Isomorphism Testing for Graphs Excluding Small Minors

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***Abstract*—We prove that there is a graph isomorphism test running in time $n^{\text{polylog}(h)}$ on n-vertex graphs excluding some h-vertex graph as a minor. Previously known bounds were $n^{\text{poly}(h)}$ (Ponomarenko, 1988) and $n^{\text{polylog}(n)}$ (Babai, STOC 2016). For the algorithm we combine recent advances in the group-theoretic graph isomorphism machinery with new graph-theoretic arguments.**

***Keywords*-graph isomorphism problem, excluded minors, structure of automorphism group**

I. Introduction

Determining the computational complexity of the Graph Isomorphism Problem (GI) is one of best-known open problems in theoretical computer science. GI is obviously in NP, but neither known to be NP-complete nor known to be solvable in polynomial time. In a recent breakthrough result, Babai [1] presented a quasipolynomial-time algorithm (i.e., an algorithm running in time $n^{\text{polylog}(n)}$) deciding isomorphism of two graphs, significantly improving over the best previous algorithm running in time $n^{\mathcal{O}(\sqrt{n/\log n})}$ [2]. For his algorithm, Babai greatly extends the group-theoretic isomorphism machinery dating back to Luks [3] as well as our understanding of combinatorial methods like the Weisfeiler-Leman algorithm (see, e.g., [4], [5]). Still, the question of whether the Graph Isomorphism Problem can be solved in polynomial time remains wide open.

Polynomial-time algorithms are known for restrictions of the Graph Isomorphism Problem to several important graph classes (e.g., [6], [7], [8], [9], [10], [3], [11], [12]). In particular, Luks [3] gave an isomorphism algorithm running in time $n^{\mathcal{O}(d)}$ on input graphs of maximum degree d. Building on Luks's techniques and refinements due to Miller [13], Ponomarenko [12] designed an isomorphism test running in time $n^{\text{poly}(h)}$ for all graph classes that exclude a fixed graph with h vertices as a minor. Later, it was shown that the polynomial-time bound can be pushed to graph classes excluding a fixed topological subgraph [8].

For the algorithms mentioned above the exponent of the polynomial always depends at least linearly on the parameter in question. In light of Babai's quasipolynomial-time algorithm it seems natural to ask for which parameters these dependencies can be improved to polylogarithmic.

In [14] it was shown that Luks's original isomorphism test for bounded-degree graphs can be combined with Babai's group-theoretic techniques. By using a novel normalization technique, Schweitzer and the first two authors of this paper provided an isomorphism algorithm for graphs of maximum degree d running in time $n^{\text{polylog}(d)}$. Recently, it was shown that the group-theoretic techniques used for bounded-degree graphs can be extended to isomorphism testing of hypergraphs [15]. This was used as an important subroutine in an isomorphism test for graphs of Euler genus g running in time $n^{\text{polylog}(g)}$. Another branch of research deals with the question how Babai's and Luks's group-theoretic techniques can be combined with graph decomposition techniques [16] (see also [17], [18]). This series of papers led to an isomorphism test for graphs of tree-width at most k running in time $n^{\text{polylog}(k)}$.

In this work, we assemble the recent advances in the group-theoretic machinery developed in [14], [15], [16] and combine it with new structural results for graphs with excluded minors. Recall that a graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. If H is not a minor of G, we say that G *excludes* H as a minor. For example, all planar graphs exclude the complete graph K_5 and the complete bipartite graph $K_{3,3}$ as a minor, and in fact this characterizes the planar graphs [19]. Other natural classes of graphs excluding some fixed graph as a minor are, for example, classes of bounded genus, bounded tree-width, and the class of graphs linklessly embeddable in 3-space [20].

We present a new isomorphism test for graph classes that exclude a fixed graph as a minor, improving the previously best algorithm for this problem due to Ponomarenko [12] running in time $n^{\text{poly}(h)}$.

Theorem I.1. *There is a graph isomorphism algorithm running in time $n^{\text{polylog}(h)}$ on n-vertex graphs that exclude some h-vertex graph as a minor.*

Note that a graph G excludes some h-vertex graph as a minor if and only if G excludes the complete graph K_h on h vertices as minor. Hence, for the remainder of this work, we restrict ourselves to the case where the input graphs exclude

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K_h as minor.

The maximum h such that K_h is a minor of G is known as the *Hadwiger number* $\mathrm{hd}(G)$ of G (this means G excludes $K_{\mathrm{hd}(G)+1}$ as a minor). Thus, an equivalent formulation of our result is that we design an isomorphism test for n-vertex graphs running in time $n^{\mathrm{polylog}(\mathrm{hd}(G))}$.

Our proof heavily builds on the recently developed group-theoretic machinery (the dependencies on the main previous results are shown in Figure 1). The main technical contributions of the present paper are of a graph-theoretic nature. However, we are not using Robertson-Seymour-style structure theory for graphs with excluded minors [21], as one may expect given the previous results for graphs of bounded genus and of bounded tree-width. Instead, our results can be viewed as a structural theory for the automorphism groups of such graphs; we find that graphs excluding K_h as a minor have an isomorphism-invariant decomposition into pieces whose automorphism groups are similar to those of bounded-degree graphs (Theorem IV.4 is the precise statement). This structural result may be of independent interest. The only deeper graph-theoretic result we use is Kostochka's and Thomason's theorem stating that graphs excluding K_h as a minor have an average degree of $\mathcal{O}(h\log h)$ [22], [23].

On a high level, our algorithm follows a decomposition strategy. Given two graphs G_1 and G_2 excluding K_h as a minor, the goal is to find isomorphism-invariant subsets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ such that one can control the interplay between the subsets and its complement and one can significantly restrict the graph automorphisms on the two subsets. Note that it is crucial to define the subsets D_1 and D_2 in an isomorphism-invariant fashion as to not compare two graphs that are decomposed in structurally different ways. To capture the restrictions on the automorphism group, we build on the well-known class of $\widehat{\Gamma}_d$-groups, which are groups all whose composition factors are isomorphic to a subgroup of S_d (the symmetric group on d points). However, to prove the restrictions on the automorphism group, we mostly use combinatorial and graph-theoretic arguments.

In particular, the algorithm heavily uses the 2-dimensional Weisfeiler-Leman algorithm, a standard combinatorial algorithm which computes an isomorphism-invariant coloring of pairs of vertices. In a lengthy case-by-case analysis depending on the color patterns computed by the 2-dimensional Weisfeiler-Leman algorithm, we are able to find initial isomorphism-invariant subsets $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ such that $(\mathrm{Aut}(G_i))_{v_i}[X_i]$ (the automorphism group of G_i restricted to X_i after fixing some vertex $v_i \in X_i$) forms a $\widehat{\Gamma}_t$-group where $t \in \mathcal{O}((h\log h)^3)$.

In order to get control of the interplay between the subsets and their complement, we define a novel closure operator that builds on *t-CR-bounded* graphs, which were recently introduced in the context of isomorphism testing for bounded genus graphs [15]. This operator increases the subsets X_1 and X_2 in an isomorphism-invariant fashion and leads to (possibly larger) sets $D_i := \mathrm{cl}_t^{G_i}(X_i) \supseteq X_i$, $i \in \{1,2\}$. A feature of this operator is that a given $\widehat{\Gamma}_t$-group defined on the initial set X_i can be extended to a $\widehat{\Gamma}_t$-group defined on the superset D_i (see Theorem III.7). This provides us a $\widehat{\Gamma}_t$-group on the closure D_i (after fixing a point) which allows the use of the group-theoretic techniques from [14], [15].

The second main feature of the closure operator is that, in a graph G that excludes an h-vertex graph as a minor, the closure $D := \mathrm{cl}_t^G(X)$ of any set $X \subseteq V(G)$ can only stop to grow at a separator of small size. More precisely, we show that for every vertex set Z of a connected component of $G-D$, it holds that $|N_G(Z)| < h$. This key result shows that the interplay between D and its complement in G is simple and allows for the application of the group-theoretic decomposition framework from [17], [18], [16].

We remark that our proof strategy is quite different from that used by Ponomarenko [12] in his isomorphism test for graphs with excluded minors, because we could not improve Miller's [13] "tower-of-$\widehat{\Gamma}_d$-groups" technique to meet our quasipolynomial time demands.

The paper is structured as follows. After introducing some basic preliminaries in the next section, we review the recent advances on the group-theoretic isomorphism machinery from [15], [16] in Section III. Then, we present the main results in Section IV and also give an overview on the proofs of the main technical theorems. For technical details we refer to the full version of this paper [24].

II. Preliminaries

A. Graphs

A *graph* is a pair $G = (V(G), E(G))$ consisting of a *vertex set* $V(G)$ and an *edge set* $E(G) \subseteq \binom{V(G)}{2} := \{\{u,v\} \mid u,v \in V(G), u \neq v\}$. All graphs considered in this paper are finite, undirected and simple (i.e., they contain no loops or multiple edges). For $v,w \in V$, we also write vw as a shorthand for $\{v,w\}$. The *neighborhood* of v is denoted by $N_G(v)$. The *degree* of v, denoted by $\deg_G(v)$, is the number of edges incident with v, i.e., $\deg_G(v) = |N_G(v)|$. For $X \subseteq V(G)$, we define $N_G(X) := \left(\bigcup_{v\in X} N(v)\right) \setminus X$. If the graph G is clear from context, we usually omit the index and simply write $N(v)$, $\deg(v)$ and $N(X)$.

For a set $A \subseteq V(G)$, we denote by $G[A]$ the *induced subgraph* of G on the vertex set A. Also, we denote by $G - A$ the subgraph induced by the complement of A, that is, the graph $G - A := G[V(G) \setminus A]$. A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A set $S \subseteq V(G)$ is a *separator* of G if $G - S$ has more connected components than G. A *k-separator* of G is a separator of G of size k.

An *isomorphism* from G to a graph H is a bijection $\varphi\colon V(G) \to V(H)$ that respects the edge relation, that is, for all $v,w \in V(G)$, it holds that $vw \in E(G)$ if and only if $\varphi(v)\varphi(w) \in E(H)$. Two graphs G and H are *isomorphic*, written $G \cong H$, if there is an isomorphism from G to H. We write $\varphi\colon G \cong H$ to denote that φ is an isomorphism from G to H. Also, $\mathrm{Iso}(G,H)$ denotes the set of all isomorphisms from G to H. The automorphism group of G is $\mathrm{Aut}(G) \coloneqq \mathrm{Iso}(G,G)$. Observe that, if $\mathrm{Iso}(G,H) \neq \emptyset$, it holds that $\mathrm{Iso}(G,H) = \mathrm{Aut}(G)\varphi \coloneqq \{\gamma\varphi \mid \gamma \in \mathrm{Aut}(G)\}$ for every isomorphism $\varphi \in \mathrm{Iso}(G,H)$.

A *vertex-colored graph* is a tuple (G,χ) where G is a graph and $\chi\colon V(G) \to C$ is a mapping into some set C of colors, called *vertex-coloring*. Similarly, an *arc-colored graph* is a tuple (G,χ), where G is a graph and $\chi\colon \{(u,v) \mid \{u,v\} \in E(G)\} \to C$ is a mapping into some color set C, called *arc-coloring*. We also consider vertex- and arc-colored graphs (G,χ_V,χ_E) where χ_V is a vertex-coloring and χ_E is an arc-coloring. Also, a *pair-colored graph* is a tuple (G,χ), where G is a graph and $\chi\colon (V(G))^2 \to C$ is a mapping into some color set C. Typically, C is chosen to be an initial segment $[n]$ of the natural numbers. Isomorphisms between vertex-, arc- and pair-colored graphs have to respect the colors of the vertices, arcs and pairs.

B. Graph Minors and Topological Subgraphs

Let G be a graph. A graph H is a *minor* of G if H can be obtained from G by deleting vertices and edges of G as well as contracting edges of G. More formally, let $\mathcal{B} = \{B_1,\dots,B_h\}$ be a partition of $V(G)$ such that $G[B_i]$ is connected for all $i \in [h]$. We define $G/\mathcal{B}$ to be the graph with vertex set $V(G/\mathcal{B}) \coloneqq \mathcal{B}$ and $E(G/\mathcal{B}) \coloneqq \{BB' \mid \exists v \in B, v' \in B' \colon vv' \in E(G)\}$. A graph H is a minor of G if there is a partition $\mathcal{B} = \{B_1,\dots,B_h\}$ of connected subsets $B_i \subseteq V(G)$ such that H is isomorphic to a subgraph of $G/\mathcal{B}$. A graph G *excludes H as a minor* if H is not a minor of G. The following theorem states the well-known fact that graphs excluding small minors have bounded average degree. This was observed by Mader before Kostochka and Thomason independently proved an optimal bound on the average degree.

Theorem II.1 ([25], [22], [23])**.** *There is an absolute constant $a \geq 1$ such that for every $h \geq 1$ and every graph G that excludes K_h as a minor, it holds that $\frac{1}{|V(G)|}\sum_{v \in V(G)} \deg_G(v) \leq ah\log h$.*

A graph H is a *topological subgraph* of G if H can be obtained from G by deleting edges, deleting vertices and dissolving degree 2 vertices (which means deleting the vertex and making its two neighbors adjacent). More formally, we say that H is a topological subgraph of G if a subdivision of H is a subgraph of G (a subdivision of a graph H is obtained by replacing each edge of H by a path of length at least 1). Note that every topological subgraph of G is also a minor of G.

C. Weisfeiler-Leman Algorithm

The Weisfeiler-Leman algorithm, originally introduced by Weisfeiler and Leman in its two-dimensional form [5], forms one of the most fundamental subroutines in the context of isomorphism testing. The algorithm presented in this work crucially builds on the 1-dimensional Weisfeiler-Leman algorithm, also known as the *Color Refinement algorithm*, as well as the 2-dimensional Weisfeiler-Leman algorithm.

Let $\chi_1,\chi_2\colon V^k \to C$ be colorings of the k-tuples of vertices of G, where C is some finite set of colors. We say χ_1 *refines* χ_2, denoted $\chi_1 \preceq \chi_2$, if $\chi_1(\bar v) = \chi_1(\bar w)$ implies $\chi_2(\bar v) = \chi_2(\bar w)$ for all $\bar v, \bar w \in V^k$. The two colorings χ_1 and χ_2 are *equivalent*, denoted $\chi_1 \equiv \chi_2$, if $\chi_1 \preceq \chi_2$ and $\chi_2 \preceq \chi_1$.

The *Color Refinement algorithm* (i.e., the 1-dimensional Weisfeiler-Leman algorithm) is a procedure that, given a graph G, iteratively computes an isomorphism-invariant coloring of the vertices of G. In this work, we actually require an extension of the Color Refinement algorithm that apart from vertex-colors also takes arc-colors into account. We describe the mechanisms of the algorithm in the following. For a vertex- and arc-colored graph (G,χ_V,χ_E) define $\chi^1_{G,0} \coloneqq \chi_V$ to be the initial coloring for the algorithm. This coloring is iteratively refined by defining $\chi^1_{G,i+1}(v) \coloneqq (\chi^1_{G,i}(v), \mathcal{M}_i(v))$ where

$$\mathcal{M}_i(v) \coloneqq \{\!\{ \big(\chi^1_{G,i}(w), \chi_E(v,w), \chi_E(w,v)\big) \bigm| w \in N_G(v) \}\!\}$$

(and $\{\!\{\dots\}\!\}$ denotes a multiset). By definition, $\chi^1_{G,i+1} \preceq \chi^1_{G,i}$ for all $i \geq 0$. Thus, there is a minimal i such that $\chi^1_{G,i+1}$ is equivalent to $\chi^1_{G,i}$. For this value of i we call the coloring $\chi^1_{G,i}$ the *stable* coloring of G and denote it by $\chi^1_{\mathsf{WL}}[G]$. The Color Refinement algorithm takes as input a vertex- and arc-colored graph (G,χ_V,χ_E) and returns (a coloring that is equivalent to) $\chi^1_{\mathsf{WL}}[G]$. The procedure can be implemented in time $\mathcal{O}((m+n)\log n)$ (see, e.g., [26]).

Next, we define the 2-*dimensional Weisfeiler-Leman algorithm*. For a vertex-colored graph (G,χ_V) let $\chi^2_{G,0}\colon (V(G))^2 \to C$ be the coloring where each pair is colored with the isomorphism type of its underlying ordered subgraph. More formally, $\chi^2_{G,0}(v_1,v_2) = \chi^2_{G,0}(v_1',v_2')$ if and only if $\chi_V(v_i) = \chi_V(v_i')$ for both $i \in \{1,2\}$, $v_1 = v_2 \Leftrightarrow v_1' = v_2'$ and $v_1v_2 \in E(G) \Leftrightarrow v_1'v_2' \in E(G)$. We then recursively define the coloring $\chi^2_{G,i}$ obtained after i rounds of the algorithm. Let $\chi^2_{G,i+1}(v_1,v_2) \coloneqq (\chi^2_{G,i}(v_1,v_2), \mathcal{M}_i(v_1,v_2))$ where

$$\mathcal{M}_i(v_1,v_2) \coloneqq \{\!\{ \big(\chi^2_{G,i}(v_1,w), \chi^2_{G,i}(w,v_2)\big) \bigm| w \in V(G) \}\!\}.$$

Again, there is a minimal i such that $\chi^2_{G,i+1}$ is equivalent to $\chi^2_{G,i}$ and for this i the coloring $\chi^2_{\mathsf{WL}}[G] \coloneqq \chi^2_{G,i}$ is the *stable* coloring of G.

Note that the algorithm can easily be extended to arc-colored and pair-colored graphs by modifying the definition of the initial coloring $\chi^2_{G,0}$ accordingly. However, in contrast to the Color Refinement algorithm, the 2-dimensional Weisfeiler-Leman algorithm is only applied to vertex-colored graphs throughout this paper.

The 2-dimensional Weisfeiler-Leman algorithm takes as input a (vertex-, arc- or pair-)colored graph G and returns (a coloring that is equivalent to) $\chi^2_{\mathsf{WL}}[G]$. This can be implemented in time $\mathcal{O}(n^3 \log n)$ (see [27]).

D. Group Theory

In this subsection, we introduce the group-theoretic notions required in this work. For a general background on group theory we refer to [28], whereas background on permutation groups can be found in [29].

Permutation groups: A *permutation group* acting on a set Ω is a subgroup $\Gamma \leq \mathrm{Sym}(\Omega)$ of the symmetric group. The size of the permutation domain Ω is called the *degree* of Γ. If $\Omega = [n]$, then we also write S_n instead of $\mathrm{Sym}(\Omega)$. For $\gamma \in \Gamma$ and $\alpha \in \Omega$ we denote by α^γ the image of α under the permutation γ. The set $\alpha^\Gamma = \{\alpha^\gamma \mid \gamma \in \Gamma\}$ is the *orbit* of α.

For $\alpha \in \Omega$ the group $\Gamma_\alpha = \{\gamma \in \Gamma \mid \alpha^\gamma = \alpha\} \leq \Gamma$ is the *stabilizer* of α in Γ. The *pointwise stabilizer* of a set $A \subseteq \Omega$ is the subgroup $\Gamma_{(A)} = \{\gamma \in \Gamma \mid \forall \alpha \in A\colon \alpha^\gamma = \alpha\}$. For $A \subseteq \Omega$ and $\gamma \in \Gamma$ let $A^\gamma = \{\alpha^\gamma \mid \alpha \in A\}$. The set A is Γ-*invariant* if $A^\gamma = A$ for all $\gamma \in \Gamma$.

For $A \subseteq \Omega$ and a bijection $\theta\colon \Omega \to \Omega'$ we denote by $\theta[A]$ the restriction of θ to the domain A. For a Γ-invariant set $A \subseteq \Omega$, we denote by $\Gamma[A] := \{\gamma[A] \mid \gamma \in \Gamma\}$ the induced action of Γ on A, i.e., the group obtained from Γ by restricting all permutations to A. More generally, for every set Λ of bijections with domain Ω, we denote by $\Lambda[A] := \{\theta[A] \mid \theta \in \Lambda\}$.

Let $\Gamma \leq \mathrm{Sym}(\Omega)$ and $\Gamma' \leq \mathrm{Sym}(\Omega')$. A *homomorphism* is a mapping $\varphi\colon \Gamma \to \Gamma'$ such that $\varphi(\gamma)\varphi(\delta) = \varphi(\gamma\delta)$ for all $\gamma, \delta \in \Gamma$. A bijective homomorphism is also called *isomorphism*. For $\gamma \in \Gamma$ we denote by γ^φ the φ-image of γ. Similarly, for $\Delta \leq \Gamma$ we denote by Δ^φ the φ-image of Δ (note that Δ^φ is a subgroup of Γ').

Algorithms for permutation groups: We review some basic facts about algorithms for permutation groups. For detailed information we refer to [30].

In order to perform computational tasks for permutation groups efficiently the groups are represented by generating sets of small size. Indeed, most algorithms are based on so-called strong generating sets, which can be chosen of size quadratic in the size of the permutation domain of the group and can be computed in polynomial time given an arbitrary generating set (see, e.g., [30]).

Theorem II.2 (cf. [30])**.** *Let $\Gamma \leq \mathrm{Sym}(\Omega)$ and let S be a generating set for Γ. Then the following tasks can be performed in time polynomial in n and $|S|$:*

1) *compute the order of Γ,*
2) *given $\gamma \in \mathrm{Sym}(\Omega)$, test whether $\gamma \in \Gamma$,*
3) *compute the orbits of Γ, and*
4) *given $A \subseteq \Omega$, compute a generating set for $\Gamma_{(A)}$.*

Groups with restricted composition factors: In this work, we shall be interested in a particular subclass of permutation groups, namely groups with restricted composition factors. Let Γ be a group. A *subnormal series* is a sequence of subgroups $\Gamma = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots \trianglerighteq \Gamma_k = \{\mathrm{id}\}$. The length of the series is k and the groups Γ_{i-1}/Γ_i are the factor groups of the series, $i \in [k]$. A *composition series* is a strictly decreasing subnormal series of maximal length. For every finite group Γ all composition series have the same family (considered as a multiset) of factor groups (cf. [28]). A *composition factor* of a finite group Γ is a factor group of a composition series of Γ.

Definition II.3. For $d \geq 2$ let $\widehat{\Gamma}_d$ denote the class of all groups Γ for which every composition factor of Γ is isomorphic to a subgroup of S_d.

We want to stress the fact that there are two similar classes of groups that have been used in the literature both typically denoted by Γ_d. One of these is the class introduced by Luks [3] that we denote by $\widehat{\Gamma}_d$, while the other one used in [31] in particular allows composition factors that are simple groups of Lie type of bounded dimension.

Lemma II.4 (Luks [3])**.** *Let $\Gamma \in \widehat{\Gamma}_d$. Then*

1) *$\Delta \in \widehat{\Gamma}_d$ for every subgroup $\Delta \leq \Gamma$, and*
2) *$\Gamma^\varphi \in \widehat{\Gamma}_d$ for every homomorphism $\varphi\colon \Gamma \to \Delta$.*

III. Group-Theoretic Techniques for Isomorphism Testing

Next, we present several group-theoretic tools in the context of isomorphism testing which are exploited by our algorithm testing isomorphism for graph classes that exclude a fixed minor. The dependencies between the main results leading to this paper are shown in Figure 1.

A. Hypergraph Isomorphism

Two hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ are isomorphic if there is a bijection $\varphi\colon V_1 \to V_2$ such that $E \in \mathcal{E}_1$ if and only if $E^\varphi \in \mathcal{E}_2$ for all $E \in 2^{V_1}$ (where $E^\varphi := \{\varphi(v) \mid v \in E\}$ and 2^{V_1} denotes the power set of V_1). We write $\varphi\colon \mathcal{H}_1 \cong \mathcal{H}_2$ to denote that φ is an isomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$. Consistent with previous notation, we denote by $\mathrm{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ the set of isomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$. More generally, for $\Gamma \leq \mathrm{Sym}(V_1)$ and a bijection $\theta\colon V_1 \to V_2$, we define

$$\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2) := \{\varphi \in \Gamma\theta \mid \varphi\colon \mathcal{H}_1 \cong \mathcal{H}_2\}.$$

The set $\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$ is either empty, or it is a coset of $\mathrm{Aut}_\Gamma(\mathcal{H}_1) := \mathrm{Iso}_\Gamma(\mathcal{H}_1, \mathcal{H}_1)$, i.e., $\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2) = \mathrm{Aut}_\Gamma(\mathcal{H}_1)\varphi$ where $\varphi \in \mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$ is an arbitrary isomorphism. As a result, the set $\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$ can be represented efficiently by a generating set for $\mathrm{Aut}_\Gamma(\mathcal{H}_1)$ and

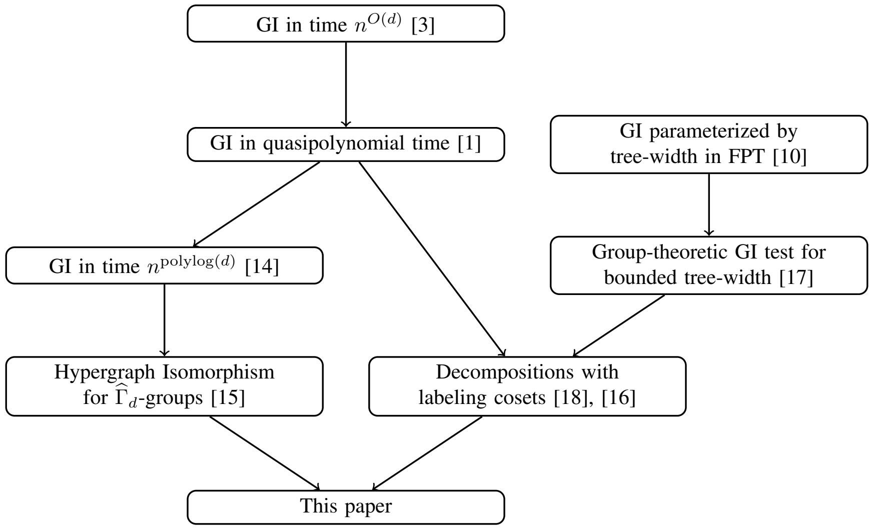

Figure 1. Dependencies between the main results leading to this paper.

a single isomorphism $\varphi \in \mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$. In the remainder of this work, all sets of isomorphisms are represented in this way.

Theorem III.1 ([15, Corollary 16])**.** *Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs and let $\Gamma \leq \mathrm{Sym}(V_1)$ be a $\widehat{\Gamma}_d$-group and $\theta\colon V_1 \to V_2$ a bijection. Then $\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$ can be computed in time $(n+m)^{\mathcal{O}((\log d)^c)}$ for some absolute constant c where $n := |V_1|$ and $m := |\mathcal{E}_1|$.*

B. Coset-Labeled Hypergraphs

Actually, for the applications in this paper, the Hypergraph Isomorphism Problem itself turns out to be insufficient. Instead, we require a generalization of the problem that is, for example, motivated by graph decomposition approaches to graph isomorphism testing (see, e.g., [17], [16]). Let G_1 and G_2 be two graphs and suppose that an algorithm has already computed sets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ in an isomorphism-invariant way, i.e., each isomorphism from G_1 to G_2 also maps D_1 to D_2. Moreover, assume that $G_1 - D_1$ is not connected and let $Z_1^i, \dots, Z_\ell^i$ be the connected components of $G_i - D_i$ (without loss of generality $G_1 - D_1$ and $G_2 - D_2$ have the same number of connected components, otherwise the graphs are non-isomorphic). Also, let $S_j^i := N_{G_i}(Z_j^i)$ for all $j \in [\ell]$ and $i \in \{1,2\}$. A natural strategy for an algorithm is to recursively compute representations for $\mathrm{Iso}(G_1[Z_{j_1}^1 \cup S_{j_1}^1], G_2[Z_{j_2}^2 \cup S_{j_2}^2])$ for all $j_1, j_2 \in [\ell]$. Then, in the second step, the algorithm needs to compute all isomorphisms $\varphi\colon G_1[D_1] \cong G_2[D_2]$ such that there is a bijection $\sigma\colon [\ell] \to [\ell]$ satisfying

1) $(S_j^1)^\varphi = S_{\sigma(j)}^2$, and
2) the restriction $\varphi[S_j^1]$ extends to an isomorphism from $G_1[Z_j^1 \cup S_j^1]$ to $G_2[Z_{\sigma(j)}^2 \cup S_{\sigma(j)}^2]$ (in the natural way)

for all $j \in [\ell]$.

Let us first discuss a simplified case where $S_{j_1}^1 \neq S_{j_2}^1$ for all distinct $j_1, j_2 \in [\ell]$. In this situation the first property naturally translates to an instance of the Hypergraph Isomorphism Problem (in particular, the bijection σ is unique for any given bijection φ). However, for the second property, we also need to be able to put restrictions on how two hyperedges can be mapped to each other. Towards this end, we consider hypergraphs with coset-labeled hyperedges where each hyperedge is additionally labeled by a coset.

A *labeling* of a set V is a bijection $\rho\colon V \to \{1, \dots, |V|\}$. A *labeling coset* of a set V is a set Λ consisting of labelings such that $\Lambda = \Delta\rho := \{\delta\rho \mid \delta \in \Delta\}$ for some group $\Delta \leq \mathrm{Sym}(V)$ and some labeling $\rho\colon V \to \{1, \dots, |V|\}$. Observe that each labeling coset $\Delta\rho$ can also be written as $\rho\Theta := \{\rho\theta \mid \theta \in \Theta\}$ where $\Theta := \rho^{-1}\Delta\rho \leq S_{|V|}$.

Definition III.2 (Coset-Labeled Hypergraph)**.** A *coset-labeled hypergraph* is a tuple $\mathcal{H} = (V, \mathcal{E}, \mathfrak{p})$ where V is a finite set of vertices, $\mathcal{E} \subseteq 2^V$ is a set of hyperedges, and $\mathfrak{p}$ is a function that associates with each $E \in \mathcal{E}$ a pair $\mathfrak{p}(E) = (\rho\Theta, c)$ consisting of a labeling coset of E and a natural number $c \in \mathbb{N}$.

Two coset-labeled hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1, \mathfrak{p}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2, \mathfrak{p}_2)$ are *isomorphic* if there is a bijection $\varphi\colon V_1 \to V_2$ such that

1) $E \in \mathcal{E}_1$ if and only if $E^\varphi \in \mathcal{E}_2$ for all $E \in 2^{V_1}$, and
2) for all $E \in \mathcal{E}_1$ with $\mathfrak{p}_1(E) = (\rho_1\Theta_1, c_1)$ and $\mathfrak{p}_2(E^\varphi) = (\rho_2\Theta_2, c_2)$ we have $c_1 = c_2$ and

$$\varphi[E]^{-1}\rho_1\Theta_1 = \rho_2\Theta_2. \tag{1}$$

In this case, φ is an *isomorphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted by $\varphi\colon \mathcal{H}_1 \cong \mathcal{H}_2$. Observe that (1) is equivalent to $c_1 = c_2$, $\Theta_1 = \Theta_2$ and $\varphi[E] \in \rho_1\Theta_1\rho_2^{-1}$. For $\Gamma \leq \mathrm{Sym}(V_1)$ and a bijection $\theta\colon V_1 \to V_2$ let

$$\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2) := \{\varphi \in \Gamma\theta \mid \varphi\colon \mathcal{H}_1 \cong \mathcal{H}_2\}.$$

Note that, for two coset-labeled hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, the set of isomorphisms $\mathrm{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ forms a coset of $\mathrm{Aut}(\mathcal{H}_1)$ and therefore, it again admits a compact representation. Indeed, this is a crucial feature of the above definition that again allows the application of group-theoretic techniques.

The next theorem is an immediate consequence of [32, Theorem 6.6.7] and Theorem III.1.

Theorem III.3. *Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1, \mathfrak{p}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2, \mathfrak{p}_2)$ be two coset-labeled hypergraphs such that for all $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ it holds $|E| \leq d$. Also let $\Gamma \leq \mathrm{Sym}(V_1)$ be a $\widehat{\Gamma}_d$-group and $\theta\colon V_1 \to V_2$ a bijection.*

Then $\mathrm{Iso}_{\Gamma\theta}(\mathcal{H}_1, \mathcal{H}_2)$ can be computed in time $(n+m)^{\mathcal{O}((\log d)^c)}$ for some absolute constant c where $n := |V_1|$ and $m := |\mathcal{E}_1|$.

C. Multiple-Labeling-Cosets

The last theorem covers the problem discussed in the beginning of the previous subsection assuming that all separators of the first graph are distinct, i.e., $S_{j_1}^1 \neq S_{j_2}^1$ for all distinct $j_1, j_2 \in [\ell]$. In this subsection, we consider the case in which $S_{j_1}^1 = S_{j_2}^1$ for all $j_1, j_2 \in [\ell]$. In order to handle the case of identical separators, we build on a framework considered in [18], [16]. (The mixed case in which some, but not all, separators coincide can be handled by a mixture of both techniques.)

Definition III.4 (Multiple-Labeling-Coset)**.** A *multiple-labeling-coset* is a tuple $\mathcal{X} = (V, L, \mathfrak{p})$ where $L = \{\rho_1\Theta_1, \dots, \rho_t\Theta_t\}$ is a set of labeling cosets $\rho_i\Theta_i$, $i \in [t]$, of the set V and $\mathfrak{p}\colon L \to \mathbb{N}$ is a function that assigns each labeling coset $\rho\Theta \in L$ a natural number $\mathfrak{p}(\rho\Theta) = c$.

Two multiple-labeling-cosets $\mathcal{X}_1 = (V_1, L_1, \mathfrak{p}_1)$ and $\mathcal{X}_2 = (V_2, L_2, \mathfrak{p}_2)$ are *isomorphic* if there is a bijection $\varphi\colon V_1 \to V_2$ such that

$$\begin{aligned} &\big(\ \rho\Theta \in L_1 \ \ \wedge \ \ \mathfrak{p}_1(\rho\Theta) = c\ \big) \\ \iff &\big(\ \varphi^{-1}\rho\Theta \in L_2 \ \ \wedge \ \ \mathfrak{p}_2(\varphi^{-1}\rho\Theta) = c\ \big) \end{aligned} \tag{2}$$

for all labeling cosets $\rho\Theta$ of V and all $c \in \mathbb{N}$. In this case, φ is an *isomorphism* from $\mathcal{X}_1$ to $\mathcal{X}_2$, denoted by $\varphi\colon \mathcal{X}_1 \cong \mathcal{X}_2$. Observe that (2) is equivalent to $|L_1| = |L_2|$ and for each $\rho_1\Theta_1 \in L_1$ there is a $\rho_2\Theta_2 \in L_2$ such that $\mathfrak{p}_1(\rho_1\Theta_1) = \mathfrak{p}_2(\rho_2\Theta_2)$ and $\Theta_1 = \Theta_2$ and $\varphi \in \rho_1\Theta_1\rho_2^{-1}$. Let

$$\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_2) := \{\varphi\colon V_1 \to V_2 \mid \varphi\colon \mathcal{X}_1 \cong \mathcal{X}_2\}$$

Again, the set of isomorphisms $\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_2)$ forms a coset of $\mathrm{Aut}(\mathcal{X}_1) := \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_1)$ and therefore, it again admits a compact representation.

Theorem III.5 ([16])**.** *Let $\mathcal{X}_1 = (V_1, L_1, \mathfrak{p}_1)$ and $\mathcal{X}_2 = (V_2, L_2, \mathfrak{p}_2)$ be two multiple-labeling cosets.*

Then $\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_2)$ can be computed in time $(n+m)^{\mathcal{O}((\log n)^c)}$ for some absolute constant c where $n := |V_1|$ and $m := |L_1|$.

D. Allowing Color Refinement to Split Small Color Classes

In order to be able to apply the decomposition framework outlined above, an algorithm first needs to compute an isomorphism-invariant subset $D \subseteq V(G)$ such that $N_G(Z)$ is sufficiently small for every connected component Z of the graph $G - D$. Moreover, the application of Theorem III.3 additionally requires a $\widehat{\Gamma}_d$-group that restricts the set of possible automorphisms for the set D. Both problems are tackled building on the notion of *t-CR-bounded graphs*. This class of graphs has been recently introduced by the second author of this paper [15] and has already been exploited for isomorphism testing of graphs of bounded genus which form an important subfamily of graph classes excluding a fixed graph as a minor.

Intuitively speaking, a vertex-colored graph (G, χ) is t-CR-bounded, $t \in \mathbb{N}$, if it possible to obtain a discrete vertex-coloring (a vertex-coloring is discrete if each vertex has a distinct color) for the graph by iteratively applying the following two operations:

- applying the Color Refinement algorithm, and
- picking a color class $[v]_\chi := \{w \in V(G) \mid \chi(v) = \chi(w)\}$ for some vertex $v \in V(G)$ where $|[v]_\chi| \leq t$ and individualizing each vertex in that class (every vertex in that color class is assigned a distinct color).

In this work, we exploit the ideas behind t-CR-bounded graphs to define a closure operator. Given an initial set $X \subseteq V(G)$, all vertices from X are first individualized before applying the operators of the t-CR-bounded definition. The closure of the set X (with respect to the parameter t) then contains all singleton vertices after the refinement procedure stabilizes.

Definition III.6. Let (G, χ_V, χ_E) be a vertex- and arc-colored graph and $X \subseteq V(G)$. Let $(\chi_i)_{i \geq 0}$ be the sequence of vertex-colorings where

$$\chi_0(v) := \begin{cases} (v, 1) & \text{if } v \in X \\ (\chi_V(v), 0) & \text{otherwise} \end{cases},$$

$\chi_{2i+1} := \chi_{\mathsf{WL}}^1[G, \chi_{2i}, \chi_E]$ and

$$\chi_{2i+2}(v) := \begin{cases} (v, 1) & \text{if } |[v]_{\chi_{2i+1}}| \leq t \\ (\chi_{2i+1}(v), 0) & \text{otherwise} \end{cases}$$

for all $i \geq 0$. Since $\chi_{i+1} \preceq \chi_i$ for all $i \geq 0$ there is some minimal i^* such that $\chi_{i^*} \equiv \chi_{i^*+1}$. We define

$$\mathrm{cl}_t^{(G,\chi_V,\chi_E)}(X) := \{v \in V(G) \mid |[v]_{\chi_{i^*}}| = 1\}.$$

For a sequence of vertices $v_1, \dots, v_k \in V(G)$ we also denote $\mathrm{cl}_t^{(G,\chi_V,\chi_E)}(v_1, \dots, v_k) := \mathrm{cl}_t^{(G,\chi_V,\chi_E)}(\{v_1, \dots, v_k\})$.

We usually omit the vertex- and arc-colorings and simply write cl_t^G instead of $\mathrm{cl}_t^{(G,\chi_V,\chi_E)}$.

For applications in graph classes with an excluded minor it turns out to be useful to combine the concept of cl_t^G with the 2-dimensional Weisfeiler-Leman algorithm. More precisely, in order to increase the scope of the set cl_t^G, information computed by the 2-dimensional Weisfeiler-Leman algorithm are taken into account. Since the 2-dimensional Weisfeiler-Leman algorithm computes a pair-coloring, we extend the definition of cl_t^G to pair-colored graphs. For a pair-colored graph (G, χ) we define $\mathrm{cl}_t^{(G,\chi)} := \mathrm{cl}_t^{(K_n,\widetilde{\chi})}$ where K_n is the complete graph on the same vertex set $V(G)$ and $\widetilde{\chi}(v, w) = (\mathrm{atp}(v, w), \chi(v, w))$ where $\mathrm{atp}(v, w) = 0$ if $v = w$, $\mathrm{atp}(v, w) = 1$ if $vw \in E(G)$, and $\mathrm{atp}(v, w) = 2$ otherwise. This allows us to take all pair-colors into account for the Color Refinement algorithm, but also still respect the edges of the input graph G.

It can be shown that for each t-CR-bounded graph G it holds that $\mathrm{Aut}(G) \in \widehat{\Gamma}_t$. Moreover, there is an algorithm that, given a graph G, computes a $\widehat{\Gamma}_t$-group $\Gamma \leq \mathrm{Sym}(V(G))$ such that $\mathrm{Aut}(G) \leq \Gamma$ in time $n^{\mathrm{polylog}(t)}$ where n is the number of vertices of G. It is important for our techniques that this statement generalizes to t-CR-bounded pairs (G, X) for which we already have a good knowledge of the structure of X in form of a $\widehat{\Gamma}_t$-group of $\Gamma \leq \mathrm{Sym}(X)$ as stated in the following theorem.

Theorem III.7 ([15]). *Let G_1, G_2 be two graphs and let $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$. Also, let $\Gamma \leq \mathrm{Sym}(X_1)$ be a $\widehat{\Gamma}_t$-group and $\theta\colon X_1 \to X_2$ a bijection. Moreover, let $D_i := \mathrm{cl}_t^{G_i}(X_i)$ for $i \in \{1, 2\}$ and define $\Gamma'\theta' := \{\varphi \in \mathrm{Iso}((G_1, X_1), (G_2, X_2)) \mid \varphi[X_1] \in \Gamma\theta\}[D_1]$.*

Then $\Gamma' \in \widehat{\Gamma}_t$. Moreover, there is an algorithm computing a $\widehat{\Gamma}_t$-group $\Delta \leq \mathrm{Sym}(D_1)$ and a bijection $\delta\colon D_1 \to D_2$ such that $\Gamma'\theta' \subseteq \Delta\delta$ in time $n^{\mathcal{O}((\log t)^c)}$ for some absolute constant c where $n := |V(G_1)|$.

IV. Exploiting the Structure of Graphs Excluding a Minor

A. Overview

In the following, we give a more detailed description of the high-level strategy for building a faster isomorphism test for graph classes that exclude a fixed minor. In particular, we state the two main technical theorems which build the groundwork for the isomorphism test.

The basic idea for our isomorphism test is to follow the decomposition framework outlined in the previous section. Let G_1 and G_2 be two connected graphs that exclude K_h as a minor (note that it is always possible to restrict to connected graphs by considering the connected components of the input graphs separately). To apply the decomposition framework outlined in the previous section, we need to compute subsets $D_i \subseteq V(G_i)$, $i \in \{1, 2\}$, such that

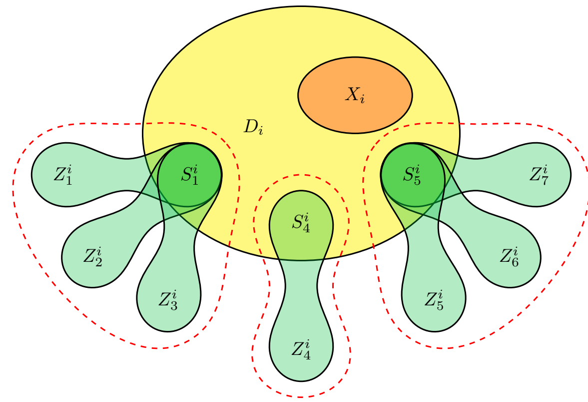

Figure 2. Visualization of the graph decomposition.

1) the subsets D_1, D_2 are isomorphism-invariant, i.e., $D_1^\varphi = D_2$ for all $\varphi \in \mathrm{Iso}(G_1, G_2)$,
2) for each connected component Z_i of $G_i - D_i$ it holds $|N_{G_i}(Z_i)| < h$ and,
3) one can efficiently compute a $\widehat{\Gamma}_d$-group $\Delta \leq \mathrm{Sym}(D_1)$ and a bijection $\delta\colon D_1 \to D_2$ such that $\mathrm{Iso}(G_1, G_2)[D_1] \subseteq \Delta\delta$.

In such a setting, the decomposition framework can be applied as follows (see also Figure 2). For every pair of connected components $Z_{j_1}^1$ and $Z_{j_2}^2$ of $G_1 - D_1$ and $G_2 - D_2$, respectively, the algorithm recursively computes the set of isomorphisms from $G_1[Z_{j_1}^1 \cup S_{j_1}^1]$ to $G_2[Z_{j_2}^2 \cup S_{j_2}^2]$ where $S_{j_i}^i := N_{G_i}(Z_{j_i}^i)$, $i \in \{1, 2\}$. Then, the set of isomorphisms from G_1 to G_2 can be computed by combining Theorem III.5 and III.3. Recall that Theorem III.5 handles the case in which $S_{j_1}^1 = S_{j_2}^1$ for all connected components $Z_{j_1}^1, Z_{j_2}^1$ of $G_1 - D_1$. To achieve the desired running time for this case, we exploit Property 2. For Theorem III.3, which handles the case of distinct separators $S_{j_1}^1 \neq S_{j_2}^1$, we require sufficient structural information of the sets D_1 and D_2. More precisely, we require Property 3 to ensure the desired time bound.

Now, we turn to the question how to find the sets D_1 and D_2 satisfying Property 1, 2 and 3. The central idea is to build on the closure operator $\mathrm{cl}_t^{G_i}$ (where t is polynomially bounded in h). We construct the sets by computing the closure $D_i := \mathrm{cl}_t^{G_i}(X_i)$ for some suitable initial set X_i. The first key insight is that this process of growing the sets X_i can only be stopped by separators of small size which ensures Property 2.

Theorem IV.1. *Let G be a graph that excludes K_h as a topological subgraph and let $X \subseteq V(G)$. Let $t \geq 3h^3$ and define $D := \mathrm{cl}_t^G(X)$. Let Z be the vertex set of a connected component of $G - D$. Then $|N_G(Z)| < h$.*

Observe that the theorem addresses graphs that only exclude K_h as a topological subgraph which is a weaker requirement than excluding K_h as a minor. As a central tool, it is argued that graphs, for which all color classes under the

Color Refinement algorithm are large, contain large numbers of vertex-disjoint trees with predefined color patterns. The vertex-disjoint trees then allow for the construction of a topological minor on the vertex set $N_G(Z)$.

In order to ensure Property 3, we need sufficient structural information for the sets D_i, $i \in \{1,2\}$. Using Theorem III.7, we are able to extend structural information in form of a $\widehat{\Gamma}_d$-group from the sets X_i to the supersets $D_i \supseteq X_i$, $i \in \{1,2\}$.

Hence, the main task that remains to be solved is the computation of the initial isomorphism-invariant sets X_1 and X_2 as well as suitable restrictions on the set $\mathrm{Iso}(G_1,G_2)[X_1] = \{\varphi[X_1] \mid \varphi \in \mathrm{Iso}(G_1,G_2)\}$. Ideally, one would like to compute a $\widehat{\Gamma}_d$-group $\Gamma \leq \mathrm{Sym}(X_1)$ and a bijection $\theta\colon X_1 \to X_2$ such that $\mathrm{Iso}(G_1,G_2)[X_1] \subseteq \Gamma\theta$. But this is not always possible. For example, for a cycle C_p of length p where p is a prime number, it is only possible to choose $X = V(C_p)$ (because C_p is vertex-transitive) and $\mathrm{Aut}(C_p) \notin \widehat{\Gamma}_d$ for all $p > d$.

However, we are able to prove that there are isomorphism-invariant sets X_1 and X_2 such that, after individualizing a single vertex $v_1 \in X_1$ and $v_2 \in X_2$ in each input graph, the set $\mathrm{Iso}((G_1,v_1),(G_2,v_2))[X_1] = \{\varphi[X_1] \mid \varphi \in \mathrm{Iso}(G_1,G_2), v_1^\varphi = v_2\}$ has the desired structure. This is achieved by the next theorem which forms the second main technical contribution of this paper and again relies on the closure operator cl_t^G. Recall the definition of the constant a from Theorem II.1. Without loss of generality assume $a \geq 2$.

Theorem IV.2. *Let $t \geq (ah\log h)^3$. There is a polynomial-time algorithm that, given a connected vertex-colored graph G, either correctly concludes that G has a minor isomorphic to K_h or computes a pair-colored graph (G',χ') and a set $X \subseteq V(G')$ such that*

1) *$X = \{v \in V(G') \mid \chi'(v,v) = c\}$ for some color $c \in \{\chi'(v,v) \mid v \in V(G')\}$,*
2) *$X \subseteq \mathrm{cl}_t^{(G',\chi')}(v)$ for every $v \in X$, and*
3) *$X \subseteq V(G)$.*

Moreover, the output of the algorithm is isomorphism-invariant with respect to G.

Observe that Property 1 and 2 of the theorem imply that $(\mathrm{Aut}(G))_v[X] \in \widehat{\Gamma}_t$ for all $v \in X$ by Theorem III.7.

For technical reasons, the theorem actually provides a second graph (G_i',χ_i') for both input graphs G_i. Intuitively speaking, one can think of G_i' as an extension of G_i which allows us to build additional structural information about G_i into the graph structure of G_i'.

Following the general strategy outlined above and building on both theorems, we can show the main result of this paper.

Theorem IV.3. *Let $h \in \mathbb{N}$. There is an algorithm that, given two connected vertex-colored graphs G_1, G_2 with n vertices, either correctly concludes that G_1 has a minor isomorphic to K_h or decides whether G_1 is isomorphic to G_2 in time $n^{\mathcal{O}((\log h)^c)}$ for some absolute constant c.*

We remark that, by standard reduction techniques, there is also an algorithm computing a representation for the set $\mathrm{Iso}(G_1,G_2)$ in time $n^{\mathcal{O}((\log h)^c)}$ assuming G_1 excludes K_h as a minor.

The proof of the last theorem also reveals some insight into the structure of the automorphism group of a graph that excludes K_h as a minor.

Let G be a graph. A *tree decomposition* for G is a pair (T,β) where T is a rooted tree and $\beta\colon V(T) \to 2^{V(G)}$ such that

(T.1) for every $e \in E(G)$ there is some $t \in V(T)$ such that $e \subseteq \beta(t)$, and
(T.2) for every $v \in V(G)$ the graph $T[\{t \in V(T) \mid v \in \beta(t)\}]$ is non-empty and connected.

The *adhesion-width* of (T,β) is $\max_{t_1t_2 \in E(T)} |\beta(t_1) \cap \beta(t_2)|$.

Theorem IV.4. *Let G be a graph that excludes K_h as a minor. Then there is an isomorphism-invariant tree decomposition (T,β) of G such that*

1) *the adhesion-width of (T,β) is at most $h-1$, and*
2) *for all $t \in V(T)$ there exists $v \in \beta(t)$ such that $(\mathrm{Aut}(G))_v[\beta(t)] \in \widehat{\Gamma}_d$ for $d := \lceil (ah\log h)^3 \rceil$.*

In the remainder of this section we provide some details on the proofs of Theorem IV.1 and IV.2 which build the main technical results of this work.

B. Finding Separators of Small Size

The proof of Theorem IV.1 relies on the following lemma.

For a vertex-colored graph (G,χ) and $W \subseteq V(G)$ define $G[[\chi,W]]$ to be the graph with vertex set $V(G[[\chi,W]]) := \chi(W)$ and edge set $E(G[[\chi,W]]) := \{\chi(w_1)\chi(w_2) \mid w_1w_2 \in E(G[W])\}$.

Lemma IV.5. *Let $h \geq 1$. Let G be a graph and $V(G) = V_1 \uplus V_2$ be a partition of the vertex set of G. Also let χ be a vertex-coloring of G and suppose that*

1) *G excludes K_h as topological subgraph,*
2) *$G[[\chi,V_2]]$ is connected,*
3) *$|V_1| \geq h$ and $N_G(V_2) = V_1$,*
4) *$|[v]_\chi| = 1$ for all $v \in V_1$, and*
5) *χ is stable with respect to the Color Refinement algorithm.*

Then there is some $w \in V_2$ such that $|[w]_\chi| < 3h^3$.

Let us first give a proof for Theorem IV.1 based on Lemma IV.5.

Proof of Theorem IV.1: Let χ be the final vertex-coloring that is stable under the t-CR-bounded algorithm with respect to the initial set X. Let Z be a connected component of $G - D$ and assume for sake of contradiction that $|N_G(Z)| \geq h$. Let $V_2 := \{v \in V(G) \mid \chi(v) \in \chi(Z)\}$

and $V_1 := N_G(V_2)$ and define $H := G[V_1 \cup V_2]$. Since H is a subgraph of G, the graph H also excludes K_h as a topological subgraph. Moreover, $|V_1| \geq |N(Z)| \geq h$. Also, $|[v]_\chi| = 1$ for all $v \in V_1 \subseteq D = \text{cl}_t^G(X)$. Finally, $\chi|_H$ is stable under the Color Refinement algorithm for the graph H and $H[[\chi|_H, V_2]]$ is connected since $G[Z]$ is connected. By Lemma IV.5 there is some $w \in V_2$ such that $|[w]_\chi| < 3h^3 \leq t$. This means that $|[w]_\chi| = 1$ since each vertex in a color class of size smaller than t is assigned a distinct color by the t-CR-bounded procedure. Therefore, $[w]_\chi \subseteq D$ which contradicts the fact that $w \in V_2 \subseteq V(G) \setminus D$. ∎

Next, let us turn to Lemma IV.5. For the proof we assume that $|[w]_\chi| \geq 3h^3$ for all $w \in V_2$ and aim to construct a topological subgraph K_h. The vertices of the topological subgraph are located in the set V_1. This leaves the task to construct disjoint paths between vertices from V_1 using the vertices from the set V_2. Actually, it turns out to be more convenient to construct a large number of disjoint trees each of which can be used to obtain a single path connecting two vertices in V_1.

Let G be a graph, let $\chi\colon V(G) \to C$ be a vertex-coloring and let T be a tree with vertex set $V(T) = C$. A subgraph $H \subseteq G$ *agrees* with T if $\chi|_{V(H)}\colon H \cong T$, i.e., the coloring χ induces an isomorphism between H and T. Equivalently, H agrees with T if $|V(H) \cap \chi^{-1}(c)| = 1$ for every $c \in C$ and $c_1c_2 \in E(T)$ if and only if $H[\chi^{-1}(c_1), \chi^{-1}(c_2)]$ contains an edge for all $c_1, c_2 \in C$. Observe that each $H \subseteq G$ that agrees with a tree T is also a tree.

The main step for proving Lemma IV.5 is the next lemma that guarantees the existence of a large number of vertex-disjoint trees with a predefined color pattern.

For a tree T we define $V_{\leq i}(T) := \{t \in V(T) \mid \deg(t) \leq i\}$ and $V_{\geq i}(T) := \{t \in V(T) \mid \deg(t) \geq i\}$. It is well known that for trees T it holds that $|V_{\geq 3}(T)| \leq |V_{\leq 1}(T)|$.

Also, for a graph G and two disjoint sets $A, B \subseteq V(G)$ let $G[A, B]$ be the bipartite graph with vertex set $V(G[A, B]) := A \cup B$ and $E(G[A, B]) := \{vw \mid v \in A, w \in B, vw \in E(G)\}$. A bipartite $G = (V, W, E)$ is *biregular* if $\deg(v_1) = \deg(v_2)$ for all $v_1, v_2 \in V$ and $\deg(w_1) = \deg(w_2)$ for all $w_1, w_2 \in W$.

Lemma IV.6. *Let G be a graph, let $\chi\colon V(G) \to C$ be a vertex-coloring and let T be a tree with vertex set $V(T) = C$. Assume that $G[\chi^{-1}(c_1), \chi^{-1}(c_2)]$ is a non-empty biregular graph for every $c_1c_2 \in E(T)$. Let $m := \min_{c \in C} |\chi^{-1}(c)|$ and let $\ell := 2|V_{\leq 1}(T)| + |V_{\geq 3}(T)|$.*

Then there are (at least) $\lfloor \frac{m}{\ell} \rfloor$ pairwise vertex-disjoint trees in G that agree with T.

A proof of this lemma can be found in the full version [24].

Proof of Lemma IV.5: Consider the graph $H := G[[\chi, V_2]]$ which is connected. Let $v_1, \dots, v_h \in V_1$ be distinct vertices and let $w_1, \dots, w_h \in V_2$ such that $v_iw_i \in E(G)$. Note that $[w_i]_\chi \subseteq N(v_i)$ for all $i \in [h]$ since χ is stable with respect to the Color Refinement algorithm. Also define $c_i = \chi(w_i)$. Now let $T \subseteq H$ be a Steiner tree for $\{c_1, \dots, c_h\}$, i.e., a tree that contains all the vertices $c_1, \dots, c_h$ and is minimal with respect to the subgraph relation. Hence, T is a tree with $c_1, \dots, c_h \in V(T)$ and $|V_{\leq 1}(T)| \leq h$. This also implies that $|V_{\geq 3}(T)| \leq h$.

Now let $\ell := 2|V_{\leq 1}(T)| + |V_{\geq 3}(T)| \leq 3h$. Assume for the sake of contradiction that $m := \min_{c \in V(T)} |\chi^{-1}(c)| \geq 3h^3$. Also note that $G[\chi^{-1}(t_1), \chi^{-1}(t_2)]$ is biregular and non-trivial for all $t_1t_2 \in E(T)$. By Lemma IV.6, there are $k := \lfloor \frac{m}{\ell} \rfloor \geq h^2$ pairwise vertex-disjoint trees $H_1, \dots, H_k$ that agree with T. But this gives a topological subgraph K_h of the graph G. For each unordered pair v_iv_j, $i, j \in [h]$, and each H_p, $p \in [k]$, there is a path in the graph H_p from a vertex $w'_i \in [w_i]_\chi \subseteq N(v_i)$ to a vertex $w'_j \in [w_j]_\chi \subseteq N(v_j)$. Therefore, for each unordered pair v_iv_j, $i, j \in [h]$, there is a path from v_i to v_j in G and these paths are internally vertex disjoint (since $H_1, \dots, H_k$ are pairwise vertex-disjoint trees). ∎

C. Finding an Initial Color Class

Next, we give an overview on the proof of Theorem IV.2. The proof builds on the 2-dimensional Weisfeiler-Leman algorithm and requires some additional notation. Let G be a graph and let $\chi := \chi^2_{\text{WL}}[G]$ the coloring computed by the 2-dimensional Weisfeiler-Leman algorithm. We refer to $C_V := C_V(G, \chi) := \{\chi(v, v) \mid v \in V(G)\}$ as the set of *vertex colors* and $C_E := C_E(G, \chi) := \{\chi(v, w) \mid vw \in E(G)\}$ as the set of *edge colors*. For a vertex color $c \in C_V(G, \chi)$, we define $V_c := V_c(G, \chi) := \{v \in V(G) \mid \chi(v, v) = c\}$ as the set of all vertices with color c. Similar, for an edge color $c \in C_E(G, \chi)$ we define $E_c := E_c(G, \chi) := \{v_1v_2 \in E(G) \mid \chi(v_1, v_2) = c\}$. Let $c \in C_E$ be an edge color. We define the graph $G[c]$ with vertex set

$$V(G[c]) := \bigcup_{e \in E_c} e$$

and edge set

$$E(G[c]) := E_c.$$

Note that the endvertices of all c-colored edges have the same vertex colors, that is, for all edges $vw, v'w' \in E(G)$ with $\chi(v, w) = \chi(v', w') = c$ we have $\chi(v, v) = \chi(v', v')$ and $\chi(w, w) = \chi(w', w')$. This implies $1 \leq |C_V(G[c], \chi)| \leq 2$. We say that $G[c]$ is *unicolored* if $|C_V(G[c], \chi)| = 1$. Otherwise $G[c]$ is called *bicolored*.

For the moment, suppose there is an edge color $c \in C_E$ such that $G[c]$ is connected. First assume $G[c]$ is unicolored. Then $G[c]$ is d-regular for some natural number d. Moreover, $d \leq ah \log h$ by Theorem II.1. Then $V(G[c]) = \text{cl}_t^{G[c]}(v) \subseteq \text{cl}_t^{(G,\chi)}(v)$ for all $v \in V(G[c])$ and $t \geq d$. Hence, setting $(G', \chi') := (G, \chi)$ and $X := V(G[c])$ proves Theorem IV.2.

A similar strategy also works if there is some edge color $c \in C_E$ such that $G[c]$ is bicolored and connected as the next lemma indicates.

Lemma IV.7. *Let $t \geq (ah \log h)^2$. Let $G = (V_1, V_2, E)$ be a connected bipartite graph that excludes K_h as a minor and define $\chi := \chi^2_{\mathsf{WL}}[G]$. Suppose that $\chi(v_1, v_2) = \chi(v_1', v_2')$ for all $(v_1, v_2), (v_1', v_2') \in V_1 \times V_2$ with $v_1v_2, v_1'v_2' \in E$. Also assume that $|V_1| \leq |V_2|$. Then $V_1 \subseteq \mathrm{cl}_t^{(G,\chi)}(v)$ for all $v \in V_1 \cup V_2$.*

Now let $c \in C_E$ be an edge color and let A be the vertex set of a connected component of $G[c]$. We define a size parameter for the graph $G[c]$ as

$$s(c) := \min_{d \in C_V(G[c],\chi)} |A \cap V_d|.$$

Note that this is well-defined since every two connected components of $G[c]$ are equivalent with respect to the 2-dimensional Weisfeiler-Leman algorithm.

Proof Sketch of Theorem IV.2: The above arguments already handle the case in which $G[c]$ is connected for some edge color c. Hence, we can assume that for all edge colors c the graph $G[c]$ is not connected. We distinguish two cases.

First suppose there is some edge color $c \in C_E$ such that $s(c) \leq ah^3$ (where a is the constant from Theorem II.1). In this case the algorithm works by recursion. Choose an edge color $c \in C_E$ such that $s(c) \leq ah^3$ (to ensure that the color in C_E is chosen in an isomorphism-invariant way, the algorithm chooses the smallest color in $C_E \subseteq \mathbb{N}$ according to the ordering of natural numbers) and let $A_1, \dots, A_\ell$ be connected components of $G[c]$. Also, let F be the graph obtained from G by contracting each of the sets A_i to a single vertex. The algorithm recursively computes an isomorphism-invariant graph (F', χ_F') and a set $X_F \subseteq V(F')$ that satisfies Properties 1, 2 and 3 with respect to the input graph F. (If F contains a minor K_h, then G also contains a minor K_h since F is a minor of G.) If $X_F \subseteq V(G)$, then the algorithm simply returns (F', χ_F') and the set X_F.

Otherwise, $X_F \cap \{A_1, \dots, A_\ell\} \neq \emptyset$. Let $d := \mathrm{argmin}_{d \in C_V(G,\chi), A_i \cap V_d \neq \emptyset} |A_i \cap V_d|$ for some $i \in [\ell]$. This means $s(c) = |A_i \cap V_d|$. The algorithm constructs G' where

$$V(G') := V(F') \uplus \bigcup_{A \in X_F \cap \{A_1, \dots, A_\ell\}} (A \cap V_d)$$

and

$$\begin{aligned} E(G') := \; & E(F') \\ & \cup \{Av \mid A \in X_F \cap \{A_1, \dots, A_\ell\}, v \in A \cap V_d\}. \end{aligned}$$

Also, $\chi'(v,w) := (\chi_F'(v,w), 0)$ for every $v, w \in V(F')$, $\chi'(w,v) = \chi'(v,w) := (1,1)$ for all distinct $v \in V(G')$, $w \in V(G') \setminus V(F')$, and $\chi'(v,v) := (0,1)$ for every $v \in V(G') \setminus V(F')$. Clearly, (G', χ') is constructed in an isomorphism-invariant manner. The algorithm returns (G', χ') together with set $X := V(G') \setminus V(F')$. It is easy to verify that all desired properties are satisfied.

In the other case $s(c) \geq ah^3$ for all edge colors $c \in C_E$. Let $d := \mathrm{argmin}_{d \in C_V} |V_d|$ and define $X := V_d$ to be a smallest color class (as before, if this color is not unique, then the algorithm chooses the smallest color in $C_V \subseteq \mathbb{N}$ with minimal color class size). Now suppose that $(G', \chi') := (G, \chi)$ together with the set X does not satisfy Property 2 (otherwise the algorithm is done since Properties 1 and 3 are clearly satisfied). The central claim is that in this situation one can compute a vertex-coloring λ that is strictly finer than the one induced by χ. The algorithm then updates the coloring χ taking the vertex-colors according to λ into account and running the 2-dimensional Weisfeiler-Leman algorithm again. Note that this procedure can only be repeated at most $n-1$ times which means that eventually one of the other cases must be satisfied giving the desired outcome.

The basic idea for computing the coloring λ builds on Theorem IV.1. Let $c \in C_E$ be an edge color such that $X \subseteq V(G[c])$ and let $A_1, \dots, A_\ell$ be the vertex sets of the connected components of $G[c]$. Also let $v \in A_i \cap X$. Then $A_i \cap X \subseteq \mathrm{cl}_t^{(G,\chi)}(v)$ by Lemma IV.7. In particular, $D_i := \mathrm{cl}_t^{(G,\chi)}(v) = \mathrm{cl}_t^{(G,\chi)}(v')$ for all $v' \in A_i \cap X$. Moreover, either $A_j \cap X \subseteq D_i$ or $A_j \cap D_i = \emptyset$ for all $j \in [\ell]$ by Lemma IV.7.

For simplicity assume that the second option is satisfied for all $i, j \in [\ell]$ with $i \neq j$. The general proof strategy is similar, but slightly more involved. Also observe that the second option has to be satisfied at least once by our assumption that $X \not\subseteq \mathrm{cl}_t^{(G,\chi)}(v)$ for some $v \in X$.

Now each pair $i, j \in [\ell]$, $i \neq j$, can be associated with the set $S_j^i := N_G(Z_j^i)$ where Z_j^i denotes the connected the component of $G - D_i$ which contains A_j. Observe that $|S_j^i| < h$ by Theorem IV.1.

In order to describe the coloring λ, the algorithm aims at collecting a small family of these sets. Towards this end, we define an isomorphism-invariant minor H of G with vertex set $V(H) := \{A_1, \dots, A_\ell\}$. In turn, this allows us to define an isomorphism-invariant set

$$Y := \bigcup_{A_iA_j \in E(H)} S_j^i \cup S_i^j.$$

Crucially,

$$|Y| < 2 \cdot |E(H)| \cdot h \leq ah^2 \log h \cdot \ell \leq ah^2 \log h \cdot \frac{|X|}{ah^3} \leq |X|$$

by Theorem II.1 and the fact that $s(c) \geq ah^3$. Since X was defined to be the smallest color class this implies that the vertex-coloring induced by χ can be refined to a vertex-coloring λ by taking membership in Y into account. More precisely, define $\lambda(v) := (\chi(v,v), 1)$ for all $v \in Y$ and $\lambda(v) := (\chi(v,v), 0)$ for all $v \in V(G) \setminus Y$. Then

$$\min_{d \in \mathrm{im}(\lambda)} |\lambda^{-1}(d)| \leq |Y| < |X| = \min_{d \in C_V(G,\chi)} |V_d|$$

which implies that λ strictly refines the vertex-coloring induced by χ as desired. ∎

As before, all missing details can be found in the full version [24].

V. Conclusion

We presented an isomorphism test for graph classes that exclude K_h as a minor running in time $n^{\text{polylog}(h)}$. The algorithm builds on group-theoretic methods from [15], [16] as well as novel insights on the isomorphism-invariant structure of graphs excluding the minor K_h.

A number of interesting questions remain. The first question concerns the isomorphism problem for graph classes that exclude K_h as a topological subgraph. We conjecture that there is an algorithm solving this problem in time $n^{\text{polylog}(h)}$. Actually, most of the techniques developed in this work also extend to classes that only exclude K_h as a topological subgraph rather than as a minor. In particular, this includes Theorem IV.1. Indeed, the only part of the algorithm that exploits closure under taking minors is the subroutine from Theorem IV.2 which provides the initial set X together with sufficient structural information on this set.

The second question is whether the graph isomorphism problem parameterized by the Hadwiger number (the maximum h such that K_h is a minor) is fixed-parameter tractable. Note that our result is independent of such an fpt result, because our algorithm is obviously not fpt, but it also has no exponential dependence on h as a typical fpt-algorithm running in time $f(h) \cdot n^c$ has. Running times of the form $n^{\text{polylog}(k)}$ for parameterized problems with input size n and parameter k so far seem to be quite specific to the isomorphism problem. It may be worthwhile to study them more systematically in a broader context.

Our final question regards the structure of the automorphism group of graphs excluding K_h as a minor. Babai [33] conjectured that all composition factors of such groups are cyclic groups, alternating groups, or their size is bounded by $f(h)$ for some function f. Our new insights, summarized in Theorem IV.4, significantly restrict the automorphism group of graphs excluding K_h and could be an important step towards proving Babai's conjecture.

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