CONSTRUCTIVE ARROWS:
AN INTRODUCTION TO CATEGORIES, TOPOSES AND LOGIC

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Abstract

Category theory, especially topos theory, admits a new perspective on the study of logic and mathematical foundations. In this dissertation, we provide an introduction to the development of logic in a topos, and show why this logic does not validate the law of excluded middle. Assuming no prior knowledge of category theory, we motivate and introduce some main concepts of categories that allow for defining a topos. We briefly provide an introduction to order theory, giving the tools needed for analysis of the subobject algebras in a topos. We introduce the domain of formal logic and define propositional logical valuations on the subobject algebras and on a topos. We end with showing how the topos logic is intuitionistic, by virtue of the subobject algebras being Heyting algebras.

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Contents

Chapter 1

Introduction

Abstraction is a powerful idea in mathematics. Taking an abstract, 'external' view of something often elicits deeper understanding — an idea that has lead to the modern success of category theory. This dissertation takes an abstract view of mathematics and logic, and through this perspective connects the two domains together.

Categories and internal languages

In the 1940s, during their study into natural transformations in algebraic geometry, Samuel Eilenburg and Saunders Mac Lane began to develop the mathematical field now known as category theory. This domain takes an extrinsic viewpoint of areas of mathematics, and concerns itself with *relationships* between things more than internal structures, for example, as in set theory. For example, when looking at topological spaces, category theory cares more about the continuous functions between them than the spaces themselves. A category is thus a collection of objects (like topological spaces) and arrows between them (like continuous functions).

Later, in the 1950s, Alexander Grothendieck began detailed investigations of a special type of category: a topos. These are categories that, in some sense, look like the category of sets. William Lawvere furthered topos theory by showing that mathematics can be founded within it, whereby an 'internal logic' of toposes is identified and used to develop an axiomatisation of set theory [Law66]. This internal logic takes the form of arrows in a category, which represent both logical propositions and logical connectives. Curiously, this logic is not classical — the name given to the everyday logic of mathematics. It is weaker than classical logic: fewer statements come out valid. This is by virtue of it lacking the law of excluded middle, which requires statements to be either true or false. Lacking this law is what defines an *intuitionistic logic* — the type of logic championed by constructive mathematicians, who maintain that a proof of existence of an object is meaningless without a process of construction.[1]

Aim and prospectus

This work aims to provide a detailed introduction to the categorial analysis of logic in toposes, and to explain why this topos logic is intuitionistic.[2]

The principle source for this dissertation is Robert Goldblatt's book *Topoi: The Categorial Analysis of Logic*, which is considered by many to be the definitive introduction to understanding logic in a topos [Gol84]. This dissertation will differ in Goldblatt's approach, however, by taking a more intuitive journey in understanding just how different ideas can be leveraged to introduce logic into categories, rather than outright defining a categorial logical system. The journey will be enriched and contextualised using Awodey's work on category theory, Rasiowa's work on algebraic logic, and Priest's work on non-classical logic [Awo06; Ras74; Pri08].

We will begin in Chapter 2 with an overview of some basic ideas and definitions in category theory, before defining the key type of category we are interested in: a topos. In Chapter 3, we will digress into an introduction to order theory, where we define lattices and algebras. These structures have established connections to logic, and so provide a natural starting point for logical analysis. We will walk through how one might identify and use this lattice structure within the context of categories in Chapter 4. Chapter 5 is where we will formally examine logic, introduce how logical analysis could be done in a topos, and answer the question of the connection between this topos logic and intuitionistic logic. Finally, in Chapter 6 we will provide a brief discussion on how this theory can be progressed, and what questions one might ask next.

[1] Constructivist mathematics was formalised over the early decades of the 20$^{\text{th}}$ century by L.E.J. Brouwer.
[2] The word 'categorial' means pertaining to categories.

Chapter 2

Categories and Toposes

To begin, we need to explain some basic concepts of category theory. We will define what a category is, and introduce some features we may encounter within one, such as products, limits and subobjects. We conclude this chapter with the definition of a topos — a special type of category whose properties will become the focus of later chapters.

2.1 Definition

If one walks into a room of mathematicians and begins to talk about sets, one assumes a 'universe of discourse' — a world that we agree upon, within which these objects known as sets exist. We understand there to be connections between sets in this universe, which we call functions. These mappings obey specific rules of structure, for example, we can compose functions to find a new one and there is always an identity function for each set. Now, if one ceases set-based discussion and begins talking about groups, the universe of discourse changes. This new universe comprises of the objects known as groups, and there are new mappings between them — the group homomorphisms.

What about if we considered the world of all topological spaces? Or all lattices? Or rings? If we looked hard enough in these worlds, we would notice structure similar to what we saw with sets and groups, where there are maps between the objects that seem to obey certain rules. When we step back from these 'worlds of mathematics', we are glimpsing *categories*.

Definition 2.1 (Category)**.** A *category* $\mathscr{C}$ is a structure comprised of the following:

(a) A collection of things called *objects*, $a, b, c, \ldots$

(b) A collection of things called *arrows*[3], $f, g, h, \ldots$

Key sources for Chapter 2: [Awo06], [Gol84].

[3] Arrows are also known as *morphisms*.

(c) For each arrow f, there are objects $\mathrm{dom}(f)$ (the *domain* of f) and $\mathrm{cod}(f)$ (the *codomain* of f). If $a = \mathrm{dom}(f)$ and $b = \mathrm{cod}(f)$, we write

$$f : a \longrightarrow b$$

(d) For each two arrows f and g with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is an arrow

$$g \circ f : \mathrm{dom}(f) \longrightarrow \mathrm{cod}(g)$$

called the *composite* of f and g. Composite arrows must satisfy associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : a \to b, g : b \to c, h : c \to d$.

(e) For each object b, there is a unique arrow $\mathrm{id}_b : b \to b$ called the *identity arrow*, such that for f with $\mathrm{cod}(f) = b$ and g with $\mathrm{dom}(g) = b$,

$$f = f \circ \mathrm{id}_b, \quad g = \mathrm{id}_b \circ g.$$

Remark 2.2. Note the use of the word 'collection' when referring to the collective noun of objects and arrows — this is purposefully not 'set' or 'class'. Most of these collections are "too big" to be sets, having to be described as proper classes in the language of set theory.[4] We do not, however, use the word 'class' to describe them for a more philosophical reason. Some mathematicians, such as Lambek and Lawvere, consider category theory to be an alternate foundation of mathematics, and through the language of categories are able to define and describe the theory of sets and classes.[5] Hence to use this terminology would be circular if one were to adopt this view — a perspective we are agnostic to as of writing. Nonetheless it is perfectly consistent to assume set theory as a foundation and use its terminology as descriptors for category theory. Thus, despite fascination over these particularities, for the time being we may assume a 'collection' of objects or arrows to be a class.

[4] The most widely accepted formal foundation for mathematics is Zermelo-Fraenkel set theory. In this theory we have to distinguish between *small classes* (which are sets) and *proper classes* (which are not). Proper classes are like sets in that they contain things, but they have cardinalities so large to keep them on equal footing as sets would cause paradoxes. An exploration of this distinction can be found in [Mad83]. See Chapter 1, section 8 of [Awo06] for more details on small and large categories.

[5] For Lawvere's approach to a categorial foundations of mathematics see [Law66]. For a recent review of the area by Lambek and Scott see [LS11].

In this theory of categories, we often use *commutative diagrams* of objects and arrows to communicate information. With objects a, b, c and arrows f, g, h between them, we say that the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd} \tag{2.1}$$

commutes if (and only if) $h = g \circ f$. More complicated diagrams are said to commute if each triangle within it commutes. As an example, requirement (e) in Definition 2.1 can rephrased as saying that

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "\mathrm{id}_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"'] & c \end{tikzcd} \tag{2.2}$$

must commute. Requirement (e) also implies that identity arrows must be unique. To see this, let id_b and id'_b be two identity arrows on the object b (i.e., $a = c = b$ and $f = g = \mathrm{id}'_b$ in Diagram (2.2)). Then $\mathrm{id}_b \circ \mathrm{id}'_b = \mathrm{id}'_b$ and $\mathrm{id}_b \circ \mathrm{id}'_b = \mathrm{id}_b$ (consider the left and right triangles of (2.2)), hence $\mathrm{id}'_b = \mathrm{id}_b$.

In addition to 'commute', another piece of categorial jargon we frequently use is 'factor'. Considering Diagram (2.1), we say that the arrow *h factors through* each of f and g. That is, h factors through e when it is equal to the composition of e and with some other arrow.

Below are some common categories that mathematicians work in. Categories are usually notated with short, boldface names.

Example 2.3. The following structures are categories.

(a) The category **1** is the category uniquely determined by the property of having exactly one object and arrow — call them a and the identity of a, id_a. This object-arrow pair can be relabelled however we want, but the categorial structure will remain the same — which is what we care about in the study of categories. Because of this, we can say that **1** is unique (at least up to some relabelling). Similarly, **2** is the category uniquely determined by having two objects and three arrows.

$$\mathbf{1}: \quad \begin{tikzcd} a \arrow[loop above, "\mathrm{id}_a"] \end{tikzcd} \qquad\qquad \mathbf{2}: \quad \begin{tikzcd} a \arrow[loop above] \arrow[r] & b \arrow[loop above] \end{tikzcd}$$

From now on, we may omit drawing any identity arrow loops in commutative diagrams.

(b) **Set** is the category of sets, with sets as objects and total functions between sets as arrows. We will frequently return to **Set** for concrete examples of important categorial concepts and definition. **Set** is also a *topos*, a concept to be elaborated on at the end of this chapter.

(c) $\mathbf{Set}^{\rightarrow}$ is the category of arrows of **Set**. Equivalently this is the category with set functions as objects. The $\mathbf{Set}^{\rightarrow}$ arrows are defined in terms of pairs of **Set** arrows. Given sets A, B, C, D with functions $f : A \to B, g : C \to D$ (so f and g are objects of $\mathbf{Set}^{\rightarrow}$), an arrow

$$f \xrightarrow{(i,j)} g \tag{2.3}$$

in $\mathbf{Set}^{\rightarrow}$ is a pair (i, j) of functions such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "j"] & D \end{tikzcd} \tag{2.4}$$

commutes.

(d) **Grp** has groups as objects and group homomorphisms as arrows, while the category **Ring** has rings as objects and ring homomorphisms as arrows.

(e) **Met** has metric spaces as objects and contraction maps as arrows. The category **Top** has topological spaces as objects and all continuous functions as arrows.

(f) **Pos** has partially ordered sets as objects and monotone increasing functions as arrows.

Before moving to an analysis of various possible structures that can appear in categories, we first define the so-called *hom-set* of two objects.

Definition 2.4. Given a category $\mathscr{C}$ with objects a and b, the collection of all arrows from a to b (i.e., all arrows f with $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$) is known as the *hom-set*, denoted $\mathscr{C}(a, b)$.

This collection of arrows will turn out to have great importance when we start talking about subobjects and truth values — but more on that later.

2.2 Arrows

We will now define and discuss various types of structures that can appear in a category. Specifically, these will be the structures required for the definition of a topos. We begin by focusing on the arrows in a category, and the different ways they can interact and exist.

2.2.1 Monic

When considering functions between sets, we call a function $f : A \to B$ *injective* if it satisfies

$$f(x) = f(y) \implies x = y.$$

If we have some further functions $g, h : C \to A$, then the equality $(f \circ g)(x) = (f \circ h)(x)$ implies that $g(x) = h(x)$. So injectivity also results in a *left cancellation* property. This becomes the basis for defining a certain type of arrow.

Definition 2.5 (Monic arrow)**.** In a category $\mathscr{C}$, an arrow $f : a \to b$ is *monic* if for any two arrows $g, h : c \to a$ the equality $f \circ g = f \circ h$ implies that $g = h$.

$$c \underset{h}{\overset{g}{\rightrightarrows}} a \overset{f}{\rightarrowtail} b \tag{2.5}$$

We draw monic arrows with a tail ($\rightarrowtail$). Monic arrows are also known as left cancellable.

Example 2.6. In **Set**, monic arrows are exactly those functions that are injective. We have seen that injective arrows must be monic. For the converse, suppose $f : A \rightarrowtail B$ is monic and let $x, y \in A$, assuming that $f(x) = f(y)$. Consider the functions $g, h : \{w\} \to A$ where $g(w) = x$

and $h(w) = y$, noting that $\{w\}$ can be any singleton object. Then $f \circ g = f \circ h$, so $g(w) = h(w)$ by left cancellation, and thus $x = y$.

A simple result that we will use later is that the composition of monic arrows is also monic.

Proposition 2.7. *Let e and f be monic arrows as shown in the diagram below. Then $e \circ f$ is monic too.*

$$d \overset{g}{\underset{h}{\rightrightarrows}} a \overset{f}{\rightarrowtail} b \overset{e}{\rightarrowtail} c \tag{2.6}$$

Proof. Suppose there exist arrows $g, h : d \to a$ such that $(e \circ f) \circ g = (e \circ f) \circ h$. Then $e \circ (f \circ g) = e \circ (f \circ h)$ by associativity of arrows, giving that $f \circ g = f \circ h$ because e is monic. But f is monic too, so $g = h$, and hence $(e \circ f)$ is monic. $\square$

2.2.2 Epic

As well as injective functions on sets we can have surjective functions. These functions turn out to have a similar cancellation property, but now they are *right cancellable*. The categorial abstraction of surjective functions are *epic* arrows.

Definition 2.8 (Epic arrow)**.** In a category $\mathscr{C}$, an arrow $f : a \to b$ is *epic* if for any two arrows $g, h : b \to c$ the equality $g \circ f = h \circ f$ implies that $g = h$.

$$a \overset{f}{\twoheadrightarrow} b \overset{g}{\underset{h}{\rightrightarrows}} c \tag{2.7}$$

We draw epic arrows with a double arrow head ($\twoheadrightarrow$). Epic arrows are also known as right cancellable.

Example 2.9. In **Set**, epic arrows are exactly those functions that are surjective.

Like monic arrows, the composition of epic arrows remains epic.

Proposition 2.10. *Let e and f be epic arrows as shown in the diagram below. Then $f \circ e$ is epic too.*

$$d \overset{e}{\twoheadrightarrow} a \overset{f}{\twoheadrightarrow} b \overset{g}{\underset{h}{\rightrightarrows}} c \tag{2.8}$$

Proof. Suppose there exist arrows $g, h : d \to a$ such that $g \circ (f \circ e) = h \circ (f \circ e)$. Then $(g \circ f) \circ e = (h \circ f) \circ e$ by associativity of arrows, giving that $g \circ f = h \circ f$ because e is epic. But f is epic too, so $g = h$, and hence $(f \circ e)$ is epic. □

2.2.3 Iso

A function that is both injective and surjective is known to be an isomorphism of sets — it is simply a relabelling of elements that does not change any structure. Abstracting to categories, we can have similar constructions called *iso arrows*.

Definition 2.11 (Iso arrows and objects)**.** In a category $\mathscr{C}$, an arrow $f^{-1} : b \to a$ is an *inverse arrow for f* if $f^{-1} \circ f = \mathrm{id}_a$ and $f \circ f^{-1} = \mathrm{id}_b$. Equivalently, the diagram

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}_a"] \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "\mathrm{id}_b"] \end{tikzcd} \tag{2.9}$$

commutes. When an arrow $f : a \to b$ has an inverse we say that it is *iso*, and objects a and b are *isomorphic*, denoted $a \cong b$. In a commutative diagram we may write an iso arrow k with two arrowheads, meaning that the arrow commutes in either direction.[6]

The inverse to an arrow f, should it exist, must be unique. To see this, suppose both f^{-1} and g are inverses of f. Then $g = \mathrm{id}_a \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ \mathrm{id}_b = f^{-1}$.

Iso arrows were motivated as abstractions of those arrows that are both monic and epic. The next proposition confirms that, like how injective and surjective functions are bijective, monic and epic arrows are iso.

Proposition 2.12. *Suppose $f : a \to b$ is iso with inverse $f^{-1} : b \to a$. Then f is monic and epic.*

Proof. (Adapted from the proof for Prop. 2.6 in [Awo06, p. 27]). Suppose there exist arrows

[6] In category theory, constructions can only be defined up to isomorphism. This is due to properties (d) and (e) in the definition of a category (Def. 2.1), and how they interact with iso arrows — any arrow-theoretic construction involving an object a will extend via composition with iso arrows to all objects isomorphic to a.

g_1, g_2, h_1, h_2 such that $f \circ g_1 = f \circ g_2$ and $h_1 \circ f^{-1} = h_2 \circ f^{-1}$.

$$\begin{tikzcd}
c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] \arrow[dr, "\mathrm{id}_a"'] & b \arrow[d, "f^{-1}"] & \\
 & & a \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & d
\end{tikzcd} \tag{2.10}$$

Then $f^{-1} \circ f \circ g_1 = f^{-1} \circ f \circ g_2$ and $h_1 \circ f^{-1} \circ f = h_2 \circ f^{-1} \circ f$. But $f^{-1} \circ f = \mathrm{id}_a$, so $\mathrm{id}_a \circ g_1 = \mathrm{id}_a \circ g_2$ and $h_1 \circ \mathrm{id}_a = h_2 \circ \mathrm{id}_a$, that is, $g_1 = g_2$ and $h_1 = h_2$. Thus f is monic and f^{-1} is epic. Swapping the order of f and f^{-1} then gives them both as epic and monic. $\square$

While all iso arrows are monic and epic, the converse does not always hold.[7]

2.2.4 Unique arrows

The remaining definitions in this chapter will all be relative to an arbitrary category $\mathscr{C}$.

Definition 2.13 (Initial object)**.** An object 0 is *initial* if for every object a there is precisely one arrow from 0 to a.

There are two notations used for this unique arrow from 0 to a — we may use either $0_a : 0 \to a$ or $! : 0 \to a$ (the exclamation mark is often used to denote unique arrows).

Dual to initial objects are terminal objects.

Definition 2.14 (Terminal object)**.** An object 1 is *terminal* if for every object a there is precisely one arrow from a to 1.

Similarly as with initial objects, we denote the unique arrow from a to 1 as either $1_a : a \to 1$ or $! : a \to 1$.

Example 2.15. In **Set**, the empty set is an initial object and any singleton — a set with a single element — is a terminal object. Recall that the arrows in **Set** are total functions between sets. I.e., if $f : X \to Y$ is an arrow then it is a subset of the Cartesian product $X \times Y$, such that each element $x \in X$ appears exactly once.

[7] For examples, see [Gol84, pp. 40–41].

If $X = \emptyset$ then the product $X \times Y$ is empty. This empty relation is known as the empty function (it still vacuously satisfies the definition of a function), and it exists for any Y. It is also the only such function that can exist between $\emptyset$ and Y. Hence there is exactly one arrow from $\emptyset$ to every set Y in **Set**, so $\emptyset$ is an initial object.

Now taking Y to be the singleton $\{\star\}$, we have the constant function $f : X \to \{\star\}$ that maps every $x \in X$ to $\star \in Y$. Notice that f is also the only subset of $X \times Y$ that satisfies the definition of a total function. So for any set X in **Set** there exactly one arrow from X to $\{\star\}$ — the constant function — and hence $\{\star\}$ is terminal.

A natural question to ask while we are defining these various categorial concepts is whether the constructions they identify are unique or not. We will show later, after considering limits and colimits, that in all cases these objects are defined up to isomorphism.

2.3 Duality

Definition 2.16. Let $\mathscr{C}$ be a category. The *dual category* of a category, $\mathscr{C}^{\mathrm{op}}$, consists of

(i) the same collection of objects as $\mathscr{C}$;

(ii) the reverse of each arrow in $\mathscr{C}$, i.e., the same arrows f as in $\mathscr{C}$ but $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ are swapped.[8]

Furthermore, if Σ is a statement using terms defined in Definition 2.1 then the *dual statement* Σ^{op} is obtained by replacing every instance of

(i) “dom" for “cod",

(ii) “cod" for “dom",

(iii) and “$h = f \circ g$" for “$h = g \circ f$".

The statement Σ can be a definition (e.g., that of a monic arrow) or a result (e.g., any two limits over the same diagram are isomorphic — although this is a result we will arrive at later). In effect, taking the dual of Σ means reversing all the arrows in a descriptive commutative diagram, like the one used in the definition of a monic arrow.

[8]Mac Lane provides a justification that $\mathscr{C}^{\mathrm{op}}$ is indeed a well-defined category in [Mac98, p. 33].

Definition 2.1 itself turns out to be an invariant statement under this operation of taking the dual statement. This result leads to the remarkable *duality principle*:

the statement Σ is true of $\mathscr{C}$ if and only if Σ^{op} is true of $\mathscr{C}$.[9]

The duality principle means that when operating in the study of categories, we need only prove half as many results, as the other half will follow by duality. We also need only provide detailed definitions of half as many categorial constructions.

Example 2.17. The following constructions exhibit duality.

(a) The definition of monic arrows is dual to the definition of epic arrows. Notice that the diagram used in Definition 2.5 is dual to the diagram used in Definition 2.8, and the left cancellation requirement ($f \circ g = f \circ h \implies g = h$) is dual to the right cancellation requirement ($g \circ f = h \circ f \implies g = h$).

(b) Because of monic-epic duality, the dual of any theorem that applies to monic arrows will apply to epic arrows (and vice versa). For example, Proposition 2.10 which applies to epic arrows is precisely the dual of 2.7, which applies to monic arrows.

(c) The definition of an initial object is dual to the definition of a terminal object.

2.4 Products and Equalisers

When defining products and equalisers we introduce the notion of a *universal construction.* This is where we define something to have a certain arrow-theoretic relationship within a category, and then demand that it is the most 'universal' version of that thing. This hazy idea should become clear with the categorial abstraction of a familiar concept: the cartesian product.

Definition 2.18 (Product)**.** Let a and b be objects in a category $\mathscr{C}$. A *product* of a and b is an object $a \times b$ with arrows

$$\mathrm{pr}_a : (a \times b) \to a, \ \mathrm{pr}_b : (a \times b) \to b$$

[9] A justification for the duality principle can be found in [Awo06].

such that for any object c and pair of arrows $f : c \to a, g : c \to b$ there is precisely one arrow $\langle f, g\rangle : c \dashrightarrow (a \times b)$ that satisfies $\mathrm{pr}_a \circ \langle f, g\rangle = f$ and $\mathrm{pr}_b \circ \langle f, g\rangle = g$.[10] Equivalently, $\langle f, g\rangle$ is the unique arrow ensuring that

$$\begin{tikzcd} & c \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, bend left, "g"] & \\ a & a \times b \arrow[l, "\mathrm{pr}_a"] \arrow[r, "\mathrm{pr}_b"'] & b \end{tikzcd} \tag{2.11}$$

commutes. The arrows pr_a and pr_b are known as *projection arrows*, and may be written pr_1 and pr_2, respectively, to denote which element of the ordered product arrow $\langle f, g\rangle$ they pick out. We shall sometimes draw unique arrows that exist by virtue of some definition with a dashed line as we have done above.

Notice there are two parts to the definition of the product p of a and b. The first requires an arrow-theoretic relationship of p to a and b, namely the existence of the projection arrows. The second part demands that any other object c that also has arrows to a and b — i.e., any other object that could *play the role of p* in this construction — has a *unique factorisation* through p. This positions the object p as the universal version of this construction, through which all other similar constructions factor.

Definition 2.19 (Equaliser)**.** An *equaliser* of a pair of arrows with common domain and codomain, $f, g : a \to b$, is an object m along with an arrow $e : m \to a$ such that $f \circ e = g \circ e$, where for any other arrow $h : c \to a$ that satisfies $f \circ h = g \circ h$ there exists a unique arrow $k : c \to m$ such that $e \circ k = h$.

$$\begin{tikzcd} m \arrow[r, "e"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ & c \arrow[ul, dashed, "k"] \arrow[u, "h"'] & \end{tikzcd} \tag{2.12}$$

As with products, notice that Definition 2.19 has two parts: the requirement that e 'equalises' the two arrows f and g ($e \circ f = e \circ g$), and that any other such equalising arrow h factors uniquely through it.

[10]The label of the object $a \times b$ is just a name that illustrates its existence in terms of the definition of the product. This is not denoting some binary operation on the objects a and b themselves.

2.5 Limits

To discuss limits, let us first formalise the arrow and object drawings we have been using to explain concepts in category theory.

Definition 2.20 (Diagram). A *diagram* D in a category $\mathscr{C}$ is a collection of objects in $\mathscr{C}$ $d_1, d_2, d_3, \ldots,$ and a collection of arrows between them $g : d_i \to d_j$, where the arrows are from $\mathscr{C}$ as well.[11]

Note that a diagram is not required to satisfy the axioms for being a category itself. It is just a 'portion' of the entire category; if we think of $\mathscr{C}$ as a directed graph then a diagram is a subgraph.

Definition 2.21 (Cone). Let D be a diagram in a category $\mathscr{C}$. A *cone* over D is an object c and a collection of arrows $f_i : c \to d_i$, one f_i for each object d_i in D, such for any arrow $g : d_i \to d_j$ in D, $g \circ f_i = f_j$. Equivalently, c and the arrows f_i are such that the diagram

$$\begin{tikzcd} & c \arrow[dl, bend right, "f_i"'] \arrow[dr, bend left, "f_j"] & \\ d_i \arrow[rr, "g"] & & d_j \end{tikzcd} \tag{2.13}$$

commutes.

A *universal cone over D* — a cone through which all others factor — is known as a limit.

Definition 2.22 (Limit). Let D be a diagram in a category $\mathscr{C}$. A *limit* over D is a cone l with arrows $h_i : l \to d_i$, such that if c with arrows f_i is also a cone for D then there exists a unique arrow $k : c \dashrightarrow l$ where, for all h_i and f_i, $h_i \circ k = f_i$. Equivalently, k is the unique arrow

[11] There is a more formal way to define a diagram than we have done in Definition 2.20. The other definition involves functors and index categories, however as this dissertation avoids discussion of functors we have opted for a simpler approach that treats a diagram as a 'portion' of $\mathscr{C}$.

allowing the diagram

$$
\begin{tikzcd}
& c \arrow[d, dashed, "k"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\
& l \arrow[dl, bend right, "h_i"'] \arrow[dr, bend left, "h_j"] & \\
d_i \arrow[rr, "g"] & & d_j
\end{tikzcd}
\tag{2.14}
$$

to commute.

Proposition 2.23. *If a limit over D exists, it is unique up to isomorphism.*

Proof. To see this, suppose that l and l' are limits over D (with arrows $h_i : l \to d_i$ and ${h_i}' : l' \to d_i$ respectively). Then, because both l and l' are cones and limits for D, they uniquely factor through each other. So there exist unique arrows $k : l' \dashrightarrow l$ and $k' : l \dashrightarrow l'$ such that ${h_i}' = h_i \circ k$ and $h_i = {h_i}' \circ k'$ for all i. This gives the following commutative diagram for each d_i in D.

$$
\begin{tikzcd}
l \arrow[rr, dashed, bend left, "k'"] \arrow[dr, bend right, "h_i"'] & & l' \arrow[ll, dashed, bend left, "k"] \arrow[dl, bend left, "{h_i}'"] \\
& d_i &
\end{tikzcd}
\tag{2.15}
$$

Since k and k' are unique with respect to the above diagram, the composition must also be the unique arrow $k \circ k' : l \to l$ such that $h_i = h_i \circ (k \circ k')$. But this composition is just the identity arrow on l, id_l, as identity arrows are also unique. A similar argument gives that $k' \circ k = \mathrm{id}_{l'}$. Hence k and k' are inverse to each other, thus l and l' are isomorphic. $\square$

Moving forward, we will now refer to *the* limit over D, should it exist.

Example 2.24. Consider the empty diagram D in $\mathscr{C}$ that consists of no objects or arrows. A cone over D will be an object c without any arrows. By Definition 2.22, a limit over D is then an object l such that for any other cone over D (i.e., any object c in $\mathscr{C}$), there is a unique arrow $k : c \dashrightarrow l$. But this is just the definition of a terminal object (Def. 2.14). So we say that a terminal object is a limit over the empty diagram.

Example 2.25. Suppose the diagram D consists of two objects a and b, and two arrows f and g between them.

$$
\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}
\tag{2.16}
$$

A cone over D is then an object c and arrows $h : c \to a, j : c \to b$ such that $j = f \circ h$ and $j = g \circ h$. Hence $f \circ h = g \circ h$, so h has the property of 'equalising' f and g.

$$\begin{tikzcd} & c \arrow[dl, bend right, "h"'] \arrow[dr, bend left, "j"] & \\ a \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & b \end{tikzcd} \tag{2.17}$$

So, the limit over D is an object l and arrow h that has this equalising property of f and g ($f \circ h = g \circ h$), such that any other such object and arrows factors uniquely through it. Note that the existence of $j = f \circ h = g \circ h$ is guaranteed by the commutativity of arrows in categories so we can drop it from the requirement of a limit over D. Then this limit is just the previously defined equaliser of f and g.

Example 2.26. For our final example of a limit, suppose D is the diagram consisting of objects a and b, and no arrows. A cone over D will then be an object c with arrows $f_a : c \to a$ and $f_b : c \to b$.

$$a \xleftarrow{f_a} c \xrightarrow{f_b} b \tag{2.18}$$

The limit over D is then the product of a and b as defined in Definition 2.18.

Remark 2.27. As demonstrated above in Proposition 2.23, limits are unique up to isomorphism. Because terminal objects, equalisers and products are special cases of limits, we conclude that these constructions are also unique up to isomorphism. As such, should these constructions exist in a category, we will now refer to *the* terminal object, *the* equaliser of f and g, and *the* product of a and b.

The dual construction to a limit is a colimit — essentially the definition of a limit with the arrows reversed. This definition requires the dual notion to a cone, unsurprisingly known as a co-cone.

Definition 2.28 (Co-cone)**.** Let D be a diagram in a category $\mathscr{C}$. A *co-cone* under D is an object c and a collection of arrows $f_i : d_i \to c$, one f_i for each object d_i in D, such for any arrow $g : d_i \to d_j$ in D, $f_j \circ g = f_i$. Equivalently, c and the arrows f_i are such that the diagram

$$
\begin{tikzcd}
d_i \arrow[rr, "g"] \arrow[dr, "f_i"'] & & d_j \arrow[dl, "f_j"] \\
& c &
\end{tikzcd}
\tag{2.19}
$$

commutes.

A colimit is then the universal co-cone under D.

Definition 2.29 (Colimit)**.** Let D be a diagram in a category $\mathscr{C}$. A *colimit* under D is a co-cone l with arrows $h_i : l \to d_i$, such that if c with arrows f_i is also a cone for D then there exists a unique arrow $k : l \dashrightarrow c$ where, for all h_i and f_i, $k \circ h_i = f_i$. Equivalently, k is the unique arrow allowing the diagram

$$
\begin{tikzcd}
d_i \arrow[rr, "g"] \arrow[dr, "h_i"'] \arrow[ddr, bend right, "f_i"'] & & d_j \arrow[dl, "h_j"] \arrow[ddl, bend left, "f_j"] \\
& l \arrow[d, dashed, "k"] & \\
& c &
\end{tikzcd}
\tag{2.20}
$$

to commute.

A category is said to be *complete* if limits exist over every diagram.[12] If limits only exist over all *finite* diagrams then the category is *finitely complete*. Dually, a category is *co-complete* if colimits exist under every small diagram, and *finitely co-complete* if colimits under every finite diagram.

2.6 Some limits and colimits of interest

2.6.1 Coproducts and coequalisers

Applying our knowledge of colimits, we can dualise two earlier constructions. First recall that the product of objects a and b is the limit over the diagram consisting of just the objects

[12] This is not quite true. The definition of completeness requires limits only over all *small* diagrams, whose collections of objects and arrows have cardinality smaller than a proper class — i.e. they are sets [Awo06, p. 167]. However, for our purposes this distinction is inconsequential.

a and b (Example 2.26). The *coproduct* is then just the colimit under this same diagram. More explicitly, it is defined the following way in [Gol84, p. 54].

Definition 2.30 (Coproduct)**.** The *coproduct* of a and b is an object $a+b$ with arrows $i_a : (a+b) \to a$, $i_b : (a+b) \to b$, such that for any object c and pair of arrows $f : a \to c$, $g : b \to c$ there is precisely one arrow $[f,g] : (a+b) \dashrightarrow c$ that satisfies $[f,g] \circ i_a = f$ and $[f,g] \circ i_b = g$.[13]

$$\begin{tikzcd}
a \arrow[r, "i_a"] \arrow[dr, bend right, "f"'] & a+b \arrow[d, dashed, "{[f,g]}"] & b \arrow[l, "i_b"'] \arrow[dl, bend left, "g"] \\
 & c &
\end{tikzcd} \tag{2.21}$$

Secondly, we can dualise the equaliser. Recalling that the equaliser of $f,g : a \to b$ is the limit over this same diagram (Example 2.25), the *coequaliser* is the colimit under $f,g : a \to b$.

Definition 2.31 (Coequaliser, [Gol84])**.** The *coequaliser* for a pair of arrows $f,g : a \to b$ is an arrow $q : b \to e$ such that $q \circ f = q \circ g$, and if there exists an $h : b \to c$ such that $h \circ f = h \circ g$ then there is a unique arrow $k : e \dashrightarrow c$ such that $h = k \circ e$.

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[rr, "q"] \arrow[dr, "h"] & & e \arrow[dl, dashed, "k"] \\
 & & c &
\end{tikzcd} \tag{2.22}$$

2.6.2 Pulling back

We now introduce a categorial construction central to our later discussion of toposes and logic.

Definition 2.32 (Pullback)**.** A *pullback* is a limit over a diagram that consists of a pair of arrows f,g with common codomain c.

$$\begin{tikzcd}
 & b \arrow[d, "g"'] \\
a \arrow[r, "f"] & c
\end{tikzcd} \tag{2.23}$$

[13]The coproduct is also known as the *sum* of a and b, hence the notation using $+$. Additionally, as in the case of the product, the object $+$ is only notation for the arrow construction and does not refer to any addition operation on the objects themselves.

Explicitly, a pullback of the above diagram is an object d along with arrows $f' : d \to b$, $g' : d \to a$ such that $g \circ f' = f \circ g'$, where if e is an object with arrows $i : e \to b$, $j : e \to a$ that satisfy $g \circ i = f \circ h$, there exists a unique arrow $k : e \dashrightarrow d$ such that $f' \circ k = i$ and $g' \circ k = h$.

$$\begin{tikzcd} e \arrow[dr, dashed, "k"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "h"'] & & \\ & d \arrow[r, "f'"] \arrow[d, "g'"] & b \arrow[d, "g"] \\ & a \arrow[r, "f"] & c \end{tikzcd} \tag{2.24}$$

The (d, b, a, c) square in the above definition is known as a *pullback square*, and we say that f' (or g') is the result of pulling f back along g (or g back along f).

Remark 2.33. One may have noticed that a limit over the diagram $a \xrightarrow{f} c \xleftarrow{g} b$ would contain an arrow $e : d \to c$ in addition to arrows $f' : d \to b$ and $g' : d \to a$. However since arrows in cones and limits commute (Definitions 2.21 and 2.22), we know that $e = f' \circ g = g' \circ f$, so we omit it when defining pullbacks. This composition arrow e will become important later, however, in the discussion of meets and joins in Chapter 4.

The following lemma will prove especially useful when proving properties of structures defined in terms of pullbacks — of which we will see many.

Lemma 2.34 (Pullback Lemma (PBL))**.** *Suppose that the diagram*

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "h"] & y \arrow[r, "v"] \arrow[d, "i"] & z \arrow[d, "j"] \\ a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd} \tag{2.25}$$

commutes.

1. *If the outer rectangle and the right square are pullbacks, then so is the left square. We say 'outer rectangle' to refer to the (x, z, a, c) commutative square, with arrows $(v \circ u, g \circ f, h, j)$.*

2. *If the left and right squares are pullbacks, then so is the outer rectangle.*

Proof. We first consider the case where the outer rectangle and right square are pullbacks. Suppose that there is an object Ξ with arrows $\xi_1 : \Xi \to y$ and $\xi_2 : \Xi \to a$ such that $i \circ \xi_1 = f \circ \xi_2$. We then have the following commutative diagram.

$$\begin{tikzcd}
\Xi \arrow[rrd, bend left, "\xi_1"] \arrow[rdd, bend right, "\xi_2"'] & & & \\
& x \arrow[r, "u"] \arrow[d, "h"] & y \arrow[r, "v"] \arrow[d, "i"] & z \arrow[d, "j"] \\
& a \arrow[r, "f"] & b \arrow[r, "g"] & c
\end{tikzcd} \tag{2.26}$$

To show that the left square is a pullback we have to find a unique arrow from Ξ to x that commutes in Diagram (2.26). We can deduce that the composition arrow $v \circ \xi_1 : \Xi \to z$ exists and commutes in (2.26). Then $j \circ v \circ \xi_1 = g \circ f \circ \xi_2$, and the outer rectangle is a pullback square so there exists a unique arrow $k : \Xi \dashrightarrow x$ such that the following diagram commutes.

$$\begin{tikzcd}
\Xi \arrow[rrrd, bend left, "v \circ \xi_1"] \arrow[rd, dashed, "k"] \arrow[rdd, bend right, "\xi_2"'] & & & \\
& x \arrow[r, "u"] \arrow[d, "h"] & y \arrow[r, "v"] \arrow[d, "i"] & z \arrow[d, "j"] \\
& a \arrow[r, "f"] & b \arrow[r, "g"] & c
\end{tikzcd} \tag{2.27}$$

For k to commute in (2.26), we need that $u \circ k = \xi_1$. Since the right square is a pullback and $j \circ v \circ \xi_1 = g \circ f \circ \xi_2$, there exists a unique arrow from Ξ to y that commutes in (2.27). But we have seen that ξ_1 satisfies this role — so it must be this unique arrow, and hence $u \circ k = \xi_1$. Thus the left square is a pullback.

We now consider the case of the left and right squares being pullbacks. As before, suppose that there is an object Ξ with arrows $\xi_1 : \Xi \to z$ and $\xi_2 : \Xi \to a$ such that $j \circ \xi_1 = g \circ f \circ \xi_2$.

This gives the following commutative diagram.

$$
\begin{tikzcd}
\Xi \arrow[rrrd, bend left, "\xi_1"] \arrow[rdd, bend right, "\xi_2"'] & & & \\
& x \arrow[r, "u"] \arrow[d, "h"] & y \arrow[r, "v"] \arrow[d, "i"] & z \arrow[d, "j"] \\
& a \arrow[r, "f"] & b \arrow[r, "g"] & c
\end{tikzcd}
\tag{2.28}
$$

To show that the outer rectangle is a pullback we have to find a unique arrow from Ξ to x that commutes in Diagram (2.28). Because the right square is a pullback there exists a unique arrow $k' : \Xi \dashrightarrow z$ that commutes in (2.28). But the left square is a pullback too, so this arrow implies the existence of a unique arrow $k : \Xi \dashrightarrow x$ making the following diagram commute.

$$
\begin{tikzcd}
\Xi \arrow[rrrd, bend left, "\xi_1"] \arrow[rdd, bend right, "\xi_2"'] \arrow[rd, dashed, "k"] \arrow[rrd, dashed, "k'"] & & & \\
& x \arrow[r, "u"] \arrow[d, "h"] & y \arrow[r, "v"] \arrow[d, "i"] & z \arrow[d, "j"] \\
& a \arrow[r, "f"] & b \arrow[r, "g"] & c
\end{tikzcd}
\tag{2.29}
$$

So the commutativity of (2.28) implies the existence of the unique commuting k. Thus the outer rectangle is a pullback.[14] □

A simple fact about pullbacks is the following proposition, which we will prove now while still in the headspace of pullbacks. [15] It will be used later in Chapter 4.

Proposition 2.35. *Suppose that*

$$
\begin{tikzcd}
a \arrow[r, tail, "f'"] \arrow[d, "g'"'] & b \arrow[d, "g"] \\
c \arrow[r, tail, "f"] & d
\end{tikzcd}
\tag{2.30}
$$

is a pullback square. If f is monic, then f' is also monic.

[14] Note that in the Pullback Lemma, it is *not* the case that the outer rectangle and left square being pullbacks implies that the right square is one too.

[15] This is given as an exercise in [Gol84, p. 68].

Proof. Assume there are arrows $h, j : e \to a$ such that $f' \circ h = f' \circ j$. We aim to show that $h = j$. By the assumption, $g \circ f' \circ h = g \circ f' \circ j$. The pullback square commutes, which means that $g \circ f' = f \circ g'$, hence $f \circ g' \circ h = f \circ g' \circ j$. Since f is monic, we can cancel on the left to obtain that $g' \circ h = g' \circ j$. This is shown in the commutative diagram below.

$$\begin{tikzcd}
e \arrow[dr, bend left, "j"] \arrow[dr, bend right, "h"'] \arrow[drr, bend left, "f' \circ h = f' \circ j"] \arrow[ddr, bend right, "g' \circ h = g' \circ j"'] & & \\
& a \arrow[r, "f'"] \arrow[d, "g'"] & b \arrow[d, "g"] \\
& c \arrow[r, tail, "f"] & d
\end{tikzcd} \tag{2.31}$$

Now we use the pullback property to conclude that the arrow from e to a must be unique, and thus $h = j$. □

2.7 Subobjects

2.7.1 Definition

At the beginning of this chapter, we claimed that categories tend to formalise the notions of 'universes of discourse' in mathematics. We have seen how various categorial constructions serve to generalise familiar objects, like Cartesian products or the empty set. What about the relation between objects that is *inclusion* — i.e., the idea of being a subset or a *subobject?*

Consider a set Y and a (proper) subset $X \subseteq Y$. Because each $x \in X$ has the property of being an element of Y too, we can associate with X its injective *inclusion map* $\iota : X \hookrightarrow Y$, where $\iota(x) = x$. Here, x is treated as a member of X while $\iota(x)$ is treated as a member of Y. Because ι is injective we know it will be monic in **Set**, so we may be tempted to define a subobject of an object d to be some object a with a monic arrow $f : a \rightarrowtail g$. This definition would turn out to serve us well, however one more observation encourages a modification.

Given our subset $X \subset Y$ there might exist some set X', where X' is a relabelling of X through a bijection $b : X' \to X$. Then the composition function $\iota \circ b : X' \to Y$ would remain injective and so be monic in **Set**, meaning X' along with $\iota \circ b$ would be a categorial subobject of Y under the above definition. But it seems that X' and X are not really different subsets as one is just a relabelling. Bijections are iso arrows in **Set**, so it then makes sense to modify the definition of a subobject to account for this isomorphism of domains.[16] By doing this, we collapse the many different monic arrows f' that 'play the same role' as f into one distinct object — an *equivalence class*.

An *equivalence relation* is a binary relation $\sim$ that is

(i) *reflexive*: $x \sim x$;

(ii) *transitive*: if $x \sim y$ and $y \sim z$ then $x \sim z$;

(iii) *symmetric*: if $x \sim y$ then $y \sim x$.

Given an element x, the equivalence class $\overline{x}$ is the collection of all elements related to x by $\sim$.

Definition 2.36 (Subobject)**.** Let d be an object in a category $\mathscr{C}$, and let $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ be monic arrows into d. We say that $f \simeq g$ if $a \cong b$, i.e., the domains a and b are isomorphic (Def. 2.11). A *subobject* of d is an equivalence class of all arrows related by $\simeq$, denoted $\overline{f}$ where f is a representative arrow of this collection. We denote the class of all subobjects of d as $\mathrm{Sub}(d)$.

$$\begin{tikzcd} a \arrow[dr, tail, "f"] \arrow[dd, leftrightarrow, "\cong"'] & \\ & d \\ b \arrow[ur, tail, "g"'] & \end{tikzcd} \tag{2.32}$$

It is straightforward to see that $\simeq$ is indeed an equivalence relation (it is a result of $\cong$ being an equivalence relation), and so this notion of subobjects is well-defined. Let $f : a \rightarrowtail d$, $g : b \rightarrowtail d$ and $h : c \rightarrowtail d$ be monic arrows into d.

(i) *Reflexive:* Since the identity arrow on a is an isomorphism, we have that $a \cong a$ so $f \simeq f$.

[16] A further justification for defining subobjects in terms of their equivalence classes will become apparent in Chapter 4, when we begin to explore the partially ordered structure of $\mathrm{Sub}(d)$.

$$
\begin{tikzcd}
a \arrow[dd, leftrightarrow, "\mathrm{id}_a"'] \arrow[dr, tail, "f"] & \\
& d \\
a \arrow[ur, tail, "f"'] &
\end{tikzcd}
\tag{2.33}
$$

(ii) *Transitive:* If $f \simeq g$ and $g \simeq h$ then $a \cong b$ and $b \cong c$. So taking the composition of iso arrows from a to c we obtain an isomorphism $a \simeq c$. Hence $f \simeq h$.

$$
\begin{tikzcd}
a \arrow[d, leftrightarrow, "\cong"'] \arrow[dd, leftrightarrow, bend right=60, "\cong"'] \arrow[dr, tail, "f"] & \\
b \arrow[r, tail, "g"] & d \\
c \arrow[u, leftrightarrow, "\cong"] \arrow[ur, tail, "h"'] &
\end{tikzcd}
\tag{2.34}
$$

(iii) *Symmetric:* If $f \simeq g$ then $a \cong b$. This is the same as $b \cong a$, so $g \simeq f$.

We thus obtain an equivalence relation.

Remark 2.37. We will often refer to 'the subobject f' in place of 'the subobject $\overline{f}$'. While technically incorrect, this poses no direct issues since categorial constructions are defined up to isomorphism of objects, and so anything involving subobjects will remain stable under the equivalence relation $\simeq$. We will, however, resort to $\overline{f}$ when the distinction is required. The important point is that $f \simeq g$ if and only if $\overline{f} = \overline{g}$.

2.7.2 Factorisation

Given a function between sets $f : A \to B$ it is always possible to decompose it into a surjection $e : A \to f(A)$ and an injection $m : f(A) \hookrightarrow B$. This is done by first mapping each $x \in A$ to $f(x)$ in $f(A)$, and then injecting this set into the codomain B through the inclusion map.

Under some conditions, this process of 'function factorisation' can be abstracted to categories. Importantly, this means that for any $f : a \to d$ we can find 'the part of f' that is a subobject of d — the monic part of the factorisation given in the following theorem.

Theorem 2.38 (Epi-monic factorisation)**.** *Let $\mathscr{C}$ be a category that admits pullbacks and coequalisers. Further suppose that the pullback of any epic arrow remains epic. Then for any* *arrow $f : a \to b$ there exist arrows e and m such that $f = m \circ e$, where e is epic and m is monic. Moreover, if we also have that $f = v \circ u$ where v is monic, then $m = v \circ k$ for some arrow k. The middle object* $\mathrm{dom}\, m = \mathrm{cod}\, e$ *is called $f(a)$, and we call the construction the epi-monic factorisation of f.*[17]

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, bend right, two heads, "e"'] & & b \\ & f(a) \arrow[ur, bend right, tail, "m"'] & \end{tikzcd} \tag{2.35}$$

The proof for this theorem introduces some interesting concepts and is required for our future discussion of lattice joins in $\mathrm{Sub}(d)$ in Chapter 4, but does not add any substantive understanding to our journey of understanding logic in categories. It is given in Appendix A.

2.7.3 Classification

A construction seen throughout many areas of mathematics is a *characteristic function of a subset*. This is where we characterise a subset $X \subseteq Y$ by mapping the elements of Y that are also in X to 1, and the remaining elements of Y to 0.[18] This mapping is known as the characteristic function of X, $\chi_X : Y \to \{0,1\}$. Given that $\{1\} \subset \{0,1\}$, we can illustrate the connection of how $\{1\}$ is to $\{0,1\}$ as X is to Y through the following pullback square.

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, "!"'] & Y \arrow[d, "\chi_X"] \\ \{1\} \arrow[r, hook] & \{0,1\} \end{tikzcd} \tag{2.36}$$

Abstracting this notion, we arrive at the categorial equivalent construction for characterising subobjects.[19]

Definition 2.39 (Subobject classifier)**.** Let $\mathscr{C}$ be a category with terminal object 1. A *subobject classifier* for $\mathscr{C}$ is an object Ω along with a monic $\boldsymbol{t} : 1 \rightarrowtail \Omega$ such that for any monic

[17] Goldblatt labels the arrows e and m by f^* and $\mathrm{im}\, f$ respectively [Gol84, p. 112].

[18] This assumes, of course, that everything in Y is either in X or not. A simple assumption that all mathematicians learn to do in their education, but nonetheless ironic considering where this study of subobjects will take us.

[19] For a full explanation of how one arrives at this abstraction, see [Gol84, pp. 79–81].

$f : a \rightarrowtail d$ there exists a unique arrow χ_f making

$$\begin{tikzcd} a \arrow[r, tail, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\ 1 \arrow[r, "\boldsymbol{t}"] & \Omega \end{tikzcd} \tag{2.37}$$

a pullback square. The arrow χ_f is called the *characteristic arrow* or *character* of f.

We call the arrow $\boldsymbol{t}$ the 'true' arrow for reasons that will become clear in Chapter 5. Furthermore, thinking intrinsically, the characteristic arrow can be interpreted as mapping the 'a part of d' to the 'true part of Ω'.

An important result that will come in use when discussing the internal structure of $\mathrm{Sub}(d)$ is that there is a one-to-one relationship between subobjects of d and character arrows from d to Ω.[20]

Proposition 2.40. *Suppose we have the monic arrows $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ in a category $\mathscr{C}$ with a subobject classifier. Then*

$$f \simeq g \textit{ if and only if } \chi_f = \chi_g.$$

That is, there is exactly one characteristic arrow for each subobject equivalence class $\bar{f}$.

Proof. First assume that $f \simeq g$. This means there exists an iso arrow k with inverse k^{-1} such that

$$\begin{tikzcd} a \arrow[dr, tail, "f"] \arrow[dd, bend left, "k"] & \\ & d \\ b \arrow[ur, tail, "g"'] \arrow[uu, bend left, "k^{-1}"] & \end{tikzcd} \tag{2.38}$$

commutes. Since f is monic, we can classify it using the subobject classifier. So, onto the above diagram, append the diagram of the characteristic arrow χ_f (in Def. 2.39) to form the following commutative diagram. (Note that the existence and commutativity of $1_b : b \to 1$ is

[20]This result essentially tells us that in an area of mathematics where we can characterise things that look like subsets (i.e., subobjects) by things that look like functions (i.e., arrows), the subobject classifier is a good way of doing it.

guaranteed by the composite arrow $k^{-1} \circ 1_a$ and uniqueness of arrows to terminal objects).

$$
\begin{tikzcd}
b \arrow[dr, bend left, "k^{-1}"] \arrow[drr, bend left=40, "g"] \arrow[ddr, bend right=40, "!"'] & & \\
& a \arrow[ul, bend left, "k"] \arrow[r, tail, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\
& 1 \arrow[r, "t"] & \Omega
\end{tikzcd}
\tag{2.39}
$$

To show that $\chi_f = \chi_g$ we need to establish χ_f as a character for g, i.e., showing that the outer square is a pullback. To this end, suppose there exists an object Ξ with arrows $\xi_1 : \Xi \to d$ and $\xi_2 : \Xi \to 1$ that commutes with the above diagram.

$$
\begin{tikzcd}
\Xi \arrow[dr, dashed, "k \circ m"] \arrow[ddr, dashed, bend right, "m"'] \arrow[ddrr, bend left=40, "\xi_1"] \arrow[dddr, bend right=50, "\xi_2"'] & & \\
& b \arrow[dr, bend left, "g"] \arrow[ddr, bend right, "!"'] & & \\
& & a \arrow[ul, "k"'] \arrow[r, tail, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\
& & 1 \arrow[r, "t"] & \Omega
\end{tikzcd}
\tag{2.40}
$$

Since the $(a, d, 1, \Omega)$ square is a pullback square (by definition of the subobject classifier), there exists a unique arrow $m : \Xi \dashrightarrow a$ where $\xi_1 = f \circ m$ and $\xi_2 = 0_a \circ m$. Therefore, taking the composition arrow $k \circ m : \Xi \to b$ we find a unique arrow (unique because m is unique) that factors the outer square through the $(b, d, 1, \Omega)$ square. Thus the $(b, d, 1, \Omega)$ square is a pullback and $\chi_f = \chi_g$ by uniqueness from Def. 2.39.

Conversely, assume that $\chi_f = \chi_g$. Then the following two squares are pullbacks.

$$
\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f = \chi_g"] \\
1 \arrow[r, "t"] & \Omega
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \arrow[r, tail, "g"] \arrow[d, "!"'] & d \arrow[d, "\chi_f = \chi_g"] \\
1 \arrow[r, "t"] & \Omega
\end{tikzcd}
\tag{2.41}
$$

Recall that from Definition 2.32, these pullbacks are both limits over the diagram

$$
1 \xrightarrow{t} \Omega \xleftarrow{\chi_f = \chi_g} d \ .
$$

But limits over the same diagram are unique up to isomorphism, as shown in Proposition 2.23. Therefore there must exist an iso arrow $k: a \to b$, i.e., $a \cong b$ so $f \simeq g$. $\square$

This discussion of subobject classifiers concludes our brief introduction to some constructions in category theory. We now have (almost) enough to define the type of category that will turn out to be central to later discussions of logic: a topos.

2.8 Toposes

Before we define a topos, we need to briefly address a missing piece of the upcoming definition. One of the conditions for a category to be a topos is that is has *exponentiation*. For the sake of brevity and relevance, this dissertation will sidestep disscussing such a concept. An understanding of exponentiation will not be required in our discussion of logic in a topos — it is simply another construction, like limits, one may demand to see in a category.

With that disclaimer said, we now turn to defining the type of category that we will focus on from now on.

Definition 2.41 (Topos)**.** An *elementary topos* is a category that satisfies the following properties:

(i) limits exist over all finite diagrams;

(ii) co-limits exist under all finite diagrams;

(iii) there exists a subobject classifier Ω;

(iv) the category has exponentiation.

We usually denote a topos by $\mathscr{E}$ (a cursive E).[21]

[21] There are various equivalent definitions for toposes. Recall from §2.5 (Limits) that categories with property (i) are *finitely complete*, while categories with (ii) are *finitely co-complete*. In fact, a category is finitely complete if and only if it has a terminal object and pullbacks for each arrow pair with common codomain, and finitely co-complete if it has an initial object and pushouts for each arrow pair with common domain [HS07]. So an equivalent list of defining properties for a topos is

(i′) there exists a terminal object and pullbacks;

(ii′) there exists an initial object and pushouts;

(iii′) there exists a subobject classifier Ω;

(iv′) the category has exponentiation.

Furthermore, categories with properties (i) and (iii) are known as being *Cartesian closed*. It has then been shown that Cartesian closed categories with a subobject classifier are necessarily finitely co-complete [KLM75]. Therefore another equivalent definition of a topos is a category where

(i″) it is Cartesian closed;

(ii″) there exists a subobject classifier Ω.

Remark 2.42. The word *elementary* in the definition identifies a specific type of topos, but this distinction is unimportant for our discussion so we will just refer to these constructions as toposes.[22]

Example 2.43. The following categories are toposes.[23]

(a) **Set**, where the subobject classifier is given by the previously discussed two-valued set $\{0,1\}$.

(b) $\mathbf{Set}^{\rightarrow}$, whose subobject classifier is rather more elaborate than the one in **Set**. It takes the form of a function between a three-valued set and $\{0,1\}$, accounting for how set functions act on subsets. This is explored in more detail in Example 5 of [Gol84, pp. 86–88]. The more exotic nature of Ω in $\mathbf{Set}^{\rightarrow}$ leads to notable difference to **Set** when considering the 'topos logic', which will be discussed in Chapter 5.

An important feature of toposes is the following. A proof can be found in [LS86, pp. 157–158], Lemma 6.5.

Theorem 2.44. *The pullback of an epic arrow in a topos remains epic.*

Using this, we can form epi-monic factorisations. These will allow for a description of lattice joins in $\mathrm{Sub}(d)$ — but this will be covered in Chapter 4

Corollary 2.45. *Any arrow f in a topos can be factored as $f = e \circ m$, where e is epic and m is monic.*

Proof. Toposes admit all finite limits and colimits, so pullbacks and coequalisers can always be constructed. Then, using the above theorem, we can apply Theorem 2.38 to obtain the result. □

[22] Another type of topos is a *Grothendieck topos*. See [Joh77] for more on the distinction.

[23] Convinced by Johnstone's 'thermoi' argument, we use the 'toposes' instead of 'topoi' as the plural. "I do so because (in its mathematical sense) the word topos is not a direct derivative of its Greek root, but rather a back-formation from topology. I have nothing further to say on the matter, except to ask those toposophers who persist in talking about topoi whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in thermoi." [Joh77, pp. 12–13].

Now that we know the definition of a topos, we move to examine another area of mathematics: order theory. The results and algebraic constructions we will encounter turn out to act as connections between toposes and logic, by virtue of the structure found in $\mathrm{Sub}(d)$ — which will be explored in Chapter 4.

Chapter 3

An Introduction to Order

In Chapter 2 we introduced the basic ideas needed to understand the inner workings of categories — structures like products, limits, and pullbacks. We then defined a special type of category: a topos. We motivated category theory as something that looks at the 'big picture' of an area of mathematical discourse. In Chapter 4, we will forgo such a wide viewpoint and instead focus on an individual object in a category and its collection of subobjects. We will see that under some simple criteria requiring existence of initial and terminal objects, a rich structure presents itself in this collection.

However, before we can begin discussing this structure we first need to establish some preliminaries of *order theory*, namely the notions of *posets* and *lattices*. A lattice, like a group or a topological space, is simply a set endowed with some special relations. One can think of a lattice as a network of points with a notion of 'up' and 'down', and a way to connect distant points together. We will first define posets, which constitute the foundation of lattices. Then we will introduce various properties one may demand to see in a lattice.

3.1 Posets

A poset is a set with some notion of *less than or equal to* and *greater than or equal to*.

Definition 3.1 (Poset)**.** A *partially ordered set* (or *poset*) is a set P equipped with a binary relation $\leq$ that obeys the following properties for all x, y, z in P:

(i) *reflexivity*: $x \leq x$;

(ii) *transitivity*: if $x \leq y$ and $y \leq z$ then $x \leq z$;

Compare this Chapter with Chapters 1 and 2 in Birkhoff [Bir67], and Chapter 1 in Rasiowa and Sikorski, [RS68]. Specifically, Chapter 1, sections §1, §2, §4, §5, §6, §9, §10, and Chapter 2, sections §10, §11 in Birkhoff; and Chapter 1, sections §5, §6, §9, §10, §11, §12 in Rasiowa and Sikorski.

(iii) *antisymmetry*: if $x \leq y$ and $y \leq x$ then $x = y$.

The relation $\leq$ is known as a partial order on P. We sometimes write a poset as the tuple $P = (P, \leq)$.

Given a poset $(P, \leq)$, a *bottom* is an element $\perp \in P$ such that $\perp \leq x$ for all $x \in P$, while a *top* is an element $\top \in P$ such that $x \leq \top$ for all $x \in P$.[24] Using the property of antisymmetry, it is clear that if such elements exist they must be unique. For example, $\top' \leq \top$ and $\top \leq \top'$ implies that $\top' = \top$.

A canonical example of a poset is the set of all subsets of some set X (also known as the power set of X, $\mathscr{P}(X)$), with the set inclusion relation $\subseteq$ being the partial order. This poset $(\mathscr{P}(X), \subseteq)$ has the top element X and the bottom element $\emptyset$.

Definition 3.2. Given a set X of elements in P, we have the following definitions.

1. $y \in X$ is the *minimum* of X if $y \leq x$ for all $x \in X$, written $y = \min X$. Note that if y' and y are both minimums, then $y \leq y'$ and $y' \leq y$ so $y = y'$ by antisymmetry, i.e., if it exists, then the minimum of X is unique.
2. $a \in P$ is a *lower bound* for X if $a \leq x$ for all $x \in X$. Notice that a is also the minimum of X if $a \in X$.
3. a is the *infimum* (greatest lower bound) if $a' \leq a$ for all other lower bounds of a' of X, written $a = \inf X$. If $\inf X$ exists it is unique by antisymmetry.
4. $z \in X$ is the *maximum* of X if $x \leq z$ for all $x \in X$, written $z = \max X$. If $\max X$ exists it is unique by antisymmetry.
5. b is an *upper bound* for X if $x \leq b$ for all $x \in X$. Notice that, similar to the lower bound definition, b is also the maximum of X if $b \in X$.
6. b is the *supremum* (least upper bound) of X if $b \leq b'$ for all other upper bounds b', written $b = \sup X$. If $\sup X$ exists it is unique by antisymmetry.

3.2 Lattices

Given a poset with infimums and supremums, we can define a lattice.

[24] The bottom (respectively top) element in a poset is also referred to as the least (greatest) element, minimum (maximum) element, or zero (unit) element.

Definition 3.3 (Lattice)**.** Let $(L, \leq)$ be a poset where for any two elements $x, y \in L$, $\inf\{x, y\}$ and $\sup\{x, y\}$ exist. Denote them as

$$x \frown y := \inf\{x, y\}, \quad x \smile y := \sup\{x, y\}.$$

These operations $\frown$ and $\smile$ are called *meet* and *join* respectively.[25] The structure $(L, \leq, \frown, \smile)$ is then called a *lattice* (which we sometimes simply write as L). The lattice is *complete* if it admits supremums and infimums for every subset X of L.

The operations of meet and join give rise to an interesting algebraic structure, as shown by the next lemma from [Bir67, p. 8].

Lemma 3.4 ([Bir67])**.** *Let $(L, \leq, \frown, \smile)$ be a lattice. The meet and join operations satisfy the following laws:*

(i) Idempotence: $x \frown x = x, \quad x \smile x = x$

(ii) Commutativity: $x \frown y = y \frown x, \quad x \smile y = y \smile x$

(iii) Associativity: $x \frown (y \frown z) = (x \frown y) \frown z, \quad x \smile (y \smile z) = (x \smile y) \smile z$

(iv) Absorption: $x \frown (x \smile y) = x \smile (x \frown y) = x$

Furthermore, $x \leq y$ is equivalent to each of the conditions

$$x \frown y = x \text{ and } x \smile y = y.$$

Proof. These laws follow directly from the definitions of $\frown$ and $\smile$. For example, $x \frown x = \inf\{x\}$, and $x \leq x$ so $\inf\{x\} = x$ by the definition of the infimum. The proof for the remaining laws can be found in [Bir67]. □

We now introduce some further requirements that one can demand of a lattice.

3.2.1 Distributivity

Definition 3.5. A *distributive* lattice $(L, \leq, \frown, \smile)$ is one that obeys the following two laws for all $x, y, z \in L$:

[25] The standard notation for meet and join is $\wedge$ and $\vee$ respectively — different to what we have introduced here. We have opted for the same shapes but with flatter corners to differentiate them from the logical operations of conjunction and disjunction, which will be introduced in Chapter 5.

(i) $x \frown (y \smile z) = (x \frown y) \smile (x \frown z)$

(ii) $x \smile (y \frown z) = (x \smile y) \frown (x \smile z)$

These two requirements are in fact equivalent, as shown in [Bir67, p. 11].

Proposition 3.6 ([Bir67])**.** *A lattice L obeys law (i) in Definition 3.5 for all $x, y, z \in L$ if and only if it obeys law (ii) for all $x, y, z \in L$.*

Proof. We prove that (i) implies (ii). Let $x, y, z \in L$ and assume property (i) in Def. 3.5. Then

$$
\begin{aligned}
(x \smile y) \frown (x \smile z) &= \big((x \smile y) \frown x\big) \smile \big((x \smile y) \frown z\big) && \text{(assumption of (i))} \\
&= x \smile \big(z \frown (x \smile y)\big) && \text{(absorption and commutativity)} \\
&= x \smile \big((z \frown x) \smile (z \frown y)\big) && \text{(assumption of (i))} \\
&= \big(x \smile (z \frown x)\big) \smile (z \frown y) && \text{(associativity)} \\
&= x \smile (z \frown y) && \text{(absorption)}
\end{aligned}
$$

The proof for the converse is of the same structure but with $\frown$ and $\smile$ operations interchanged. □

3.2.2 Boundedness

Given a lattice structure $(L, \leq, \frown, \smile)$, if a bottom element $\bot$ exists in the poset $(L, \leq)$ then we say that it is a bottom element for the lattice too. Similarly, if a top element $\top$ exists for the poset then it is a top element for the lattice.

Definition 3.7. If the poset $(L, \leq)$ constituting a lattice $(L, \leq, \frown, \smile)$ has both a bottom and a top element, then the lattice is *bounded.* Such a lattice may be written as the tuple $(L, \leq, \frown, \smile, \top, \bot)$.

Top and bottom elements interact with the meet and join operators in the following way.

Corollary 3.8. *If a lattice L has a bottom $\bot$ then for all $x \in L$,*

$$x \frown \bot = \bot, \quad x \smile \bot = x.$$

If a lattice L has a top $\top$ then for all $x \in L$,

$$x \frown \top = x, \quad x \smile \top = \top.$$

Proof. This follows from the final result in Lemma 3.4, noting that $\bot \le x \le \top$ for all x. $\square$

3.2.3 Complementation

Complementation in lattices gives a notion of an 'opposite element'.

Definition 3.9 (Varieties of complementation in a lattice)**.** Let $(L, \le, \frown, \smile)$ be a lattice and let $x, y, z \in L$. We then have the following definitions.

1. Suppose that L is a bounded lattice. Then y is a *complement* of x if $x \frown y = \bot$ and $x \smile y = \top$. If a complement of x exists we denote it $x^\star$. If every element has a complement then we say that the lattice is *complemented.*
2. Next suppose that L has a bottom $\bot$ but not necessarily a top. Then y is a *pseudo-complement* of x if it the maximum of all elements y' such that $x \frown y' = \bot$. If a pseudo-complement of x exists we denote it $-x$, and we can write it as
$$-x = \max\{y \in L \mid x \frown y = \bot\}.$$
If every element has a pseudo-complement then the lattice is *pseudo-complemented.*
3. Finally, making no assumptions on the existence of top or bottom elements in L, we say that z is a *relative pseudo-complement of x with respect to y* if it is the greatest element such that $(x \frown z) \le y$. If a pseudo-complement of x relative to y exists we denote it $x \sqsupset y$, and we can write
$$x \sqsupset y = \max\{z \in L \mid x \frown z \le y\}.^{26}$$
If for every pair x, y of elements the relative pseudo-complement $x \sqsupset y$ exists then the lattice is called *Brouwerian.*[27]

There are some interesting immediate facts that arise from the above definition. First, notice from the above Corollary 3.8 that $\bot = \top^\star$ and $\top = \bot^\star$ in a bounded lattice. Secondly, as we define $-x$ and $x \sqsupset y$ to be maximum elements of a set, they must be unique if they

[26] Some mathematicians use the notation $y : x$ [Bir67, p. 45] or $x \to y$ [RS68, p. 54] in place of $x \sqsupset y$.

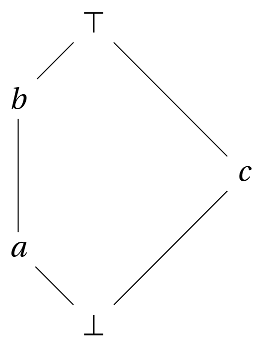
[27] These lattices were originally simply called pseudo-complemented lattices (like in [RS68]), however they are now more widely known as Brouwerian (as in [Bir67]).

exist. And thirdly, while a pseudo-complemented lattice is only assumed to be bounded below, it turns out to be necessarily bounded above as well.

Proposition 3.10. *A pseudo-complemented lattice L has a top element.*

Proof. We know that L has a bottom element $\bot$. Consider the pseudo-complement of $\bot$, $-\bot = \max\{y \in L \mid x \frown \bot = \bot\}$. But all $y \in L$ meet this criterion (Cor. 3.8) so $-\bot = \max L$, which is simply the definition of a top element $\top$. □

One may wonder what the difference between complements and pseudo-complements is, since both complemented and pseudo-complemented lattices are bounded (complemented by definition, pseudo-complemented by the above Proposition 3.10) and have similar definitions. A simple example that illustrates the difference is the pentagon lattice N_5. Diagram (3.1) represents the lattice N_5, where, for example, the line connecting the elements $\bot$ and c means that $\bot \leq c$ since $\bot$ is positioned lower than c. Given x, y on branches in the diagram, where the two branches first intersect below we have the meet $x \frown y$ and where they first intersect above we have the join $x \smile y$. E.g., $a \frown c$ is the first intersection of branches below a and c, which is $\bot$.



(3.1)

The lattice N_5 is complemented. This can be seen by looking at each element in turn:

$\top, \bot$: these are always the complement of each other, as we saw above;

a: since $a \frown c = \bot$ and $a \smile c = \top$ we have $c = a^\star$;

b: similarly, $b \frown c = \bot$ and $b \smile c = \top$ means that $c = b^\star$;

c: from the above two cases, $a = c^\star$ and $b = c^\star$.

So all elements have complements, and in the case of c it has two. All elements also have pseudo-complements, which for a and b remain the same as their complements: $c = -a$

and $c = -b$. For c, its pseudo-complement is $b = -c$ — the greatest of its two complements. Thus, from this example we see how the account of pseudo-complementation and (standard) complementation can differ. It also reveals another fact: that double complementation (and double pseudo-complementation) does not necessarily cancel, that is, it is not always the case that $(x^\star)^\star = x$ or $-(-x) = x$. When we manoeuvre from algebras to logic, double complementation will manifest as double negation: "it is not the case that it is not raining". In only some cases will we be able to deduce from this that is it indeed raining — a result of how sometimes double complementation does not cancel.

Returning from this digression, we introduce a different characterisation of relative pseudo-complementation.

Proposition 3.11. *The element $x \sqsupset y$ is the relative pseudo-complement of x with respect to y if and only if the following statement holds for all elements a in the lattice:*

$$a \leq (x \sqsupset y) \text{ if and only if } (x \frown a) \leq y.$$

Proof. This is simply a result of expanding the definition of a 'greatest element' in the definition of a relative pseudo-complement. I.e., $(x \sqsupset y)$ is the greatest of all elements a in the lattice that satisfy $(x \frown a) \leq y$. □

If a bottom element is added to a Brouwerian lattice then pseudo-complementation is gained, as the pseudo-complement of x is just its relative pseudo-complement with respect to $\perp$. This can be seen by comparing the definitions of $-x$ and $x \sqsupset \perp$ (noting that $a \leq \perp$ is equivalent to $a = \perp$ since it is the bottom). This fact and the above Proposition 3.11 give the following corollary.

Corollary 3.12. *The element $-x$ is the pseudo-complement of x if and only if the following statement holds for all elements a in the lattice:*

$$a \leq (-x) \text{ if and only if } (x \frown a) = \perp.$$

3.3 Boolean and Heyting algebras

We have taken an excursion into lattice theory for the precise reason of introducing the following two types of lattices: Boolean and Heyting algebras. They have deep established connections to logic which we will utilise in Chapter 5, when we find a surprising connection between the subobject structure of the terminal object 1 in a topos, and the structure of the entire topos itself. But for now, let us define these lattices and leave the logic until later.

Definition 3.13. Let $(L, \leq, \frown, \smile)$ be a lattice.

(i) L is a *Boolean algebra* if it is distributive, bounded and complemented.

(ii) L is a *Heyting algebra* if it a Brouwerian lattice and has a bottom element.[28]

Remark 3.14. Throughout this dissertation we will assume any algebra being referenced to be *non-degenerate* — containing more than a single element — unless specified otherwise.

It turns out that Heyting algebras are necessarily distributive and bounded. Distributivity follows from the following theorem proved in [Bir67, pp. 45–46] (since Heyting algebras are Brouwerian).

Theorem 3.15 ([Bir67])**.** *If L is a Brouwerian lattice then it is distributive.*

For a top, take the relative pseudo-complement of any element x with respect to itself, $x \sqsupset x$. By considering Definition 3 in 3.9, we have that $x \sqsupset x = \max\{z \in L \mid x \frown z \leq x\}$. But by Lemma 3.4, $x \frown z \leq x$ is equivalent to $x \leq z$. Therefore $x \sqsupset x = \max\{z \in L \mid x \leq z\}$ which defines a top element of L. So if we require that all pseudo-complements exist in L, we must have that for any $x, y \in L$, $x \sqsupset x = y \sqsupset y = \top$ since top elements are unique.

Taking this into account, we then see that a Heyting algebra is like a Boolean algebra but with relative pseudo-complementation instead of complementation. The next proposition results in further connections between Heyting and Boolean algebras, for which a proof can be found in [Gol84, p. 166].

Proposition 3.16. *In a Boolean algebra, for all elements a,*

[28] An lattice is called an algebra typically when it has some form of complementation on it. The two lattices here have some notions of complements (Boolean by definition, Heyting by relative pseudo-complementation with respect to $\perp$) so we call them algebras.

$$a \leq (x^\star \smile y) \textit{ if and only if } (x \frown a) \leq y.$$

This results in the following corollary, letting us think of a Heyting algebra as a generalised Boolean algebra.

Corollary 3.17. *In a Boolean algebra L we have the following:*

(i) $x \sqsupset y = x^\star \smile y$*;*

(ii) the lattice is Brouwerian;

(iii) L is a Heyting algebra

(iv) $x^\star = -x$*;*

(v) complements are unique.

Proof. By Proposition 3.16, for all $a \in L$, $a \leq (x^\star \smile y)$ if and only if $(x \frown a) \leq y$. Then Proposition 3.11 gives that $x \sqsupset y = x^\star \smile y$ (i). Therefore a Boolean algebra admits all relative pseudo-complements and so is Brouwerian (ii), and has a bottom element so is Heyting (iii). We saw on page 42 that in a Brouwerian lattice with a bottom element it is possible to write pseudo-complements as $-x = x \sqsupset \bot$. So using (i), $-x = x \sqsupset \bot = x^\star \smile \bot = x^\star$, giving (iv). Finally, since pseudo-complements are unique we therefore have that the complements $x^\star$ are unique (v). □

If all Boolean algebras are Heyting algebra, when are Heyting algebras Boolean? The following proposition from [Joh82, p. 8] clarifies a precise answer.

Proposition 3.18. *Let* $(L, \frown, \smile, \sqsupset, \bot)$ *be a Heyting algebra. Then the following three conditions are equivalent.*

(i) L is a Boolean algebra,

(ii) $x \smile -x = \top$ *for all* $x \in L$*,*

(iii) $-(-x) = x$ *for all* $x \in L$*,*

where $-x$ *is defined as* $x \sqsupset \bot$*.*

Partial proof. If (i) holds then the pseudo-complementation given by $-$ is equivalent to complementation by the above Corollary 3.17. Thus by the definition of complementation (ii) holds. Conversely, if (ii) holds then $-$ is a complementation operator (as we know that $x \frown -x = \bot$ already), which means the algebra is Boolean (i). The proof for (i) $\implies$ (iii) and

(iii) $\implies$ (ii) is given in [Joh82, p. 9]. These implications establish the equivalence of all three statements. $\square$

We will see in the next chapter how these algebras manifest in a previously met structure: $\mathrm{Sub}(d)$.

Chapter 4

The Structure in Sub(d)

Having introduced some order theory concepts, we now turn to applying them to Sub(d) — the collection of subobjects of an object d in a category. We will first see that Sub(d) forms a poset when endowed with a rather natural ordering relation. We will then walk through the development of finding meet, join and relative pseudo-complement operators. The structure that emerges will turn out to have deep connections to logic, which we will discuss in Chapter 5.

4.1 Subobjects form a poset

First, we will define an ordering relation on Sub(d). Recall that when discussing universal constructions, we said that 'the most universal X' is the one through which other things that 'play the role of X' factor. Continuing this idea, we can say that a subobject f is 'lesser than' the subobject g if it factors through it. However since subobjects are not just arrows but equivalence classes of arrows, we need to amend this definition slightly.

Definition 4.1. Let d be an object in a category $\mathscr{C}$ and let $f : a \rightarrowtail d, g : b \rightarrowtail d$ be representatives of the subobjects $\overline{f}, \overline{g}$ (i.e., a is isomorphic to the domain of any other monic in $\overline{f}$, and similarly for g). Then we say that $\overline{f} \leq \overline{g}$ when there exists an arrow $h : b \to a$ in $\mathscr{C}$ such that $f = g \circ h$. That is, the diagram

$$\begin{tikzcd} a \arrow[dr, tail, "f"] \arrow[dd, "h"'] & \\ & d \\ b \arrow[ur, tail, "g"'] & \end{tikzcd} \tag{4.1}$$

commutes.

Key source for Chapter 4: [Gol84, pp. 146–166].

Importantly, this definition makes sense as an extension from representatives to equivalence classes.

Proposition 4.2. *Definition 4.1 is well-defined: the statement $\overline{f} \le \overline{g}$ does not depend on the choice of representatives. That is, if $f, f' \in \overline{f}$ and $g, g' \in \overline{g}$ then there exists an h such that $f = g \circ h$ iff there exists an h' such that $f' = g' \circ h'$.*

Proof. Let $f, f' \in \overline{f}$ and $g, g' \in \overline{g}$ where f and g are defined as in Definition 4.1, and $f' : a' \rightarrowtail d$, $g' : b' \rightarrowtail d$. First suppose there exists an arrow $h : b \to a$ such that $f = g \circ h$. We know that f and f' have isomorphic domains, as do g and g', so we can form the commutative diagram

$$\begin{tikzcd}
 & a' \arrow[dl, "v"'] \arrow[dd, tail, "f'"] \\
a \arrow[dr, tail, "f"] \arrow[dd, "h"] & \\
 & d \\
b \arrow[ur, tail, "g"] \arrow[dr, "w"'] & \\
 & b' \arrow[uu, tail, "g'"']
\end{tikzcd} \tag{4.2}$$

where v and w are isomorphisms. So by taking $h' = w \circ h \circ v$ we have that $f' = g' \circ h'$ as required. The proof of the converse is the same but with f, g and f', g' swapped. □

Now we reach the first substantive lemma for this section. The ordering relation we found on $\mathrm{Sub}(d)$ is specifically a partial ordering, bounded above and below.

Lemma 4.3. *Let d be an object in a topos $\mathscr{E}$. Then $\mathrm{Sub}(d)$ equipped with $\le$ forms a bounded poset with top id_d and bottom 0_d.*

Proof. There are five aspects to this: showing that $\le$ is (a) reflexive, (b) transitive, (c) antisymmetric, and that (d) id_d is a top element and (e) 0_d a bottom. We progress through these properties one-by-one. Let $\overline{f}, \overline{g}, \overline{h}$ be subobjects of d with representatives $f : a \rightarrowtail d$, $g : b \rightarrowtail d, h : c \rightarrowtail d$.

(a) *Reflexivity.* Clearly we can write f as $f = f \circ \mathrm{id}_a$, so $\overline{f} \leq \overline{f}$.

$$\begin{tikzcd}
a \arrow[dr, tail, "f"] \arrow[dd, "\mathrm{id}_a"'] & \\
& d \\
a \arrow[ur, tail, "f"'] &
\end{tikzcd} \tag{4.3}$$

(b) *Transitivity.* Suppose $\overline{f} \leq \overline{g}$ and $\overline{g} \leq \overline{h}$. Then there exist arrows v, w such that $f = g \circ v$ and $g = h \circ w$. Combining these gives that $f = (h \circ w) \circ v = h \circ (w \circ v)$, thus $\overline{f} \leq \overline{h}$.

$$\begin{tikzcd}
a \arrow[dr, tail, "f"] \arrow[dd, "v"] & \\
& d \\
b \arrow[ur, tail, "g"'] \arrow[dr, "w"'] & \\
& c \arrow[uu, tail, "h"']
\end{tikzcd} \tag{4.4}$$

(c) *Antisymmetry.* Suppose $\overline{f} \leq \overline{g}$ and $\overline{g} \leq \overline{f}$. Then there exist arrows v, w such that $f = g \circ v$ and $g = f \circ w$. Combining these gives that

$$g \circ \mathrm{id}_b = g = (g \circ v) \circ w = g \circ (v \circ w).$$

Since g is monic we can cancel on the left, giving that $\mathrm{id}_b = v \circ w$. A similar process with f leads to $\mathrm{id}_a = w \circ v$. This is illustrated in the following commutative diagram.

$$\begin{tikzcd}
& b \arrow[dl, "w"'] \arrow[dd, tail, "g"] \arrow[ddl, bend right=60, "\mathrm{id}_b"'] \\
a \arrow[dr, tail, "f"] \arrow[dd, "v"] \arrow[ddd, bend right=70, "\mathrm{id}_a"'] & \\
& d \\
b \arrow[ur, tail, "g"'] \arrow[dr, "w"'] & \\
& a \arrow[uu, tail, "f"']
\end{tikzcd} \tag{4.5}$$

Hence, by definition, v and w are iso arrows with each other as inverses (Def. 2.11).

Thus the objects a and b are isomorphic and f and g are representatives of the same subobjects. That is, $\overline{f} = \overline{g}$. From these three results ((a)–(c)) we conclude that $(\mathrm{Sub}(d), \leq)$ is a poset (Def. 3.1).

(d) By the definition of the identity arrow on d we have that $f = \mathrm{id}_d \circ f$, so $f \leq \mathrm{id}_d$. Hence id_a is the top element of the poset.

$$\begin{tikzcd} a \arrow[dr, tail, "f"] \arrow[dd, "f"'] & \\ & d \\ d \arrow[ur, tail, "\mathrm{id}_d"'] & \end{tikzcd} \tag{4.6}$$

(e) Since 0 is initial there exist unique arrows $0_a : 0 \to a$ and $0_d : 0 \to d$. The composition $f \circ 0_a$ is also then unique, and so can only be one arrow: 0_d. Hence $0_d = f \circ 0_a$, so $0_d \leq f$ and 0_d is the bottom element of the poset.

$$\begin{tikzcd} 0 \arrow[dr, tail, "0_d"] \arrow[dd, "0_a"'] & \\ & d \\ a \arrow[ur, tail, "f"'] & \end{tikzcd} \tag{4.7}$$

□

Remark 4.4. Observe that in the above proof, the only categorial requirement that came from $\mathscr{E}$ being a topos was the existence of an initial object. So Lemma 4.3 in fact applies to any category with an initial object. We can generalise to all categories by removing the final condition for a bottom element, that is, in all categories $\mathscr{C}$, for any object d in $\mathscr{C}$, $(\mathrm{Sub}(d), \leq)$ forms a poset with a top element.

From now on, for representative monic arrows f, g, the statement $f \leq g$ shall mean that $\overline{f} \leq \overline{g}$. I.e., that there exists an arrow h in such that $f = g \circ h$. Notice that when we interpret $\leq$ as an ordering on the monic arrows into d (rather than on the set of equivalence classes) we still obtain reflexivity and transitivity, by the same proofs as above. We do not, however, have antisymmetry. Rather, the proof for (c) above shows that $f \leq g$ and $g \leq f$ implies that

$f \simeq g$. This is one of the motivating reasons behind redefining subobjects to be equivalence classes under $\simeq$, as then $f \simeq g$ implies that $\overline{f} = \overline{g}$, so we achieve antisymmetry and thus a partial order on $\mathrm{Sub}(d)$.

As warned about earlier in Remark 2.37, we will tend to refer to 'the subobject f' instead of 'the subobject $\overline{f}$' when there is no risk of confusion.

4.2 The Poset forms a Lattice

We have just seen that for any object d, the structure $(\mathrm{Sub}(d), \leq)$ is a poset. We saw in §3.2 how a poset can be extended to a lattice structure if we can find infimums and supremums for any two elements. So, given subobjects f, g, can we find other subobjects that represent $f \frown g = \inf\{f, g\}$ and $f \smile g = \sup\{f, g\}$?

4.2.1 Infimums

For an object d in a topos $\mathscr{E}$, suppose we have monic arrows $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$. Recall from Definition 3.3 that the *meet*, $l = f \frown g$, is the infimum of $\{f, g\}$. That is, $l : m \rightarrowtail d$ is monic; $l \leq f$ and $l \leq g$; and for any other l' where $l' \leq f$ and $l' \leq g$, we have that $l' \leq l$. Unpacking this definition further, for $l \leq f$ and $l \leq g$ we need the existence of arrows h_1, h_2 such that

$$\begin{tikzcd} & m \arrow[dl, bend right, "h_1"'] \arrow[d, tail, "l"] \arrow[dr, bend left, "h_2"] & \\ a \arrow[r, tail, "f"'] & d & b \arrow[l, tail, "g"] \end{tikzcd} \tag{4.8}$$

commutes (so $l = f \circ h_1$ and $l = g \circ h_2$, i.e., $l \leq f$ and $l \leq g$). For l to be the infimum means that any other l' factors through it. That is, if the arrows l', h'_1 and h'_2 exist as in the following diagram, then there exists an arrow $k : m' \dashrightarrow m$ so $l' = l \circ k$.

$$
\begin{tikzcd}
 & m' \arrow[dl, tail, bend right=60, "h_1'"'] \arrow[d, tail, bend right, "l'"'] \arrow[d, dashed, shift left=2, "k"] \arrow[dr, tail, bend left=60, "h_2'"] & \\
 & m \arrow[dl, tail, bend right, "h_1"'] \arrow[d, tail, "l"] \arrow[dr, tail, bend left, "h_2"] & \\
a \arrow[r, tail, "f"'] & d & b \arrow[l, tail, "g"]
\end{tikzcd}
\tag{4.9}
$$

But recall from Definition 2.22 that this is the *limit* over the diagram $a \xrightarrow{f} c \xleftarrow{g} b$. This type of diagram has a specific name for the limit: the pullback (Def. 2.32). Hence we define $f \frown g$ as the diagonal arrow in the pullback (which was discussed in Remark 2.33).

Definition 4.5 (Meet in $\mathrm{Sub}(d)$)**.** Given monic arrows $f : a \rightarrowtail d$, $g : b \rightarrowtail d$ in a topos $\mathscr{E}$, the *meet* of f and g, $f \frown g : (a \frown b) \longrightarrow d$, is defined as the diagonal arrow in the following pullback square.[29]

$$
\begin{tikzcd}
(a \frown b) \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, tail, "f \frown g"] & b \arrow[d, tail, "g"] \\
a \arrow[r, tail, "f"] & d
\end{tikzcd}
\tag{4.10}
$$

Recall that the arrow f' (or g') is monic if f (or g) is (Prop. 2.35), and that the composition of two monic arrows gives a monic arrow (Prop. 2.7). Hence the arrow $f \frown g$ certainly is a monic arrow into d, so this is well-defined.

Lemma 4.6. *Let $\overline{f}$ and $\overline{g}$ be subobjects of d in a topos $\mathscr{E}$, with representatives $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$. Then the subobject $\overline{f \frown g}$ exists and is the infimum of $\{\overline{f}, \overline{g}\}$ under the partial ordering $\leq$.*[30]

Proof. Since $\mathscr{E}$ is finitely complete (Def. 2.41), the arrow $f \frown g$ defined in Def. 4.5 exists. Furthermore, it is monic so the equivalence class $\overline{f \frown g}$ exists. By definition, $(f \frown g) = f \circ g' = g \circ f'$, so $\overline{f \frown g}$ is a lower bound for $\overline{f}$ and $\overline{g}$.

[29] The label of the object $(a \frown b)$ is just a name that illustrates its existence in terms of arrows from a and b to d. This is *not* denoting some lattice structure on the objects of the topos.

[30] Recall that $\overline{f \frown g}$ is defined to be the collection of all monic arrows into d with domain isomorphic to the domain of $f \frown g$, the arrow defined in Def 4.5.

Now to see that it is the greatest such bound, suppose there exists a subobject $\overline{\xi}$ that also satisfies $\overline{\xi} \leq \overline{f}$ and $\overline{\xi} \leq \overline{g}$. Let $\xi : \Xi \rightarrowtail d$ be a representative of this subobject. Then $\xi = f \circ \xi_1$ and $\xi = g \circ \xi_2$ for some ξ_1, ξ_2. But by the definition of a pullback square (Def. 2.32), there must exist a $k : \Xi \dashrightarrow (a \frown b)$ that ξ_1 and ξ_2 (and hence ξ) factor through. This is illustrated in the following commutative diagram.

$$\begin{tikzcd}
\Xi \arrow[dr, dashed, "k"] \arrow[rrd, bend left, "\xi_2"] \arrow[rrdd, bend left, "\xi"] \arrow[rdd, bend right, "\xi_1"'] & & \\
& (a \frown b) \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, tail, "f \frown g"] & b \arrow[d, tail, "g"] \\
& a \arrow[r, tail, "f"'] & d
\end{tikzcd} \tag{4.11}$$

Therefore, since $\xi = (f \frown g) \circ k$ we have that $\overline{\xi} \leq \overline{f \frown g}$. The lower bound $\overline{\xi}$ was chosen arbitrarily, so this is true for all lower bounds of $\{\overline{f}, \overline{g}\}$. Thus $\overline{f \frown g} = \inf\{\overline{f}, \overline{g}\}$. □

4.2.2 Supremums

We have identified a meet operation on subobjects, but can we identify a join? Given $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$, we want a monic $l = a \smile b : m \to d$ where $l \geq f$ and $l \geq g$, and for any other such upper bound l' we require that $l \leq l'$. For l to be an upper bound for f and g we need arrows h_1, h_2 such that

$$\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[dr, bend right, "h_1"'] & d & b \arrow[l, tail, "g"'] \arrow[dl, bend left, "h_2"] \\
& m \arrow[u, tail, "l"] &
\end{tikzcd} \tag{4.12}$$

commutes. For l to be the least upper bound, we need that if l' is also an upper bound then $l = l' \circ k$ for some arrow k (so $l \leq l'$). That is, if the arrows l', h'_1, h'_2 exist in the following diagram, then so does the arrow k.

$$\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[dr, bend right, "h_1"'] \arrow[ddr, bend right=60, "h_1'"'] & d & b \arrow[l, tail, "g"'] \arrow[dl, bend left, "h_2"] \arrow[ddl, bend left=60, "h_2'"] \\
& m \arrow[u, tail, "l"'] \arrow[d, dashed, "k"] & \\
& m' \arrow[uu, tail, bend left=40, "l'"] &
\end{tikzcd} \tag{4.13}$$

Notice that if the arrows l and l' were to be reversed then this construction would be the colimit under the diagram $a \xrightarrow{f} c \xleftarrow{g} b$. Nonetheless, we certainly are looking for some sort of colimit construction that ensures existence of an arrow k whenever we have another upper bound. Considering Diagram (4.12), one may notice similarities with the diagram given for the definition of a coproduct (Def. 2.30), where the sum object $(a+b)$ is the object m, h_1 and h_2 are the arrows i_1 and i_2, and $[f,g]$ is the supremum arrow l. Furthermore, the colimiting behaviour of the coproduct does indeed ensure the existence of such an arrow k as given in Diagram (4.13). An issue arises, however, with the nature of arrow $[f,g]$ — it is not guaranteed to be monic. This is why we introduced the concept of epi-monic factorisations in Chapter 2: by invoking Theorem 2.38 we can factorise $[f,g]$ into its epic and monic parts, and then take the monic part as $f \smile g$.

Definition 4.7 (Join in Sub(d))**.** Given monic arrows $f : a \rightarrowtail d$, $g : b \rightarrowtail d$ in a topos $\mathscr{E}$, the *join* of f and g, $f \smile g : (a \smile b) \longrightarrow d$, is defined as the monic part in the epi-monic factorisation of the coproduct arrow $[f,g]$.

$$\begin{tikzcd}
a \arrow[r, "i_a"] \arrow[dr, tail, bend right, "f"'] & a+b \arrow[d, "{[f,g]}"] & b \arrow[l, "i_b"'] \arrow[dl, tail, bend left, "g"] \\
& c &
\end{tikzcd}
\qquad
\begin{tikzcd}
a+b \arrow[rr, "{[f,g]}"] \arrow[dr, two heads, bend right, "e"'] & & d \\
& a \smile b \arrow[ur, tail, bend right, "f \smile g"'] &
\end{tikzcd} \tag{4.14}$$

The following lemma confirms that this gives the kind of arrow we desire. The proof is adapted from [Gol84, pp. 151–153], part (2) of Theorem 1.

Lemma 4.8. *Let* $\overline{f}$ *and* $\overline{g}$ *be subobjects of* d *in a topos* $\mathscr{E}$*, with representatives* $f : a \rightarrowtail d$ *and* $g : b \rightarrowtail d$*. Then the subobject* $\overline{f \smile g}$ *exists and is the supremum of* $\{\overline{f}, \overline{g}\}$ *under the partial ordering* $\leq$*.*

Proof. As in the proof to the previous lemma, since $\mathscr{E}$ is finitely co-complete by Definition 2.41, the arrow $f \smile g$ exists. By its own definition it is monic, so the equivalence class $\overline{f \smile g}$ exists. Furthermore, its definition has that $f = (f \smile g) \circ (e \circ i_a)$ and $g = (f \smile g) \circ (e \circ i_b)$, so $f \leq (f \smile g)$ and $g \leq (f \smile g)$: $\overline{f \smile g}$ is an upper bound bound for $\overline{f}$ and $\overline{g}$.

Now, to show that it is the lowest such bound, suppose there exists a subobject $\overline{\xi}$ such that $\overline{f} \leq \overline{\xi}$ and $\overline{g} \leq \overline{\xi}$. Let $\xi : \Xi \rightarrowtail d$ be a representative of this other upper bound. Then $f = \xi \circ \xi_1$ and $g = \xi \circ \xi_2$ for some arrow ξ_1, ξ_2. By the definition of the coproduct (Def. 2.30), there exists a unique arrow $[\xi_1, \xi_2] : (a+b) \dashrightarrow \Xi$ such that $[\xi_1, \xi_2] \circ i_a = \xi_1$ and $[\xi_1, \xi_2] \circ i_b = \xi_2$. Now, since $[\xi_1, \xi_2]$ is unique, the composition arrow $\xi \circ [\xi_1, \xi_2]$ must also be unique to Diagram (4.15). However, we have defined $[f, g]$ to be this unique arrow, so $[f, g] = \xi \circ [\xi_1, \xi_2]$.

$$\begin{tikzcd}
a \arrow[r, "i_a"] \arrow[dr, tail, bend right, "\xi_1"] \arrow[ddr, tail, bend right=60, "f"'] & a+b \arrow[d, dashed, "{[\xi_1,\xi_2]}"] \arrow[dd, bend right=40, "{[f,g]}"'] & b \arrow[l, "i_b"'] \arrow[dl, tail, bend left, "\xi_2"'] \arrow[ddl, tail, bend left=60, "g"] \\
& \Xi \arrow[d, tail, "\xi"] & \\
& d &
\end{tikzcd} \tag{4.15}$$

But by the definition of $f \smile g$, we already have the epi-monic factorisation $[f, g] = (f \smile g) \circ e$. Then, since ξ is monic, from Theorem 2.38 there must exist a $k : (a \smile b) \to \Xi$ where $(f \smile g) = \xi \circ k$.

$$\begin{tikzcd}
& \Xi \arrow[dr, tail, bend left, "\xi"] & \\
a+b \arrow[ur, bend left, "{[\xi_1,\xi_2]}"] \arrow[rr, "{[f,g]}"] \arrow[dr, two heads, bend right, "e"'] & & d \\
& a \smile b \arrow[uu, dashed, "k"'] \arrow[ur, tail, bend right, "f \smile g"'] &
\end{tikzcd} \tag{4.16}$$

Hence $\overline{f \smile g} \leq \overline{\xi}$. Thus $\overline{f \smile g} = \sup\{\overline{f}, \overline{g}\}$. $\square$

Putting the results from this section and last all together, we arrive at the following theorem.

Theorem 4.9. *Let d be an object in a topos $\mathscr{E}$. Then $(\mathrm{Sub}(d), \leq)$ equipped with the operations $\frown$ and $\smile$, where $\overline{f} \frown \overline{g} := \overline{f \frown g}$ and $\overline{f} \smile \overline{g} := \overline{f \smile g}$, forms a bounded lattice. We write this as $(\mathrm{Sub}(d), \leq, \frown, \smile)$, but we may simply refer to it as $\mathrm{Sub}(d)$ or as the subobject lattice.*

Proof. By Lemma 4.3, $(\mathrm{Sub}(d), \leq)$ forms a bounded poset. By Lemmas 4.6 and 4.8, infimums and supremums exist for any two elements in $\mathrm{Sub}(d)$. Therefore this structure $(\mathrm{Sub}(d), \leq, \frown, \smile)$ is a lattice under Definition 3.3. □

Remark 4.10. As in Remark 4.4, we make a brief note of how this result can generalise to any category. For all categories, $(\mathrm{Sub}(d), \leq)$ forms a poset with a top. What specific conditions are then required for this to be a lattice? The meet operation required pullbacks for its definition, so a category must have pullbacks of monics into d for infimums to exist on $(\mathrm{Sub}(d), \leq)$.[31] The requirements for the join operation are somewhat more substantive. We required coproducts for the $[f, g]$ arrow, and then to allow for epi-monic factorisation we needed pullbacks and equalisers, and the pullbacks of epic arrows needed to remain epic.[32] These requirements are all used in the constructive proof of Theorem 2.38 in Appendix A. Finally, if we require the category to have an initial object then the poset, and thus lattice, is bounded.

4.3 The Lattice forms an Algebra

In Section 4.1 we saw that in a topos, for any object d, $\mathrm{Sub}(d)$ forms a poset. Then in the previous section we saw how we can go further and identify a meet and join structure. Can we complete what was discussed in Chapter 3 and find complements, pseudo-complements, and relative pseudo-complements? The answer, it turns out, is sometimes, yes, and yes. In this section we will show how one can identify the relative pseudo-complement of f with respect to g in $\mathrm{Sub}(d)$. From having relative pseudo-complements and a bottom (0_d), pseudo-complements can be found (Corollary 3.12). Sometimes these pseudo-complements will be complements in the standard sense as well, and sometimes they will not. The fact that there is a distinction here is what makes toposes so interesting from a logical perspective, as this results in the 'topos logic' (which we will examine in the next chapter) to be non-classical.

[31] A poset with a meet operation (and not necessarily a join) is called a *meet-semilattice*. So, any category with pullbacks has that $\mathrm{Sub}(d)$ always forms a meet-semilattice.

[32] A poset with joins (and not necessarily meets) is called a *join-semilattice*. But because joins in $\mathrm{Sub}(d)$ required pullbacks, if we have joins we necessarily have meets.

4.3.1 Relative pseudo-complementation

Suppose we have monic arrows $f : a \rightarrowtail d$, $g : b \rightarrowtail d$. Recall from Chapter 3 that the relative pseudo-complement of f with respect to g is an element $l : m \rightarrowtail d$ such that for any other arrow h, $h \leq l$ iff $(f \frown h) \leq g$ (Prop. 3.11). We can, however, characterise the statement $(f \frown h) \leq g$ in an easier way by the following lemma from Goldblatt.

Lemma 4.11. *If f, g and h are monic arrows into the object d in a topos, then $f \frown h \simeq g \frown h$ if and only if $\chi_f \circ h = \chi_g \circ h$.*

Proof. This is a simple application of Proposition 2.40 (uniqueness of characteristic arrows using a subobject classifier) to pullback squares constructed from the definition of meet. A full proof is provided in [Gol84, p. 163]. □

This gives the following corollary.

Corollary 4.12. *With f, g, h defined as above,*

$$(f \frown h) \leq g \textit{ if and only if } \chi_{f \frown g} \circ h = \chi_f \circ h$$

Proof. (From [Gol84, p. 164].)[33]

$$\begin{aligned}
(f \frown h) \leq g \quad & \text{iff} \quad (f \frown h) \frown g \simeq f \frown h && \text{(Lemma 3.4)} \\
& \text{iff} \quad (f \frown g) \frown h \simeq f \frown h && \text{(commutativity and associativity)} \\
& \text{iff} \quad \chi_{f \frown g} \circ h = \chi_f \circ h && \text{(Lemma 4.11)}
\end{aligned}$$
□

Now, we see that we want an arrow l where $\chi_{f \frown g} \circ h = \chi_f \circ h$ iff there exists a k such that $h = l \circ k$ (i.e., $h \leq l$). This condition is just that of an equaliser (Def. 2.19).

Definition 4.13 (Relative pseudo-complementation in $\mathrm{Sub}(d)$)**.** Given monic arrows $f : a \rightarrowtail d$, $g : b \rightarrowtail d$ in a topos $\mathscr{E}$, the *relative pseudo-complement* of f with respect to g, $f \sqsupset g$:

[33] Note that this proof makes use of the identification of $\simeq$ and $=$ on the $\mathrm{Sub}(d)$ lattice. I.e., Lem. 3.4 applies to the lattice structure of equivalence classes of arrows, not the arrows themselves. However, because $f \simeq g$ iff $\overline{f} = \overline{g}$, this can be applied to the arrows directly.

$(a \sqsupset b) \longrightarrow d$, is defined as equaliser of the characteristic arrows of f and $f \frown g$: χ_f and $\chi_{f\frown g} : d \longrightarrow \Omega$.

$$(a \sqsupset b) \xrightarrow{f \sqsupset g} d \overset{\chi_f}{\underset{\chi_{f\frown g}}{\rightrightarrows}} \Omega \tag{4.17}$$

Lemma 4.14. *Let $\overline{f}$ and $\overline{g}$ be subobjects of d in a topos $\mathscr{E}$, with representatives $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$. Then the subobject $\overline{f \sqsupset g}$ exists and is the relative pseudo-complement of $\overline{f}$ with respect to $\overline{g}$ under the partial ordering $\leq$.*

Before proving this lemma, we note a simple proposition proved in [Gol84, p. 57].

Proposition 4.15. *Suppose e is an equaliser. Then e is monic.*

Given this fact, we can move to the proof concerning relative pseudo-complements.

Proof of Lemma 4.14. By virtue of $\mathscr{E}$ being a topos, the arrow $f \frown g$ exists. Hence χ_f and $\chi_{f\frown g}$ exist as we have a subobject classifier, and then their equaliser $f \sqsupset g$ exists as we have all finite limits. Every equaliser is monic (by the above Proposition 4.15), so the subobject $\overline{f \sqsupset g}$ exists.

For $\overline{f \sqsupset g}$ to be the relative pseudo-complement of $\overline{f}$ with respect to $\overline{g}$, by Prop. 3.11 we need to show that for all other subobjects $\overline{\xi}$,

$$\overline{\xi} \leq \overline{f \sqsupset g} \text{ if and only if } \overline{f \frown \xi} \leq \overline{g}. \quad (\star)$$

Let $\xi : \Xi \rightarrowtail d$ be a representative of $\overline{\xi}$. Then $(\star)$ is equivalent to showing that $\xi \leq (f \sqsupset g)$ if and only if $(f \frown \xi) \leq g$. By the above Corollary 4.12, $(f \frown \xi) \leq g$ iff $\chi_{f\frown g} \circ \xi = \chi_f \circ \xi$. So we need to establish that

$$\xi \leq (f \sqsupset g) \text{ if and only if } \chi_{f\frown g} \circ \xi = \chi_f \circ \xi, \quad (\star\star)$$

which follows from the definition of $f \sqsupset g$ as the equaliser of χ_f and $\chi_{f\frown g}$.

$$\begin{tikzcd} (a \sqsupset b) \arrow[r, "f \sqsupset g"] & d \arrow[r, shift left, "\chi_f"] \arrow[r, shift right, "\chi_{f\frown g}"'] & \Omega \\ \Xi \arrow[u, dashed, "k"] \arrow[ur, "\xi"'] & & \end{tikzcd} \tag{4.18}$$

First suppose that $\chi_{f\frown g} \circ \xi = \chi_f \circ \xi$. Then by the definition of equalisers (Def. 2.19) there exists an arrow $k : \Xi \to (a \sqsupset b)$ such that $\xi = (f \sqsupset g) \circ k$, that is, $\xi \leq (f \sqsupset g)$. Next suppose that $\xi \leq (f \sqsupset g)$, so $\xi = (f \sqsupset g) \circ k$ for some k. Then, as $\chi_{f \frown g} \circ (f \sqsupset g) = \chi_f \circ (f \sqsupset g)$ (Def. 4.13),

$$\chi_f \circ \xi = \chi_f \circ (f \sqsupset g) \circ k = \chi_{f \frown g} \circ (f \sqsupset g) \circ k = \chi_{f \frown g} \circ \xi.$$

Hence ($\star\star$) has been established, and thus $\overline{f \sqsupset g}$ is the relative pseudo-complement of $\overline{f}$ with respect to $\overline{g}$. □

Given this lemma, we can use some of the definitions for specific lattices given in Chapter 3.

Corollary 4.16. *Given an object d in a topos, the lattice* $(\mathrm{Sub}(d), \leq, \frown, \smile)$ *is Brouwerian.*

Proof. Lemma 4.14 confirms that relative pseudo-complements exist for any two subobjects in $\mathrm{Sub}(d)$. This is precisely the definition of a Brouwerian lattice (Def. 3.9). □

Noting that $\mathrm{Sub}(d)$ has a bottom element gives a further refinement to this corollary.

Corollary 4.17. *Let d be an object in a topos $\mathscr{E}$. Then* $(\mathrm{Sub}(d), \leq, \frown, \smile)$ *is a Heyting algebra.*

Proof. $\mathrm{Sub}(d)$ is Brouwerian by Corollary 4.16. Toposes have initial objects, so the lattice has a bottom. This defines a Heyting algebra (Def. 3.13). □

4.3.2 Pseudo-complementation and complementation

In §3.2.3, we mentioned how pseudo-complements can be formed using relative pseudo-complements and a bottom element of the lattice. Since $\mathrm{Sub}(d)$ forms a Heyting algebra, we can carry out this construction and define $-f := f \sqsupset 0_d$. From the above Definition 4.13, this is the equaliser of χ_f and $\chi_{f \frown 0_d}$. But since 0_d is the lattice bottom, using the lattice properties in Corollary 3.8 we have that $f \frown 0_d = 0_d$. This leads to the following definition.

Definition 4.18 (Pseudo-complementation in $\mathrm{Sub}(d)$)**.** Given the monic arrow $f : a \rightarrowtail d$ in a topos $\mathscr{E}$, the *pseudo-complement* of f, denoted $-f$, is defined as equaliser of the characteristic arrows of f and 0_d, $\chi_f, \chi_{0_d} : d \longrightarrow \Omega$.

$$-a \xrightarrow{-f} d \overset{\chi_f}{\underset{\chi_{0_d}}{\rightrightarrows}} \Omega \tag{4.19}$$

Similar to with meets, joins and relative pseudo-complements, the pseudo-complement of the subobject $\overline{f}$ is then $-\overline{f} = \overline{-f}$.

Remark 4.19. As with meets and joins, we note what constructions in the definition of a topos were actually used for identifying relative pseudo-complementation (and hence pseudo-complementation). We required equalisers to define $\sqsupset$, and one of the arrows being equalised used the meet operation, so this in turn requires pullbacks (as discussed in Remark 4.10). Furthermore, Lemma 4.11 requires the subobject classifier and its one-to-one relationship of subobjects to characteristic arrows. So, a category with a subobject classifier, pullbacks and equalisers will admit relative pseudo-complements in every $\mathrm{Sub}(d)$. If it has an initial object then it will further admit pseudo-complements using the above Definition 4.18.

To see when these pseudo-complements become complements in the Boolean algebra sense, we can use Proposition 3.18 — this tells us that $\mathrm{Sub}(d)$ is a Boolean algebra if and only if $-\left(-\overline{f}\right) = \overline{f}$ for all subobjects $\overline{f}$.

With meets, join, relative pseudo-complements and pseudo-complements identified, we are ready to do logic in $\mathrm{Sub}(d)$.

Chapter 5

Logic in Toposes

In Chapter 2 we introduced categories (specifically toposes) and some possible structures within them. Using this, and the concepts introduced in Chapter 3, we were able to identify a Heyting algebra structure on $\mathrm{Sub}(d)$, where d is any object in a topos $\mathscr{E}$. Additionally, we have frequently remarked on the connections between Heyting and Boolean algebras and logic. Now, is there a way to lift this subobject structure out of the land of equivalence classes and into the the arrow and object framework of $\mathscr{E}$? Moreover, are there connections between toposes and logic?

The first section in this chapter will introduce formal mathematical logic and valuations on algebras. Using established results from Helena Rasiowa, we will precisely state what is meant by the connection between logic and our algebras of interest. We will then show how the subobject structure in toposes can be lifted into the arrow structure using the subobject classifier and the terminal object. This will provide everything we need for the major result of this dissertation: that topos logic is intuitionistic.

5.1 Formalised logic

To study logic is to study the validity of arguments. Given a statement, what other statements does it entail? What are the rules of this entailment — in essence, what logically follows from what? The logician examines different accounts of this entailment, and may argue for different particular systems of logic as the 'correct' one. As we shall see, there is more than the one option that we may be used to.

Before we can begin thinking about different logics and their entailments, we must first

Key sources for Chapter 5: [Pri08], [Ras74], [RS68], [Gol84].

agree upon what a *statement* actually is.

5.1.1 Syntax

Formalised logic is done with so-called *well-formed formulas* (wffs), also known as *statements*. The collection of wffs is defined recursively using an alphabet and a set of formation rules, giving the correct way to construct new wffs from existing ones.

Definition 5.1. The *propositional language* (PL) is defined as follows. There is a set called the *alphabet* for PL, comprised of the union of the following sets.

(a) An infinite number of *propositional variables* or *atoms*, $P = \{p_0, p_1, p_2, \ldots\}$;

(b) The logical connectives $\{\neg, \wedge, \vee, \supset\}$;

(c) The brackets $\{(,)\}$.

The collection of well-formed formulas $\mathscr{W}$ is then recursively constructed with the rules

1. Each atom p_i is a member of $\mathscr{W}$;
2. For every wff α in $\mathscr{W}$, $(\neg\alpha)$ is also a wff.
3. For every two wffs α, β in $\mathscr{W}$ (including when $\beta = \alpha$), each of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, and $(\alpha \supset \beta)$ are also wffs.

For example, '$((p_1 \wedge p_{12}) \supset (\neg p_3))$' is a wff, while '$(\neg p_4 \wedge) \vee p_0(p_1 \neg$' is not. When writing wffs we will often neglect to write the outermost brackets .

The logical connectives introduced above in the PL alphabet are more than just ink marks. They represent the operations of negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and the material conditional ($\supset$). These are read as *not, and, or,* and *implies,* respectively. For example, $\alpha \wedge \beta$ is read as α *and* β. To understand the operation of these connectives, we need to build up a notion of semantics.

Remark 5.2. The propositional language we have introduced is *zeroth order*. A *higher order* language is one that involves quantification ("for all", "there exists"). Analysis without quantifiers still provides a suitably meaningful distinction between classical and intuitionistic logic, via the zeroth order statement of *excluded middle*. We will introduce this in more detail shortly.

5.1.2 Semantics

A classical truth value is an element of $\{0,1\}$, where 0 corresponds to 'false' and 1 to 'true'. Non-classical truth values can also comprise just the $\{0,1\}$ set, or they can be more exotic, featuring intermediate values or even the whole $[0,1]$ continuum.[34] We limit ourselves to the classical case for now.

For our purposes, we can take the logical connectives as acting on truth values to output truth values.[35] Using x and y as variables over $\{0,1\}$, the following two tables list the output of each connective given the possible inputs.

x	$\neg x$
1	0
0	1

x	y	$x \wedge y$	$x \vee y$	$x \supset y$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

Table 5.1: Standard logical connectives

For example, we say that x *implies* y is false if x is true and y is false, but true in all other cases.

We can extend the above truth tables to all sentences using a recursive truth assignment function, V. This works similarly to how we recursively constructed the collection of sentences.

Definition 5.3. A *classical truth assignment* or *valuation* is a function $V : \mathscr{W} \longrightarrow \{0,1\}$ that maps each sentence to a truth value. This is done by first assigning each atom $p_i \in P$ a truth value. Then, given the assignment for the atoms, V is extended to all of $\mathscr{W}$ with the rules

(a) $V(\neg\alpha) = \neg V(\alpha)$

(b) $V(\alpha \wedge \beta) = V(\alpha) \wedge V(\beta)$

(c) $V(\alpha \vee \beta) = V(\alpha) \vee V(\beta)$

(d) $V(\alpha \supset \beta) = V(\alpha) \supset V(\beta)$

where the connectives operate on the truth values as in Table 5.1. Given a wff α, we say that

[34] For examples of these more exotic logics, see [Pri08]. Specifically, see Chapter 7 for examples of logics with multiple truth values, and Chapter 11 for logics that use a continuum of truth values.

[35] More formally, the connectives assign statements to truth values, but only after a valuation has been carried out on $\mathscr{W}$. For an example of how such functions work, see [Pri08, p. 120].

(i) α is *satisfied* by the valuation V if $V(\alpha) = 1$;

(ii) α is a *tautology* or *classically valid* if every valuation V satisfies α, written as $\vDash_{\mathbf{CL}} \alpha$.[36]

The relation $\vDash_{\mathbf{CL}}$ is called the *entailment relation* of the semantics.

This definition means that, in practice, we can replace each atom in a wff with its truth value as assigned by V. Then using the rules in the truth tables we can evaluate the entire compositional wff.

Example 5.4. Take the wff $(\neg p_1) \supset (p_1 \vee p_2)$ and suppose there is some valuation V on the atoms p_1, p_2. By using the rules in Definition 5.3 we have that

$$\begin{aligned} V\big((\neg p_1) \supset (p_1 \vee p_2)\big) &= V(\neg p_1) \supset V(p_1 \vee p_2) \\ &= \big(\neg V(p_1)\big) \supset \big(V(p_1) \vee V(p_2)\big). \end{aligned}$$

So if $V(p_1) = 1$ and $V(p_2) = 0$, we would have

$$\begin{aligned} V\big((\neg p_1) \supset (p_1 \vee p_2)\big) &= (\neg 1) \supset (1 \vee 0) \\ &= 0 \supset 1 \\ &= 1. \end{aligned}$$

Example 5.5. Now consider the wff $p_3 \vee \neg p_3$ and suppose that $V(p_3) = 0$. Then

$$\begin{aligned} V\big(p_3 \vee (\neg p_3)\big) &= V(p_3) \vee V(\neg p_3) \\ &= V(p_3) \vee \neg(V(p_3)) \\ &= 0 \vee \neg(0) \\ &= 0 \vee 1 \\ &= 1 \end{aligned}$$

This wff is, in fact, always going to evaluate to $V\big(p_3 \vee (\neg p_3)\big) = 1$ no matter the assign-

[36] The ‘CL’ refers to the classical system of logic that we have defined using Table 5.1. Later we will discuss a different system of logic whose valid statements do not align with the valid statements of classical logic. For example, $\vDash_{\mathbf{CL}} \neg\neg p \supset p$ (double negations cancel in classical logic) while $\nvDash_{\mathbf{I}} \neg\neg p \supset p$ (double negations do not always cancel in an intuitionistic logic I).

ment to the atoms. It is known as the *law of excluded middle* — it is saying that either α or its negation must be true; there is no middle ground. It is precisely this statement that comes under question in developing intuitionistic logic.

5.1.3 Logical consequence

We have so far seen how statements may be true or false with respect to a truth assignment. There is another more practical method of establishing statements we take to be true — *proof.*

Proving a statement γ involves taking earlier true statements and combining them in ways that we know preserve truth, resulting in γ. One account of this idea are *Hilbert systems.* In these systems, we start with a list of generalised statements called *axioms* which we take to be true, and from the axioms we derive new statements via *rules of inference.*[37]

An example of a Hilbert system is the system $\mathfrak{L}$, given in [RS68, p. 188].

Definition 5.6. The Hilbert system $\mathfrak{L}$ is comprised of the axioms (where α, β, γ are arbitrary statements)

(A1) $(\alpha \supset \beta) \supset ((\beta \supset \gamma) \supset (\alpha \supset \gamma))$,

(A2) $\alpha \supset (\alpha \vee \beta)$,

(A3) $\beta \supset (\alpha \vee \beta)$,

(A4) $(\alpha \supset \gamma) \supset ((\beta \supset \gamma) \supset ((\alpha \vee \beta) \supset \gamma))$,

(A5) $(\alpha \wedge \beta) \supset \alpha$,

(A6) $(\alpha \wedge \beta) \supset \beta$,

(A7) $(\gamma \supset \alpha) \supset ((\gamma \supset \beta) \supset (\gamma \supset (\alpha \wedge \beta)))$,

(A8) $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \wedge \beta) \supset \gamma)$,

(A9) $((\alpha \wedge \beta) \supset \gamma) \supset (\alpha \supset (\beta \supset \gamma))$,

(A10) $(\alpha \wedge \neg\alpha) \supset \beta$,

(A11) $(\alpha \supset (\alpha \wedge \neg\alpha)) \supset \neg\alpha$,

(A12) $\alpha \vee \neg\alpha$,

along with the single rule of inference *modus ponens.*

Modus ponens: *From the statements α and $\alpha \supset \beta$, the statement β may be derived.*

If α is derivable from the axioms alone then it is a *theorem* of $\mathfrak{L}$, denoted $\vdash_{\mathfrak{L}} \alpha$, where $\vdash$ is the *consequence relation* of the proof system. This is distinct from the notion of valid-

[37] An alternative type of deductive system is a *Gentzen system,* which typically feature more rules of inference and fewer (or no) axioms than a Hilbert system.

ity under entailment, which is concerned with valuation functions. In a proof system, we methodically deduce new statements from established ones using a set machinery.

The important relationship between logical consequence and logical entailment comes in the form of *soundness* and *completeness*. Given a proof system with consequence relation $\vdash$ and a valuation with entailment relation $\vDash$,

(i) $\vdash$ is *sound* with respect to $\vDash$ when $\vdash \alpha$ implies $\vDash \alpha$, i.e., all theorems are valid statements;

(ii) $\vdash$ is *complete* with respect to $\vDash$ when $\vDash \alpha$ implies $\vdash \alpha$, i.e., all valid statements are theorems.

The reason for introducing the proof system $\mathfrak{L}$ is its connection to classical logic, shown by the following theorem.

Theorem 5.7. *The system $\mathfrak{L}$ with $\vdash_{\mathfrak{L}}$ is sound and complete with respect to the classical logic relation $\vDash_{\boldsymbol{CL}}$.*[38]

A proof can be found in [RS68, p. 189].

5.2 Putting values on algebras

We now illustrate the reason behind our earlier focus on lattices and algebras. Algebras with meets, joins, a form of complementation, and a two place relation can be used to evaluate logical statements.

5.2.1 Boolean validity

In the case of a Boolean algebra, a logical valuation is defined as below.

Definition 5.8. Let $(B, \frown, \smile, \sqsupset, \star)$ be a non-degenerate Boolean algebra where $x \sqsupset y = x^\star \smile y$ as in Corollary 3.17. Then, a *B-valuation* is a mapping $V_B : \mathscr{W} \to B$ constructed as follows. First assign each atom $p_i \in P \subset \mathscr{W}$ an element in B to obtain a mapping $V_B : P \to B$. This is extended to all of $\mathscr{W}$ by the rules

[38]This system $\mathfrak{L}$ is by no means unique in this fact. There are many Hilbert systems that give equivalent theorems, such as Frege's original axiomatisation for classical logic.

(a) $V_B(\neg\alpha) = V_B(\alpha)^\star$;

(b) $V_B(\alpha \wedge \beta) = V_B(\alpha) \frown V_B(\beta)$;

(c) $V_B(\alpha \vee \beta) = V_B(\alpha) \smile V_B(\beta)$;

(d) $V_B(\alpha \supset \beta) = V_B(\alpha) \sqsupset V_B(\beta)$.

In this way, given an initial assignment of atoms to elements B, every possible wff is mapped to an element in the algebra. Given a wff α, we say that

(i) α is *satisfied* by the B-valuation V_B if $V_B(\alpha) = \top$;

(ii) α is a *tautology of B* or *valid in B* if every B-valuation V_B satisfies α, written as $\vDash_B \alpha$.[39]

Boolean algebras are of interest to logicians due to their association with classical logic: classical logic is sound and complete with respect to any (non-degenerate) Boolean algebra.

Theorem 5.9. *Let α be a statement in the propositional language. Then for all non-degenerate Boolean algebras B,*

$$\vdash_{\mathfrak{L}} \alpha \textit{ if and only if } \vDash_B \alpha.$$

That is, the proof system $\mathfrak{L}$ is sound and complete with respect to the semantic system on any Boolean algebra.

Sketch of proof. To show soundness, we need to establish that each axiom of $\mathfrak{L}$ is a tautology of any Boolean algebra B, which is done by using lattice properties of meet and join. As an example, consider (A2) under a valuation V_B. By the rules in Definition 5.8, we have that

$$\begin{aligned}
V_B(\alpha \supset (\alpha \vee \beta)) &= V_B(\alpha) \sqsupset \big(V_B(\alpha) \smile V_B(\beta)\big) \\
&= V_B(\alpha)^\star \smile \big(V_B(\alpha) \smile V_B(\beta)\big) && \text{(definition of } \sqsupset \text{ in a Boolean algebra)} \\
&= \big(V_B(\alpha)^\star \smile V_B(\alpha)\big) \smile V_B(\beta) && \text{(associativity in a lattice — Lem. 3.4)} \\
&= \top \smile V_B(\beta) && \text{(definition of a complement)} \\
&= \top && \text{(Cor. 3.8)}
\end{aligned}$$

So V_B satisfies (A2). This valuation was arbitrary, so (A2) is a tautology of B. After showing each axiom is a tautology, we need to show that the rule of modus ponens preserves validity. The details of this can be found in [RS68, p. 258], Theorem 1.3.

[39] We interpret the top to mean ‘true’. Dually, the bottom will be interpreted ‘false’.

Proving completeness involves manipulating structures in Boolean algebras that we have not explored, namely ideals and algebra homomorphisms. The relationship to classical logic comes from mapping any B to the Boolean algebra of two elements, and then interpreting those elements through the truth tables in 5.1. The details can be found in [RS68, pp. 259–260], Theorem 2.2. $\square$

Note that since the classical truth tables also provide a sound and complete account of the tautologies of $\mathfrak{L}$ (Thm. 5.7), we have that $\models_{CL} \alpha$ if and only if $\models_B \alpha$ for any Boolean algebra. I.e., the tautologies of any Boolean algebra are exactly the same tautologies of classical logic.

5.2.2 Heyting validity

If, instead of a Boolean algebra B we have a more general Heyting algebra H, then the above definition of a B-valuation (Def. 5.8) can be modified to obtain an H-valuation, V_H. Again, we begin with an assignment of the atoms $p_i \in P$ to elements of H. This is extended with the rules

(a) $V_H(\neg\alpha) = -V_H(\alpha)$;

(b) $V_H(\alpha \wedge \beta) = V_H(\alpha) \frown V_H(\beta)$;

(c) $V_H(\alpha \vee \beta) = V_H(\alpha) \smile V_H(\beta)$;

(d) $V_H(\alpha \supset \beta) = V_H(\alpha) \sqsupset V_H(\beta)$.

Given a wff α, we say that

(i) α is *satisfied* by the H-valuation V_H if $V_H(\alpha) = \top$;

(ii) α is a *tautology of* H or *valid in* H if every H-valuation V_H satisfies α, written as $\models_H \alpha$.

The difference to Boolean valuations is in rule (a), where complements are replaced by pseudo-complements defined as $-x = x \sqsupset \bot$. Because Boolean algebras are a specific type of Heyting algebra where pseudo-complements are also complements (Cor. 3.17), when H is Boolean this V_H valuation becomes the previously defined V_B valuation.

We saw in Theorem 5.9 that Boolean algebras provide an account of classical logic. What, then, is the logic that corresponds to the more general Heyting algebras? A pointer comes from the difference in between complement and pseudo-complement. Complements satisfy $x \frown x^\star = \bot$ and $x \smile x^\star = \top$, while pseudo-complements are only guaranteed to satisfy the former expression: $x \frown -x = \bot$. Now consider an H-valuation of axiom A12 for the system $\mathfrak{L}$: $V_H(\alpha \vee \neg\alpha) = V_H(\alpha) \smile -V_H(\alpha)$. In a Boolean algebra we would know this to be equal to $\top$ by the definition of complementation. However, in a Heyting algebra with pseudo-complementation, this is not guaranteed. Hence the logic that is represented by Heyting algebras is something like $\mathfrak{L}$ with A12 removed.

The axiom A12 is the *law of excluded middle* (LEM). As discussed earlier, this law requires that any statement α must be true or not true. A classical logic system without LEM is said to be *intuitionistic*. Removing LEM has dramatic effects on the nature of what can be proved; it results in a mathematics where statements must be *constructed*, where proof of existence of objects is impossible without outlining a practical method for their formation.[40]

Let $\mathfrak{I}$ denote the Hilbert system of $\mathfrak{L}$ with LEM removed, but retaining all other axioms and the modus ponens rule of inference. This, then, is the system that corresponds to Heyting validity.

Theorem 5.10. *Let α be a statement in the propositional language. Then for all non-degenerate Heyting algebras H,*

$$\vdash_{\mathfrak{I}} \alpha \text{ if and only if} \vDash_H \alpha$$

That is, the intuitionistic proof system $\mathfrak{I}$ is sound and complete with respect to the semantic system on any Heyting algebra.

Sketch of proof. The process of this proof is similar to that of Theorem 5.9.

To show soundness, we need to establish that each axiom of $\mathfrak{I}$ is a tautology of any Boolean algebra H — this is done using lattice properties of $\frown, \smile, \sqsupset$ and $-$. Next, we need to show that the rule of modus ponens preserves validity. The details of this can be found in [RS68, p. 384], Theorem 2.5.

Proving completeness involves relating the Heyting algebra H to the topological notions of open sets in a dense metric space. The details of the proof can be found in [RS68, pp. 385–387], Theorem 3.2. □

[40]This way of reasoning was championed by Luitzen E. J. Brouwer. His student, Arend Heyting, devised Heyting algebras for the explicit purpose of modelling intuitionistic logic.

Note that any statement derivable in $\mathfrak{I}$ is derivable in $\mathfrak{L}$ (since every axiom and rule in $\mathfrak{I}$ is in $\mathfrak{L}$), so intuitionistic logic is a *sublogic* of classical logic, as any intuitionistic theorem will also be a classical theorem.

5.2.3 Sub(d) validity

Using the above notions of valuations on algebras, we can consider the case of Sub(d).

Definition 5.11. A Sub(d)-*valuation* is a function $V_d : \mathscr{W} \to \mathrm{Sub}(d)$ that maps each sentence to a subobject of d. First, each atom p_i is assigned a subobject $\overline{f_i}$. The mapping is then extended inductively over the following rules.[41]

(a) $V_d(\neg\alpha) = \overline{-V_d(\alpha)}$;

(b) $V_d(\alpha \wedge \beta) = \overline{V_d(\alpha) \frown V_d(\beta)}$;

(c) $V_d(\alpha \vee \beta) = \overline{V_d(\alpha) \smile V_d(\beta)}$;

(d) $V_d(\alpha \supset \beta) = \overline{V_d(\alpha) \sqsupset V_d(\beta)}$.

Furthermore, as in the previous definitions of valuations, given a wff α, we say that

(i) α is *satisfied* by the Sub(d)-valuation V_d if $V_d(\alpha) = \overline{\mathrm{id}_1}$;

(ii) α is a *tautology of* Sub(d) or *valid in* Sub(d) if every Sub(d)-valuation V_d satisfies α, written as $\vDash_d \alpha$.

Theorem 5.12. *In a topos, given a object d, the intuitionistic proof system $\vdash_{\mathfrak{I}}$ is sound and complete with respect to the entailment relation $\vDash_d$ on* Sub(d).

Proof. By Corollary 4.17, Sub(d) is a Heyting algebra. Theorem 5.10 then gives the result. □

So we see that intuitionistic logic has a strong connection with the logic defined on the subobjects of a topos. Yet, our main question was concerning a different logical structure — which we develop in the next section.

5.3 Lifting out of Sub(d)

We have developed a theory of lattices within the collection of subobjects of some object d, and we have seen how logic can be encoded on this lattice. However, subobjects are not

[41] When we define valuations inductively we assume that $V_d(\alpha)$ and $V_d(\beta)$ have already been assigned some subobjects, like $\overline{g}$ and $\overline{h}$. The operations given in rules (a)–(d) are then the algebraic operations on Sub(d) as detailed in Lemmas 4.6, 4.8 and 4.14, and Definition 4.18.

technically *in* a topos — a subobject is an equivalence class of monic arrows. The arrows are parts of the category, but the equivalence classes are not. So, can we find a way to lift the structure of the subobjects out of $\mathrm{Sub}(d)$ into the categorial arrow and object makeup of $\mathscr{E}$?

5.3.1 Connecting arrows in $\mathscr{E}$

Recall that each subobject $\overline{f}$ in $\mathrm{Sub}(d)$ corresponds to exactly one characteristic arrow $\chi_f : d \to \Omega$. This property will provide the suitable machinery for our desired lifting.

Definition 5.13. Given a topos $\mathscr{E}$, let $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ be representatives of the subobjects $\overline{f}$ and $\overline{g}$ in the subobject algebra $(\mathrm{Sub}(d), \frown, \smile, \sqsupset, -)$. Then the operations $\neg, \wedge, \vee$ and $\supset$ are defined on the characters χ_f and χ_g as follows.

1. $\neg\chi_f \coloneqq \chi_{-f}$

$$\begin{tikzcd} -a \arrow[r, tail, "-f"] \arrow[d, "!"'] & d \arrow[d, "\chi_{-f}=\neg\chi_f"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.1}$$

2. $\chi_f \wedge \chi_g \coloneqq \chi_{f\frown g}$

$$\begin{tikzcd} a \frown b \arrow[r, tail, "f\frown g"] \arrow[d, "!"'] & d \arrow[d, "\chi_{f\frown g}=\chi_f\wedge\chi_g"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.2}$$

3. $\chi_f \vee \chi_g \coloneqq \chi_{f\smile g}$

$$\begin{tikzcd} a \smile b \arrow[r, tail, "f\smile g"] \arrow[d, "!"'] & d \arrow[d, "\chi_{f\smile g}=\chi_f\vee\chi_g"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.3}$$

4. $\chi_f \supset \chi_g \coloneqq \chi_{f\sqsupset g}$

$$\begin{tikzcd} a \sqsupset b \arrow[r, tail, "f\sqsupset g"] \arrow[d, "!"'] & d \arrow[d, "\chi_{f\sqsupset g}=\chi_f\supset\chi_g"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.4}$$

Each of these arrows is a member of the hom-set $\mathscr{E}(d, \Omega)$ — the collection of arrows from d to Ω. Additionally, note that Proposition 2.40 (characters are unique to their subobjects) ensures this construction is independent on the choice of representative.

5.3.2 Logical arrows

The connectives defined in Definition 5.13 are essentially mappings on the arrows of $\mathscr{E}$ — $\wedge, \vee$ and $\supset$ map an ordered pair of two arrows in $\mathscr{E}$ to a third, while $\neg$ maps one arrow to another. Considering the earlier motivation for lifting the $\mathrm{Sub}(d)$ into the topos (that it would be nice to have structures that are actually *in* the topos), this has not gained much ground. These mappings are externally defined relationships between arrows, only made possible from an external viewpoint to the topos. There is, however, a way to have an internal mapping in a category: composition. The composition arrow $f \circ g = h$ can be read as f mapping g to h. With this motivation, we define the following logical connective arrows that give this desired property.

Definition 5.14 (Logical connective arrows)**.** The arrows $\neg, \wedge, \vee$ and $\supset$ in a topos $\mathscr{E}$ are defined as follows.[42]

$\neg$: The arrow $\neg : \Omega \to \Omega$ is the character of the character of 0_1, $\chi_{\chi_{0_1}}$.[43]

$$\begin{tikzcd} 1 \arrow[r, "\chi_{0_1}"] \arrow[d, "!"'] & \Omega \arrow[d, "\neg := \chi_{\chi_{0_1}}"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.5}$$

$\wedge$: The arrow $\wedge : \Omega \times \Omega \to \Omega$ is the character of the product arrow $\langle t, t\rangle : 1 \to \Omega \times \Omega$.[44]

$$\begin{tikzcd} 1 \arrow[r, "\langle t,t\rangle"] \arrow[d, "!"'] & \Omega \times \Omega \arrow[d, "\wedge := \chi_{\langle t,t\rangle}"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.6}$$

$\vee$: The arrow $\vee : \Omega \times \Omega \to \Omega$ is the character of the coproduct arrow $[\langle t \circ 1_\Omega, \mathrm{id}_\Omega\rangle, \langle \mathrm{id}_\Omega, t \circ 1_\Omega\rangle]$.[45]

$$\begin{tikzcd} \Omega + \Omega \arrow[r, "{[\langle t\circ 1_\Omega, \mathrm{id}_\Omega\rangle, \langle \mathrm{id}_\Omega, t\circ 1_\Omega\rangle]}"] \arrow[d, "!"'] & \Omega \times \Omega \arrow[d, "\vee := \chi_{[\langle t\circ 1_\Omega, \mathrm{id}_\Omega\rangle, \langle \mathrm{id}_\Omega, t\circ 1_\Omega\rangle]}"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.7}$$

[42] For a full walkthrough of the development of these arrows see §6.6 in [Gol84].

[43] Taking the character of a character may seem rather elaborate, however the reason will become apparent when considering truth values. We will interpret the character of 0_1, χ_{0_1}, as 'false'. Recalling the intuition behind the subobject classifier, the character of f into d is what maps the 'f part of d' to the 'true part of Ω'. In this context, the character of χ_{0_1} is what maps the 'false part of Ω' to the 'true part of Ω' — i.e. it flips true and false, which is exactly what we would expect of negation.

[44] Elaborating from the previous footnote, the point of this construction is to map the 'part of $\Omega \times \Omega$ that is true under conjunction' to the 'true part of Ω'. In the case of **Set**, the $\langle t, t\rangle$ subobject is the subset $\{(1,1)\}$ in $\Omega \times \Omega = \{(0,0),(0,1),(1,0),(1,1)\}$. Note that in Table 5.1, $1 \wedge 1 = 1$ while all other combinations give 0. Hence $(1,1)$ is the 'part of $\Omega \times \Omega$ that is true under conjunction' in **Set**.

[45] In **Set**, this complicated coproduct arrow picks out the subset $\{(0,1),(1,0),(1,1)\}$, which are the truth value combinations that return 1 under $\vee$ in Table 5.1.

$\supset$: The arrow $\supset : \Omega \times \Omega \to \Omega$ is the character of $e : \leqslant \to \Omega \times \Omega$, the equaliser of $\wedge$ and pr_1.[46]

$$\begin{tikzcd}
\leqslant \arrow[r, "e"] & \Omega \times \Omega \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "\mathrm{pr}_1"'] & \Omega \\
\bullet \arrow[u, dashed] \arrow[ur] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\leqslant \arrow[r, "e"] \arrow[d, "!"'] & \Omega \times \Omega \arrow[d, "\supset := \chi_e"] \\
1 \arrow[r, "t"] & \Omega
\end{tikzcd} \tag{5.8}$$

The next theorem establishes the desired properties of these arrows.

Theorem 5.15. *Given a topos $\mathscr{E}$ with the arrows $\neg, \wedge, \vee$ and $\supset$ defined as in Definition 5.14, the following identities hold.*

(i) $\wedge \circ \langle \chi_f, \chi_g \rangle = \chi_{f \frown g}$

(ii) $\vee \circ \langle \chi_f, \chi_g \rangle = \chi_{f \smile g}$

(iii) $\supset \circ \langle \chi_f, \chi_g \rangle = \chi_{f \sqsupset g}$

(iv) $\neg \circ \chi_f = \chi_{-f}$

Proof. We prove (iii) and (iv). The proofs for (i) and (ii) can be found in [Gol84, pp. 148–151], Theorems 2 and 3.

(iii) By the definition of $f \sqsupset g$ (Def. 4.13), $\chi_f \circ (f \sqsupset g) = \chi_{f \frown g} \circ (f \sqsupset g)$. By the earlier property (i) in this theorem, $\wedge \circ \langle \chi_f, \chi_g \rangle = \chi_{f \frown g}$. Also note that $\mathrm{pr}_1 \circ \langle \chi_f, \chi_g \rangle = \chi_f$. So we obtain that

$$\mathrm{pr}_1 \circ \langle \chi_f, \chi_g \rangle \circ (f \sqsupset g) = \wedge \circ \langle \chi_f, \chi_g \rangle \circ (f \sqsupset g).$$

Then $\langle \chi_f, \chi_g \rangle \circ (f \sqsupset g)$ equalises pr_1 and $\wedge$. But the arrow $e : \leqslant \to \Omega \times \Omega$ has been defined to be this equaliser, so there must exist a unique $k : (a \sqsupset b) \dashrightarrow \leqslant$ that ensures the square in (5.9) commutes.

$$\begin{tikzcd}
\Xi \arrow[dr, dashed, "j"] \arrow[drr, bend left, "\xi_1"] \arrow[ddr, bend right, "\xi_2"'] & & & \\
& a \sqsupset b \arrow[r, "f \sqsupset g"] \arrow[d, dashed, "k"] & d \arrow[d, "\langle \chi_f, \chi_g \rangle"] \arrow[r, shift left, bend left, "\wedge \circ \langle \chi_f, \chi_g \rangle"] \arrow[r, shift right, bend left, "\chi_f"'] & \Omega \\
& \leqslant \arrow[r, "e"] & \Omega \times \Omega \arrow[ur, shift left, bend right, "\mathrm{pr}_1"] \arrow[ur, shift right, bend right, "\wedge"'] &
\end{tikzcd} \tag{5.9}$$

[46] In **Set**, this equaliser e picks out the subset $\{(0,0), (0,1), (1,1)\}$ — hence the notation $\leqslant$. These are the combinations that return 1 under $\supset$ in Table 5.1.

To see that this square is a pullback, suppose there exist arrows ξ_1, ξ_2 as in the above diagram, such that $\langle \chi_f, \chi_g \rangle \circ \xi_1 = e \circ \xi_2$. Then

$$\begin{aligned}
\chi_f \circ \xi_1 &= \mathrm{pr}_1 \circ \langle \chi_f, \chi_g \rangle \circ \xi_1 \\
&= \mathrm{pr}_1 \circ e \circ \xi_2 \\
&= \wedge \circ e \circ \xi_2 \quad \text{(as } e \text{ equalises } \mathrm{pr}_1 \text{ and } \wedge\text{)} \\
&= \wedge \circ \langle \chi_f, \chi_g \rangle \circ \xi_1
\end{aligned}$$

So ξ_1 equalises χ_f and $\wedge \circ \langle \chi_f, \chi_g \rangle$. But $f \sqsupset g$ is defined to be this equaliser, so there exists a unique arrow $j : \Xi \dashrightarrow a \sqsupset b$ such that the entire diagram (5.9) commutes. Thus the central square is a pullback.

Now, consider the following diagram (5.10). The top square is a pullback by the above reasoning, while the bottom square is a pullback by definition of $\supset$.

$$\begin{tikzcd}
a \sqsupset b \arrow[r, "f \sqsupset g"] \arrow[d, dashed, "k"] \arrow[dd, bend right, "!"'] & d \arrow[d, "\langle \chi_f, \chi_g \rangle"] \\
\leqslant \arrow[r, "e"] \arrow[d, "!"] & \Omega \times \Omega \arrow[d, "\supset"] \\
1 \arrow[r, "t"] & \Omega
\end{tikzcd} \tag{5.10}$$

Hence by the Pullback Lemma (2.34), the outer rectangle is a pullback. Since the outer rectangle defines the subobject classifier acting on $f \sqsupset g$, we get that $\supset \circ \langle \chi_f, \chi_g \rangle = \chi_{f \sqsupset g}$.

(iv) As $-f$ is the pseudo-complement of f in $\mathrm{Sub}(d)$, by Corollary 3.12 we have that for all monic arrows $g : b \rightarrowtail d$,

$$g \leq -f \quad \text{iff} \quad f \frown g \simeq 0_d. \tag{5.11}$$

From the lattice properties we encountered in Chapter 3 (Lem. 3.4 and Cor. 3.8), $g \leq -f$ is equivalent to $g \frown -f \simeq g$, $g \simeq g \frown g$ and $f \frown g \simeq 0_d \simeq 0_d \frown g$. Then (5.11) becomes

$$g \frown -f \simeq g \frown g \quad \text{iff} \quad f \frown g \simeq 0_d \frown g. \tag{5.12}$$

By Lemma 4.11, this is equivalent to

$$\chi_{-f} \circ g = \chi_g \circ g \quad \text{iff} \quad \chi_f \circ g = \chi_{0_d} \circ g. \tag{5.13}$$

This implies that

$$\neg \circ \chi_{-f} \circ g = \neg \circ \chi_g \circ g \quad \text{iff} \quad \chi_f \circ g = \chi_{0_d} \circ g. \tag{5.14}$$

We now establish four small facts.

First, consider Diagram (5.15) which shows the commutative square that defines χ_g.

$$\begin{tikzcd} b \arrow[r, tail, "g"] \arrow[d, "1_b"'] & d \arrow[d, "\chi_g"] \\ 1 \arrow[r, "\boldsymbol{t}"] & \Omega \end{tikzcd} \tag{5.15}$$

From this we see that $\chi_g \circ g = \boldsymbol{t} \circ 1_b$ ($\star$). Next, consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r, "!=0_1"] \arrow[d, "0_1"'] & 1 \arrow[d, "\boldsymbol{t}"] \\ 1 \arrow[r, "\chi_{0_1}"] \arrow[d, "!=\mathrm{id}_1"'] & \Omega \arrow[d, "\neg=\chi_{\chi_{0_1}}"] \\ 1 \arrow[r, "\boldsymbol{t}"] & \Omega \end{tikzcd} \tag{5.16}$$

The bottom square is the pullback that defines $\neg$, and the top is the pullback that defines χ_{0_1}. Hence by the Pullback Lemma the outer rectangle is a pullback, and so $\neg \circ \boldsymbol{t} = \chi_{0_1}$ ($\star\star$).

Thirdly, note that the composition arrow $1_d \circ g : b \to 1$ must exist, but 1_b is defined to be the unique arrow satisfying this condition, so $1_b = 1_d \circ g$ $(\star\star\star)$.

$$\begin{tikzcd} b \arrow[r, tail, "g"] \arrow[rr, dashed, bend right, "1_b"'] & d \arrow[r, dashed, "1_d"] & 1 \end{tikzcd} \tag{5.17}$$

Finally, an examination of Diagram (5.18) shows that $\chi_{0_1} \circ 1_d = \chi_{0_d}$.

$$\begin{tikzcd} & d \arrow[d, "1_d"] \arrow[dd, bend left=60, "\chi_{0_1} \circ 1_d"] \\ 0 \arrow[ur, "0_d"] \arrow[r, "0_1"] \arrow[d, "!"'] & 1 \arrow[d, "\chi_{0_1}"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd} \tag{5.18}$$

To see this, first note that the bottom square is the pullback defining χ_{0_1}. The outer square is thus a pullback as well, as any arrow ξ_1 from some object Ξ to d will commute with 1_d to be give an arrow to 1, and so the pullback property of the $(0, 1, 1, \Omega)$ square will guarantee an arrow $k : \Xi \dashrightarrow 0$, ensuring that the outer square is a pullback as well. Therefore, $\chi_{0_1} \circ 1_d = \chi_{0_d}$ $(\star\star\star\star)$.

Using these four results,

$$\begin{aligned} \neg \circ \chi_g \circ g &= \neg \circ \boldsymbol{t} \circ 1_b && (\star) \\ &= \chi_{0_1} \circ 1_b && (\star\star) \\ &= \chi_{0_1} \circ 1_d \circ g && (\star\star\star) \\ &= \chi_{0_d} \circ g && (\star\star\star\star) \end{aligned}$$

Now, (5.19) becomes

$$\neg \circ \chi_{-f} \circ g = \chi_{0_d} \circ g \quad \text{iff} \quad \chi_f \circ g = \chi_{0_d} \circ g. \tag{5.19}$$

But this is just equivalent to the equality

$$\neg \circ \chi_{-f} \circ g = \chi_f \circ g. \tag{5.20}$$

Recall that this holds for all monic arrows g into d. By Theorem 2.38, every arrow into d can be factorised into an epic arrow and a monic arrow, where the monic part has codomain d. Thus (5.19) holds for every arrow g into d, regardless of whether it is monic. Therefore, $\neg \circ \chi_{-f} = \chi_f$. $\square$

Thus, by Definition 5.13,

(i) $\neg \circ \chi_f = \neg \chi_f$;

(ii) $\wedge \circ \langle \chi_f, \chi_g \rangle = \chi_f \wedge \chi_g$;

(iii) $\vee \circ \langle \chi_f, \chi_g \rangle = \chi_f \vee \chi_g$;

(iv) $\supset \circ \langle \chi_f, \chi_g \rangle = \chi_f \supset \chi_g$.

5.3.3 The logic of a topos

So far, we have been undiscerning with the choice of the object d in $\mathscr{E}(d, \Omega)$. While our zeroth-order logic of propositions and connectives could be carried out in any $\mathscr{E}(d, \Omega)$, what makes 'topos logic' special is its higher order capabilities with internalised forms of existential and universal quantification ("there exists" and "for all").[47] For reasons that we are unable to explain within this dissertation, these methods require constructions only available on $\mathscr{E}(1, \Omega)$.[48]

To move from Sub(1) to $\mathscr{E}(1, \Omega)$ we use the characteristic arrows of subobjects. Doing this, we can transfer over the bottom and top of the Sub(1) algebra. The top element of Sub(1) is id_1. The character of this arrow is then the previously defined classifying arrow $\boldsymbol{t}$ for the subobject classifier (hence the name 'true' for this arrow).

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{id}_1"] \arrow[d, "!=\mathrm{id}_1"'] & 1 \arrow[d, "\chi_{\mathrm{id}_1}"] \\ 1 \arrow[r, "\boldsymbol{t}"] & \Omega \end{tikzcd} \tag{5.21}$$

If the character of the top element is 'true', we call the character of the bottom element 'false'.

[47]These higher order constructions are what exponential objects are used for. Further, they utilise *functors* and *adjoints* — two concepts we have not discussed. For details of the quantification arrows in a topos and how $\mathscr{E}(1, \Omega)$ is relevant to their construction see [Gol84, pp. 245–248].

[48]Additionally, $\mathscr{E}(1, \Omega)$ gives a nice internal version of truth values. Observe that $\mathscr{E}(d, \Omega)$ is an *external* construction: it does not exist anywhere within $\mathscr{E}$ as an object. There is an internal representation of it though, given in terms of exponential objects (the material we skipped for brevity in Def. 2.41). In the case of $\mathscr{E}(1, \Omega)$ this exponential object (whatever that may be) collapses to be isomorphic to Ω itself — so Ω represents the truth values of the logic.

Hence, $\boldsymbol{f} := \chi_{0_d}$.

$$\begin{tikzcd} 0 \arrow[r, "0_1"] \arrow[d, "!=\mathrm{id}_1"'] & 1 \arrow[d, "\boldsymbol{f}:=\chi_{0_1}"] \\ 1 \arrow[r, "\boldsymbol{t}"] & \Omega \end{tikzcd} \tag{5.22}$$

This sheds some light on the earlier definition of the $\neg$ arrow. We now see that $\neg := \chi_{\boldsymbol{f}}$; see the earlier footnote 43 on page 71 for more discussion on this. Furthermore, we saw in the proof of Theorem 5.15 that $\neg \circ \boldsymbol{t} = \chi_{0_1}$ (Diagram (5.16)), hence $\neg \circ \boldsymbol{t} = \boldsymbol{f}$ — which is what we would expect for negation.

Given a suitable structure on the topos itself, we can finally define validity in $\mathscr{E}$.

Definition 5.16. An *$\mathscr{E}$-valuation* is a function $V_{\mathscr{E}} : \mathscr{W} \to \mathscr{E}(1,\Omega)$ that maps each sentence to an arrow in $\mathscr{E}(1,\Omega)$, which we call a truth value. Similar to the previous valuation function definitions, this is done by first assigning each atom $p_i \in P$ a truth value $V_{\mathscr{E}}(p_i) : 1 \to \Omega$. Then, given the assignment for the atoms, $V_{\mathscr{E}}$ is extended to all of $\mathscr{W}$ with the rules

(a) $V_{\mathscr{E}}(\neg\alpha) = \neg V_{\mathscr{E}}(\alpha) = \neg \circ V_{\mathscr{E}}(\alpha)$. This is illustrated with the diagram:

$$\begin{tikzcd} 1 \arrow[rr, "V(\alpha)"] \arrow[dr, "V(\neg\alpha)"'] & & \Omega \arrow[dl, "\neg"] \\ & \Omega & \end{tikzcd} \tag{5.23}$$

(b) $V_{\mathscr{E}}(\alpha \wedge \beta) = V_{\mathscr{E}}(\alpha) \wedge V_{\mathscr{E}}(\beta) = \wedge \circ \langle V_{\mathscr{E}}(\alpha), V_{\mathscr{E}}(\beta)\rangle$

(c) $V_{\mathscr{E}}(\alpha \vee \beta) = V_{\mathscr{E}}(\alpha) \vee V_{\mathscr{E}}(\beta) = \vee \circ \langle V_{\mathscr{E}}(\alpha), V_{\mathscr{E}}(\beta)\rangle$

(d) $V_{\mathscr{E}}(\alpha \supset \beta) = V_{\mathscr{E}}(\alpha) \supset V_{\mathscr{E}}(\beta) = \supset \circ \langle V_{\mathscr{E}}(\alpha), V_{\mathscr{E}}(\beta)\rangle$.

Rules (b), (c) and (d) are illustrated in the diagram:

$$\begin{tikzcd} & 1 \arrow[dl, bend right, "V(\alpha)"'] \arrow[d, dashed, "\langle V(\alpha), V(\beta)\rangle"] \arrow[dr, bend left, "V(\beta)"] & \\ \Omega & \Omega \times \Omega \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] \arrow[dl, "\wedge"] \arrow[d, "\vee"] \arrow[dr, "\supset"] & \Omega \\ \Omega & \Omega & \Omega \end{tikzcd} \tag{5.24}$$

Furthermore, given a wff α, we say that

(i) α is *satisfied* by the $\mathscr{E}$-valuation $V_{\mathscr{E}}$ if $V_{\mathscr{E}}(\alpha) = \boldsymbol{t}$;

(ii) α is a *tautology of* $\mathscr{E}$ or *valid in* $\mathscr{E}$ if every $\mathscr{E}$-valuation $V_{\mathscr{E}}$ satisfies α, written as $\vDash_{\mathscr{E}} \alpha$.

The system of logical connective arrows in a topos along with the above semantics is what we have been referring to throughout this dissertation as 'topos logic'. Now that we have a grasp on how logic in a topos arises, we move to analysing its structure. The next theorem provides the final piece of information to answer the question we began with: what does it mean for topos logic to be intuitionistic?

Theorem 5.17. *Let γ be a statement in the propositional language, and let $\mathscr{E}$ be a topos. Then*

$$\vDash_{\mathscr{E}} \gamma \quad \textit{iff} \quad \vDash_1 \gamma.$$

That is, the logic defined in $\mathscr{E}$ has the same tautologies as the language on $\mathrm{Sub}(1)$.

Proof. We show that there exists an $\mathscr{E}$-valuation that does not satisfy γ if and only if there exists a $\mathrm{Sub}(1)$-valuation that does not satisfy γ.[49]

Suppose that there is a $\mathrm{Sub}(1)$-valuation V_1 such that $V_1(\gamma) \neq \mathrm{id}_1$. By definition, V_1 first assigns the atoms p_i as such: $V_1(p_i) = \overline{f_i}$. Construct the $\mathscr{E}$-valuation $V_{\mathscr{E}}$ by setting the assignment of the atoms as the characters of the V_1 assignments. That is, $V_{\mathscr{E}}(p_i) = \chi_{V_1(p_i)} = \chi_{f_i} : 1 \to \Omega$, where $f_i : a_i \to 1$ is a representative of $\overline{f_i}$.

$$\begin{tikzcd}
a_i \arrow[r, tail, "V_1(p_i)=f_i"] \arrow[d, "!"'] & 1 \arrow[d, "V_{\mathscr{E}}(p_i)=\chi_{f_i}"] \\
1 \arrow[r, "t"] & \Omega
\end{tikzcd} \tag{5.25}$$

We aim to show that $V_{\mathscr{E}}(\alpha) = \chi_{V_1(\alpha)}$ for all statements α, not only the atoms. To this end we proceed inductively through the logical connectives. Assume that for statements α and β, it is the case that $V_{\mathscr{E}}(\alpha) = \chi_{V_1(\alpha)}$ and $V_{\mathscr{E}}(\beta) = \chi_{V_1(\beta)}$. Then, we have the following equalities over the four logical connectives.

[49] This uses that a statement of contraposition ($\neg p \implies \neg q$) is the same as a statement of implication ($q \implies p$). Ironically, this equivalence is not always valid in an intuitionistic logic, so we find ourselves proving the intuitionism of topos logic using non-intuitionistic methods.

Negation:

$$\begin{aligned} V_{\mathscr{E}}(\neg\alpha) &= \neg V_{\mathscr{E}}(\alpha) && \text{(Def. 5.16 (a))} \\ &= \neg\chi_{V_1(\alpha)} && \text{(assumption)} \\ &= \chi_{-V_1(\alpha)} && \text{(Def. 5.13 (1))} \\ &= \chi_{V_1(\neg\alpha)} && \text{(Def. 5.11 (a))} \end{aligned}$$

Conjunction:

$$\begin{aligned} V_{\mathscr{E}}(\alpha\wedge\beta) &= V_{\mathscr{E}}(\alpha)\wedge V_{\mathscr{E}}(\beta) && \text{(Def. 5.16 (b))} \\ &= \chi_{V_1(\alpha)}\wedge\chi_{V_1(\beta)} && \text{(assumption)} \\ &= \chi_{[V_1(\alpha)\frown V_1(\beta)]} && \text{(Def. 5.13 (2))} \\ &= \chi_{V_1(\alpha\wedge\beta)} && \text{(Def. 5.11 (b))} \end{aligned}$$

Disjunction:

$$\begin{aligned} V_{\mathscr{E}}(\alpha\vee\beta) &= V_{\mathscr{E}}(\alpha)\vee V_{\mathscr{E}}(\beta) && \text{(Def. 5.16 (c))} \\ &= \chi_{V_1(\alpha)}\vee\chi_{V_1(\beta)} && \text{(assumption)} \\ &= \chi_{[V_1(\alpha)\smile V_1(\beta)]} && \text{(Def. 5.13 (3))} \\ &= \chi_{V_1(\alpha\vee\beta)} && \text{(Def. 5.11 (c))} \end{aligned}$$

Material conditional:

$$\begin{aligned} V_{\mathscr{E}}(\alpha\supset\beta) &= V_{\mathscr{E}}(\alpha)\supset V_{\mathscr{E}}(\beta); && \text{(Def. 5.16 (d))} \\ &= \chi_{V_1(\alpha)}\supset\chi_{V_1(\beta)} && \text{(assumption)} \\ &= \chi_{[V_1(\alpha)\sqsupset V_1(\beta)]} && \text{(Def. 5.13 (4))} \\ &= \chi_{V_1(\alpha\supset\beta)} && \text{(Def. 5.11 (d))} \end{aligned}$$

Thus, starting with the base case of the atoms (where we know that $V_{\mathscr{E}}(p_i) = \chi_{V_1(p_i)}$), we have by induction that $V_{\mathscr{E}}(\alpha) = \chi_{V_1(\alpha)}$ for all statements α. Then, if $V_1(\gamma) \neq \mathrm{id}_1$, by uniqueness of characteristic arrows to their subobjects (Prop. 2.40) we know that $\chi_{V_1(\gamma)} \neq \chi_{\mathrm{id}_1}$. We saw earlier that $\boldsymbol{t} = \chi_{\mathrm{id}_1}$, and furthermore $V_{\mathscr{E}}(\gamma) = \chi_{V_1(\gamma)}$ by the above inductive argument. Therefore, $V_{\mathscr{E}}(\gamma) \neq \boldsymbol{t}$, i.e., the valuation $V_{\mathscr{E}}$ does not satisfy γ.

Now suppose there exists a $\mathscr{E}$-valuation $V_{\mathscr{E}}$ that does not satisfy γ, so $V_{\mathscr{E}}(\gamma) \neq \boldsymbol{t}$. Using a similar process as above, we can construct an $\mathrm{Sub}(1)$-valuation that does not satisfy γ. Consider the assignment of each atom $V_{\mathscr{E}}(p_i)$, and take its pullback along $\boldsymbol{t}$ to obtain a monic $f_i : a_i \rightarrowtail 1$ (this is monic by Prop. 2.35, because $\boldsymbol{t}$ is monic). This gives that $V_{\mathscr{E}}(p_i) = \chi_{f_i}$. Construct V_1 by assigning to each p_i the corresponding pullback of $V_{\mathscr{E}}(p_i)$ along $\boldsymbol{t}$, f_i. Then $V_{\mathscr{E}}(p_i) = \chi_{V_1(p_i)}$. The same induction argument as before gives that $V_{\mathscr{E}}(\alpha) = \chi_{V_1(\alpha)}$ for all statements α, and hence $\chi_{V_1(\gamma)} = V_{\mathscr{E}}(\gamma) \neq \boldsymbol{t} = \chi_{\mathrm{id}_1}$, so $V_1(\gamma) \neq \mathrm{id}_1$ again by uniqueness of characters to subobjects (Prop. 2.40). That is, V_1 does not satisfy γ.

Thus, it is not the case that $\vDash_{\mathscr{E}} \gamma$ iff it is not the case that $\vDash_1 \gamma$. In other words, $\vDash_{\mathscr{E}} \gamma$ iff $\vDash_1 \gamma$. □

Before we conclude with the final theorem, let us pause to review. We have seen that 'topos logic' is a form of reasoning definable in the structure of arrows in $\mathscr{E}(1,\Omega)$ in a topos, where propositions are connected via the composition of product arrows and logical connective arrows.

However, this structure does not exist on its own. There is a deeper, external structure beneath it — $\mathrm{Sub}(1)$. We saw that the subobjects of the terminal object 1 form, in general, a Heyting algebra. We have seen how valuation functions can be applied to Heyting algebras in general, and how that results in intuitionistic logic. And so valuation functions applied to $\mathrm{Sub}(1)$ will be intuitionistic too. Furthermore, the previous theorem has established a one-to-one correspondence of tautologies of $\mathscr{E}$ with tautologies of $\mathrm{Sub}(1)$. This leads to our final theorem.

Theorem 5.18. *Given a topos $\mathscr{E}$, the intuitionistic proof system $\vdash_{\mathfrak{J}}$ is sound and complete with respect to the semantic relation $\vDash_{\mathscr{E}}$.*

Proof. By Theorem 5.12, $\vdash_{\mathfrak{J}}$ is sound and complete with respect to $\vDash_1$. Then by the previous theorem, $\vDash_1$ gives the same tautologies as $\vDash_{\mathscr{E}}$. Thus $\vdash_{\mathfrak{J}}$ is sound and complete with respect to $\vDash_{\mathscr{E}}$. $\square$

So the logic of a topos is, in general, an intuitionistic logic. We say 'in general' as it is certainly possible for the logic to be classical. This will happen when the logic on Sub(1) is classical, which happens when Sub(1) is a Boolean algebra. Recall from §4.3.2 that Sub(d) is Boolean precisely when $-\big(-f\big) = f$ for all $f : a \rightarrowtail d$. From Theorem 5.15, $\chi_{-(-f)} = \neg \circ \chi_{-f} = \neg \circ \neg \circ \chi_f$. Then by uniqueness of characters (Prop. 2.40) $-(-f) = f$ iff $\neg \circ \neg \circ \chi_f = \chi_f$, i.e. $\neg \circ \neg = \mathrm{id}_\Omega$. This is precisely the condition for the logic in $\mathscr{E}$ to be classical (and for *every* subobject algebra in $\mathscr{E}$ to be Boolean).[50]

We conclude this chapter with examples of toposes that exhibit classical and non-classical logics.

Example 5.19. The following toposes have different logics.

(a) Recall that in **Set** the subobject classifier is the two-valued set $\{0,1\}$. The negation arrow $\neg : \{0,1\} \to \{0,1\}$ acts by swapping the values; i.e., $\neg(0) = 1$ and $\neg(1) = 0$. Clearly this means that $\neg(\neg(x)) = x$, so $\neg \circ \neg = \mathrm{id}_\Omega$ and hence the logic on **Set** is classical.

(b) Our other example of a topos in §2.8 was $\mathbf{Set}^{\to}$. This topos has that $\neg \circ \neg \neq \mathrm{id}_\Omega$, and so the logic is intuitionistic.[51]

[50] The semantics we defined in Definition 5.16 are external to the topos. They utilise a mapping $V_{\mathscr{E}}$ that has no representation *in* $\mathscr{E}$. Furthermore, the set of wffs $\mathscr{W}$ is external. An internal structure can be built up, however. This is done by composing the logical connective arrows with each other to build more complex statements, but they remain as arrows from $\Omega \times \Omega \times \cdots \times \Omega$ to Ω (a new Ω is introduced in the domain every time we utilise a product arrow, like how $V_{\mathscr{E}}$ is recursively established in diagram (5.24)). The ability to construct statements *within* a topos without a reliance on external elements is what makes topos logic special from a foundational point of view. It still carries the same properties as our external semantics we have discussed, in that $\neg \circ \neg = \mathrm{id}_\Omega$ precisely when the logic validates the law of excluded middle.

[51] $\mathbf{Set}^{\to}$ can also be viewed as a category of functors (which we have not introduced) from the category **2** to **Set**, written as $\mathbf{Set}^{\mathbf{2}}$. It turns out that for any (small) category $\mathscr{C}$, the category $\mathbf{Set}^{\mathscr{C}}$ of functors from $\mathscr{C}$ to **Set** is a topos. Considering the logic on this topos, the interpretation is that each **Set** gives a site for classical logic, but the overall structure $\mathbf{Set}^{\mathscr{C}}$ resembles an intuitionistic logic. This resembles the notion of a *Kripke frame* as a model for non-classical logics.

Chapter 6

Conclusion

Review

The logic in a topos is a powerful tool; so much so that mathematicians like Lawvere and Lambek have posited it as an alternate foundation for the entirety of mathematics [Law66; LS11]. This dissertation began with two questions: What is topos logic? Why is it said to be intuitionistic? Answering these questions required introducing a great deal of mathematical machinery.

We began with an introduction to the domain of categories in Chapter 2. We encountered some important concepts like duality and universal constructions, before moving to define the specific type of category that forms the basis for our later discussion: a topos.

In Chapter 3 we moved from the arrows and objects of category theory, and looked at concepts of order. We introduced what it means for a set to form a poset, a lattice, and a Boolean or Heyting algebra. We examined the connection between these two algebras, and saw that a Boolean algebra is precisely a Heyting algebra where double complementation eliminates.

We returned to arrow-theoretic constructions in Chapter 4. A partial ordering on the subobjects of an object was identified, and we were able to further identify meets, joins, relative pseudo-complements and pseuo-complements, giving a Heyting algebra structure.

In Chapter 5 we began to think about logic. We introduced the basics of formal propositional logic: an alphabet, semantics, proof, and how these ideas relate through soundness and completeness. We saw how truth is evaluated through a valuation function, and how such valuation functions can be defined on the subobject algebras in a topos. This gave an intuitionistic logic; it did not validate the law of excluded middle.

The subobject algebra is a structure external to the topos, so we lifted the logic out of the subobjects and into the topos through the subobject classifier and specific arrows that take the role of logical connectives. From here, we could define a valuation function on the topos itself, giving an answer to the question: what is topos logic? Despite lifting the logic out of the subobject structure, we saw how validity on the topos was still intimately connected to validity on the subobject algebra — they have the same tautologies. And thus because the subobject logic is intuitionistic, so too is the topos logic. This answered our second question: why is topos logic intuitionistic? We concluded with an account of when this topos logic becomes classical: it is precisely when double negation eliminates.

A difference in logics

As mathematicians, we often take our choice of logic for granted. Our definitions and results are built upon a logical foundation that is commonly taken to be standard and proper — classical logic. While it may be considered the standard logical system, some logicians are of the view that it is no more (or even less) entitled than any other logical system, of which there are many.[52]

An intuitionistic logic allows statements to be neither true nor false. What about a logic that allows statements to be both true and false? This is nearing what is known as a *paraconistent logic*, where statements can be inconsistent without causing problems throughout the entire logical structure. This has real use when structures may be known to contain inconsistencies, such as databases or human beliefs. Given that intuitionistic logics can be represented in toposes, are there categories that represent paraconsistent logics?

The intuitionistic nature of topos logic came from the Heyting algebra structure on Sub(1). Therefore, for the categorial logic to be paraconsistent we would expect the structure on Sub(1) to be something that models paraconsistent logic. The Heyting structure on Sub(1) arose from the account of relative pseudo-complementation and pseudo-complementation, so these would need to be modified to achieve a different type of algebra. Mortensen and Lavers have proceeded this way in [ML95] by defining a *paraconsistent algebra* which is

[52] See [Pri08] for many different such logics, each with benefits and weaknesses of reasoning.

somewhat dual to a Heyting algebra, and showing how one can arise in a category by using a so-called ‘complement-classifier’ — a modification of the subobject classifier. Then, using some modified definitions for logical connective arrows and a valuation function, a paraconsistent logic can be found in a type of category they call a ‘complement-topos’. They further argue that a complement-topos is, in fact, a different way of looking at a topos, and so toposes themselves admit a paraconsistent logic. This analysis is extended by Estrada-González in [Est10] and [Est15], who concludes that the logic in a topos “is a truly protean categorial creature which can accommodate the most diverse descriptions and support an enormous variety of logics”, be them intuitionistic or paraconsistent.

Some further questions of categorial paraconsistent logic remain, however. The logic in a topos can fully axiomatise set theory and thus provide an alternate foundation for mathematics. Can the paraconsistent logic in a complement-topos do the same? Could it form a foundation for a paraconsistent theory of mathematics? Furthermore, the logic developed by Mortensen and Lavers is not a standard paraconsistent logic. So, given a commonly studied paraconsistent logic, such as Priest’s *Logic of Paradox* [Pri79], does there exist a category that where this logic can be identified on the arrows? Answers to these questions would provide an interesting method of analysis of these non-classical logics. If many different types of mathematics (classical, intuitionistic, paraconsistent) could be internalised in categories, it would give us reason to question the tenet of classical foundations, as, in this setup, a classical foundation of mathematics would be but one choice of categorial foundation — namely, **Set**.

Bibliography

[Awo06] Steve Awodey. *Category theory*. Vol. 49. Oxford Logic Guides. New York: The Clarendon Press, Oxford University Press, 2006. ISBN: 0-19-856861-4; 978-0-19-856861-2. DOI: 10.1093/acprof:oso/9780198568612.001.0001.

[Bir67] Garrett Birkhoff. *Lattice theory*. Third. Vol. XXV. American Mathematical Society Colloquium Publications. Providence, Rhode Island: American Mathematical Society, 1967.

[Est10] Luis Estrada-González. "Complement-topoi and dual intuitionistic logic". In: *The Australasian Journal of Logic* 9 (2010), pp. 26–44. ISSN: 1448-5052. DOI: 10.26686/ajl.v9i0.1819.

[Est15] Luis Estrada-González. "From (Paraconsistent) Topos Logic to Universal (Topos) Logic". In: *The Road to Universal Logic*. Ed. by Arnold Koslow and Arthur Buchsbaum. Studies in Universal Logic. Cham: Birkhäuser, 2015. Chap. 12, pp. 263–295. ISBN: 978-3-319-15367-4. DOI: 10.1007/978-3-319-15368-1_12.

[Gol84] Robert Goldblatt. *Topoi*. Second. Vol. 98. Studies in Logic and the Foundations of Mathematics. The categorial analysis of logic. Amsterdam: North-Holland Publishing Company, 1984. ISBN: 0-444-86711-2.

[HS07] Horst Herrlich and George E. Strecker. *Category theory*. Third. Vol. 1. Sigma Series in Pure Mathematics. An introduction. Lemgo: Heldermann Verlag, 2007. ISBN: 978-3-88538-001-6.

[Joh77] P. T. Johnstone. *Topos theory*. Vol. 10. London Mathematical Society Monographs. New York: Academic Press, 1977. ISBN: 0-12-387850-0.

[Joh82] Peter T. Johnstone. *Stone spaces*. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1982. ISBN: 0-521-23893-5.

[KLM75] A. Kock, P. Lecouturier, and C. J. Mikkelsen. "Some topos theoretic concepts of finiteness". In: *Model theory and topoi (Conf., Berlin, 1973)*. 1975, 209–283. Lecture Notes in Math., Vol. 445.

[Law66] F. William Lawvere. "The category of categories as a foundation for mathematics". In: *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*. New York: Springer, 1966, pp. 1–20.

[LS11] Joachim Lambek and Philip J. Scott. "Reflections on the categorical foundations of mathematics". In: *Foundational theories of classical and constructive mathematics*. Vol. 76. Western Ontario Series in Philosophy of Science. Dordrecht: Springer, 2011, pp. 171–186. DOI: 10.1007/978-94-007-0431-2_9.

[LS86] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*. Vol. 7. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1986. ISBN: 0-521-24665-2.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. New York: Springer-Verlag, 1998. ISBN: 0-387-98403-8.

[Mad83] Penelope Maddy. "Proper classes". In: *The Journal of Symbolic Logic* 48.1 (1983), pp. 113–139. ISSN: 0022-4812. DOI: 10.2307/2273327.

[ML95] Chris Mortensen and Peter Lavers. "Category theory". In: *Inconsistent mathematics*. Ed. by Chris Mortensen. Vol. 312. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers Group, 1995. Chap. 11. ISBN: 0-7923-3186-9. DOI: 10.1007/978-94-015-8453-1.

[Pri08] Graham Priest. *An introduction to non-classical logic*. Second. Cambridge Introductions to Philosophy. Cambridge: Cambridge University Press, 2008. ISBN: 978-0-521-67026-5. DOI: 10.1017/CBO9780511801174.

[Pri79] Graham Priest. "The logic of paradox". In: *Journal of Philosophical Logic* 8.2 (1979), pp. 219–241. ISSN: 0022-3611. DOI: 10.1007/BF00258428.

[Ras74] Helena Rasiowa. *An algebraic approach to non-classical logics*. Studies in Logic and the Foundations of Mathematics, Vol. 78. Amsterdam: North-Holland Publishing Company, 1974. ISBN: 0-7204-22647.

[RS68] Helena Rasiowa and Roman Sikorski. *The mathematics of metamathematics*. Second, revised. Vol. 41. Monografie matematyczne. Warsaw: Polish Scientific Publishers, 1968.

Appendix A

Epic-monic factorisations

We first need a smaller fact concerning coequalisers.

Proposition A.1. *Suppose $q : b \to c$ coequalises $f, g : a \to b$. Then q is epic. That is, for any two arrows $h_1, h_2 : c \to d$, if $h_1 \circ q = h_2 \circ q$ then $h_1 = h_2$.*

This is Proposition 3.18 in [Awo06, p. 58], where the proof is just the dual of the proof given for Prop. 3.15 in [Awo06, p. 56]. Having established this fact, we move to the proof for Theorem 2.38.

Theorem A.2 (Epi-monic factorisation)**.** *Let $\mathscr{C}$ be a category that admits pullbacks and coequalisers. Further suppose that the pullback of any epic arrow remains epic. Then for any arrow $f : a \to b$ there exist arrows e and m such that $f = m \circ e$, where e is epic and m is monic. Moreover, if we also have that $f = v \circ u$ where v is monic, then $m = v \circ k$ for some arrow k. The middle object* $\operatorname{dom} m = \operatorname{cod} e$ *is called $f(a)$, and we call the construction the epi-monic factorisation of f.*

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, bend right, two heads, "e"'] & & b \\ & f(a) \arrow[ur, bend right, tail, "m"'] & \end{tikzcd} \tag{A.1}$$

Proof for Theorem 2.38. The proof is constructive. Begin by forming the pullback of f along itself to obtain two arrows f_1 and f_2.[53]

$$\begin{tikzcd} F \arrow[r, "f_2"] \arrow[d, "f_1"'] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd} \tag{A.2}$$

[53] (F, f_1, f_2) is known as the *kernel pair* of f.

Now let $e : a \to f(a)$ be the coequaliser of f_1 and f_2. Because f also coequalises these arrows ($f \circ f_1 = f \circ f_2$ by the definition of f_1, f_2) there exists a unique arrow $m : f(a) \dashrightarrow b$ such that $f = m \circ e$.

$$
\begin{tikzcd}
F \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & a \arrow[r, two heads, "e"] \arrow[dr, "f"'] & f(a) \arrow[d, dashed, tail, "m"] \\
& & b
\end{tikzcd}
\tag{A.3}
$$

Because e is a coequaliser it is epic by the above proposition.

Now we turn to showing that m is monic. To do this, first take the pullback of m along itself to obtain the arrows $m_1, m_2 : M \to f(a)$. We show that $m_1 = m_2$.

$$
\begin{tikzcd}
M \arrow[r, "m_2"] \arrow[d, "m_1"'] & f(a) \arrow[d, "m"] \\
f(a) \arrow[r, "m"'] & m
\end{tikzcd}
\tag{A.4}
$$

We can append the arrows $e : a \to f(a)$ onto this diagram. Then form three pullbacks:

(1) First, the pullback of e and m_1, shown as the bottom left square in Diagram (A.5). The pullbacks of m_1 and e are labelled as ϕ_1 and ϕ_2 respectively. Because e is epic and the pullback of epic arrows are epic we have that ϕ_2 is epic too.

(2) Second, the pullback of e and m_2, shown as the top right square in (A.5). The pullbacks of e and m_2 are labelled as π_1 and π_2 respectively. Similar to above, because e is epic we have that π_1 is epic.

(3) Finally, the pullback of π_1 and ϕ_2, shown in the top left in (A.5). The pullbacks of π_1 and ϕ_2 are labelled as γ_1 and γ_2 respectively. Because π_1 and ϕ_2 are epic we have that γ_1 and γ_2 are epic, and so the composition $\alpha = \pi_1 \circ \gamma_2 = \phi_2 \circ \gamma_1$ is epic. (Prop. 2.10).

$$
\begin{tikzcd}
\Gamma \arrow[r, two heads, "\gamma_2"] \arrow[d, two heads, "\gamma_1"'] \arrow[dr, two heads, "\alpha"] & \Pi \arrow[r, "\pi_2"] \arrow[d, two heads, "\pi_1"] & a \arrow[d, two heads, "e"] \arrow[dd, bend left, "f"] \\
\Phi \arrow[r, two heads, "\phi_2"'] \arrow[d, "\phi_1"'] & M \arrow[r, "m_2"'] \arrow[d, "m_1"] & f(a) \arrow[d, "m"] \\
a \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & f(a) \arrow[r, "m"] & m
\end{tikzcd}
\tag{A.5}
$$

Because all of these squares are pullbacks, by the Pullback Lemma (Lem. 2.34) we have that the entire outer square (Γ, a, a, m) is a pullback. But this is then the pullback of $f = m \circ e$ along itself, which we have already defined to give the arrows f_1, f_2 in Diagram (A.2). Pullbacks are a type of limit, and limits are unique up to isomorphism by Proposition 2.23. Hence there exists an iso arrow k between F and Γ such that the diagram

$$\begin{tikzcd}
F \arrow[dr, leftrightarrow, "k"] \arrow[drrr, bend left, "f_2"] \arrow[ddd, bend right=60, "f_1"'] \arrow[dddr, bend right, "f_1"'] & & & \\
& \Gamma \arrow[r, two heads, "\gamma_2"] \arrow[d, two heads, "\gamma_1"] \arrow[dr, two heads, "\alpha"] & \Pi \arrow[r, "\pi_2"] \arrow[d, two heads, "\pi_1"] & a \arrow[d, two heads, "e"] \arrow[dd, bend left, "f"] \\
& \Phi \arrow[r, two heads, "\phi_2"'] \arrow[d, "\phi_1"] & M \arrow[r, "m_2"'] \arrow[d, "m_1"] & f(a) \arrow[d, "m"] \\
& a \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & f(a) \arrow[r, "m"] & m
\end{tikzcd} \tag{A.6}$$

commutes.[54] Examining (A.6), we find that

$$\begin{aligned}
m_1 \circ \alpha \circ k = m_1 \circ \phi_2 \circ \gamma_1 \circ k = e \circ f_1 & \\
&= e \circ f_2 \quad \text{(as } e \text{ coequalises } f_1 \text{ and } f_2\text{)} \\
&= e \circ \pi_2 \circ \gamma_2 \circ k \\
&= m_2 \circ \pi_1 \circ \gamma_2 \circ k \\
&= m_2 \circ \alpha \circ k.
\end{aligned}$$

Since α is epic we know that $\alpha \circ k$ is epic (iso arrows are epic and monic, Prop. 2.12), so we conclude that $m_1 = m_2$.

Now, suppose there exist arrows $g, h : c \to f(a)$ where $m \circ g = m \circ h$.

$$\begin{tikzcd}
c \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & f(a) \arrow[r, "m"] & m
\end{tikzcd} \tag{A.7}$$

[54]As mentioned in Def. 2.11, the arrow k having a double arrowhead means that it commutes in either direction.

By the definition of m_1 and m_2 as the pullbacks of m along itself (A.4), there must exist a unique arrow $k : c \dashrightarrow M$ such that

$$
\begin{tikzcd}
c \arrow[drr, bend left, "g"] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "h"'] & & \\
& M \arrow[r, "m_2"] \arrow[d, "m_1"'] & f(a) \arrow[d, "m"] \\
& f(a) \arrow[r, "m"'] & m
\end{tikzcd}
\tag{A.8}
$$

commutes. Hence, noting that $m_1 = m_2$, $h = m_1 \circ k = m_2 \circ k = g$. Thus m is monic.

We have so far established the existence of m and e as a factorisation for f, and shown that m is monic and e is epic. We now show that m factors through any other monic arrow into b that makes up a factorisation of f.

Suppose there exist arrows u, v that form some other factorisation $f = v \circ u$, where v is monic. Recalling that $f \circ f_1 = f \circ f_2$ (A.2), we have that $v \circ u \circ f_1 = v \circ u \circ f_2$. Since v is monic (left cancellable) this gives that $u \circ f_1 = u \circ f_2$, i.e., u coequalises f_1 and f_2. But we have defined e to be this coequaliser, so there exists a unique arrow $k : f(a) \dashrightarrow c$ such that

$$
\begin{tikzcd}
F \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & a \arrow[r, two heads, "e"] \arrow[d, "u"'] & f(a) \arrow[dl, dashed, "k"] \arrow[d, tail, "m"] \\
& c \arrow[r, tail, "v"'] & b
\end{tikzcd}
\tag{A.9}
$$

commutes. Hence $m = v \circ k$, completing the proof. □