

$\mathbb{C}\mathbb{P}^2$ -STABLE CLASSIFICATION OF 4-MANIFOLDS WITH FINITE FUNDAMENTAL GROUP

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ABSTRACT. We show that two closed, connected 4-manifolds with finite fundamental groups are $\mathbb{C}\mathbb{P}^2$ -stably homeomorphic if and only if their quadratic 2-types are stably isomorphic and their Kirby-Siebenmann invariants agree.

1. INTRODUCTION

Two 4-manifolds are $\mathbb{C}\mathbb{P}^2$ -stably homeomorphic if they become homeomorphic after taking connected sum with finitely many copies of $\mathbb{C}\mathbb{P}^2$, where we allow different numbers of copies for the two manifolds. If the manifolds are orientable, we allow connected sums with both orientations of $\mathbb{C}\mathbb{P}^2$. We will show that the $\mathbb{C}\mathbb{P}^2$ -stable classification for closed connected 4-manifolds with finite fundamental group is determined by their (stable) quadratic 2-type.

The quadratic 2-type Q_M consists of the Postnikov 2-type of the 4-manifold M , determined by $\pi_1 M, \pi_2 M$ (as a $\pi_1 M$ -module) and the k -invariant in $H^3(\pi_1 M; \pi_2 M)$, together with the orientation character $w_1 M : \pi_1 M \rightarrow \{\pm 1\}$ and the equivariant intersection form $\lambda_M : \pi_2 M \times \pi_2 M \rightarrow \mathbb{Z}[\pi_1 M]$:

$$Q_M := (\pi_1 M, w_1 M, \pi_2 M, k_M, \lambda_M).$$

We say that the quadratic 2-types of two manifolds are stably isomorphic if they become isomorphic after finitely many stabilizations as follows:

$$Q_M \mapsto Q_{M \# \pm \mathbb{C}\mathbb{P}^2} \cong (\pi_1 M, w_1 M, \pi_2 M \oplus \mathbb{Z}[\pi_1 M], (k_M, 0), \lambda_M \perp \langle \pm 1 \rangle)$$

For a manifold M with an isomorphism $\pi_1 M \cong \pi$, we pick a continuous map $c : M \rightarrow B\pi$ inducing the isomorphism on π_1 . Then c is uniquely determined up to based homotopy by $\pi_1(c)$.

Theorem 1.1. *Let M_1, M_2 be closed, connected 4-manifolds with finite fundamental group π , orientation character $w : \pi \rightarrow \{\pm 1\}$ and equal Kirby-Siebenmann invariants. Then the following are equivalent:*

- (1) $c_*[M_1] = c_*[M_2] \in H_4(\pi; \mathbb{Z}^w) / \pm \text{Out}(\pi)$;
- (2) M_1 and M_2 are $\mathbb{C}\mathbb{P}^2$ -stably homeomorphic;
- (3) the quadratic 2-types Q_{M_1} and Q_{M_2} are stably isomorphic.

Remark 1.2. Since stably homeomorphic smooth 4-manifolds are stably diffeomorphic, [Theorem 1.1](#) remains true in the smooth category (where Kirby-Siebenmann vanishes). In the topological case, one could also allow taking connected sums with

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Freedman's manifold $*\mathbb{C}\mathbb{P}^2$. Then the Kirby-Siebenmann invariant can be changed and [Theorem 1.1](#) has a version where one does not have to control it.

Since $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2 \# (-\mathbb{C}\mathbb{P}^2) \cong \mathbb{C}\mathbb{P}^2 \# (S^2 \times S^2)$, two manifolds are $\mathbb{C}\mathbb{P}^2$ -stably homomorphic if they are even *stably homeomorphic*, i.e. homeomorphic after taking connected sum with finitely many copies of $S^2 \times S^2$.

It follows from Kreck's modified surgery [[Kre99](#)] that (1) and (2) are equivalent for arbitrary fundamental groups, see also [[KPT](#), Thm 1.1]. Moreover, (2) implies (3) by definition and hence the interesting implication for us is (3) \Rightarrow (1).

In [[KPT](#), Thm A] we showed that already the quadruple (π_1, π_2, k, w_1) determines the $\mathbb{C}\mathbb{P}^2$ -stable classification, for groups that have one end or are torsion-free.

We also described two 4-manifolds of the form lens space times circle which both have fundamental group $\mathbb{Z}/n \times \mathbb{Z}$ and $\pi_2 = 0$ but are not $\mathbb{C}\mathbb{P}^2$ -stably homeomorphic [[KPT](#), Ex. 1.4]. Hence for infinite groups (with two ends and torsion), the implication (3) \Rightarrow (1) is false in general.

The following question remains open, compare [[KPT](#), Question 1.5], even though we show there that the answer is "no" if $H_4(\pi; \mathbb{Z}^w)$ is annihilated by 4 or 6.

Question 1.3. Are there finite groups such that the equivariant intersection form λ is needed in [Theorem 1.1](#).

Our proof is based on the following results on *finite Poincaré complexes*. Let X be a finite connected Poincaré 4-complex with finite fundamental group π , orientation character w and chosen fundamental class $[X] \in H_4(X; \mathbb{Z}^w) \cong \mathbb{Z}$. If B is a 3-coconnected CW-complex then X is *B-polarized* if it is equipped with a 3-equivalence $f: X \rightarrow B$. The set of 4-dimensional *B-polarized* Poincaré complexes with orientation character $w: \pi_1 B \rightarrow \{\pm 1\}$, up to homotopy equivalence over B , is denoted by $\mathcal{S}_4^{\text{PD}}(B, w)$.

Theorem 1.4. *Assume that $\mathcal{S}_4^{\text{PD}}(B, w)$ is non-empty and that $\pi := \pi_1 B$ is finite. Then there is an exact sequence*

$$0 \rightarrow \text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(B))) \rightarrow \mathcal{S}_4^{\text{PD}}(B, w) \rightarrow \mathbb{Z}/|\pi| \times \text{Herm}(\pi_2(B)).$$

Hambleton-Kreck [[HK88](#), Thm 1.1] prove this result in the oriented case $w = 0$ but we need the above generalization to the non-orientable case. The general case as well as [Theorem 1.5](#) below were obtained by the second author in his PhD-thesis [[Tei92](#)] but have not yet been published, so we'll review them in this paper.

For a hermitian form λ on $\pi_2 B$, let $\mathcal{S}_4^{\text{PD}}(B, w, \lambda)$ be the subset of $\mathcal{S}_4^{\text{PD}}(B, w)$ of *B-polarized* Poincaré complexes (X, f) such that λ is mapped to the intersection form λ_X via f^* .

Theorem 1.5. *Assume that $\mathcal{S}_4^{\text{PD}}(B, w, \lambda)$ is non-empty. Then there is a bijection*

$$\mathcal{S}_4^{\text{PD}}(B, w, \lambda) \longleftrightarrow \text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B)).$$

Corollary 1.6. *If $\text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X)) = 0$, the quadratic 2-type Q_X of X determines the homotopy type of X .*

In the orientable case, the above torsion group vanishes if π has cyclic or quaternion 2-Sylow-subgroups [[HK88](#), [Bau88](#)] whereas in the non-orientable case, the following is a simple non-vanishing result:

Proposition 1.7. *Let (X, w) be a Poincaré 4-complex with fundamental group π . If $\pi_2(X)$ splits off a free module and there exists an element $g \in \pi$ with $g^2 = 1$ and $w(g) = -1$, then $\text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X)) \neq 0$.*

Remark 1.8. This applies to simple 4-manifolds such as $X = \mathbb{R}\mathbb{P}^4 \# k\mathbb{C}\mathbb{P}^2$ and $X = (\mathbb{R}\mathbb{P}^2 \times S^2) \# k\mathbb{C}\mathbb{P}^2$ for $k \geq 1$. In these cases the quadratic 2-type still determines the homotopy type by [HKT94, Theorem 3] when we restrict to manifolds instead of Poincaré complexes.

Considering $X = \mathbb{R}\mathbb{P}^4 \# \mathbb{C}\mathbb{P}^2$ we have $\pi_2(X) \cong \mathbb{Z}[\mathbb{Z}/2]$ and by the proof of Proposition 1.7 we have $\text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2])) \cong \mathbb{Z}/2$. We will show in Proposition 3.13 that there is a Poincaré complex with the same quadratic 2-type as X which is not homotopy equivalent to X . By the above this implies that it cannot be homotopy equivalent to a manifold.

Remark 1.9. For $X = \mathbb{R}\mathbb{P}^2 \times S^2$ we have that $\pi_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}^w$ and thus $\Gamma(\pi_2(X)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^w$ and $\mathbb{Z}^w \otimes \Gamma(\pi_2(X)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$. Hence up to homotopy equivalence there are at most four Poincaré complexes with the same quadratic 2-type as X . For manifolds this case was studied by Kim, Kojima and Raymond [KKR92] and Hambleton and Kreck together with the second author [HKT94]. Among the four homotopy types of Poincaré complexes there are three that are realized by manifolds. These are distinguished by their intersection form on $\mathbb{Z}/2$ -homology and either a $\mathbb{Z}/4$ -valued quadratic refinement of the equivariant intersection form [KKR92] or a $\mathbb{Z}/8$ -valued Arf invariant [HKT94]. Hambleton and Milgram [HM78, §3] showed that there is a Poincaré complex with this quadratic 2-type which is not homotopy equivalent to a manifold. Its Spivak normal fibration admits no reduction to a vector bundle [HM78, §4]. Hence up to homotopy equivalence there are precisely four Poincaré with the quadratic 2-type of $\mathbb{R}\mathbb{P}^2 \times S^2$.

We will recall the necessary background on the equivariant intersection form and its connection to Whitehead’s quadratic functor in Section 2. In Section 3, we will then prove the statements from the introduction.

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2. BACKGROUND

Let X be a finite connected Poincaré 4-complex with finite fundamental group π , orientation character $w : \pi_1 X \rightarrow \{\pm 1\}$ and chosen fundamental class $[X] \in H_4(X; \mathbb{Z}^w) \cong \mathbb{Z}$. Here the coefficient group \mathbb{Z}^w is \mathbb{Z} viewed as a $\mathbb{Z}\pi$ -module with action $gz = w(g)z$

We have an involution on $\mathbb{Z}\pi$ given by $g \mapsto \bar{g} := w(g)g^{-1}$ and we use this involution to consider a right $\mathbb{Z}\pi$ -module as a left $\mathbb{Z}\pi$ -module if necessary. We fix a point $x_0 \in X$ and a lift $\tilde{x}_0 \in \tilde{X}$. We abbreviate $\pi_2(X, x_0)$ by $\pi_2(X)$ and write $H_*(-)$ for $H_*(-; \mathbb{Z})$. Using the isomorphism $H_{cs}^2(\tilde{X}) \xrightarrow{-\cap[X]} H_2(\tilde{X}) \xleftarrow{\cong} \pi_2(\tilde{X}) \xrightarrow{\cong} \pi_2(X)$, we can view the equivariant intersection form as a pairing on $\pi_2 X$, or, as we prefer in this paper, as a pairing $H_{cs}^2(\tilde{X}) \times H_{cs}^2(\tilde{X}) \rightarrow \mathbb{Z}\pi$. It is given by

$$\lambda_X(\alpha, \beta) = \lambda_X(\beta)(\alpha) = \sum_{g \in \pi} (g\alpha \cup \beta) \cap [X] g^{-1} = \sum_{g \in \pi} g\alpha \cap ((\beta \cap [X])) g^{-1}.$$

The form λ_X is hermitian with respect to our involution: For $h_i \in \pi$ we have

$$\begin{aligned} \lambda_X(h_1\alpha, h_2\beta) &= \sum_{g \in \pi} ((gh_1\alpha \cup h_2\beta) \cap [X])g^{-1} \\ &= \sum_{g \in \pi} ((h_2^{-1}gh_1\alpha \cup \beta) \cap w(h_2)[X])g^{-1} \\ &= \sum_{g \in \pi} ((g\alpha \cup \beta) \cap w(h_2)[X])h_1g^{-1}h_2^{-1} \\ &= h_1\lambda(\alpha, \beta)w(h_2)h_2^{-1} = h_1\lambda(\alpha, \beta)\bar{h}_2 \end{aligned}$$

and $\lambda_X(\alpha, \beta) = \overline{\lambda_X(\beta, \alpha)}$.

The 2-type B of X is the 2nd stage of a Postnikov tower for X . This means that there is a 3-equivalence $f: X \rightarrow B$ and B fibers over $K(\pi, 1)$ with fiber $K(\pi_2(X), 2)$. Such fibrations are classified by the unique obstruction k_X for finding a section.

$$k_X \in H^3(K(\pi, 1); \pi_2(K(\pi_2(X), 2))) = H^3(\pi; \pi_2(X))$$

is by definition the k -invariant of X . The *quadratic 2-type* of X is

$$Q_X := (\pi, w, \pi_2(X), k_X, \lambda_X).$$

Definition 2.1. Let A be an abelian group. Then $\Gamma(A)$ is the abelian group with generators $v(a)$ for all $a \in A$ and the following relations:

$$\{v(-a) - v(a) \mid a \in A\} \quad \text{and}$$

$$\{v(a+b+c) - v(a+b) - v(b+c) - v(c+a) + v(a) + v(b) + v(c) \mid a, b, c \in A\}.$$

Lemma 2.2 ([Whi50, page 62]). *If A is free abelian with basis B , then $\Gamma(A)$ is free abelian with basis $\{v(b), v(b+b') - v(b) - v(b') \mid b, b' \in B\}$. Another important case is $\Gamma(\mathbb{Z}/2) \cong \mathbb{Z}/4$.*

Let Y be a simply-connected CW-complex and let $\eta: S^3 \rightarrow S^2$ be the Hopf map. It induces a map $\hat{\eta}: \Gamma(\pi_2(Y)) \rightarrow \pi_3(Y)$ given by $v(\alpha) \mapsto \alpha \circ \eta$.

Theorem 2.3 ([Whi50, Sections 10 and 13]). *For every simply-connected CW-complex Y we have Whitehead's exact sequence*

$$\dots \pi_4(Y) \rightarrow H_4(Y) \xrightarrow{b_4} \Gamma(\pi_2(Y)) \xrightarrow{\hat{\eta}} \pi_3(Y) \rightarrow H_3(Y; \mathbb{Z}) \rightarrow 0.$$

We will now explain how the equivariant intersection form λ_X for a finite Poincaré 4-complex with finite fundamental group can be viewed as an element of $\Gamma(\pi_2(X))$. This is well-known, but for convenience and to fix notation we sketch the argument here.

Let A free abelian. An element $\varphi \in \text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}), \mathbb{Z}, A)$ is symmetric if for all $f, g \in \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$ we have $f(\varphi(g)) = g(\varphi(f))$. And we denote the subgroup of symmetric homomorphisms by $\text{Sym}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}), A)$. Let Y be a finite simply-connected CW-complex with $H_2(Y)$ free abelian. We define $\text{Sym}_{\mathbb{Z}}(H^2(Y), H_2(Y))$ as $\text{Sym}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(H_2(Y), \mathbb{Z}), H_2(Y))$ using the canonical isomorphism $H^2(Y) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(Y), \mathbb{Z})$.

Theorem 2.4 ([Whi50, p. 96]). *Let Y be a finite simply-connected CW-complex with $H_2(Y)$ free abelian. There is an isomorphism*

$$\zeta: \text{Sym}_{\mathbb{Z}}(H^2(Y), H_2(Y)) \rightarrow \Gamma(\pi_2(Y))$$

such that for every class $z \in H_4(Y)$ we have $\zeta(- \cap z) = b_4(z)$.

Let X be a finite Poincaré 4-complex with finite fundamental group π and orientation character w . Note that by Poincaré duality $H_2(\tilde{X}) \cong \text{Hom}_{\mathbb{Z}}(H_2(\tilde{X}), \mathbb{Z})$ and thus $H_2(\tilde{X})$ is free abelian. Let $\text{Herm}(H^2(\tilde{X}))$ denote the hermitian forms on $H^2(\tilde{X})$. We have a map

$$\text{Herm}(H^2(\tilde{X})) \rightarrow \text{Sym}_{\mathbb{Z}}(H^2(\tilde{X}), \text{Hom}_{\mathbb{Z}}(H^2(\tilde{X}), \mathbb{Z}))$$

given by

$$\lambda \mapsto (\beta \mapsto (\alpha \mapsto \text{ev}_0 \lambda(\alpha, \beta))),$$

where $\text{ev}_0: \mathbb{Z}\pi \rightarrow \mathbb{Z}$ takes the coefficient at the identity element. This map is injective with image the π -linear homomorphisms, where we view $\text{Hom}_{\mathbb{Z}}(H^2(\tilde{X}), \mathbb{Z})$ as a left $\mathbb{Z}\pi$ -module with the involution on $\mathbb{Z}\pi$ coming from w . After identifying $\text{Hom}_{\mathbb{Z}}(H^2(\tilde{X}), \mathbb{Z})$ with $H_2(\tilde{X})$, the intersection form $\lambda_X \in \text{Herm}(H^2(\tilde{X}))$ maps to $-\cap[\tilde{X}]$. Therefore, we consider the intersection form as an element of $\Gamma(\pi_2(X))$ as follows.

Definition 2.5. Let X be a finite Poincaré 4-complex with finite fundamental group π and orientation character w . Then we can consider the element

$$\lambda_X := \zeta(-\cap[\tilde{X}]) = b_4([\tilde{X}]) \in \Gamma(\pi_2(X)).$$

Corollary 2.6. Let X be a finite Poincaré 4-complex with finite fundamental group and 2-type B . Let $f: X \rightarrow B$ be a 3-equivalence. Then there is an isomorphism $\Gamma(\pi_2(X)) \cong H_4(\tilde{B}, \mathbb{Z})$ mapping λ_X to $\tilde{f}_*[\tilde{X}]$.

Proof. Since B is the 2-type of X , we have $\pi_3(B) = \pi_4(B) = 0$. Consider the diagram:

$$\begin{array}{ccccccc} \pi_4(X) & \longrightarrow & H_4(\tilde{X}) & \longrightarrow & \Gamma(\pi_2(X, x_0)) & \longrightarrow & \pi_3(X) \longrightarrow 0 \\ \downarrow & & \downarrow \tilde{f}_* & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & H_4(\tilde{B}) & \xrightarrow{\cong} & \Gamma(\pi_2(B, f(x_0))) & \longrightarrow & 0 \end{array}$$

From the commutativity of the diagram and [Definition 2.5](#) it follows that λ_X is mapped to $\tilde{f}_*[\tilde{X}]$ under the composition of the two isomorphisms. \square

Corollary 2.7. Let $X = K \cup_{\alpha} D^4$ be a finite Poincaré complex with finite fundamental group, with orientation character w and with 3-skeleton K . Under the map $\Gamma(\pi_2 X) = \Gamma(\pi_2 K) \rightarrow \pi_3(K)$ the element λ_X is mapped to $N^w \alpha$ (up to a unit), where $N^w = \sum_{g \in \pi} w(g)g \in \mathbb{Z}\pi$.

Proof. We can assume that $\pi_2 X$ is nontrivial since otherwise X is homotopy equivalent to either a sphere S^4 or a projective plane $\mathbb{R}P^4$. The Whitehead sequences for X and K fit into the following commutative diagram:

$$\begin{array}{ccccccccc}
H_4(\tilde{X}) & \xrightarrow{b_4} & \Gamma(\pi_2 X) & \xrightarrow{\hat{\eta}} & \pi_3(X) & \longrightarrow & H_3(\tilde{X}) & \longrightarrow & 0 \\
\uparrow & & \parallel & & \uparrow & & \uparrow & & \\
0 = H_4(\tilde{K}) & \longrightarrow & \Gamma(\pi_2 K) & \xrightarrow{\hat{\eta}} & \pi_3(K) & \longrightarrow & H_3(\tilde{K}) & \longrightarrow & 0 \\
& & & & \uparrow & & \uparrow & & \\
& & & & \pi_4(X, K) & \xrightarrow{\cong} & H_4(\tilde{X}, \tilde{K}) \cong \mathbb{Z}\pi & & \\
& & & & \uparrow & & \uparrow & & \\
& & & & \pi_4(X) & \longrightarrow & H_4(\tilde{X}) \cong \mathbb{Z} & & \\
& & & & & & \uparrow & & \\
& & & & & & N^w & &
\end{array}$$

There is a choice of isomorphism $\mathbb{Z}\pi \cong \pi_4(X, K)$ such that 1 is mapped to α under $\pi_4(X, K) \rightarrow \pi_3(K)$. Since λ_X comes from $H_4(\tilde{X})$, the image $\hat{\eta}(\lambda_X) \in \pi_3(K)$ lies in the kernel of the maps to $\pi_3(X)$ and $H_3(\tilde{K})$. Therefore, by a quick diagram chase, $\hat{\eta}(\lambda_X)$ is a multiple of $N^w\alpha$ and since $(-\cap[\tilde{X}]) \in \text{Sym}_{\mathbb{Z}}(H^2(\tilde{X}), H_2(\tilde{X}, \mathbb{Z}))$ is an isomorphism and hence primitive, it is mapped to $N^w\alpha$ up to a unit. \square

In [Section 3](#) we will need the following results about the transfer map.

Let $U \leq \pi$ have finite index. Let X be a CW-complex with fundamental group π and let \hat{X} denote the finite covering of X with fundamental group U . Then X and \hat{X} have the same universal covering and let C_* denote its cellular $\mathbb{Z}\pi$ -chain complex. We can consider it as a right $\mathbb{Z}\pi$ -module using the involution on $\mathbb{Z}\pi$. If M is a $\mathbb{Z}\pi$ -module we can restrict the action to U and consider the projection

$$C_* \otimes_{\mathbb{Z}U} M \xrightarrow{p} C_* \otimes_{\mathbb{Z}\pi} M$$

inducing

$$p_* : H_*(\hat{X}; M) \rightarrow H_*(X; M).$$

On the chain level we obtain a map in the other direction by

$$\text{tr} : C_* \otimes_{\mathbb{Z}\pi} M \rightarrow C_* \otimes_{\mathbb{Z}U} M, \quad c \otimes m \mapsto \sum_{Ug \in U \setminus \pi} cg^{-1} \otimes gm.$$

This map induces a map

$$\text{tr}_* : H_*(X; M) \rightarrow H_*(\hat{X}; M)$$

and similarly one constructs a map

$$\text{tr}^* : H^*(\hat{X}; M) \rightarrow H^*(X; M).$$

Lemma 2.8. *These transfer maps have the following properties:*

- (1) $p_* \circ \text{tr}_*$ and $\text{tr}^* \circ p^*$ are multiplication by the index $[U : \pi]$.
- (2) If $U \leq \pi$ is normal, then π/U acts on \hat{X} and also on $H_*(\hat{X}; M)$ and $H^*(\hat{X}; M)$. In this case, $\text{tr}_* \circ p_*$ and $p^* \circ \text{tr}^*$ are multiplication by $\sum_{g \in \pi/U} g$.
- (3) If π acts trivially on a commutative ring R , then tr^* is an $H^*(X; R)$ -module homomorphism, i.e.

$$\text{tr}^*(p^*(x) \cup y) = x \cup \text{tr}^*(y)$$

for all $x \in H^*(X; R)$ and $y \in H^*(\hat{X}; R)$.

(4) *The transfer commutes with Kronecker-products and Steenrod-squares:*

$$\langle tr^*(y), b \rangle = \langle y, tr_* b \rangle \in R$$

and

$$Sq^i(tr^*(y)) = tr^*(Sq^i(y)) \in H^*(X; \mathbb{Z}/2).$$

Proof. (1) and (2) follow directly from the definitions. For (3) and (4) one uses that for trivial modules tr_* and tr^* are induced by a stable map $tr: \Sigma^\infty X \rightarrow \Sigma^\infty \widehat{X}$ of suspension spectra, see [Ada78, Chapter 4]. \square

3. THE CP²-STABLE CLASSIFICATION

In this section we will recall the proof of [HK88, Theorem 1.1] and then prove [Theorems 1.1, 1.4](#) and [1.5](#). At the end of the section we will prove [Proposition 1.7](#) and [Proposition 3.13](#) which was mentioned in the introduction.

Let X, Y be finite Poincaré 4-complexes with finite fundamental group π . Assume that X and Y have isomorphic quadratic 2-types:

$$Q_X = (\pi, w, \pi_2(X), k, \lambda_X) \cong Q_Y.$$

In the following we will always assume that $\pi_2(X) \neq 0$ since otherwise X is homotopy equivalent to either a sphere S^4 or a projective plane $\mathbb{R}P^4$. Let B be the 2-type of X and let $f: X \rightarrow B$ be a 3-equivalence. From the results of [Section 2](#), in particular [Corollary 2.6](#), it follows that there is a 3-equivalence $g: Y \rightarrow B$ such that $\tilde{g}_*[\tilde{Y}] = \tilde{f}_*[\tilde{X}]$.

If in this situation there is a map $h: X \rightarrow Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & & B \end{array}$$

homotopy commutative, then h is an isomorphism on π_1, π_2 . By [Corollary 2.6](#) there is an isomorphism $\Gamma(\pi_2(Y)) \cong H_4(\tilde{B})$ mapping λ_Y to $\tilde{g}_*[\tilde{Y}]$ and by assumption, $\tilde{g}_*[\tilde{Y}] = \tilde{f}_*[\tilde{X}] = \tilde{g}_*\tilde{h}_*[\tilde{X}]$. Since $H_4(\tilde{Y}) \cong \mathbb{Z}$, $H_4(\tilde{B})$ is torsion-free and $\tilde{g}_*[\tilde{Y}]$ is non-trivial, this implies $\tilde{h}_*[\tilde{X}] = [\tilde{Y}]$. Thus by Poincaré duality \tilde{h} is an isomorphism on homology of universal coverings in all degrees and hence h is a homotopy equivalence.

Lemma 3.1. *A map h as above exists if and only if $f_*[X] = g_*[Y] \in H_4(B; \mathbb{Z}^w)$.*

Proof. This result is proved by Hambleton and Kreck [HK88, Lemma 1.3] in the oriented case. Since the proof in the non-oriented case works the same we will not give any details here. However, we want to point out the following two things.

Firstly, there is a minor mistake at the beginning of the proof of [HK88, Lemma 1.3], where the authors construct B by attaching cells of dimension 4 and higher to Y , i.e. $g: Y \rightarrow B$ is the inclusion. Then they say that they want to extend

$$X^{(3)} \xrightarrow{f^{(3)}} B^{(3)} = Y^{(3)} \subseteq Y$$

over the 4-cell of X .

In fact, using the vanishing of obstructions in cohomology they can only extend the map $X^{(2)} \rightarrow Y$, because they might have to change the given map on the 3-cells in order to extend over the 4-cell. But also such an extension h suffices for the proof

because the given (trivial) homotopy between f and $g \circ h$ on $X^{(2)} \times I$ extends to $X \times I$ using $\pi_3(B) = \pi_4(B) = 0$.

Secondly, to get the non-oriented case, note that Hambleton and Kreck show for $w \equiv 1$ that the obstruction in

$$H^4(X; \pi_3 Y) \cong H_0(X; \pi_3 Y \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w) \cong \pi_3 Y \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \cong \pi_4(B, Y) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \cong H_4(B, Y; \mathbb{Z}^w)$$

is given by the image of $f_*[X]$ under $H_4(B; \mathbb{Z}^w) \rightarrow H_4(B, Y; \mathbb{Z}^w)$ and therefore vanishes if $f_*[X] = g_*[Y]$. All these isomorphisms continue to hold for non-trivial w . \square

Let X be a finite Poincaré complex with orientation character w , finite fundamental group π , intersection form λ_X and with a fixed 3-equivalence $f: X \rightarrow B$, where B is 3-coconnected.

Let λ be a hermitian form on $\pi_2(B)$. Recall that $\mathcal{S}_4^{\text{PD}}(B, w, \lambda)$ denotes the set of 4-dimensional B -polarized Poincaré complexes (Y, g) with orientation character w , such that λ maps to λ_Y , up to homotopy equivalence over B . Assume $(X, f) \in \mathcal{S}_4^{\text{PD}}(B, w, \lambda)$. By Lemma 3.1, sending (Y, g) to $g_*[Y] - f_*[X]$ defines an injection

$$\mathcal{S}_4^{\text{PD}}(B, w, \lambda) \rightarrow H_4^w(B) := H_4(B; \mathbb{Z}^w).$$

We want to compute the image of this map. Under the isomorphism $H_4(\tilde{B}) \cong \Gamma\pi_2(B)$, the element $\tilde{f}_*[\tilde{X}]$ is determined by $\lambda = f_*\lambda_X$ by Corollary 2.6. Furthermore, recall that $\text{tr}_*[X] = [\tilde{X}]$. Therefore $|\pi|(g_*[Y] - f_*[X]) = p_* \text{tr}_*(g_*[Y] - f_*[X]) = 0$, i.e. the above image lies in the torsion subgroup of $H_4^w(B)$.

Write X as $K \cup_{\alpha} D^4$, where K is a 3-complex. Since K is a 3-dimensional complex, $H_3^w(K)$ is a subgroup of a free abelian group and thus torsion-free. Furthermore, $H^1(X) \cong H^1(\pi) \cong \text{Hom}_{\mathbb{Z}}(\pi, \mathbb{Z}) = 0$. Using this, from the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_4^w(X) & \longrightarrow & H_4^w(X, K) & \longrightarrow & H_3^w(K) \longrightarrow H_3^w(X) \\ & & \parallel & & \parallel & & \text{PD}^{-1} \downarrow \cong \\ & & \mathbb{Z} & & \mathbb{Z} & & H^1(X) \end{array}$$

we obtain $H_3^w(K) = 0$. Therefore, $H_4^w(B) \rightarrow H_4^w(B, K)$ is an isomorphism. Since B is 3-coconnected, $\pi_4(B, K) \rightarrow \pi_3(K)$ is an isomorphism. Combining these, we obtain the very useful isomorphism

$$(3.2) \quad H_4^w(B) \xrightarrow{\cong} H_4^w(B, K) \xleftarrow{\cong} \pi_4(B, K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \xrightarrow{\cong} \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w.$$

Consider the commutative diagram

$$\begin{array}{ccccc} H_4^w(X) & \longrightarrow & H_4^w(X, K) & \longleftarrow & \pi_4(X, K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_4^w(B) & \xrightarrow{\cong} & H^w(B, K) & \xleftarrow{\cong} & \pi_4(B, K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \xrightarrow{\cong} \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \end{array}$$

\(\searrow \partial\)

and view the top cell of X as an element in $\pi_4(X, K)$. Then it has boundary α and its image in $H_4^w(X, K)$ is also the image of $[X] \in H_4^w(X)$. Hence the composition of the horizontal isomorphisms (3.2) maps $f_*[X]$ to $[\alpha] \otimes 1$.

Finally, we use Whitehead sequences for \tilde{X} and \tilde{K} :

$$\begin{array}{ccccccc}
 \pi_4(\tilde{X}) & \xrightarrow{0} & H_4(\tilde{X}) & \xrightarrow{b_4} & \Gamma(\pi_2(X)) & \xrightarrow{\hat{\eta}} & \pi_3(X) \longrightarrow H_3(\tilde{X}) \xlongequal{=} 0 \\
 & & \uparrow & & \parallel & & \uparrow \\
 & & 0 & \longrightarrow & \Gamma(\pi_2(K)) & \xrightarrow{\hat{\eta}} & \pi_3(K) \longrightarrow H_3(\tilde{K}) \longrightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & \pi_4(\tilde{X}, \tilde{K}) \xrightarrow{\cong} H_4(\tilde{X}, \tilde{K}) \xrightarrow{\cong} \mathbb{Z}\pi \\
 & & & & & & \uparrow \\
 & & & & & & 0 \\
 & & & & & & \uparrow \\
 \pi_4(\tilde{X}) & \xrightarrow{0} & H_4(\tilde{X}) & & & &
 \end{array}$$

The image of the composition $H_4(\tilde{X}) \xrightarrow{b_4} \Gamma(\pi_2(X)) = \Gamma(\pi_2(K)) \xrightarrow{\hat{\eta}} \pi_3(K)$ lies in the kernel of the map to $\pi_3(X)$ and thus inside $\pi_4(\tilde{X}, \tilde{K}) \cong H_4(\tilde{X}, \tilde{K}) \cong \mathbb{Z}\pi$. We see that it fits into a short exact sequence

$$0 \rightarrow H_4(\tilde{X}) \rightarrow H_4(\tilde{X}, \tilde{K}) \rightarrow H_3(\tilde{K}) \rightarrow 0$$

and we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_4(\tilde{X}) & \longrightarrow & \mathbb{Z}\pi & \longrightarrow & H_3(\tilde{K}) \longrightarrow 0 \\
 & & \downarrow b_4 & & \downarrow & & \parallel \\
 0 & \longrightarrow & \Gamma(\pi_2(K)) & \longrightarrow & \pi_3(K) & \longrightarrow & H_3(\tilde{K}) \longrightarrow 0
 \end{array}$$

The fundamental class $[\tilde{X}]$ maps downward to $\lambda_X \in \Gamma(\pi_2 X) = \Gamma(\pi_2 K)$ and, by [Corollary 2.7](#), to the right to $\pm N^w := \sum_{g \in \pi} w(g)g$. Note that $H_4(\tilde{X}) \cong \mathbb{Z}^w$. By the upper short exact sequence above, $H_3(\tilde{K}) \cong \mathbb{Z}\pi/N^w$. Tensoring the diagram with \mathbb{Z}^w over $\mathbb{Z}\pi$ we obtain the following diagram of the long exact Tor-sequences:

$$\begin{array}{ccccccc}
 & & \mathbb{Z} & \xrightarrow{|\pi|} & \mathbb{Z} & & \\
 & & \parallel & & \parallel & & \\
 0 \rightarrow & \text{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w, \mathbb{Z}^w) & \longrightarrow & \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \xrightarrow{N^w \otimes \text{Id}} & \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \longrightarrow \mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \rightarrow 0 \\
 & \parallel & & \downarrow & & \downarrow & \parallel \\
 0 \rightarrow & \text{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w, \mathbb{Z}^w) & \rightarrow & \Gamma(\pi_2(K)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \rightarrow & \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \rightarrow \mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \rightarrow 0
 \end{array}$$

The top row shows that $\text{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w, \mathbb{Z}^w) = 0$ and $\mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w = \mathbb{Z}/|\pi|\mathbb{Z}$. Using that $\pi_3 K \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \cong H_4^w(B)$ and $\pi_2(B) \cong \pi_2(K)$, the bottom row can be identified with the short exact sequence:

$$(3.3) \quad 0 \rightarrow \Gamma(\pi_2(B)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \xrightarrow{i} H_4^w(B) \xrightarrow{q} \mathbb{Z}/|\pi|\mathbb{Z} \rightarrow 0$$

with $i(\lambda \otimes 1) = |\pi|f_*[X]$ and $q(f_*[X]) = 1 + |\pi|\mathbb{Z}$, exactly as in the oriented case [[HK88](#), page 89]. From here [Theorem 1.4](#) follows as in the proof of [[HK88](#), Theorem 1.1].

Now we are nearly ready to prove the following theorem which completes the proof of [Theorem 1.5](#). Recall that $(Y, g) \mapsto g_*[Y] - f_*[X]$ defines an injection $\mathcal{S}_4^{\text{PD}}(B, w, f_*\lambda_X) \rightarrow H_4^w(B)$ with image inside the torsion subgroup.

Theorem 3.4. *For all $(Y, g) \in \mathcal{S}_4^{\text{PD}}(B, w, \lambda)$ we have $q(g_*[Y] - f_*[X]) = 0$ and $(Y, g) \mapsto i^{-1}(g_*[Y] - f_*[X])$ defines a bijection*

$$\mathcal{S}_4^{\text{PD}}(B, w, \lambda) \leftrightarrow \text{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w).$$

Before proving [Theorem 3.4](#) we will first show how it implies [Theorem 1.1](#).

Proof of [Theorem 1.1](#). It remains to show that for a given closed, connected 4-manifold M with fundamental group π and orientation character w the element $c_*[M] \in H_4(\pi; \mathbb{Z}^w)$ is determined by the stable quadratic 2-type of M . Let B denote the Postnikov 2-type of M and consider the sequence $M \xrightarrow{f} B \xrightarrow{c} B\pi$. Considering the following diagram and following the construction of [\(3.3\)](#), we see that the map i in [\(3.3\)](#) can be identified with the canonical map $H_4(\tilde{B}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \rightarrow H_4^w(B)$.

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \uparrow & & \uparrow & \\
0 & \longrightarrow & \Gamma(\pi_2(\tilde{K})) & \xrightarrow{\hat{\eta}} & \pi_3(\tilde{K}) & \longrightarrow & H_3(\tilde{K}) \longrightarrow 0 \\
& & & \uparrow \cong & & \uparrow & \\
& & & \pi_4(\tilde{B}, \tilde{K}) & \xrightarrow{\cong} & H_4(\tilde{B}, \tilde{K}) & \\
& & & \uparrow & & \uparrow & \\
& & & 0 & \longrightarrow & H_4(\tilde{B}) & \\
& & & & & \uparrow & \\
& & & & & 0 &
\end{array}$$

Now given another closed, connected 4-manifold N with orientation character w and a 3-equivalence $g: N \rightarrow B$ such that $g^*\lambda_N = f^*\lambda_M$, then $f_*[M] - g_*[N]$ is contained in $\text{Tors}(\Gamma(\pi_2(B)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w) \cong \text{Tors}(H_4(\tilde{B}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ by [Theorem 3.4](#). Since the composition $\tilde{B} \rightarrow B \xrightarrow{c} B\pi$ is trivial, we thus have $c_*f_*[M] = c_*g_*[N] \in H_4^w(B\pi)$ as claimed. \square

For the proof of [Theorem 3.4](#) we need the following lemma.

Lemma 3.5. *If $w \equiv 1$ or if the 2-Sylow subgroup of π has order bigger than 2, there exists a homomorphism*

$$\kappa: \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \rightarrow \mathbb{Z} \text{ with } \lambda \otimes 1 \mapsto 1.$$

We will first prove [Theorem 3.4](#) assuming [Lemma 3.5](#), and then we will prove [Lemma 3.5](#).

Proof of [Theorem 3.4](#). We will first show that $q(g_*[Y] - f_*[X]) = 0$.

In the case that $w \equiv 1$ or the 2-Sylow subgroup of π has order more than 2, we will show that $i(\text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B))) = \text{Tors}(H_4^w(B))$. This proves $q(g_*[Y] - f_*[X]) = 0$ in this case. The image of $\text{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B))$ is obviously contained in $\text{Tors}(H_4^w(B))$ and we have to show the other inclusion.

Let $x \in \text{Tors}(H_4^w(B))$ be given. Because $H_4(\tilde{B}) \cong \Gamma(\pi_2 B)$ is torsion free, we see that $|\pi|x = p_* \text{tr}_*(x) = 0$. From [\(3.3\)](#) and $q(f_*[X]) = 1 \in \mathbb{Z}/|\pi|$ we see that there

exists $k \in \mathbb{N}$ and $y \in \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w$ such that $x + kf_*[X] = i(y)$. It follows that

$$|\pi|i(y) = |\pi|x + k|\pi|f_*[X] = ki(\lambda \otimes 1)$$

Thus, $|\pi|y = k(\lambda \otimes 1)$ and by [Lemma 3.5](#) we have $|\pi|\kappa(y) = k\kappa(\lambda \otimes 1) = k$. Therefore, $|\pi|$ divides k and

$$x = i(y - \frac{k}{|\pi|}(\lambda \otimes 1)) \in \text{im}(i).$$

We now consider the case $\pi = Q \rtimes \mathbb{Z}/2$ with $|Q|$ odd and $w = p_2$ the projection onto $\mathbb{Z}/2 \cong \{\pm 1\}$. Since $f_*[X] \in H_4^w(B)$ maps to a generator of $\mathbb{Z}/|\pi| \cong \mathbb{Z}/|Q| \oplus \mathbb{Z}/2$ under q there is $x \in \mathbb{Z}/|Q|$ with $q(f_*[X]) = (x, 1 + 2\mathbb{Z}) \in \mathbb{Z}/|Q| \oplus \mathbb{Z}/2$. For any $(Y, g) \in \mathcal{S}_4^{\text{PD}}(B, w, \lambda)$ there is in the same way $y \in \mathbb{Z}/|Q|$ with $q(g_*[Y]) = (y, 1 + 2\mathbb{Z})$ and hence $q(g_*[Y] - f_*[X]) = (y - x, 0)$.

Taking double coverings with respect to the subgroup $Q \leq \pi$ we get a commutative square

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{B} \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where $p_*\widehat{f}_*[\widehat{X}] = 2f_*[X]$. So we may conclude from the following commutative diagram that $\widehat{q}(g_*[\widehat{Y}]) = y$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2 \widehat{B}) \otimes_{\mathbb{Z}Q} \mathbb{Z} & \xrightarrow{\widehat{i}} & H_4(\widehat{B}) & \xrightarrow{\widehat{q}} & \mathbb{Z}/|Q| \longrightarrow 0 \\ & & \downarrow & & \downarrow p_* & & \downarrow \cdot 2 \\ 0 & \longrightarrow & \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \xrightarrow{i} & H_4^w(B) & \xrightarrow{q} & \mathbb{Z}/|\pi| \longrightarrow 0 \end{array}$$

Since \widehat{X} is an orientable Poincaré complex we already know that [Theorem 3.4](#) holds for \widehat{X} . Therefore, $0 = \widehat{q}(\widehat{g}_*[\widehat{Y}] - \widehat{f}_*[\widehat{X}]) = y - x \in \mathbb{Z}/|Q|$ and thus also $q(g_*[Y] - f_*[X]) = 0$.

By the above, the map $\mathcal{S}_4^{\text{PD}}(B, w, \lambda) \rightarrow \text{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ is well defined and we already now that it is injective. Showing surjectivity is easy since we can use [\[HK88, page 89\]](#) to reattach the top cell of X as follows. Recall that $X = K \cup_{\alpha} D^4$. For a given element $\beta \in \text{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ pick a preimage $\widehat{\beta} \in \Gamma(\pi_2 B) \subseteq \pi_3(K)$ and attach a 4-cell to K along a representative of $\widehat{\beta} + \alpha$. The resulting complex X_{β} has the same intersection form as X and also $H_3(\widetilde{X}_{\beta}) = 0$ because $\widehat{\beta}$ maps trivially to $H_3(\widetilde{K})$. As in [\[HK88\]](#), the map $f|_K$ extends to a 3-equivalence $f_{\beta}: X_{\beta} \rightarrow B$ such that $f_{\beta*}[X_{\beta}] = f_*[X] + \beta$. \square

For the proof of [Lemma 3.5](#) we need the following constructions of maps from $\Gamma(\pi_2(X)) = \Gamma(\pi_2(B))$ to \mathbb{Z}^w . Consider the isomorphism

$$\zeta^{-1}: \Gamma(\pi_2(X)) \rightarrow \text{Sym}_{\mathbb{Z}}(H^2(\widetilde{X}), H_2(\widetilde{X}))$$

from [Theorem 2.4](#). Precomposing with the inverse of PD: $H^2(\widetilde{X}) \xrightarrow{-\cap[\widetilde{X}]} H_2(\widetilde{X})$, we get a map

$$(3.6) \quad \kappa': \Gamma(\pi_2(X)) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(\widetilde{X}), H_2(\widetilde{X})).$$

The construction of κ' and the following two lemmas below are due to Stefan Bauer [Bau88].

Lemma 3.7. *If we let π act on $\text{Hom}_{\mathbb{Z}}(H_2(\tilde{X}), H_2(\tilde{X}))$ by $g\varphi = w(g)(g_* \circ \varphi \circ g_*^{-1})$, then κ' is equivariant. In particular,*

$$\text{trace} \circ \kappa': \Gamma(\pi_2(X)) \rightarrow \mathbb{Z}^w$$

is equivariant.

Proof. If we let π act on $\text{Sym}(H^2(\tilde{X}), H_2(\tilde{X}))$ by $g\psi = g_* \circ \psi \circ g^*$, then by naturality ζ is equivariant. We have

$$g_*(g^* \alpha \cap [\tilde{X}]) = \alpha \cap g_*[\tilde{X}] = w(g)(\alpha \cap [\tilde{X}])$$

and thus $g^* \circ \text{PD}^{-1} = w(g)(\text{PD}^{-1} \circ g_*^{-1})$. This proves the lemma since $\kappa'(x) = \zeta(x) \circ \text{PD}^{-1}$. \square

Lemma 3.8. *When we view λ_X as an element of $\Gamma(\pi_2(X))$ as in Definition 2.5, then*

$$\kappa'(\lambda_X) = \text{Id}_{H_2(\tilde{X})}.$$

Proof. Recall that λ_X is the image of $-\cap[\tilde{X}]$ under ζ . Thus, precomposing with the inverse of $\text{PD} = -\cap[\tilde{X}]$ gives the identity on $H_2(\tilde{X})$. \square

A central involution in a group G is an element of order 2 in the center of G .

Lemma 3.9. *Let G be a 2-group, $w: G \rightarrow \mathbb{Z}/2$. If $w \equiv 1$ or $|G| > 2$, there exists a central involution $\tau \in G$ with $w(\tau) = 1 \in \{\pm 1\}$.*

Proof. Since G is a 2-group there exists a central involution. We are done if $w(\tau) = 1$. Otherwise the map w splits and since τ is central, we have $G = \langle \tau \rangle \times \ker(w)$. Since $|G| > 2$ there again exists a central involution $\tau' \in \ker(w)$ and this is still central in $G = \langle \tau \rangle \times \ker(w)$. \square

Let $\tau_*: \text{Hom}_{\mathbb{Z}}(H_2(\tilde{X}), H_2(\tilde{X})) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(\tilde{X}), H_2(\tilde{X}))$ be given by precomposition with the action of τ on \tilde{X} .

Lemma 3.10. *The map*

$$\text{trace} \circ \tau_* \circ \kappa': \Gamma(\pi_2(X)) \rightarrow \mathbb{Z}^w$$

is equivariant and the element λ_X is mapped to -2 .

Proof. Since κ' and the trace are equivariant, we have to show that τ_* is equivariant. This is the case, because the group element τ is central. Since $\kappa'(\lambda_X) = \text{Id}_{H_2(\tilde{X})}$ by Lemma 3.8, we have $\tau_*(\kappa'(\lambda_X)) = \tau: H_2(\tilde{X}) \rightarrow H_2(\tilde{X}; \mathbb{Z})$. Since $w(\tau) = 1$ and τ acts fixed point free on \tilde{X} , the Lefschetz formula yields

$$0 = \text{trace}(\tau_{H_0(\tilde{X})}) + \text{trace}(\tau_{H_4(\tilde{X})}) + \text{trace}(\tau_{H_2(\tilde{X})}) = 2 + \text{trace}(\tau_*(\kappa'(\lambda_X)))$$

and therefore $\text{trace}(\tau_*(\kappa'(\lambda_X))) = -2$. \square

We want to show that the image of $\text{trace} \circ \tau_* \circ \kappa'$ is always even. For this we need the following lemma. This is well-known and for example follows from [Bre72, VII Theorem 7.4]. For the reader's convenience we give a proof.

Lemma 3.11. *Let $\tau \in \pi$ have order 2. Then the symmetric bilinear form*

$$H^2(\tilde{X}; \mathbb{Z}/2) \otimes H^2(\tilde{X}; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2, \quad x \otimes y \mapsto \langle \tau x \cup y, [\tilde{X}] \rangle$$

is even, i.e. $\langle \tau x, x \rangle = 0$ for all $x \in H^2(\tilde{X}; \mathbb{Z}/2)$.

Proof. Let $q: \tilde{X} \rightarrow \hat{X} := \tilde{X}/\tau$ be the double covering. Then $q^* \circ \text{tr}^*(x) = x + \tau x$ and

$$\begin{aligned} \langle q^* \circ \text{tr}^*(x) \cup x, [\tilde{X}] \rangle &= \langle \text{tr}^*(q^* \circ \text{tr}^*(x) \cup x), [\hat{X}] \rangle \\ &= \langle \text{tr}^*(x) \cup \text{tr}^*(x), [\hat{X}] \rangle \\ &= \langle Sq^2(\text{tr}^*(x)), [\hat{X}] \rangle \\ &= \langle Sq^2(x), \text{tr}_*[\hat{X}] \rangle \\ &= \langle x \cup x, [\tilde{X}] \rangle. \end{aligned}$$

Here we used the identities from Lemma 2.8 (with $p = q$). It follows that

$$\langle x \cup \tau x, [\tilde{X}] \rangle = \langle x \cup (q^* \circ \text{tr}^*(x) + x), [\tilde{X}] \rangle = 2\langle x \cup x, [\tilde{X}] \rangle = 0$$

for all $x \in H^2(\tilde{X}; \mathbb{Z}/2)$. \square

Lemma 3.12. *The image of $\text{trace} \circ \tau_* \circ \kappa'$ is contained in $2\mathbb{Z}^w$.*

Proof. The trace fits into the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathbb{Z}}(H_2(\tilde{X}), H_2(\tilde{X})) & \xrightarrow{\text{trace}} & \mathbb{Z} \\ & \searrow \varphi_1 & \nearrow \varphi_2 \\ & H^2(\tilde{X}) \otimes H_2(\tilde{X}) & \end{array}$$

where φ_1 is the slant map and φ_2 the evaluation, i.e. $\varphi_1(\alpha \otimes x) := (y \mapsto (\alpha \cap y)x)$ and $\varphi_2(\alpha \otimes x) := \alpha \cap x$.

Let $\{\beta_i\}$ be a \mathbb{Z} -basis of $H_2(\tilde{X})$. Every element in $H^2(\tilde{X}) \otimes H_2(\tilde{X})$ can be written as $\sum_{i,j} \lambda_{i,j} \beta_i \otimes (\beta_j \cap [\tilde{X}])$ for some $\lambda_{i,j} \in \mathbb{Z}$. The map

$$\Psi := \varphi_1\left(\sum_{i,j} \lambda_{i,j} \beta_i \otimes (\beta_j \cap [\tilde{X}])\right) \circ \text{PD}: H^2(\tilde{X}) \rightarrow H_2(\tilde{X})$$

is given by $\Psi(\alpha) = \sum_{i,j} \lambda_{i,j} ((\alpha \cup \beta_i) \cap [\tilde{X}])(\beta_j \cap [\tilde{X}])$. Recall that it is symmetric if $\alpha' \cap \Psi(\alpha) = \alpha \cap \Psi(\alpha')$. Let $\{\beta_i^*\}$ be the basis dual to $\{\beta_i\}$ in the sense that $(\beta_i^* \cup \beta_j) \cap [\tilde{X}] = \delta_{ij}$. We see that if Ψ is symmetric, then

$$\begin{aligned} \lambda_{i',j'} &= \sum_{i,j} \lambda_{i,j} ((\beta_{i'}^* \cup \beta_i) \cap [\tilde{X}])(\beta_j \cap [\tilde{X}]) \\ &= \beta_{j'}^* \cap \Psi(\beta_{i'}^*) \\ &= \beta_{i'}^* \cap \Psi(\beta_{j'}^*) \\ &= \sum_{i,j} \lambda_{i,j} ((\beta_{j'}^* \cup \beta_i) \cap [\tilde{X}])(\beta_i \cap [\tilde{X}]) \\ &= \lambda_{j',i'} \end{aligned}$$

One easily computes

$$\tau_*(\varphi_1(\sum_{i,j} \lambda_{i,j} \beta_i \otimes (\beta_j \cap [\tilde{X}]))) = \varphi_1(\sum_{i,j} \lambda_{i,j} \tau \beta_i \otimes (\beta_j \cap [\tilde{X}])))$$

and using that τ has order two and $w(\tau) = 1$, we get

$$(\tau \beta_i \cup \beta_j) \cap [\tilde{X}] = (\tau^2 \beta_i \cup \tau \beta_j) \cap w(\tau)[\tilde{X}] = (\beta_i \cup \tau \beta_j) \cap [\tilde{X}].$$

Let Ψ be symmetric. By [Lemma 3.11](#) we then get

$$\begin{aligned} \varphi_2(\sum_{i,j} \lambda_{i,j} \tau \beta_i \otimes (\beta_j \cap [\tilde{X}]))) &= \sum_{i,j} \lambda_{i,j} (\tau \beta_i \cup \beta_j) \cap [\tilde{X}] \\ &= \sum_i (\tau \beta_i \cup \beta_i) \cap [\tilde{X}] + 2 \sum_{i < j} \lambda_{i,j} (\tau \beta_i \cup \beta_j) \cap [\tilde{X}] \\ &= 2 \sum_i (\beta_i \cup \beta_i) \cap [\tilde{X}] + 2 \sum_{i < j} \lambda_{i,j} (\tau \beta_i \cup \beta_j) \cap [\tilde{X}] \end{aligned}$$

This implies the lemma since by definition of κ' every element in its image is coming from $\text{Sym}_{\mathbb{Z}}(H^2(\tilde{X}), H_2(\tilde{X}))$. \square

Proof of [Lemma 3.5](#). Since $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w = \mathbb{Z}$, $f_*: \pi_2(X) \rightarrow \pi_2(B)$ is an isomorphism and $f^* \lambda = \lambda_X$, it suffices to show that there exists a $\mathbb{Z}\pi$ -homomorphism

$$\kappa: \Gamma(\pi_2(X)) \rightarrow \mathbb{Z}^w$$

with $\kappa(\lambda_X) = 1$.

We have $\text{rank}(H_2(\tilde{X})) = \chi(\tilde{X}) - 2 = |\pi| \chi(X) - 2$, where χ denotes the Euler characteristic. Thus it suffices to show that there exist homomorphisms $\kappa_i: \Gamma(\pi_2(X)) \rightarrow \mathbb{Z}^w$ such that:

- (1) $\kappa_1(\lambda_X) = |\pi|$,
- (2) $\kappa_2(\lambda_X) = \text{rank}(H_2(\tilde{X}))$ and
- (3) $\kappa_3(\lambda_X)$ is odd,

because if $\kappa_3(\lambda_X) = 2n + 1$, then for $\kappa = \kappa_3 + n\kappa_2 - n\chi(X)\kappa_1$ we have $\kappa(\lambda_X) = 1$. **(1):** For a $\mathbb{Z}\pi$ -module M let $M^{\pi w}$ denote the submodule consisting of those objects m with $gm = w(g)m$ for all $g \in \pi$. We have $\lambda_X \in \Gamma(\pi_2(X))^{\pi w}$. Any \mathbb{Z} -homomorphism $\Gamma(\pi_2(X))^{\pi w} \rightarrow \mathbb{Z}^w$ is already a $\mathbb{Z}\pi$ -homomorphism, since π acts on both sides via w . Since X is a finite 4-dimensional Poincaré complex with finite fundamental group, $\pi_2(X)$ is free abelian. By [Lemma 2.2](#), $\Gamma(\pi_2(X))$ is therefore free abelian. Capping with $[\tilde{X}]$ gives an isomorphism $H^2(\tilde{X}) \rightarrow H_2(\tilde{X})$ and hence $\lambda_X \in \Gamma(\pi_2(X))^{\pi w}$ is primitive. Therefore, there exists a \mathbb{Z} - and thus $\mathbb{Z}\pi$ -linear map

$$d: \Gamma(\pi_2(X))^{\pi w} \rightarrow \mathbb{Z}^w$$

with $d(\lambda_X) = 1$. We define κ_1 as the composition

$$\kappa_1: \Gamma(\pi_2(X)) \xrightarrow{\cdot N^w} (\Gamma(\pi_2(X))^w)^{\pi} \xrightarrow{d} \mathbb{Z}^w.$$

(2): Define $\kappa_2 := \text{trace} \circ \kappa'$. By [Lemma 3.8](#),

$$\kappa_2(\lambda_X) = \text{trace}(\text{Id}_{H_2(\tilde{X})}) = \text{rank}(H_2(\tilde{X})).$$

(3): Let $G \leq \pi$ be a 2-Sylow subgroup and $p: X' \rightarrow X$ the corresponding covering. Then the map

$$\text{tr}_G^{\pi}: \Gamma(\pi_2 X) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \rightarrow \Gamma(\pi_2 X) \otimes_{\mathbb{Z}G} \mathbb{Z}^{p^* w}$$

has the property that

$$\mathrm{tr}_G^\pi(\lambda_X \otimes 1) = \sum_{k \in G \setminus \pi} (k(\lambda_X) \otimes k \cdot 1) = [\pi : G](\lambda_X \otimes 1),$$

where the last equation follows from $k(\lambda_X) = w(k) \cdot \lambda_X \in \Gamma(\pi_2 X)$ for all $k \in \pi$. Since $[\pi : G]$ is odd and $\lambda_X = \lambda_{X'}$, we can therefore assume that π is a 2-group by composing κ_3 for G with tr_G^π . Note that $p^* : H^1(X; \mathbb{Z}/2) \rightarrow H^1(X'; \mathbb{Z}/2)$ is injective and thus p^*w is non trivial if and only if w is non trivial.

Let κ' be the homomorphism from (3.6). By Lemma 3.12, the image of $\mathrm{trace} \circ \tau_* \circ \kappa'$ is always even and we can define $\kappa_3 := -\frac{1}{2}(\mathrm{trace} \circ \tau_* \circ \kappa')$. By Lemma 3.10, $\kappa_3(\lambda_X) = 1$. \square

Proof of Proposition 1.7. By [HK88, Theorem 2.1] we have

$$\Gamma(\mathbb{Z}\pi) \cong F \oplus \bigoplus_{g \in \pi \setminus \{1\}, g^2=1} \mathbb{Z}\pi/\mathbb{Z}\pi(1-g),$$

for some free module F . It is an easy calculation that this implies

$$\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi)) \cong (\mathbb{Z}/2)^r,$$

where r is the number of $g \in \pi \setminus \{1\}$ with $g^2 = 1$ and $w(g) = -1$. Since

$$\Gamma(\pi_2 X) \cong \Gamma(\mathbb{Z}\pi) \oplus \Gamma(M) \oplus \mathbb{Z}\pi \otimes_{\mathbb{Z}} M,$$

$\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi))$ is a summand of $\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X))$. \square

Proposition 3.13. *There is a finite Poincaré complex with the same quadratic 2-type as $\mathbb{R}\mathbb{P}^4 \# \mathbb{C}\mathbb{P}^2$ which is not homotopy equivalent to $\mathbb{R}\mathbb{P}^4 \# \mathbb{C}\mathbb{P}^2$.*

Proof. Let B be the Postnikov 2-type of $\mathbb{R}\mathbb{P}^4 \# \mathbb{C}\mathbb{P}^2$. By Lemma 3.1, B -polarized Poincaré complexes up to homotopy equivalence over B are classified by the image of their fundamental class in $H_4(B; \mathbb{Z}^w)$. Hence to understand them up to homotopy equivalence, we have to study the action of all self-homotopy equivalences of B on $H_4(B; \mathbb{Z}^w)$. As $\pi_2(B)$ is free, the k -invariant of B is trivial and the map $B \rightarrow B\mathbb{Z}/2$ admits a splitting s . Using the Serre spectral sequence and the proof of Proposition 1.7, one computes

$$\begin{aligned} H_4(B; \mathbb{Z}^w) &\cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} H_4(\tilde{B}; \mathbb{Z}) \oplus H_4(B\mathbb{Z}/2; \mathbb{Z}^w) \\ &\cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2]) \oplus \mathbb{Z}/2 \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \end{aligned}$$

By obstruction theory, for every self-homotopy equivalence φ of B we have $\varphi \circ s \simeq s$. Hence φ acts diagonally on $H_4(B; \mathbb{Z}^w)$ with the identity on $H_4(B\mathbb{Z}/2; \mathbb{Z}^w)$ and on $\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} H_4(\tilde{B}; \mathbb{Z})$ by the map on $\pi_2(B)$ induced by φ . The automorphisms of $\pi_2(B) \cong \mathbb{Z}[\mathbb{Z}/2]$ are given by multiplication with $\{\pm 1, \pm g\}$ where $g \in \mathbb{Z}/2$ is the generator. It follows that the action of φ on $H_4(B; \mathbb{Z}^w)$ is given by ± 1 and thus is the identity on the torsion subgroup. In particular, it is the identity on the torsion subgroup $\mathbb{Z}/2$ of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2])$. It thus follows from Theorem 1.5 that there is precisely one Poincaré complex which has the same quadratic 2-type as $\mathbb{R}\mathbb{P}^4 \# \mathbb{C}\mathbb{P}^2$ but is not homotopy equivalent to it. \square

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