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Solvability for a system of Hadamard fractional multi-point boundary value problems*

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Abstract. In this paper, we study a system of Hadamard fractional multi-point boundary value problems. We first obtain triple positive solutions when the nonlinearities satisfy some bounded conditions. Next, we also obtain a nontrivial solution when the nonlinearities can be asymptotically linear growth. Furthermore, we provide two examples to illustrate our main results.

Keywords: Hadamard fractional differential equations, multi-point boundary value problems, multiple solutions, asymptotically linear growth.

1 Introduction

In this paper, we use some fixed point theorems to study the existence of solutions for the system of Hadamard fractional multi-point boundary value problems

$$\begin{aligned}
&D^q u(t) + f_1\big(t, u(t), v(t)\big) = 0, \quad 1 < t < \mathrm{e},\\
&D^q v(t) + f_2\big(t, u(t), v(t)\big) = 0, \quad 1 < t < \mathrm{e},\\
&u(1) = \delta u(1) = 0, \qquad u(\mathrm{e}) = \sum_{i=1}^{m-1} a_i u(\xi_i),\\
&v(1) = \delta v(1) = 0, \qquad v(\mathrm{e}) = \sum_{j=1}^{n-1} b_j v(\eta_j),
\end{aligned} \tag{1}$$

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where $q \in (2,3]$ is a real number, D^q is the q-order Hadamard fractional derivatives, and δ means the delta derivative, i.e., $\delta u(1) = (t\,\mathrm{d}u/\mathrm{d}t)|_{t=1}$, $\delta v(1) = (t\,\mathrm{d}v/\mathrm{d}t)|_{t=1}$. The constants a_i, b_j, ξ_i, η_j ($i = 1,2,\dots,m-1$, $j = 1,2,\dots,n-1, m,n \geqslant 2$) and f_1, f_2 satisfy the conditions:

(H0) $a_i, b_j \geqslant 0$, $\xi_i, \eta_j \in (1,\mathrm{e})$ with $\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1} \in [0,1)$, and $\sum_{j=1}^{n-1} b_j(\log\eta_j)^{q-1} \in [0,1)$;

(H1) $f_i \in C([1,\mathrm{e}] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$, $i = 1,2$.

Fractional-order equations, as a generalization of the case of integer order, can accurately characterize some complex phenomena in nature. It has been proved that there are many special advantages in some fields, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, which has become a hot research topic of common concern in the world. For example, in [5], the authors investigated the following fractional-order advection–diffusion–reaction boundary value problem:

$$\varepsilon^C D^\alpha x + \gamma x' + f(x) = S(t), \quad t \in [0,1], \qquad x(0) = x_L, \quad x(1) = x_R,$$

where $1 < \alpha \leqslant 2$, $0 < \varepsilon \leqslant 1$, $\gamma \in \mathbb{R}$, ${}^C D^\alpha$ is the fractional derivative of Caputo sense, and $S(t)$ is a spatially dependent source term.

It has been observed that most of papers in the literature on the fractional-order equations involves either Riemann–Liouville- or Caputo-type fractional derivative. Apart from the two derivatives, Hadamard derivative is another kind of fractional derivative that was introduced by Hadamard [9]. This fractional derivative differs from the other ones in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. For detailed materials of Hadamard fractional derivative and integral, we refer to the papers [1–4, 8, 10, 11, 13–23, 25–29] and references therein. In [14], the authors studied the Riemann–Liouville fractional differential inclusion with Hadamard fractional integral boundary conditions

$$\begin{aligned} &{}_{RL}D^q x(t) \in F\bigl(t, x(t)\bigr), \quad 0 < t < T,\ 1 < q \leqslant 2, \\ &x(0) = 0, \qquad x(T) = \sum_{i=1}^{n} \alpha_{i\,H} I^{p_i} x(\eta_i), \end{aligned}$$

where $1 < q \leqslant 2$, ${}_{RL}D^q$ is the Riemann–Liouville fractional derivative, ${}_H I^{p_i}$ is the Hadamard fractional integral, $\eta_i \in (0,T)$ with $\sum_{i=1}^{n} \alpha_i \eta_i^{q-1}/(q-1)^{p_i} \neq T^{q-1}$. In [29], Zhang et al. utilized the Guo–Krasnosel'skii fixed point theorem to obtain the multiple positive solutions for the Hadamard fractional integral boundary value problems

$$\begin{aligned} &D^\beta\bigl(\varphi_p\bigl(D^\alpha u(t)\bigr)\bigr) = f\bigl(t, u(t)\bigr), \quad 1 < t < \mathrm{e}, \\ &u(1) = u'(1) = u'(\mathrm{e}) = D^\alpha u(1) = 0, \\ &\varphi_p\bigl(D^\alpha u(\mathrm{e})\bigr) = \mu \int_1^{\mathrm{e}} \varphi_p\bigl(D^\alpha u(t)\bigr)\frac{\mathrm{d}t}{t}, \end{aligned}$$

where $\alpha \in (2,3]$, $\beta \in (1,2]$, $\mu \in [0,\beta]$, φ_p is the p-Laplacian, and the nonlinearity f grows $(p-1)$-superlinearly and $(p-1)$-sublinearly.

On the other hand, we note that coupled systems of fractional-order equations have also been investigated by many authors, we refer to [2, 4, 8, 11, 13, 17–21, 23, 25–27]. Ahmad and Ntouyas in [2] investigated some results for the system of Hadamard fractional differential equations

$$\begin{aligned}
&D^{\alpha}u(t) = f\bigl(t,u(t),v(t)\bigr), \quad 1<t<\mathrm{e},\ 1<\alpha\leqslant 2,\\
&D^{\beta}u(t) = g\bigl(t,u(t),v(t)\bigr), \quad 1<t<\mathrm{e},\ 1<\beta\leqslant 2,\\
&u(1)=0, \quad u(\mathrm{e}) = I^{\gamma}u(\sigma_1),\\
&v(1)=0, \qquad v(\mathrm{e}) = I^{\gamma}v(\sigma_2),
\end{aligned}$$

where I^{γ} is the Hadamard fractional integral with $\gamma > 0$. By using Leray–Schauder's alternative and Banach's contraction principle the authors obtained the existence and uniqueness of solutions, respectively. In [11], Jiang et al. adopted the fixed point index to study the existence of positive solutions for the system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions

$$\begin{aligned}
&D^{\beta}u(t) + f_1\bigl(t,u(t),v(t)\bigr) = 0, \quad 1<t<\mathrm{e},\\
&D^{\beta}v(t) + f_2\bigl(t,u(t),v(t)\bigr) = 0, \quad 1<t<\mathrm{e},\\
&u(1)=v(1)=u'(1)=v'(1)=0,\\
&u(\mathrm{e}) = \int_1^{\mathrm{e}} h(s)v(s)\frac{\mathrm{d}s}{s}, \qquad v(\mathrm{e}) = \int_1^{\mathrm{e}} g(s)u(s)\frac{\mathrm{d}s}{s},
\end{aligned}$$

where the nonlinearities f_i $(i=1,2)$ can grow superlinearly and sublinearly.

Inspired by the works above, in this paper, we use some fixed point methods to study the existence of solutions for (1). We first obtain triple positive solutions when the nonlinearities satisfy some bounded conditions. Next, we also obtain a nontrivial solution when the nonlinearities can be asymptotically linear growth. Finally, we offer two examples to illustrate our main results.

The outline of the paper is organized as follows. In Section 2, we give revelent definitions and lemmas, and some important properties of the corresponding Green's function are also obtained. In Section 3, we give the detailed proofs for the existence theorems. In Section 4, we present two examples to illustrate our main results.

2 Preliminaries

In this section, we only provide the definition of the Hadamard fractional derivative, for more details we refer the reader to [1].

Definition 1. (See [1].) The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{\mathrm{d}s}{s}, \quad n-1 < q < n,$$

where $n = [q]+1$, $[q]$ denotes the integer part of the real number q, and $\log(\cdot) = \log_{\mathrm{e}}(\cdot)$.

Lemma 1. *Let $y \in C[1,\mathrm{e}]$. Then the Hadamard fractional multi-point boundary value problems*

$$\begin{aligned} &D^q u(t) + y(t) = 0, \quad 1 < t < \mathrm{e}, \\ &u(1) = \delta u(1) = 0, \qquad u(\mathrm{e}) = \sum_{i=1}^{m-1} a_i u(\xi_i) \end{aligned}$$

has a solution, which can take the form $u(t) = \int_1^{\mathrm{e}} G_1(t,s)y(s)(\mathrm{d}s/s)$, where

$$G_1(t,s) = G_0(t,s) + \frac{(\log t)^{q-1}}{1 - \sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1}} \sum_{i=1}^{m-1} a_i G_0(\xi_i, s), \quad t,s \in [1,\mathrm{e}],$$

$$G_0(t,s) = \frac{1}{\Gamma(q)}\begin{cases} (\log t)^{q-1}(1-\log s)^{q-1} - (\log t - \log s)^{q-1}, & 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ (\log t)^{q-1}(1-\log s)^{q-1}, & 1 \leqslant t \leqslant s \leqslant \mathrm{e}. \end{cases}$$

Proof. Using Lemmas 2–3 of [27], we have

$$\begin{aligned} u(t) &= c_1(\log t)^{q-1} + c_2(\log t)^{q-2} + c_3(\log t)^{q-3} \\ &\quad - \frac{1}{\Gamma(q)}\int_1^t (\log t - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s}, \end{aligned}$$

where $c_i \in \mathbb{R}$ ($i = 1,2,3$). Note that from $u(1) = \delta u(1) = 0$ we have $c_2 = c_3 = 0$. Consequently, we obtain

$$u(t) = c_1(\log t)^{q-1} - \frac{1}{\Gamma(q)}\int_1^t (\log t - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s}. \tag{2}$$

From $u(\mathrm{e}) = \sum_{i=1}^{m-1} a_i u(\xi_i)$ we obtain

$$\begin{aligned} &c_1 - \frac{1}{\Gamma(q)}\int_1^{\mathrm{e}} (1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\ &\quad = c_1 \sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1} - \frac{1}{\Gamma(q)}\sum_{i=1}^{m-1} a_i \int_1^{\xi_i} (\log \xi_i - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s}. \end{aligned}$$

This, together with (H0), implies that

$$c_1 = \frac{1}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \int_1^{\mathrm{e}} (1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} - \frac{1}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} a_i \int_1^{\xi_i} (\log\xi_i - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s}.$$

Therefore, by (2) we have

$$\begin{aligned}
u(t) &= \frac{1}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \int_1^{\mathrm{e}} (\log t)^{q-1}(1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{(\log t)^{q-1}}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} a_i \int_1^{\xi_i} (\log\xi_i - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(q)} \int_1^{t} (\log t - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&= \frac{1}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \int_1^{\mathrm{e}} (\log t)^{q-1}(1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(q)} \int_1^{\mathrm{e}} (\log t)^{q-1}(1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{(\log t)^{q-1}}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} a_i \int_1^{\xi_i} (\log\xi_i - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(q)} \int_1^{t} (\log t - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(q)} \int_1^{\mathrm{e}} (\log t)^{q-1}(1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&= \int_1^{\mathrm{e}} G_0(t,s) y(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \int_1^{\mathrm{e}} (\log t)^{q-1}(1-\log s)^{q-1} y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{(\log t)^{q-1}}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} a_i \int_1^{\xi_i} (\log\xi_i - \log s)^{q-1} y(s)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^{\mathrm{e}} G_0(t,s)y(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{q-1}}{1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}\Bigg[\frac{1}{\Gamma(q)}\sum_{i=1}^{m-1} a_i\int_1^{\mathrm{e}} (\log\xi_i)^{q-1}(1-\log s)^{q-1}y(s)\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(q)}\sum_{i=1}^{m-1} a_i\int_1^{\xi_i}(\log\xi_i-\log s)^{q-1}y(s)\frac{\mathrm{d}s}{s}\Bigg] \\
&= \int_1^{\mathrm{e}} G_0(t,s)y(s)\frac{\mathrm{d}s}{s} + \frac{(\log t)^{q-1}}{1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}\sum_{i=1}^{m-1} a_i\int_1^{\mathrm{e}} G_0(\xi_i,s)y(s)\frac{\mathrm{d}s}{s} \\
&= \int_1^{\mathrm{e}} G_1(t,s)y(s)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

This completes the proof. □

Lemma 2. (*See* [24].) *The function G_0 has the following inequalities*:

$$
\begin{aligned}
&\text{(i)}\quad (\log t)^{q-1}(1-\log t)\log s(1-\log s)^{q-1} \\
&\qquad\leqslant \Gamma(q)G_0(t,s) \leqslant (q-1)\log s(1-\log s)^{q-1}, \quad t,s\in[1,\mathrm{e}], \\
&\text{(ii)}\quad (\log t)^{q-1}(1-\log t)\log s(1-\log s)^{q-1} \\
&\qquad\leqslant \Gamma(q)G_0(t,s) \leqslant (q-1)(\log t)^{q-1}(1-\log t), \quad t,s\in[1,\mathrm{e}].
\end{aligned}
$$

Lemma 3.

$$
G_1(t,s) \geqslant \omega(t)G_1(\tau,s), \quad \omega(t) := \frac{1}{q-1}(\log t)^{q-1}(1-\log t), \quad t,s,\tau\in[1,\mathrm{e}].
$$

Proof. From Lemma 2(i) we have

$$
\begin{aligned}
G_1(t,s) &\leqslant \frac{q-1}{\Gamma(q)}\log s(1-\log s)^{q-1} \\
&\quad + \frac{q-1}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)}\sum_{i=1}^{m-1}\Gamma(q)a_iG_0(\xi_i,s)
\end{aligned}
$$

and

$$
\begin{aligned}
G_1(t,s) &\geqslant \frac{1}{\Gamma(q)}(\log t)^{q-1}(1-\log t)\log s(1-\log s)^{q-1} \\
&\quad + \frac{(\log t)^{q-1}}{(1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1})\Gamma(q)}\sum_{i=1}^{m-1}\Gamma(q)a_iG_0(\xi_i,s)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{1}{\Gamma(q)}(\log t)^{q-1}(1-\log t)\log s(1-\log s)^{q-1} \\
&\quad + \frac{(\log t)^{q-1}(1-\log t)}{(1-\sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} \Gamma(q) a_i G_0(\xi_i, s) \\
&= \frac{1}{q-1}(\log t)^{q-1}(1-\log t)\Biggl[\frac{q-1}{\Gamma(q)}\log s(1-\log s)^{q-1} \\
&\quad + \frac{q-1}{(1-\sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1})\Gamma(q)} \sum_{i=1}^{m-1} \Gamma(q) a_i G_0(\xi_i, s)\Biggr] \\
&\geqslant \frac{1}{q-1}(\log t)^{q-1}(1-\log t) G_1(\tau, s), \quad t, s, \tau \in [1, \mathrm{e}].
\end{aligned}
$$

This completes the proof. □

Let $E := C[1,\mathrm{e}]$, $\|u\| = \max_{t\in[1,\mathrm{e}]} |u(t)|$, and $P := \{u \in E\colon\ u(t) \geqslant 0\ \forall t \in [1,\mathrm{e}]\}$. Then $(E, \|\cdot\|)$ becomes a real Banach space, and P is a cone on E. Moreover, $E \times E$ is a Banach space with the norm $\|(u,v)\| = \|u\| + \|v\|$, and $P \times P$ is a cone on $E \times E$. Let

$$
G_2(t,s) = G_0(t,s) + \frac{(\log t)^{q-1}}{1-\sum_{j=1}^{n-1} b_j(\log \eta_j)^{q-1}} \sum_{j=1}^{n-1} b_j G_0(\eta_j, s), \quad t, s \in [1,\mathrm{e}].
$$

Then from Lemma 1 we obtain that (1) is equivalent to the following system of Hammerstein-type integral equations:

$$
\begin{aligned}
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} \int_1^{\mathrm{e}} G_1(t,s) f_1(s, u(s), v(s)) \frac{\mathrm{d}s}{s} \\ \int_1^{\mathrm{e}} G_2(t,s) f_2(s, u(s), v(s)) \frac{\mathrm{d}s}{s} \end{pmatrix} \\
&:= \begin{pmatrix} A_1(u,v)(t) \\ A_2(u,v)(t) \end{pmatrix}, \quad u, v \in P,\ t \in [1,\mathrm{e}].
\end{aligned}
$$

Therefore, we can define an operator $A : P \times P \to P \times P$ as follows:

$$
A(u,v)(t) = (A_1, A_2)(u,v)(t), \quad u, v \in P,\ t \in [1,\mathrm{e}].
$$

Note that G_i and f_i $(i = 1,2)$ are nonnegative continuous functions, so the operators A_i: $P \times P \to P$ $(i = 1,2)$ and A: $P \times P \to P \times P$ are three completely continuous operators. Moreover, if $(\overline{u}, \overline{v}) \in (P \times P) \setminus \{\mathbf{0}\}$ is a fixed point of A, then $(\overline{u}, \overline{v})$ is a positive solution for (1). Therefore, in what follows, we turn to study the existence of fixed points of the operator A.

Lemma 4. *Let* $\eta = \min_{t\in[5/4,3\mathrm{e}/4]} \omega(t)$. *Then* $A_i(P,P) \subset P_0$ $(i = 1,2)$*, where*

$$
P_0 = \Bigl\{ y \in P\colon \min_{t\in[5/4,3\mathrm{e}/4]} y(t) \geqslant \eta \|y\| \Bigr\}.
$$

Proof. From Lemma 3 we have

$$\begin{aligned} A_1(u,v)(t) &= \int_1^{\mathrm{e}} G_1(t,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} \\ &\geqslant \omega(t)\int_1^{\mathrm{e}} G_1(\tau,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} \\ &\geqslant \omega(t)A_1(u,v)(\tau), \quad u,v\in P,\ t,\tau\in[1,\mathrm{e}]. \end{aligned}$$

This implies that

$$A_1(u,v)(t) \geqslant \omega(t)\|A_1(u,v)\|, \quad t\in[1,\mathrm{e}].$$

Consequently, if $t\in[5/4,3\mathrm{e}/4]$, we obtain

$$\omega(t)\big\|A_1(u,v)\big\| \geqslant \min_{t\in[5/4,3\mathrm{e}/4]}\omega(t)\big\|A_1(u,v)\big\| = \eta\big\|A_1(u,v)\big\|,$$

and then

$$\min_{t\in[5/4,3\mathrm{e}/4]} A_1(u,v)(t) \geqslant \eta\big\|A_1(u,v)\big\|.$$

On the other hand, using the method of Lemma 3, we also have $G_2(t,s)\geqslant\omega(t)G_2(\tau,s)$ for $t,s,\tau\in[1,\mathrm{e}]$, and thus $A_2(P,P)\subset P_0$. This completes the proof. □

Let γ, β, θ be nonnegative continuous convex functionals on P, and let α, ψ be nonnegative continuous concave functionals on P; then for nonnegative numbers h', a', b', d' and c', convex sets are defined:

$$\begin{aligned} P(\gamma,c') &= \big\{y\in P\colon\ \gamma(y)<c'\big\}, \\ P(\gamma,\alpha,a',c') &= \big\{y\in P\colon\ a'\leqslant\alpha(y);\ \gamma(y)\leqslant c'\big\}, \\ Q(\gamma,\beta,d',c') &= \big\{y\in P\colon\ \beta(y)\leqslant d';\ \gamma(y)\leqslant c'\big\}, \\ P(\gamma,\theta,\alpha,a',b',c') &= \big\{y\in P\colon\ a'<\alpha(y);\ \theta(y)\leqslant b';\ \gamma(y)\leqslant c'\big\}, \\ Q(\gamma,\beta,\psi,h',d',c') &= \big\{y\in P\colon\ h'\leqslant\psi(y);\ \beta(y)\leqslant d';\ \gamma(y)\leqslant c'\big\}. \end{aligned}$$

Lemma 5. (*See* [6].) *Let P be a cone in the real Banach space E. Suppose that α and ψ are nonnegative continuous concave functionals on P and γ,β,θ are nonnegative continuous convex functionals on P such that, for some positive numbers c' and e', $\alpha(y)\leqslant\beta(y)$ and $\|y\|\leqslant\mathrm{e}'\gamma(y)$ for all $y\in\overline{P(\gamma,c')}$. Suppose further that $T\colon\ \overline{P(\gamma,c')}\to\overline{P(\gamma,c')}$ is completely continuous and there exist constants h', d', a', and $b'\geqslant 0$ with $0<d'<a'$ such that each of the following is satisfied*:

- (B1) $\{y\in P(\gamma,\theta,\alpha,a',b',c')\colon\ \alpha(y)>a'\}\neq\emptyset$ *and* $\alpha(Ty)>a'$ *for* $y\in P(\gamma,\theta,\alpha,a',b',c')$;
- (B2) $\{y\in Q(\gamma,\beta,\psi,h',d',c')\colon\ \beta(y)>d'\}\neq\emptyset$ *and* $\beta(Ty)>d'$ *for* $y\in Q(\gamma,\beta,\psi,h',d',c')$;

(B3) $\alpha(Ty) > a'$ *provided that* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$;
(B4) $\beta(Ty) < d'$ *provided that* $y \in Q(\gamma, \beta, \psi, h', d', c')$ *with* $\psi(Ty) < h'$.

Then T *has at least three fixed points* $y_1, y_2, y_3 \in \overline{P(y, c')}$ *such that* $\beta(y_1) < d'$, $a' < \alpha(y_2)$ *and* $d' < \beta(y_3)$ *with* $\alpha(y_3) < a'$.

Lemma 6. (*See* [12].) *Let* E *be a Banach space, and* A: $E \to E$ *be a completely continuous operator. Assume that* T: $E \to E$ *is a bounded linear operator such that* 1 *is not an eigenvalue of* T *and* $\lim_{\|u\|\to\infty} \|Au - Tu\|/\|u\| = 0$. *Then* A *has a fixed point in* E.

Remark 1. (i) If we use t, s to replace $\log t$, $\log s$ in G_0 of Lemma 1, respectively, we can obtain a function

$$\widetilde{G}_0(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{q-1}(1-s)^{q-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

This function happens to be the Green's function for the Riemann–Liouville fractional boundary value problem

$$\begin{aligned} &D_{0+}^q u(t) + y(t) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u(1) = 0, \end{aligned}$$

where $q \in (2,3]$, and D_{0+}^q is the Riemann–Liouville fractional derivative, $y \in C[0,1]$. For details, please refer to Lemma 3.1 in [24].

(ii) Note that for multi-point boundary value problems, the Green's functions may be complicated. For example, in [7], Bai studied the fractional three-point boundary value problem

$$\begin{aligned} &D_{0+}^\alpha \chi(x) + f\big(x, \chi(x)\big) = 0, \quad 0 < x < 1, \\ &\chi(0) = 0, \qquad \beta\chi(\eta) = \chi(1), \end{aligned} \tag{3}$$

where $\alpha \in (1,2]$, $\beta\eta^{\alpha-1}$, $\eta \in (0,1)$. The Green's function is

$$G(x,y) = \begin{cases} \frac{[x(1-y)]^{\alpha-1} - \beta x^{\alpha-1}(\eta-y)^{\alpha-1}(x-y)^{\alpha-1}(1-\beta\eta^{\alpha-1})}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leqslant y \leqslant x \leqslant 1,\ y \leqslant \eta, \\ \frac{[x(1-y)]^{\alpha-1} - (x-y)^{\alpha-1}(1-\beta\eta^{\alpha-1})}{(1-\beta\eta^{\alpha-1}))\Gamma(\alpha)}, & 0 < \eta \leqslant y \leqslant x \leqslant 1, \\ \frac{[x(1-y)]^{\alpha-1} - \beta x^{\alpha-1}(\eta-y)^{\alpha-1}}{(1-\beta\eta^{n-1})\Gamma(\alpha)}, & 0 \leqslant x \leqslant y \leqslant \eta < 1, \\ \frac{[x(1-y)]^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leqslant x \leqslant y \leqslant 1,\ \eta \leqslant y. \end{cases} \tag{4}$$

Note that if $\beta = 0$, then (3) reduces to the problem

$$\begin{aligned} &D_{0+}^\alpha \chi(x) + f\big(x, \chi(x)\big) = 0, \quad 0 < x < 1, \\ &\chi(0) = \chi(1) = 0. \end{aligned} \tag{5}$$

The Green's function is

$$g(x,y)=\frac{1}{\Gamma(\alpha)}\begin{cases}[x(1-y)]^{\alpha-1}-(x-y)^{\alpha-1}, & 0\leqslant y\leqslant x\leqslant 1,\\ [x(1-y)]^{\alpha-1}, & 0\leqslant x\leqslant y\leqslant 1.\end{cases}\tag{6}$$

Now, if the three-point problem (3) is considered as a perturbation of the two-point problem (5), we can use (6) to obtain (4), i.e.,

$$G(x,y)=g(x,y)+\frac{\beta x^{\alpha-1}}{1-\beta\eta^{\alpha-1}}g(\eta,y).$$

This simple idea motivates our study in Lemma 1.

Combining the above, we do not need to construct new Green's functions to obtain the equivalent Hammerstein-type integral equations for our problem (1).

3 Main results

Now, we state our main theorems, and provide their proofs.

Theorem 1. *Let* $0<a'<b'<b'/\eta<c'$, (H0)–(H1) *and the following conditions hold*:

- (H2) $f_1(t,u(t),v(t))<a'L_1$, $f_2(t,u(t),v(t))<a'L_2$, $t\in[1,\mathrm{e}]$ *and* $u+v\in[\eta a',a']$;
- (H3) $f_1(t,u(t),v(t))>b'M_1$, $f_2(t,u(t),v(t))>b'M_2$, $t\in I$ *and* $u+v\in[b',b'/\eta]$;
- (H4) $f_1(t,u(t),v(t))<c'L_1$, $f_2(t,u(t),v(t))<c'L_2$, $t\in[1,\mathrm{e}]$ *and* $u+v\in[0,c']$, *where*

$$L_1=\frac{\Gamma(q+2)}{2(q-1)}\left[1+\frac{\sum_{i=1}^{m-1}a_i}{1-\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}}\right]^{-1},$$

$$L_2=\frac{\Gamma(q+2)}{2(q-1)}\left[1+\frac{\sum_{j=1}^{n-1}b_j}{1-\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}}\right]^{-1},$$

$$M_1=\frac{1}{2\eta\int_{5/4}^{3\mathrm{e}/4}k(s)\frac{\mathrm{d}s}{s}}\left[1+\frac{\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}(1-\log\xi_i)}{1-\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}}\right]^{-1},$$

$$M_2=\frac{1}{2\eta\int_{5/4}^{3\mathrm{e}/4}k(s)\frac{\mathrm{d}s}{s}}\left[1+\frac{\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}(1-\log\eta_j)}{1-\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}}\right]^{-1},$$

and

$$k(s)=\frac{q-1}{\Gamma(q)}\log s(1-\log s)^{q-1},\quad s\in[1,\mathrm{e}].$$

Then (1) *has at least triple positive solutions.*

Proof. From Lemma 4 we have

$$\begin{aligned}&\min_{t\in[5/4,3\mathrm{e}/4]}\bigl\{A_1(u,v)(t)+A_2(u,v)(t)\bigr\}\\&\quad\geqslant\eta\bigl(\bigl\|A_1(u,v)\bigr\|+\bigl\|A_2(u,v)\bigr\|\bigr)=\eta\bigl\|A(u,v)\bigr\|.\end{aligned}$$

Therefore, for our conclusions, we need to define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P_0 by

$$\begin{aligned}\alpha(u,v)&=\min_{t\in I}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\},\\\psi(u,v)&=\min_{t\in I_1}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\},\\\gamma(u,v)&=\max_{t\in[1,\mathrm{e}]}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\},\\\beta(u,v)&=\max_{t\in I_1}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\},\\\theta(u,v)&=\max_{t\in I}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\},\end{aligned}$$

where $I=[5/4,3\mathrm{e}/4]$, $I_1=[3/2,2]$. For any $(u,v)\in P_0$, we have

$$\begin{aligned}\alpha(u,v)&=\min_{t\in I}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\}\leqslant\max_{t\in I_1}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\}=\beta(u,v),\\\bigl\|(u,v)\bigr\|&\leqslant\frac{1}{\eta}\min_{t\in I}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\}\leqslant\frac{1}{\eta}\max_{t\in[1,\mathrm{e}]}\bigl\{\bigl|u(t)\bigr|+\bigl|v(t)\bigr|\bigr\}=\frac{1}{\eta}\gamma(u,v).\end{aligned}$$

We show that A: $\overline{P(\gamma,c')}\to\overline{P(\gamma,c')}$. Indeed, if $(u,v)\in\overline{P(\gamma,c')}$, then we have $0\leqslant u(t)+v(t)\leqslant c'$ for $t\in[1,\mathrm{e}]$. Consequently, (H4) is used to obtain

$$\begin{aligned}\gamma\bigl(A(u,v)\bigr)(t)&=\max_{t\in[1,\mathrm{e}]}\bigl\{A_1(u,v)(t)+A_2(u,v)(t)\bigr\}\\&\leqslant\int_1^{\mathrm{e}}k(s)\left[1+\frac{\sum_{i=1}^{m-1}a_i}{1-\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}}\right]f_1\bigl(s,u(s),v(s)\bigr)\frac{\mathrm{d}s}{s}\\&\quad+\int_1^{\mathrm{e}}k(s)\left[1+\frac{\sum_{j=1}^{n-1}b_j}{1-\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}}\right]f_2\bigl(s,u(s),v(s)\bigr)\frac{\mathrm{d}s}{s}\\&<\frac{q-1}{\Gamma(q+2)}\left[1+\frac{\sum_{i=1}^{m-1}a_i}{1-\sum_{i=1}^{m-1}a_i(\log\xi_i)^{q-1}}\right]c'L_1\\&\quad+\frac{q-1}{\Gamma(q+2)}\left[1+\frac{\sum_{j=1}^{n-1}b_j}{1-\sum_{j=1}^{n-1}b_j(\log\eta_j)^{q-1}}\right]c'L_2\\&=c'.\end{aligned}$$

Now, (B1) and (B2) of Lemma 5 are to be verified. Note that

$$\frac{b' + (b'/\eta)}{2} \in \left\{(u,v) \in P\left(\gamma, \theta, \alpha, b', \frac{b'}{\eta}, c'\right)\colon\ \alpha(u,v) > b'\right\} \neq \emptyset,$$
$$\frac{\eta a' + a'}{2} \in \left\{(u,v) \in Q(\gamma, \beta, \psi, \eta a', a', c')\colon\ \beta(u,v) < a'\right\} \neq \emptyset.$$

Therefore, if $(u,v) \in P(\gamma,\theta,\alpha,b',b'/\eta,c')$, then $b' \leqslant u(t)+v(t) \leqslant b'/\eta$ for $t \in I$; if $(u,v) \in Q(\gamma,\beta,\psi,\eta a',a',c')$, then $\eta a' \leqslant u(t)+v(t) \leqslant a'$ for $t \in I_1$. As a result, (H3) enables us to find

$$\begin{aligned}
&\alpha\big(A(u,v)\big)(t)\\
&\quad= \min_{t\in I}\left\{\big|A_1(u,v)(t)\big| + \big|A_2(u,v)(t)\big|\right\}\\
&\quad\geqslant \min_{t\in I}\left\{\int_1^{\mathrm{e}} G_1(t,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(t,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}\\
&\quad\geqslant \min_{t\in I}(\log t)^{q-1}(1-\log t)\\
&\qquad\times\left\{\int_1^{\mathrm{e}}\left[1+\frac{\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}(1-\log\xi_i)}{1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}\right]\frac{\log s(1-\log s)^{q-1}}{\Gamma(q)}\right.\\
&\qquad\qquad\times f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\\
&\qquad+\int_1^{\mathrm{e}}\left[1+\frac{\sum_{j=1}^{n-1} b_j(\log\eta_j)^{q-1}(1-\log\eta_j)}{1-\sum_{j=1}^{n-1} b_j(\log\eta_j)^{q-1}}\right]\frac{\log s(1-\log s)^{q-1}}{\Gamma(q)}\\
&\qquad\qquad\left.\times f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}\\
&\quad> (q-1)\eta\left\{\int_I\left[1+\frac{\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}(1-\log\xi_i)}{1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}\right]\frac{\log s(1-\log s)^{q-1}}{\Gamma(q)}b'M_1\frac{\mathrm{d}s}{s}\right.\\
&\qquad\left.+\int_I\left[1+\frac{\sum_{j=1}^{n-1} b_j(\log\eta_j)^{q-1}(1-\log\eta_j)}{1-\sum_{j=1}^{n-1} b_j(\log\eta_j)^{q-1}}\right]\frac{\log s(1-\log s)^{q-1}}{\Gamma(q)}b'M_2\frac{\mathrm{d}s}{s}\right\}\\
&\quad= b'.
\end{aligned}$$

Moreover, (H2) implies that

$$\begin{aligned}
\beta\big(A(u,v)\big)(t) &= \max_{t\in I_1}\left\{A_1(u,v)(t) + A_2(u,v)(t)\right\}\\
&\leqslant \int_1^{\mathrm{e}} k(s)\left[1+\frac{\sum_{i=1}^{m-1} a_i}{1-\sum_{i=1}^{m-1} a_i(\log\xi_i)^{q-1}}\right] f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_1^{\mathrm{e}} k(s)\left[1 + \frac{\sum_{j=1}^{n-1} b_j}{1 - \sum_{j=1}^{n-1} b_j (\log \eta_j)^{q-1}}\right] f_2\big(s, u(s), v(s)\big)\frac{\mathrm{d}s}{s} \\
&< \frac{q-1}{\Gamma(q+2)}\left[1 + \frac{\sum_{i=1}^{m-1} a_i}{1 - \sum_{i=1}^{m-1} a_i (\log \xi_i)^{q-1}}\right] a' L_1 \\
&\quad + \frac{q-1}{\Gamma(q+2)}\left[1 + \frac{\sum_{j=1}^{n-1} b_j}{1 - \sum_{j=1}^{n-1} b_j (\log \eta_j)^{q-1}}\right] a' L_2 \\
&= a'.
\end{aligned}
$$

Next, (B3) of Lemma 5 is satisfied. Let $(u, v) \in P(\gamma, \alpha, b', c')$ with $\theta(A(u, v))(t) > b'/\eta$. Therefore, for all $\tau \in [1, \mathrm{e}]$, we have

$$
\begin{aligned}
&\alpha\big(A(u,v)\big)(t) \\
&\quad = \min_{t\in I} \big\{\big|A_1(u,v)(t)\big| + \big|A_2(u,v)(t)\big|\big\} \\
&\quad \geqslant \min_{t\in I} \omega(t)\left\{\int_1^{\mathrm{e}} G_1(\tau,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(\tau,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\} \\
&\quad = \eta\left\{\int_1^{\mathrm{e}} G_1(\tau,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(\tau,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}.
\end{aligned}
$$

Note that the last line of the above is independent of the variable t, and thus we obtain

$$
\begin{aligned}
&\alpha\big(A(u,v)\big)(t) \\
&\quad \geqslant \eta \max_{\tau\in[1,\mathrm{e}]} \left\{\int_1^{\mathrm{e}} G_1(\tau,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(\tau,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\} \\
&\quad \geqslant \eta \max_{\tau\in I} \left\{\int_1^{\mathrm{e}} G_1(\tau,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(\tau,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\} \\
&\quad \geqslant \eta\theta\big(A(u,v)\big)(\tau) > b'.
\end{aligned}
$$

Finally, we prove that (B4) holds. Let $(u, v) \in Q(\gamma, \beta, a', c')$ with $\psi(A(u, v))(t) < \eta a'$. Note that $\min_{t\in I} G_i(t, s) \geqslant \eta G_i(\tau, s)$ for $\tau, s \in [1, \mathrm{e}]$, and thus $\min_{t\in I} G_i(t, s) \geqslant \eta \max_{\tau\in[1,\mathrm{e}]} G_i(\tau, s)$ for $s \in [1, \mathrm{e}]$, $i = 1, 2$. Therefore, we have

$$
\begin{aligned}
&\beta\big(A(u,v)\big)(t) \\
&\quad = \max_{t\in I_1} \big\{A_1(u,v)(t) + A_2(u,v)(t)\big\} \\
&\quad = \max_{t\in I_1} \left\{\int_1^{\mathrm{e}} G_1(t,s) f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} G_2(t,s) f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \max_{t\in[1,\mathrm{e}]}\left\{\int_1^{\mathrm{e}} G_1(t,s)f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}+\int_1^{\mathrm{e}} G_2(t,s)f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}\\
&\leqslant \frac{1}{\eta}\min_{t\in I}\left\{\int_1^{\mathrm{e}} G_1(t,s)f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}+\int_1^{\mathrm{e}} G_2(t,s)f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}\\
&\leqslant \frac{1}{\eta}\min_{t\in I_1}\left\{\int_1^{\mathrm{e}} G_1(t,s)f_1\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}+\int_1^{\mathrm{e}} G_2(t,s)f_2\big(s,u(s),v(s)\big)\frac{\mathrm{d}s}{s}\right\}\\
&= \frac{1}{\eta}\psi\big(A(u,v)\big)(t) < a'.
\end{aligned}
$$

Up to now, we have proved that all the assumptions of Lemma 5 are satisfied. Therefore, (1) has at least triple positive solutions, (x_1,x_2), (y_1,y_2), and (z_1,z_2) such that $\beta(x_1,x_2) < a'$, $b' < \alpha(y_1,y_2)$, and $a' < \beta(z_1,z_2)$ with $\alpha(z_1,z_2) < b'$. This completes the proof. □

Theorem 2. *Suppose that* (H0) *and the following conditions hold*:

- (H1′) $f_i \in C([1,\mathrm{e}]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $i=1,2$;
- (H5) $\lim_{|u+v|\to\infty} f_i(t,u,v)/(u+v)=\sigma_i$ *uniformly for* $t\in[1,\mathrm{e}]$, *where* $|\sigma_i|<2L_i$, $i=1,2$;
- (H6) $f_i(t,0,0)\not\equiv 0$ *for* $t\in[1,\mathrm{e}]$, $i=1,2$.

Then (1) *has at least one nontrivial solution.*

Proof. Define operators $T_i : E\times E\to E$ as follows:

$$
T_i(u,v)(t)=\sigma_i\int_1^{\mathrm{e}} G_i(t,s)\big(u(s)+v(s)\big)\frac{\mathrm{d}s}{s},\quad u,v\in E,\ t\in[1,\mathrm{e}],\ i=1,2.
$$

Now, we prove that 1 is not an eigenvalue of T_i ($i=1,2$), and we only need to consider the case $i=1$ (the case $i=2$ can be dealt with a similar method). Argument by contrary. If let $u+v=w$, then we have

$$
\sigma_1\int_1^{\mathrm{e}} G_1(t,s)w(s)\frac{\mathrm{d}s}{s}=w(t), \tag{7}
$$

and by Lemma 1 we obtain

$$
\begin{gathered}
D^q w(t)+\sigma_1 w(t)=0,\quad 1<t<\mathrm{e},\\
w(1)=\delta w(1)=0,\qquad w(\mathrm{e})=\sum_{i=1}^{m-1}a_i w(\xi_i),
\end{gathered} \tag{8}
$$

where q, δ, a_i, ξ_i ($i=1,2,\dots,m-1$) satisfy (H0). We distinguish two cases.

Case 1. $\sigma_1 = 0$. From (8) and Lemma 1 of [23] we have

$$D^q w(t) = 0 \quad \text{and} \quad w(t) = c_1(\log t)^{q-1} + c_2(\log t)^{q-2} + c_3(\log t)^{q-3},$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, 3$). By the boundary conditions in (8) we have $c_2 = c_3 = 0$. Therefore, we find

$$c_1 = c_1 \sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1},$$

and (H0) indicates that $c_1 = 0$. Consequently, we have $w(t) \equiv 0$ for $t \in [1, \mathrm{e}]$. This contradicts to the definition of eigenvalue and eigenfunction.

Case 2. $\sigma_1 \neq 0$. From (7) we have

$$\begin{aligned}
\|w\| &= |\sigma_1| \max_{t\in[1,\mathrm{e}]} \left| \int_1^{\mathrm{e}} G_1(t,s)w(s)\frac{\mathrm{d}s}{s} \right| \\
&\leqslant |\sigma_1| \max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_1(t,s)\big|w(s)\big|\frac{\mathrm{d}s}{s} \leqslant |\sigma_1|\|w\| \max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_1(t,s)\frac{\mathrm{d}s}{s} \\
&\leqslant |\sigma_1|\|w\| \int_1^{\mathrm{e}} k(s)\left[1 + \frac{\sum_{i=1}^{m-1} a_i}{1 - \sum_{i=1}^{m-1} a_i(\log \xi_i)^{q-1}}\right]\frac{\mathrm{d}s}{s} \\
&= |\sigma_1|\|w\|\frac{1}{2L_1} < \|w\|.
\end{aligned}$$

This has a contradiction.

Above all, 1 is not an eigenvalue of T_i ($i = 1, 2$) as required. Hence, if we let the operator $T: E \times E \to E \times E$ as follows:

$$T(u,v)(t) = (T_1, T_2)(u,v)(t), \quad u, v \in E, t \in [1, \mathrm{e}],$$

we know $\mathbf{1} := (1, 1)$ is not an eigenvalue of T.

From (H5), for all $\varepsilon > 0$, there exist $\widetilde{M_i} > 0$ ($i = 1, 2$) such that

$$\begin{aligned}
\big|f_1(t,u,v) - \sigma_1(u+v)\big| &\leqslant \varepsilon|u+v|, \quad |u+v| \geqslant \widetilde{M_1},\ t \in [1, \mathrm{e}], \\
\big|f_2(t,u,v) - \sigma_2(u+v)\big| &\leqslant \varepsilon|u+v|, \quad |u+v| \geqslant \widetilde{M_2},\ t \in [1, \mathrm{e}].
\end{aligned}$$

Consequently, there are $\zeta_1 > 0, \zeta_2 > 0$ such that

$$\begin{aligned}
\big|f_1(t,u,v) - \sigma_1(u+v)\big| &\leqslant \varepsilon|u+v| + \zeta_1, \quad u, v \in \mathbb{R},\ t \in [1, \mathrm{e}], \\
\big|f_2(t,u,v) - \sigma_2(u+v)\big| &\leqslant \varepsilon|u+v| + \zeta_2, \quad u, v \in \mathbb{R},\ t \in [1, \mathrm{e}].
\end{aligned}$$

As a result, we have

$$\begin{aligned}
&\lim_{\|(u,v)\|\to\infty} \frac{\|A(u,v)-T(u,v)\|}{\|(u,v)\|} \\
&\quad = \lim_{\|u\|+\|v\|\to\infty} \frac{\|A_1(u,v)-T_1(u,v)\|+\|A_2(u,v)-T_2(u,v)\|}{\|u\|+\|v\|} \\
&\quad \leqslant \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} |\int_1^{\mathrm{e}} G_1(t,s)(f_1(s,u(s),v(s))-\sigma_1(u(s)+v(s)))\frac{\mathrm{d}s}{s}|}{\|u\|+\|v\|} \\
&\qquad + \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} |\int_1^{\mathrm{e}} G_2(t,s)(f_2(s,u(s),v(s))-\sigma_2(u(s)+v(s)))\frac{\mathrm{d}s}{s}|}{\|u\|+\|v\|} \\
&\quad \leqslant \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_1(t,s)|f_1(s,u(s),v(s))-\sigma_1(u(s)+v(s))|\frac{\mathrm{d}s}{s}}{\|u\|+\|v\|} \\
&\qquad + \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_2(t,s)|f_2(s,u(s),v(s))-\sigma_2(u(s)+v(s))|\frac{\mathrm{d}s}{s}}{\|u\|+\|v\|} \\
&\quad \leqslant \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_1(t,s)(\varepsilon|u(s)+v(s)|+\zeta_1)\frac{\mathrm{d}s}{s}}{\|u\|+\|v\|} \\
&\qquad + \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_2(t,s)(\varepsilon|u(s)+v(s)|+\zeta_2)\frac{\mathrm{d}s}{s}}{\|u\|+\|v\|} \\
&\quad \leqslant \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_1(t,s)\frac{\mathrm{d}s}{s}(\varepsilon\|u+v\|+\zeta_1)}{\|u\|+\|v\|} \\
&\qquad + \lim_{\|u\|+\|v\|\to\infty} \frac{\max_{t\in[1,\mathrm{e}]} \int_1^{\mathrm{e}} G_2(t,s)\frac{\mathrm{d}s}{s}(\varepsilon\|u+v\|+\zeta_2)}{\|u\|+\|v\|} \\
&\quad \leqslant \lim_{\|u\|+\|v\|\to\infty} \frac{\frac{1}{2L_1}(\varepsilon\|u+v\|+\zeta_1)}{\|u\|+\|v\|} + \lim_{\|u\|+\|v\|\to\infty} \frac{\frac{1}{2L_2}(\varepsilon\|u+v\|+\zeta_2)}{\|u\|+\|v\|} \\
&\quad \leqslant \frac{\varepsilon}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right).
\end{aligned}$$

For the arbitrariness of ε, we have $\lim_{\|(u,v)\|\to\infty} \|A(u,v)-T(u,v)\|/\|(u,v)\| = 0$. Note from (H6) that $\mathbf{0} = (0,0)$ is not a fixed point of A. Hence, from Lemma 6 we have that A has a fixed point in E, and this fixed point is nontrivial, i.e., (1) has at least one nontrivial solution. This completes the proof. □

4 Examples

In (1), let $q = 2.5$, $m = n = 2$, $a_1 = b_1 = 2$, $\xi_1 = \eta_1 = 1.5$, and then we obtain the system of Hadamard fractional three-point boundary value problems

$$\begin{aligned}
&D^{2.5}u(t) + f_1\bigl(t,u(t),v(t)\bigr) = 0, \quad 1<t<\mathrm{e}, \\
&D^{2.5}v(t) + f_2\bigl(t,u(t),v(t)\bigr) = 0, \quad 1<t<\mathrm{e},
\end{aligned} \tag{9_1}$$

$$
\begin{aligned}
&u(1) = \delta u(1) = 0, \qquad u(\mathrm{e}) = 2u(1.5),\\
&v(1) = \delta v(1) = 0, \qquad v(\mathrm{e}) = 2v(1.5).
\end{aligned} \tag{9_2}
$$

By direct calculation we obtain

$$
\begin{gathered}
\eta = 0.055, \qquad a_1(\log\xi_1)^{q-1} = b_1(\log\eta_1)^{q-1} = 0.516,\\
a_1(\log\xi_1)^{q-1}(1-\log\xi_1) = b_1(\log\eta_1)^{q-1}(1-\log\eta_1) = 0.307,\\
\int\limits_{5/4}^{3\mathrm{e}/4} \log s(1-\log s)^{1.5}\frac{\mathrm{d}s}{s} = 0.081.
\end{gathered}
$$

Therefore, we obtain

$$
L_1 = L_2 = 0.755, \qquad M_1 = M_2 = 60.86.
$$

Example 1. If we choose $a' = 1$, $b' = 10$, $c' = 900$, then $0 < a' < b' < b'/\eta < c'$. Moreover, let

$$
f_1(t,u,v) = \begin{cases} \frac{t}{60} + \frac{|\sin(u+v)|}{70} + \frac{6.5(u+v)^3}{10}, & 0 \leqslant u+v \leqslant 10,\ t \in [1,\mathrm{e}],\\ \frac{t}{60} + 650 + \frac{|\sin(u+v)|}{70}, & u+v > 10,\ t\in[1,\mathrm{e}], \end{cases}
$$

and

$$
f_2(t,u,v) = \begin{cases} \frac{t}{70} + \frac{|\cos(u+v)|}{80} + \frac{6.3(u+v)^3}{10}, & 0 \leqslant u+v \leqslant 10,\ t \in [1,\mathrm{e}],\\ \frac{t}{70} + 630 + \frac{|\cos(u+v)|}{80}, & u+v > 10,\ t\in[1,\mathrm{e}]. \end{cases}
$$

Then we obtain

$$
\begin{aligned}
\text{(i)}\quad & f_1\big(t,u(t),v(t)\big) \leqslant \frac{\mathrm{e}}{60} + \frac{1}{70} + 0.65 < a'L_1 = 0.755,\\
& f_2\big(t,u(t),v(t)\big) \leqslant \frac{\mathrm{e}}{70} + \frac{1}{80} + 0.63 < a'L_2 = 0.755, \quad t\in[1,\mathrm{e}],\\
& u+v \in [0.055, 1];\\
\text{(ii)}\quad & f_1\big(t,u(t),v(t)\big) \geqslant 650 > b'M_1 = 608.6,\\
& f_2\big(t,u(t),v(t)\big) \geqslant 630 > b'M_2 = 608.6, \quad t \in \left[\frac{5}{4}, \frac{3\mathrm{e}}{4}\right],\\
& u+v \in [10, 181.82];\\
\text{(iii)}\quad & f_1\big(t,u(t),v(t)\big) \leqslant \frac{\mathrm{e}}{60} + \frac{1}{70} + 650 < c'L_1 = 679.5,\\
& f_2\big(t,u(t),v(t)\big) \leqslant \frac{\mathrm{e}}{70} + \frac{1}{80} + 630 < c'L_2 = 679.5, \quad t\in[1,\mathrm{e}],\\
& u+v \in [0, 900].
\end{aligned}
$$

Then all the assumptions of Theorem 1 are satisfied. So, (9) has at least triple positive solutions.

Example 2. Let

$$f_1(t,u,v) = \sigma_1(u+v) + \mu_1 t + \nu_1,$$
$$f_2(t,u,v) = \sigma_2(u+v) + \mu_2 t + \nu_2,$$

where $|\sigma_i| < 1.51$, $\mu_i \neq 0$, $\nu_i \neq 0$ for $u,v \in \mathbb{R}$, $t \in [1,\mathrm{e}]$, $i = 1,2$. Then all the conditions of Theorem 2 hold. Hence, (9) has at least one nontrivial solution.

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