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Scattering of charged particles by random electromagnetic fields

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Summary: A general, relativistic formalism for particle scattering by random electromagnetic fields is developed in the weak-interaction approximation. The scattering is described by a FOKKER-PLANCK equation, the coefficients of which are derived as linear spectral integrals over an infinite set of resonance surfaces. The formulae are evaluated explicitly for three field models. The theory is applied to the spatial diffusion parallel to the mean field, the equilibrium pitch-angle distribution, and FERMI acceleration processes. Detailed comparisons with observations will be given in subsequent papers.

Zusammenfassung: Es wird eine allgemeine, relativistische Theorie für die Streuung geladener Teilchen durch statistische elektromagnetische Störfelder entwickelt. Die Streuung wird durch eine FOKKER-PLANCK-Gleichung beschrieben, deren Koeffizienten durch Spektralintegrale über eine unendliche Schar von Resonanzflächen dargestellt werden. Die Integrale werden für drei Feldmodelle explizit ausgewertet. Als Anwendungsbeispiele werden die räumliche Diffusion parallel zum mittleren Feld, die Gleichgewichts-Pitchwinkelverteilung und FERMI Beschleunigungsprozesse untersucht. Der Vergleich mit Beobachtungen wird in späteren Arbeiten ausführlicher dargestellt.

1. Introduction

Interactions between charged particles and fluctuating electromagnetic fields play an important role in a number of geo- and astrophysical problems. The particle distributions of the solar wind and the magnetosphere represent collision less plasmas, for which the details of distributions are determined primarily by wave-particle rather than particle-particle interactions. Various scattering processes of high-energy particles in interplanetary space, as infered from solar-flare particle propagation or the solar modulation of the galactic radiation, are also believed to have their origin in random irregularities in the interplanetary fields.

It appears worthwhile, therefore, to investigate in a general manner the influence of an arbitrary random electromagnetic field on the particle distributions of a magnetized plasma. Various aspects of the problem have been treated in recent papers by JOKIPII (1966), ROELOF (1966), KENNEL and PETSCHEK (1966), and others. In particular, it has been recognized that the evolution of the particle distributions can be described by a

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FOKKER-PLANCK equation, the coefficients of which depend linearly on the field covariance functions or spectra. However, the functional dependence on the spectra has not been derived in closed form, so that estimates of the coefficients have usually been based on various rather questionable approximations. By using spectral representations throughout and applying the asymptotic resonance relations of wave scattering theory (c.f. HASSELMANN, 1967), we shall show in the following that the FOKKER-PLANCK coefficients can be derived in closed form as spectral integrals over an infinite series of equidistant resonance surfaces.

Resonant interactions occur between individual particles and FOURIER components of the electromagnetic field whenever the "parallel" frequency (i. e., the DOPPLER shifted and LORENTZ dilated frequency of the field component relative to the parallel particle motion) is equal to a multiple of the particle's LARMOR frequency. Special cases include the Landau resonance and the MHD- and Whistler-mode resonances considered by DRAGT (1961), WENZEL (1961), KENNEL and PETSCHEK (1966) and, in a somewhat different context, by JOKIPII (1966) and ROELOF (1966). The wave number in these cases is parallel to the mean field. If the wave number is non-parallel, an infinite number of resonances occur. They are essentially the same resonances that occur in the theory of plasma waves (c. f. STIX, 1962), except that we are concerned here with their effect on the particle distributions rather than on the wave motion. Normally, the FOKKER-PLANCK coefficients include comparable contributions from many resonances.

The theory is based on the weak-interaction approximation, which requires that $\partial H/\partial\omega \ll HT$, where H is the electromagnetic spectrum, ω is the frequency, and T is a characteristic transfer time. The condition does not depend explicitly on the LARMOR frequency; it involves only the "smoothness" of the spectrum. The analysis is carried through explicitly as a perturbation about the guiding centre description, which requires additionally that the transfer time is large compared with the gyration period. In many applications, both conditions are satisfied. However, the guiding-centre approximation is not a basic limitation of the theory. In the case of a non-magnetized plasma, or a plasma with a weak mean field, an alternative FOKKER-PLANCK equation can be derived by perturbing about the zero'th order distributions for a non-magnetized plasma (Appendix).

A covariant tensor notation is used to provide a unified treatment for electric and magnetic interactions and space-time variations. A further advantage is the natural derivation of interrelationships between different transfer processes, such as pitch-angle scattering and FERMI acceleration (Section 8).

The FOKKER-PLANCK coefficients are evaluated explicitly for two axisymmetric magnetic field models and one isotropic model, all three of which include arbitrary circular polarisation factors. The theory is then applied to pitch-angle scattering, spatial diffusion parallel to the mean field, gradient-induced anisotropies and FERMI acceleration processes. Comparison with observations will be given in later papers.

The random electromagnetic field is regarded throughout as given. Normally, the electromagnetic fluctuations will be associated with plasma waves, whose dispersion and stability characteristics depend on the particle distributions. To predict both the particle distributions and the electromagnetic spectrum, the FOKKER-PLANCK equation describing the effect of the field on the particle distributions must be considered together with the complementary transport equation describing the back-interaction of the particle distributions on the plasma-wave spectra. However, we shall not treat the full cross-coupled problem here. The question has been discussed for the case of the radiation belts by KENNEL and PETSCHEK (1966) and for conditions in interplanetary space by SCARF (1966).

2. Representation of the field and equations of motion

Consider the relativistic motion of a particle of charge e and rest mass m in an electromagnetic field which consists of a uniform magnetic component B in the x^3-direction and superimposed random fluctuations.

We assume that the random field has zero mean and is statistically stationary and homogeneous. Its four-potential φ_i $(i = 0, 1, 2, 3)$ may then be represented by a FOURIER-STIELTJES integral

$$\varphi_i(x) = \int d\varphi_i(k)\, e^{\iota k_j x^j}, \qquad d\varphi_i(-k) = d\varphi_i(k)^*, \qquad \iota = \sqrt{-1}, \tag{2.1}$$

with statistically orthogonal FOURIER increments,

$$\langle d\varphi_i^*(k')\, d\varphi_j(k)\rangle = H_{ij}(k)\,\delta(k'-k)\, dk\, dk', \tag{2.2}$$

where $H_{ij}(k)$ is the (four-dimensional) electromagnetic spectral density matrix. Cornered parentheses denote ensemble expectation values.

The LORENTZ condition $\partial\varphi_i/\partial x_i = 0$ yields

$$k^i H_{ij} = k^j H_{ij} = 0 \tag{2.3}$$

Covariant and contravariant components are related through

$$x_i = g_{ij} x^j$$

where

$$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -c^{-2},\ g_{ij} = 0 \text{ for } i \neq j.$$

The equations of motion of the particle may then be written

$$\frac{du^i}{d\tau} - \Omega M^i_j u^j = \frac{e}{mc} F^i_j u^j \tag{2.4}$$

where $u^i = dr^i/d\tau$, r^i is the particle position, τ is the eigen-time, $d\tau^2 = dx_i\,dx^i$, $\Omega = eB/mc$ is the cyclotron frequency[1]),

$$M^i_j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

and F_{ij} is the random electromagnetic field tensor,

$$F_{ij} = \frac{\partial \varphi_j}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^j} = \int dF_{ij}(k)\, e^{\iota k_l x^l}$$

with

$$dF_{ij}(k) = \iota(k_i d\varphi_j(k) - k_j d\varphi_i(k)) \tag{2.6}$$

We shall assume that the random field can be regarded as a small perturbation. The zero'th order particle motion $\overset{0}{r}{}^i$, $\overset{0}{u}{}^i$ is then given by the equations

$$\frac{d\overset{0}{u}{}^i}{d\tau} - \Omega M^i_j \overset{0}{u}{}^j = 0 \tag{2.7}$$

The general solution is

$$\overset{0}{u}{}^i = a^i_j\, \overset{0}{\mathrm{u}}{}^j \tag{2.8}$$

where

$$\overset{0}{\mathrm{u}}{}^j = \mathrm{U}^j e^{\iota\omega_{(j)}\tau}, \quad \mathrm{U}^i = \text{const} \tag{2.9}$$

and

$$a^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\iota}{\sqrt{2}} & -\frac{\iota}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.10}$$

is the eigen-vector matrix of equation (2.7); the eigen-frequencies are

$$\begin{aligned} \omega_{(0)} &= \omega_{(3)} = 0 \\ \omega_{(1)} &= -\omega_{(2)} = \Omega \end{aligned} \tag{2.11}$$

[1]) Ω can be positive or negative, according to the sign of the charge.

We note that according to equation (2.9) the amplitudes U^j and eigen-frequencies $\omega_{(j)}$ are not vector variables. Parentheses are used here to denote indices which are excluded from the sum convention.

The amplitudes satisfy the reality conditions

$$\begin{aligned} &\mathrm{U}^1=(\mathrm{U}^2)^* \\ &\mathrm{U}^0, \mathrm{U}^3 \text{ real} \end{aligned} \tag{2.12}$$

and the relation

$$(\mathrm{U}^0)^2-\frac{1}{c^2}(2|\mathrm{U}^1|^2+(\mathrm{U}^3)^2)=1, \tag{2.13}$$

which follows from the normalisation $u^i u_i = 1$.

The usual parallel and perpendicular velocities are given by

$$u_{\|}=\mathrm{U}^3,\ u_{\perp}=\sqrt{2}\,|\mathrm{U}^1|.$$

Besides velocities u^i, $u_{\|}$, $u_{\perp}$ with respect to eigen-time we shall use velocities $v^i = dx^i/dt$, $v_{\|}$, $v_{\perp}$ with respect to real time, where

$$v^i = u^i/\gamma,\ \text{etc.};$$

$$\gamma=u^0=\left(1-\frac{v_\perp^2+v_\|^2}{c^2}\right)^{-\frac{1}{2}}=\left(1+\frac{u_\perp^2+u_\|^2}{c^2}\right)^{\frac{1}{2}}$$

represents the ratio of the total energy of a particle to its rest energy.

For future reference, we write down the zero'th order particle position

$$\overset{0}{\mathrm{r}}{}^i=(a_0^i\mathrm{U}^0+a_3^i\mathrm{U}^3)\tau+\frac{a_1^i\mathrm{U}^1 e^{\iota\Omega\tau}}{\iota\Omega}-\frac{a_2^i\mathrm{U}^2 e^{-\iota\Omega\tau}}{\iota\Omega} \tag{2.14}$$

In the following, we shall be concerned primarily with the non-stationary response of the linear system (2.7) to various forms of resonant excitation. For this purpose, we need to resolve the excitation into its normal-mode constituents, i. e. to transform from variables u^i in a cartesian basis to helical variables u^i (non-cursive symbols) in the eigen-vector basis. The transformation is given by equations (2.8), (2.10); it affects only the perpendicular components,

$$\begin{aligned} u^1&=\frac{1}{\sqrt{2}}(\mathrm{u}^1+\mathrm{u}^2), & \mathrm{u}^1&=\frac{1}{\sqrt{2}}(u^1-\iota u^2) \\ u^2&=\frac{\iota}{\sqrt{2}}(\mathrm{u}^1-\mathrm{u}^2), & \mathrm{u}^2&=\frac{1}{\sqrt{2}}(u^1+\iota u^2) \end{aligned} \tag{2.15}$$

The corresponding transformation for the covariant components is given by

$$u_1 = \frac{1}{\sqrt{2}}(\mathrm{u}_1 + \mathrm{u}_2), \qquad \mathrm{u}_1 = \frac{1}{\sqrt{2}}(u_1 + \iota u_2)$$
$$u_2 = \frac{-\iota}{\sqrt{2}}(\mathrm{u}_1 - \mathrm{u}_2), \qquad \mathrm{u}_2 = \frac{1}{\sqrt{2}}(u_1 - \iota u_2) \tag{2.16}$$

We note that co- and contravariant helical components satisfy the relation

$$\mathrm{u}_i^* = g_{ij}\,\mathrm{u}^j \tag{2.17}$$

Although the amplitudes U^j are non-tensor variables, it is convenient to introduce "covariant" amplitudes U_j defined by the transformation (2.17). Equation (2.9) can then be written in the covariant form

$$\mathrm{u}_i = \mathrm{U}_i\, e^{-\iota\omega_{(i)}\tau} \tag{2.18}$$

In helical coordinates, the equations of motion now become

$$\left(\frac{d}{d\tau} - \iota\omega_{(i)}\right)\mathrm{u}^i = \frac{e\mathrm{u}_j}{mc}\int d\mathrm{F}^i_j(k)\, e^{\iota \mathrm{k}_l \mathrm{x}^l} \tag{2.19}$$

where the operator on the left-hand side is diagonal.

3. Perturbation expansion

We attempt to construct a solution of the full equations of motion by expanding the particle motion in a perturbation series,

$$r^i = \overset{0}{r}{}^i + \overset{1}{r}{}^i + \overset{2}{r}{}^i + \ldots \tag{3.1}$$

where $\overset{n}{r}{}^i$ is of order n in the random field amplitudes.

The local field at the position of the particle may be similarly expanded,

$$\varphi_i(r(\tau)) = \overset{1}{\varphi}_i + \overset{2}{\varphi}_i + \ldots \tag{3.2}$$

where

$$\overset{1}{\varphi}_i(\tau) = \varphi_i(\overset{0}{r}(\tau)) \tag{3.3}$$

$$\overset{2}{\varphi}_i(\tau) = \overset{1}{r}{}^j \frac{\partial}{\partial x^j}\varphi_i(\overset{0}{r}(\tau)) \tag{3.4}$$

etc.

Substituting the FOURIER expansion (2.1) and the zero'th order solution (2.14) in equation (3.3), we obtain for the first-order field

$$\overset{1}{\varphi}_i = \int d\varphi_i(k) \exp\left\{\iota(\mathrm{k}_0 \mathrm{U}^0 + \mathrm{k}_3 U^3)\tau + \frac{\mathrm{k}_1 \mathrm{U}^1 e^{\iota\Omega\tau}}{\Omega} - \frac{\mathrm{k}_2 \mathrm{U}^2 e^{-\iota\Omega\tau}}{\Omega}\right\}$$

On expanding the exponential in a BESSEL series, this may be written

$$\overset{1}{\varphi}_i = \int d\varphi_i(k)\, e^{\iota\omega_{\|}\tau} \sum_{p=-\infty}^{\infty} c_k^p\, e^{\iota p\Omega\tau} \tag{3.5}$$

where

$$\omega_{\|} = \mathrm{k}_0 u^0 + \mathrm{k}_3 u^3 \tag{3.6}$$

$$c_k^p = J_p(\varrho_\perp)\, e^{\iota p\Theta} \tag{3.7}$$

and $\varrho_\perp$ and Θ are defined by

$$\frac{2\mathrm{k}_1 \mathrm{U}^1}{\Omega} = \varrho_\perp e^{\iota\Theta} \tag{3.8}$$

The phase angle Θ is found later to be irrelevant; the modulus is given by

$$\varrho_\perp = k_\perp r_\perp \tag{3.9}$$

where

$$k_\perp = (k_1^2 + k_2^2)^{\frac{1}{2}}$$

and $r_\perp = u_\perp/\Omega$ is the LARMOR radius.

We note that the reality condition (2.12) yields the relation

$$c_k^p = (c_{-k}^{-p})^* \tag{3.10}$$

Equation (3.5) states that to first order a particle sees a given FOURIER component $d\varphi_i(k)$ as a series of sinusoidal components of frequency $\omega_{\|} + p\Omega, p = 0, \pm 1, \pm 2, \ldots$ The "parallel" frequency $\omega_{\|}$ represents the original frequency k_0 of the FOURIER component modified by the relativistic dilatation factor $u^0 = \gamma$ and the DOPPLER shift $k_3 u^3$ induced by the particle motion parallel to the field lines. It is clearly invariant with respect to LORENTZ transformations parallel to the mean field. A FOURIER component whose spatial variation is parallel to the mean field is seen by the particle as a single sinusoid of frequency $\omega_{\|}$. However, if the component varies also in the perpendicular direction, the particle's gyration motion leads to a splitting of the parallel frequency into a series of components separated by multiples of the cyclotron frequency.

If the transverse wave length $2\pi/k_\perp$ of the field component is comparable to the LARMOR radius $r_\perp$, the periodic lateral variations encountered by the particle along its helical path are no longer sinusoidal, so that all harmonics of the gyration frequency arise. The strength of the side lines is determined by the parameter $\varrho_\perp = k_\perp r_\perp$. For scattering involving a spectrum with a finite angular spread, $\varrho_\perp$ is typically of order unity, so that a number of higher harmonics normally need to be considered.

For the first-order particle motion we obtain from the equations of motion (2.19) and the perturbation expansions (3.1) and (3.3)

$$\left(\frac{d}{d\tau} - \iota\omega_{(i)}\right)\overset{1}{\mathrm{u}}{}^i = \frac{e\mathrm{U}^j e^{\iota\omega_{(j)}\tau}}{mc} \int d\mathrm{F}^i_j(k)\, e^{\iota \mathrm{k}_l \overset{\circ}{r}{}^l}$$

Introducing the BESSEL expansion (3.5), this yields

$$\left(\frac{d}{d\tau} - \iota\omega_{(i)}\right)\overset{1}{\mathrm{u}}{}^i = \frac{e\mathrm{U}^j}{mc} \int d\mathrm{F}^i_j(k) \sum_{p=-\infty}^{\infty} c^p_k \exp\{\iota(\omega_{(j)} + \omega_{\parallel} + p\Omega)\tau\} \tag{3.11}$$

If the initial perturbation is taken as zero, the solution at time τ is then given by

$$\overset{1}{\mathrm{u}}{}^i = \frac{e\mathrm{U}^j}{mc} \int d\mathrm{F}^i_j(k) \sum_{p=-\infty}^{\infty} c^p_k e^{\iota(\omega_{(j)} + \omega_{\parallel} + p\Omega)\tau} \Delta_1(\omega_{(j)} + \omega_{\parallel} + p\Omega - \omega_{(i)}) \tag{3.12}$$

where

$$\Delta_1(\omega') = \frac{1 - e^{-\iota\omega'\tau}}{\iota\omega'} \tag{3.13}$$

is the response function of the operator $(d/d\tau - \iota\omega)$ to the sinusoidal excitation $\exp\{\iota(\omega + \omega')\tau\}$. We note that the definition of the response function includes the homogeneous solution needed to satisfy the initial condition. In this manner, the singularity which would otherwise occur at the resonance point $\omega' = 0$ is removed, enabling the investigation of the secular behaviour at $\omega' = 0$. The solution contains resonances whenever the parallel frequency is equal to a multiple of the cyclotron frequency.

Integration of equation (3.12) yields the first-order particle positions,

$$\overset{1}{\mathrm{r}}{}^i = \frac{e\mathrm{U}^j}{mc} \int d\mathrm{F}^i_j(k) \sum_{p=-\infty}^{\infty} c^p_k e^{\iota(\omega_{(j)} + \omega_{\parallel} + p\Omega)\tau} \Delta_2(\omega_{(j)} + \omega_{\parallel} + p\Omega, \omega_{(j)} + \omega_{\parallel} + p\Omega - \omega_{(i)}) \tag{3.14}$$

where

$$e^{\iota\omega\tau} \Delta_2(\omega, \omega') = \int_0^{\tau} e^{\iota\omega\tau'} \Delta_1(\omega', \tau')\, d\tau' \tag{3.15}$$

or

$$\Delta_2(\omega, \omega') = \frac{\Delta_1(\omega) - e^{-\iota\omega'\tau} \Delta_1(\omega - \omega')}{\iota\omega'} \tag{3.16}$$

4. The transition moments

Consider now an ensemble of identical, non-interacting particles characterized by a number density with respect to eight-dimensional $x - u$ space (which in the present treatment replaces the usual non-covariant number density in six-dimensional $\boldsymbol{x} - \boldsymbol{u}$ phase space). Since $u_i u^i = 1$, only three velocity variables need be given. We assume that in the undisturbed mean field the circular-phase distribution is uniform. The number of velocity variables can then be further reduced to two, so that we may introduce a six-dimensional number density n $(x, u^0, u^3; \tau)$, say, where the perpendicular velocity is determined by u^0 and u^3 through (2.13). In order that the velocity distributions remain strictly axisymmetric for spatially varying n, we shall interpret x here as referring to the guiding-centres rather than the instantaneous particle positions.

Interactions with the random field affect the distribution in two ways. The non-resonant perturbations of the particle motion lead to small, stationary distortions. Although important for the theory of plasma oscillations, these are not of interest in the present problem. The resonant interactions, on the other hand, produce secular variations of the particles zero'th order motion which lead ultimately to a finite modification of the initial particle distribution. The secular variations can be described by a FOKKER-PLANCK equation, in which the transport coefficients are determined by the first and second moments of the transition probabilities from one zero'th order particle state to another.

Let δx^i, δu^0 and δu^3 represent the variations of the zero'th order (guiding centre) parameters of a particle during the eigen-time τ.

We have

$$\begin{aligned}\delta u^i &= u^i - \overset{0}{u}{}^i = \overset{1}{u}{}^i + \overset{2}{u}{}^i + \dots \\ \delta x^i &= r^i - \overset{0}{r}{}^i = \overset{1}{r}{}^i + \overset{2}{r}{}^i + \dots\end{aligned} \tag{4.1}$$

Considering first the second moments, the expansion begins with the terms

$$\langle \delta u^i \delta u^j \rangle = \langle \overset{1}{u}{}^i \overset{1}{u}{}^j \rangle + \dots \tag{4.2}$$

$$\langle \delta u^i \delta x^j \rangle = \langle \overset{1}{u}{}^i \overset{1}{r}{}^j \rangle + \dots \tag{4.3}$$

$$\langle \delta x^i \delta x^j \rangle = \langle \overset{1}{r}{}^i \overset{1}{r}{}^j \rangle + \dots \tag{4.4}$$

As equations (4.1)—(4.4) stand, the right hand sides cannot strictly be identified with variations in the zero'th order particle variables, since they include both resonant and non-resonant perturbations. However, we shall be concerned primarily with the asymptotic form of the transition moments for large τ. In this limit, only the resonant terms contribute, and these may then be interpreted as secular changes in the zero'th order particle motions.

Substituting the solution (3.12) in equation (4.2), we obtain for the second velocity moment

$$\langle \delta u^i \delta u_j \rangle = \left(\frac{e}{mc}\right)^2 \int dk\, G^{in}{}_{jm}(k) \langle U_n U^m \rangle \cdot \sum_{p,\, p'=-\infty}^{\infty} c_k^p c_k^{-p'}$$

$$\cdot \{\exp[\iota(\omega_{(m)} - \omega_{(n)} + (p-p')\,\Omega)\tau] \cdot \Delta_1(\omega_{(m)} + \omega_{\|} + p\Omega + \omega_{(j)})$$

$$\cdot \Delta_1^*(\omega_{(n)} + \omega'_{\|} + p'\Omega + \omega_{(i)}) \tag{4.5}$$

where

$$G_{injm}(k) = \frac{\langle dF_{in}(-k)\, dF_{jm}(k)\rangle}{dk} = k_i k_j H_{nm} + k_n k_m H_{ij} - k_i k_m H_{nj} - k_j k_n H_{im} \tag{4.6}$$

For large τ, it is readily seen that

$$\Delta_1^*(\omega - \omega_1)\, \Delta_1(\omega - \omega_2) \to \begin{cases} 2\pi\tau\delta(\omega - \omega_1) + 0(1) \text{ for } \omega_1 = \omega_2 \\ 0(1) \text{ for } \omega_1 \neq \omega_2 \end{cases} \tag{4.7}$$

and one obtains asymptotically

$$\langle \delta u^i \delta u_j \rangle = -\frac{\langle \delta u^i \delta u^j \rangle}{c^2} \to 2\pi\tau \left(\frac{e}{mc}\right)^2 \langle U_n U^m \rangle$$

$$\cdot \int dk\, G^{in}{}_{jm}(k) \sum_{p=-\infty}^{\infty} |c_k^p|^2\, \delta(\omega_{\|} + p\Omega + \omega_{(m)}) \tag{4.8}$$

for $i, j = 0$ or 3.

For a uniform circular-phase distribution, the amplitude moments on the right hand side of (4.8) are given by

$$\langle U_n U^m \rangle = \begin{cases} -\delta_n^m \dfrac{|U^1|^2}{c^2} \text{ for } n = 1, 2;\ m = 0, 1, 2, 3 \\ -\dfrac{U^n U^m}{c^2} \quad \text{for } n, m = 0 \text{ or } 3 \end{cases} \tag{4.9}$$

To determine the remaining second moments (4.3) and (4.4), we need consider only the zero frequency resonances of $\overset{1}{r}{}^i$, since the resonances at $\pm\Omega$ correspond to variations of the LARMOR radius rather than the guiding centre. Furthermore, we can restrict the analysis to the transverse components of $\overset{1}{r}{}^i$, since the diffusion parallel to the mean field is governed to lowest order by the convection due to the zero'th order parallel velocities. In the same manner as above, we obtain then from (4.3), (4.4) and (3.14)

$$\langle \delta u^i \delta x^j \rangle = 0 \qquad (i=0,3;\ j=1,2) \tag{4.10}$$

and

$$\langle \delta x^i \delta x_j \rangle = \frac{2\pi\tau}{\omega_{(i)}\omega_{(j)}} \left(\frac{e}{mc}\right)^2 \langle U_n U^m \rangle \int dk\, G^{in}{}_{jm}(k) \sum_{p=-\infty}^{\infty} |c_k^p|^2\, \delta(\omega_{(m)} + \omega_{\|} + p\Omega) \qquad (i, j = 1 \text{ or } 2) \tag{4.11}$$

The second transition moments depend only on the spectral densities on the resonance surfaces $\omega_{(m)} + \omega_{\|} + p\Omega = 0$. Formally,

$$\langle \delta x^i \delta x_j \rangle = \frac{\langle \delta u^i \delta u_j \rangle}{\omega_{(i)}\omega_{(j)}},$$

although equations (4.8) and (4.11) apply for different indices.

The asymptotic expressions for the transition moments remain valid so long as the perturbations can still be regarded as small. This implies small τ, as opposed to the large τ required for the asymptotic response relation (4.7). Clearly, the two conditions will be compatible only if the interactions are sufficiently weak. To determine the precise form of this restriction, we need to investigate the limiting process involved in the transition from equation (4.5) to (4.8) (the derivation of (4.10) and (4.11) follows analogously).

For large τ, the response function $\Delta_1(\omega - \omega_1)$ develops a resonance peak at $\omega = \omega_1$ of amplitude τ and width $\delta\omega = 2\pi/\tau$. The quadratic product $\Delta_1(\omega - \omega_1).\ \Delta_1^*(\omega - \omega_2)$ in equation (4.5) will therefore have either two such peaks, if $\omega_1 \neq \omega_2$, or a single peak of amplitude τ^2 and width $\approx \delta\omega$, if $\omega_1 = \omega_2$. In the latter case, $|\Delta_1(\omega - \omega_1)|^2$ can be replaced in (4.8) by $2\pi\tau . \delta(\omega - \omega_1)$, provided the variation δH_{ij} of the spectrum H_{ij} within the frequency band $\delta\omega = \delta\omega_{\|}$ about the resonance frequency remains small. A first restriction on our analysis is thus that the velocity perturbations within the time τ and the spectral perturbations δH_{ij} within the conjugate frequency interval $\delta\omega = 2\pi/\tau$ can both be made small for a finite range of τ. In terms of the characteristic transfer term $T = \tau u^2/\langle \delta u^2 \rangle$, this may be written

$$\frac{\partial H_{ij}}{\partial \omega_{\|}} \ll \left(\frac{H_{ij}}{\delta\omega} = \frac{H_{ij}\tau}{2\pi}\right) \ll \frac{H_{ij}T}{2\pi}$$

where $\partial H_{ij}/\partial \omega_{\|}$ represents the spectral derivative normal to the resonance surface.

The condition is essentially a randomness criterion. Interpreted in terms of time-series analysis, it requires that the spectral densities on the resonance surfaces can be resolved from a finite field record of duration short compared with the transfer time[1]).

[1]) We note that the condition involves local spectral properties. It cannot be expressed, as has sometimes been assumed, in terms of correlation scales, which are integral moments of the spectrum.

If this is not the case, the field cannot be meaningfully described as a random spectral continuum. For example, if the power spectrum consists of a number of narrow peaks of width T^{-1} or less, the interaction analysis should be based on a deterministic field description in terms of a discrete line spectrum. The quadratic transition products are then either proportional to τ^2, for resonant particles, or constant, for non-resonant particles, and a FOKKER-PLANCK description is not applicable.

In the case $\omega_1 \neq \omega_2$, the integration over two separate peaks of amplitude τ and width $2\pi/\tau$ yields only a bounded contribution to (4.5), which may be neglected compared with the single-peak terms proportional to τ. This presumes, however, that τ is sufficiently large for the separate peaks to be resolved, or that $(\omega_1 - \omega_2)\tau \gg 1$. Since $\omega_1 - \omega_2$ in equation (4.5) is an integral multiple of Ω, the inequality is satisfied if the variations of the particle velocities within a gyration period remain small. This is the case if the field fluctuations in the appropriate wave number range are small compared with the mean field, as already assumed.

Apart from these conditions, a statistical closure relation has been invoked in the derivation of (4.8). In taking expectation values, the particle velocities and field amplitudes were treated as statistically independent (more precisely, the joint fourth cumulants were neglected). This is closely analogous to the "random-phase" or GAUSSIAN hypothesis in the theory of wave-wave and wave-external field interactions (cf. PEIERLS, 1929, HASSELMANN, 1967) and to BOLTZMANNS "molecular chaos" hypothesis for the case of interacting particles in a dilute gas. The closure hypothesis for weakly interacting systems has been investigated extensively in the classical and quantum-statistical theory of irreversible processes. Various methods of proof have been developed. Derivations based on the master equation method may be found, for example, in PRIGOGINE (1962) and in articles by KAMPEN and VAN HOVE in COHEN (1962).

5. The FOKKER-PLANCK equation

Since the second infinitesimal transition moments are proportional to τ, the evolution of the particle distribution n $(x, u^0, u^3; \tau)$ is governed by a FOKKER-PLANCK equation (cf. CHANDRASEKHAR 1943)

$$\frac{\partial \mathrm{n}}{\partial \tau}+\frac{\partial}{\partial x^i}(\bar{u}^i \mathrm{n})-\frac{\partial}{\partial x^i}\left(\kappa^{ij}\frac{\partial \mathrm{n}}{\partial x^j}\right)-\frac{\partial}{\partial u^i}\left(D^{ij}\frac{\partial \mathrm{n}}{\partial u^j}\right)=0 \tag{5.1}$$

where

$$\kappa^{ij} = \begin{cases} \dfrac{\langle \delta x^i \delta x^j \rangle}{2\tau} \text{ for } i,j=1 \text{ or } 2 \text{ (equation 4.10)} \\ 0 \text{ for } i \text{ or } j \neq 1,2 \\ \text{(negligible compared with the zero'th order convective term)} \end{cases} \tag{5.2}$$

$$D^{ij}=\frac{\langle\delta u^i\delta u^j\rangle}{2\tau}\quad\text{for } i,j=0 \text{ or } 3 \text{ (equation 4.8)}\tag{5.3}$$

and

$$\bar{u}^i=\begin{cases}u^i & \text{for } i=0 \text{ or } 3\\ 0 & \text{for } i=1 \text{ or } 2\end{cases}\tag{5.4}$$

Equation (5.1) does not include mean-acceleration terms

$$\frac{\partial}{\partial u^i}(\bar{a}^i\mathrm{n}),\quad i=0,3,$$

or transverse convective terms

$$\frac{\partial}{\partial x^i}(\bar{u}^i\mathrm{n}),\quad i=1,2.$$

It has been pointed out by DUNGEY (1965) that these must vanish on account of LIOUVILLE's theorem, which requires that n = const is a possible solution of (5.1). (See also FÄLTHAMMAR, 1966 and JOKIPII, 1966)[1]). The coefficients are given by

$$\bar{u}^i=\frac{\langle\delta x^i\rangle}{\tau}-\frac{\partial}{\partial x^j}\frac{\langle\delta x^i\delta x^j\rangle}{2\tau}\quad(i,j=1 \text{ or } 2)\tag{5.5}$$

$$\bar{a}^i=\frac{\langle\delta u^i\rangle}{\tau}-\frac{\partial}{\partial u^j}\frac{\langle\delta u^i\delta u^j\rangle}{2\tau}\quad(i,j=0 \text{ or } 3),\tag{5.6}$$

which in the present case imply an interrelationship between the first and second moments. It may be mentioned here that the evaluation of the first moments in terms of the perturbation expansion (which requires an extension of the present analysis to second order) does *not* yield zero $\bar{u}^i$ ($i=1,2$) and $\bar{a}^i$ ($i=0,3$) according to equations (5.5) and (5.6). Presumably, this is because the equations are strictly valid only for rigorous solutions of the equations of motion. Their derivation depends on the property that the particle flux in phase space is incompressible for each realisation of the ensemble of fields; it can be readily seen that this is not the case for the truncated perturbation solutions. Rather than follow up this question, however, we have given in the appendix an alternative derivation of the FOKKER-PLANCK equation based on a perturbation expansion of the LIOUVILLE equation as outlined by ROELOF (1967). It confirms $\bar{u}^i=0$ ($i=1,2$), $\bar{a}^i=0$ ($i=0,3$) and the relations (5.2), (5.3).

[1]) DUNGEY's argument is normally expressed for the mean acceleration terms, but applies equally to the transverse convection terms.

The FOKKER-PLANCK equation (5.1) describes the diffusion of particles (more precisely, guiding centres) as a function of eigen-time τ in the space $R_4(x)$. $R_2(u^0, u^3)$. Of interest physically is the number density $\tilde{n}(\boldsymbol{x}, u^0, u^3; x^0)$ in $R_3(x)$. $R_2(u^0, u^3)$ space, with $x^0 = t$ as parameter. This is given by the zero'th component

$$\tilde{n} = u^0 \cdot n = \gamma \cdot n$$

of the flux vector $S^i = u^i . n$, where

$$n = \int_{-\infty}^{\infty} \mathrm{n} d\tau .$$

$S^i\eta_i$ is the number of particles whose paths $x(\tau)$ pass with velocity u^0, u^3 through a unit surface element of normal η_i at x. The remaining three components represent the particle flux $\boldsymbol{S} = \boldsymbol{u}\, n = \boldsymbol{v}\, \tilde{n}$ in R_3. We shall use the number density n rather than $\tilde{n}$ in the following, since it is a scalar invariant.

The FOKKER-PLANCK equation for n follows by integrating (5.1) over τ,

$$\frac{\partial}{\partial x^i}(\bar{u}^i n) - \frac{\partial}{\partial x^i}\left(\kappa^{ij}\frac{\partial n}{\partial x^j}\right) - \frac{\partial}{\partial u^i}\left(D^{ij}\frac{\partial n}{\partial u^j}\right) = 0 \tag{5.7}$$

Equation (5.7), together with the expressions (5.2) to (5.4), (4.8) and (4.11) for the diffusion coefficients, complete the main part of our analysis. In the following sections we investigate in more detail particular solutions of the FOKKER-PLANCK equation (5.7) under simple conditions. In most applications, equation (5.7) needs to be extended to include adiabatic drift and accelerations terms due to slow variations of the mean field. However, we shall consider in this paper only effects arising from the random-field interactions.

In section 6, the diffusion coefficients are determined for the case of a time-independent, random magnetic field. The coefficients are evaluated explicitly for three models: (a) an axisymmetric transverse field varying parallel to the mean field, (b) an axisymmetric field in which the energy is concentrated in the perpendicular wave-number directions, and (c) an isotropic field. All models include an arbitrary circular polarisation.

For spatially uniform distributions, the FOKKER-PLANCK equation reduces in the case of a time-independent magnetic field to a one-dimensional diffusion equation for the pitch angles. All solutions tend for $\tau \to \infty$ to an isotropic equilibrium distribution. If the particle distribution is non-uniform, or the field time dependent, the degenerate one-dimensional diffusion character is destroyed, and an equilibrium distribution no longer exists. However, in many cases the time scales of the spatial diffusion or acceleration processes are large compared with the pitch-angle relaxation time, so that the processes can be regarded as small perturbations about the pitch-angle equilibrium. The spatial diffusion is treated in this manner in section 7, and the FERMI acceleration processes in section 8.

6. Diffusion coefficients for a random magnetic field

In many cases, the electric components of the random field can be neglected compared with the magnetic components. The ratio of electric to magnetic forces is of the order $E'c/B'v$, where E' and B' are typical field amplitudes and v is the particle velocity. Since $\dot{B} = -c\nabla \times E$, this is of the order a/v, where a is a characteristic phase velocity of the random field. Hence the electric forces are small if $v \gg a$. To the same approximation, the magnetic fields can be regarded as time independent, since the field frequencies are small compared with the DOPPLER shifts induced by the particle motion. This yields no notational simplification, however, and has been ignored in the general expressions (6.3) and (6.4) (the time-independent case is obtained simply by replacing k in (6.3) and (6.4) by $\boldsymbol{k}$).

The field tensor reduces in the case of a magnetic field B_γ ($\gamma = 1, 2, 3$) to

$$F_{\alpha\beta} = c^{-4} \varepsilon_{\alpha\beta}{}^{\gamma} B_\gamma \tag{6.1}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the LEVI-CIVITA symbol. The field spectrum G_{ijnm} can then be expressed in terms of the magnetic spectrum

$$B_{\alpha\beta}(k) = \frac{\langle dB_\alpha(-k)\, dB_\beta(k)\rangle}{dk} = C_{\alpha\beta} + iQ_{\alpha\beta} \tag{6.2}$$

where $dB_\alpha(k)$ represents a FOURIER-STIELTJES increment,

$$B_\alpha = \int dB_\alpha(k)\, e^{ik_i x^i}$$

and $C_{\alpha\beta}$, $Q_{\alpha\beta}$ are the co- and quadrature spectra, respectively.

Since the particle energy is not affected by the magnetic field, the diffusion coefficients D^{00} and D^{03} vanish. The remaining coefficients are found to reduce to

$$D^{33} = \frac{\pi}{2}\left(\frac{e}{mc}\right)^2 \frac{u_\perp^2}{c^4} \int \sum_{p=-\infty}^{\infty} |c_k^{p+1}|^2 (C^{11} + C^{22} + 2Q^{12})\, \delta(\omega_\parallel + p\Omega)\, dk \tag{6.3}$$

and

$$\kappa^{ij} = \frac{\pi}{2\Omega^2 c^4}\left(\frac{e}{mc}\right)^2 \int \sum_{p=-\infty}^{\infty} \delta(\omega_\parallel + p\Omega) \left\{ |c_k^{p+1}|^2 \left(\frac{-\delta^{ij}}{c^2}\right) u_\perp^2 C^{33} + 2\, |c_k^p|^2\, u_\parallel^2 C^{ij} \right\} dk \tag{6.4}$$

It is difficult to determine the magnetic spectrum experimentally, since single-satellite observations yield only the projection of the spectrum in a particular wave-number direction. For later applications, we shall accordingly consider three idealised models:

(a)

$$B_{ij}(k)=\begin{cases}\frac{1}{4}\delta(k_0)\,\delta(k_1)\,\delta(k_2)\,f(|k_3|)\left(\delta_{ij}+\iota\sigma(|k_3|)\,\varepsilon_{ij3}\frac{k_3}{|k_3|}\right) & \text{for } i,j=1,2\\ 0 \text{ for } i \text{ or } j=3\end{cases} \tag{6.5}$$

(b)

$$B_{ij}(k)=\begin{cases}0 \text{ for } |k_3|>|k_3^{\max}|\\ \frac{\delta(k_0)}{8\pi k_\perp^2}\left\{g_\perp\left(\delta_{ij}-\frac{k_i k_j}{k_\perp^2}\right)+(g_\parallel-g_\perp)\,\delta_{i3}\delta_{j3}+\iota\sigma\sqrt{g_\perp g_\parallel}\,\varepsilon_{ijn}\frac{k_n}{k_\perp}\right\} & \text{for } k_3=0\\ \quad (g_\perp=g_\perp(k_\perp),\ g_\parallel=g_\parallel(k_\perp),\ \sigma=\sigma(k_\perp))\end{cases} \tag{6.6}$$

(c)

$$B_{ij}(k)=\frac{\delta(k_0)}{8\pi\mathfrak{k}^2}f(\mathfrak{k})\left(\delta_{ij}-\frac{k_i k_j}{\mathfrak{k}^2}+\iota\sigma(\mathfrak{k})\,\varepsilon_{ijn}\,k_n/\mathfrak{k}\right)\quad(\mathfrak{k}=|k|) \tag{6.7}$$

For notational convenience δ_{ij} is defined here as the KRONECKER symbol rather than δ^i_j. We assume that in the range of interest the scalar spectra can be represented by power laws,

$$\begin{aligned} f(\varkappa)&=C\varkappa^q\,(\varkappa=\mathfrak{k} \text{ or } k_3)\\ g_{\perp,\parallel}&=C_{\perp,\parallel}k_\perp^q \end{aligned} \tag{6.8}$$

with constant C, $C_\perp$, $C_\parallel$ and q.

The spectrum (a) represents the general form for an axisymmetric, transverse field varying only in the direction parallel to the mean field. The spectrum (b) is also axisymmetric, but in this case the energy is concentrated in the perpendicular rather than parallel wave-number directions. It is zero for $|k_3| > |k_3^{\max}|$ (which is taken to be small). Thus for velocities in the range $|u_\parallel| < \Omega/k_3$, only the $p = 0$ resonance surface $k_\parallel u_\parallel + p\Omega = 0$ contributes to the transport coefficients. We need therefore consider only the general form (6.6) of an axisymmetric spectrum on the plane $k_3 = 0$. The spectrum (c), finally, is isotropic.

The spectra (a) and (c) are normalised such that

$$\int_0^\infty f(\varkappa)\,d\varkappa=\int B_{\alpha\alpha}(k)\,dk=\langle B_\alpha B_\alpha\rangle\,(\varkappa=\mathfrak{k} \text{ or } |k_3|).$$

The functions $g_\perp$ and $g_\parallel$ in (b) represent, respectively, the spectral density of the transverse field intensity $B_{11} + B_{22}$ and the longitudinal field intensity B_{33} on the surface $k_3 = 0$.

All three models are invariant under the rotations of their respective groups, but not reflections (cf. CHANDRASEKHAR, 1953). The circular polarisation is determined by the functions σ, where $-1 \leq \sigma \leq +1$. The limits correspond to pure left or right polarisation. For simplicity, we take σ in the following as constant. In case (b) the diffusion coefficients are found to be independent of the circular polarisation.

The models yield diffusion coefficients of the form

$$D^{33} = \left(\frac{e}{mc}\right)^2 u_\perp^2 (\alpha_1 + \sigma\alpha_2) \tag{6.9}$$

$$\kappa^{11} = \kappa^{22} = \left(\frac{e}{mc}\right)^2 \frac{u_\perp^2}{\Omega^2} \beta$$

$$\kappa^{12} = 0 \tag{6.10}$$

where in case (a)[1])

$$\begin{Bmatrix}\alpha_1 \\ \alpha_2\end{Bmatrix} = \frac{\pi}{4|u_\parallel|} f(|k_\parallel|) \begin{Bmatrix}1 \\ k_\parallel/|k_\parallel|\end{Bmatrix} = \frac{\pi C |\Omega|^q |u_\parallel|^{-q-1}}{4} \begin{Bmatrix}1 \\ \dfrac{\Omega}{|\Omega|} \dfrac{u_\parallel}{|u_\parallel|}\end{Bmatrix} \tag{6.11}$$

with $k_\parallel = \Omega/u_\parallel$,

$$\beta = \frac{\pi |u_\parallel|}{4 u_\perp^2} f(0), \tag{6.12}$$

in case (b)

$$\alpha_1 = \frac{\pi}{8|u_\parallel|} \int_0^\infty J_1^2(k_\perp r_\perp) q_\perp(k_\perp) \frac{dk_\perp}{k_\perp} = \frac{\pi |\Omega|^q C_\perp I_{1q}}{8 |u_\parallel| u_\perp^q}, \tag{6.13}$$

$$\alpha_2 = 0,$$

$$\beta = \frac{\pi}{8|u_\parallel|} \int_0^\infty \left\{ J_1^2(k_\perp r_\perp) g_\parallel(k_\perp) + J_0^2(k_\perp r_\perp) \frac{u_\parallel^2}{u_\perp^2} g_\perp(k_\perp) \right\} \frac{dk_\perp}{k_\perp}$$

$$= \frac{\pi \Omega^q}{8|u_\parallel| u_\perp^q} \left(C_\parallel I_{1q} + \frac{u_\parallel^2}{u_\perp^2} C_\perp I_{0q} \right) \tag{6.14}$$

where

$$I_{pq} = \int_0^\infty J_p^2(x) x^{q-1} \, dx,$$

---

[1]) In applications not involving covariance considerations we shall change from a tensor notation to the usual notation $u_\parallel = u^3$, $k_\parallel = k_3$, etc.

and in case (c)

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}=\frac{\pi}{8\,|u_{\parallel}|}\sum_{p=-\infty}^{\infty}\int_0^{\infty} dk_{\perp}\frac{k_{\perp}}{\mathfrak{k}^2}f(\mathfrak{k})\,J_{p+1}^2(k_{\perp}r_{\perp})\begin{Bmatrix}2-k_{\perp}^2/\mathfrak{k}^2\\ 2\,k_{\parallel}/\mathfrak{k}\end{Bmatrix} \tag{6.15}$$

$$\beta=\frac{\pi}{8\,|u_{\parallel}|}\sum_{p=-\infty}^{\infty}\int_0^{\infty}\frac{dk_{\perp}k_{\perp}f(\mathfrak{k})}{\mathfrak{k}^4}\left[J_{p+1}^2(k_{\perp}r_{\perp})\,k_{\perp}^2+\frac{u_{\parallel}^2}{u_{\perp}^2}J_p^2(k_{\perp}r_{\perp})(2\,\mathfrak{k}^2-k_{\perp}^2)\right] \tag{6.16}$$

with $k_{\parallel} = -\,p\Omega/u_{\parallel}$.

The sign of the particle charge enters in equation (6.9) through the polarisation term $\sigma\alpha_2$; the term α_1 is sign independent. Formally, the charge dependence is contained in Ω (see footnote on page 356).

The discussion of the transport coefficients will be restricted to power law scalar spectra with constant exponent q, although the expressions for arbitrary spectra are also given. Experimentally, the spectrum slope is often found to increase with wavenumber (see e. g. COLEMAN (1966), SISCOE et al. (1968) for fluctuations of magnetic fields in interplanetary space). However, the form of $D^{33}(u_{\parallel})$ in these cases can be readily infered from the subsequent discussion.

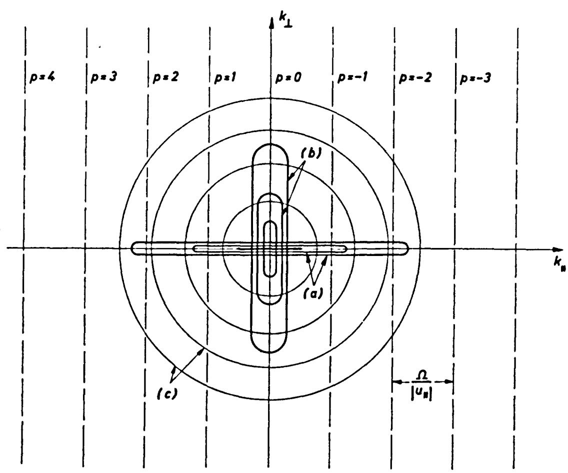

Fig. 1: Schematic distributions of the scalar spectral densities in the $k_{\parallel} - k_{\perp}$-plane for the magnetic field models (a), (b) and (c). The distributions are indicated by contours of equal power density. For case (a) the curves should lie directly on the $k_{\parallel}$-axis. The resonance surfaces for different p are also shown, for a given value of $|\Omega/u_{\parallel}|$.

The three models (a), (b) and (c) differ in the distribution of spectral power in the $k_\parallel - k_\perp$-plane, as indicated in Figure 1. The resonance surfaces are drawn for a fixed value of $|\Omega/u_\parallel|$. For model (a), the power is concentrated entirely on the $k_\parallel$-axis; it is distributed about the $k_\perp$-plane (but within the resonance surfaces $p = \pm 1$) for model (b), whereas in the isotropic case (c) the surfaces of constant power densities are spherical shells centered in the origin. The locations of these isocontours govern the contribution of the various resonance surfaces to the transport coefficients.

In the axisymmetric cases, only a single resonance surface contributes to the velocity diffusion coefficient. This is immediately apparent in case (b), where the only resonance surface contained in the power density contours is $p = 0$, corresponding to $k_\parallel = 0$. In case (a), a particle moving along the field with velocity $u_\parallel$ "sees" only the fundamental frequency $\omega_\parallel = k_\parallel u_\parallel$, since there are no transverse field variations. This yields only a contribution from the $p = -1$ resonance, $k_\parallel = \Omega/u_\parallel$, although the power density is distributed along the entire $k_\parallel$-axis. In the isotropic case (c), all resonance surfaces enter.

Figures 2 to 5 show the pitch-angle diffusion coefficient D^{33} for various values of the spectral parameter q and the polarisation σ as a function of $u_\parallel/\mathrm{u}$, where

$$\mathrm{u} = (u_\parallel^2 + u_\perp^2)^{\frac{1}{2}} = c\sqrt{\gamma^2 - 1}\,.$$

Case (a):

The coefficient D^{33} exists for all q (Figure 2). There is a strong dependence on the polarisation. The behaviour near $u_\parallel = 0$ is finite if $f(k_\parallel) \sim k_\parallel^{-1}$ and is zero or infinite according as the spectrum falls off more rapidly or slowly than this rate. For asymptotic values $q \leq -2$ the pitch angle scattering becomes so ineffective near 90°, that the coefficient $\kappa_\parallel$ for longitudinal diffusion (see the discussion in section 7) tends to infinity. This singular behaviour, however, results only from a too restrictive idealisation. If a finite angular spread is allowed for, the coefficient D^{33} ($u_\parallel = 0$) is always non-zero, and $\kappa_\parallel$ is finite. In general, very little energy is needed in components with wavenumbers perpendicular to the field to produce appreciable scattering at pitch-angles of 90°. This will be seen more clearly from the other two models.

Case (b):

The pitch-angle diffusion coefficient exists if the scalar spectra approach infinity less rapidly than k^{-2} as $k \to 0$ (Figure 3). Otherwise the integral in (6.13) does not exist. There is no polarisation dependence. The pitch angle scattering is seen to be strongly peaked at large pitch-angles ($u_\parallel = 0$). This is a special feature of the $p = 0$ resonance. The singularity at $u_\parallel = 0$ is due to a coincidence of the $p = 0$ resonance term $\delta(k_0\gamma + k_\parallel u_\parallel)$ and the δ-function factor $\delta(k_0)$ in the spectrum; it is removed if the finite phase velocities of the field components are taken into account.

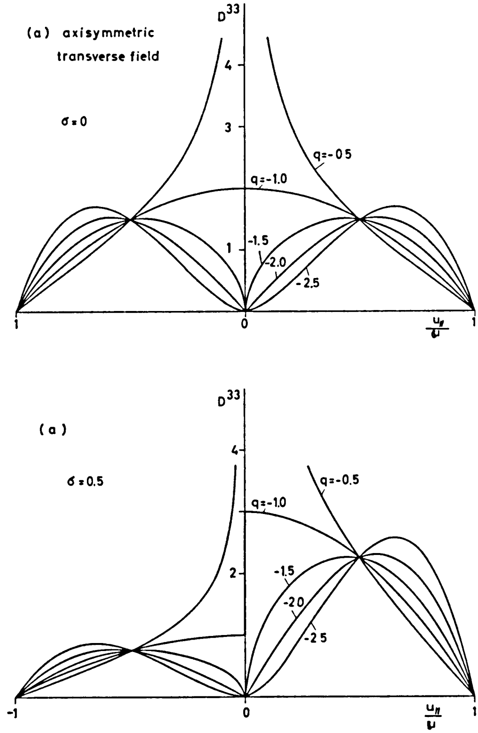

Fig. 2: Pitch-angle diffusion coefficients D^{33} in units of $\frac{\pi u}{4}\left(\frac{e}{mc}\right)^2 f_0$ for an axisymmetric transverse field (a) and the polarisations $\sigma = 0$ and $\sigma = 0.5$. The scalar spectra $f(|k_{\|}|) = C\,|k_{\|}|^{\,q}$ are normalized to the same value f_0 at $k_{\|} = 2\,\Omega/u$ for all q.

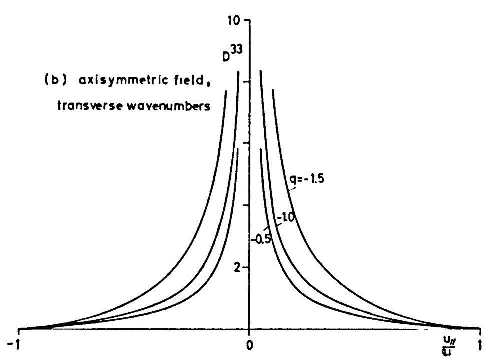

Fig. 3: Pitch-angle diffusion coefficients D^{33} in units of $\frac{\pi u}{4}\left(\frac{e}{mc}\right)^2 g_{\perp 0}$ for an axisymmetric transverse field (b) containing only perpendicular wave numbers. The coefficient is independent of the polarization. The scalar spectra $g_{\perp}(k_{\perp})$ are normalized to the same value $g_{\perp 0}$ at $k_{\perp} = 2\,\Omega/u$ for all q.

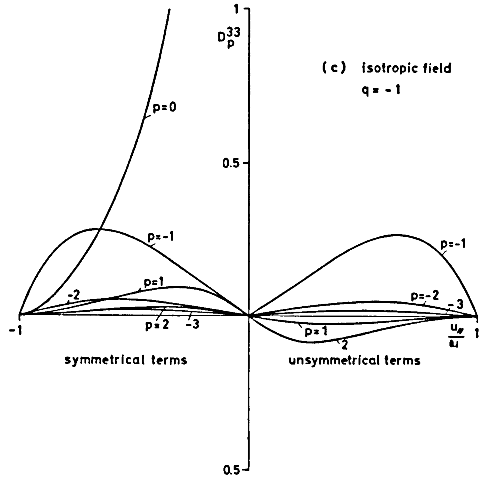

Fig. 4: Contributions D^{33}_p from the resonance surface p to the pitch-angle diffusion coefficient D^{33} in the isotropic case (c) for $q = -1$. The symmetrical terms (shown only for negative $u_{||}$) are independent of the polarization; the asymmetrical terms are proportional to σ (shown only for positive $u_{||}$ and $\sigma = 1$).

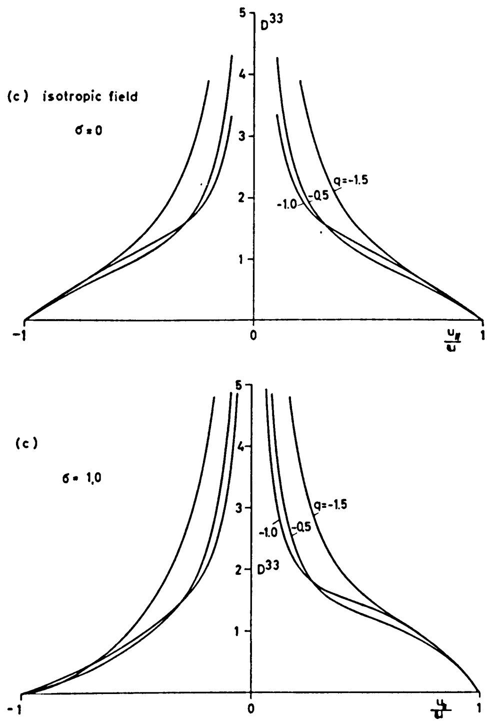

Fig. 5: Pitch-angle diffusion coefficients D^{33}, in units of $\frac{\pi u}{4}\left(\frac{e}{mc}\right)^2 f_0$ for an isotropic field (c) and the polarizations $\sigma = 0$ and $\sigma = 1$. The scalar spectra $f(\mathfrak{k}) = C\mathfrak{k}^q$ are normalized to the same value f_0 at $\mathfrak{k} = 2\,\Omega/u$.

Case (*c*):

The isotropic case lies between the axisymmetric cases (a) and (b), as may have been anticipated from the distribution of spectral densities, Figure 1. This is best seen from the contributions from the seperate resonances, Figure 4. For small $u_{\|}$, the $p = 0$ resonance dominates, and the diffusion coefficient becomes identical with the expression (6.13) for case (b). In the other limit $u_{\|} \to \pm \mathfrak{u}$, only the $p = -1$ resonance remains, and we recover case (a). The net coefficient obtained by summing over all resonances is shown in Figure 5. The polarisation effect is seen to be much smaller than in case (a). This is largely due to the dominance of the polarisation-independent $p = 0$ contribution (see Figure 4); for a power law isotropic spectrum, the $p = 0$ resonance surface passes through the region of highest spectral density at zero wavenumber (Figure 1).

Spatial diffusion coefficient:

The coefficient κ^{ij} for spatial diffusion perpendicular to the magnetic field (see eq. (6.9)) is determined in case (a) by the spectral density at zero wave number; this may not always be defined. The lateral diffusion arises in this case from particles following slow meanders of the mean field, which can be described by a random process only if $f(0)$ exists. Similarly, in cases (b) and (c) the $p = 0$ contribution of the second terms in the integral expressions for β converge only if $g_{\perp}$ and f approach zero at zero wave number.

Equilibrium pitch angle distribution:

If the particle distribution is spatially uniform, the FOKKER-PLANCK equation reduces in the case of magnetic-field interactions to a one-dimensional diffusion equation

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial u_{\|}} \left(\frac{D^{33}}{\gamma} \frac{\partial n}{\partial u_{\|}} \right) = 0 \tag{6.17}$$

for the pitch-angle. The solutions tend asymptotically to an equilibrium distribution $n_e =$ const. in a time of the order of the relaxation time

$$t_{rel} = 0(\gamma \mathfrak{u}^2 / D^{33}) \tag{6.18}$$

The solution $n_e =$ const implies also a constant (isotropic) distribution with respect to the solid angle $\tilde{\omega}$, since

$$\frac{\partial u_{\|}}{\partial \tilde{\omega}} = \frac{\mathfrak{u}}{2\pi} = \text{const.}$$

The functional form of the diffusion coefficient is irrelevant for the equilibrium distribution, which is always isotropic. However, in practice deviations from equilibrium arise, for example, from pitch-angle dependent loss terms (the "loss cone" in radiation belt problems), adiabatic pitch-angle variations, spatial density gradients (section 7) or acceleration processes (section 8). All these effects compete with the tendency to isotropy due to pitch-angle scattering. The resulting balance is strongly influenced by the functional dependence of $D^{33}(u_{\|})$, which we see from Figures 1—4 varies considerably with the magnetic-field model.

7. Spatial diffusion parallel to the mean field

We consider now particle distributions which vary slowly in space and time. We assume that the scattering field is again a magnetic field, and that the particle distributions are close to the isotropic pitch-angle equilibrium.

The particles travel a distance of the order $\lambda = \mathfrak{u} t_{rel}/\gamma$ before their velocities are appreciably changed by pitch-angle scattering. If the time and spatial variations are so slow that

$$\partial n/\partial x_{\|} \ll n/\lambda, \quad \partial n/\partial t \ll n/t_{rel},$$

we may expect the evolution of the omnidirectional number density

$$\varrho(x,\gamma;t) = \int_{-\mathfrak{u}}^{+\mathfrak{u}} \gamma n \, du_{\|} \tag{7.1}$$

to be governed in the appropriate limit by a diffusion equation, in which the longitudinal diffusion coefficient $\kappa_{\|}$ is of the order λ^2/t_{rel} (JOKIPII, 1966, ROELOF, 1966).

Since the distribution is close to equilibrium, we may write

$$n(x,\gamma,u_{\|}) = \varrho(x,\gamma;t)/2\gamma\mathfrak{u} + n'(x,\gamma,u_{\|}) \tag{7.2}$$

where $n' \ll n$.

Substituting equation (7.2) in the FOKKER-PLANCK equation and treating n' as a perturbation, we obtain

$$\frac{\partial \varrho}{\partial t} - \frac{\partial}{\partial x^i}\left(\frac{\bar{\kappa}^{ij}}{\gamma}\frac{\partial \varrho}{\partial x^j}\right) - \frac{\partial}{\partial x_{\|}}\left(\kappa_{\|}\frac{\partial \varrho}{\partial x_{\|}}\right) = 0 \quad (i,j=1,2) \tag{7.3}$$

where

$$\bar{\kappa}_{ij} = \frac{1}{2\mathfrak{u}} \int_{-\mathfrak{u}}^{+\mathfrak{u}} \kappa^{ij} \, du_{\|} \tag{7.4}$$

and

$$\kappa_{\|} = \frac{1}{4\gamma\mathfrak{u}} \int_{-\mathfrak{u}}^{\mathfrak{u}} \left[\int_0^{u_{\|}} \frac{(u'_{\perp})^2}{D^{33}} du'_{\|}\right] u_{\|} \, du_{\|} \tag{7.5}$$

which is identical in the non-relativistic limit with JOKIPII's (1966) result[1]). The anisotropy induced by the spatial gradient is given by

$$n' = -\frac{\partial \varrho}{\partial x_{\|}}\left[\frac{1}{4\gamma\mathfrak{u}}\int_0^{u_{\|}} \frac{(u'_{\perp})^2}{D^{33}} du'_{\|} + \alpha\right] \tag{7.6}$$

---

[1]) A factor $1/2$ should be inserted in the right hand side of JOKIPII's equation (28).

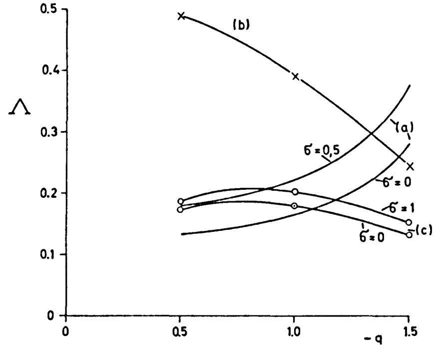

Fig. 6: The proportionality factor Λ of the expression for the parallel diffusion coefficient $\kappa_{\|} = \Lambda \left(\frac{u}{c}\right)^{q+3} \gamma^{-1}$ in units of $\frac{2c^3}{\pi f_0}\left(\frac{mc}{e}\right)^2$. All scalar spectra are normalizėd to have the same value f_0 at scalar wave numbers $k_{\|}$, $k_{\perp}$ or $\mathfrak{k} = 2\,\Omega/c$.
(The same normalization is applied in Figures 7, 8 and 9).

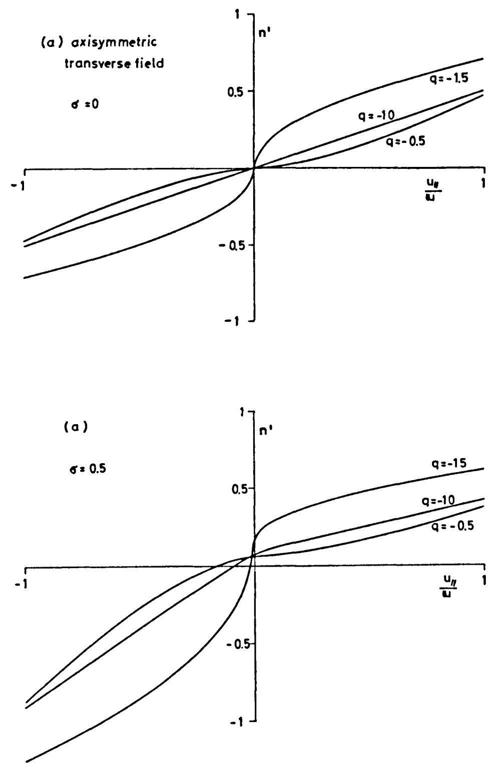

Fig. 7: Gradient induced anisotropies n' for the axisymmetric case (a) in units of $-\frac{\partial \varrho}{\partial x_{\|}}\left(\frac{mc}{e}\right)^2 \frac{u\gamma^{-1}}{\pi f_0}$.

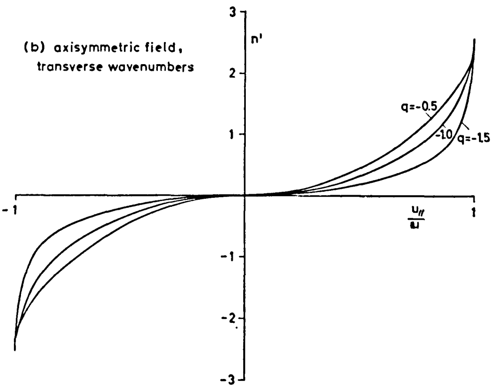

Fig. 8: Same as Figure 7 for the axisymmetric case (b).

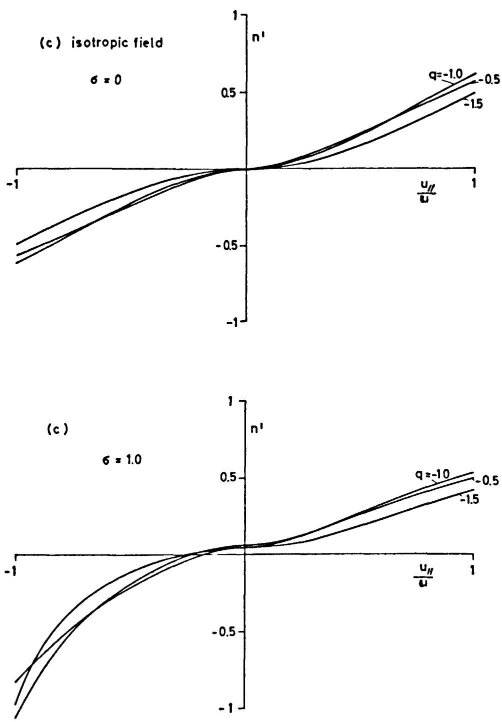

Fig. 9: Same as Figure 7 for the isotropic case (c).

where the integration constant α is determined by the side condition

$$\int_{-\mathfrak{u}}^{\mathfrak{u}} n' du_{\parallel} = 0 \text{ (cf. (7.1) and (7.2))}.$$

For the axisymmetric models (a) and (b), equations (7.5) and (7.6) yield

(a)

$$\kappa_{\parallel} = \frac{2}{\pi C}\left(\frac{mc}{e}\right)^2 \frac{\Omega^{-q}\gamma^{-1}\mathfrak{u}^{q+3}}{(q+2)(q+4)(1-\sigma^2)} \tag{7.7}$$

$$n' = -\frac{\partial \varrho}{\partial x_{\parallel}} \frac{1}{\pi C}\left(\frac{mc}{e}\right)^2 \frac{\Omega^{-q}\gamma^{-1}\mathfrak{u}^{-1}}{(q+2)(1-\sigma^2)} \left\{ |u_{\parallel}|^{q+2} \left(\frac{u_{\parallel}}{|u_{\parallel}|} - \sigma \frac{\Omega}{|\Omega|}\right) + \sigma \frac{\Omega}{|\Omega|} \frac{\mathfrak{u}^{q+2}}{q+3} \right\} \tag{7.8}$$

(b)

$$\kappa_{\parallel} = \frac{2}{\pi C_{\perp}}\left(\frac{mc}{e}\right)^2 \frac{\Omega^{-q}\gamma^{-1}\mathfrak{u}^{q+3}}{(q+4) I_{1q}} \tag{7.9}$$

$$n' = -\frac{\partial \varrho}{\partial x_{\parallel}} \frac{2}{\pi C_{\perp}}\left(\frac{mc}{e}\right)^2 \frac{\Omega^{-q}\gamma^{-1}\mathfrak{u}^{-1}}{(q+2) I_{1q}} \frac{u_{\parallel}}{|u_{\parallel}|} [\mathfrak{u}^{q+2} - u_{\perp}^{q+2}] \tag{7.10}$$

The isotropic model (c) does not lead to closed expressions, but results were obtained numerically.

For power-law spectra, the dependence of $\kappa_{\parallel}$ on the energy γ is the same for all three models, $\kappa_{\parallel} = \Lambda (\mathfrak{u}/c)^{q+3} \gamma^{-1}$. However, the proportionality factor Λ depends strongly on the model, as is indicated in figure 6. The associated anisotropies are shown in figures 7–9. Figures 6–9 demonstrate again the significance of the field structure in the discussion of scattering processes.

The longitudinal diffusion coefficients $\kappa_{\parallel}$ are meaningful in the axisymmetric cases only if $-q < 2$; if the spectra fall off more steeply for large wave numbers, the diffusion coefficient D^{33} approaches zero too rapidly at $u_{\parallel} = 0$, (case (a)) or $u_{\parallel} = \pm \mathfrak{u}$ (case (b)) for the integral (7.5) to converge[1]). Physically, this is due to the mean free path becoming infinite as D^{33} approaches zero; the diffusion coefficient diverges when "escape holes" develop at $u_{\parallel} = 0$ or $u_{\parallel} = \pm \mathfrak{u}$. The singularities at $\sigma = \pm 1$ in case (a) is similarly due to the diffusion coefficient D^{33} vanishing. In this case, there is an "escape band" for $0 \leq \mp u_{\parallel} \leq \mathfrak{u}$, cf. equations (6.9) and (6.11). The zeros in D^{33} arise only for idealised spectral distributions containing a single resonance term. Normal spectral distributions involving more than one resonance surface, such as the isotropic model (c), yield finite $\kappa_{\parallel}$.

---

[1]) This is independent of the fact that I_{1q} diverges for $-q \geq 2$ due to the singularity of the power law at zero wave number. We assume for the present discussion that the true spectrum yields finite coefficients I_{1q}.

For $q = -1$, which is appropriate for certain energy ranges (cf. COLEMAN, 1966), the parallel diffusion coefficient for the axisymmetric case (a) becomes

$$\kappa_{\|} = \frac{2}{3\pi C}\left(\frac{mc}{e}\right)^2 \frac{\Omega u^2}{(1-\sigma^2)\gamma} \tag{7.11}$$

This is identical for $\sigma = 0$ with JOKIPII's (1966) equation (39)[1]), derived for an isotropic spectrum. JOKIPII introduced approximations which were equivalent to retaining only the $p = -1$ resonance; the correspondence to our case (a) is therefore not surprising. However, for a power-law isotropic spectrum there is no range in which the $p = -1$ resonance is asymptotically dominate; the isotropic case is quantitatively different from the axisymmetric cases (a) or (b), cf. Figures 4 and 5.

The anisotropy of the pitch-angle distributions induced by the density gradient is one of various anisotropies caused by small deviations from the pitch-angle equilibrium. We shall discuss these in relation to observations in more detail in a subsequent paper.

8. Acceleration processes

In the general case of time-dependent field fluctuations, the field-particle interactions affect both the pitch-angle and the energy. A stationary equilibrium distribution will normally not exist, since the particles can diffuse away continually to higher energies.

An exception, however, is the class of random fields for which the parallel phase velocity $-k_0/k_3$ is the same for all FOURIER components. In this case, the field can be transformed to a time-independent magnetic field in a coordinate system $\tilde{x}$ by a LORENTZ transformation $L_{\|}(w)$ parallel to the mean field,

$$\begin{aligned} \tilde{x}^0 &= \gamma_w\left(x^0 - \frac{w}{c^2}x^3\right) \\ \tilde{x}^3 &= \gamma_w(x^3 - wx^0), \quad \text{where } w = -k_0/k_3, \quad \gamma_w = (1 - w^2/c^2)^{-\frac{1}{2}} \end{aligned} \tag{8.1}$$

Since LORENTZ transformations leave the phase element invariant, the number density n transforms as a scalar. Hence the equilibrium distribution in x follows immediately from the isotropic equilibrium distribution in $\tilde{x}$,

$$n_e(u^0, u^3) = \tilde{n}_e(\tilde{u}^0, \tilde{u}^3) = \tilde{\varrho}(\tilde{u}^0)/2\,\tilde{u}^0 c\sqrt{(\tilde{u}^0)^2 - 1} \quad (\tilde{u}^0 = \gamma_w(u^0 + wu^3/c^2)) \tag{8.2}$$

The velocity diffusion in the u^0, u^3 plane occurs along the lines (Figure 10)

$$u^0 - \frac{w}{c^2}u^3 = \tilde{u}^0/\gamma_w = \text{const} \tag{8.3}$$

[1]) Except for a factor 1/2, see footnote p. 375.

The relation (8.3) was deduced by BRICE (1964) from classical energy-momentum considerations and by KENNEL and PETSCHEK (1966) from de BROGLIES relation and the correspondence principle (see also JOKIPII and PARKER, 1967).

The velocity diffusion coefficients transform as components of a tensor,

$$D^{33}(u^0, u^3) = \gamma_w^2 \tilde{D}^{33}(\tilde{u}^0, \tilde{u}^3) \tag{8.4}$$

$$D^{03}(u^0, u^3) = \gamma_w^2 \frac{w}{c^2} \tilde{D}^{33}(\tilde{u}^0, \tilde{u}^3) \tag{8.5}$$

$$D^{00}(u^0, u^3) = \gamma_w^2 \frac{w^2}{c^4} \tilde{D}^{33}(\tilde{u}^0, \tilde{u}^3) \tag{8.6}$$

Although the transformation induces energy diffusion terms, the resultant diffusion in the velocity plane u^0, u^3 remains one-dimensional, $|D^{ij}| = D^{33}D^{00} - (D^{03})^2 = 0$. We are concerned here simply with pitch-angle scattering viewed from a different coordinate system, and the energy transfer ceases once equilibrium is established.

For example, in a solar wind model in which the electromagnetic fluctuations are regarded as "frozen" into the plasma, the random field appears as a time-independent magnetic field in the system moving with the plasma. Energetic particles traversing interplanetary space will therefore attain an isotropic equilibrium distribution relative to the local solar wind in a time equal to the pitch-angle relaxation time. One recovers

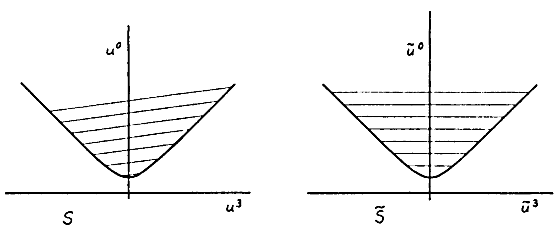

Fig. 10: LORENTZ transformation of the isotropic pitch-angle equilibrium in the system $\tilde{S}$ to a system S moving with velocity w. The scattering in S occurs along the lines $u^0 - \frac{w}{c^2} u^3 = \text{const}$.

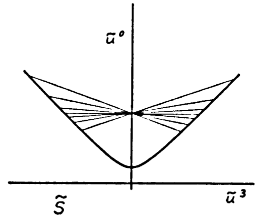

Fig. 11: Removal of the one-dimensional scattering degeneracy of Figure 10 through an ensemble of waves travelling with different phase velocities a. The scattering direction for each phase velocity is indicated by straight lines. The net scattering is two-dimensional with $D^{00} \sim a^2$.

in this approximation the usual picture of the solar wind "sweeping" particles away from the sun, but no acceleration.

To obtain a continuous energy transfer, the one-dimensional degeneracy has to be removed, for example, by inclusion of adiabatic deceleration terms (cf. PARKER, 1966), or by allowing for variations of the phase velocities in the diffusion coefficients (Figure 11).

For an arbitrary time dependent field model, the diffusion coefficient can be evaluated using the general formulae of section 4. However, until more detailed observations of the space-time structure of the fields become available, it appears adequate to determine the coefficients only for a simple example which can be constructed from one of the magnet field models discussed previously.

Consider the electromagnetic field obtained by superimposing a set of fields, each member of which is derived from the same time-independent magnetic field by a different LORENTZ transformation $L_{\parallel}(a)$. Assuming no two transformations to be identical, the spectrum of the resultant field is given by the sum of the spectra of the individual fields. Since the transport coefficients are linear in the spectra, they are then also obtained by superposition. Hence the velocity diffusion coefficients are given by averaging the expressions (8.4)—(8.6) over the set of LORENTZ transformations $L_{\parallel}(a)$. We assume that $a/v \ll 1$, so that the expressions may be expanded in powers of a/v. Without loss of generality, the mean value $\bar{a}$ may also be taken as zero. (If $\bar{a} \neq 0$, we can first carry out a LORENTZ transformation to the system in which $\bar{a}=0$ and then refer the ensemble of transformations to this new system. It is readily seen that this does not affect the final result (8.9) to lowest order.) We obtain in this manner

$$
\begin{aligned}
D^{33}(u^0,u^3) &= \gamma_w^2 \tilde{D}^{33}(u^0,u^3)+\dots \\
D^{03}(u^0,u^3) &= \overline{a^2}(\dots)+\dots \\
D^{00}(u^0,u^3) &= \frac{\overline{a^2}}{c^4}\tilde{D}^{33}(u^0,u^3)+\dots
\end{aligned}
\tag{8.7}
$$

where, for lowest order, the arguments $\tilde{u}^0$, $\tilde{u}^3$ in the right hand side have been replaced by u^0, u^3.

To determine the effect of the energy scattering on the omnidirectional distribution

$$
\varrho(u^0)=\int_{-u}^{u} n(u^0,u^3)\,u^0 du^3,
$$

we average the FOKKER-PLANCK equation (5.7) over the pitch angles. Taking, for simplicity, the distributions to be spatially uniform, this yields

$$
\frac{\partial\varrho}{\partial t}-\frac{\partial}{\partial u^0}\left[\int_{-u}^{+u} D^{03}\frac{\partial n}{\partial u^3}du^3\right]-\frac{\partial}{\partial u^0}\left[\int_{-u}^{u} D^{00}\frac{\partial n}{\partial u^0}du^3\right]=0 \tag{8.8}
$$

To lowest order, the distribution is isotropic, $n = \varrho(u^0)/2\,\mathfrak{u}u^0$. For $\tilde{a} = 0$, D^{03} is of the same order as D^{00}, so that the second term in (8.8) is negligible. The equation then becomes

$$\frac{\partial \varrho}{\partial t}+\frac{\partial}{\partial \gamma}(A_\gamma \varrho)-\frac{\partial}{\partial \gamma}\left(D_\gamma \frac{\partial \varrho}{\partial \gamma}\right)=0 \tag{8.9}$$

where the energy diffusion coefficient

$$D_\gamma=\frac{\overline{a^2}}{2\gamma\mathfrak{u}c^4}\int_{-\mathfrak{u}}^{\mathfrak{u}}\tilde{D}^{33}(u^0,u^3)\,du^3 \tag{8.10}$$

and the acceleration

$$A_\gamma=\frac{D_\gamma}{\mathfrak{u}\gamma}\cdot\frac{d}{d\gamma}(\mathfrak{u}\gamma) \tag{8.11}$$

The proportionality of D_γ and A_γ to $\overline{a^2}$ agrees with previous estimates of the FERMI effect based on statistical models of discrete interaction processes (FERMI, 1949, 1954; cf. JOKIPII and PARKER, 1967 for a summary of other papers).

For the axisymmetric model (a) we obtain from (8.10) and (6.9), (6.11)

$$D_\gamma=\left(\frac{e}{mc}\right)^2\overline{a^2}\,\frac{\pi C\Omega^q\mathfrak{u}^{-q+1}}{2c^4\gamma(2-q)(-q)} \tag{8.12}$$

The relation (8.10) diverges if applied to the models (b) and (c) or, in general, to any field model which has a finite energy density on the $p = 0$ resonance surface $k_{\parallel} = 0$. This is due to the occurence of infinite phase velocities $-k_0/k_3$ at $k_3 = 0$, for which the above transformation procedure has to be modified.

If the FERMI processes are combined with spatial diffusion (section 7), we obtain the transport equation

$$\frac{\partial \varrho}{\partial t}+\frac{\partial}{\partial \gamma}(A_\gamma \varrho)-\frac{\partial}{\partial \gamma}\left(D_\gamma \frac{\partial \varrho}{\partial \gamma}\right)-\frac{\partial}{\partial x_{\parallel}}\left(\kappa_{\parallel}\frac{\partial \varrho}{\partial x_{\parallel}}\right)-\frac{\partial}{\partial x^i}\left(\kappa^{ij}\frac{\partial \varrho}{\partial x^j}\right)=0 \tag{8.13}$$

The equation applies to a reference frame in which the mean phase velocity vanishes (or is at least not larger than the rms variations in the phase velocity). In interplanetary space, this corresponds to a system moving with the solar wind; $(\overline{a^2})^{\frac{1}{2}}$ is then presumably of the order of the mhd-wave velocities. Applications of an extended form of equation (8.13) including adiabatic terms will be discussed in a subsequent paper.

Appendix

Derivation of the FOKKER-PLANCK equation

The following derivation of the FOKKER-PLANCK equation starting from the LIOUVILLE equation is similar to an approach outlined by ROELOF (1967), except that we shall use spectral representations throughout. The diffusion coefficients obtained in this way are found to be identical with the coefficients infered in section 5 from the quadratic transition moments. In the case of a magnetised plasma, the mean acceleration and convection terms vanish, as anticipated by DUNGEY (1965).

To illustrate the method, we consider first an unmagnetised plasma. The equations of motion (2.4) reduce in this case to

$$\frac{du^i}{d\tau}=\frac{e}{mc}F^i_j u^j$$

The LIOUVILLE equation for the number density $n\ (x, u; \tau)$ is then

$$\frac{\partial n}{\partial \tau}+u^i\frac{\partial n}{\partial x^i}+\frac{e}{mc}F^i_j u_j\frac{\partial n}{\partial u^i}=0 \tag{A 1}$$

We regard n here as a distribution in eight-dimensional $x - u$ phase space. On account of the side condition $u^i u_i = 1$, n contains a δ-function factor $\delta\ (u^i u_i - 1)$. In particular, an axisymmetric distribution about the x^3-axis is given by

$$n=\frac{\mathrm{n}}{\pi c^2}\delta\,(u^i u_i-1) \tag{A 2}$$

where $\mathrm{n}\ (x, u^0, u^3; \tau)$ is the six-dimensional distribution introduced in section 4.

Assuming that the random field can be treated as a perturbation, we may expand n in a series

$$n=n^0+n^1+n^2+\cdots \tag{A 3}$$

where n^0 is a solution of the zero'th order LIOUVILLE (convection) equation

$$\frac{\partial \overset{0}{n}}{\partial \tau}+u^i\frac{\partial \overset{0}{n}}{\partial x^i}=0 \tag{A 4}$$

and $\overset{r}{n}$ is of order r in the random field.

Subtracting equation (A 3) from (A 1) and writing

$$\overset{1}{n} = \int d\overset{1}{n}(k,u)\, e^{\iota kx},$$

the first order perturbation is given by

$$\left(\frac{\partial}{\partial \tau} + \iota\omega\right) d\overset{1}{n}(k,u) = -\frac{e}{mc}\, dF^i_j(k)\, u^j \frac{\partial \overset{0}{n}}{\partial u^i}$$

where $\omega = k_i u^i$. The solution may be written

$$d\overset{1}{n}(k,u) = -\frac{e}{mc}\, dF^i_j(k)\, u^j \Delta_1(\omega) \tag{A 5}$$

where Δ_1 is defined by equation (3.13).

The expectation value of $\overset{1}{n}$ vanishes, since $\langle dF^i_j \rangle = 0$. To determine the mean perturbation of the particle distribution, we must therefore go to second order. From equations (A 1), (A 3) and (A 5) we obtain for $\overset{2}{n}$

$$\left(\frac{\partial}{\partial \tau} + u^i \frac{\partial}{\partial x^i}\right) \overset{2}{n} = \left(\frac{e}{mc}\right)^2 \frac{\partial}{\partial u^i} \int\int dF^i_n(k')\, dF^j_m(k)\, u^n u^m \Delta_1(\omega) \frac{\partial \overset{0}{n}}{\partial u^j} \tag{A 6}$$

For large τ, Δ_1 develops a δ-function at the resonance frequency $\omega = 0$,

$$\lim_{\tau \to \infty} \Delta_1(\omega) = \pi\delta(\omega) - \iota\{P\}\frac{1}{\omega} \tag{A 7}$$

where $\{P\}$ denotes the principal value. Thus the expectation value of equation (A 6) yields, asymptotically,

$$\left(\frac{\partial}{\partial \tau} + u^i \frac{\partial}{\partial x^i}\right) \langle \overset{2}{n} \rangle = \frac{\partial}{\partial u^i}\left(D^{ij} \frac{\partial \overset{0}{n}}{\partial u^j}\right) + \frac{\partial}{\partial u^i}\left(A^{ij} \frac{\partial \overset{0}{n}}{\partial u^j}\right) \tag{A 8}$$

where

$$D^{ij} = \pi \left(\frac{e}{mc}\right)^2 u^n u^m \int G^{i\ j}_{\ n\ m}(k)\, \delta(\omega)\, dk \tag{A 9}$$

and

$$A^{ij} = -\iota \left(\frac{e}{mc}\right)^2 u^n u^m \int \{P\}\, G^{i\ j}_{\ n\ m}(k)\, \omega^{-1}\, dk \tag{A 10}$$

From the symmetry and reality relations

$$G^{i\ j}_{\ n\ m}(k) = (G^{j\ i}_{\ m\ n}(k))^* = (G^{i\ j}_{\ n\ m}(-k))^*$$

it follows that D^{ij} is a symmetrical tensor, whereas A^{ij} is antisymmetrical.

Adding equations (A 4) and (A 8) we obtain the FOKKER-PLANCK equation

$$\frac{\partial}{\partial \tau}\langle n\rangle + u^i \frac{\partial}{\partial x^i}\langle n\rangle + \frac{\partial}{\partial u^i}(\overline{a}^i\langle n\rangle) - \frac{\partial}{\partial u^i}\left(D^{ij}\frac{\partial\langle n\rangle}{\partial u^j}\right) = 0 \tag{A 11}$$

where the mean acceleration term is defined as

$$\overline{a}^i = \frac{\partial}{\partial u^j} A^{ij} \tag{A 12}$$

Since A^{ij} is antisymmetrical, $\partial \bar{a}^i/\partial u^i = 0$. Hence $n =$ const is a possible solution of (A 11), as required by LIOUVILLE's theorem. In the case of a magnetised plasma, the mean accelerations will be found to vanish identically.

If the random field is superimposed on a uniform magnetic field, the equations of motion are given by (2.4) and the LIOUVILLE equation becomes

$$\left\{\frac{\partial}{\partial \tau} + u^i \frac{\partial}{\partial x^i} + \left(\Omega M^i_j + \frac{e}{mc} F^i_j\right) u^j \frac{\partial}{\partial u^i}\right\} n = 0 \tag{A 13}$$

The equation can be brought into a form similar to (A 1) by transforming first to helical coordinates, $x, u \to \mathrm{x}, \mathrm{u}$ (equations 2.15), and then to a co-rotating (semi-characteristic) coordinate system X, U defined by

$$\begin{aligned} &\mathrm{u}^i = \mathrm{U}^i, \quad \mathrm{x}^i = \mathrm{X}^i && \text{for } i = 0,3 \\ &\mathrm{u}^i = \mathrm{U}^i e^{\iota \omega_{(i)} \tau}, \quad \mathrm{x}^i = \mathrm{X}^i + \frac{\mathrm{U}^i}{\iota \omega_{(i)}} e^{\iota \omega_{(i)} \tau} && \text{for } i = 1,2 \end{aligned} \tag{A 14}$$

This yields, with $n\,(x, u; \tau) = N(X, U; \tau)$,

$$\left\{\frac{\partial}{\partial \tau} + (1-\lambda_{(i)})\,\mathrm{U}^i \frac{\partial}{\partial \mathrm{X}^i} + \frac{e}{mc}\mathrm{F}^i_j \mathrm{U}^j e^{\iota\omega_{(j)}\tau}\left(e^{-\iota\omega_{(i)}\tau}\frac{\partial}{\partial \mathrm{U}^i} + \iota \frac{\lambda_{(i)}}{\omega_{(i)}}\frac{\partial}{\partial \mathrm{X}^i}\right)\right\} N = 0 \tag{A 15}$$

where the variable

$$\lambda_{(i)} = \begin{cases} 1 & \text{for } i = 1,2 \\ 0 & \text{for } i = 0,3 \end{cases}$$

is introduced to distinguish operations applying to the indices 1,2 from those applying to 0,3.

In terms of X, U, the FOURIER decomposition of the field becomes

$$F^i_j = \int dF^i_j(k)\, e^{\iota kx} = \int d\mathrm{F}^i_j(k)\, e^{\iota \mathrm{kX}} \sum_{p=-\infty}^{\infty} c^p_k \; e^{\iota p \Omega \tau} \tag{A 16}$$

Substituting (A 16) in (A 15), introducing the expansion (A 3) and writing

$$N=\int dN(k)\,e^{\iota kX},$$

we obtain for the first order solution

$$d\overset{1}{N}(k)=-\frac{e}{mc}dF^i_j(k)\,U^j\sum_{p=-\infty}^{\infty}c_k^p\Big\{e^{\iota(\omega_{(j)}-\omega_{(i)}+p\Omega)\tau}$$

$$\cdot\Delta_1(\omega_{(j)}-\omega_{(i)}+\omega_{\|}+p\Omega)\frac{\partial}{\partial U^i}$$

$$+\iota\frac{\lambda_{(i)}}{\omega_{(i)}}e^{\iota(\omega_{(j)}+p\Omega)\tau}\cdot\Delta_1(\omega_{(j)}+\omega_{\|}+p\Omega)\frac{\partial}{\partial X^i}\Big\}\overset{0}{N} \tag{A 17}$$

The second order solution is then given by

$$\left(\frac{\partial}{\partial\tau}+\iota(1-\lambda_{(i)})\,U_i\frac{\partial}{\partial X^i}\right)\overset{2}{N}=\left(\frac{e}{mc}\right)^2\left(e^{-\iota\omega_{(i)}\tau}\frac{\partial}{\partial U^i}+\iota\frac{\lambda_{(i)}}{\omega_{(i)}}\frac{\partial}{\partial X^i}\right)$$

$$\cdot\int\int dF^i_n(k')\,dF^j_m(k)\,e^{\iota(k+k')X}\sum_{p,p'=-\infty}^{\infty}c_{k'}^{p'}c_k^p\,U^m\,U^n\,e^{\iota(\omega_{(n)}+p'\Omega)\tau}$$

$$\cdot\Big\{\Delta_1(\omega_{(m)}-\omega_{(j)}+\omega_{\|}+p\Omega)\,e^{\iota(\omega_{(m)}-\omega_{(j)}+p\Omega)\tau}\cdot\frac{\partial}{\partial U^j}$$

$$+\iota\frac{\lambda_{(j)}}{\omega_{(j)}}e^{\iota(\omega_{(m)}+p\Omega)\tau}\cdot\Delta_1(\omega_{(m)}+\omega_{\|}+p\Omega)\frac{\partial}{\partial X^j}\Big\}\overset{0}{N} \tag{A 18}$$

For large τ, the expectation value of (A 18) yields

$$\left(\frac{\partial}{\partial\tau}+\iota(1-\lambda_{(i)})\,U_i\frac{\partial}{\partial X^i}\right)\langle\overset{2}{N}\rangle=\sum_{i,j=1,2}\frac{\partial}{\partial X^i}\left(K^{ij}\frac{\partial\overset{0}{N}}{\partial X^j}\right)+\sum_{i,j=0,3}\frac{\partial}{\partial U^i}\left(D^{ij}\frac{\partial\overset{0}{N}}{\partial U^j}\right) \tag{A 19}$$

where

$$K^i_j=\left(\frac{e}{mc}\right)^2\frac{\pi}{\omega_{(i)}\omega_{(j)}}\langle U_nU^m\rangle\int dk\,G^{in}{}_{jm}(k)\sum_{p=-\infty}^{\infty}|c_k^p|^2\,\delta(\omega_{(m)}+\omega_{\|}+p\Omega) \tag{A 20}$$

$$(i,j=1,2)$$

$$D^i_j=\left(\frac{e}{mc}\right)^2\pi\,\langle U_nU^m\rangle\int dk\,G^{in}{}_{jm}(k)\sum_{p=-\infty}^{\infty}|c_k^p|^2\,\delta(\omega_{(m)}+\omega_{\|}+p\Omega) \tag{A 21}$$

$$(i,j=0,3)$$

and $\langle U_n U^m \rangle$ represents the mean value over the phase angle (cf. equation (4.9)). In deriving (A 19) to (A 21), we have assumed an axisymmetric distribution about the x^3 axis and have retained only the zero-frequency terms that yield a secular contribution on integration of (A 19) with respect to τ.

Combining equations (A 19), (A 3) and (A 13), transforming back to the original variables, and introducing the distribution n according to (A 2), we regain the Fokker-Planck equation (5.1). The diffusion coefficients (A 20) and (A 21) are identical with the coefficients (5.3), (5.4) derived from the transition moments.

References

BRICE, N.: Fundamentals of Very Low Frequency Emission Generating Mechanisms. J. Geophys. Res. 69 (21), 4515—4522, 1964.

CHANDRASEKHAR, S.: Stochastic Problems in Physics and Astronomy. Rev. Mod. Phys. 15, 1—89, 1943.

—: Some Aspects of the Statistical Theory of Turbulence. Proc. 4th Symp. in Appl. Math., Amer. Math. Soc., 1, 1953.

COHEN, E. G. D.: Fundamental Problems in Statistical Mechanics. N. Holland Publ. Co., Amsterdam, 1962.

COLEMAN, P. J., JR.: Variations in the Interplanetary Magnetic Field: Mariner 2. J. Geophys. Res. 71, 5509—5531, 1966.

DRAGT, A. J.: Effect of Hydromagnetic Waves on the Lifetime of Van Allen Radiation Protons. J. Geophys. Res. 66, 1641—1649, 1961.

DUNGEY, J. W.: Effects of electromagnetic perturbations on particles trapped in the radiation belts. Space Sci. Rev. 4, 199—222, 1965.

FÄLTHAMMAR, C.-G.: On the Transport of Trapped Particles in the Outer Magnetosphere. J. Geophys. Res. 71, 1487—1491, 1966.

FERMI, E.: Galatic Magnetic Fields and the Origin of Cosmic Radiation. Astrophys. J. 119, 1—6, 1954.

—: On the Origin of Cosmic Radiation. Phys. Rev. 75, 1169—1174, 1949.

HASSELMANN, K.: Non-linear Interactions Treated by the Methods of Theoretical Physics (with application to the generation of waves by wind). Proc. Roy. Soc. A 299, 77—100, 1967.

JOKIPII, J. R.: Cosmic-Ray Propagation. I. Charged Particles in a Random Magnetic Field. Astrophys. J. 146, 480—487, 1966.

JOKIPII, J. R., and E. N. PARKER: Energy Changes of Cosmic Rays in the Solar System. Planet. Space Sci. 15, 1375—1386, 1967.

KENNEL, C. F., and H. E. PETSCHEK: Limit on Stably Trapped Particle Fluxes. J. Geophys. Res. 71, 1—28, 1966.

PARKER, E. N.: The Effect of Adiabatic Deceleration on the Cosmic Ray Spectrum in the Solar System. Planet. Space Sci. 14, 371—380, 1966.

PEIERLS, R. E.: Zur kinetischen Theorie der Wärmeleitung in Kristallen. Ann. Physik 3, 1055—1101, 1929.

PRIGOGINE, I.: Non-Equilibrium Statistical Mechanics. Interscience, John Wiley & Sons, New York—London, 319 pp, 1962.

ROELOF, E. C.: Transport of Cosmic Rays in the Interplanetary Medium. 10th Internat. Conf. Cosmic Rays, Calgary, Canada, 1967.

—: Statistical Theory of Charged Particle Transport in Disordered Magnetic Fields. Thesis, University of California, Berkeley 1966.

SCARF, F. L.: Wave-Particle Interactions in the Solar Wind. Raumfahrtforschung, Heft 3, 114—122, 1966.

SISCOE, G. L., L. DAVIS JR., P. J. COLEMAN JR., E. J. SMITH and D. E. JONES: Power Spectra and Discontinuities of the Interplanetary Magnetic Field: Mariner 4. J. Geophys. Res. 73, 61—82, 1968.

STIX, T. H.: The Theory of Plasma Waves. McGraw-Hill, New York, San Francisco, Toronto, London 283 pp., 1962.

WENTZEL, D. G.: Hydromagnetic Waves and the Trapped Radiation. Part I: Breakdown of the Adiabatic Invariant. J. Geophys. Res. 66, 359—362, 1961.