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Research Article

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Lions-type theorem of the p-Laplacian and applications

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Abstract: In this article, our aim is to establish a generalized version of Lions-type theorem for the p-Laplacian. As an application of this theorem, we consider the existence of ground state solution for the quasilinear elliptic equation with the critical growth.

Keywords: Lions theorem; quasilinear elliptic equation; singular potential; critical exponent; ground state solution

MSC: 35A15; 35R11; 35J92

1 Introduction

Consider the quasilinear elliptic equation:

$$-\Delta_p u + V(|x|)|u|^{p-2}u = f(u), \ \ x \in \mathbb{R}^N, \tag{$\mathcal{Q}$}$$

where $N \geqslant 3$, $p \in \left(1, \sqrt{N}\right)$, $V(|x|) = \frac{A}{|x|^\alpha}$, $\alpha \in (0, p)$, $A > 0$ is a real constant, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. Here, the nonlinearity f is given by

(F_1) $f(u) = |u|^{p^\star-2}u + |u|^{q-2}u + \lambda|u|^{p^\star_\alpha-2}u$, where $\lambda > 0$, $p^\star_\alpha = p + \frac{p^2\alpha}{p(N-1)-\alpha(p-1)}$, $p^\star = \frac{Np}{N-p}$ and $q \in \left(\frac{p(N+\alpha-p)}{N-p}, p^\star\right)$.

Remark 1.1. *The generalized version of the Berestycki-Lions conditions for the nonlinearity f is given as follows:*

(F_2) $f \in C(\mathbb{R}, \mathbb{R})$, *there exists* $C > 0$ *such that* $|sf(s)| \leqslant C(|s|^{p^\star_\alpha} + |s|^{p^\star})$ *for* $s \in \mathbb{R}$;

(F_3) $\lim\limits_{s\to 0} \frac{F(s)}{|s|^{p^\star_\alpha}} = 1$ and $\lim\limits_{s\to 0} \frac{F(s)}{|s|^{p^\star}} = 1$, *where* $F(s) := \int_0^s f(t)\mathrm{d}t$; *and*

(F_4) *there exists an* $s_0 \in \mathbb{R}\backslash\{0\}$ *such that* $F(s_0) \neq 0$.

In view of (F_1), *clearly, the nonlinearity f satisfies the generalized version of the Berestycki-Lions conditions* (F_2)-(F_4). *This is the "almost optimal" choice of the nonlinearity f.*

As is well known, for $p = 2$, equation ($\mathcal{Q}$) is a model for describing the stationary state of reaction-diffusion equations in population dynamics [7]. It also arises in several other scientific fields such as plasma physics, condensed matter physics and cosmology [6]. The existence of solution of equation ($\mathcal{Q}$) has been studied extensively by modern variational methods under various hypotheses on the singular potential V and the nonlinearity f. Let us briefly recall some related results. For $p = 2$ and $V(|x|) = \frac{1}{|x|^\alpha}$, the existence and nonexistence of solutions to equation ($\mathcal{Q}$) have been studied in [3, 4, 15, 17, 20, 23]. For $p \neq 2$, the nonexistence results of equation ($\mathcal{Q}$) were presented in [1, 5, 8, 9, 14, 16, 18] and the references therein.

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For $p \in (1, N)$ and $V(x) = \frac{1}{|x|^p}$, Ghoussoub-Yuan [9] investigated the equation:

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = |u|^{p^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $p \in (1, N)$ and $\mu \in \left(0, \left(\frac{N-p}{p}\right)^p\right)$, and established the existence of solutions to equation (1.1) by using the variational methods. Abdellaoui-Peral [1] considered the equation:

$$-\Delta_p u - \frac{\mu + k(x)}{|x|^p}|u|^{p-2}u = |u|^{p^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

and discussed the existence and nonexistence of solutions to equation (1.2) under different assumptions on $k(x)$ by applying the concentration compactness principle and Pohožaev-type identity. Filippucci-Pucci-Robert [8] considered the problem:

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u = |u|^{p^*-2}u + \frac{|u|^{p_s^*}}{|x|^s}, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $s \in (0, N)$ and $p_s^* = \frac{p(N-s)}{N-p}$ is the Hardy-Sobolev critical exponent, and obtained the existence results of equation (1.3) by the choice of a suitable energy level for the mountain pass theorem and analysis of concentration.

Su-Wang-Willem [18] dealt with a generalized version with the singular potential:

$$-\Delta_p u + V(|x|)|u|^{p-2}u = Q(x)f(u), \quad x \in \mathbb{R}^N, \tag{$\mathcal{GQ}$}$$

where $1 < p < N$, and V and Q satisfy

(V_1) $V \in C(0, \infty)$, $V > 0$ and there exist real numbers a and a_0 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0 \text{ and } \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0.$$

(Q_1) $Q \in C(0, \infty)$, $Q > 0$ and there exist real numbers b and b_0 such that

$$\limsup_{r\to\infty} \frac{Q(r)}{r^b} < \infty \text{ and } \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

They attained the radial inequalities with respect to the parameters a, a_0, b, b_0, then established main results on continuous and compact embeddings and the existence of solution to equation ($\mathcal{GQ}$). Badiale-Guida-Rolando [5] generalized the embedding results under different conditions on V and Q, and explored the existence of solution to equation ($\mathcal{GQ}$) with the sub-critical and super-critical growth.

If $a = a_0 = -\alpha$ and $b = b_0 = 0$ in conditions (V_1) and (Q_1), equation ($\mathcal{GQ}$) reduces to ($\mathcal{Q}$). Let us introduce the result on continuous and compact embeddings described in [18].

Proposition 1.1. *Suppose that $N \geqslant 3$ and $p \in (1, N)$. Then we have*

$$\begin{cases} W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \hookrightarrow L^r(\mathbb{R}^N), & r \in [p_\alpha^*, p^*], \ \alpha \in (0, p), \\ W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \hookrightarrow L^r(\mathbb{R}^N), & r \in [p^*, p_\alpha^*], \ \alpha \in \left(p, \frac{N-1}{p-1}p\right), \\ W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \hookrightarrow L^r(\mathbb{R}^N), & r \in [p^*, \infty), \ \alpha \in \left[\frac{N-1}{p-1}p, \infty\right). \end{cases}$$

Furthermore, the embeddings are compact if $r \neq p_\alpha^*$ and $r \neq p^*$, where $p_\alpha^* = \frac{p^2(N-1)+p\alpha}{p(N-1)-\alpha(p-1)} = p + \frac{p^2\alpha}{p(N-1)-\alpha(p-1)}$, $p^* = \frac{Np}{N-p}$, and $W^{1,p}_{rad}(\mathbb{R}^N, \alpha) := D^{1,p}_{rad}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, \alpha)$ is the set of radial functions in $W^{1,p}(\mathbb{R}^N, \alpha)$.

It is very natural to ask whether there exists a solution to equation ($\mathcal{Q}$) with the embedding top index p^* and bottom index p_α^*? To the best of our knowledge, it seems that so far there is no affirmative answer in the literature.

From Proposition 1.1, the embeddings

$$W^{1,p}_{rad}(\mathbb{R}^N,\alpha) \not\hookrightarrow L^{p^*}(\mathbb{R}^N) \text{ and } W^{1,p}_{rad}(\mathbb{R}^N,\alpha) \not\hookrightarrow L^{p^*_\alpha}(\mathbb{R}^N)$$

are not compact. As a result, it is difficult to prove that the Palais-Smale (minimizing) sequence is strongly convergent if we seek solutions of equation ($\mathcal{Q}$) with the critical exponent.

Lions [10] considered the noncompact embedding problem by the concentration-compactness principle: *Only vanishing, dichotomy or tightness are possible*. If one can exclude vanishing and dichotomy, then tightness occurs. It is not difficult to rule out vanishing. But sometimes it is hard to exclude the dichotomy. Therefore, it becomes interesting to ask under what conditions dichotomy cannot occur? In [11, pp. 232], Lions gave a useful answer.

Proposition 1.2. (Lions Theorem) *Let $N \geqslant 3$ and $p \in (1,N)$. Let $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ be any bounded sequence satisfying*
(Condition A) $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^r \mathrm{d}x > 0$ *for $r \in (p, p^*)$.*
Then there exists $\{y_n\} \subset \mathbb{R}^N$ such that the sequence $\bar{u}_n(x) = u_n(x+y_n)$ converges weakly and a.e. to $\bar{u} \not\equiv 0$ in $L^r_{loc}\left(\mathbb{R}^N\right)$.

Following [10, 11], we can derive a similar result as follows immediately.

Proposition 1.3. (Lions-type theorem I) *Let $N \geqslant 3$ and $p \in (1,N)$. Let $\{u_n\} \subset W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$ be any bounded sequence satisfying*
(Condition B) $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^r \mathrm{d}x > 0$, *where*

$$\begin{cases} r \in (p^*_\alpha, p^*) & \alpha \in (0,p), \\ r \in (p^*, p^*_\alpha) & \alpha \in \left(p, \frac{N-1}{p-1}p\right), \\ r \in (p^*, \infty) & \alpha \in \left[\frac{N-1}{p-1}p, \infty\right). \end{cases}$$

Then the sequence $\{u_n\}$ converges weakly and a.e. to $u \not\equiv 0$ in $L^r_{loc}\left(\mathbb{R}^N\right)$.

From Conditions A and B, we see that Propositions 1.2 and 1.3 provide a technical tool to the cases: (a) the nonlinearity f has neither the embedding top index nor embedding bottom index, and (b) the nonlinearity f has either the embedding top index or embedding bottom index. However, Propositions 1.2 and 1.3 become invalid when f contains both embedding top and bottom indices. Hence, we shall establish a more generalized result for $\alpha \in (0,p)$ as follows.

Theorem 1.1. *Suppose that $N \geqslant 3$, $p \in (1,N)$ and $\alpha \in (0,p)$. Let $\{u_n\} \subset W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$ be any bounded sequence satisfying*
(Condition C) $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \mathrm{d}x > 0$ and $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*_\alpha} \mathrm{d}x > 0$.
Then the sequence $\{u_n\}$ converges weakly and a.e. to $u \not\equiv 0$ in $L^p_{loc}(\mathbb{R}^N)$.

Before presenting the existence of ground state solution to equation ($\mathcal{Q}$), let us state the regularity properties of any nonnegative weak solutions of equation ($\mathcal{Q}$) with $\alpha \in (0,p)$.

Theorem 1.2. *Suppose that $N \geqslant 3$, $p \in (1,N)$, $\alpha \in (0,p)$ and condition (F_2) holds. If U is a nonnegative weak solution of equation ($\mathcal{Q}$), then the following statements are true.*
(i) $U \in L^r(\mathbb{R}^N)$ for $r \in [p^*_\alpha, \infty]$.
(ii) U is a positive solution.
(iii) U satisfies the Pohožaev-type identity:

$$\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla U|^p \mathrm{d}x + \frac{N-\alpha}{p}\int_{\mathbb{R}^N} \frac{A}{|x|^\alpha}|U|^p \mathrm{d}x = N\int_{\mathbb{R}^N} F(U)\mathrm{d}x.$$

As an application of Theorems 1.1 and 1.2, when the nonlinearity f satisfies condition (F_1), i.e. equation $(\mathcal{Q})$ takes the form

$$-\Delta u + \frac{A}{|x|^\alpha}|u|^{p-2}u = |u|^{p^*-2}u + |u|^{q-2}u + \lambda|u|^{p_\alpha^\star-2}u, \ \ x \in \mathbb{R}^N, \qquad (\mathcal{S}_{p_\alpha^\star})$$

we have

Theorem 1.3. *Assume that* $N \geqslant 3$, $p \in (1, \sqrt{N})$, $\alpha \in (0, p)$, $q \in \left(\frac{p(N-p+\alpha)}{N-p}, p^\star\right)$ *and condition* (F_1) *holds. Suppose that*

$$\left(\frac{\alpha}{p} S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star - p}} S^{-\frac{p^\star}{p^\star - p}}\right)^{\frac{p_\alpha^\star - p}{p}} \geqslant \lambda > 0,$$

where λ *is a constant,* S_α *and* S *are the best constants of the following inequalities [2, 19]:*

$$S_\alpha \left(\int_{\mathbb{R}^N} |u|^{p_\alpha^\star} \mathrm{d}x\right)^{\frac{p}{p_\alpha^\star}} \leqslant \|u\|^p_{W^{1,p}(\mathbb{R}^N,\alpha)}, \ \ u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha), \tag{1.4}$$

and

$$S \left(\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} \leqslant \|u\|^p_{D^{1,p}(\mathbb{R}^N)}, \ \ u \in D^{1,p}(\mathbb{R}^N). \tag{1.5}$$

Then equation $(\mathcal{S}_{p_\alpha^\star})$ *has a positive ground state solution* $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$.

The rest of this paper is organized as follows. In Section 2, we briefly introduce some useful notations and inequalities. In Sections 3-5, we prove Theorems 1.1-1.3, respectively.

2 Preliminaries

For $N \geqslant 3$ and $p \in (1, N)$, let

$$D^{1,p}(\mathbb{R}^N) := \left\{u \in L^{p^*}(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x < \infty\right\}$$

with the semi-norm

$$\|u\|^p_{D^{1,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x.$$

Let

$$W^{1,p}(\mathbb{R}^N, \alpha) := \left\{u \in D^{1,p}(\mathbb{R}^N) \,\middle|\, \|u\|^p_{L^p(\mathbb{R}^N,\alpha)} = \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha}|u|^p \mathrm{d}x < \infty\right\} = D^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, \alpha)$$

with the norm

$$\|u\|^p_{W^{1,p}(\mathbb{R}^N,\alpha)} := \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha}|u|^p \mathrm{d}x.$$

The following inequalities will play a crucial role in the proof of Theorem 1.3:

$$S_\alpha \left(\int_{\mathbb{R}^N} |u|^{p_\alpha^\star} \mathrm{d}x\right)^{\frac{p}{p_\alpha^\star}} \leqslant \|u\|^p_{W^{1,p}(\mathbb{R}^N,\alpha)}, \ \ u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha), \tag{2.1}$$

and

$$S \left(\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} \leqslant \|u\|^p_{D^{1,p}(\mathbb{R}^N)}, \ \ u \in D^{1,p}(\mathbb{R}^N). \tag{2.2}$$

A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space with the norm $\|u\|_{\mathcal{M}^{q,\varpi}(\mathbb{R}^N)}$, where $q \in [1, \infty)$ and $\varpi \in (0, N]$, if and only if

$$\|u\|^q_{\mathcal{M}^{q,\varpi}(\mathbb{R}^N)} := \sup_{R>0, x\in\mathbb{R}^N} R^{\varpi-N} \int_{B(x,R)} |u(y)|^q \mathrm{d}y < \infty.$$

Lemma 2.1. ([12], Refined Sobolev inequality with the Morrey norm) *For $N \geqslant 3$ and $p \in (1, N)$, there exists $C > 0$ such that for ι and ϑ satisfying $\frac{p}{p^\star} \leqslant \iota < 1$ and $1 \leqslant \vartheta < p^\star$, we have*

$$\left(\int_{\mathbb{R}^N} |u|^{p^\star} \mathrm{d}x\right)^{\frac{1}{p^\star}} \leqslant C\|u\|^{\iota}_{D^{1,p}(\mathbb{R}^N)}\|u\|^{1-\iota}_{\mathcal{M}^{\vartheta,\frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)},$$

for $u \in D^{1,p}(\mathbb{R}^N)$.

It follows from Lemma 2.1 and $W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \subset D^{1,p}_{rad}(\mathbb{R}^N) \subset D^{1,p}(\mathbb{R}^N)$ that

$$\left(\int_{\mathbb{R}^N} |u|^{p^\star} \mathrm{d}x\right)^{\frac{1}{p^\star}} \leqslant C\|u\|^{\iota}_{W^{1,p}(\mathbb{R}^N,\alpha)}\|u\|^{1-\iota}_{\mathcal{M}^{\vartheta,\frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)}, \quad u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha). \tag{2.3}$$

To prove the generalized version of Lions-type theorem, we need the following technical lemma.

Lemma 2.2 ([18]). *Let $N \geqslant 3$, $p \in (1, N)$ and $\alpha \in (0, p)$. Then the inequality*

$$\sup_{|x|>0} |u(x)| \leqslant \frac{C}{|x|^{\frac{p(N-1)-\alpha(p-1)}{p^2}}}\|u\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}$$

holds for $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$.

Throughout this article, we will use the symbol C to denote a generic constant, possibly varying from line to line. However, special occurrences will be denoted by C_1, $\bar{C}$ or the like.

3 Proof of Theorem 1.1

In this section, by applying the refined Sobolev inequality with the Morrey norm and Lemma 2.2, we prove a generalized version of Lions-type theorem.

Proof of Theorem 1.1. We separate the proof into four steps.

Step 1. Note that $\{u_n\}$ is a bounded sequence in $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$. Then, up to a subsequence, we assume

$$u_n \rightharpoonup u \text{ in } W^{1,p}_{rad}(\mathbb{R}^N, \alpha), \ u_n \to u \text{ a.e. in } \mathbb{R}^N, \ u_n \to u \text{ in } L^p_{loc}(\mathbb{R}^N).$$

According to (2.3) and **Condition C**, there exists a positive constant C such that for any n there holds

$$\|u_n\|_{\mathcal{M}^{p,N-p}(\mathbb{R}^N)} \geqslant C > 0.$$

On the other hand, from [12, pp 809] we note that $\{u_n\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ and

$$W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \hookrightarrow D^{1,p}_{rad}(\mathbb{R}^N) \hookrightarrow L^{p^\star}(\mathbb{R}^N) \hookrightarrow \mathcal{M}^{p,N-p}(\mathbb{R}^N).$$

Then we have

$$\|u_n\|_{\mathcal{M}^{p,N-p}(\mathbb{R}^N)} \leqslant C$$

for some $C > 0$ independent of n. Hence, there exists a positive constant C_0 such that for any n there holds

$$C_0 \leqslant \|u_n\|_{\mathcal{M}^{p,N-p}(\mathbb{R}^N)} \leqslant C_0^{-1}.$$

From this inequality, we deduce that for any $n \in \mathbb{N}$ there exist $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\sigma_n^{-p} \int_{B(x_n,\sigma_n)} |u_n(y)|^p \mathrm{d}y \geqslant \|u_n\|^p_{\mathcal{M}^{p,N-p}(\mathbb{R}^N)} - \frac{C}{2n} \geqslant C_1 > 0. \tag{3.1}$$

Step 2. We show $\lim_{n\to\infty} \sigma_n = \bar{\sigma} \neq 0$, where $\bar{\sigma} \in (0, \infty)$.

It suffices to show that $\lim_{n\to\infty}\sigma_n \neq \infty$. Otherwise, we suppose $\lim_{n\to\infty}\sigma_n = \infty$. In view of the boundedness of $\{u_n\}$ and **Condition C**, we get

$$0 < \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_\alpha^\star}\mathrm{d}y \leqslant C.$$

Since $N \geqslant 3$, $p \in (1,N)$ and $\alpha < p$, we have

$$-p + N\left(1 - \frac{p}{p_\alpha^\star}\right) < 0 \Leftrightarrow p_\alpha^\star < p^\star. \tag{3.2}$$

It follows from (3.1)-(3.2) that

$$\begin{aligned}
0 < C_1 \leqslant & \sigma_n^{-p}\int_{B(x_n,\sigma_n)}|u_n(y)|^p\mathrm{d}y \\
\leqslant & \sigma_n^{-p}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p_\alpha^\star - p}{p_\alpha^\star}}\left(\int_{B(x_n,\sigma_n)}|u_n(y)|^{p_\alpha^\star}\mathrm{d}y\right)^{\frac{p}{p_\alpha^\star}} \\
= & \sigma_n^{-p}\left(\frac{\omega_{N-1}}{N}\sigma_n^N\right)^{\frac{p_\alpha^\star - p}{p_\alpha^\star}}\left(\int_{B(x_n,\sigma_n)}|u_n(y)|^{p_\alpha^\star}\mathrm{d}y\right)^{\frac{p}{p_\alpha^\star}} \\
\leqslant & \sigma_n^{-p+N(1-\frac{p}{p_\alpha^\star})}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{p_\alpha^\star - p}{p_\alpha^\star}}\left(\int_{\mathbb{R}^N}|u_n(y)|^{p_\alpha^\star}\mathrm{d}y\right)^{\frac{p}{p_\alpha^\star}} \\
\leqslant & C\sigma_n^{-p+N(1-\frac{p}{p_\alpha^\star})} \\
\to & 0, \text{ as } n \to \infty,
\end{aligned}$$

where ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$. This is a contradiction.

By the Bolzano-Weierstrass theorem, up to a subsequence, still denoted by $\{\sigma_n\}$, there exists $\bar{\sigma} \in [0,\infty)$ such that

$$\lim_{n\to\infty}\sigma_n = \bar{\sigma}.$$

We now show $\lim_{n\to\infty}\sigma_n = \bar{\sigma} \neq 0$. Suppose on the contrary that $\lim_{n\to\infty}\sigma_n = \bar{\sigma} = 0$. By using the uniform boundedness of $\{u_n\}$ in $W_{rad}^{1,p}(\mathbb{R}^N,\alpha)$, we have

$$C\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^{p^\star}\mathrm{d}y\right)^{\frac{p}{p^\star}} \leqslant \lim_{n\to\infty}\|u_n\|^p_{W_{rad}^{1,p}(\mathbb{R}^N,\alpha)} \leqslant \bar{C}.$$

It follows from Hölder's and Sobolev's inequalities that

$$\begin{aligned}
\int_{B(0,\sigma_n)}|u_n|^{p_\alpha^\star}\mathrm{d}y \leqslant & \left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star - p_\alpha^\star}{p^\star}}\left(\int_{B(0,\sigma_n)}|u_n|^{p^\star}\mathrm{d}y\right)^{\frac{p_\alpha^\star}{p^\star}} \\
\leqslant & S^{-\frac{p_\alpha^\star}{p}}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star - p_\alpha^\star}{p^\star}}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}y\right)^{\frac{p_\alpha^\star - p}{p}}\int_{B(0,\sigma_n)}|\nabla u_n|^p\mathrm{d}y \\
\leqslant & CS^{-\frac{p_\alpha^\star}{p}}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star - p_\alpha^\star}{p^\star}}\int_{B(0,\sigma_n)}|\nabla u_n|^p\mathrm{d}y.
\end{aligned}$$

Similarly, for each $z \in \mathbb{R}^N$ we have

$$\int_{B(z,\sigma_n)}|u_n|^{p_\alpha^\star}\mathrm{d}y \leqslant CS^{-\frac{p_\alpha^\star}{p}}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star - p_\alpha^\star}{p^\star}}\int_{B(z,\sigma_n)}|\nabla u_n|^p\mathrm{d}y.$$

Covering $\mathbb{R}^N$ by balls of radius σ_n, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{p_\alpha^\star}\mathrm{d}y \leqslant & C(N+1)S^{-\frac{p_\alpha^\star}{p}}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star-p_\alpha^\star}{p^\star}}\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}y\\ \leqslant & C(N+1)S^{-\frac{p_\alpha^\star}{p}}\left(\int_{B(0,\sigma_n)}\mathrm{d}y\right)^{\frac{p^\star-p_\alpha^\star}{p^\star}}\\ = & C(N+1)S^{-\frac{p_\alpha^\star}{p}}\left(\frac{\omega_{N-1}}{N}\right)^{1-\frac{p_\alpha^\star}{p^\star}}\sigma_n^{N(1-\frac{p_\alpha^\star}{p^\star})},\end{aligned}$$

where ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$.

Taking $n\to\infty$ and applying $\lim\limits_{n\to\infty}\sigma_n=0$ leads to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_\alpha^\star}\mathrm{d}y \leqslant C(N+1)S^{-\frac{p_\alpha^\star}{p}}\left(\frac{\omega_{N-1}}{N}\right)^{1-\frac{p_\alpha^\star}{p^\star}}\lim_{n\to\infty}\sigma_n^{N(1-\frac{p_\alpha^\star}{p^\star})}=0,$$

which yields a contradiction to $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\mathrm{d}x>0$ given in **Condition C**.

Using $\lim\limits_{n\to\infty}\sigma_n=\bar{\sigma}\neq 0$, up to a subsequence, we have $\sigma_n\in(\frac{\bar{\sigma}}{2},2\bar{\sigma})$ and

$$\frac{2^p}{\bar{\sigma}^p}\int_{B(x_n,\sigma_n)}|u_n(y)|^p\mathrm{d}y\geqslant C_1>0,$$

which gives

$$\int_{B(x_n,\sigma_n)}|u_n(y)|^2\mathrm{d}y\geqslant\frac{C_1\bar{\sigma}^p}{2^p}>0. \tag{3.3}$$

Step 3. We show that $\{x_n\}$ is a bounded sequence.

By way of contradiction, we can assume $|x_n|\to\infty$ as $n\to\infty$. According to Lemma 2.2, we have

$$\sup_{|x|>0}|u_n(x)|\leqslant\frac{C}{|x|^{\frac{p(N-1)-\alpha(p-1)}{p^2}}}\|u_n\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}\leqslant\frac{C}{|x|^{\frac{p(N-1)-\alpha(p-1)}{p^2}}}.$$

For any $\left(\frac{C_1\bar{\sigma}^p}{2^p|B(0,2\bar{\sigma})|}\right)^{\frac{1}{p}}>\varepsilon>0$, there exists an $M>0$ such that for any $n>M$ there holds

$$|u_n(x)|\leqslant\frac{C}{||x_n|-\sigma_n|^{\frac{p(N-1)-\alpha(p-1)}{p^2}}}\leqslant\varepsilon,\ \ x\in B^c(0,||x_n|-\sigma_n|).$$

From $B(x_n,\sigma_n)\subset B^c(0,||x_n|-\sigma_n|)$, it follows that

$$\int_{B(x_n,\sigma_n)}|u_n(y)|^p\mathrm{d}y\leqslant\varepsilon^p\int_{B(x_n,\sigma_n)}\mathrm{d}y=\varepsilon^p|B(x_n,\sigma_n)|=\varepsilon^p|B(0,\sigma_n)|\leqslant\varepsilon^p|B(0,2\bar{\sigma})|<\frac{C_1\bar{\sigma}^p}{2^p}.$$

Step 4. Note that $\{x_n\}$ is bounded. There exists $0<\tilde{C}<\infty$ such that $0\leqslant|x_n|<\tilde{C}$. In view of $\lim\limits_{n\to\infty}\sigma_n=\bar{\sigma}\neq 0$, up to a subsequence, we have

$$B(x_n,\sigma_n)\subset B(0,|x_n|+\sigma_n)\subset B(0,\tilde{C}+2\bar{\sigma}).$$

From (3.1), we deduce that

$$\begin{aligned}0<C_1\bar{\sigma}^p=&C_1\lim_{n\to\infty}\sigma_n^p\\ \leqslant&\lim_{n\to\infty}(\sigma_n^p)\lim_{n\to\infty}\left(\sigma_n^{-p}\int_{B(x_n,\sigma_n)}|u_n|^p\mathrm{d}y\right)\\ =&\lim_{n\to\infty}\int_{B(x_n,\sigma_n)}|u_n|^p\mathrm{d}y\\ \leqslant&\lim_{n\to\infty}\int_{B(0,\tilde{C}+2\bar{\sigma})}|u_n|^p\mathrm{d}y.\end{aligned}$$

Applying the embedding $W^{1,p}_{rad}(\mathbb{R}^N,\alpha)\hookrightarrow D^{1,p}_{rad}(\mathbb{R}^N)\hookrightarrow L^p_{loc}(\mathbb{R}^N)$, we obtain $u\not\equiv 0$. □

4 Proof of Theorem 1.2

In this section, we prove that any nonnegative weak solutions of equation (Ω) with $\alpha \in (0, p)$ have additional regularity properties.

Lemma 4.1. *Assume that all the conditions described in Theorem 1.2 hold. For each $L > 2$, define*

$$U_L(x) = \begin{cases} U(x) & \text{if } U(x) \leqslant L, \\ L & \text{if } U(x) > L. \end{cases}$$

Set $\bar{U}_L = UU_L^{p(\mu-1)}$, where $\mu > 1$. Let $\tilde{C}_t$ be the best embedding constant from $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ to $L^t(\mathbb{R}^N)$ for $t \in [p^\star_\alpha, p^\star]$. Then we have

$$\left(\int_{\mathbb{R}^N} |UU_L^{\mu-1}|^t \mathrm{d}x\right)^{\frac{p}{t}} \leqslant \frac{C\mu^p}{\tilde{C}_t} \left[\int_{\mathbb{R}^N} U^{p^\star_\alpha - p} |UU_L^{\mu-1}|^p \mathrm{d}x + \int_{\mathbb{R}^N} U^{p^\star - p} |UU_L^{\mu-1}|^p \mathrm{d}x\right]. \tag{4.1}$$

Proof. Let U be a nonnegative ground state solution of equation (Ω). We show that $\bar{U}_L \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$. By a straightforward calculation, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(UU_L^{p(\mu-1)})|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^N} |U_L^{p(\mu-1)}\nabla U + p(\mu-1)UU_L^{p(\mu-1)-1}\nabla U_L|^p \mathrm{d}x \\
&\leqslant 2^p \int_{\mathbb{R}^N} |U_L^{p(\mu-1)}|^p |\nabla U|^p \mathrm{d}x + (2p)^p(\mu-1)^p \int_{\mathbb{R}^N} |UU_L^{p(\mu-1)-1}|^p |\nabla U_L|^p \mathrm{d}x \\
&= 2^p \int_{\mathbb{R}^N} |U_L^{p(\mu-1)}|^p |\nabla U|^p \mathrm{d}x + (2p)^p(\mu-1)^p \int_{\{x|U(x)\leqslant L\}} |U_L^{p(\mu-1)}|^p |\nabla U|^p \mathrm{d}x \\
&\leqslant 2^p L^{p^2(\mu-1)}(1 + (\mu-1)^p) \int_{\mathbb{R}^N} |\nabla U|^p \mathrm{d}x,
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |UU_L^{p(\mu-1)}|^p \mathrm{d}x \leqslant L^{p^2(\mu-1)} \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |U|^p \mathrm{d}x.$$

This implies that $\bar{U}_L \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$.

Note that U is a nonnegative ground state solution of equation (Ω). Then

$$\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |U|^{p-2} U\varphi \mathrm{d}x = \int_{\mathbb{R}^N} f(U)\varphi \mathrm{d}x.$$

Substituting $\bar{U}_L$ into the above equation, we get

$$\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U \nabla \bar{U}_L \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |UU_L^{\mu-1}|^p \mathrm{d}x = \int_{\mathbb{R}^N} f(U)\bar{U}_L \mathrm{d}x. \tag{4.2}$$

Since

$$\int_{\mathbb{R}^N} UU_L^{p(\mu-1)-1} |\nabla U|^{p-2}\nabla U \nabla U_L \mathrm{d}x = \int_{\{x|U(x)\leqslant L\}} U^{p(\mu-1)} |\nabla U|^p \mathrm{d}x \geqslant 0,$$

it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U \nabla \bar{U}_L \mathrm{d}x \\
&= \int_{\mathbb{R}^N} U_L^{p(\mu-1)} |\nabla U|^p \mathrm{d}x + p(\mu-1) \int_{\mathbb{R}^N} UU_L^{p(\mu-1)-1} |\nabla U|^{p-2}\nabla U \nabla U_L \mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^N} U_L^{p(\mu-1)} |\nabla U|^p \mathrm{d}x.
\end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla(UU_L^{\mu-1})|^p \mathrm{d}x \leqslant&[2^p + (2p)^p(\mu-1)^p] \int_{\mathbb{R}^N} U_L^{p(\mu-1)} |\nabla U|^p \mathrm{d}x \\ \leqslant&(2p)^p \mu^p \int_{\mathbb{R}^N} U_L^{p(\mu-1)} |\nabla U|^p \mathrm{d}x.\end{aligned} \tag{4.4}$$

It follows from (4.2)-(4.4) and Proposition 1.1 that

$$\begin{aligned}\tilde{C}_t \left(\int_{\mathbb{R}^N} |UU_L^{\mu-1}|^t \mathrm{d}x\right)^{\frac{p}{t}} \leqslant&(2p)^p\mu^p \left(\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U \nabla \bar{U}_L \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |UU_L^{\mu-1}|^p \mathrm{d}x\right) \\ \leqslant&C\mu^p \left[\int_{\mathbb{R}^N} U^{p^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x + \int_{\mathbb{R}^N} U^{p_\alpha^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x\right].\end{aligned}$$

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into three parts.

Part (i). We first show the L^∞ estimate of U by the following four steps.

Step 1. We claim that

$$\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_1} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1-1)}} < \infty,$$

where $\mu_1 := 1 + \frac{p^\star - p_\alpha^\star}{p}$.

Taking $\bar{d} \in \mathbb{R}^+$ and using Hölder's inequality, we can derive

$$\begin{aligned}&\int_{\mathbb{R}^N} U^{p^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x \\ \leqslant&\bar{d}^{p^\star-p_\alpha^\star} \int_{\{x|U(x)\leqslant\bar{d}\}} U^{p_\alpha^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x + \int_{\{x|U(x)>\bar{d}\}} U^{p^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x \\ \leqslant&\bar{d}^{p^\star-p_\alpha^\star} \int_{\mathbb{R}^N} |U|^{p_\alpha^\star+p\mu-p} \mathrm{d}x + \left(\int_{\{x|U(x)>\bar{d}\}} U^{p^\star} \mathrm{d}x\right)^{\frac{p^\star-p}{p^\star}} \left(\int_{\mathbb{R}^N} |UU_L^{\mu-1}|^{p^\star} \mathrm{d}x\right)^{\frac{p}{p^\star}}.\end{aligned}$$

We choose $\bar{d}$ such that

$$\left(\int_{\{x|U(x)>\bar{d}\}} U^{p^\star} \mathrm{d}x\right)^{\frac{p^\star-p}{p^\star}} \leqslant \frac{\tilde{C}_{p^\star}}{2C\mu^p}.$$

Substituting the above two inequalities into (4.1) with the choice of $t = p^\star$, we get

$$\left(\int_{\mathbb{R}^N} |UU_L^{\mu-1}|^{p^\star} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant \frac{2C\mu^p}{\tilde{C}_{p^\star}} \left[\int_{\mathbb{R}^N} U^{p_\alpha^\star-p} |UU_L^{\mu-1}|^p \mathrm{d}x + \bar{d}^{p^\star-p_\alpha^\star} \int_{\mathbb{R}^N} |U|^{p_\alpha^\star+p\mu-p} \mathrm{d}x\right].$$

Taking the limit as $L \to \infty$ in the above inequality leads to

$$\left(\int_{\mathbb{R}^N} |U|^{p^\star\mu} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant \frac{2C\mu^p}{\tilde{C}_{p^\star}} \left[1 + \bar{d}^{p^\star-p_\alpha^\star}\right] \int_{\mathbb{R}^N} |U|^{p_\alpha^\star+p\mu-p} \mathrm{d}x.$$

Let $p_\alpha^\star + p\mu - p \in (p_\alpha^\star, p^\star]$ and choose $\mu \in \left(1, 1 + \frac{p^\star-p_\alpha^\star}{p}\right]$. Then

$$U \in L^{p^\star\mu}(\mathbb{R}^N).$$

Hence, we have

$$U \in L^{\check{p}_1}(\mathbb{R}^N), \quad \check{p}_1 \in [p_\alpha^\star, p^\star] \cup \left(p^\star, p^\star\left(1 + \frac{p^\star - p_\alpha^\star}{p}\right)\right].$$

We now choose $\mu_1 := 1 + \frac{p^\star - p_\alpha^\star}{p}$. Then we obtain $p^\star \mu_1 \in \left[p^\star, p^\star\left(1 + \frac{p^\star - p_\alpha^\star}{p}\right)\right]$ and

$$\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_1} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1 - 1)}} < \infty.$$

Step 2. We show that

$$\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_2 - 1)}} \leqslant (C\mu_2)^{\frac{p}{\mu_2 - 1}} \left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_1} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1 - 1)}},$$

where $\mu_2 - 1 = (\frac{p^\star}{p})(\mu_1 - 1)$.

Let $\mu_2 := 1 + \frac{p^\star}{p} \cdot (\mu_1 - 1)$. Then $p^\star + p\mu_2 - p = p^\star \mu_1$ and $\mu_2 - 1 = (\frac{p^\star}{p})(\mu_1 - 1)$. Let $\mu \in [\mu_1, \mu_2]$. We find

$$p_\alpha^\star < p_\alpha^\star + p\mu - p < p^\star + p\mu - p \leqslant p^\star \mu_1.$$

From Lemma 4.1, we get

$$\left(\int_{\mathbb{R}^N} |U|^{p^\star \mu} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant \frac{C\mu^p}{\tilde{C}_{p^\star}} \left[\int_{\mathbb{R}^N} |U|^{p_\alpha^\star + p\mu - p} \mathrm{d}x + \int_{\mathbb{R}^N} |U|^{p^\star + p\mu - p} \mathrm{d}x\right] < \infty.$$

For each $\mu \in [\mu_1, \mu_2]$, we have $U \in L^{p^\star \mu}(\mathbb{R}^N)$ and

$$U \in L^{\check{p}_2}(\mathbb{R}^N), \quad \check{p}_2 \in [p_\alpha^\star, p^\star \mu_1] \cup [p^\star \mu_1, p^\star \mu_2].$$

Set $\mu = \mu_2$. Then

$$\left(\int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant \frac{C\mu_2^p}{\tilde{C}_{p^\star}} \left[\int_{\mathbb{R}^N} |U|^{p_\alpha^\star + p\mu_2 - p} \mathrm{d}x + \int_{\mathbb{R}^N} |U|^{p^\star + p\mu_2 - p} \mathrm{d}x\right] < \infty.$$

Let $a_2 = \frac{p^\star(p^\star - p_\alpha^\star)}{p(\mu_2 - 1)}$ and $b_2 = p_\alpha^\star + p\mu_2 - p - a_2$. Then $\frac{p^\star b_2}{p^\star - a_2} = p^\star + p\mu_2 - p$. It follows from Young's inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} |U|^{p_\alpha^\star + p\mu_2 - p} dx &= \int_{\mathbb{R}^N} |U|^{a_2} |U|^{b_2} dx \\ &\leqslant \frac{a_2}{p^\star} \int_{\mathbb{R}^N} |U|^{p^\star} dx + \frac{p^\star - a_2}{p^\star} \int_{\mathbb{R}^N} |U|^{p^\star + p\mu_2 - p} dx \\ &\leqslant C\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star + p\mu_2 - p} \mathrm{d}x\right).\end{aligned}$$

Thus, we deduce

$$\left(\int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant C\mu_2^p \left(1 + \int_{\mathbb{R}^N} |U|^{p^\star + p\mu_2 - p} \mathrm{d}x\right).$$

For every $x_1, x_2 > 0$, we know that

$$(x_1 + x_2)^{\frac{p}{p^\star}} \leqslant x_1^{\frac{p}{p^\star}} + x_2^{\frac{p}{p^\star}}.$$

We then obtain

$$\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant 1 + \left(\int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star}} \leqslant C\mu_2^p \left(1 + \int_{\mathbb{R}^N} |U|^{p^\star + p\mu_2 - p} \mathrm{d}x\right).$$

That is,

$$\begin{aligned}\left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_2} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_2 - 1)}} &\leqslant (C\mu_2)^{\frac{p}{\mu_2 - 1}} \left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_1} \mathrm{d}x\right)^{\frac{1}{\mu_2 - 1}} \\ &= (C\mu_2)^{\frac{p}{\mu_2 - 1}} \left(1 + \int_{\mathbb{R}^N} |U|^{p^\star \mu_1} \mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1 - 1)}}.\end{aligned}$$

Step 3. We show that

$$\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_3}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_3-1)}}\leqslant (C\mu_3)^{\frac{p}{\mu_3-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_2}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_2-1)}},$$

where $\mu_3-1=(\frac{p^*}{p})(\mu_2-1)$.

Let $\mu_3:=1+\frac{p^*}{p}\cdot(\mu_2-1)$. Then $p^*+p\mu_3-p=p^*\mu_2$ and $\mu_3-1=(\frac{p^*}{p})(\mu_2-1)$. Let $\mu\in[\mu_2,\mu_3]$. Then

$$p_\alpha^\star<p_\alpha^\star+p\mu-p<p^*+p\mu-p\leqslant p^*\mu_2.$$

So we get

$$\left(\int_{\mathbb{R}^N}|U|^{p^*\mu}\mathrm{d}x\right)^{\frac{p}{p^*}}\leqslant\frac{C\mu^p}{\tilde{C}_{p^*}}\left[\int_{\mathbb{R}^N}|U|^{p_\alpha^\star+p\mu-p}\mathrm{d}x+\int_{\mathbb{R}^N}|U|^{p^*+p\mu-p}\mathrm{d}x\right]<\infty.$$

For each $\mu\in[\mu_2,\mu_3]$, we have $U\in L^{p^*\mu}(\mathbb{R}^N)$ and

$$U\in L^{\check{p}_3}(\mathbb{R}^N),\quad \check{p}_3\in[p_\alpha^\star,p^*\mu_2]\cup[p^*\mu_2,p^*\mu_3].$$

Set $\mu=\mu_3$. Then

$$\left(\int_{\mathbb{R}^N}|U|^{p^*\mu_3}\mathrm{d}x\right)^{\frac{p}{p^*}}\leqslant\frac{C\mu_3^p}{\tilde{C}_{p^*}}\left[\int_{\mathbb{R}^N}|U|^{p_\alpha^\star+p\mu_3-p}\mathrm{d}x+\int_{\mathbb{R}^N}|U|^{p^*+p\mu_3-p}\mathrm{d}x\right]<\infty.$$

Let $a_3=\frac{p^*(p^*-p_\alpha^\star)}{p(\mu_3-1)}$ and $b_3=p_\alpha^\star+p\mu_3-p-a_3$. Then $\frac{p^*b_3}{p^*-a_3}=p^*+p\mu_3-p$. It follows from Young's inequality that

$$\int_{\mathbb{R}^N}|U|^{p_\alpha^\star+p\mu_3-p}dx\leqslant C\left(1+\int_{\mathbb{R}^N}|U|^{p^*+p\mu_3-p}\mathrm{d}x\right).$$

Hence, we obtain

$$\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_3}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_3-1)}}\leqslant (C\mu_3)^{\frac{p}{\mu_3-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_2}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_2-1)}}.$$

Step 4. Iterating the above process and recalling that

$$p^*+p\mu_{i+1}-p=p^*\mu_i,\quad i\geqslant 1\text{ and }i\in\mathbb{N},$$

we have

$$\mu_{i+1}-1=\left(\frac{p^*}{p}\right)^i(\mu_1-1)$$

and

$$\begin{aligned}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_{i+1}}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_{i+1}-1)}}&\leqslant(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_i}\mathrm{d}x\right)^{\frac{1}{\mu_{i+1}-1}}\\&=(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_i}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_i-1)}}.\end{aligned}$$

For $m\in\mathbb{N}$, we further get

$$\begin{aligned}&\prod_{i=1}^m(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_1}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_1-1)}}\\&\geqslant\left(1+\int_{\mathbb{R}^N}|U|^{p^*\mu_{m+1}}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_{m+1}-1)}}\\&\geqslant\left(\int_{\mathbb{R}^N}|U|^{p^*\mu_{m+1}}\mathrm{d}x\right)^{\frac{p}{p^*(\mu_{m+1}-1)}}.\end{aligned}$$

If $\int_{\mathbb{R}^N}|U|^{p^\star\mu_{m+1}}\mathrm{d}x\leqslant 1$, then

$$\prod_{i=1}^{m}(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^\star\mu_1}\mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1-1)}}\geqslant\left(\int_{\mathbb{R}^N}|U|^{p^\star\mu_{m+1}}\mathrm{d}x\right)^{\frac{p+\frac{p^2}{p^\star-p_\alpha^\star}}{p^\star\mu_{m+1}}}=\|U\|_{L^{p^\star\mu_{m+1}}(\mathbb{R}^N)}^{p+\frac{p^2}{p^\star-p_\alpha^\star}}.$$

That is,

$$\|U\|_{L^{p^\star\mu_{m+1}}(\mathbb{R}^N)}\leqslant\left[\prod_{i=1}^{m}(C\mu_{i+1})^{\frac{1}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^\star\mu_1}\mathrm{d}x\right)^{\frac{1}{p^\star(\mu_1-1)}}\right]^{\frac{p^\star-p_\alpha^\star}{p^\star-p_\alpha^\star+p}}. \tag{4.5}$$

If $\int_{\mathbb{R}^N}|U|^{p^\star\mu_{m+1}}\mathrm{d}x>1$, then

$$\prod_{i=1}^{m}(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^\star\mu_1}\mathrm{d}x\right)^{\frac{p}{p^\star(\mu_1-1)}}\geqslant\left(\int_{\mathbb{R}^N}|U|^{p^\star\mu_{m+1}}\mathrm{d}x\right)^{\frac{p}{p^\star\mu_{m+1}}}=\|U\|_{L^{p^\star\mu_{m+1}}(\mathbb{R}^N)}^{p}.$$

That is,

$$\|U\|_{L^{p^\star\mu_{m+1}}(\mathbb{R}^N)}\leqslant\prod_{i=1}^{m}(C\mu_{i+1})^{\frac{1}{\mu_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|U|^{p^\star\mu_1}\mathrm{d}x\right)^{\frac{1}{p^\star(\mu_1-1)}}. \tag{4.6}$$

Note that

$$\lim_{m\to\infty}\prod_{i=1}^{m}(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}=\lim_{m\to\infty}e^{p\sum_{i=1}^{m}\left(\frac{\ln C}{\mu_{i+1}-1}+\frac{\ln\mu_{i+1}}{\mu_{i+1}-1}\right)}.$$

For the series $\sum_{i=1}^{\infty}\frac{\ln C}{\mu_{i+1}-1}$, using the root test, we get

$$\lim_{i\to\infty}\sqrt[i]{\frac{\ln C}{\mu_{i+1}-1}}=\lim_{i\to\infty}\sqrt[i]{\frac{\ln C}{(\frac{p^\star}{p})^i(\mu_1-1)}}=\frac{p}{p^\star}<1,$$

which indicates $\sum_{i=1}^{\infty}\frac{\ln C}{\mu_{i+1}-1}<\infty$.

For the series $\sum_{i=1}^{\infty}\frac{\ln\mu_{i+1}}{\mu_{i+1}-1}$, by using the ratio test, we find

$$\begin{aligned}\lim_{i\to\infty}\frac{\ln\mu_{i+2}}{\mu_{i+2}-1}\cdot\frac{\mu_{i+1}-1}{\ln\mu_{i+1}}&=\frac{p}{p^\star}\lim_{i\to\infty}\frac{\ln\left(1+\frac{p^\star}{p}(\mu_{i+1}-1)\right)}{\ln\mu_{i+1}}\\&\leqslant\frac{p}{p^\star}\lim_{i\to\infty}\frac{\ln\left(\frac{p^\star}{p}+\frac{p^\star}{p}(\mu_{i+1}-1)\right)}{\ln\mu_{i+1}}\\&=\frac{p}{p^\star}\lim_{i\to\infty}\left(\frac{\ln\frac{p^\star}{p}}{\ln\mu_{i+1}}+\frac{\ln\mu_{i+1}}{\ln\mu_{i+1}}\right)\\&<1,\end{aligned}$$

which implies $\sum_{i=1}^{\infty}\frac{\ln\mu_{i+1}}{\mu_{i+1}-1}<\infty$. Hence, we have $\prod_{i=1}^{\infty}(C\mu_{i+1})^{\frac{p}{\mu_{i+1}-1}}<\infty$.

Letting $m\to\infty$ in (4.5) and (4.6), we obtain

$$\|U\|_{L^\infty(\mathbb{R}^N)}<\infty.$$

Part (ii). We rewrite equation $(\mathcal{Q})$ as $-\Delta_p U = g(x, U)$, where

$$g(x, U) = -\frac{A}{|x|^\alpha}U + f(U).$$

For all $\Omega \subset\subset \mathbb{R}^N \setminus \{0\}$, there exists $C(\Omega) > 0$ such that $|g(x, U)| \leqslant C(\Omega)(1 + |U|^{p^*-1})$ for $x \in \Omega$. It follows from Theorem 1.2 (i) and [21, Theorem 1 and Proposition 1] that $U \in C^1(\mathbb{R}^N \setminus \{0\})$. Finally, the strict positivity follows form the strong maximum principle [22].

Part (iii). Applying Theorem 1.2 (ii) and following [8, Claim 5.3], we can derive the Pohožaev-type identity

$$\frac{N-p}{p}\int_{\mathbb{R}^N}|\nabla U|^p dx + \frac{N-\alpha}{p}\int_{\mathbb{R}^N}\frac{A}{|x|^\alpha}|U|^p dx = N\int_{\mathbb{R}^N}F(U)dx.$$

Consequently, the proof is completed. □

5 Proof of Theorem 1.3

As we see, equation $(\mathcal{S}_{p_\alpha^\star})$ is variational and its solutions are the critical points of the functional defined in $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ by

$$J(u) := \frac{1}{p}\|u\|^p_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)} - \frac{\lambda}{p_\alpha^\star}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star}dx - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx - \frac{1}{p^\star}\int_{\mathbb{R}^N}|u|^{p^\star}dx.$$

From Proposition 1.1, we know that $J \in C^1(W^{1,p}_{rad}(\mathbb{R}^N, \alpha), \mathbb{R})$. It is easy to see that if $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ is a critical point of J, i.e.

$$\begin{aligned}0 = \langle J'(u), \varphi\rangle &= \int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u\nabla\varphi + \frac{A}{|x|^\alpha}|u|^{p-2}u\varphi)dx\\ &\quad - \lambda\int_{\mathbb{R}^N}|u|^{p_\alpha^\star-2}u\varphi dx - \int_{\mathbb{R}^N}|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^N}|u|^{p^\star-2}u\varphi dx\end{aligned}$$

for all $\varphi \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$, then u is a weak solution of equation $(\mathcal{S}_{p_\alpha^\star})$.

Define

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)) > 0,$$

where

$$\Gamma = \left\{\gamma \in C\left([0, 1], W^{1,p}_{rad}(\mathbb{R}^N, \alpha)\right) |\gamma(0) = 0, J(\gamma(1)) < 0\right\}.$$

Define

$$\mathcal{N} = \{u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)|\langle J'(u), u\rangle = 0,\ u \neq 0\},$$

and let

$$\bar{c} = \inf_{u\in\mathcal{N}} J(u).$$

It is easy to check that $c = \bar{c} = \bar{\bar{c}} := \inf_{u\in W^{1,p}_{rad}(\mathbb{R}^N,\alpha)\setminus\{0\}}\sup_{t\geqslant 0} J(tu) > 0.$

Lemma 5.1. *Assume that all the conditions described in Theorem 1.3 hold. Then*

$$0 < c < c^\star := \min\left\{\frac{\alpha}{Np}\left(\frac{1}{\lambda}\right)^{\frac{p}{p_\alpha^\star-p}}S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star-p}}, \frac{1}{N}S^{\frac{p^\star}{p^\star-p}}\right\}.$$

Proof. If $\left(\frac{\alpha}{p}S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star-p}}S^{-\frac{p^\star}{p^\star-p}}\right)^{\frac{p_\alpha^\star-p}{p}} \geqslant \lambda > 0$, then

$$\frac{\alpha}{Np}\left(\frac{1}{\lambda}\right)^{\frac{p}{p_\alpha^\star-p}}S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star-p}} \geqslant \frac{1}{N}S^{\frac{p^\star}{p^\star-p}}.$$

We choose

$$\omega_\sigma(x) = \frac{C\sigma^{\frac{N-p}{p(p-1)}}}{(\sigma^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}$$

and

$$\|w_\sigma\|^p_{D^{1,p}(\mathbb{R}^N)} = \|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |w_1|^{p^*} \mathrm{d}x = \int_{\mathbb{R}^N} |w_\sigma|^{p^*} \mathrm{d}x = S^{\frac{p^*}{p^*-p}}.$$

A straightforward calculation gives

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|w_\sigma|^p}{|x|^\alpha} \mathrm{d}x &\leqslant C \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} \frac{\sigma^{N-p}}{(\sigma^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} \mathrm{d}x \\
&\leqslant C \int_0^\infty \frac{1}{\rho^\alpha} \frac{1}{(\sigma^{\frac{p}{p-1}} + \rho^{\frac{p}{p-1}})^{N-p}} \rho^{N-1} \mathrm{d}\rho \\
&= C \int_0^1 \frac{\rho^{N-1-\alpha}}{(1 + \rho^{\frac{p}{p-1}})^{N-p}} \mathrm{d}\rho + C \int_1^\infty \frac{\rho^{N-1-\alpha}}{(\sigma^{\frac{p}{p-1}} + \rho^{\frac{p}{p-1}})^{N-p}} \mathrm{d}\rho \\
&\leqslant C \int_0^1 \frac{\rho^{N-1-\alpha}}{(\sigma^{\frac{p}{p-1}} + \rho^{\frac{p}{p-1}})^{N-p}} \mathrm{d}\rho + C \int_1^\infty \rho^{N-1-\alpha-\frac{p(N-p)}{p-1}} \mathrm{d}\rho.
\end{aligned}$$

It is not difficult to see that

$$\int_0^1 \frac{\rho^{N-1-\alpha}}{(\sigma^{\frac{p}{p-1}} + \rho^{\frac{p}{p-1}})^{N-p}} \mathrm{d}\rho \leqslant \int_0^1 \frac{\rho^{N-1-\alpha}}{(\sigma^{\frac{p}{p-1}})^{N-p}} \mathrm{d}\rho < \infty.$$

In view of $p \in (1, \sqrt{N})$, we have $N - 1 - \alpha - \frac{p(N-p)}{p-1} < -1$ and

$$\int_1^\infty \rho^{N-1-\alpha-\frac{p(N-p)}{p-1}} \mathrm{d}\rho < \infty.$$

This implies that $w_\sigma \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$. So we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |w_\sigma|^{p^*_\alpha} \mathrm{d}x &= \sigma^{p-\frac{p\alpha(N-p)}{p(N-1)-\alpha(p-1)}} \int_{\mathbb{R}^N} |w_1|^{p^*_\alpha} \mathrm{d}x, \\
\int_{\mathbb{R}^N} |w_\sigma|^q \mathrm{d}x &= \sigma^{q+\frac{(p-q)N}{p}} \int_{\mathbb{R}^N} |w_1|^q \mathrm{d}x, \\
\int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} |w_\sigma|^p \mathrm{d}x &= \sigma^{p-\alpha} \int_{\mathbb{R}^N} \frac{1}{|x|^\alpha} |w_1|^p \mathrm{d}x,
\end{aligned}$$

and $\lim_{t\to\infty} J(tw_\sigma) = -\infty$.

Let $\bar{t}_\sigma > t_\sigma > 0$ satisfy

$$\sup_{t\geqslant 0} J(tw_\sigma) = J(t_\sigma w_\sigma) \text{ and } J(\bar{t}_\sigma w_\sigma) < 0.$$

Taking $\gamma(t) = t\bar{t}_\sigma w_\sigma$, we get

$$c \leqslant \max_{t\in[0,1]} J(\gamma(t)) = J(t_\sigma w_\sigma).$$

Note that

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=t_\sigma} J(tw_\sigma) \\
&= \left(t_\sigma^{p-1} - t_\sigma^{p^*-1}\right) \|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \sigma^{p-\alpha} t_\sigma^{p-1} \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} |w_1|^p \mathrm{d}x \\
&\quad - \lambda \sigma^{p-\frac{p\alpha(N-p)}{p(N-1)-\alpha(p-1)}} t_\sigma^{p^*_\alpha-1} \int_{\mathbb{R}^N} |w_1|^{p^*_\alpha} \mathrm{d}x - \sigma^{q+\frac{(p-q)N}{p}} t_\sigma^{q-1} \int_{\mathbb{R}^N} |w_1|^q \mathrm{d}x.
\end{aligned} \tag{5.1}$$

Let $t_0 := \limsup_{\sigma\to 0} t_\sigma$. We claim that $t_0 < \infty$. Otherwise, we assume that $t_0 = \infty$. Taking an upper limit as $\sigma \to 0$ in (5.1), we get

$$\begin{aligned}
&\limsup_{\sigma\to 0}\left(t_\sigma^{p-1}\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \sigma^{p-\alpha} t_\sigma^{p-1}\int_{\mathbb{R}^N}\frac{A}{|x|^\alpha}|w_1|^p \mathrm{d}x\right)\\
&=\limsup_{\sigma\to 0}\left(t_\sigma^{p^\star-1}\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \lambda\sigma^{p-\frac{p\alpha(N-p)}{p(N-1)-\alpha(p-1)}} t_\sigma^{p^\star_\alpha-1}\int_{\mathbb{R}^N}|w_1|^{p^\star_\alpha}\mathrm{d}x\right.\\
&\left.+\sigma^{q+\frac{(p-q)N}{p}} t_\sigma^{q-1}\int_{\mathbb{R}^N}|w_1|^q \mathrm{d}x\right).
\end{aligned}\tag{5.2}$$

It follows from $p < p^\star$ and $\limsup_{\sigma\to 0} t_\sigma = \infty$ that

$$\begin{aligned}
&\limsup_{\sigma\to 0} t_\sigma^{p-1}\left(\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \sigma^{p-\alpha}\int_{\mathbb{R}^N}\frac{A}{|x|^\alpha}|w_1|^p \mathrm{d}x\right)\\
&<\limsup_{\sigma\to 0} t_\sigma^{p^\star-1}\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)}\\
&\leqslant\limsup_{\sigma\to 0}\left(t_\sigma^{p^\star-1}\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \lambda\sigma^{p-\frac{p\alpha(N-p)}{p(N-1)-\alpha(p-1)}} t_\sigma^{p^\star_\alpha-1}\int_{\mathbb{R}^N}|w_1|^{p^\star_\alpha}\mathrm{d}x\right.\\
&\left.+\sigma^{q+\frac{(p-q)N}{p}} t_\sigma^{q-1}\int_{\mathbb{R}^N}|w_1|^q \mathrm{d}x\right).
\end{aligned}$$

This contradicts (5.2). That is, $t_0 < \infty$.

Passing to an upper limit as $\sigma \to 0$ in (5.1) leads to

$$0 = \left(t_0^{p-1} - t_0^{p^\star-1}\right)\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)},$$

which implies

$$t_0^{p-1} - t_0^{p^\star-1} = 0 \text{ and } t_0 = 1.$$

Let $\{\sigma_n\}$ be a sequence such that $\sigma \to 0$ as $n \to \infty$. Then, up to a subsequence, still defined by $\{\sigma_n\}$, we have

$$\frac{t_0}{2} < t_{\sigma_n}.\tag{5.3}$$

Hence, we can choose $\tilde\sigma > 0$ small enough such that

$$\frac{t_0}{2} < t_{\tilde\sigma} \neq t_0.$$

Set

$$g(t) = \frac{t^p}{p} - \frac{t^{p^\star}}{p^\star} \text{ and } g'(t) = t^{p-1} - t^{p^\star-1}.$$

Then, we have $g'(t_0) = 0$, $g'(t) < 0$ for $t > t_0$, and $g'(t) > 0$ for $t < t_0$. Hence, $g(t)$ attains its maximum at t_0. That is,

$$g(t) < g(t_0) = \frac{1}{N}\tag{5.4}$$

for any $t \neq t_0$.

It follows from $q > \frac{p(N+\alpha-p)}{N-p}$ and (5.3)-(5.4) that

$$\begin{aligned}
J(t_{\tilde\sigma}w_{\tilde\sigma}) &\leqslant \left(\frac{t_{\tilde\sigma}^p}{p} - \frac{t_{\tilde\sigma}^{p^\star}}{p^\star}\right)\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \tilde\sigma^{p-\alpha} t_{\tilde\sigma}\int_{\mathbb{R}^N}\frac{A}{|x|^\alpha}|w_1|^p\mathrm{d}x - \frac{\tilde\sigma^{q+\frac{(p-q)N}{p}} t_{\tilde\sigma}^q}{q}\int_{\mathbb{R}^N}|w_1|^q\mathrm{d}x\\
&\leqslant \left(\frac{t_{\tilde\sigma}^p}{p} - \frac{t_{\tilde\sigma}^{p^\star}}{p^\star}\right)\|w_1\|^p_{D^{1,p}(\mathbb{R}^N)}\\
&= \left(\frac{t_{\tilde\sigma}^p}{p} - \frac{t_{\tilde\sigma}^{p^\star}}{p^\star}\right) S^{\frac{p^\star}{p^\star-p}}\\
&< \frac{1}{N} S^{\frac{p^\star}{p^\star-p}}
\end{aligned}$$

for sufficiently small $\tilde\sigma$. Consequently, we arrive at the desired result. □

5.1 Perturbation Equation

Applying Theorems 1.1 and 1.2, it is easy to prove the existence of positive ground state solution of the following equation (with small $\varepsilon > 0$, see [18]):

$$-\Delta_p u + \frac{A}{|x|^\alpha}|u|^{p-2}u = |u|^{p^*-\varepsilon-2}u + |u|^{q-2}u + \lambda|u|^{p_\alpha^*+\varepsilon-2}u, \ \ x \in \mathbb{R}^N. \qquad (\mathcal{S}_{p_\alpha^*+\varepsilon})$$

Set the energy functional of equation $(\mathcal{S}_{p_\alpha^*+\varepsilon})$ as follows:

$$J_\varepsilon(u) := \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \frac{A}{|x|^\alpha}|u|^p)\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x - \frac{1}{p^*-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^*-\varepsilon}\mathrm{d}x - \frac{\lambda}{p_\alpha^*+\varepsilon}\int_{\mathbb{R}^N}|u|^{p_\alpha^*+\varepsilon}\mathrm{d}x.$$

Let v_ε be a positive ground state solution of equation $(\mathcal{S}_{p_\alpha^*+\varepsilon})$. For all $\varphi \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$, it follows that

$$\begin{aligned} 0 = \langle J_\varepsilon'(v_\varepsilon), \varphi\rangle &= \int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\nabla\varphi + \frac{A}{|x|^\alpha}|v_\varepsilon|^{p-2}v_\varepsilon\varphi)\mathrm{d}x - \int_{\mathbb{R}^N}|v_\varepsilon|^{q-2}v_\varepsilon\varphi\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N}|v_\varepsilon|^{p^*-\varepsilon-2}v_\varepsilon\varphi\mathrm{d}x - \lambda\int_{\mathbb{R}^N}|v_\varepsilon|^{p_\alpha^*+\varepsilon-2}v_\varepsilon\varphi\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} 0 = P_\varepsilon(v_\varepsilon) &= \frac{1}{p^*}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p\mathrm{d}x + \frac{N-\alpha}{pN}\int_{\mathbb{R}^N}\frac{A|v_\varepsilon|^p}{|x|^\alpha}\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N}|v_\varepsilon|^q\mathrm{d}x \\ &\quad - \frac{1}{p^*-\varepsilon}\int_{\mathbb{R}^N}|v_\varepsilon|^{p^*-\varepsilon}\mathrm{d}x - \frac{\lambda}{p_\alpha^*+\varepsilon}\int_{\mathbb{R}^N}|v_\varepsilon|^{p_\alpha^*+\varepsilon}\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} c_\varepsilon = J_\varepsilon(v_\varepsilon) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + \frac{A}{|x|^\alpha}|v_\varepsilon|^p)\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N}|v_\varepsilon|^q\mathrm{d}x \\ &\quad - \frac{1}{p^*-\varepsilon}\int_{\mathbb{R}^N}|v_\varepsilon|^{p^*-\varepsilon}\mathrm{d}x - \frac{\lambda}{p_\alpha^*+\varepsilon}\int_{\mathbb{R}^N}|v_\varepsilon|^{p_\alpha^*+\varepsilon}\mathrm{d}x. \end{aligned}$$

We then have the following lemma for equation $(\mathcal{S}_{p_\alpha^*+\varepsilon})$.

Lemma 5.2. *Assume that all the conditions described in Theorem 1.3 hold. Then the following statements are true.*

(i) For each $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \setminus \{0\}$, there exists a unique $\tau_\varepsilon > 0$ such that $P_\varepsilon(u_{\tau_\varepsilon}) = 0$ for $\varepsilon \in (0, \varepsilon_0]$, where

$$u_\tau(x) = \begin{cases} u(\frac{x}{\tau}), & \tau > 0, \\ 0, & \tau = 0. \end{cases}$$

Moreover, we have $J_\varepsilon(u_{\tau_\varepsilon}) = \max_{\tau\geqslant 0} J_\varepsilon(u_\tau)$.

(ii) $c_\varepsilon = c_\varepsilon^P$ for $\varepsilon \in (0, \varepsilon_0]$, where

$$c_\varepsilon = \inf\{J_\varepsilon(u) | u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \text{ and } J_\varepsilon'(u) = 0\},$$

and

$$c_\varepsilon^P = \inf\{J_\varepsilon(u) | u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \text{ and } P_\varepsilon(u) = 0\}.$$

(iii) $\limsup_{\varepsilon\to 0} c_\varepsilon \leqslant c$.

(iv) $c_\varepsilon \geqslant 0$ for $\varepsilon \in [0, \varepsilon_0]$.

(v) Let $\varepsilon_n \to 0^+$ and $\{v_{\varepsilon_n}\} \subset W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ satisfy

$$J_{\varepsilon_n}(v_{\varepsilon_n}) = c_{\varepsilon_n}, \ \ P_{\varepsilon_n}(v_{\varepsilon_n}) = 0, \ \ J_{\varepsilon_n}'(v_{\varepsilon_n}) = 0.$$

Then, $\{v_{\varepsilon_n}\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N, \alpha)$ and $\liminf_{n\to\infty} c_{\varepsilon_n} > 0$.

Proof. **(i)** For each $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \setminus \{0\}$, we set

$$\varphi_\varepsilon(\tau) = J_\varepsilon(u_\tau) = \frac{\tau^{N-p}}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{\tau^{N-\alpha}}{p}\int_{\mathbb{R}^N} \frac{A|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{\tau^N}{q}\int_{\mathbb{R}^N}|u|^q \mathrm{d}x - \lambda\frac{\tau^N}{p^\star_\alpha+\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon}\mathrm{d}x - \frac{\tau^N}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x.$$

A direct calculation gives

$$\varphi_\varepsilon'(\tau) = \frac{N-p}{p}\tau^{N-p-1}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{N-\alpha}{p}\tau^{N-\alpha-1}\int_{\mathbb{R}^N} \frac{A|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{N\tau^{N-1}}{q}\int_{\mathbb{R}^N}|u|^q \mathrm{d}x - \frac{\lambda N\tau^{N-1}}{p^\star_\alpha+\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon}\mathrm{d}x - \frac{N\tau^{N-1}}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x.$$

In view of $N \geqslant 3$, $p \in (1, \sqrt{N})$ and $\alpha \in (0, p)$, we find that $\varphi_\varepsilon'(\tau) > 0$ for small $\tau > 0$ and $\lim_{\tau\to\infty} \varphi_\varepsilon'(\tau) < 0$. Then there exists $\tau_\varepsilon > 0$ such that $\varphi_\varepsilon'(\tau_\varepsilon) = 0$ and $J_\varepsilon(u_{\tau_\varepsilon}) = \max_{\tau\geqslant 0} J_\varepsilon(u_\tau)$. Moreover, $P_\varepsilon(u_{\tau_\varepsilon}) = \frac{1}{N}\tau_\varepsilon\varphi_\varepsilon'(\tau_\varepsilon) = 0$.

(ii). On one hand, Theorem 1.2 implies that $c_\varepsilon \geqslant c_\varepsilon^P$ for $\varepsilon \in [0, \varepsilon_0]$. On the other hand, we have

$$c_\varepsilon = c_\varepsilon^{mp} := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J_\varepsilon(\gamma(t)) > 0,$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], W^{1,p}_{rad}(\mathbb{R}^N,\alpha)\right) | \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0\right\}.$$

It is easy to see that there exists τ_1 large enough such that $J_\varepsilon(u_{\tau_1}) < 0$. Hence, we can choose $\gamma(t) = u_{t\tau_1}$.

Using Lemma 5.2 (i), we have $c_\varepsilon^{mp} \leqslant \max_{\tau\geqslant 0} J_\varepsilon(u_\tau) = J_\varepsilon(u_{\tau_\varepsilon})$. Since u is arbitrary, we obtain $c_\varepsilon^{mp} \leqslant c_\varepsilon^P$ and $c_\varepsilon = c_\varepsilon^P$ for $\varepsilon \in (0, \varepsilon_0]$.

(iii). For any $\delta \in (0, \frac{1}{2})$, there exists $u \in W^{1,p}_{rad}(\mathbb{R}^N, \alpha) \setminus \{0\}$ with $P(u) = 0$ such that $J(u) < c + \delta$. In view of $P(u) = 0$, we get

$$\frac{N-p}{p}\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x + \frac{N-\alpha}{p}\int_{\mathbb{R}^N}\frac{A|u|^p}{|x|^\alpha}\mathrm{d}x = \frac{N}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + \frac{\lambda N}{p^\star_\alpha}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha}\mathrm{d}x + \frac{N}{p^\star}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x > 0.$$

Then there exists $\bar\tau > 0$ large enough such that

$$J(u_{\bar\tau}) = \frac{\bar\tau^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x + \frac{\bar\tau^{N-\alpha}}{p}\int_{\mathbb{R}^N}\frac{A|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{\bar\tau^N}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x - \bar\tau^N\left(\frac{\lambda}{p^\star_\alpha}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha}\mathrm{d}x + \frac{1}{p^\star}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x\right) \leqslant -1.$$

We now show the continuity of $\frac{\tau^N}{p^\star_\alpha+\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon}\mathrm{d}x$ and $\frac{\tau^N}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x$ on $(\tau, \varepsilon) \in [0, \bar\tau] \times (0, \varepsilon_0)$.

Firstly, it is easy to check the continuity of $\frac{\tau^N}{p^\star_\alpha+\varepsilon}$ and $\frac{\tau^N}{p^\star-\varepsilon}$ on $(\tau, \varepsilon) \in [0, \bar\tau] \times (0, \varepsilon_0)$.

Secondly, let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$. Then $p^\star_\alpha + \varepsilon_1 < p^\star_\alpha + \varepsilon_2 < p^\star$. It follows from Hölder's and Young's inequalities that

$$\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_2}\mathrm{d}x \leqslant \frac{p^\star - p^\star_\alpha - \varepsilon_2}{p^\star - p^\star_\alpha - \varepsilon_1}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_1}\mathrm{d}x + \frac{\varepsilon_2-\varepsilon_1}{p^\star - p^\star_\alpha - \varepsilon_1}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x,$$

which gives

$$\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_2}\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_1}\mathrm{d}x \leqslant \frac{\varepsilon_1-\varepsilon_2}{p^\star - p^\star_\alpha - \varepsilon_1}\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_1}\mathrm{d}x + \frac{\varepsilon_2-\varepsilon_1}{p^\star - p^\star_\alpha - \varepsilon_1}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x.$$

That is,

$$\left|\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_2}\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_1}\mathrm{d}x\right| \leqslant \frac{\varepsilon_1-\varepsilon_2}{p^\star - p^\star_\alpha - \varepsilon_1}\left|\int_{\mathbb{R}^N}|u|^{p^\star_\alpha+\varepsilon_1}\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x\right|. \quad (5.5)$$

From (5.5), it is not difficult to see the continuity of $\int_{\mathbb{R}^N}|u|^{p_\alpha^\star+\varepsilon}\mathrm{d}x$ on $\varepsilon\in(0,\varepsilon_0)$. Similarly, we can prove the continuity of $\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x$ on $\varepsilon\in(0,\varepsilon_0)$ too.

Thirdly, let $f_1(\tau,\varepsilon)=\frac{\tau^N}{p_\alpha^\star+\varepsilon}$, $f_2(\tau,\varepsilon)=\frac{\tau^N}{p^\star-\varepsilon}$, $g_1(\varepsilon)=\int_{\mathbb{R}^N}|u|^{p_\alpha^\star+\varepsilon}\mathrm{d}x$ and $g_2(\varepsilon)=\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x$. Then $f_1(\tau,\varepsilon)\cdot g_1(\varepsilon)$ and $f_2(\tau,\varepsilon)\cdot g_2(\varepsilon)$ are continuous on $(\tau,\varepsilon)\in[0,\bar\tau]\times(0,\varepsilon_0)$.

Finally, by using the continuity of $\frac{\tau^N}{p_\alpha^\star+\varepsilon}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star+\varepsilon}\mathrm{d}x$ and $\frac{\tau^N}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x$ on $(\tau,\varepsilon)\in[0,\bar\tau]\times(0,\varepsilon_0)$, there exists $\bar\varepsilon>0$ such that for all $\varepsilon\in(0,\bar\varepsilon)$ and $\tau\in[0,\bar\tau]$ there holds

$$\begin{aligned}&|J_\varepsilon(u_\tau)-J(u_\tau)|\\&=\tau^N\left|\frac{\lambda}{p_\alpha^\star+\varepsilon}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star+\varepsilon}\mathrm{d}x-\frac{\lambda}{p_\alpha^\star}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star}\mathrm{d}x+\frac{1}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x-\frac{1}{p^\star}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x\right|\\&\leqslant\tau^N\left|\frac{\lambda}{p_\alpha^\star+\varepsilon}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star+\varepsilon}\mathrm{d}x-\frac{\lambda}{p_\alpha^\star}\int_{\mathbb{R}^N}|u|^{p_\alpha^\star}\mathrm{d}x\right|+\tau^N\left|\frac{1}{p^\star-\varepsilon}\int_{\mathbb{R}^N}|u|^{p^\star-\varepsilon}\mathrm{d}x-\frac{1}{p^\star}\int_{\mathbb{R}^N}|u|^{p^\star}\mathrm{d}x\right|\\&<\delta,\end{aligned}$$

which implies

$$J_\varepsilon(u_{\bar\tau})\leqslant-\frac{1}{2},\quad\varepsilon\in(0,\bar\varepsilon).$$

Note that $J_\varepsilon(u_\tau)>0$ for τ small enough. Then there exists $\bar\tau_\varepsilon\in(0,\bar\tau)$ such that $\frac{d}{d\tau}J_\varepsilon(u_\tau)|_{\tau=\bar\tau_\varepsilon}$, and $P_\varepsilon(u_{\bar\tau_\varepsilon})=0$. By Lemma 5.2 (i), we know $J(u_{\bar\tau_\varepsilon})\leqslant J(u)$. Thus, for any $\varepsilon\in(0,\bar\varepsilon)$ there holds

$$c_\varepsilon\leqslant J_\varepsilon(u_{\bar\tau_\varepsilon})\leqslant J(u_{\bar\tau_\varepsilon})+\delta\leqslant J(u)+\delta<c+2\delta.$$

Hence, $\limsup\limits_{\varepsilon\to0}c_\varepsilon\leqslant c$.

(iv). By a direct calculation, we have

$$c_\varepsilon=J_\varepsilon(v_\varepsilon)-P_\varepsilon(v_\varepsilon)=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^2\mathrm{d}x+\frac{\alpha}{pN}\int_{\mathbb{R}^N}\frac{A|v_\varepsilon|^2}{|x|^\alpha}\mathrm{d}x\geqslant0.$$

(v). By virtue of Lemma 5.2 (iii), we have

$$c+1\geqslant c_{\varepsilon_n}=J_{\varepsilon_n}(v_{\varepsilon_n})-P_{\varepsilon_n}(v_{\varepsilon_n})=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x+\frac{\alpha}{pN}\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\geqslant C\|v_{\varepsilon_n}\|^p_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}.$$

Namely, $\{v_{\varepsilon_n}\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$.

It follows from (2.1)-(2.2) that

$$0=P_{\varepsilon_n}(v_{\varepsilon_n})\geqslant C\|v_{\varepsilon_n}\|^p_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}-C\|v_{\varepsilon_n}\|^q_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}-C\|v_{\varepsilon_n}\|^{p_\alpha^\star+\varepsilon_n}_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}-C\|v_{\varepsilon_n}\|^{p^\star-\varepsilon_n}_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)},$$

which implies that there exists $C>0$ independent of n such that

$$\|v_{\varepsilon_n}\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}\geqslant C.$$

Hence, we obtain $\liminf\limits_{n\to\infty}c_{\varepsilon_n}>0$. □

5.2 Ground State Solution

In this subsection, by using the perturbation method [13] and Pohožaev-type identity [1], we present the proof of Theorem 1.3.

Proof of Theorem 1.3. We separate our proof into two steps.

Step 1. We take $\varepsilon\to0$ in equation $(\mathcal{S}_{p_\alpha^\star+\varepsilon})$. For each small ε_n, there exists a positive ground state solution v_{ε_n}. Using Lemma 5.2 (iii), we have

$$c+1\geqslant c_{\varepsilon_n}=J_{\varepsilon_n}(v_{\varepsilon_n})-\frac{1}{p_\alpha^\star+\varepsilon_n}\langle J'_{\varepsilon_n}(v_{\varepsilon_n}),v_{\varepsilon_n}\rangle\geqslant\left(\frac{1}{p}-\frac{1}{p_\alpha^\star+\varepsilon_n}\right)\|v_{\varepsilon_n}\|^p_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}.$$

This implies that $\{v_{\varepsilon_n}\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$. Then, up to a subsequence, we assume that

$$v_{\varepsilon_n} \rightharpoonup v \text{ in } W^{1,p}_{rad}(\mathbb{R}^N,\alpha),\ v_{\varepsilon_n} \to v \text{ a.e. in } \mathbb{R}^N,\ v_{\varepsilon_n} \to v \text{ in } L^r(\mathbb{R}^N),\ r \in (p^\star_\alpha, p^\star).$$

For any $\varphi \in W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$, as $n \to \infty$, we claim that

$$\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^\star-\varepsilon_n-2} v_{\varepsilon_n}\varphi \mathrm{d}x = \int_{\mathbb{R}^N} |v|^{p^\star-2} v\varphi \mathrm{d}x + o(1) \tag{5.6}$$

and

$$\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^\star_\alpha+\varepsilon_n-2} v_{\varepsilon_n}\varphi \mathrm{d}x = \int_{\mathbb{R}^N} |v|^{p^\star_\alpha-2} v\varphi \mathrm{d}x + o(1). \tag{5.7}$$

Here, we only show (5.6), because the proof of (5.7) can be processed in a similar manner. For any $\epsilon > 0$, there exists a sufficiently large $R > 0$ such that

$$\begin{aligned}
&\int_{|x|>R} |v_{\varepsilon_n}|^{p^\star-\varepsilon_n-2} v_{\varepsilon_n}\varphi \mathrm{d}x - \int_{|x|>R} |v|^{p^\star-2} v\varphi \mathrm{d}x \\
&\leqslant \int_{|x|>R} |v_{\varepsilon_n}|^{p^\star-\varepsilon_n-1} |\varphi| \mathrm{d}x + \int_{|x|>R} |v|^{p^\star-1} |\varphi| \mathrm{d}x \\
&\leqslant \left(\int_{|x|>R} |v_{\varepsilon_n}|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{1-\frac{1}{p^\star-\varepsilon_n}} \left(\int_{|x|>R} |\varphi|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{\frac{1}{p^\star-\varepsilon_n}} \\
&\quad + \left(\int_{|x|>R} |v|^{p^\star} \mathrm{d}x\right)^{1-\frac{1}{p^\star}} \left(\int_{|x|>R} |\varphi|^{p^\star} \mathrm{d}x\right)^{\frac{1}{p^\star}} < \frac{\epsilon}{2}.
\end{aligned}$$

On the other hand, note that $\{v_{\varepsilon_n}\}$ is bounded in $W^{1,p}_{rad}(\mathbb{R}^N,\alpha)$. There exists $C > 0$ such that

$$\left(\int_{|x|\leqslant R} |v_{\varepsilon_n}|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{1-\frac{1}{p^\star-\varepsilon_n}} < C.$$

In view of $E \subset \mathbb{R}^N$ and small $\varepsilon_n > 0$, it follows from Holder's and Young's inequalities that

$$\begin{aligned}
&\int_E |\varphi|^{p^\star-\varepsilon_n} \mathrm{d}x \\
&\leqslant \left(\int_E |\varphi|^{p^\star_\alpha} \mathrm{d}x\right)^{\frac{\varepsilon_n}{p^\star-p^\star_\alpha}} \left(\int_E |\varphi|^{p^\star} \mathrm{d}x\right)^{\frac{p^\star-p^\star_\alpha-\varepsilon_n}{p^\star-p^\star_\alpha}} \\
&\leqslant \frac{\varepsilon_n}{p^\star-p^\star_\alpha} \int_E |\varphi|^{p^\star_\alpha} \mathrm{d}x + \frac{p^\star-p^\star_\alpha-\varepsilon_n}{p^\star-p^\star_\alpha} \int_E |\varphi|^{p^\star} \mathrm{d}x \\
&\leqslant \int_E |\varphi|^{p^\star_\alpha} \mathrm{d}x + \int_E |\varphi|^{p^\star} \mathrm{d}x.
\end{aligned}$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that when $E \subset \{x \in \mathbb{R}^N \,|\, |x| \leqslant R\}$ with $|E| < \delta$ there holds

$$\begin{aligned}
&\int_E |v_{\varepsilon_n}|^{p^\star-\varepsilon_n-2} v_{\varepsilon_n}\varphi \mathrm{d}x \\
&\leqslant \left(\int_E |v_{\varepsilon_n}|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{1-\frac{1}{p^\star-\varepsilon_n}} \left(\int_E |\varphi|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{\frac{1}{p^\star-\varepsilon_n}} \\
&\leqslant \left(\int_E |v_{\varepsilon_n}|^{p^\star-\varepsilon_n} \mathrm{d}x\right)^{1-\frac{1}{p^\star-\varepsilon_n}} \left(\int_E |\varphi|^{p^\star_\alpha} \mathrm{d}x + \int_E |\varphi|^{p^\star} \mathrm{d}x\right)^{\frac{1}{p^\star-\varepsilon_n}} \\
&< C\epsilon,
\end{aligned}$$

where the last inequality is true due to the absolute continuity of $\int_E |\varphi|^{p^\star_\alpha} \mathrm{d}x$ and $\int_E |\varphi|^{p^\star} \mathrm{d}x$.

Making use of the fact $|v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \to |v|^{p^*-2}v\varphi$ a.e. in $\mathbb{R}^N$, by Vitali's convergence Theorem, we have

$$\int_{|x|\leqslant R} |v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x = \int_{|x|\leqslant R} |v|^{p^*-2}v\varphi \mathrm{d}x + \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x - \int_{\mathbb{R}^N} |v|^{p^*-2}v\varphi \mathrm{d}x \\
\leqslant &\int_{|x|>R} |v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x - \int_{|x|>R} |v|^{p^*-2}v\varphi \mathrm{d}x \\
&+ \int_{|x|\leqslant R} |v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x - \int_{|x|\leqslant R} |v|^{p^*-2}v\varphi \mathrm{d}x \\
\leqslant &\epsilon.
\end{aligned}$$

Hence, we arrive at (5.6).

It follows from (5.6) and (5.7) that

$$\begin{aligned}
0 =&\langle J'_{\varepsilon_n}(v_{\varepsilon_n}), \varphi\rangle \\
=&\int_{\mathbb{R}^N} \left(|\nabla v_{\varepsilon_n}|^{p-2}\nabla v_{\varepsilon_n}\nabla\varphi + \frac{A}{|x|^\alpha}|v_{\varepsilon_n}|^{p-2}v_{\varepsilon_n}\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{q-2}v_{\varepsilon_n}\varphi \mathrm{d}x \\
&- \int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*-\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x - \lambda\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*+\varepsilon_n-2}v_{\varepsilon_n}\varphi \mathrm{d}x \\
=&\int_{\mathbb{R}^N} \left(|\nabla v|^{p-2}\nabla v\nabla\varphi + \frac{A}{|x|^\alpha}|v|^{p-2}v\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^N} |v|^{q-2}v\varphi \mathrm{d}x \\
&- \int_{\mathbb{R}^N} |v|^{p^*-2}v\varphi \mathrm{d}x - \lambda\int_{\mathbb{R}^N} |v|^{p_\alpha^*-2}v\varphi \mathrm{d}x \\
=&\langle J'(v), \varphi\rangle.
\end{aligned}$$

This indicates that v is a weak solution of equation (Ω).

Step 2. We claim that $v \not\equiv 0$.

In view of $\langle J'_{\varepsilon_n}(v_{\varepsilon_n}), v_{\varepsilon_n}\rangle = 0$, we get

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_n}|^p \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x = \int_{\mathbb{R}^N} |v_{\varepsilon_n}|^q \mathrm{d}x + \int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*-\varepsilon_n}\mathrm{d}x + \lambda\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*+\varepsilon_n}\mathrm{d}x. \tag{5.8}$$

It follows from Holder's and Young's inequalities that

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*+\varepsilon_n}\mathrm{d}x \leqslant& \left(\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*}\mathrm{d}x\right)^{\frac{p^*-p_\alpha^*-\varepsilon_n}{p^*-p_\alpha^*}} \left(\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*}\mathrm{d}x\right)^{\frac{\varepsilon_n}{p^*-p_\alpha^*}} \\
\leqslant& \frac{p^*-p_\alpha^*-\varepsilon_n}{p^*-p_\alpha^*}\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*}\mathrm{d}x + \frac{\varepsilon_n}{p^*-p_\alpha^*}\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*}\mathrm{d}x,
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*-\varepsilon_n}\mathrm{d}x \leqslant& \left(\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*}\mathrm{d}x\right)^{\frac{\varepsilon_n}{p^*-p_\alpha^*}} \left(\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*}\mathrm{d}x\right)^{\frac{p^*-p_\alpha^*-\varepsilon_n}{p^*-p_\alpha^*}} \\
\leqslant& \frac{\varepsilon_n}{p^*-p_\alpha^*}\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*}\mathrm{d}x + \frac{p^*-p_\alpha^*-\varepsilon_n}{p^*-p_\alpha^*}\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*}\mathrm{d}x.
\end{aligned} \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8) leads to

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_n}|^p \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x \\
\leqslant& \left(1 + \frac{\varepsilon_n(\lambda-1)}{p^*-p_\alpha^*}\right)\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p^*}\mathrm{d}x + \left(\lambda + \frac{\varepsilon_n(1-\lambda)}{p^*-p_\alpha^*}\right)\int_{\mathbb{R}^N} |v_{\varepsilon_n}|^{p_\alpha^*}\mathrm{d}x + \int_{\mathbb{R}^N} |v_{\varepsilon_n}|^q \mathrm{d}x.
\end{aligned} \tag{5.11}$$

It suffices to show that there exists $C > 0$ such that $C \leqslant \|v_{\varepsilon_n}\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}$. Otherwise, we assume that $\|v_{\varepsilon_n}\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)} \to 0$. Then it yields $c_{\varepsilon_n} \to 0$, which contradicts $\liminf_{n\to\infty} c_{\varepsilon_n} > 0$, see Lemma 5.2 (v).

It follows from (5.11) and $0 < C \leqslant \|v_{\varepsilon_n}\|_{W^{1,p}_{rad}(\mathbb{R}^N,\alpha)}$ that only one of the following statements is true.

$$\begin{aligned}&(i)\ \int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^\star_\alpha}\mathrm{d}x\to 0\ \text{ and }\ C\leqslant\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^*}\mathrm{d}x;\ \text{ or}\\&(ii)\ C\leqslant\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^\star_\alpha}\mathrm{d}x\ \text{ and }\ \int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^*}\mathrm{d}x\to 0;\ \text{ or}\\&(iii)\ C\leqslant\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^\star_\alpha}\mathrm{d}x\ \text{ and }\ C\leqslant\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^*}\mathrm{d}x.\end{aligned}\tag{5.12}$$

We first exclude (i) in (5.12). Suppose on the contrary that (5.12) (i) holds. It follows from (5.12) (i) and (5.11) that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x&\leqslant\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\\&\leqslant\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^*}\mathrm{d}x\\&\leqslant S^{-\frac{p^*}{p}}\left(\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x\right)^{\frac{p^*}{p}},\end{aligned}$$

which gives

$$S^{\frac{p^*}{p^*-p}}\leqslant\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x.$$

In view of Lemma 5.2, we get

$$\begin{aligned}c\geqslant&c_{\varepsilon_n}\\=&J_{\varepsilon_n}(v_{\varepsilon_n})-\frac{1}{N}P_{\varepsilon_n}(v_{\varepsilon_n})\\=&\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^2\mathrm{d}x+\frac{\alpha}{pN}\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\\\geqslant&\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x\\\geqslant&\frac{1}{N}S^{\frac{p^*}{p^*-p}}.\end{aligned}$$

This yields a contradiction with the fact of $c < \min\left\{\frac{1}{N}S^{\frac{p^*}{p^*-p}},\ \frac{\alpha}{pN}\left(\frac{1}{\lambda}\right)^{\frac{p}{p^\star_\alpha-p}}S_\alpha^{\frac{p^\star_\alpha}{p^\star_\alpha-p}}\right\}$, see Lemma 5.1. Hence, (5.12) (i) can not occur.

We now exclude (ii) in (5.12). Suppose on the contrary that (5.12) (ii) holds. It follows form (5.12) (ii) and (5.11) that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x&\leqslant\lambda\int_{\mathbb{R}^N}|v_{\varepsilon_n}|^{p^\star_\alpha}\mathrm{d}x\\&\leqslant\lambda S_\alpha^{-\frac{p^\star_\alpha}{p}}\left(\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p^\star_\alpha}{p}},\end{aligned}$$

which gives

$$\left(\frac{1}{\lambda}\right)^{\frac{p}{p^\star_\alpha-p}}S_\alpha^{\frac{p^\star_\alpha}{p^\star_\alpha-p}}\leqslant\int_{\mathbb{R}^N}|\nabla v_{\varepsilon_n}|^p\mathrm{d}x+\int_{\mathbb{R}^N}\frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x.$$

Using Lemma 5.2 yields

$$\begin{aligned}c \geqslant & c_{\varepsilon_n}\\ =& J_{\varepsilon_n}(v_{\varepsilon_n}) - P_{\varepsilon_n}(v_{\varepsilon_n})\\ =& \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_n}|^p \mathrm{d}x + \frac{\alpha}{pN}\int_{\mathbb{R}^N} \frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\\ \geqslant & \frac{\alpha}{pN}\left(\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_n}|^p \mathrm{d}x + \int_{\mathbb{R}^N} \frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\right)\\ \geqslant & \frac{\alpha}{pN}\left(\frac{1}{\lambda}\right)^{\frac{p}{p_\alpha^\star - p}} S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star - p}}.\end{aligned}$$

This contradicts the fact of $c < \min\left\{\frac{1}{N}S^{\frac{p^\star}{p^\star - p}}, \frac{\alpha}{pN}\left(\frac{1}{\lambda}\right)^{\frac{p}{p_\alpha^\star - p}} S_\alpha^{\frac{p_\alpha^\star}{p_\alpha^\star - p}}\right\}$, see Lemma 5.1. Hence, (5.12) (ii) can not occur either.

We now draw a conclusion that (5.12) (iii) is true. By virtue of Theorem 1.1, we have $v \not\equiv 0$. In view of Theorem 1.2, $P(v) = 0$ and the weakly lower semi-continuity of the norm, we obtain

$$\begin{aligned}c \leqslant & J(v)\\ =& J(v) - P(v)\\ =& \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^p \mathrm{d}x + \frac{\alpha}{pN}\int_{\mathbb{R}^N} \frac{A|v|^p}{|x|^\alpha}\mathrm{d}x\\ \leqslant & \liminf_{n\to\infty} \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_{\varepsilon_n}|^p \mathrm{d}x + \liminf_{n\to\infty} \frac{\alpha}{pN}\int_{\mathbb{R}^N} \frac{A|v_{\varepsilon_n}|^p}{|x|^\alpha}\mathrm{d}x\\ =& J_{\varepsilon_n}(v_{\varepsilon_n}) - \frac{1}{N}P_{\varepsilon_n}(v_{\varepsilon_n})\\ =& c_{\varepsilon_n}\\ \leqslant & c.\end{aligned}$$

Consequently, v is a positive ground state solution. □

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