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Some Properties of Chaotic Modified of Bogdanov Map

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Abstract

In this research to the modified dynamics of Bogdanov's map studied, and the found sensitivity to the initial conditions of the modified map found as well as the Lyapunov exponent .the general characteristics of the map by the diffeomorpism. Finally we boosted my research with matlab to find chaotic areas

Keywords: fixed point, sensitive of the initial condition, Lyapunove exponent .

Introduction

One of the most recent theories in mathematics is chaos theory, which is not older than several decades, which deals with the non-linear motor system that shows a kind of chaotic behavior, and that this behavior is either through the inability to determine. The initial conditions or physical nature show us that the hidden system in this apparent chaos lays the foundations for studying weather forecasting and overpopulation systems. Chaos theory is the theory that the newest, however trivial and small, can evolve and become something unexpected.

$$\begin{pmatrix} x+y \\ y+ay+kx(x-1)+mxy \end{pmatrix}$$

The Bogdanov map provides a good approximation to the dynamics of the Poincaré map of periodically forced oscillators, first considered by Bogdanov [9], Takens and Arnold [10] in their study of the double zero eigenvalue singularity, they proved the existence of a dimension-two fixed point at the origin, called a Bogdanov-Takens cusp with a nonzero Jordan canonical form but the best contribution come from Arrowsmith and etal. [3, 4].

Then we studied modified bogdanov map of 2-D discrete dynamical system , this form is $B^1_{k,a,m}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y+ay+k(|x|-x)+myx \end{pmatrix}$, in the research we have replaced x^2 by $|x|$, We proved the modified Bogdanov map is almost linear map, we found all fixed point in this map and proved some chaotic properties of it, one of them is sensitive dependence on initial conditions and the second property is Lyapounov exponents finally we showed some properties of modified Bogdanov map

2. Preliminaries:-

A dynamical system is a map $K\times S \xrightarrow{\phi} S$ where S is an open set of Euclidean space and writing by $\phi\begin{pmatrix} t \\ x \end{pmatrix} = \phi_t(x)$, the map ϕ_t: $S \longrightarrow S$ satisfies (a) ϕ : $S \longrightarrow S$ is the identity , that is $\phi(x)= x$ for all x in S (b) The composition $\phi_t \circ \phi_s = \phi_{t+s}$, for each t, s in K . In case K is $B^1_{k,a,m}$ the dynamical system is described to be discrete dynamical system. In case K is real line the dynamical system is described to be continuous[2]. The map $B^1_{k,a,m}$:$R^2 \longrightarrow R^2$ is a linear map if for all s and u in R^2

$F(as+bu) = aF(s) + bF(u)$,for all real numbers a and b[7]. A map $B^1_{k,a,m}$:$R^2 \longrightarrow R^2$ is C^1, if all of its first partial derivatives exist and are continuous. $B^1_{k,a,m}$ is C^∞ , if its mixed k^{th} partial derivatives exist and are continuous for all $k\in Z^+$ [8].Let $B^1_{k,a,m}$:$R^2 \rightarrow R^2$ be a map and h is a fixed point then $B^1_{k,a,m}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h_1 \\ k_1 \end{pmatrix}$ that's H=$\begin{bmatrix} h \\ k \end{bmatrix}$ for which $F_1(h)=h$, $F_2(k)=k$ is say fixed point . let u be a subset of R^2 a map $B^1_{k,a,m}$ an always be in the form $B^1_{k,a,m}$ (u)=$\begin{bmatrix} F_1(u) \\ F_2(u) \end{bmatrix}$, for all u in U where F_1, F_2 are real valued coordinate map of F. let $B^1_{k,a,m}$:$R^2 \rightarrow R^2$ be a map and $u_0 \varepsilon R^2$, if $|JB^1_{k,a,m}(v0)| < 1$ then $B^1_{k,a,m}$ is area expansion at v_0, if $|JB^1_{k,a,m}(v0)| > 1$ then $B^1_{k,a,m}$ is area of constriction at v_0. A map$B^{1}_{k,a,m}$:$R^2 \rightarrow R^2$ is diffeomorpism proved as follows "one to one , onto , C^∞ and its inverse $(B^{1}_{k,a,m})^{-1}$: $R^2 \rightarrow R^2$.

3. General properties of modified Bogdanov map

We discuss here the general characteristics of modified Bogdanov map

Proportion(3.1):-

Modified of Bogdanov map has unique fixed point if x<0 and k≠0.

Proof:-

By the definition of fixed point we have

$\begin{pmatrix} x+y \\ y+ay+k(|x|-x)+myx \end{pmatrix}$, y=0 in above equation. We get $k(|x|-x)=0$

hance $(|x|-x)=0$. if x<0 and k≠ 0 we get x=0 therefore modified of Bogdanov map $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *has unique fixed point*

Remark(3.2):-

1.if x>0 and k≠0 we get infinite fixed point.

2.if k=0,y=0,x=0 we get infinite fixed point.

Proportion(3.3):-

If x<0 ,$k \neq 0$ and fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Then Jacobian of the modified Bogdanov map is 1+a+2k

D $B^{1}_{k,a,m}$ (V_0)=$\begin{bmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f2}{\partial y} \\ \frac{\partial g1}{\partial x} & \frac{\partial g2}{\partial y} \end{bmatrix}$ =$\begin{bmatrix} 1 & 1 \\ -2k+ym & 1+a+mx \end{bmatrix}$, so $J = detDB^{1}_{k,a,m}(V0) = 1+a+2k$

Proposition (3.4):-

if $x<0$ *and* $-2-2k<a<-2k$ then $B^{1}_{k,a,m}$ is area contracting map and If $-2k<a<-2-2k$ then $B^{1}_{k,a,m}$ is area expanding.

Proof:-

by proposition (3.3) then $|1+a+2k|<1$ hence $-2-2k<a<-2k$

therefore the $B^{1}_{k,a,m}\begin{pmatrix} x \\ y \end{pmatrix}$ is area contracting map and if $x<0$ *therefore* $-2k<a<-2-2k$ this implies that$B^{1}_{k,a,m}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area expanding.

Proposition (3.5):-

1. If x<0 and k $\neq 0$ then $B^{1}_{k,a,m}$ is one to one and onto

Proof:-

1. Let $T(x,y)=(y+x, y+ay-2kx+myx)$

T(1,0)=(1,-2k)

T(0,1)=(1,1+a)

Then $\begin{pmatrix} 1 & -2k \\ 1 & 1+a \end{pmatrix}$ if x<0 ,k≠ 0 so $B^{1}_{k,a,m}\begin{pmatrix} 1 & -2k \\ 1 & 1+a \end{pmatrix}$ using Row Echelon from ,we get $B^{1}_{k,a,m}\begin{pmatrix} 1 & -2k \\ 0 & 1+2k+a \end{pmatrix}$ map have a pivot position in every column the $B^{1}_{k,a,m}$is one to one and has a pivot position in every row then $B^{1}_{k,a,m}$is onto.

Proposition (3.6):-

If x<0 ,k≠ 0 then $B^{1}_{k,a,m}$is C^∞

Proof:-

If x<0 , $k \neq 0$ then $B^{1}_{k,a,m}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+x \\ y+ay+k|x|-kx+mxy \end{pmatrix}$

Then all first partial derivatives continues and exist, then $\frac{\partial^n f(x,y)}{\partial x^n} = 0\ \forall n \epsilon N$ *and* $\frac{\partial^n f(x,y)}{\partial y^n} = 0$, $\frac{\partial^n g(x,y)}{\partial x^n} = 0\ \forall n \geq 2$, *and* $\frac{\partial^n g(x,y)}{\partial y^n} = 0$ we have that all its $B^1_{k,a,m}$ exist $k - th$ partial derivatives continues and exist.

Remark(3.7):- The modified Bogdanov map is diffeomorphism

Proposition (3.8):-

If x<0 ,k≠ 0 and fixed point $\binom{0}{0}$ then The eigenvalues of modified Bogdanov map is $\lambda_{1,2} = \frac{(2+a)\pm\sqrt{a^2-2a-8k}}{2}$

Proof:-

To find the eigenvalues of modified Bogdanov map

$$DB^1_{k,a,m}(V_0)=\begin{bmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f2}{\partial y} \\ \frac{\partial g1}{\partial x} & \frac{\partial g2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ -2k & 1+a-\lambda \end{bmatrix} = 0\ therefore\ (1-\lambda)(1+a-\lambda)+2k = 0$$

Since $\lambda_{1,2} = \frac{(2+a)\pm\sqrt{a^2-2a-8k}}{2}$ are the eigenvalues of modified Bogdanov map.

1. Sensitive Dependence On Initial Conditions :-

In the behavior of an chaotic system which depends very accurately on the initial circumstances and therefore if the system undergoes minor changes in those circumstances resulting in significant changes in the system and its behavior and thus make long-term prophecy impossible

Definition (4-1)[1,5]

The $f: X \to X$ is said to be *sensitive dependence on initial conditions* if there exists $\varepsilon > 0$ such that for any $x_0 \in X$ and any open set $U \subset X$ containing x_0 there exists $y_0 \in U$ and $n \in Z^+$ such that $d(f^n(x_0), f^n(y_0)) > \varepsilon$. That is $\exists \varepsilon > 0, \forall x, \forall \delta > 0, \exists y \in B_\delta(x), \exists n : d(f^n(x_0), f^n(y_0)) \geq \varepsilon$ .A dynamical system has sensitive dependence on initial conditions on subset $X' \subseteq X$ if there is $\epsilon > 0$ such that for every $x \in X'$ and $\delta > 0$ there are $y \in Y$ and $n \in N$ for which $d(x,y) < \delta$ and $d(f^n(x_0), f^n(y_n)) > \varepsilon$. Although there is no universal agreement on definition of chaos, its generally agreed .The anarchic system is fundamentally dependent on initial condition of allergies

5.The Transitivity :-

Topological transitivity is a global characteristic of a dynamical system. Though the local structure of a topologically transitive dynamical system fulfills certain conditions , for example, the absence of attracting invariant sets, there is a variety of such system Say, some of them have dense periodic points while some of them may be minimal and so without any periodic point[6]

Definition (5.1) [2]:-

Let F:X→X be a dynamical system . If for every pair of nonempty open sets U and V in X , there is a n ∈ N such that , $f^n(U) \cap V \neq \emptyset$, then f has topologically transitive. Many times, the system is used to be transitive if there is an $x_0 \in X$ such that $\overline{O(x_0)}$ =X (i.e) f has a dense orbit). Both of as these definitions of transitivity are equivalent, in a wide class of spaces, including all connected compact metric spaces.

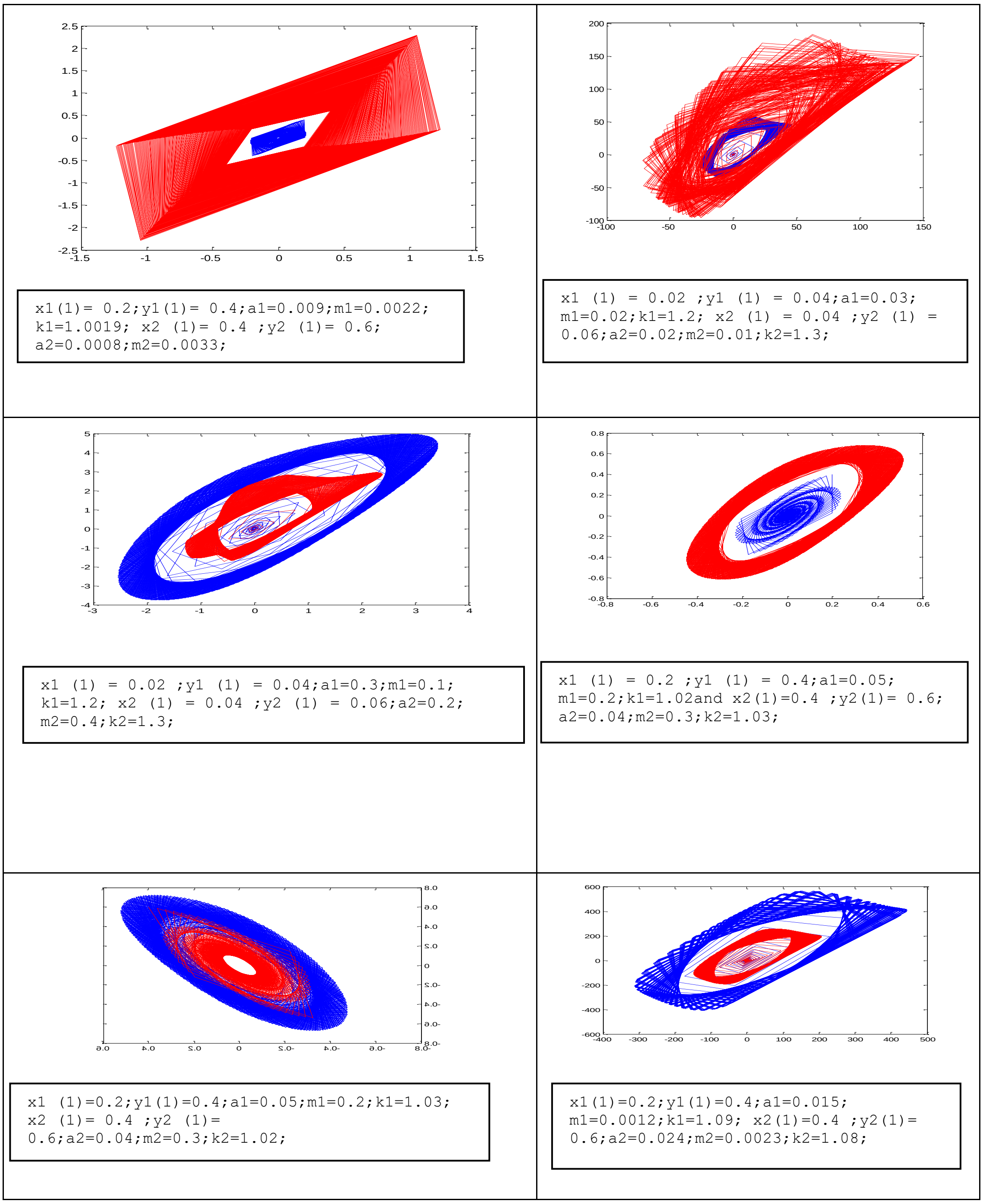

Fig (1):- Sensitive Dependence on Initial Conditions of The Modified Bogdanov map.

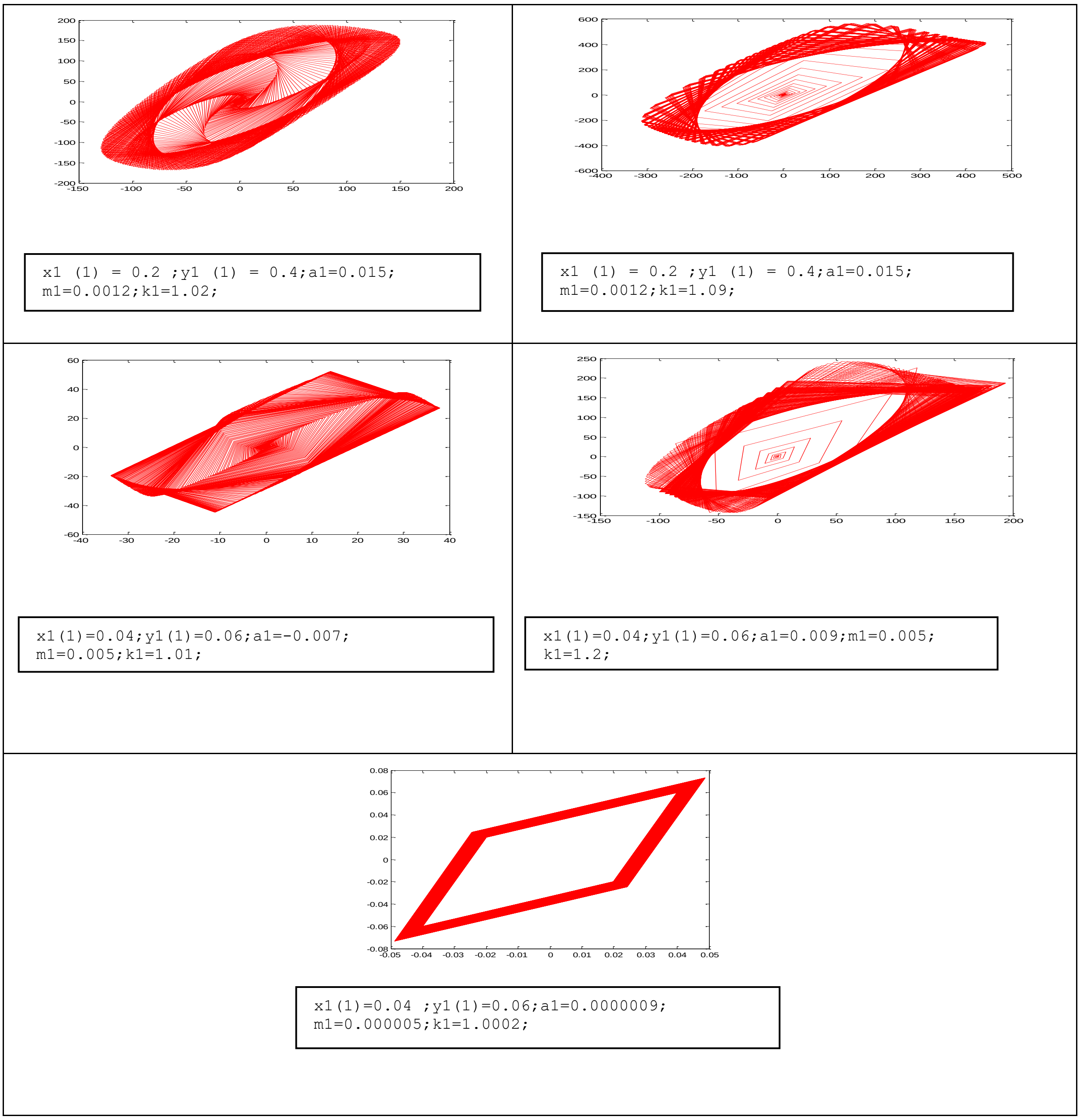

Fig (2):- The Transitive of modified of $B^1_{k,a,m}$.

6. Lyapunov exponent:-

It is a tool used to find out the chaos of the system, which provides us with a measure of the convergence or spacing of paths

Definition (6.1)[4]:-

Let F: X→X be continuous differential map, where X is any metric space. Then all x in X in direction V the Lyapunov exponent was defined of a map F at X by L(x,v)= $\lim_{n\to\infty} \frac{1}{n} \ln\| DF_x^n v\|$ whenever the limit exists in higher dimensions for example in R^n the map F will have n Lyapunov exponents, say $L_1^{\pm}(x,v_1)$,$L_2^{\pm}(x,v_2)$, … , $L_n^{\pm}(x,v_n)$, for a maximum Lyapunov exponent that is

$L_{\pm}(x,v) = Max\,\{L_1^{\pm}(x,v_1), L_2^{\pm}(x,v_2), L_3^{\pm}(x,v_3), \dots, L_n^{\pm}(x,v_n)\}$, where v=(v_1 ,v_2,…,v_n)

Proposition (6.2):-

If x<0, k≠ 0 and fixed point $\binom{0}{0}$ then $B^{1}_{k,a,m}\binom{x}{y}$ has positive Lyapunov exponent

proof:-

let $x=\binom{x}{y}\epsilon R^2$, the Lyapunov exponent of $B^{1}_{k,a,m}$ is given by the formula

$L_1\left(\binom{x}{y},v_1\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left\|DB^{1}_{k,a,m}\binom{x}{y},v_1\right\|$by proposition (3.8), we have $B^{1}_{k,a,m}$ has two eigenvalues such that $|\lambda_1|=\frac{1}{|\lambda_2|}$ and since $if\ |\lambda_1|<1\ then\ L_1\left(\binom{x}{y},v_1\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left\|\left(DB^{1}_{k,a,m}\binom{x}{y},v_1\right)^n\right\|>\ln\left\|\frac{(a+2)+\sqrt{(2+a)^2-4(1+a+2k)}}{2}\right\|$

By hypothesis $L_1>0$, so if $|\lambda_1|<1$ then $L_2\left(\binom{x}{y},v_2\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left\|\left(DB^{1}_{k,a,m}\binom{x}{y},v_2\right)^n\right\|<\ln\left\|\frac{(2+a)+\sqrt{(2+a)^2-4(1+a+2k}}{2}\right\|$

Thus the Lyapunov exponent , $L_1(x,y)=max\{L_1(x,y),L_2(x,y)\}$ hence the Lyapunov exponent map is positive .**Definition (6.3) [4]:-**

A map $B^{1}_{k,a,m}$ is chaotic in sense of Gulick if it satisfies at least one of the following conditions:-

1. $B^{1}_{k,a,m}$ has a positive Lyapunov at each point in its domain that is not eventually periodic.
2. $B^{1}_{k,a,m}$ has *sensitive dependence on initial conditions* on its domain.

By draw of sensitive dependence on initial condition figure (1) and Proposition (6.2) then map is chaotic in sense of Gulick.

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