders of accuracy for FOEU, SOEU and SOEM schemes.

		FOEU		SOEU		SOEM	
N	L	L^1 error	Order	L^1 error	Order	L^1 error	Order
10	40	2.51E−1		3.68E−3		6.30E−3	
20	80	1.15E−1	1.12	9.63E−4	1.94	1.66E−3	1.92
40	160	5.56E−2	1.05	2.50E−4	1.95	4.33E−4	1.94
80	320	2.74E−2	1.02	6.39E−5	1.97	1.11E−4	1.97
160	640	1.36E−2	1.01	1.62E−5	1.98	2.81E−5	1.98
320	1280	6.78E−3	1.00	4.07E−6	1.99	7.07E−5	1.99
640	2560	3.39E−3	1.00	1.02E−6	2.00	1.77E−6	1.99

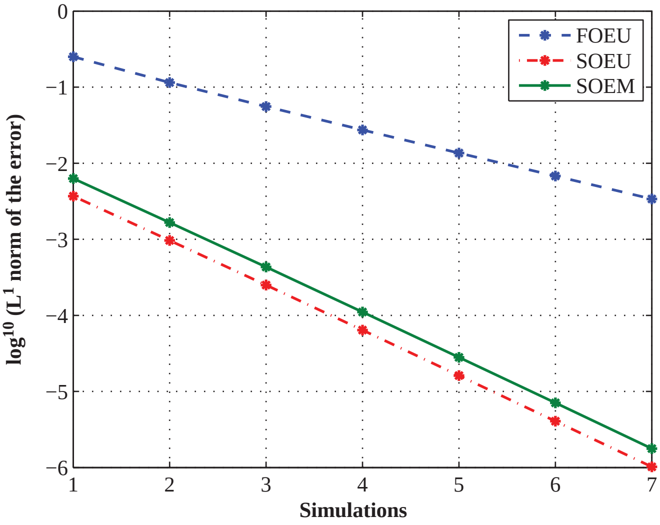

Figure 1. The logarithmic value of L^1 norm of the errors for FOEU, SOEU and SOEM schemes in seven simulations.

To have a better understanding of the order of accuracy, we plot the logarithmic value of the L_1 norm of the errors for all three schemes in Figure 1. Combining Table 1 and Figure 1, one can see clearly that the two second-order methods SOEU and SOEM perform almost equally well in this case when the model parameters and solutions are smooth functions. Also it seems that to get a similar accuracy as obtained in the second-order methods, the first-order method requires to adopt step sizes at least 32 times smaller.

6.2. *Behaviour at discontinuity*

The superiority of the SOEM scheme over both FOEU and SOEU methods is clear once solutions become discontinuous. To show this, we set the initial condition in the DSSM as

$$p^0(s) = \begin{cases} 0.5, & 0 \le s < 0.25, \\ 1, & 0.25 \le s \le 0.75, \\ 0.5, & 0.75 < s \le 1, \end{cases}$$

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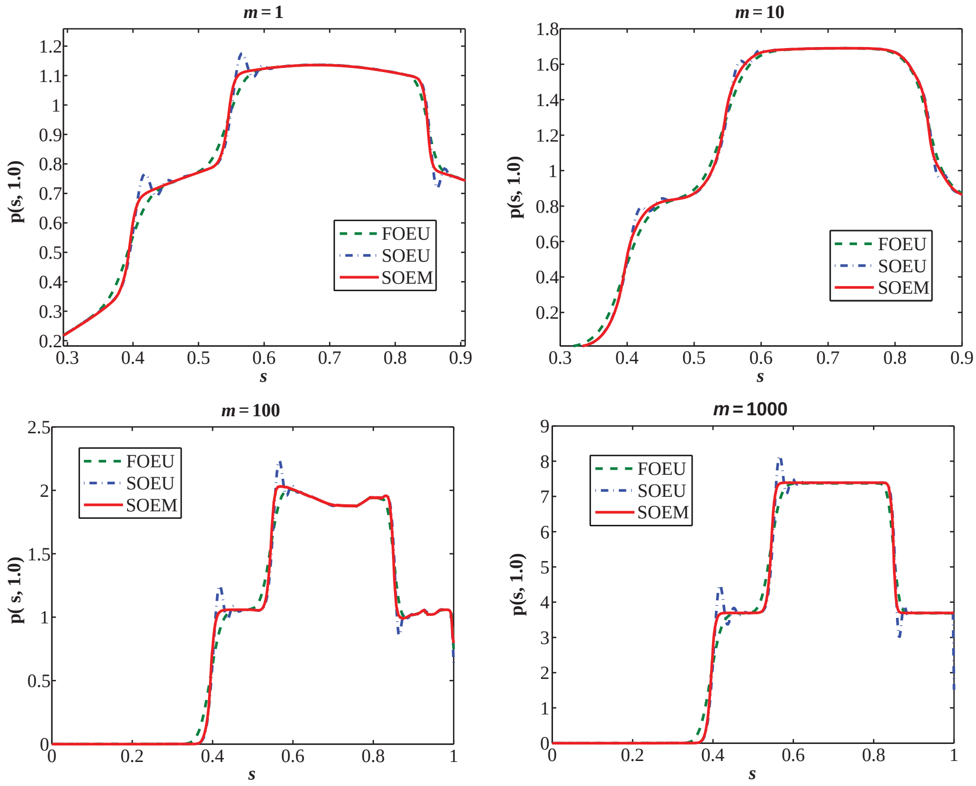

Figure 2. The size distributions at time $t = 1$ are plotted for all three schemes for $m = 1, 10, 100$ and 1000.

and choose the following model ingredients:

$$\beta(s,y,Q) = \begin{cases} 0, & s \leq y - \dfrac{1}{2m}, \\ m, & y - \dfrac{1}{2m} \leq s \leq y + \dfrac{1}{2m}, \\ 0, & s > y + \dfrac{1}{2m}, \end{cases}$$

$$\gamma(s,Q) = \frac{(1-s)}{2},$$

$$\mu(s,Q) = 2\exp(0.1Q),$$

with m being a positive constant.

The above parameter choices introduce several discontinuities in the solution: two that arise from the initial condition and another that arises from the incompatibility of the boundary and initial condition at the origin. In the numerical simulations we use $T = 1.0, N = 400$ and $L = 800$. The results corresponding to different values of m for all three methods are shown in Figure 2. One can observe that SOEM scheme performs better than both the FOEU and SOEU schemes. It demonstrates sharper accuracy in capturing the discontinuity in the solution than both upwind schemes without generating (decaying) spurious oscillations.

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6.3. *Numerical verification of the convergence of solutions of DSSM to CSSM*

In this section we provide some numerical corroboration to Theorem 5.1. To this end, we set the initial condition to be $p^0(s) = s^3$ and the parameters involved in the DSSM as follows:

$$\gamma(s,Q) = \tfrac{1}{2}(1-s),$$
$$\mu(s,Q) = 1.$$

To invoke Theorem 5.1 let

$$\beta_1(s,a,b) = \frac{s^{a-1}(1-s)^{b-1}}{B(a,b)}, \quad s \in [0,1], \tag{54}$$

be the Beta probability density function with parameters a and b; while

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$$

is the Beta function.

Note that the expected value (mean) of the Beta distribution random variable s with two parameters a and b given in Equation (54) is $1/(1+b/a)$. Therefore, as $b/a \to \infty$, the mean is located at the left boundary, $s = 0$. That is, when $b/a \to \infty$, the beta distribution becomes a one point Degenerate distribution with a Dirac delta function spike at $s = 0$ with 100% probability (absolute certainty) [18]. Based on this property of Beta distribution, we fix $a > 1$ and choose a sequence $b_n \to \infty$. It is easy to check that the sequence of fertility functions of DSSM $\beta_n(s,y,Q) = \beta_1(s;a,b_n)\beta_2(y,Q)$ with $\beta_2 \equiv 1$ satisfies the conditions in Theorem 5.1. Thus, this theorem states that the solutions of the DSSM will converge to the solution of CSSM with fertility $\beta_2 = 1$ in the weak* topology. The numerical results we present below demonstrate that this convergence may actually hold in a stronger topology, namely L^1.

In the numerical simulations presented below, we choose $a = 1.01$ and $b = 50, 75, 100$, respectively. The graphs of the fertility function β corresponding to these values of b are shown in Figure 3 (left). To simulate the CSSM, we let the fertility $\beta_2 = 1$ and for all other parameters we use the same values given above for DSSM. We then apply SOEM for solving the DSSM and CSSM. The results of the densities of DSSM and CSSM at $T = 0.8$ are presented in Figure 3 (right).

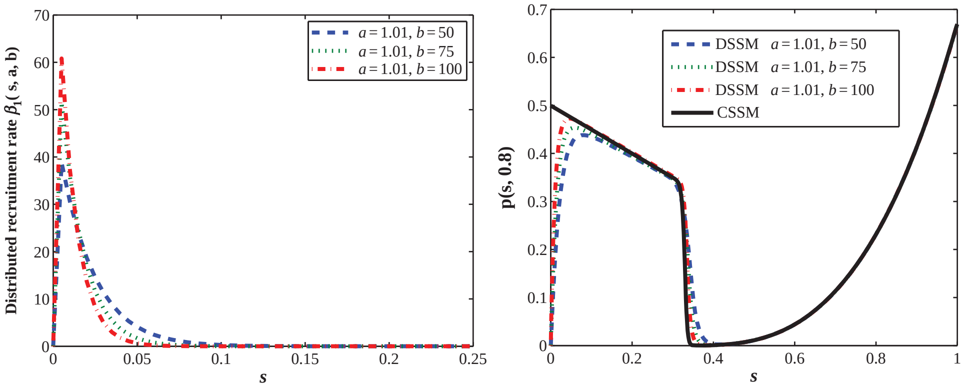

Figure 3. Left: The graph of the distributed recruitment rate $\beta_1(s,a,b)$ for $a = 1.01$ and $b = 50, 75, 100$. Right: The solution $p(s, 0.8)$ for DSSM corresponding to $a = 1.01$ and $b = 50, 75, 100$ against the solution $p(s, 0.8)$ for CSSM.

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6.4. *Supercritical Hopf-bifurcation*

We present a ‘toy model’ here, in which a unique positive steady-state loses its stability via Hopf-bifurcation. This example is further interesting, since as we will see, the net reproduction function is decreasing at the steady state (i.e. its derivative is negative) but the steady state is unstable. In fact this is the only case when stability can be lost via Hopf-bifurcation, since if the model ingredients are such that the derivative of the net reproduction function is positive then the governing linear semigroup is positive, see e.g. [10]. To illustrate the main ideas first we introduce a simple example for the classical Gurtin–MacCamy-type model, and then we perform numerical simulations to show that supercritical Hopf-bifurcation occurs in a corresponding distributed states-at-birth model, too.

Let $\gamma \equiv 1$ and the mortality rate $\mu = \mu(s)$. Assume that the survival probability $\pi(s) = \exp\{-\int_0^s \mu(\tau)\,\mathrm{d}\tau\}$ is given by

$$\pi(s) = \begin{cases} 1, & s \in [0, s_c], \\ 0, & s \in [s_c, 1], \end{cases} \tag{55}$$

$$\beta(s, Q) = \begin{cases} 0, & s \in [0, q) \cup (q+\varepsilon, 1],\ Q \in [0, \infty), \\ \mathrm{e}^{-Q}\tilde{R}\varepsilon^{-1}, & s \in [q, q+\varepsilon],\ Q \in [0, \infty), \end{cases} \tag{56}$$

where $\tilde{R} > 1$, $\varepsilon > 0$, and $0 < q < s_c < 1$. Note that in this example both the fertility and mortality functions are discontinuous. With this choice of the survival probability and fertility function the net reproduction function reads:

$$R(Q) = \int_q^{q+\varepsilon} \mathrm{e}^{-Q}\tilde{R}\varepsilon^{-1}\,\mathrm{d}s = \tilde{R}\,\mathrm{e}^{-Q}.$$

Hence for any $\tilde{R} > 1$ there is a unique positive steady state with total population size $Q_* = \ln(\tilde{R})$. We have

$$p_*(0) = \frac{Q_*}{\int_0^1 \pi(s)\,\mathrm{d}s} = s_c{}^{-1}\ln(\tilde{R}) \quad \text{and} \quad p_*(s) = s_c{}^{-1}\ln(\tilde{R})\pi(s).$$

The characteristic equation corresponding to the linearized system at the positive steady state reads (see e.g. [10]):

$$\begin{aligned} 1 = K(\lambda) &= \int_0^1 \beta(s, Q_*)\pi(s)\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s + \int_0^1 \pi(s)\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s \int_0^1 s_c{}^{-1}\ln(R)\beta_Q(s, Q_*)\pi(s)\,\mathrm{d}s \\ &= \mathrm{e}^{-\lambda q}\frac{1-\mathrm{e}^{-\lambda\varepsilon}}{\lambda\varepsilon} - s_c{}^{-1}\ln(\tilde{R})\frac{1-\mathrm{e}^{-\lambda s_c}}{\lambda}. \end{aligned} \tag{57}$$

In the limit as $\varepsilon \to 0$ the characteristic equation (57) reduces to:

$$1 = \mathrm{e}^{-\lambda q} - s_c{}^{-1}\ln(\tilde{R})\frac{1-\mathrm{e}^{-\lambda s_c}}{\lambda}. \tag{58}$$

We look first for pure imaginary roots of the characteristic equation (58), i.e. assume that $\lambda = \mathrm{i}\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. For such an eigenvalue equation (58) reads:

$$1 = (\cos(\alpha q) - \mathrm{i}\sin(\alpha q)) + \mathrm{i}\alpha^{-1}s_c{}^{-1}\ln(\tilde{R})(1 - \cos(\alpha s_c) + \mathrm{i}\sin(\alpha s_c)), \tag{59}$$

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which is equivalent to

$$1 = \cos(\alpha q) - \alpha^{-1} {s_c}^{-1} \ln(\tilde{R}) \sin(\alpha s_c), \tag{60}$$

$$0 = -\sin(\alpha q) - \alpha^{-1} {s_c}^{-1} \ln(\tilde{R})(\cos(\alpha s_c) - 1). \tag{61}$$

Straightforward calculations show that for $q = \frac{1}{6}$, $s_c = \frac{1}{2}$ and for $\ln(\tilde{R}) = 3\pi/2$ equations (60)–(61) admit the solution $\lambda = 3\pi\mathrm{i}$. Next we would like to show that the pair of purely imaginary eigenvalues $\lambda_{1/2} = \pm 3\pi\mathrm{i}$ cross the y-axis to the right. To this end we write ${s_c}^{-1} \ln(\tilde{R}) = 3\pi + r$ and $\lambda = q + 3\pi\mathrm{i}$, where $r, q \in \mathbb{R}$. The characteristic equation (58) reads:

$$1 = -\mathrm{i}\mathrm{e}^{-q/6} - \frac{r + 3\pi}{q + 3\pi\mathrm{i}} (1 - \mathrm{i}\mathrm{e}^{-r/2}),$$

which leads to

$$3\pi = -q\,\mathrm{e}^{-q/6} + (r + 3\pi)\,\mathrm{e}^{-r/2},$$

$$q = 3\pi\,\mathrm{e}^{-q/6} - r - 3\pi \Rightarrow r + 3\pi = 3\pi\,\mathrm{e}^{-q/6} - q.$$

Hence for $r < 0$ ($\iff s_c > 0.5$) small enough the eigenvalue $\lambda = q + 3\pi\mathrm{i}$ will have a positive real part. Next we note that the continuous dependence of the eigenvalues on the parameter ε (see e.g. [13, Ch. IV.3.5]) implies that for $\varepsilon > 0$ small enough the eigenvalue will still have a positive real part.

Our next numerical example demonstrates, for the first time as far as we know, that such bifurcation may also occur in the DSSM. This is somewhat surprising mainly because the integral operator representing the distributed states-at-birth may have a smoothing effect, in general. We let $p^0 = s$ and

$$\gamma = 1,$$

$$\mu = \frac{160}{(250000s^2 - 250000s + 62505)(0.32 \arctan(250 - 500s) + 2)},$$

$$\beta = \beta_1(s, Q)\beta_2(y),$$

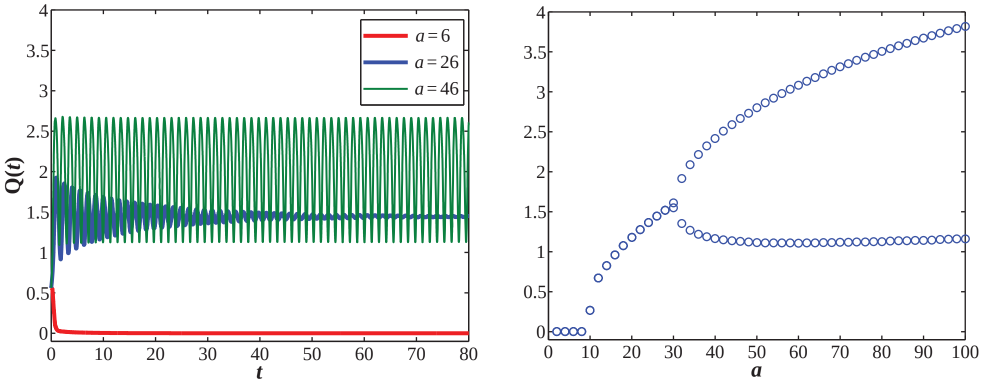

Figure 4. Left: A comparison of the total population sizes $Q(t)$ for $a = 6, 26, 46$; Right: Bifurcation graph of Q with respect to parameter a.

where

$$\beta_1(s,Q) = a\exp(-Q)(10\arctan(5-1000s)+15.7),$$

$$\beta_2(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-0.5\left(100\left(y-\frac{1}{6}+0.005\right)\right)^2\right)\exp\left(\frac{3\pi}{2}\right).$$

Here a is a positive constant. The dynamics of total population $Q(t)$ for different values of a are shown in Figure 4 (left). Only the maximum and minimum values of Q are plotted in the bifurcation graph of the dynamics of $Q(t)$ with respect to a in Figure 4 (right).

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7. Conclusion

We have developed a first-order upwind scheme and a second-order finite difference scheme to approximate the solution of a size-structured population model with distributed states-at-birth. Convergence of both schemes to the unique bounded TV weak solution has been proved. Numerical results are provided to demonstrate the capability of the numerical methods in resolving smooth as well as discontinuous solutions. For smooth solutions both schemes achieve the designed order of accuracy. For discontinuous solutions, the second-order scheme demonstrates better accuracy in capturing the discontinuity compared to upwind schemes. The second-order scheme is also applied to the distributed states-at-birth model (DSSM) to show that supercritical Hopf-bifurcation may occur in such models.

We also proved the convergence of the weak solution for the DSSM in the weak* topology to that of the CSSM under certain conditions on the fertility function and used the second order scheme to demonstrate this convergence.

Funding

The work of A.S. Ackleh, X. Li and B. Ma is partially supported by the National Science Foundation under grant # DMS-1312963.

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