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Fuzzy Formal Concept Analysis

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Abstract. Formal Context Analysis is a mathematical theory that enables us to find concepts from a given set of objects, a set of attributes and a relation on them. There is a hierarchy of such concepts, from which a complete lattice can be made. In this paper we present a generalization of these ideas using fuzzy subsets and fuzzy implications defined from lower semicontinuous t-norms which, under suitable conditions, also results in a complete lattice.

Keywords: Formal Concept Analysis · FCA
Fuzzy Formal Concept Analysis · Fuzzy attributes · Concept lattice
Fuzzy concept lattice

1 Introduction

Formal Concept Analysis (FCA) constitutes a powerful tool for acquisition and representation of knowledge as well as for conceptual data analysis, based on notions from general lattice theory. In FCA, data is represented as a conceptual hierarchy, organized as a concept lattice that relates objects and their properties (see Ganter and Wille 1999). FCA has applications in several fields.

According to Hardy-Vallée, a concept is "a general knowledge that [...] represents a category of objects, events or situations."[1] For example, the concept "Library" represents each individual library. One such "general knowledge" ("Library") abstracts attributes (e.g. having a *catalogue* of its *books*) common to all objects (libraries). A concept in FCA is defined by a set of objects and a corresponding set of attributes.

Nevertheless, real life knowledge is seldom precise. For instance, an automobile manufacturer may construct "concepts" that relate car features (objects) and consumer profiles (attributes). These concepts would be useful if they are

[1] "une connaissance générale qui [...] représente une catégorie d'objets, d'événements ou de situations." See https://www.researchgate.net/profile/Benoit_Hardy-Vallee/publication/228799196.

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interested for example in selling more cars to young women. Because "youngness" is a linguistic imprecise idea, a good strategy would be to use fuzzy FCA.

Fuzzy FCA has been previously proposed in the literature and there exists a vast body of literature on the field, a fair portion of which has been surveyed by Poelmans et al. (2014). Most authors, such as Belohlávek and Vychodil (2007), consider left continuous t-norms in order to define fuzzy implications underlying the fuzzy FCA notions. In this paper we propose the use of fuzzy implications defined from lower semicontinuous t-norms. It should be observed that, due to monotonicity (and commutativity), both notions of continuity are equivalent for t-norms. However, the techniques used in the proofs are different.

2 Formal Concept Analysis

The definitions and theorems presented in this section follow those presented by Ganter and Wille (1999) with a different notation and slightly different proofs.

Definition 1. *A* formal context *is an ordered triple* $\mathbb{C} := \langle \mathcal{O}, \mathcal{A}, I \rangle$*, in which* $\mathcal{O}$ *and* $\mathcal{A}$ *are non-empty sets, and* $I \subseteq \mathcal{O} \times \mathcal{A}$ *is a binary relation.*

The elements of $\mathcal{O}$ are called *objects*, and the elements of $\mathcal{A}$ *attributes*. We say that, in the context $\mathbb{C}$, object *o has attribute a* iff oIa. A finite context can be represented as a table, indexing rows by objects and columns by attributes, and marking cell (o, a) iff oIa, as in Table 1.

Table 1. A formal context of animals

$\mathcal{O}$	$\mathcal{A}$						
	Vertebrate	Lay eggs	Carnivorous	Has wings	Flies	Quadruped	Crawls
Eagle	X	X	X	X	X		
Snake	X	X	X				X
Goose	X	X		X	X		
Swan	X	X		X	X		
Lion	X		X			X	

Definition 2. *Let* $\mathbb{C} = \langle \mathcal{O}, \mathcal{A}, I \rangle$ *be a formal context. We define two maps,* $^* : 2^{\mathcal{O}} \to 2^{\mathcal{A}}$ *and* $^\wedge : 2^{\mathcal{A}} \to 2^{\mathcal{O}}$ *(and write* O^* *and* $A^\wedge$*) as follows:*

$$O^* := \{a \in \mathcal{A} : oIa \ \mathit{for\ all}\ o \in O\} \tag{1}$$

$$A^\wedge := \{o \in \mathcal{O} : oIa \ \mathit{for\ all}\ a \in A\}. \tag{2}$$

Finally, we define the central object of FCA:

Definition 3. *Let* $\mathbb{C} = \langle \mathcal{O}, \mathcal{A}, I \rangle$ *be a formal context. A* formal concept *(or simply* concept*) of* $\mathbb{C}$ *is an ordered pair* $C = \langle O, A \rangle$ *such that* $O \subseteq \mathcal{O}$, $A \subseteq \mathcal{A}$, $O^* = A$ *and* $A^\wedge = O$. *The sets* O *and* A *are called the* extent *and* intent *of the concept* C *respectively.*

Thus, $C = \langle O, A \rangle$ is a concept when O is precisely the subset of objects that has all the attributes of A in common.

Example 4. The following are formal concepts of the context presented in Table 1:

$$C_{\text{Lion}} = \langle \{\text{Lion}\}, \{\text{Quadruped, Carnivorous, Vertebrate}\} \rangle, \quad (3)$$

$$C_{\text{Carnivorous}} = \langle \{\text{Eagle, Snake, Lion}\}, \{\text{Carnivorous, Vertebrate}\} \rangle. \quad (4)$$

Notice that there is an inverse relation between the numbers of elements in the intent and the extent of a concept. If we increase the number of elements in the extent (we added "Eagle" and "Snake" to it), the number of elements in the intent is reduced ("Quadrupede" is not in the intent of $C_{\text{Carnivorous}}$). In fact, the following useful properties hold:

Theorem 5. *Let* $O, O_1, O_2 \subseteq \mathcal{O}$ *and* $A, A_1, A_2 \subseteq \mathcal{A}$. *Then*

1. If $O_1 \subseteq O_2$ *then* $O_2^* \subseteq O_1^*$ *1'. If* $A_1 \subseteq A_2$ *then* $A_2^\wedge \subseteq A_1^\wedge$
2. $O \subseteq O^{*\wedge}$ *2'.* $A \subseteq A^{\wedge *}$
3. $O^* = O^{*\wedge *}$ *3'.* $A^\wedge = A^{\wedge * \wedge}$
4. $O \subseteq A^\wedge$ *iff* $A \subseteq O^*$ *iff* $O \times A \subseteq I$

Proof. We shall prove items 1., 2., 3. and 4.. Items with a prime can be proved analogously.

1. Let $a \in O_2^*$. Then oIa for all $o \in O_2$. In particular, oIa for all $o \in O_1$. Thus, $a \in O_1^*$.
2. Let $o \in O$. Then oIa for all $a \in O^*$, by definition of O^*. Thus, by definition of $O^{*\wedge}$, we have $o \in O^{*\wedge}$.
3. From 2'. with $A = O^*$ we already know that $O^* \subseteq O^{*\wedge *}$. Let $a \in O^{*\wedge *}$. Then

   (i) oIa for all $o \in O^{*\wedge}$,

   by definition of $O^{*\wedge *}$. Now let $\tilde{o} \in O^{*\wedge}$ be fixed. For all $\tilde{a} \in \mathcal{A}$,

   (ii) if $\tilde{o}I\tilde{a}$ then $\tilde{a} \in O^*$,

   by definition of $O^{*\wedge}$. From **(i)** we have $\tilde{o}Ia$. Thus, using **(ii)** we conclude that $a \in O^*$.
4. Suppose $O \subseteq A^\wedge$. By 1., $A^{\wedge *} \subseteq O^*$. Using 2'. and transitivity of $\subseteq$, we have $A \subseteq O^*$.
   Now suppose $A \subseteq O^*$. By 1'. and 2., we have $O \subseteq O^{*\wedge} \subseteq A^\wedge$.
   Assuming $A \subseteq O^*$, let $o \in O$ and $a \in A$. By definition of O^*, $oI\tilde{a}$ for all $\tilde{a} \in O^*$. By hypothesis, $a \in A \subseteq O^*$. Thus, oIa. Since $o \in O$ and $a \in A$ are arbitrary, $O \times A \subseteq I$. Hence, $A \subseteq O^* \Rightarrow O \times A \subseteq I$.

Finally, suppose $O \times A \subseteq I$. Let $o \in O$. By hypothesis, for all $a \in A$ we have oIa. By definition of O^*, if oIa then $a \in O^*$. Thus if $a \in A$ then $a \in O^*$. This completes the proof.

From properties 3. and 3'. we see that, given $O \subseteq \mathcal{O}$ and $A \subseteq \mathcal{A}$, $\langle O^{*\wedge}, O^* \rangle$ and $\langle A^\wedge, A^{\wedge *} \rangle$ are concepts. On the other hand if $C = \langle O, A \rangle$ is a concept then by definition

$$C = \langle A^\wedge, A \rangle = \langle (O^*)^\wedge, O^* \rangle.$$

Consequently every concept has either form $\langle O^{*\wedge}, O^* \rangle$ or $\langle A^\wedge, A^{\wedge *} \rangle$, where $O \subseteq \mathcal{O}$ and $A \subseteq \mathcal{A}$ are arbitrary.

This gives us a procedure for finding concepts. Choose a subset of objects (or attributes), apply * to get an intent, and then apply $^\wedge$ to get an extent (in fact, the smallest extent containing the original object subset). To find all the concepts of a given context, simply list all the subsets of objects (or attributes), and then apply the maps * and $^\wedge$ respectively (or $^\wedge$ and *).

Example 6. Notice that

$$\{\text{Eagle}\}^{*\wedge} = \{\text{Vertebrate, Lay eggs, Carnivorous, Has wings, Flies}\}^\wedge = \{\text{Eagle}\}.$$

Thus, the following is a concept:

$$C_{\text{Eagle}} = \langle \{\text{Eagle}\}, \{\text{Flies, Has Wings, Lay eggs, Carnivorous, Vertebrate}\} \rangle. \tag{5}$$

There are also interesting properties regarding intersections of extents and intents. For instance, the intersection of the intents of (3) and (5) is the intent of (4). This is a particular case of a more general fact, presented in the following proposition.

Proposition 7. *Let J be an index set and, for each $\alpha \in J$, let $O_\alpha \subseteq \mathcal{O}$ and let $A_\alpha \subseteq \mathcal{A}$. Then*

$$\bigcap_{\alpha \in J} O_\alpha^* = \left(\bigcup_{\alpha \in J} O_\alpha \right)^*, \tag{6}$$

$$\bigcap_{\alpha \in J} A_\alpha^\wedge = \left(\bigcup_{\alpha \in J} A_\alpha \right)^\wedge. \tag{7}$$

Proof. We shall prove only (6). The proof of (7) is analogous.

$$\begin{aligned} a \in \bigcap_{\alpha \in J} O_\alpha^* \; & \text{iff } a \in O_\alpha^* \text{ for all } \alpha \in J \\ & \text{iff } oIa \text{ for all } o \in O_\alpha, \text{ for all } \alpha \in J \\ & \text{iff } oIa \text{ for all } o \in \bigcup_{\alpha \in J} O_\alpha \\ & \text{iff } a \in \left(\bigcup_{\alpha \in J} O_\alpha \right)^*. \end{aligned}$$

Given two sets $Y \subseteq X$ and a partial order $\leq$ on X, the *infimum* of Y (if it exists) is the element i_Y such that $i_y \leq y$ for any $y \in Y$ and if $x \leq y$ for all $y \in Y$ then $x \leq i_y$. Replacing $\leq$ by $\geq$ and i_Y by s_Y we get the definition of the supremum s_Y of Y. $\mathcal{L} = \langle X, \leq \rangle$ is called a *lattice* if every finite subset of X has both an infimum and a supremum. If every subset of X has an infimum and a supremum, then $\mathcal{L}$ is said to be *complete*.

From Proposition 7 we can show that, with an order induced by set inclusion, the set of all formal concepts of any context constitutes a complete lattice.

Theorem 8. *Let $\mathbb{C}$ be a formal context. Let $\mathfrak{B}(\mathbb{C})$ be the set of all concepts of $\mathbb{C}$. Define the relation $\leq$ on $\mathfrak{B}(\mathbb{C})^2$ by $\langle O_1, A_1 \rangle \leq \langle O_2, A_2 \rangle$ iff $O_1 \subseteq O_2$. Then $\leq$ is an order on $\mathfrak{B}(\mathbb{C})$. If $C_1 \leq C_2$ we say that C_1 is a* subconcept *of C_2. Correspondingly, C_2 is a* superconcept *of C_1. Furthermore, $\mathcal{L}_{\mathbb{C}} := \langle \mathfrak{B}(\mathbb{C}), \leq \rangle$ is a complete lattice, called the* concept lattice *of $\mathbb{C}$.*

If J is an index set and $C_\alpha = \langle O_\alpha, A_\alpha \rangle \in \mathfrak{B}(\mathbb{C})$ for each $\alpha \in J$ then

$$\inf_{\alpha \in J} C_\alpha = \left\langle \bigcap_{\alpha \in J} O_\alpha, \left(\bigcap_{\alpha \in J} O_\alpha \right)^* \right\rangle \tag{8}$$

$$\sup_{\alpha \in J} C_\alpha = \left\langle \left(\bigcap_{\alpha \in J} A_\alpha \right)^\wedge, \bigcap_{\alpha \in J} A_\alpha \right\rangle . \tag{9}$$

Proof. That $\leq$ is an order is clear from the fact that $\subseteq$ is an order. Now we prove (9). For each $\alpha \in J$ we have $A_\alpha = O_\alpha^*$. Thus, by (6),

$$\bigcap_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in J} O_\alpha^* = \left(\bigcup_{\alpha \in J} O_\alpha \right)^*$$

is the intent of a concept (applying $^\wedge$ gives the concept's extent). Hence, by properties of set intersection, $\cap_{\alpha \in J} A_\alpha$ is the greatest intent smaller than all the C_α. Using 1'. of Theorem 5, $(\cap_{\alpha \in J} A_\alpha)^\wedge$ is the smallest extent greater than all the O_α, and so the supremum of $\leq$ is as stated.

The proof of (8) is similar to that of (9), only working with the extents of the C_α rather than their intents, and applying (7) instead of (6).

Theorem 8 allows us to use lattice theory for finding out many properties that come from a formal context. In particular, a finite concept lattice has an easy visual representation (see Example 9 below). In order to interpret the concept lattice from the diagram, one may write, for each concept on the diagram, the elements of its intent and extent. However, from the order $\leq$ of the concept lattice a tidier manner of presenting the diagram can be devised: for a given concept, instead of writing every element of its extent (or intent), we write only those objects (attributes) that did not appear below (above) in the concept lattice.

Example 9. The concept lattice of the context presented in Table 1 is shown in Fig. 1. Each concept is represented by a circle. Here animals are represented by numbers 1–5 in the order they appear in Table 1. Attributes are represented by letters a-g, also in the order they appear in the table. The extent (intent) of a given concept C has an object (attribute) iff that object (attribute) appears near a concept $\tilde{C}$ such that there is a descending (ascending) path from C to $\tilde{C}$.

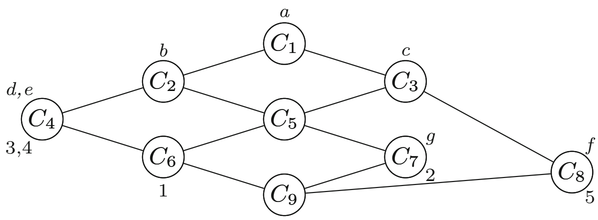

Fig. 1. Concept lattice of animals

Take, for instance, the concept C_3. It has ascending paths to concepts C_1 and C_3, and descending paths to concepts C_8, C_5, C_6, C_7 and C_9. On the other hand, C_3 has no (strictly ascending or descending) paths to C_2 or C_4. Thus, $C_3 = \langle\{1, 2, 5\}, \{a, c\}\rangle$.

Concept lattices are visual tools that allow us to find out relations on attributes and objects (for example, any animal with property e also has property b). However, limitations may arise on the theory, as, for example, the definition of formal context allows us only to work only with precise relations. For example, a chicken can fly for short distances, but this information could not be expressed on a formal context as defined earlier. In the next section we generalize these ideas to allow fuzzy objects, attributes and relations.

3 Fuzzy Formal Concept Analysis

We first state some basic definitions of fuzzy logic and then proceed to a fuzzy generalization of FCA. Definition of triangular norm corresponds to that presented by Klement et al. (2000), whereas definitions of fuzzy implication, fuzzy subset and fuzzy (binary) relation correspond to definitions by de Barros et al. (2017).

Definition 10. *A* triangular norm *(or* t-norm*) is a map* $\triangle\colon [0,1]^2 \to [0,1]$ *satisfying, for all* $x, y, z \in [0,1]$*:*

1. $x \triangle y = y \triangle x$ *(commutativity)*
2. $x \triangle (y \triangle z) = (x \triangle y) \triangle z$ *(associativity)*
3. *If* $y \leq z$ *then* $x \triangle y \leq x \triangle z$ *(monotonicity)*
4. $x \triangle 1 = x$ *(boundary condition)*

Definition 11. *A* fuzzy implication *is a map* $\Rightarrow\colon [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$*:*

1. $(0 \Rightarrow 0) = 1$, $(1 \Rightarrow 0) = 0$, *(boundary conditions)*
   $(0 \Rightarrow 1) = 1$ *and* $(1 \Rightarrow 1) = 1$
2. *If* $y \leq x$ *then* $(x \Rightarrow z) \leq (y \Rightarrow z)$ *(monotonicity in the first component)*
3. *If* $x \leq y$ *then* $(z \Rightarrow x) \leq (z \Rightarrow y)$ *(monotonicity in the second component)*

Definition 12. *A* R-implication *is a fuzzy implication* $\Rightarrow_R$ *defined by*

$$(x \Rightarrow_R y) = \bigvee \{z \in [0,1] : x \vartriangle z \leq y\}, \tag{10}$$

where $\vartriangle$ *is a t-norm and* $\bigvee$ *stands for supremum. We say that* $\Rightarrow_R$ *is* induced *by* $\vartriangle$*.*

Definition 13. *Let* U *be a (classical universal) set. A* fuzzy subset F *of* U *is defined by a function* $\phi_F : U \longrightarrow [0,1]$*, called the* membership function *of* F*.*

Given two fuzzy subsets F_1, F_2 *of* U*, we say that* F_1 *is a* (fuzzy) subset *of* F_2 *if* $\phi_{F_1}(u) \leq \phi_{F_2}(u)$ *for all* $u \in U$*, and in this case we write* $F_1 \subseteq F_2$*.*

If $\langle F_\alpha \rangle_{\alpha \in J}$ *is a family of fuzzy subsets of* U*, then their union and intersection are defined respectively by the membership functions*

$$\phi_{\cup_{\alpha \in J} F_\alpha}(u) = \bigvee_{\alpha \in J} \phi_\alpha(u) \tag{11}$$

$$\phi_{\cap_{\alpha \in J} F_\alpha}(u) = \bigwedge_{\alpha \in J} \phi_\alpha(u), \tag{12}$$

where $\bigvee$ *and* $\bigwedge$ *stand for supremum and infimum respectively.*

Definition 14. *A* fuzzy (binary) relation *on a pair of (classical) sets* U_1, U_2 *is a fuzzy subset of* $U_1 \times U_2$*. In particular, if* A_1, A_2 *are fuzzy subsets of* U_1, U_2 *respectively, and* $\vartriangle$ *is a t-norm, then the* (fuzzy) cartesian product of A_1, A_2 induced by $\vartriangle$ *is the fuzzy relation* $A_1 \times_\vartriangle A_2$ *on* $U_1 \times U_2$ *defined by the membership function*

$$\phi_{A_1 \times_\vartriangle A_2}(x_1, x_2) = \phi_1(x_1) \vartriangle \phi_2(x_2). \tag{13}$$

Now we can start defining the objects of Fuzzy Formal Concept Analysis (fFCA).

Definition 15. *A* fuzzy formal context *is an ordered triple* $\mathbb{C}_\mathrm{f} := \langle \mathcal{O}, \mathcal{A}, I_\mathrm{f} \rangle$*, in which* $\mathcal{O}$ *and* $\mathcal{A}$ *are non-empty (classical) sets, and* $I_\mathrm{f} \subseteq \mathcal{O} \times \mathcal{A}$ *is a fuzzy binary relation.*

Notice that, in (1) and (2), the characteristic functions of O^* and $A^\wedge$ can be expressed respectively by

$$\chi_{O^*}(\tilde{a}) = \forall o \in \mathcal{O}(o \in O \longrightarrow oI\tilde{a}),$$
$$\chi_{A^\wedge}(\tilde{o}) = \forall a \in \mathcal{A}(a \in A \longrightarrow \tilde{o}Ia).$$

Since an application of the universal quantifier, $\forall$, returns the smallest truth value a predicate assumes ($\forall x P(x) = 1$ iff $P(u) = 1$ for each $u \in U$), we generalize $\forall$ as an infimum.

Definition 16. *Let $\mathbb{C}_\mathrm{f} = \langle \mathcal{O}, \mathcal{A}, I_\mathrm{f} \rangle$ be a fuzzy formal context. Let $\Rightarrow$ be a fuzzy implication. Then we define, for all fuzzy subsets $O \subseteq \mathcal{O}$ and $A \subseteq \mathcal{A}$, the fuzzy subsets $O^* \subseteq \mathcal{A}$ and $A^\wedge \subseteq \mathcal{O}$ by their membership functions defined as*

$$\phi_{O^*}(a) = \inf_{o \in \mathcal{O}} \left[\phi_O(o) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right], \tag{14}$$

$$\phi_{A^\wedge}(o) = \inf_{a \in \mathcal{A}} \left[\phi_A(a) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right]. \tag{15}$$

With these maps defined we can finally generalize the idea of formal concept.

Definition 17. *Let $\mathbb{C}_\mathrm{f} = \langle \mathcal{O}, \mathcal{A}, I_\mathrm{f} \rangle$ be a fuzzy formal context and let $\Rightarrow$ be a fuzzy implication. Let O, A be fuzzy subsets of $\mathcal{O}$ and $\mathcal{A}$, respectively. We say that $C_\mathrm{f} = \langle O, A \rangle$ is a* fuzzy formal concept *of $\mathbb{C}_\mathrm{f}$ if $O^* = A$ and $A^\wedge = O$.*

Our goal is to show that, as in the classical case, it is possible to produce a complete lattice of fuzzy formal concepts. Our definitions of semicontinuous functions are proven by Bourbaki (1966) to be equivalent with the definitions of semicontinuity he presents. The upper semicontinuous R-implication induced by a lower semicontinuous t-norm is precisely what allows us to have a complete lattice of fuzzy concepts.

In what follows, a sequence $(x_1, x_2, ...)$ is denoted by (x_n), and so x_0 is not an element of the sequence (x_n).

Definition 18. *Let $X, Y \subseteq \mathbb{R}^2$ be non-empty (classical) sets. Let $f : X \to Y$ be a function. We say that f is* upper semicontinuous at *$x_0 \in X$ if, for any sequence (x_n) converging to x_0,*

$$\inf_{n>0} \left(\sup_{m \geq n} f(x_m) \right) =: \limsup_{n\to\infty} f(x_n) \leq f(x_0). \tag{16}$$

If f is upper semicontinuous at every $x_0 \in X$ then f is upper semicontinuous*.*

Similarly, f is lower semicontinuous at *$x_0 \in X$ if for any sequence (x_n) converging to x_0,*

$$\sup_{n>0} \left(\inf_{m \geq n} f(x_m) \right) =: \liminf_{n\to\infty} f(x_n) \geq f(x_0). \tag{17}$$

If f is lower semicontinuous at every $x_0 \in X$ then f is lower semicontinuous*.*

If f is both upper and lower semicontinuous at x_0 then f is continuous at *x_0, and*

$$\limsup_{n\to\infty} f(x_n) = f(x_0) = \liminf_{n\to\infty} f(x_n). \tag{18}$$

If f is continuous at every $x_0 \in X$ then it is continuous.

We now derive a useful expression concerning lower semicontinuous t-norms in Lemma 20, and for its proof we use Proposition 19.

In the following, we shall write $x_n \nearrow x_0$ meaning that the increasing sequence (x_n) converges to x_0. Similarly, $x_n \searrow x_0$ means that the decreasing sequence (x_n) converges to x_0.

Proposition 19. *Let $\triangle$ be a lower semicontinuous t-norm. Let $x, y \in [0,1]$. Define*

$$S = \{z \in [0,1] : x \triangle z \leq y\}.$$

Then $\sup S \in S$.

Proof. Let $z_0 = \sup S$, and let (z_n) be a sequence on $[0,1]$ such that $z_n \nearrow z_0$. Then

$$x \triangle z_m \leq y \text{ for all } m > 0,$$

as $z_m \leq z_0$ for each $m > 0$. Thus, for each $n > 0$,

$$\inf_{m \geq n} (x \triangle z_m) \leq x \triangle z_n \leq y.$$

Taking the supremum for $n > 0$ on the left-hand side we have, by lower semicontinuity of $\triangle$,

$$x \triangle z_0 \leq \liminf_{n \to \infty} (x \triangle z_n) = \sup_{n>0} \left(\inf_{m \geq n} (x \triangle z_m) \right) \leq y.$$

Therefore, $z_0 \in A$.

Lemma 20. *Let $\triangle$ be a lower semicontinuous t-norm and let $\Rightarrow$ be the R-implication induced by $\triangle$. Then, for all $x, y \in [0,1]$ we have $x \leq [(x \Rightarrow y) \Rightarrow y]$.*

Proof. Let $x, y \in [0,1]$. By Proposition 19, $x \triangle (x \Rightarrow y) \leq y$. Using commutativity of $\triangle$, it is clear that $x \in S := \{z \in [0,1] : (x \Rightarrow y) \triangle z \leq y\}$. Hence

$$x \leq \sup S = [(x \Rightarrow y) \Rightarrow y].$$

Now we have what is necessary to generalize Theorem 5, establishing dual relations for the maps * and $^\wedge$ in the fuzzy case.

Theorem 21. *Let $\mathbb{C}_f = \langle \mathcal{O}, \mathcal{A}, I_f \rangle$ be a fuzzy context. Let O, O_1, O_2 be fuzzy subsets of $\mathcal{O}$ and A, A_1, A_2 be fuzzy subsets of $\mathcal{A}$. Let $\triangle$: $[0,1]^2 \to [0,1]$ be a lower semicontinuous t-norm, and let $\Rightarrow$ be the R-implication induced by $\triangle$. Then:*

1. If $O_1 \subseteq O_2$ then $O_2^* \subseteq O_1^*$	*1'. If $A_1 \subseteq A_2$ then $A_2^\wedge \subseteq A_1^\wedge$*
2. $O \subseteq O^{*\wedge}$	*2'. $A \subseteq A^{\wedge *}$*
3. $O^* = O^{*\wedge *}$	*3'. $A^\wedge = A^{\wedge * \wedge}$*

4. $O \subseteq A^\wedge$ iff $A \subseteq O^*$ iff $O \times_\triangle A \subseteq I_f$

Proof. We shall prove items 1., 2., 3. and 4.. Items with a prime can be proved analogously.

1. Suppose that $O_1 \subseteq O_2$. Let $a \in \mathcal{A}$. Let $o \in \mathcal{O}$. By hypothesis, $\phi_{O_1}(o) \leq \phi_{O_2}(o)$. Since $\Rightarrow$ is decreasing in its first component we have

$$(\phi_{O_2}(o) \Rightarrow \phi_{I_f}(o,a)) \leq (\phi_{O_1}(o) \Rightarrow \phi_{I_f}(o,a)).$$

   Taking the infimum over o on both sides, we see that

$$\phi_{O_2^*}(a) \leq \phi_{O_1^*}(a)$$

   by definition of the map *. But $a \in \mathcal{A}$ is arbitrary. Thus $O_2^* \subseteq O_1^*$.

2. Let $o \in \mathcal{O}$. By hypothesis, $\vartriangle$ is lower semicontinuous and so Lemma 20 holds. Using monotonicity of $\Rightarrow$ in its first component we get

$$\begin{aligned}\phi_O(o) &\leq [(\phi_O(o) \Rightarrow \phi_{I_\mathrm{f}}(o,a)) \Rightarrow \phi_{I_\mathrm{f}}(o,a)] \\ &\leq \left[\inf_{\tilde{o}\in\mathcal{O}} (\phi_O(\tilde{o}) \Rightarrow \phi_{I_\mathrm{f}}(\tilde{o},a)) \Rightarrow \phi_{I_\mathrm{f}}(o,a)\right] \\ &= [\phi_{O^*}(a) \Rightarrow \phi_{I_\mathrm{f}}(o,a)]\,.\end{aligned}$$

Taking the infimum over a on the right-hand side, we get

$$\phi_O(o) \leq \phi_{O^{*\wedge}}(o).$$

Since $o \in \mathcal{O}$ is arbitrary, $O \subseteq O^{*\wedge}$.

3. From 2., we have $O \subseteq O^{*\wedge}$. Thus, using 1., $O^{*\wedge *} \subseteq O^*$. On the other hand, using $A = O^*$ in 2'., we have $O^* \subseteq O^{*\wedge *}$. Hence, $O^* = O^{*\wedge *}$.
4. We shall first prove that if $A \subseteq O^*$ then $O \subseteq A^\wedge$; then we show that $O \subseteq A^\wedge$ implies $O \times_\vartriangle A \subseteq I_\mathrm{f}$; and finally we prove that if $O \times_\vartriangle A \subseteq I_\mathrm{f}$ then $A \subseteq O^*$, concluding that the three properties are equivalent.
   (a) Suppose that $A \subseteq O^*$. By 1'., $O^{*\wedge} \subseteq A^\wedge$. Using 2. and transitivity of $\subseteq$, we have $O \subseteq A^\wedge$.
   (b) Suppose that $O \subseteq A^\wedge$. Let $o \in \mathcal{O}$ and $a \in \mathcal{A}$. Then

$$\begin{aligned}\phi_O(o) \leq \phi_{A^\wedge}(o) &= \inf_{\tilde{a}\in\mathcal{A}} [\phi_A(\tilde{a}) \Rightarrow \phi_{I_\mathrm{f}}(o,\tilde{a}))] \\ &\leq [\phi_A(a) \Rightarrow \phi_{I_\mathrm{f}}(o,a)] \\ &= \sup\{z \in [0,1] : \phi_A(a) \vartriangle z \leq \phi_{I_\mathrm{f}}(o,a)\}.\end{aligned}$$

   By lower semicontinuity of $\vartriangle$, Proposition 19 holds and so

$$\phi_O(o) \in \{z \in [0,1] : \phi_A(a) \vartriangle z \leq \phi_{I_\mathrm{f}}(o,a)\}.$$

   Using commutativity of $\vartriangle$, we have

$$\phi_{O\times_\vartriangle A}(o,a) = \phi_O(o) \vartriangle \phi_A(a) \leq \phi_{I_\mathrm{f}}(o,a).$$

   But $o \in \mathcal{O}$ and $a \in \mathcal{A}$ are arbitrary and so $O \times_\vartriangle A \subseteq I_\mathrm{f}$.
   (c) Suppose that $O \times_\vartriangle A \subseteq I_\mathrm{f}$. Then for all $o \in \mathcal{O}$ and for all $a \in \mathcal{A}$,

$$\phi_A(a) \in \{z \in [0,1] : \phi_O(o) \vartriangle z \leq \phi_{I_\mathrm{f}}(o,a)\},$$

   whence

$$\phi_A(a) \leq \sup\{z \in [0,1] : \phi_O(o) \vartriangle z \leq \phi_{I_\mathrm{f}}(o,a)\} = [\phi_O(o) \Rightarrow \phi_{I_\mathrm{f}}(o,a)]\,.$$

   Taking the infimum over o on the right-hand side, we see that

$$\phi_A(a) \leq \phi_{O^*}(a).$$

   But $a \in \mathcal{A}$ is arbitrary, and so $A \subseteq O^*$.

As stated earlier, we want to prove that a lattice of fuzzy formal concepts is complete. As we shall see, the fuzzy implication we use when defining * and $\wedge$ has to be upper semicontinuous. It turns out that for R-implicationsm this property follows from lower semicontinuity of the t-norm, as we show in Proposition 23. Before that we make an intermediate step.

Lemma 22. *Let $X, Y \subseteq \mathbb{R}$. Let $f : X \to Y$ be an increasing function. Let (x_n) be a sequence on X such that $x_n \searrow x \in X$. Let (y_n) be a sequence on Y such that, for all $n > 0$, $f(x_n) \leq y_n$. Then*

$$f(x) \leq \liminf_{n\to\infty} y_n. \tag{19}$$

Proof. By monotonicity of (x_n), we have $\sup_{m\geq n_0} x_m = x_{n_0}$ for all $n_0 > 0$, whence for each $n_0 > 0$ fixed, $\inf_{n>0} \left(\sup_{m\geq n} x_m\right) \leq x_{n_0}$. But (x_n) converges to x, and so for all $n_0 > 0$ we have $x = \limsup_{n\to\infty} x_n \leq x_{n_0}$. Because f is increasing we have, for each $n_0 > 0$, $f(x) \leq f(x_{n_0}) \leq y_{n_0}$. Now, taking the limit inferior over n_0 on the right-hand side and changing the variable n_0 to n, we complete the proof.

Proposition 23. *Let $\triangle$ be a lower semicontinuous t-norm, and let $\Rightarrow$ be the R-implication induced by $\triangle$. Then for all $x_0, y_0 \in [0, 1]$ fixed, the maps $x \mapsto (x \Rightarrow y_0)$ and $y \mapsto (x_0 \Rightarrow y)$ are upper semicontinuous.*

Proof. Let $x', y' \in [0, 1]$. We want to show that $x \mapsto (x \Rightarrow y_0)$ and $y \mapsto (x_0 \Rightarrow y)$ are upper semicontinuous at x' and y' respectively. Let $(x_n), (y_n)$ be sequences on $[0, 1]$ converging respectively to x' and y'. For each $n > 0$, consider the following definitions:

$$z_n^{(1)} = (x_n \Rightarrow y_0), \qquad z_n^{(2)} = (x_0 \Rightarrow y_n), \tag{20}$$

$$\tilde{x}^{(1)} = x', \qquad \tilde{x}^{(2)} = x_0, \tag{21}$$

$$\tilde{x}_n^{(1)} = \inf_{m\geq n} x_m, \qquad \tilde{x}_n^{(2)} = x_0, \tag{22}$$

$$\tilde{y}^{(1)} = y_0, \qquad \tilde{y}^{(2)} = y', \tag{23}$$

$$\tilde{y}_n^{(1)} = y_0, \qquad \tilde{y}_n^{(2)} = \sup_{m\geq n} y_m, \tag{24}$$

$$\tilde{z}_n^{(1)} = \left(\tilde{x}_n^{(1)} \Rightarrow \tilde{y}_n^{(1)}\right), \qquad \tilde{z}_n^{(2)} = \left(\tilde{x}_n^{(2)} \Rightarrow \tilde{y}_n^{(2)}\right). \tag{25}$$

Notice that, for $i = 1, 2$ the following hold:

1. $\tilde{x}_n^{(i)} \nearrow \tilde{x}^{(i)}$, as the $\tilde{x}_n^{(1)}$ are infima of decreasing sets and $\left(\tilde{x}_n^{(2)}\right)$ is constant;
2. $\tilde{y}_n^{(i)} \searrow \tilde{y}^{(i)}$, because $\left(\tilde{y}_n^{(1)}\right)$ is constant and the $\tilde{x}_n^{(2)}$ are suprema of decreasing sets;
3. For all $n > 0$ monotonicity of $\Rightarrow$ yields $z_n^{(i)} \leq \tilde{z}_n^{(i)}$, whence $\limsup_{n\to\infty} z_n^{(i)} \leq \limsup_{n\to\infty} \tilde{z}_n^{(i)}$.
4. The sequence $(\tilde{z}_n^{(i)})$ is decreasing, for $i = 1, 2$.

Now let $n_0 > 0$ be fixed, and let $n > n_0$. By Proposition 19 and (25), we have

$$\tilde{x}_{n_0}^{(i)} \vartriangle \tilde{z}_n^{(i)} \leq \tilde{x}_n^{(i)} \vartriangle \tilde{z}_n^{(i)} \leq \tilde{y}_n^{(i)}.$$

Thus, (19) yields

$$\tilde{x}_{n_0}^{(i)} \vartriangle \left(\limsup_{n\to\infty} \tilde{z}_n^{(i)} \right) \leq \liminf_{n\to\infty} \tilde{y}_n^{(i)} = \tilde{y}^{(i)}.$$

Taking the limit inferior over n_0 on the left-hand side and using lower semicontinuity of $\vartriangle$,

$$\tilde{x}^{(i)} \vartriangle \left(\limsup_{n\to\infty} \tilde{z}_n^{(i)} \right) \leq \liminf_{n_0\to\infty} \left[\tilde{x}_{n_0}^{(i)} \vartriangle \left(\limsup_{n\to\infty} \tilde{z}_n^{(i)} \right) \right] \leq \tilde{y}^{(i)}.$$

Hence by definition of $\left(\tilde{x}^{(i)} \Rightarrow \tilde{y}^{(i)}\right)$ and 3. above,

$$\limsup_{n\to\infty} z_n^{(i)} \leq \limsup_{n\to\infty} \tilde{z}_n^{(i)} \leq \left(\tilde{x}^{(i)} \Rightarrow \tilde{y}^{(i)} \right).$$

For $i = 1$ this means $x \mapsto (x \Rightarrow y_0)$ is upper semicontinuous at x', and for $i = 2$, $y \mapsto (x_0 \Rightarrow y)$ is upper semicontinuous at y'. But $x', y' \in [0, 1]$ are arbitrary. Therefore, both the maps $x \mapsto (x \Rightarrow y_0)$ and $y \mapsto (x_0 \Rightarrow y)$ are upper semicontinuous.

Proposition 24. *Let $\mathbb{C}_\mathrm{f} = \langle \mathcal{O}, \mathcal{A}, I_\mathrm{f} \rangle$ be a fuzzy context. Let $\vartriangle$ be a lower semicontinuous t-norm, and let $\Rightarrow$ be the implication induced by it. Let J be an index set and, for each $\alpha \in J$, let $O_\alpha \subseteq \mathcal{O}$ and $A_\alpha \subseteq \mathcal{A}$. Then*

$$\bigcap_{\alpha\in J} O_\alpha^* = \left(\bigcup_{\alpha\in J} O_\alpha \right)^*, \tag{26}$$

$$\bigcap_{\alpha\in J} A_\alpha^\wedge = \left(\bigcup_{\alpha\in J} A_\alpha \right)^\wedge. \tag{27}$$

Proof. We prove (26). The proof of (27) is analogous. Let $a \in \mathcal{A}$. For each $\alpha_0 \in J$ we have

$$\begin{aligned}
\phi_{(\cup_{\alpha\in J} O_\alpha)^*}(a) &= \inf_{o\in\mathcal{O}} \left[\phi_{\cup_{\alpha\in J} O_\alpha}(o) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right] \\
&= \inf_{o\in\mathcal{O}} \left[\left(\sup_{\alpha\in J} \phi_{O_\alpha}(o) \right) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right] \\
&\leq \left[\left(\sup_{\alpha\in J} \phi_{O_\alpha}(o) \right) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right] \\
&\leq \left[\phi_{O_{\alpha_0}}(o) \Rightarrow \phi_{I_\mathrm{f}}(o,a) \right],
\end{aligned}$$

as $\Rightarrow$ is decreasing in the first component. Applying the infimum over $o \in \mathcal{O}$ and then the infimum over $\alpha \in J$ on the right-hand side yields $\phi_{(\cup_{\alpha\in J} O_\alpha)^*}(a) \leq \phi_{\cap_{\alpha\in J} O_\alpha^*}(a)$. Since $a \in \mathcal{A}$ is arbitrary, $\left(\bigcup_{\alpha\in J} O_\alpha\right)^* \subseteq \bigcap_{\alpha\in J} O_\alpha^*$.

On the other hand, suppose for the sake of contradiction that for some $o \in \mathcal{O}$ and $a \in \mathcal{A}$,

$$\kappa := \inf_{\alpha\in J}\left[\inf_{\tilde{o}\in\mathcal{O}}\left(\phi_{O_\alpha}(\tilde{o}) \Rightarrow \phi_{I_{\mathrm{f}}}(\tilde{o},a)\right)\right] > \left[\left(\sup_{\alpha\in J}\phi_{O_\alpha}(o)\right) \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right].$$

Define $x_0 = \sup_{\alpha\in J}\phi_{O_\alpha}(o)$. Let (α_n) be a sequence on J such that $\left(\phi_{O_{\alpha_n}}(o)\right)$ converges to x_0 and, for each $n > 0$, define $x_n = \phi_{O_{\alpha_n}}(o)$. By hypothesis, for each $n > 0$,

$$\begin{aligned}\left(x_0 \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right) &< \inf_{\alpha\in J}\left[\inf_{\tilde{o}\in\mathcal{O}}\left(\phi_{O_\alpha}(\tilde{o}) \Rightarrow \phi_{I_{\mathrm{f}}}(\tilde{o},a)\right)\right] \quad (=\kappa)\\ &\leq \inf_{\tilde{o}\in\mathcal{O}}\left(\phi_{O_{\alpha_n}}(\tilde{o}) \Rightarrow \phi_{I_{\mathrm{f}}}(\tilde{o},a)\right)\\ &\leq \left(\phi_{O_{\alpha_n}}(o) \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right)\\ &= \left(x_n \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right).\end{aligned}$$

Thus,

$$\left(x_0 \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right) < \kappa \leq \limsup_{n\to\infty}\left(x_n \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right),$$

and so $x \mapsto \left(x \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right)$ is not upper semicontinuous, contradicting Proposition 23.

Hence, for all $o \in \mathcal{O}$ and for all $a \in \mathcal{A}$,

$$\phi_{\bigcap_{\alpha\in J}O^*_\alpha}(a) = \kappa \leq \left[\left(\phi_{\bigcup_{\alpha\in J}O_\alpha}(o)\right) \Rightarrow \phi_{I_{\mathrm{f}}}(o,a)\right].$$

Taking the infimum over o on the right-hand side, and since $a \in \mathcal{A}$ is arbitrary, we conclude that $\bigcap_{\alpha\in J}O^*_\alpha \subseteq \left(\bigcup_{\alpha\in J}O_\alpha\right)^*$.

Theorem 25. *Let $\mathbb{C}_{\mathrm{f}} = \langle\mathcal{O},\mathcal{A},I_{\mathrm{f}}\rangle$ be a fuzzy formal context. Using a lower semicontinuous t-norm and the R-implication induced by it to define the maps * and $^\wedge$, define the order $\leq$ on the set $\mathfrak{B}_{\mathrm{f}}(\mathbb{C}_{\mathrm{f}})$ of all the fuzzy formal concepts of $\mathbb{C}_{\mathrm{f}}$ by*

$$\langle O_1,A_1\rangle \leq \langle O_2,A_2\rangle \text{ iff } O_1 \subseteq O_2 \qquad (\text{iff } A_2 = O^*_2 \subseteq O^*_1 = A_1). \tag{28}$$

Then $\mathcal{L}_{\mathbb{C}_{\mathrm{f}}} := \langle\mathfrak{B}_{\mathrm{f}}(\mathbb{C}_{\mathrm{f}}),\leq\rangle$ is a complete lattice, called the fuzzy concept lattice *of $\mathbb{C}_{\mathrm{f}}$.*

Proof. Similar to that of Theorem 8, using Proposition 24 rather than Proposition 7.

4 Final Remarks

As mentioned in the Introduction, there exists a vast body of literature on fuzzy Formal Context Analysis. Burusco and Fuentes-González (1994) introduced fuzzy concept lattices. Belohlávek and Vychodil (2007) have shown that it is possible to define a complete lattice of fuzzy concepts, by means of fuzzy implications obtained from left continuous t-norms. These t-norms are equivalent to lower semicontinuous t-norms. In this paper we recast some basic notions and results from fuzzy FCA in terms of lower semicontinuous t-norms. Given the difference between both definitions for t-norms (left continuous and lower semicontinuous), proving the results requires somewhat different techniques. Within our framework it is shown that the set of fuzzy concepts is a complete lattice.

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