Improving Computational Efficiency of Communication for Omniscience and Successive Omniscience

Ni Ding, *Member, IEEE*, Parastoo Sadeghi, *Senior Member, IEEE*, and Thierry Rakotoarivelo, *Member, IEEE*

***Abstract*—Communication for omniscience (CO) refers to the problem where the users in a finite set V observe a discrete multiple random source and want to exchange data over broadcast channels to reach omniscience, the state where everyone recovers the entire source. This paper studies how to improve the computational complexity for the problem of minimizing the sum-rate for attaining omniscience in V. While the existing algorithms rely on the submodular function minimization (SFM) techniques and complete in $O(|V|^2 \cdot \text{SFM}(|V|)$ time, we prove the strict strong map property of the nesting SFM problem. We propose a parametric (PAR) algorithm that utilizes the parametric SFM techniques and reduces the complexity to $O(|V| \cdot \text{SFM}(|V|)$.**

We propose efficient solutions to the successive omniscience (SO): attaining omniscience successively in user subsets. We first focus on how to determine a complimentary subset $X_* \subsetneq V$ in the existing two-stage SO such that if the local omniscience in X_* is reached first, the global omniscience whereafter can still be attained with the minimum sum-rate. It is shown that such a subset can be extracted at one of the iterations of the PAR algorithm. We then propose a novel multi-stage SO strategy: a nesting sequence of complimentary user subsets $X_*^{(1)} \subsetneq \ldots \subsetneq X_*^{(K)} = V$, the omniscience in which is attained progressively by the monotonic rate vectors $\mathbf{r}_V^{(1)} \leq \ldots \leq \mathbf{r}_V^{(K)}$. We propose algorithms to obtain this K-stage SO from the returned results by the PAR algorithm. The run time of these algorithms is the same as the PAR algorithm.

***Index Terms*—communication for omniscience, Dilworth truncation, submodularity.**

I. INTRODUCTION

Communication for Omniscience (CO): Let there be a finite number of users indexed by the set V. Each user observes a distinct component of a discrete memoryless multiple random source in private. The users are allowed to exchange their observations over public noiseless broadcast channels so as to attain *omniscience*, the state that each user reconstructs all components in the multiple source. This process is called CO [3], where the fundamental problem is how to attain omniscience with the minimum sum of broadcast rates. While the CO problem formulated in [3] considers the asymptotic limits as the observation length goes to infinity, a non-asymptotic model is studied in [4]–[6]. In this non-asymptotic model, the number of observations is assumed to be finite and the communication rates are restricted to be integer-valued. The CO problem has a wide range of important applications, special cases, extensions, duals and interpretations.

The CO problem in the asymptotic model is dual with the secret capacity [3], which is the maximum amount of secret key that can be generated by the users in V and equals to the amount of information in the entire source, $H(V)$, subtracted by the minimum sum-rate in CO. The non-asymptotic model is equivalent to the finite linear source model in some network coding problems. For example, in the coded cooperative data exchange (CCDE) [7]–[14], a group of users obtain parts of a packet set, say, via base-to-peer (B2P) transmissions. By broadcasting linear combinations of packets over peer-to-peer (P2P) channels, the users help each other recover the entire packet set based on a suitable network coding scheme, e.g., the random linear network coding [10]. It is shown in [14]–[18] that the solutions to the secret key agreement problem, CO and CCDE rely on the submodular function minimization (SFM) techniques in combinatorial optimization [19]. In a nutshell, all solutions in [14]–[18] require $O(|V|^2)$ calls of solving the SFM problem. Since the polynomial order of solving the SFM is still high [19, Chapter VI]: ranging from $|V|^4$ to $|V|^8$, it is important to study whether the order-wise complexity $|V|^2$ in the computational complexity can be further reduced. This requires a deep understanding of the structure of the CO problem and its optimal solution. It is known from previous works [16], [18] that the first critical/turning point in the principal sequence of partitions (PSP), a partition chain that is induced by the segmented Dilworth truncation of the residual entropy function, plays a central role in solving the CO problem. This is essentially the first or coarsest partition in the PSP that is strictly finer than the partition $\{V\}$.

Another important interpretation of CO is in the extension of the Shannon's mutual information to the multivariate case and is called the multivariate mutual information $I(V)$ [16]: $I(V)$ equals to the secret capacity. This measure was used in [20] to interpret the PSP as a hierarchical clustering result: the partitions in the PSP contain the largest user subsets X with $I(X)$ strictly greater than a given similarity threshold and get coarser from bottom to top as this similarity threshold decreases. This coincides with a more general combinatorial clustering framework, the minimum average clustering (MAC) in [21], where both the entropy and cut functions are viewed as the inhomogeneity measure of a dataset. For the cut function, the first critical value in the PSP identifies the network strength

The preliminary results of this paper have been partly published in [1], [2].

Ni Ding (email: ni.ding@unimelb.edu.au) is a Doreen Thomas Postdoctoral Fellow at the Melbourne School of Engineering, The University of Melbourne, Parkville, VIC 3010.

Parastoo Sadeghi (email: p.sadeghi@unsw.edu.au) is with the School of Engineering and Information Technology, University of New South Wales, Canberra, ACT 2601.

Thierry Rakotoarivelo (email: thierry.rakotoarivelo@data61.csiro.au) is with Data61, 5/13 Garden Street, Eveleigh, NSW 2015.

[21], [22] and the maximum number of edge-disjoint spanning trees [23]. This, in return, well explains why the secret agreement problem in the pairwise independent network (PIN) source model, which has a graphical representation, can be solved by the tree packing algorithms in [24]–[26]. Thus, it is also worth studying how to improve the existing complexity $O(|V|^2 \cdot \text{SFM}(|V|))$ for determining the whole PSP.

Successive Omniscience (SO): Instead of attaining the omniscience in a one-off manner, the idea of SO is proposed in [6], [27], [28] revealing that the state of omniscience can be reached in a two-stage manner: let a user subset $X \subsetneq V$ exchange the data first to attain omniscience and the rest of the users overhear the communications; then solve the global omniscience problem in V. By recursively applying the two-stage SO approach, the omniscience in V can be attained in a multi-stage manner. This idea has been applied to CCDE in [29], where a multi-stage SO process is scheduled by a sequence of user subsets that can transmit in order to attain omniscience. The problem of determining a local omniscience achievable rate vector for each stage was formulated and solved as a constrained multi-objective optimization problem.

However, it is shown in [6], [27] that there is a particular group of *complimentary* user subsets such that the local omniscience can be attained in any of them first, while the overall communication rates for the global omniscience whereafter still remains minimized. This also means that, if a non-complimentary subset reaches local omniscience first, e.g., in the predetermined multi-stage SO strategy [29], the users might need to transmit more than the minimum sum-rate to attain the global omniscience eventually. Therefore, the *essential problem* in SO is not to determine the transmission rate for a specific user group, but how to choose a user subset $X_* \subsetneq V$ that is complimentary in order to preserve the optimality of the global omniscience. The necessary and sufficient condition for $X_* \subsetneq V$ to be complimentary was derived in [6, Theorems 4.2 and 5.2] for the asymptotic and non-asymptotic models, respectively. But, they are based on the value of the minimum sum-rate for the global omniscience.[1]

Meanwhile, the studies on the universal multi-party data exchange problem in [30]–[32] suggest letting users adaptively increase their transmission rates and running an ideal decoder at the same time to keep searching for the user subset that reaches the omniscience state. The recursive application of this process in [31, Protocol 3] results in a multi-stage SO. This method does not require the system information, e.g., the distribution of the source. However, it requires extra scheduling overheads, e.g., ordering transmission turns based on the amount of information (entropy) in individual users' observations and repetitively checking a so-called constant difference property to determine when a user should transmit. In addition, the ideal decoder needs to be run online, which also incurs communication overheads between users, e.g., sending ACK/NACK signals. Thus, the current literature is missing an efficient overall scheduling of the multi-stage SO, before the transmissions actually take place. More specifically, this scheduling refers to the design of the K stages, for each of which, a complimentary user subset $X_*^{(k)}$ that holds the condition in [6, Theorems 4.2 and 5.2] is selected and a rate vector $\mathbf{r}_V^{(k)} = (r_i^{(k)} : i \in V)$ is determined with its reduction/projection $\mathbf{r}_{X_*^{(k)}}^{(k)}$ on $X_*^{(k)}$ being an achievable local omniscience vector. In addition, $X_*^{(K)}$ in the last stage must equal V and $\mathbf{r}_{X_*^{(K)}}$ must be an optimal rate vector that attains global omniscience with the minimum sum-rate.

[1]This means that we need to solve the CO problem first, say, by the PAR algorithm in $O(|V| \cdot \text{SFM}(|V|))$ time. This makes SO less attractive: since $|X_*| \leq |V|$, the local omniscience problem is less complex and therefore it is preferred that the complimentary subset can be determined *before* the global omniscience problem is solved.

A. Contributions

In this paper, we propose a parametric (PAR) algorithm that reduces the complexity for solving the minimum sum-rate problem in CO to $O(|V| \cdot \text{SFM}(|V|))$. We propose an efficient algorithm for searching the complimentary user subset X_* and a local omniscience achievable rate vector for the two-stage SO [6], [27], [28]. We propose a novel multi-stage SO strategy $\{(X_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1, \dots, K\}\}$ and propose algorithms to obtain this K-stage SO from the returned results by the PAR algorithm for both asymptotic and non-asymptotic models. The results indicate that CO, two-stage SO and multi-stage SO can all be solved in $O(|V| \cdot \text{SFM}(|V|))$ time.

1) Strict strong map property and PAR algorithm: The study starts with a review of the coordinate saturation capacity (CoordSatCap) algorithm in [18, Algorithm 3], a nesting algorithm in the modified decomposition algorithm (MDA) algorithm [18, Algorithm 1]. The CoordSatCap algorithm determines the Dilworth truncation, a partition $\mathcal{Q}_{\alpha,V}$ of V, for only one specific value of the minimum sum-rate estimate α. We rewrite CoordSatCap as a function of α and prove that a nesting SFM problem[2] exhibits the strict strong map property in α. We show the solution of this SFM is segmented in α and prove that its critical/turning points can be searched by a finite number of recursions. We use this proof to propose a StrMap algorithm that reduces the original $O(|V|)$ calls of the SFM algorithm to $O(1)$ calls by adopting the existing parametric SFM (PSFM) techniques in [33]–[35].

We propose a PAR algorithm that iteratively calls the subroutine StrMap to update the segmented Dilworth truncation $\mathcal{Q}_{\alpha,V}$ for all values of α. The critical points of α, as well as the corresponding partitions, which characterize the segmented $\mathcal{Q}_{\alpha,V}$, converge to the PSP of V, where the first critical value determines the minimum sum-rate of CO for both asymptotic and non-asymptotic models. The PAR algorithm also outputs a rate vector $\mathbf{r}_{\alpha,V} = (r_{\alpha,i} : i \in V)$, which is piecewise linear in α that determines an optimal rate vector for both asymptotic and non-asymptotic source models. The PAR algorithm invokes $|V|$ calls of StrMap and therefore its run time is $O(|V| \cdot \text{SFM}(|V|))$. This indicates a complexity reduction of factor $|V|$ for not only solving the minimum sum-rate problem, but also obtaining the network strength [22] and MAC clustering result [21]. The PAR algorithm also allows distributed computation, which only incurs the computation complexity $O(\text{SFM}(|V|))$ at each user $i \in V$.

[2]The solution to CO essentially relies on solving this SFM algorithm. See (6a).

2) Two-stage SO: For SO, we first focus on the problem of how to efficiently search a complimentary user subset $X_* \subsetneq V$ in the existing two-stage SO. We relax the necessary and sufficient condition in [6, Theorems 4.2 and 5.2] to a sufficient condition on $\underline{\alpha}$, a lower bound on the minimum sum-rate for the global omniscience. This lower bound can be determined in $O(|V|)$ time. This sufficient condition is used to prove that, at each iteration i of the PAR algorithm, any nonsingleton user subset contained in the partition $\mathcal{Q}_{\underline{\alpha},V_i}$ is complimentary. Here, $\mathcal{Q}_{\underline{\alpha},V}$ is the value of $\mathcal{Q}_{\alpha,V}$ at $\alpha = \underline{\alpha}$. We propose an algorithm to search for the complimentary subset $X_* \subsetneq V$ as any nonsingleton subset in $\mathcal{Q}_{\underline{\alpha},V}$ and a corresponding local omniscience achievable rate vector $\mathbf{r}_{X_*}$. This algorithm either solves the two-stage SO or, if there is no complimentary user subset, determines the solution to the minimum sum-rate for the global omniscience in $O(|V| \cdot \text{SFM}(|V|))$ time.

3) Multi-stage SO: We denote the multi-stage SO by a sequence of two-tuples $\{(X_*^{(k)}, \mathbf{r}_V^{(k)}) \colon k \in \{1, \dots, K\}\}$ with $K \leq |V| - 1$: the local omniscience is attained progressively in $X_*^{(k)}$ by transmission rate $\mathbf{r}_V^{(k)}$ from $k = 1$ to K. We propose two sufficient conditions ensuring the achievability of this K-stage SO. One is the inclusion-wise expanding subset sequence: $\emptyset \subsetneq X_*^{(1)} \subsetneq \dots \subsetneq X_*^{(K)}$, where $X_*^{(k)}$ is complimentary for all $k \in \{1, \dots, K-1\}$ and $X_*^{(K)} = V$ guarantees global omniscience is reached at the final stage; the other condition is the rate vectors form a monotonic sequence: $\mathbf{r}_V^{(1)} \leq \dots \leq \mathbf{r}_V^{(k)}$, where each $X_*^{(k)}$ achieves local omniscience in $X_*^{(k)}$. We use these sufficient conditions to propose two algorithms to extract an achievable K-stage SO from the critical points of $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ returned by the PAR algorithm for both asymptotic and non-asymptotic models. The algorithm for the non-asymptotic model needs to run the PAR algorithm twice to guarantee the monotonicity of the rate vector. The complexity of the proposed two algorithms is $O(|V| \cdot \text{SFM}(|V|))$: if there exists a complimentary subset, they return an achievable multi-stage SO strategy; otherwise, they directly return the optimal solution for global omniscience.

B. Organization

The rest of paper is organized as follows. The system model is described in Section II, where we introduce the notation, review the existing results and derive the properties of the CoordSatCap algorithm. In Section III, we prove the strict strong map property and propose the PAR algorithm and its subroutine StrMap algorithm, where we also show the complexity reduction and distributed implementation of the PAR algorithm. Section IV presents the solutions to SO: Section IV-A proposes an algorithm for searching for the complimentary subset in the two-stage SO; Section IV-B proposes algorithms for determining an achievable multi-stage SO for asymptotic and non-asymptotic models.

II. System Model

Let V with $|V| > 1$ be a finite set that contains all users in the system. We call V the *ground set*. Let $\mathsf{Z}_V = (\mathsf{Z}_i : i \in V)$ be a vector of discrete random variables indexed by V. For each $i \in V$, user i privately observes an n-sequence Z_i^n of the random source Z_i that is i.i.d. generated according to the joint distribution P_{Z_V}. We allow users to exchange their observed data directly to recover the source sequence Z_V^n. The state that each user obtains the total information in the entire multiple source is called *omniscience*, and the process that users communicate with each other to attain omniscience is called *communication for omniscience* (CO) [3].

Let $\mathbf{r}_V = (r_i : i \in V)$ be a rate vector indexed by V. We call $\mathbf{r}_V$ an *achievable rate vector* if the omniscience can be attained by letting users communicate at the rates designated by $\mathbf{r}_V$. For the original CO problem formulated in [3] considering the asymptotic limits as the *block length* n goes to infinity, each dimension r_i is the compression rate denoting the expected code length at which user i encode their observations. We also study a *non-asymptotic model*, where n is assumed to be finite. The *finite linear source model* [5] is one of the non-asymptotic models, in which the multiple random source is represented by a vector that belongs to a finite field and each r_i denotes the integer number of linear combinations of observations transmitted by user i. This finite linear source model is of particular interest in that it models the CCDE problem [7]–[9], where the users communicate over P2P channels to help each other recover a packet set. In this paper, for the omniscience problem in the non-asymptotic model, we focus on the finite linear source model. Therefore, we use the term non-asymptotic model, finite linear source model and CCDE interchangeably. The main notation in this paper is listed in Table I.

A. Minimum Sum-rate Problem

For a given rate vector $\mathbf{r}_V$, let $r \colon 2^V \mapsto \mathbb{R}_+$ be the *sum-rate function* such that

$$r(X) = \sum_{i \in X} r_i, \quad \forall X \subseteq V,$$

with the convention $r(\emptyset) = 0$. The *achievable rate region* is characterized in [3] by the set of multiterminal Slepian-Wolf constraints [36], [37]:

$$\mathscr{R}_{\text{CO}}(V) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} \colon r(X) \geq H(X|V \setminus X), \forall X \subsetneq V\},$$

where $H(X)$ is the amount of randomness in Z_X measured by the Shannon entropy [38] and $H(X|Y) = H(X \cup Y) - H(Y)$ is the conditional entropy of Z_X given Z_Y. In a finite linear source model, the entropy function H reduces to the rank of a matrix that only takes integral values.

The fundamental problem in CO is to minimize the sum-rate in the achievable rate region [3, Proposition 1]

$$R_{\text{ACO}}(V) = \min\{r(V) \colon \mathbf{r}_V \in \mathscr{R}_{\text{CO}}(V)\}, \tag{1a}$$

$$R_{\text{NCO}}(V) = \min\{r(V) \colon \mathbf{r}_V \in \mathscr{R}_{\text{CO}}(V) \cap \mathbb{Z}^{|V|}\}, \tag{1b}$$

for the asymptotic and non-asymptotic models, respectively. Denote by

$$\mathscr{R}^*_{\text{ACO}}(V) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} \colon r(V) = R_{\text{ACO}}(V)\},$$
$$\mathscr{R}^*_{\text{NCO}}(V) = \{\mathbf{r}_V \in \mathbb{Z}^{|V|} \colon r(V) = R_{\text{NCO}}(V)\}$$

the *optimal rate vector set* for the asymptotic and non-asymptotic models, respectively. We say that the minimum

TABLE I
MAIN NOTATION

Notation	Description
V, X, Y	a finite user set V and its subsets $X, Y \subseteq V$
$\mathcal{P}$, $\mathcal{X}$, $\mathcal{Y}$	calligraphic notation denoting a set of disjoint sets, e.g., a partition
$\tilde{\mathcal{P}}$, $\tilde{\mathcal{X}}$, $\tilde{\mathcal{Y}}$	the fusion of a set of disjoint sets, e.g., $\tilde{\mathcal{X}} = \sqcup_{C \in \mathcal{X}} C$
Φ	a linear ordering of all users $i \in V$
V_i	the first i users in V
$\Pi(V)$	the set of all partitions of V
$\mathbf{r}_X$, $r(\cdot)$	a rate vector $\mathbf{r}_X = (r_i : i \in X)$ and its sum-rate function $r(X) = \sum_{i \in X} r_i$
$\mathscr{R}_{\text{CO}}(X)$	the set of all omniscience achievable rate vectors
$\mathscr{R}^*_{\text{ACO}}(X)$, $R_{\text{ACO}}(X)$	the optimal rate vector set for the asymptotic model, containing omniscience achievable rate vectors having minimum the sum-rate $R_{\text{ACO}}(V)$
$\mathscr{R}^*_{\text{NCO}}(X)$, $R_{\text{NCO}}(X)$	the optimal rate vector set for the non-asymptotic model, containing omniscience achievable rate vectors having the minimum sum-rate $R_{\text{NCO}}(V)$
α	an estimation of the minimum sum-rate
$\mathbf{r}_{\alpha,X}$, $r_\alpha(\cdot)$	a rate vector $\mathbf{r}_{\alpha,X} = (r_{\alpha,i} : i \in X)$, where each $r_{\alpha,i}$ is a function of α. So is its sum-rate function $r_\alpha(X) = \sum_{i \in X} r_{\alpha,i}$
$f_\alpha(X)$	the set function $f(X) = \alpha - H(V) + H(X)$
$P(f_\alpha)$	the polyhedron of the set function f_α
$B(f_\alpha)$	the base polyhedron of the set function f_α
$f_\alpha[\mathcal{P}]$	the partition function $f_\alpha[\mathcal{P}] = \sum_{C \in \mathcal{P}} f_\alpha(C)$
$\hat{f}_\alpha$, $\mathcal{Q}_{\alpha,V}$	for a specific α, the Dilworth truncation of f_α: $\hat{f}_\alpha(V) = \min_{\mathcal{P} \in \Pi(V)} f_\alpha[\mathcal{P}]$. The minimizer, denoted by $\mathcal{Q}_{\alpha,V}$, is a partition of V.
$f_\alpha^{V_i}$	the reduction of f_α on V_i
$\langle X \rangle_{\mathcal{P}}$	the decomposition of subset X by the partition $\mathcal{P}$: $\langle X \rangle_{\mathcal{P}} = \{X \cap C : C \in \mathcal{P}\}$

sum-rate problem is solved if the value of the minimum sum-rate in (1), as well as an optimal rate vector are determined.

To solve the *minimum sum-rate problem* without dealing with the exponentially growing number of constraints in the linear programming, (1a) and (1b) are respectively converted to [3, Example 4] [39] [18, Corollary 6]

$$R_{\text{ACO}}(V) = \max_{\mathcal{P} \in \Pi(V) \colon |\mathcal{P}| > 1} \sum_{C \in \mathcal{P}} \frac{H(V) - H(C)}{|\mathcal{P}| - 1}, \tag{2a}$$

$$R_{\text{NCO}}(V) = \Big\lceil \max_{\mathcal{P} \in \Pi(V) \colon |\mathcal{P}| > 1} \sum_{C \in \mathcal{P}} \frac{H(V) - H(C)}{|\mathcal{P}| - 1} \Big\rceil, \tag{2b}$$

where $\Pi(V)$ denotes the set containing all partitions of V. It is shown in [16]–[18] that the combinatorial optimization problem in (2) can be solved based on the existing SFM techniques in polynomial time.

B. Existing Results Parameterized by the Minimum Sum-rate Estimate α

The efficiency for solving the minimum sum-rate problems in (2) relies on the submodularity of the entropy function H and the induced structure in the partition lattice. It is shown in [18] that the validity of the algorithms proposed in [14, Appendix F] and [15, Algorithm 3] for solving (2b) in CCDE and [18, Algorithm 1] for solving both (2a) and (2b) can be explained by the Dilworth truncation and the partition chain it forms in the estimation of the minimum sum-rate, which is called the *principal sequence of partitions (PSP)*.

In this section, we introduce the notation and review the Dilworth truncation, PSP and the coordinate-wise saturation capacity (CoordSatCap) algorithm, an essential nesting algorithm in [14, Appendix F], [15, Algorithm 3] and [18, Algorithm 1]. We rewrite these results as variables or functions of the minimum sum-rate estimate α. The purpose is to introduce the notation and existing statements that will be used to present and prove the strict strong map property in Section III.

1) Preliminaries: For $X \subseteq V$, let $\chi_X = (e_i : i \in V)$ be the *characteristic vector* of the subset X such that $e_i = 1$ if $i \in X$ and $e_i = 0$ if $i \notin X$. The notation $\chi_{\{i\}}$ is simplified by χ_i. Let $\sqcup$ denote the disjoint union. For $\mathcal{X}$ that contains disjoint subsets of V, we denote by $\tilde{\mathcal{X}} = \sqcup_{C \in \mathcal{X}} C$ the *fusion* of $\mathcal{X}$. For example, for $\mathcal{X} = \{\{3,4\},\{2\},\{8\}\}$, $\tilde{\mathcal{X}} = \{2,3,4,8\}$.

For partitions $\mathcal{P}, \mathcal{P}' \in \Pi(V)$, we denote by $\mathcal{P} \preceq \mathcal{P}'$ if $\mathcal{P}$ is finer than $\mathcal{P}'$ and $\mathcal{P} \prec \mathcal{P}'$ if $\mathcal{P}$ is strictly finer than $\mathcal{P}'$.[3] For any $X \subseteq V$ and $\mathcal{P} \in \Pi(V)$, $\langle X \rangle_{\mathcal{P}} = \{X \cap C : C \in \mathcal{P}\}$ denotes the *decomposition of* X *by* $\mathcal{P}$. For example, for $X = \{1,2,4\}$ and $\mathcal{P} = \{\{1,2,3\},\{4\}\}$, $\langle X \rangle_{\mathcal{P}} = \{\{1,2\},\{4\}\}$.

A function $f \colon 2^V \mapsto \mathbb{R}$ is *submodular* if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ for all $X, Y \subseteq V$. The problem $\min\{f(X) \colon X \subseteq V\}$ is an SFM problem. It can be solved in strongly polynomial time (see Appendix D) and the set of minimizers $\text{argmin}\{f(X) \colon X \subseteq V\}$ form a set lattice such that the smallest/minimal minimizer $\bigcap \text{argmin}\{f(X) \colon X \subseteq V\}$ and largest/maximal minimizer $\bigcup \text{argmin}\{f(X) \colon X \subseteq V\}$ uniquely exist and can be determined at the same time when the SFM problem is solved [19, Chapter VI].

We call $\Phi = (\phi_1, \dots, \phi_{|V|})$ a *linear ordering/permutation* of the indices in V if $\phi_i \in V$ and $\phi_i \neq \phi_{i'}$ for all $i, i' \in \{1, \dots, |V|\}$ such that $i \neq i'$. For $i \in V$, let $V_i = \{\phi_1, \dots, \phi_i\}$ be the set of the first i users in the linear ordering Φ. We call $f^{V_i} \colon 2^{V_i} \mapsto \mathbb{R}$ the *reduction* of f on V_i such that $f^{V_i}(X) = f(X)$ for all $X \subseteq V_i$ [19, Section 3.1(a)]. For example, for $\Phi = (2,3,1,4)$, $V_2 = \{2,3\}$, the reduction of f on V_2 is $f^{V_2}(X) = f(X)$ for all $X \subseteq \{2,3\}$.

2) Principal Sequence of Partitions (PSP): Let $\alpha \in \mathbb{R}_+$ be an estimation of the minimum sum-rate and define a set function $f_\alpha \colon 2^V \mapsto \mathbb{R}$ such that

$$f_\alpha(X) = \alpha - H(V) + H(X), \quad \forall X \subseteq V,$$

except that $f(\emptyset) = 0$. This function is the same as the residual entropy function in [16] in that it offsets/subtracts the amount of information in each nonempty subset X by $H(V) - \alpha$. Let

[3]The partition $\mathcal{P}$ is finer than $\mathcal{P}'$, if each subset in $\mathcal{P}$ is contained in some subset in $\mathcal{P}'$.

$f_\alpha[\cdot]$ be a partition function such that $f_\alpha[\mathcal{P}] = \sum_{C\in\mathcal{P}} f_\alpha(C)$ for all $\mathcal{P} \in \Pi(V)$, where $f_\alpha(C) = \alpha - H(V) + H(C)$. The Dilworth truncation of f_α is [40]

$$\hat{f}_\alpha(V) = \min_{\mathcal{P}\in\Pi(V)} f_\alpha[\mathcal{P}]. \tag{3}$$

For a given α, let $\mathcal{Q}_{\alpha,V} = \bigwedge \operatorname{argmin}_{\mathcal{P}\in\Pi(V)} f_\alpha[\mathcal{P}]$ be the finest minimizer of (3). The Dilworth truncation (3) exhibits a strong structure in α that is characterized by the PSP, which provides the solution to the minimum sum-rate problem.

Lemma 1. *The value of $\hat{f}_\alpha(V)$ is piecewise linear and strictly increasing in α. It is determined by $p < |V|$ critical points*

$$0 \leq \alpha^{(p)} < \ldots < \alpha^{(1)} < \alpha^{(0)} = H(V) \tag{4}$$

with the corresponding finest minimizer $\mathcal{P}^{(j)} = \mathcal{Q}_{\alpha^{(j)},V} = \bigwedge \operatorname{argmin}_{\mathcal{P}\in\Pi(V)} f_{\alpha^{(j)}}[\mathcal{P}]$ for all $j \in \{0,\ldots,p\}$ forming a partition chain

$$\{\{i\}: i \in V\} = \mathcal{P}^{(p)} \prec \ldots \prec \mathcal{P}^{(1)} \prec \mathcal{P}^{(0)} = \{V\} \tag{5}$$

such that [21], [41]

$$\mathcal{Q}_{\alpha,V} = \begin{cases} \mathcal{P}^{(p)} & \alpha \in [0, \alpha^{(p)}], \\ \mathcal{P}^{(p-1)} & \alpha \in (\alpha^{(p)}, \alpha^{(p-1)}], \\ & \vdots \\ \mathcal{P}^{(0)} & \alpha \in (\alpha^{(1)}, \alpha^{(0)}]. \end{cases}$$

The partitions in (5)*, together with the corresponding critical values $\alpha^{(j)}$, is called the Principal Sequence of Partitions (PSP) of the ground set V.* ∎

The first critical point of the PSP provides the solution to the minimum sum-rate problem.

Lemma 2 ([18, Corollary A.3])**.** *$R_{ACO}(V) = \alpha^{(1)}$ and $R_{NCO}(V) = \lceil \alpha^{(1)} \rceil$ for the asymptotic and non-asymptotic models, respectively.* ∎

3) CoordSatCap Algorithm: All of the existing algorithms in [14], [15], [18] for solving the minimum sum-rate problem in (2) run a subroutine that determines the minimum and/or the finest minimizer of the Dilworth truncation (3) for a given value of α. This subroutine is outlined by the CoordSatCap algorithm in Algorithm 1.

The idea of CoordSatCap algorithm is to keep increasing each dimension of a rate vector $\mathbf{r}_{\alpha,V}$ in the *submodular polyhedron* of f_α

$$P(f_\alpha) = \{\mathbf{r}_{\alpha,V} \in \mathbb{R}^{|V|} : r_\alpha(X) \leq f_\alpha(X), X \subseteq V\}$$

until it reaches the *base polyhedron* of the Dilworth truncation[4] $\hat{f}_\alpha$

$$B(\hat{f}_\alpha) = \{\mathbf{r}_{\alpha,V} \in P(f_\alpha) : r_\alpha(V) = \hat{f}_\alpha(V)\}.$$

Here, $\mathbf{r}_{\alpha,V} = (r_{\alpha,i} : i \in V)$ is a $|V|$-dimensional rate vector that is parameterized by the input minimum sum-rate estimate α and $r_\alpha(X) = \sum_{i\in X} r_{\alpha,i}, \forall X \subseteq V$ is the sum-rate function of this rate vector. The amount of the rate increment is determined by [19, Section 2.3] [18, Lemmas 22 and 23]

$$\min\{f_\alpha(\tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{X}}) : \{i'\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\} \tag{6a}$$
$$= \max\{\xi : \mathbf{r}_{\alpha,V} + \xi\chi_{i'} \in P(f_\alpha)\} \tag{6b}$$

where (6a) is the minimization problem in step 5 of Algorithm 1 and is a SFM problem [18, Section V-B]. The maximum of (6b) is called the *saturation capacity*. At the end of Algorithm 1, the partition $\mathcal{Q}_{\alpha,V}$ is updated to the finest minimizer of $\min_{\mathcal{P}\in\Pi(V)} f_\alpha[\mathcal{P}]$ [18, Section V-B] so that $r_\alpha(V) = \hat{f}_\alpha(V)$.

For solving the minimum sum-rate problem, [18, Algorithm 1] utilizes the outputs of the CoordSatCap algorithm and the properties of the PSP in Lemma 17 in Appendix A to update α until it reaches $R_{\text{ACO}}(V)$. Due to the equivalence

$$B(\hat{f}_\alpha) = \{\mathbf{r}_V \in \mathscr{R}_{\text{CO}}(V) : r_\alpha(V) = \hat{f}_\alpha(V) = \alpha\}$$

for all $\alpha \geq R_{\text{ACO}}(V)$ [18, Section III-B and Theorem 4], in the final call of the CoordSatCap, an optimal rate vector

$$\mathbf{r}_{R_{\text{ACO}}(V),V} \in B(\hat{f}_{R_{\text{ACO}}(V)}) = \mathscr{R}^*_{\text{ACO}}(V),$$
$$\mathbf{r}_{R_{\text{NCO}}(V),V} \in B(\hat{f}_{R_{\text{NCO}}(V)}) \cap \mathbb{Z}^{|V|} = \mathscr{R}^*_{\text{NCO}}(V)$$

is also returned for asymptotic and non-asymptotic models, respectively.

For the input $\alpha = R_{\text{ACO}}(V)$, the CoordSatCap algorithm also outputs the *fundamental partition* $\mathcal{Q}_{R_{\text{ACO}}(V),V} = \mathcal{P}^{(1)}$. This is the partition that corresponds to the first critical point in (5) and equals the finest maximizer of (2a). This is an important parameter in CCDE in that $|\mathcal{P}^{(1)}| - 1$ is the least common multiple (LCM) of $\mathbf{r}_{R_{\text{ACO}}(V),V}$ [18, Corollary 28]: By letting each packet be broken into $|\mathcal{P}^{(1)}| - 1$ chunks, the optimal rate vector $\mathbf{r}_{R_{\text{ACO}}(V),V}$ is implementable based on linear codes, which saves the overall transmission rates by no more than 1 from the optimal rate vector $\mathbf{r}_{R_{\text{NCO}}(V),V} \in \mathscr{R}^*_{\text{NCO}}(V)$.

C. Current Complexity

The complexity of the CoordSatCap algorithm is $O(|V| \cdot \text{SFM}(|V|))$. It is required to call the CoordSatCap algorithm $O(|V|)$ times to find $\alpha^{(1)}$ and $\mathcal{P}^{(1)}$ [18, Section V-D]. Therefore, the current complexity for solving the minimum sum-rate problem in CO is $O(|V|^2 \cdot \text{SFM}(|V|))$.

III. Strict Strong Map

While the CoordSatCap algorithm determines the Dilworth truncation $\hat{f}_\alpha(V)$ for only one value of α, we reveal the structural properties of the SFM problem (6a) in α and propose a parametric (PAR) algorithm that iteratively determines $\mathcal{Q}_{\alpha,V_i}$ and $\mathbf{r}_{\alpha,V}$ for all values of the minimum sum-rate estimate α. This PAR algorithm reduces the computational complexity for solving the minimum sum-rate problem in both asymptotic and non-asymptotic models and allows distributed computation. Note that, in this paper, when we say for all α, we mean for all $\alpha \in [0, H(V)]$ since the minimum sum-rates $R_{\text{ACO}}(V), R_{\text{NCO}}(V) \in [0, H(V)]$.

[4] The original purpose of the CoordSatCap algorithm is to determine the value of $\hat{f}_\alpha(V)$ by tightening the upper bound $f_\alpha(X)$ in $P(f_\alpha)$. See [18, Appendix B]. Also note that, since $\hat{f}_\alpha(V) \leq f_\alpha(V)$, $B(\hat{f}_\alpha)$ and $B(f_\alpha) = \{\mathbf{r}_{\alpha,V} \in P(f_\alpha) : r_\alpha(V) = f_\alpha(V)\}$ are not equivalent in general.

Algorithm 1: CoordSatCap Algorithm [18, Algorithm 3]

input : α, H, V and Φ
output: $\mathbf{r}_{\alpha,V} \in B(\hat{f}_\alpha)$ and $\mathcal{Q}_{\alpha,V} = \bigwedge \operatorname{argmin}_{\mathcal{P}\in\Pi(V)} f_\alpha[\mathcal{P}]$

1 Let $\mathbf{r}_{\alpha,V} := (\alpha - H(V))\chi_V$ so that $\mathbf{r}_{\alpha,V} \in P(f_\alpha)$;
2 Initiate $r_{\alpha,\phi_1} := f_\alpha(\{\phi_1\})$ and $\mathcal{Q}_{\alpha,V_1} := \{\{\phi_1\}\}$;
3 **for** $i = 2$ **to** $|V|$ **do**
4 $\quad \mathcal{Q}_{\alpha,V_i} := \mathcal{Q}_{\alpha,V_{i-1}} \sqcup \{\{\phi_i\}\}$;
5 $\quad \mathcal{U}_{\alpha,V_i} := \bigcap \operatorname{argmin}\{f_\alpha(\tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{X}}) \colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$;
6 $\quad$ Update $\mathbf{r}_{\alpha,V}$ and $\mathcal{Q}_{\alpha,V_i}$:

$$\mathbf{r}_{\alpha,V} := \mathbf{r}_{\alpha,V} + \big(f_\alpha(\tilde{\mathcal{U}}_{\alpha,V_i}) - r_\alpha(\tilde{\mathcal{U}}_{\alpha,V_i})\big)\chi_{\phi_i};$$
$$\mathcal{Q}_{\alpha,V_i} := (\mathcal{Q}_{\alpha,V_i} \setminus \mathcal{U}_{\alpha,V_i}) \sqcup \{\tilde{\mathcal{U}}_{\alpha,V_i}\};$$

7 **endfor**
8 **return** $\mathbf{r}_{\alpha,V}$ and $\mathcal{Q}_{\alpha,V}$;

A. Observations

For $f_\alpha^{V_i}$, we have the base polyhedron

$$B(\hat{f}_\alpha^{V_i}) = \{\mathbf{r}_{\alpha,V_i} \in P(f_\alpha^{V_i}) \colon r_\alpha(V_i) = \hat{f}_\alpha^{V_i}(V_i) = \hat{f}_\alpha(V_i)\}$$

where $P(f_\alpha^{V_i}) = \{\mathbf{r}_{\alpha,V_i} \in \mathbb{R}^i \colon r_\alpha(X) \leq f_\alpha^{V_i}(X) = f_\alpha(X), X \subseteq V_i\}$. Considering the values of $\mathcal{Q}_{\alpha,V_i}$ and $\mathbf{r}_{\alpha,V_i}$ in α in the CoordSatCap algorithm as the iteration index i grows, we have the following result.

Proposition 3. *After step 6 in each iteration i of Algorithm 1, $\mathcal{Q}_{\alpha,V_i} = \bigwedge \operatorname{argmin}_{\mathcal{P}\in\Pi(V_i)} f_\alpha[\mathcal{P}]$ and $\mathbf{r}_{\alpha,V_i} \in B(\hat{f}_\alpha^{V_i})$ for all α.* ■

The proof is omitted since it is a direct result that $\mathcal{Q}_{\alpha,V_i} = \bigwedge \operatorname{argmin}_{\mathcal{P}\in\Pi(V_i)} f_\alpha[\mathcal{P}]$. The fact that $\mathbf{r}_{\alpha,V_i} \in B(\hat{f}_\alpha^{V_i})$ can be seen in [18, Appendix B]. Proposition 3 suggests that we could obtain $\mathbf{r}_{\alpha,V_i}$ and $\mathcal{Q}_{\alpha,V_i}$ for all values of α for each iteration i until the final results for $i = |V|$ constitute the complete solution to the minimum sum-rate problem. We call it the *parametric method* for that the results remain functions of α. However, the question is whether this parametric approach is efficient. It is clear that $\mathcal{Q}_{\alpha,V_i}$ for all α is again characterized by the PSP of V_i,[5] the determination of which completes in $O(|V_i|^2 \cdot \text{SFM}(|V_i|))$ time. If using the existing methods [14, Appendix F], [15, Algorithm 3] or [18, Algorithm 1], the overall complexity of the parametric method is higher: $O(|V|^3 \cdot \text{SFM}(|V|))$. Instead, we prove the strict strong map property of the minimization problem (6a) in the CoordSatCap algorithm and show that the parametric method is simpler than the existing algorithms.

B. Strong Map Property

We first derive some properties of the CoordSatCap algorithm below. The proof of Lemma 4 is in Appendix B. The properties in Lemma 4 play important roles in the proof of the strict strong map property and validity of the PAR algorithm in this section. The monotonicity of the sum-rate in Lemma 4(c) also guarantees the feasibility of a multi-stage SO in Section IV.

Lemma 4. *At the end of each iteration i of Algorithm 1, the rate vector $\mathbf{r}_{\alpha,V} \in P(f_\alpha)$, where $P(f_\alpha) = P(\hat{f}_\alpha)$, and the following hold for all α:*

(a) $r_\alpha(V_i) = r_\alpha[\mathcal{Q}_{\alpha,V_i}] = f_\alpha[\mathcal{Q}_{\alpha,V_i}] = \hat{f}_\alpha(V_i)$, where $r_\alpha[\mathcal{Q}_{\alpha,V_i}] = \sum_{C\in\mathcal{Q}_{\alpha,V_i}} r_\alpha(C)$;
(b) $r_\alpha(\tilde{\mathcal{X}}) = r_\alpha[\mathcal{X}] = f_\alpha[\mathcal{X}] = \hat{f}_\alpha(\tilde{\mathcal{X}})$ for all $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$;
(c) For all $\alpha < \alpha'$, $\mathcal{Q}_{\alpha,V_i} \preceq \mathcal{Q}_{\alpha',V_i}$ and, for all $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$ and $\mathcal{X}' \subseteq \mathcal{Q}_{\alpha',V_i}$ such that $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}'$,

$$r_\alpha[\mathcal{X}] = \hat{f}_\alpha(\tilde{\mathcal{X}}) < \hat{f}_{\alpha'}(\tilde{\mathcal{X}}') = r_{\alpha'}[\mathcal{X}'].$$ ■

Recall that $\mathcal{Q}_{\alpha,V_i}$ is determined by $\mathcal{U}_{\alpha,V_i}$ in step 6 of Algorithm 1. The complexity of the parametric algorithm relies on the hardness of solving the minimization problem (6a) for all α. While the existing algorithms only utilize the submodularity of this problem for a specific value of α, we reveal a strict strong map property below. Base on the objective function in (6a), we define

$$g_\alpha(\tilde{\mathcal{X}}) = f_\alpha(\tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{X}}), \quad \forall \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}. \tag{7}$$

Definition 5 (strong map [42, Section 4.1])**.** *For two distributive lattices $\mathcal{L}_1, \mathcal{L}_2 \subseteq 2^V$,[6] and submodular functions $h_1 \colon \mathcal{L}_1 \mapsto \mathbb{R}$ and $h_2 \colon \mathcal{L}_2 \mapsto \mathbb{R}$, h_1 and h_2 form a strong map, denoted by $h_1 \to h_2$, if*

$$h_1(Y) - h_1(X) \geq h_2(Y) - h_2(X) \tag{8}$$

for all $X, Y \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that $X \subseteq Y$. The strong map is strict, denoted by $h_1 \twoheadrightarrow h_2$, if $h_1(Y) - h_1(X) > h_2(Y) - h_2(X)$ for all $X \subsetneq Y$.

Theorem 6. *In each iteration i of Algorithm 1, g_α forms a **strict strong map** in α:*

$$g_\alpha \twoheadrightarrow g_{\alpha'}, \quad \forall \alpha, \alpha' \colon \alpha < \alpha'.$$

Proof: First, for all α, all subsets of $\mathcal{Q}_{\alpha,V}$ form a lattice. For any $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$ and $\mathcal{Y} \subseteq \mathcal{Q}_{\alpha',V_i}$ such that $i \in \tilde{\mathcal{X}} \subseteq \tilde{\mathcal{Y}}$ and g_α and $g_{\alpha'}$ are both defined on $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$, we have $i \notin \tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}$. Also, there exist $\mathcal{M} \subseteq \mathcal{Q}_{\alpha,V_i}$ and $\mathcal{N} \subseteq \mathcal{Q}_{\alpha',V_i}$ (with $\mathcal{M} \preceq \mathcal{N}$) such that $\tilde{\mathcal{M}} = \tilde{\mathcal{N}} = \tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}$. According to Lemma 4(b) and (c), $r_\alpha(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) = f_\alpha[\mathcal{M}] = \hat{f}_\alpha(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}})$ and $r_{\alpha'}(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) = f_{\alpha'}[\mathcal{N}] = \hat{f}_{\alpha'}(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}})$. Then,

$$\begin{aligned} g_\alpha(\tilde{\mathcal{Y}}) &- g_\alpha(\tilde{\mathcal{X}}) - g_{\alpha'}(\tilde{\mathcal{Y}}) + g_{\alpha'}(\tilde{\mathcal{X}}) \\ &= r_{\alpha'}(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) \\ &= \begin{cases} 0 & \tilde{\mathcal{X}} = \tilde{\mathcal{Y}}, \\ \hat{f}_{\alpha'}(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) - \hat{f}_\alpha(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) & \tilde{\mathcal{X}} \subsetneq \tilde{\mathcal{Y}}, \end{cases} \end{aligned}$$

where $\hat{f}_{\alpha'}(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) - \hat{f}_\alpha(\tilde{\mathcal{Y}} \setminus \tilde{\mathcal{X}}) > 0$ for all α and α' such that $\alpha < \alpha'$ based on Lemma 4(c). Referring to Definition 5, this proves the theorem. ■

We have the minimizer $\tilde{\mathcal{U}}_{\alpha,V_i}$ also segmented in α based on the results in [42].

[5] This fact will be utilized in Section III-E to propose a distributed algorithm for solving the CO problem and in Section IV for solving the SO problem.

[6] A group of sets $\mathcal{L}$ form a distributive lattice if, for all $X, Y \in \mathcal{L}$, $X \cap Y \in \mathcal{L}$ and $X \cup Y \in \mathcal{L}$ [19, Section 3.2].

Lemma 7. *[42, Theorems 26 to 28] In each iteration i of Algorithm 1, the minimal minimizer $\mathcal{U}_{\alpha,V_i}$ of $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ satisfies $\tilde{\mathcal{U}}_{\alpha,V_i} \subseteq \tilde{\mathcal{U}}_{\alpha',V_i}$ for all $\alpha < \alpha'$. In addition, $\tilde{\mathcal{U}}_{\alpha,V_i}$ for all α is fully characterized by $q < |V_i| - 1$ critical points*[7]

$$0 \leq \alpha_q < \ldots < \alpha_1 < \alpha_0 = H(V) \tag{9}$$

and the corresponding minimal minimizer $\tilde{\mathcal{S}}_j = \tilde{\mathcal{U}}_{\alpha_j,V_i}$ for all $j \in \{0,\ldots,q\}$ forms a set chain

$$\{\phi_i\} = \tilde{\mathcal{S}}_q \subsetneq \ldots \subsetneq \tilde{\mathcal{S}}_1 \subsetneq \tilde{\mathcal{S}}_0 = V_i \tag{10}$$

and $\tilde{\mathcal{U}}_{\alpha,V_i} = \tilde{\mathcal{S}}_q = \{\phi_i\}$ for all $\alpha \in [0,\alpha_q]$ and $\tilde{\mathcal{U}}_{\alpha,V_i} = \tilde{\mathcal{S}}_j$ for all $\alpha \in (\alpha_{j+1},\alpha_j]$ such that $j \in \{0,\ldots,q-1\}$.[8] ■

To completely solve the problem $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ for all α, the remaining problem is how to determine all α_j's and $\tilde{\mathcal{S}}_j$'s, which is neither specified in Lemma 7 nor solvable directly based on the existing parametric SFM (PSFM) algorithms or other combinatorial optimization techniques (see the explanation in Appendix D). For this purpose, we derive Lemma 8 below. The proof is in Appendix C.

Lemma 8. *For all α_j's and $\tilde{\mathcal{S}}_j$'s that characterize $\tilde{\mathcal{U}}_{\alpha,V_i}$ of the minimal minimizer of $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ in Lemma 7, the following holds:*

(a) $g_{\alpha_j}(\tilde{\mathcal{S}}_{j-1}) = g_{\alpha_j}(\tilde{\mathcal{S}}_j)$, i.e., $r_{\alpha_j}(\tilde{\mathcal{S}}_{j-1} \setminus \tilde{\mathcal{S}}_j) = H(\tilde{\mathcal{S}}_{j-1}) - H(\tilde{\mathcal{S}}_j)$, for all $j \in \{1,\ldots,q\}$;

(b) for any $j,j' \in \{0,\ldots,q\}$ such that $j < j'$, let

$$\alpha = H(V) - \frac{H[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] + H(\tilde{\mathcal{S}}_{j'}) - H(\tilde{\mathcal{S}}_j)}{|\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}|}, \tag{11}$$

where $\mathcal{P}_d \preceq \mathcal{Q}_{\alpha_{j'},V_i}$.

(i) If $\mathcal{P}_d \prec \mathcal{Q}_{\alpha_{j'},V_i}$, let $\mathcal{P}^{(l)}$ in the PSP of V_{i-1} such that $\mathcal{P}_d = \mathcal{P}^{(l)} \sqcup \{\{\phi_i\}\}$. Then, the corresponding critical value $\alpha^{(l)} < \alpha_{j'}$ and $\alpha_{j+1} \geq \alpha > \alpha^{(l)}$;

(ii) if $\mathcal{P}_d = \mathcal{Q}_{\alpha_{j'},V_i}$, then $\alpha_{j+1} \geq \alpha > \alpha_{j'}$ for $j+1 < j'$ and $\alpha = \alpha_{j+1}$ for $j+1 = j'$. ■

Based on Lemma 8, we propose the StrMap algorithm (Algorithm 2) that searches for α_j's and $\tilde{\mathcal{S}}_j$'s through a finite number of recursions: the call StrMap$(V_i, \{\phi_i\}, \{\{m\}\colon m \in V_i\})$ returns $\{\tilde{\mathcal{S}}_j\colon j \in \{0,\ldots,q\}\}$ in Lemma 7.[9] Based on Lemma 4(c), $r_\alpha(\tilde{\mathcal{S}}_{j-1} \backslash \tilde{\mathcal{S}}_j)$ is piecewise linear in α. So, according to Lemma 8(a), the value of each α_j can be determined by solving the linear equation $r_\alpha(\tilde{\mathcal{S}}_{j-1} \backslash \tilde{\mathcal{S}}_j) = H(\tilde{\mathcal{S}}_{j-1}) - H(\tilde{\mathcal{S}}_j)$. The SFM in step 4 in all recursions of the StrMap algorithm can be solved efficiently by the PSFM algorithms in [33]–[35] such that the overall complexity of StrMap is $O(\text{SFM}(|V|))$.

[7]The maximal critical point is $H(V)$ instead of $H(V_i)$ because in the function $f_\alpha(X) = \alpha - H(V) + H(X)$, the offset of the entropy is $\alpha - H(V)$. If we change the offset as $f_\alpha(X) = \alpha - H(V_i) + H(X)$, we should have $\alpha_0 = H(V_i)$. See Section III-E.

[8]It should be noted that the value of $\alpha^{(j)}$'s in the PSP and α_j's in Lemma 7 do not necessarily coincide. The critical points α_j's for $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ for each iteration i also vary with the linear ordering Φ.

[9]In each recursion of StrMap, $\mathcal{P}_d \preceq \mathcal{Q}_{\alpha_{j'},V_i}$ always holds.

Algorithm 2: StrMap$(\tilde{\mathcal{S}}_j, \tilde{\mathcal{S}}_{j'}, \mathcal{P}_d)$

input : $\tilde{\mathcal{S}}_j, \tilde{\mathcal{S}}_{j'}$ such that $\tilde{\mathcal{S}}_j \supseteq \tilde{\mathcal{S}}_{j'}$ and $\mathcal{P}_d$ such that $\mathcal{P}_d \preceq \mathcal{Q}_{\alpha_{j'},V_i}$.
output: $\{\tilde{\mathcal{S}}_j, \tilde{\mathcal{S}}_{j+1}, \ldots, \tilde{\mathcal{S}}_{j'}\}$.

1 **if** $\tilde{\mathcal{S}}_j = \tilde{\mathcal{S}}_{j'}$ **then return** $\{\tilde{\mathcal{S}}_j\}$;
2 **else**
3 $\quad \alpha \coloneqq H(V) - \frac{H[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] + H(\tilde{\mathcal{S}}_{j'}) - H(\tilde{\mathcal{S}}_j)}{|\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}|}$;
4 $\quad \mathcal{U}_{\alpha,V_i} \coloneqq \bigcap \operatorname{argmin}\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$;
5 $\quad$ **if** $\tilde{\mathcal{S}}_{j'} = \tilde{\mathcal{U}}_{\alpha,V_i}$ *and* $g_\alpha(\tilde{\mathcal{S}}_j) = g_\alpha(\tilde{\mathcal{U}}_{\alpha,V_i})$ **then return** $\{\tilde{\mathcal{S}}_j, \tilde{\mathcal{S}}_{j'}\}$;
6 $\quad$ **else return** StrMap$(\tilde{\mathcal{S}}_j, \tilde{\mathcal{U}}_{\alpha,V_i}, \mathcal{Q}_{\alpha,V_i}) \cup$ StrMap$(\tilde{\mathcal{U}}_{\alpha,V_i}, \tilde{\mathcal{S}}_{j'}, \mathcal{P}_d)$;
7 **endif**

Algorithm 3: Parametric (PAR) Algorithm

input : f, V and Φ
output: segmented variables $\mathbf{r}_{\alpha,V} \in B(\hat{f}_\alpha)$ and $\mathcal{Q}_{\alpha,V} = \bigwedge \operatorname{argmin}_{\mathcal{P} \in \Pi(V)} f_\alpha[\mathcal{P}]$ for all α

$\mathbf{r}_{\alpha,V} \coloneqq (\alpha - H(V))\chi_V$ for all α;
$r_{\alpha,\phi_1} \coloneqq f_\alpha(\{\phi_1\})$ and $\mathcal{Q}_{\alpha,V_1} \coloneqq \{\{\phi_1\}\}$ for all α;
for $i = 2$ **to** $|V|$ **do**
$\quad \mathcal{Q}_{\alpha,V_i} \coloneqq \mathcal{Q}_{\alpha,V_{i-1}} \sqcup \{\{\phi_i\}\}$ for all α ;
$\quad$ Call StrMap$(V_i, \{\phi_i\}, \{\{m\}\colon m \in V_i\})$ to obtain the critical points $\{\alpha_j\colon j \in \{0,\ldots,q\}\}$ and $\{\tilde{\mathcal{S}}_j\colon j \in \{0,\ldots,q\}\}$ that determine the minimal minimizer $\tilde{\mathcal{U}}_{\alpha,V_i}$ of $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ for all α;
$\quad$ Let $\Gamma_j \coloneqq (\alpha_{j+1}, \alpha_j]$ for all $j \in \{0,\ldots,q-1\}$ and $\Gamma_q \coloneqq [0,\alpha_q]$. For each $j \in \{0,\ldots,q\}$, update $\mathbf{r}_V$ and $\mathcal{Q}_{\alpha,V_i}$ by

$$\mathbf{r}_{\alpha,V} \coloneqq \mathbf{r}_{\alpha,V} + g_\alpha(\tilde{\mathcal{S}}_j)\chi_{\phi_i};$$
$$\mathcal{Q}_{\alpha,V_i} \coloneqq (\mathcal{Q}_{\alpha,V_i} \setminus \mathcal{S}_j) \sqcup \{\tilde{\mathcal{S}}_j\};$$

$\quad$ for all $\alpha \in \Gamma_j$;
endfor
return $\mathbf{r}_V$ and $\mathcal{Q}_{\alpha,V}$ for all α;

C. Parametric Algorithm

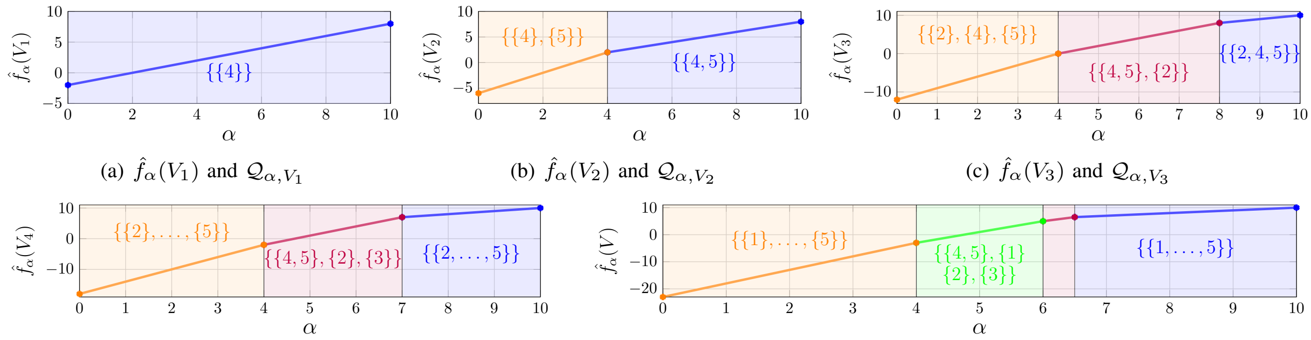
We propose the PAR algorithm in Algorithm 3. We show an example below of applying the PAR algorithm to an actual CO problem and then discuss its complexity and distributed implementation.

Example 9. *Consider a 5-user system with*

$$\mathsf{Z}_1 = (\mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_d, \mathsf{W}_h, \mathsf{W}_i),$$
$$\mathsf{Z}_2 = (\mathsf{W}_e, \mathsf{W}_f, \mathsf{W}_h, \mathsf{W}_i),$$
$$\mathsf{Z}_3 = (\mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_e, \mathsf{W}_j),$$
$$\mathsf{Z}_4 = (\mathsf{W}_a, \mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_d, \mathsf{W}_f, \mathsf{W}_g, \mathsf{W}_i, \mathsf{W}_j),$$
$$\mathsf{Z}_5 = (\mathsf{W}_a, \mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_f, \mathsf{W}_i, \mathsf{W}_j),$$

where each W_m is an independent uniformly distributed random bit. Choose the linear ordering $\Phi = (4,5,2,3,1)$ and apply the PAR algorithm.

First, initiate $r_{\alpha,i} = \alpha - H(V) = \alpha - 10$ for all $i \in V$ and α. For $i = 1$, we get $\tilde{\mathcal{U}}_{\alpha,V_1} = \{\phi_1\} = \{4\}$, $\mathcal{Q}_{\alpha,V_1} = \{\{4\}\}$ and $r_{\alpha,4} = f_\alpha(\{4\}) = \alpha - 2$ for all α.



(a) $\hat{f}_\alpha(V_1)$ and $\mathcal{Q}_{\alpha,V_1}$ (b) $\hat{f}_\alpha(V_2)$ and $\mathcal{Q}_{\alpha,V_2}$ (c) $\hat{f}_\alpha(V_3)$ and $\mathcal{Q}_{\alpha,V_3}$

(d) $\hat{f}_\alpha(V_4)$ and $\mathcal{Q}_{\alpha,V_4}$ (e) $\hat{f}_\alpha(V)$ and $\mathcal{Q}_{\alpha,V}$, where $\mathcal{Q}_{\alpha,V} = \{\{1,4,5\},\{2\},\{3\}\}, \forall \alpha \in (6, 6.5]$

Fig. 1. The piecewise linear increasing Dilworth truncation $\hat{f}_\alpha(V_i)$ in α and the segmented partition $\mathcal{Q}_{\alpha,V_i}$ obtained at the end of each iteration i of the PAR Algorithm when it is applied to the 5-user system in Example 9.

For $i = 2$, we have $\phi_2 = 5$ and $V_2 = \{4,5\}$. We first set $\mathcal{Q}_{\alpha,V_2} = \mathcal{Q}_{\alpha,V_1} \sqcup \{\{\phi_2\}\} = \{\{4\},\{5\}\}$ for all α. By the call StrMap$(\{4,5\},\{5\},\{\{4\},\{5\}\})$, we get $\tilde{\mathcal{S}}_1 = \{5\}$ and $\tilde{\mathcal{S}}_0 = \{4,5\}$ with the critical points $\alpha_1 = 4$ and $\alpha_0 = H(V) = 10$. So the minimal minimizer $\mathcal{U}_{\alpha,V_2}$ of $\min\{g_\alpha(\tilde{\mathcal{X}}) \colon \{\phi_2\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_2}\}$

$$\tilde{\mathcal{U}}_{\alpha,V_2} = \begin{cases} \{5\} & \alpha \in [0,4], \\ \{4,5\} & \alpha \in (4,10]. \end{cases} \tag{12}$$

The updated $\mathbf{r}_{\alpha,V_2}$ and $\mathcal{Q}_{\alpha,V_2}$ after step 6 are

$$\begin{aligned} r_{\alpha,4} &= \alpha - 2, \quad \forall \alpha \in [0,10], \\ r_{\alpha,5} &= \begin{cases} \alpha - 4 & \alpha \in [0,4], \\ 0 & \alpha \in (4,10], \end{cases} \\ \mathcal{Q}_{\alpha,V_2} &= \begin{cases} \{\{4\},\{5\}\} & \alpha \in [0,4], \\ \{\{4,5\}\} & \alpha \in (4,10], \end{cases} \end{aligned} \tag{13}$$

For $i = 3$, we get

$$\tilde{\mathcal{U}}_{\alpha,V_3} = \begin{cases} \{2\} & \alpha \in [0,8], \\ \{2,4,5\} & \alpha \in (8,10] \end{cases} \tag{14}$$

and

$$\begin{aligned} \mathbf{r}_{\alpha,V} &= \begin{cases} (\alpha-10, \alpha-6, \alpha-10, \alpha-2, \alpha-4) & \alpha \in [0,4], \\ (\alpha-10, \alpha-6, \alpha-10, \alpha-2, 0) & \alpha \in (4,8], \\ (\alpha-10, 2, \alpha-10, \alpha-2, 0) & \alpha \in (8,10], \end{cases} \\ \mathcal{Q}_{\alpha,V_3} &= \begin{cases} \{\{2\},\{4\},\{5\}\} & \alpha \in [0,4], \\ \{\{4,5\},\{2\}\} & \alpha \in (4,8], \\ \{\{2,4,5\}\} & \alpha \in (8,10]. \end{cases} \end{aligned} \tag{15}$$

at the end of iteration.

Repeating the same procedure, we have the following returned.[10]

$$\begin{aligned} \mathbf{r}_{\alpha,V} &= \begin{cases} (\alpha-5, \alpha-6, \alpha-6, \alpha-2, \alpha-4) & \alpha \in [0,4], \\ (\alpha-5, \alpha-6, \alpha-6, \alpha-2, 0) & \alpha \in (4,6], \\ (1, \alpha-6, \alpha-6, \alpha-2, 0) & \alpha \in (6,6.5], \\ (14-2\alpha, \alpha-6, \alpha-6, \alpha-2, 0) & \alpha \in (6.5,7], \\ (0, \alpha-6, 8-\alpha, \alpha-2, 0) & \alpha \in (7,8], \\ (0, 2, 0, \alpha-2, 0) & \alpha \in (8,10], \end{cases} \\ \mathcal{Q}_{\alpha,V} &= \begin{cases} \{\{1\},\dots,\{5\}\} & \alpha \in [0,4], \\ \{\{4,5\},\{1\},\{2\},\{3\}\} & \alpha \in (4,6], \\ \{\{1,4,5\},\{2\},\{3\}\} & \alpha \in (6,6.5], \\ \{\{1,\dots,5\}\} & \alpha \in (6.5,10]. \end{cases} \end{aligned} \tag{16}$$

See Fig. 1 for the plot of $\hat{f}_\alpha(V_i)$ as a function of α at the end of each iteration. For the final segmented partition $\mathcal{Q}_{\alpha,V}$ in (16), the corresponding PSP has the critical points $\alpha^{(3)} = 4$, $\alpha^{(2)} = 6$, $\alpha^{(1)} = 6.5$ and $\alpha^{(0)} = H(V) = 10$ with $\mathcal{P}^{(3)} = \{\{1\},\dots,\{5\}\}$, $\mathcal{P}^{(2)} = \{\{4,5\},\{1\},\{2\},\{3\}\}$, $\mathcal{P}^{(1)} = \{\{1,4,5\},\{2\},\{3\}\}$ and $\mathcal{P}^{(0)} = \{\{1,\dots,5\}\}$ so that we know $R_{ACO}(V) = \alpha^{(1)} = 6.5$ is the minimum sum-rate for the asymptotic model and $\mathcal{P}^{(1)} = \{\{4,5,1\},\{2\},\{3\}\}$ is fundamental partition. We also know an optimal achievable rate vector $\mathbf{r}_{6.5,V} = (1, 0.5, 0.5, 4.5, 0) \in \mathscr{R}^*_{ACO}(V)$, which has the LCM $|\mathcal{P}^{(1)}| - 1 = 2$ so that it is implementable by network coding schemes with 2-packet-splitting in CCDE. For the non-asymptotic model, the minimum sum-rate is $R_{NCO}(V) = \lceil R_{ACO}(V) \rceil = 7$ and $\mathbf{r}_{7,V} = (0,1,1,5,0) \in \mathscr{R}^*_{NCO}(V)$ is an optimal achievable rate vector.

It should be noted that we automatically know $\mathcal{U}_{\alpha,V_i}$ in step 6 if $\tilde{\mathcal{U}}_{\alpha,V_i}$ is obtained in that $\mathcal{U}_{\alpha,V_i} = \{C \in \mathcal{Q}_{\alpha,V_i} \colon C \subseteq \tilde{\mathcal{U}}_{\alpha,V_i}\} = \langle \tilde{\mathcal{U}}_{\alpha,V_i} \rangle_{\mathcal{Q}_{\alpha,V_i}}$.[11] For example, for $\tilde{\mathcal{U}}_{\alpha,V_2}$ in (12) and

[10] The rate vector $\mathbf{r}_{\alpha,V}$ may have critical points more than $\mathcal{Q}_{\alpha,V}$. It means that the rate update in step 6 needs to be done for more segments than q. But, this will not increase the complexity of PAR because the number of critical points of $\mathbf{r}_{\alpha,V}$ is upper bounded by $2|V|$ and so, as explained in Appendix D, the update in step 6 is less complex than SFM algorithm.

[11] This means that $\mathcal{U}_{\alpha,V_i}$ is the decomposition of $\tilde{\mathcal{U}}_{\alpha,V_i}$ by $\mathcal{Q}_{\alpha,V_i}$. Here, we should use the value of $\mathcal{Q}_{\alpha,V_i}$ in the minimization problem $\min\{g_\alpha(\tilde{\mathcal{X}}) \colon \{i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ before the updates in step 6.

$\mathcal{Q}_{\alpha,V_2} = \{\{4\},\{5\}\}$ *for all* α,

$$\mathcal{U}_{\alpha,V_2} = \begin{cases} \{\{5\}\} & \alpha \in [0,4], \\ \{\{4\},\{5\}\} & \alpha \in (4,10]. \end{cases} \tag{17}$$

We show an example of how to apply the StrMap algorithm in step 5. For $i = 3$, $\phi_3 = 2$ *and* $V_3 = \{2,4,5\}$, *consider the problem* $\min\{g_\alpha(\tilde{\mathcal{X}}) \colon \{\phi_3\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_3}\}$. *We call StrMap*$(V_3, \{\phi_3\}, \{\{m\} \colon m \in V_i\})$ *and have*

$$\alpha = H(V) - \frac{H(\{2\}) + H(\{4\}) + H(\{5\}) - H(\{2,4,5\})}{2} = 6.$$

The minimal minimizer of $\min\{g_6(\tilde{\mathcal{X}}) \colon \{\phi_4\} \in \mathcal{X} \subseteq \mathcal{Q}_{6,V_3}\}$ *is* $\tilde{\mathcal{U}}_{6,V_3} = \{2\}$. *We have* $\tilde{\mathcal{S}}_{j'} = \tilde{\mathcal{U}}_{6,V_3}$ *but* $g_6(\{2,4,5\}) \neq g_6(\{2\})$. *So, we run StrMap*$(\{2,4,5\}, \{2\}, \mathcal{Q}_{6,V_3})$ *and StrMap*$(\{2\}, \{2\}, \{\{2\},\{4\},\{5\}\})$ *where* $\mathcal{Q}_{6,V_3} = \{\{2\},\{4,5\}\}$. *In the call StrMap*$(\{2,4,5\}, \{2\}, \mathcal{Q}_{6,V_2})$, *we have*

$$\alpha = H(V) - (H(\{2\}) + H(\{4,5\}) - H(\{2,4,5\})) = 8$$

and $\tilde{\mathcal{U}}_{8,V_3} = \{2\}$. *Since* $\tilde{\mathcal{U}}_{8,V_3} = \tilde{\mathcal{S}}_{j'}$ *and* $g_8(\{2,4,5\}) = g_8(\{2\})$, *the recursion returns* $\{\{2,4,5\},\{2\}\}$. *The call StrMap*$(\{2\}, \{2\}, \{\{2\},\{4\},\{5\}\})$ *directly returns* $\{\{2\}\}$. *Finally, we have* $\{\{2\},\{2,4,5\}\}$ *returned indicating* $\tilde{\mathcal{S}}_1 = \{2\}$ *and* $\tilde{\mathcal{S}}_0 = \{2,4,5\}$ *with the critical points* $\alpha_1 = 8$ *and* $\alpha_0 = H(V) = 10$. *They determine the minimal minimizer* $\tilde{\mathcal{U}}_{\alpha,V_3}$ *in* (14) *for all* α. *Note that the SFM problems in all recursions of the StrMap algorithm can be solved in* $O(1)$ *call of a PSFM algorithm. The resulting complexity is still* $O(\text{SFM}(|V|))$. *See Appendix D for a brief on PSFM and [43, Algorithm 7] showing how to run StrMap by the PSFM algorithm in [33].*

D. Complexity

The PAR algorithm invokes $|V|$ calls of the StrMap algorithm. As explained in Appendix D, the complexity of the StrMap algorithm is $O(\text{SFM}(|V|))$. Therefore, the minimum sum-rate problem in (2) for both asymptotic and non-asymptotic models can be solved by the PAR algorithm in $O(|V| \cdot \text{SFM}(|V|))$ time. The complexity is reduced by a factor of $|V|$ from the existing computation time $O(|V|^2 \cdot \text{SFM}(|V|))$ by the MDA algorithm in [18] and the algorithms in [14], [15].

1) Related Problems: Due to the duality between CO and multi-terminal secret capacity [3, Example 4] [39], the PAR algorithm provides a tool for computing the secret capacity that is more efficient than the existing method in [16]. Replacing the entropy function H with the cut function, PAR determines the PSP and the network strength of a graph [22]. This is because the maximum number of edge-disjoint spanning trees is the largest integer that is no greater than the network strength according to [23, Section 5.1] [44], [45]. This is the reason why the secret key sharing problem in the PIN model in [24]–[26] can be solved efficiently by the tree packing algorithms. In fact, [24]–[26] utilize the parametric max-flow algorithm in [46] (see Appendix D), while we prove in this paper that the PSFM techniques reduce the complexity in obtaining the secret capacity in any multiple random source Z_V.

The PAR algorithm also efficiently solves the information-theoretic clustering problem in [20]: the returned $\mathcal{Q}_{\alpha,V}$ is a hierarchical clustering dendrogram.[12] By setting $\alpha = H(V)$, the inequalities in the polyhedron $P(f_\alpha)$ are the Slepian-Wolf constraints [36] for lossless data compression. In this case, the fundamental partition $\mathcal{P}^{(1)}$ decomposes users in V into mutually independent groups. This idea was utilized in [47] to reduce the complexity of computing a fair rate vector for the multi-terminal lossless data compression problem. Similarly, the results derived in the next section for SO can be applied to compute the incremental secret capacity in [48] and derive a staged data forwarding and sharing procedure in multi-terminal source coding and CCDE, respectively.

E. Distributed Computation

In iteration i of the PAR algorithm, the updates of the partition $\mathcal{Q}_{\alpha,V_i}$ and rate vector $\mathbf{r}_{\alpha,V_i}$ for the subsystem V_i are based on $\mathcal{Q}_{\alpha,V_{i-1}}$ and $\mathbf{r}_{\alpha,V_{i-1}}$ obtained from the previous iteration and the information only in V_i. This suggests a distributed implementation. Remove steps 1 and 2 and let the first user ϕ_1 initiate $r_{\alpha,\phi_1} = \alpha - H(\{\phi_1\})$ and $\mathcal{Q}_{\alpha,V_1} = \{\{\phi_1\}\}$ and pass them to user ϕ_2. For $i \in \{2,\ldots,|V|\}$, user ϕ_i replaces α in $r_{\alpha,V_{i-1}}$ and $\mathcal{Q}_{\alpha,V_{i-1}}$ by $\alpha := \alpha - H(V_{i-1}) + H(V_i)$ and initiates $r_{\alpha,\phi_i} = \alpha - H(V_i)$. Defining $f_\alpha(X) = \alpha - H(V_i) + H(X)$ such that $g_\alpha(\tilde{\mathcal{X}}) = f_\alpha(\tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{X}}) = \alpha - H(V_i) + H(\tilde{\mathcal{X}}) - r_\alpha(\tilde{\mathcal{X}}), \forall \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$, user ϕ_i implements steps 4 to 6 of the PAR algorithm.[13] This method distributes the computation load of the PAR algorithm to the users. The run time at each user is $O(\text{SFM}(|V|))$.[14]

IV. Successive Omniscience

The concept of SO is proposed in [6], [28]: instead of the one-off approach, the communications between the users in V can be organized in a way such that global omniscience in V can be attained in a two-stage manner. First, let the users in a subset X broadcast to attain the local omniscience and the remaining users $i \in V \setminus X$ overhear these transmissions; then, solve the global omniscience problem in V. There is a particular type of nonsingleton subsets X_*, called *complimentary subset*, such that if $X = X_*$, the final sum-rate for attaining the global omniscience in V at the second stage remains minimized [6]. But, the fact is that not all nontrivial subsets of V are complimentary. For example, for the non-complimentary user subset $\{2,4\}$ in Example 9, $R_{\text{NCO}}(\{2,4\}) = 8$, which is already greater than the minimum sum-rate for attaining the global omniscience $R_{\text{NCO}}(\{1,\ldots,5\}) = 7$. This means that, if the local omniscience is first attained in a non-complimentary subset, the overall transmission cost may increase. Therefore,

[12] The cluster tree coincides with the multi-stage SO returned by Algorithm 5 in Section IV-B. In Fig. 3, the multi-stage SO for the asymptotic model is the dendrogram solution of the information-theoretic clustering problem in [20] for the 5-user system in Example 9.

[13] In this case, we will get the same set chain (10) as in the PAR algorithm for each i, but the critical points in (9) are bounded by $H(V_i)$, i.e., $0 \leq \alpha_q < \ldots < \alpha_1 < \alpha_0 = H(V_i)$. Here, the function definition $f_\alpha(X) = \alpha - H(V_i) + H(X)$ is in fact for the purpose of attaining the local omniscience in V_i.

[14] See [43] for a detailed distributed PAR algorithm.

for the two-stage SO, we need to study how to efficiently search a complimentary subset X_* and determine the local omniscience achievable rate vectors $\mathbf{r}_{X_*} \in \mathscr{R}_{\text{CO}}(X_*)$ and $\mathbf{r}_{X_*} \in \mathscr{R}_{\text{CO}}(X_*) \cap \mathbb{Z}^{|X_*|}$ for the asymptotic and non-asymptotic models, respectively.

A. Two-stage Successive Omniscience

The following theorem states that the existence of the complimentary subset and at least one such subset can be determined by applying the lower bound $\underline{\alpha}$ on the minimum sum-rate for the global omniscience to the PAR algorithm. A local omniscience achievable rate vector can be searched at the same time. Theorem 10 is proved in Appendix E by relaxing the necessary and sufficient condition for X_* to be complementary in [6, Theorems 4.2 and 5.2] to a sufficient condition on $\underline{\alpha}$. This lower bound $\underline{\alpha}$ in Theorem 10 is determined through (2) by the singleton partition $\{\{i\} : i \in V\}$.

Theorem 10. *Let $\mathcal{Q}_{\alpha,V_i}$ and $\mathbf{r}_{\alpha,V_i}$ be the segmented partition and rate vector, respectively, obtained at the end of any iteration $i \in \{2,\dots,|V|\}$ of the PAR algorithm.*

(a) For the asymptotic model, let $\underline{\alpha} = \sum_{i\in V} \frac{H(V)-H(\{i\})}{|V|-1}$. Any nonsingleton $C \in \mathcal{Q}_{\underline{\alpha},V_i}$ is a complimentary subset and $\mathbf{r}_{\hat{\alpha},C}$ for $\hat{\alpha} = \min\{\alpha \in \mathbb{R} : f_\alpha(C) = \hat{f}_\alpha(C)\}$ is an optimal rate vector that attains local omniscience in C with the minimum sum-rate $R_{ACO}(C)$.

(b) For the non-asymptotic model, let $\underline{\alpha} = \left\lceil \sum_{i\in V} \frac{H(V)-H(\{i\})}{|V|-1} \right\rceil$. Any nonsingleton $C \in \mathcal{Q}_{\underline{\alpha},V_i}$ is a complimentary subset and $\mathbf{r}_{\hat{\alpha},C}$ for $\hat{\alpha} = \min\{\alpha \in \mathbb{Z} : f_\alpha(C) = \hat{f}_\alpha(C)\}$ is an optimal rate vector that attains local omniscience in C with the minimum sum-rate $R_{NCO}(C)$.

For both asymptotic and non-asymptotic models, if all subsets in $\mathcal{Q}_{\underline{\alpha},V_i}$ remain singleton $\mathcal{Q}_{\underline{\alpha},V_i} = \{\{m\} : m \in V_i\}$ until the $|V|$-th iteration, there does not exist a complimentary subset. ∎

Theorem 10 is implemented by Algorithm 4. For a system having at least one complimentary subset, we can find such subset X_* and the local omniscience achievable rate vector $\mathbf{r}_{X_*}$ for the two-stage SO in the first $|V|-1$ iterations of the PAR algorithm. In this case, the run time is $(|V|-1)\cdot \text{SFM}(|V|-1)$; when there does not exist a complimentary subset, we need to wait until all $|V|$ iterations of the PAR algorithm finish and Algorithm 4 outputs the results $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ denoting the minimum sum-rate and an optimal rate vector for the global omniscience problem in V. In this case, the run time is $|V|\cdot\text{SFM}(|V|)$. Therefore, the worst case complexity of Algorithm 4 is $O(|V|\cdot\text{SFM}(|V|))$.[15]

Remark 11 (Determining $\hat{\alpha}$). *In Theorem 10, to obtain the value of $\hat{\alpha}$, we just need to consider the range $[0,\underline{\alpha}]$ because $\hat{\alpha} \leq \underline{\alpha}$ in both asymptotic and non-asymptotic models. For the asymptotic model, $\hat{\alpha} = \min\{\alpha \in \mathbb{R} : f_\alpha(C) = \hat{f}_\alpha(C)\}$ is the smallest value of α such that C appears as an intact subset in $\mathcal{Q}_{\alpha,V_i}$. The reason is that: (i) for all $\alpha < \hat{\alpha}$, $f_\alpha(C) < \hat{f}_\alpha(C)$ and therefore $C \notin \bigwedge \text{argmin}_{\mathcal{P}\in\Pi(V_i)} f_\alpha[\mathcal{P}] = \mathcal{Q}_{\alpha,V_i}$; (ii) for all $\alpha \geq \hat{\alpha}$, $f_\alpha(C) = \hat{f}_\alpha(C)$ so that $C \subseteq C'$ for some $C' \in \mathcal{Q}_{\alpha,V_i}$. That is, $\hat{\alpha}$ must coincide with one of the critical values $\alpha^{(j)}$'s that segment $\mathcal{Q}_{\alpha,V_i}$ and can be searched over the regions $(\alpha^{(j)},\alpha^{(j-1)}]$ such that $\alpha^{(j)} < \underline{\alpha}$. See Fig. 2. Taking the least integer value that is no less than this critical value, we obtain $\hat{\alpha}$ for the non-asymptotic model.*

[15] Algorithm 4 terminates as soon as a complimentary set X_* is determined. Since $|X_*| < |V|$, the actual complexity of Algorithm 4 is usually much less than $O(|V|\cdot\text{SFM}(|V|))$.

Algorithm 4: Two-stage Successive Omniscience (SO) by PAR Algorithm

---

input : H, V and (an arbitrarily chosen linear ordering) Φ.
output: a complimentary subset C and an optimal rate vector $\mathbf{r}_{\hat{\alpha},C}$ for attaining local omniscience in C; if $C = \emptyset$, there is no complimentary subset and the returned $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ constitute the optimal solution to the global omniscience.

Obtain $\underline{\alpha}_{\text{ACO}} \leftarrow \sum_{i\in V} \frac{H(V)-H(\{i\})}{|V|-1}$ for the asymptotic model or $\underline{\alpha}_{\text{NCO}} \leftarrow \left\lceil \sum_{i\in V} \frac{H(V)-H(\{i\})}{|V|-1} \right\rceil$ for the non-asymptotic model;
Call PAR(H,V,Φ) and do the following at the end of iterations from 2 to $|V|$;
for $i = 2$ **to** $|V|$ **do**
  if $\exists C \in \mathcal{Q}_{\underline{\alpha},V_i} : |C| > 1$ *at some iteration i of PAR* **then**
    break and **return** C and $\mathbf{r}_{\hat{\alpha},C}$ after the update in step 6 of the PAR algorithm, where

$$\hat{\alpha} = \min\{\alpha \in \mathbb{R} : f_\alpha(C) = \hat{f}_\alpha(C)\},$$

$$\hat{\alpha} = \min\{\alpha \in \mathbb{Z} : f_\alpha(C) = \hat{f}_\alpha(C)\},$$

    for the asymptotic and non-asymptotic models, respectively (see Remark 11 for how to obtain $\hat{\alpha}$) ;
endfor
return $C = \emptyset$ and $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ returned at the end of the PAR algorithm;

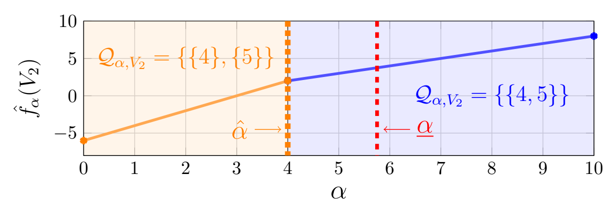


Fig. 2. For the 5-user system in Example 9, consider $\mathcal{Q}_{\alpha,V_2}$ obtained at the end of the second iteration of the PAR algorithm, we have the lower bound on the minimum sum-rate in Theorem 10 being $\underline{\alpha} = 5.75$, by which $\{4,5\} \in \mathcal{Q}_{5.75,V_2}$ is a complimentary subset for the SO in the asymptotic model. The minimum α such that users 4 and 5 appear as an intact subset in $\mathcal{Q}_{\alpha,V_2}$ is $\hat{\alpha} = 4$. For $\mathbf{r}_{\alpha,V_2}$ in (13), $\mathbf{r}_{4,V_2} = \mathbf{r}_{4,\{4,5\}} = (2,0)$ is an optimal rate vector for attaining the local omniscience in $\{4,5\}$ with the minimum sum-rate $R_{\text{ACO}}(\{4,5\}) = 2$.

B. Multi-stage Successive Omniscience

The two-stage SO can be recursively implemented. After the local omniscience in the complimentary subset X_* is attained, the users in $i \in X_*$ can be treated as a super-user $\tilde{X}_*$ that observes the source $\mathsf{Z}_{\tilde{X}_*} = \mathsf{Z}_{X_*}$. For all $i \in V \setminus X_*$, we need to update $\mathsf{Z}_i \leftarrow (\mathsf{Z}_i, \Gamma)$ with Γ being the broadcast transmissions overheard by user i when the users in X_*

are communicating to attain the local omniscience. The new system $V' = \{\tilde{X}_*\} \sqcup \{i : i \in V \setminus X_*\}$ poses a new omniscience problem, which can again be solved successively. An example of this recursive two-stage SO can be found in [1]. This method, however, requires a decoding function running at the same time to reconstruct the Z_{X_*} for each user in X_* in each recursion. The full multi-stage SO strategy is not known until the communications actually happen.

In this section, we show that a multi-stage SO can be searched without any transmission or encoding-decoding process. We first state the achievability of multi-stage SO and then present how to extract solutions from the PAR algorithm for asymptotic and non-asymptotic models, respectively.

Proposition 12. *In the asymptotic model, a K-stage SO $\{(X_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1, \dots, K\}\}$ is achievable if $X_*^{(k)}$ for all $k \in \{1, \dots, K\}$ is complimentary and forms a set sequence/chain*

$$\emptyset \subsetneq X_*^{(1)} \subsetneq X_*^{(2)} \subsetneq \dots \subsetneq X_*^{(K)} = V, \tag{18}$$

and the rate vector $\mathbf{r}_V^{(k)}$ satisfies the following conditions for all $k \in \{1, \dots, K-1\}$:

(a) $X_*^{(k)}$ attains local omniscience in $X_*^{(k)}$: $\mathbf{r}_{X_*^{(k)}}^{(k)} \in \mathscr{R}_{CO}(X_*^{(k)})$ and $\mathbf{r}_V^{(K)} \in \mathscr{R}_{ACO}^*(V)$ for the asymptotic model and $\mathbf{r}_{X_*^{(k)}} \in \mathscr{R}_{CO}(X_*^{(k)}) \cap \mathbb{Z}^{|V|}$ and $\mathbf{r}_V^{(K)} \in \mathscr{R}_{NCO}^*(V)$ for the non-asymptotic model;

(b) all other users are on standby: $r_i^{(k)} = 0$ for all $i \in V \setminus X_*^{(k)}$;

(c) the sum-rate in $X_*^{(k)}$ is nondecreasing: $r^{(k+1)}(X_*^{(k)}) \geq r^{(k)}(X_*^{(k)})$.[16]

Proof: The nesting subset chain (18) and monotonicity of $r^{(k)}(X_*^{(k)})$ in k in (c) are the necessary conditions for local omniscience in $X_*^{(k)}$, and the global omniscience in the last stage K, to be implemented subsequently. Conditions (a) and (b) ensure the local omniscience in $X_*^{(k)}$ is attained by $\mathbf{r}_{X_*^{(k)}}$ while the rest of users are overhearing. ■

The multi-stage SO in Proposition 12 should be implemented in the increasing order of $k = 1, 2, \dots, K$, where the local omniscience in $X_*^{(k-1)}$ is attained before $X_*^{(k)}$. It should be noted that a multi-stage SO strategy that satisfies the achievability in Proposition 12 cannot be found by Algorithm 4, because there is no guarantee that $X_*^{(k-1)} \subsetneq X_*^{(k)}$ and $\mathbf{r}_V^{(k-1)} \leq \mathbf{r}_V^{(k)}$ for each k.

1) Asymptotic Model: We propose Algorithm 5 that uses $\mathcal{P}^{(j)}$'s in the PSP and the corresponding rate vectors $\mathbf{r}_{\alpha^{(j)},V}$'s to build a p-stage SO that iteratively attains the local omniscience in all nonsingleton subsets in each partition $\mathcal{P}^{(j)}$. The achievability of this p-stage SO is stated in the corollary below. It is proved in Appendix F, essentially by showing the monotonic sum-rate in Lemma 4(c) for all $\mathcal{X} \in \mathcal{P}^{(j)}$.

[16]The users in $X_*^{(k)}$ reach omniscience and can be treated as one dimension at stage $k+1$. Therefore, Proposition 12(c) is in fact a relaxed condition of the dimension-wise monotonicity $\mathbf{r}_V^{(k+1)} \geq \mathbf{r}_V^{(k)}$ for all $k \in 1, \dots, K-1$. We say $\mathbf{r}_V \geq \mathbf{r}'_V$ if $r_i \geq r'_i$ for all $i \in V$ with at least one of these inequalities holding strictly. See also Remark 14(a).

Algorithm 5: Multi-stage Successive Omniscience (SO) by the PAR Algorithm for the Asymptotic Model

input : H, V and Φ.
output: an achievable p-stage SO $\{(\mathcal{X}_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1, \dots, p\}\}$.

1 Call PAR(H, V, Φ) to obtain the segmented $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ for all α, where $\mathcal{Q}_{\alpha,V}$ is segmented by p critical points $\alpha^{(1)}, \dots, \alpha^{(p)}$ and $\alpha^{(0)} = H(V)$ (see Lemma 1);
2 Initiate $\mathbf{r}_V^{(0)} \leftarrow (0, \dots, 0)$;
3 **for** $k = 1$ **to** p **do**
4 $\quad \mathbf{r}_V^{(k)} \leftarrow \mathbf{r}_V^{(k-1)}$;
5 $\quad \mathcal{X}_*^{(k)} \leftarrow \{C \in \mathcal{P}^{(p-k)} : |C| > 1\}$;
6 $\quad$ **foreach** $C \in \mathcal{X}_*^{(k)}$ **do**
7 $\quad\quad$ **if** $C \neq \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$ **then** for each $C' \in \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$ randomly select user $i \in C'$ and let $\Delta r \leftarrow r_{\alpha^{(p-k+1)}}(C') - r^{(k)}(C')$ and $r_i^{(k)} \leftarrow r_i^{(k)} + \Delta r$;
8 $\quad$ **end**
9 **endfor**
10 **return** $\mathcal{X}_*^{(k)}$ and $\mathbf{r}_V^{(k)}$ for all $k \in \{1, \dots, p\}$;

Corollary 13. *For the asymptotic model, all $\mathcal{X}_*^{(k)}$ and $\mathbf{r}_V^{(k)}$ at the end of Algorithm 5 constitute an achievable p-stage SO $\{(\mathcal{X}_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1, \dots, p\}\}$, where $\mathbf{r}_V^{(p)} \in R_{ACO}(V)$,*

$$\emptyset \subsetneq \tilde{\mathcal{X}}_*^{(1)} \subsetneq \dots \subsetneq \tilde{\mathcal{X}}_*^{(p)} = V,$$

and, for all $k \in \{1, \dots, p\}$, all $C \in \mathcal{X}_*^{(k)}$ are complimentary. For each $C \in \mathcal{X}_*^{(k)}$, the local omniscience in each C is attained by an optimal rate vector $\mathbf{r}_C^{(k)} \in R_{ACO}(C)$. ■

Remark 14. *We remark the following about Algorithm 5.*

(a) The output $\mathbf{r}_V^{(k)}$ is nondecreasing in k, i.e., $\mathbf{r}_V^{(k)} \geq \mathbf{r}_V^{(k-1)}$ for all $k \in \{2, \dots, p\}$.Therefore, $\mathbf{r}_V^{(k)}$ is not necessarily the same as $\mathbf{r}_{\alpha^{(p-k+1)},V}$ in that $\mathbf{r}_{\alpha,V}$ returned by the PAR algorithm is not monotonic in general, e.g., $r_{\alpha,3}$ in (16) is not nondecreasing in α.

(b) Algorithm 5 allows more than one complimentary subset to attain local omniscience at each stage. Since all $C \in \mathcal{X}_*^{(k)}$ are disjoint, the local omniscience in step 7 can be attained simultaneously if the broadcast transmissions between subsets do not cause interference, e.g., via orthogonal wireless channels in CCDE.[17]

(c) The interpretation of Δr in step 7 of the algorithm is: in addition to the rates for attaining the local omniscience in C', how many transmissions is required from the super-user C' for attaining the local omniscience in C. Since all users in C' have recovered $\mathsf{Z}_{C'}$ in previous stages, Δr can be assigned to any one of them. Apart from the random selection in step 7, we can moderate Δr to the users i with the lowest $r_i^{(k)}$ to improve the fairness. See Example 16.

[17]For example, if $\mathcal{P}^{(p-1)} = \{\{1,2\},\{3\},\{4,5\}\}$ at the 1st stage of SO, the users 1, 2, 4 and 5 can transmit at the same time to attain the local omniscience in $\{1,2\}$ and $\{4,5\}$, respectively. The purpose of the decomposition $\langle C \rangle_{\mathcal{P}^{(p-k+1)}}$ in step 7 is to search all users/super-users that are supposed to take part in the local omniscience in C.

This p-stage SO $\{(\mathcal{X}_*^{(k)}, \mathbf{r}_V^{(k)})\colon k \in \{1,\dots,p\}\}$ results in an agglomerative SO tree that converges to the global omniscience. This bottom-up approach can also be considered as an opposite process of the divide-and-conquer (DC) algorithm in [49], where the ground set V is recursively split into subsets until the optimal rates $r_{R_{\text{ACO}}(V),i}$ are determined for all users $i \in V$. The complexity of this DC algorithm is $O(|V|^3 \cdot \text{SFM}(|V|))$. See [18, Appendix E] for the explanation of the DC algorithm and its complexity. Algorithm 5 completes in $O(|V| \cdot \text{SFM}(|V|))$ time,[18] a reduction by a factor of $|V|^2$ compared with the DC algorithm. This reduction could be significant in large systems.

2) Non-asymptotic Model: Although all nonsingleton $C \in \mathcal{Q}_{\alpha,V}$ for each integer-valued $\alpha \in \{0,\dots,R_{\text{NCO}}(V)\}$ are complimentary based on Lemma 19, the rate vector $\mathbf{r}_{\alpha,V}$ is not necessarily nondecreasing in integer-valued α.[19] This means that Algorithm 5 cannot be applied to the non-asymptotic model by simply running each stage k of Algorithm 5 at the integer-valued critical points $\lceil \alpha^{(p-k+1)} \rceil$. In this section, we show that, an achievable K-stage SO $\{(X_*^{(k)}, \mathbf{r}_V^{(k)})\colon k \in \{1,\dots,K\}\}$ with the integer-valued $\mathbf{r}_V^{(k)}$ attaining local omniscience in each complimentary subset $X_*^{(k)}$ can be searched by no more than two calls of the PAR algorithm.[20]

That is, the complexity of solving the multi-stage SO problem in a non-asymptotic model is essentially $O(|V| \cdot \text{SFM}(|V|))$.

Corollary 15. *In the non-asymptotic model, for any nonsingleton subset sequence $X_*^{(1)} \subsetneq \dots \subsetneq X_*^{(K)} = V$ and the integer-valued sequence $\bar{\alpha}^{(1)} < \dots < \bar{\alpha}^{(K)} = R_{NCO}(V)$ such that $\bar{\alpha}^{(k)} \in [\alpha^{(j)}, \alpha^{(j-1)})$ for some $j \in \{1,\dots,p\}$ and $X_*^{(k)} \in \mathcal{P}^{(j-1)}$, let $\bar{\Phi} = (\bar{\phi}_1,\dots,\bar{\phi}_{|V|})$ be a linear ordering such that*[21]

$$\bar{\phi}_i < \bar{\phi}_{i'}, \quad \forall \bar{\phi}_i \in X_*^{(k)}, \bar{\phi}_{i'} \in X_*^{(k')} \setminus X_*^{(k)}\colon k < k'. \tag{19}$$

and $\bar{\mathbf{r}}_{\alpha,V}$ be the rate vector returned by the call PAR$(V, H, \bar{\Phi})$. For all $k \in \{1,\dots,K\}$, the integer-valued $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)}, X_*^{(k)}}$ attains local omniscience in $X_*^{(k)}$; for all $k \in \{1,\dots,K-1\}$, $\bar{r}_{\bar{\alpha}^{(k+1)}}(X_*^{(k)}) \geq \bar{r}_{\bar{\alpha}^{(k)}}(X_*^{(k)})$. ■

The proof is in Appendix G. Corollary 15 states that if the complimentary subset sequence $X_*^{(1)},\dots,X_*^{(K)}$ is known, we can obtain the linear ordering $\bar{\Phi}$ and the monotonic rate vectors $\mathbf{r}_V^{(1)},\dots,\mathbf{r}_V^{(K)}$ ensuring the achievability in Proposition 12(c) of the K-stage SO. This suggests a two-step method for solving the multi-stage SO problem in the non-asymptotic model: run the PAR algorithm with an arbitrary linear ordering Φ to obtain all $X_*^{(k)}$'s; construct the linear ordering $\bar{\Phi}$ from $X_*^{(k)}$'s and rerun the PAR algorithm with $\bar{\Phi}$ to get all $\mathbf{r}_V^{(k)}$'s.

This method is implemented in Algorithm 6. The first for-loop is to extract $X_*^{(k)}$'s and $\bar{\alpha}^{(k)}$'s from the returned results of the first run of the PAR algorithm in step 1. Here, $\bar{\alpha}^{(k)}$ can be any integer in the region $[\alpha^{(j)}, \alpha^{(j-1)})$. We chose $\bar{\alpha}^{(k)} = \min\{\alpha\colon \alpha \in [\alpha^{(j)}, \alpha^{(j-1)}) \cap \mathbb{Z}\}$. The purpose is to ensure $\bar{\alpha}^{(K)} = \min\{\alpha\colon \alpha \in [\alpha^{(1)}, \alpha^{(0)}) \cap \mathbb{Z}\} = R_{\text{NCO}}(V)$ at the last iteration. At the end of the first for-loop, we get $X_*^{(k)}$'s that form a nesting sequence $X_*^{(1)} \subsetneq \dots \subsetneq X_*^{(K)} = V$ and a linear ordering $\bar{\Phi}$ that satisfies (19). The second for-loop determines the rate vector $\mathbf{r}_V^{(k)}$ for each k based on the returned results of the second run of the PAR algorithm in step 11. The method is to allocate rates Δr to any user in $X_*^{(k-1)}$. As explained in Remark 14(c), Δr denotes the rate at which the users in $X_*^{(k-1)}$ should transmit to attain the omniscience in $X_*^{(k)}$. For the rest of the users in $X_*^{(k)} \setminus X_*^{(k-1)}$ that have not transmitted in the previous stages, we directly assign rates $\bar{r}_{\bar{\alpha}^{(k)},i}$.

The linear ordering $\bar{\Phi} = (\bar{\phi}_1,\dots,\bar{\phi}_{|V|})$ satisfying (19) can be constructed by assigning each $X_*^{(k)}$ the first $|X_*^{(k)}|$ elements $\bar{\phi}_1,\dots,\bar{\phi}_{|X_*^{(k)}|}$. This can be done by letting $\{\bar{\phi}_1,\dots,\bar{\phi}_{|X_*^{(1)}|}\} = X_*^{(1)}$ and $\{\bar{\phi}_{|X_*^{(k)}|+1},\dots,\bar{\phi}_{|X_*^{(k+1)}|}\} = X_*^{(k+1)} \setminus X_*^{(k)}$ for all $k \in \{1,\dots,K-1\}$. For example, if $X_*^{(1)} = \{3,4\}$ and $X_*^{(2)} = \{1,\dots,4\}$, then we could have $\bar{\Phi}$ being $(3,4,1,2)$, $(3,4,2,1)$, $(4,3,1,2)$ or $(4,3,2,1)$, since all of them satisfy $V_2 = X_*^{(1)} = \{3,4\}$ and $V_4 = X_*^{(2)} = \{1,\dots,4\}$.

Example 16. *We apply Algorithm 5 to the 5-user system in Example 9. The call PAR$(H, V, (4,5,2,3,1))$ returns $\mathbf{r}_{\alpha,V}$ and $\mathcal{Q}_{\alpha,V}$ in (16). Let $\mathbf{r}_V^{(0)} = (0,\dots,0)$.*

For $k = 1$, first assign $\mathbf{r}_V^{(1)} = (0,\dots,0)$. We get $\mathcal{X}_*^{(1)} = \{C \in \mathcal{P}^{(2)}\colon |C| > 1\} = \{\{4,5\}\}$ such that $\langle\{4,5\}\rangle_{\mathcal{P}^{(3)}} = \{\{4\},\{5\}\} \neq \{4,5\}$. This means local omniscience has not been attained in $\{4,5\}$ before. We then assign rates in step 7 as $r_4^{(1)} = r_{\alpha^{(3)},4} = 2$ and $r_5^{(1)} = r_{\alpha^{(3)},5} = 0$ so that $\mathbf{r}_V^{(1)} = (0,0,0,2,0)$.

For $k = 2$, assign $\mathbf{r}_V^{(2)} = \mathbf{r}_V^{(1)} = (0,0,0,2,0)$ and get $\mathcal{X}_*^{(2)} = \{C \in \mathcal{P}^{(1)}\colon |C| > 1\} = \{\{1,4,5\}\}$, where $\langle\{1,4,5\}\rangle_{\mathcal{P}^{(2)}} = \{\{1\},\{4,5\}\} \neq \{1,4,5\}$. Note that, in this case, we can merge $C' = \{4,5\}$ as a super-user because users 4 and 5 reconstruct $\mathsf{Z}_{4,5}$ in the first stage. By doing so, the dimension of the system is reduced to 4, which contains individual users in $\{1,2,3\}$ and a super-user formed by users 4 and 5. Since $r_{\alpha^{(2)}}(\{4,5\}) = 4$ so that $\Delta r = r_{\alpha^{(2)}}(\{4,5\}) - r^{(2)}(\{4,5\}) = 2$. This means in addition to $\mathbf{r}_V^{(1)} = (0,0,0,2,0)$ that attains the local omniscience in $\{4,5\}$, users 4 and 5 need to transmit 2 more times for attaining the local omniscience in $\{1,4,5\}$. In this case, we choose user 4 to transmit Δr so that $r_4^{(2)} = 2 + 2 = 4$; for $C' = \{1\}$ being singleton, we haven't assigned any rates to user 1 before and therefore $r_1^{(2)} = r_{\alpha^{(2)},1} = 1$. So, $\mathbf{r}_V^{(2)}$ is updated to $(1,0,0,4,0)$. See Fig. 3.

Repeating the same procedure, we have $\mathbf{r}_V^{(3)} = (1,0.5,0.5,4.5,0) \in \mathscr{R}_{ACO}^*(V)$ at the end of iteration $k = 3$.

[18]We neglect the computations after step 2 of Algorithm 5 because they are much less complex than the SFM algorithm. See Appendix D for the explanation. Therefore, the complexity of Algorithm 5 is the same as the PAR algorithm.

[19]An example is the rate $r_{\alpha,3}$ in (16), where $r_{7,3} = 1$ but $r_{9,3} = 0$ so that the monotonicity in Proposition 12(b) does not hold.

[20]The K-stage SO for both asymptotic and non-asymptotic models is extracted from the p segments of $\mathcal{Q}_{\alpha,V}\colon [\alpha^{(j)}, \alpha^{(j-1)}), \forall j \in \{1,\dots,p\}$. While $K = p$ for the asymptotic model, $K \leq p$ for the non-asymptotic model, because there could be no integer in segment $[\alpha^{(j)}, \alpha^{(j-1)})$ if $\alpha^{(j-1)} - \alpha^{(j)} \leq 1$.

[21]The linear ordering $\bar{\Phi}$ satisfying (19) ensures $V_{|X_*^{(k)}|} = X_*^{(k)}$ for all $k \in \{1,\dots,K\}$.

Algorithm 6: Multi-stage Successive Omniscience (SO) by the PAR Algorithm for the Non-Asymptotic Model

input : H, V and Φ.
output: an achievable K-stage SO $\{(X_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1,\dots,K\}\}$ in the non-asymptotic model.

1 Arbitrarily choose a linear ordering Φ and call $\text{PAR}(H,V,\Phi)$ to obtain the segmented $\mathcal{Q}_{\alpha,V}$ and $\mathbf{r}_{\alpha,V}$ for all α, where $\mathcal{Q}_{\alpha,V}$ is segmented by p critical points $\alpha^{(1)},\dots,\alpha^{(p)}$ and $\alpha^{(0)} = H(V)$ (see Lemma 1);
2 $k \leftarrow 1$ and $X_*^{(0)} \leftarrow \emptyset$;
3 **for** j decreasing from p **to** 1 **do**
4   **if** $[\alpha^{(j)}, \alpha^{(j-1)}) \cap \mathbb{Z} \neq \emptyset$ **then**
5     $\bar{\alpha}^{(k)} \leftarrow \min\{\alpha : \alpha \in [\alpha^{(j)}, \alpha^{(j-1)}) \cap \mathbb{Z}\}$;
6     $X_*^{(k)} \leftarrow C$, where $C \in \mathcal{P}^{(j-1)}$ such that $X_*^{(k-1)} \subsetneq C$;
7     $k \leftarrow k+1$;
8     $(\bar{\phi}_{|X_*^{(k-1)}|+1}, \dots, \bar{\phi}_{|X_*^{(k)}|}) \leftarrow X_*^{(k)} \setminus X_*^{(k-1)}$;
9   **endif**
10 **endfor**
11 $(\mathcal{Q}_{\alpha,V}, \bar{\mathbf{r}}_{\alpha,V}) \leftarrow \text{PAR}(H,V,\bar{\Phi})$;
12 Initiate $\mathbf{r}_V^{(0)} \leftarrow (0,\dots,0)$;
13 **for** $k=1$ **to** K **do**
14   $\mathbf{r}_V^{(k)} \leftarrow \mathbf{r}_V^{(k-1)}$;
15   Randomly select user $i \in X_*^{(k-1)}$ and let $\Delta r \leftarrow \bar{r}_{\bar{\alpha}^{(k)}}(X_*^{(k-1)}) - r^{(k)}(X_*^{(k-1)})$ and $r_i^{(k)} \leftarrow r_i^{(k)} + \Delta r$;
16   **foreach** $i \in X_*^{(k)} \setminus X_*^{(k-1)}$ **do** let user i transmit at rate $\bar{r}_{\bar{\alpha}^{(k)},i}$: $r_i^{(k)} \leftarrow \bar{r}_{\bar{\alpha}^{(k)},i}$;
17 **endfor**
18 **return** $\mathcal{X}_*^{(k)}$ and $\mathbf{r}_V^{(k)}$ for all $k \in \{1,\dots,K\}$;

Finally, we have a 3-stage SO $\{(\mathcal{X}_*^{(k)}, \mathbf{r}_V^{(k)}) : k \in \{1,\dots,3\}\}$.

Since all users in C' have recovered $\mathsf{Z}_{C'}$ in previous stages, Δr can be assigned to any one of them. We show a fairer allocation of Δr as follows. If we assign $\Delta r = r_{\alpha^{(2)}}(\{4,5\}) - r^{(2)}(\{4,5\}) = 2$ to user 5 in stage $k=2$ and $\Delta r = r_{\alpha^{(1)}}(\{1,4,5\}) - r^{(3)}(\{1,4,5\}) = 0.5$ to user 1 in stage $k=3$, we have a fairer rate vector sequence $\mathbf{r}_V^{(1)} = (0,0,0,2,0)$, $\mathbf{r}_V^{(2)} = (1,0,0,2,2)$ and $\mathbf{r}_V^{(3)} = (1.5,0.5,0.5,2,2) \in \mathscr{R}^*_{ACO}(V)$. In general, this approach does not necessarily result in the fairest optimal rate vector $\mathscr{R}^*_{ACO}(V)$ at the end of final stage $k=p$.[22]

We then apply Algorithm 6 for the non-asymptotic model. For three critical points $\alpha^{(3)} = 4$, $\alpha^{(2)} = 6$ and $\alpha^{(1)} = 6.5$, consider three regions $[4,6)$, $[6,6.5)$ and $[6.5,10)$. We extract three integers $\bar{\alpha}^{(1)} = 4$, $\bar{\alpha}^{(2)} = 6$ and $\bar{\alpha}^{(3)} = 7 = R_{NCO}(V)$ corresponding to complimentary subsets $X_*^{(1)} = \{4,5\}$, $X_*^{(2)} = \{1,4,5\}$ and $X_*^{(3)} = \{1,\dots,5\} = V$, respectively and the linear ordering $\bar{\Phi} = (4,5,1,2,3)$ at step 10. We run

[22]It is because in the first stage $k=1$, the rate vector $\mathbf{r}_C^{(1)}$ is already an extreme point or vertex in $\mathscr{R}^*_{\text{ACO}}(C)$ for all $C \in \mathcal{X}_*^{(1)}$ [18, Theorem 27]. Thus, Remark 14(c) only attains some level of fairness in the optimal rate vector set $\mathscr{R}^*_{\text{ACO}}(V)$. Independent from this successive approach, there are several algorithms proposed in [50] for searching the fairest optimal rate vector for both asymptotic and non-asymptotic models.

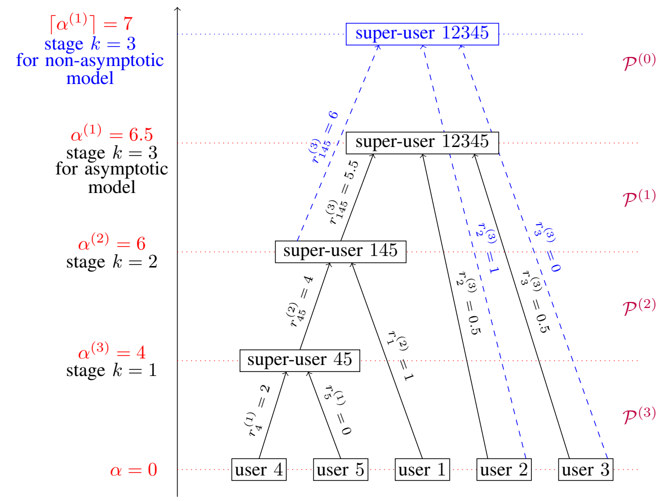


Fig. 3. The 3-stage agglomerative SO tree outlined in Example 16 by applying Algorithm 5 to the 5-user system for the asymptotic model in Example 9. Here, the users/super-users at each stage k correspond to a $\mathcal{P}^{(j)}$ in the PSP of V, which characterizes the segmented partition $\mathcal{Q}_{\alpha,V}$. The rates $r_i^{(k)}$ are determined by the segmented rate vector $\mathbf{r}_{\alpha,V}$ in (16). The 3-stage SO for the non-asymptotic model determined in Example 16 by Algorithm 6 is also shown, which only differs from the one for the asymptotic model at the last stage $k=3$, where the super-user 145 and users 2 and 3 merges at $\alpha = 7$ instead of $\alpha = 6.5$.

the PAR algorithm again with $\bar{\Phi}$ and get a new rate vector

$$\bar{\mathbf{r}}_{\alpha,V} = \begin{cases} (\alpha-5, \alpha-6, \alpha-6, \alpha-2, \alpha-4) & \alpha \in [0,4], \\ (\alpha-5, \alpha-6, \alpha-6, \alpha-2, 0) & \alpha \in (4,6], \\ (\alpha-5, \alpha-6, \alpha-6, \alpha-2, 0) & \alpha \in (6,6.5], \\ (1, \alpha-6, 7-\alpha, \alpha-2, 0) & \alpha \in (6.5,7], \\ (1,1,0,\alpha-2,0) & \alpha \in (7,10]. \end{cases}$$

where $\bar{r}_\alpha(\{4,5\}) < \bar{r}_\alpha(\{1,4,5\}) < \bar{r}_\alpha(\{1,\dots,5\})$ based on Lemma 4(c). The rate allocation procedure in step 12 to 18 is similar to Algorithm 5. We choose a fair method to allocate Δr and get $\mathbf{r}_V^{(1)} = (0,0,0,2,0)$, $\mathbf{r}_V^{(2)} = (1,0,0,2,2)$ and $\mathbf{r}_V^{(3)} = (2,1,0,2,2) \in \mathscr{R}^*_{NCO}(V)$. There is an example in [1] showing how to implement a multi-stage SO in CCDE using random linear network coding [51].

According to Theorem 10 and Algorithm 4, if there does not exist a complimentary subset $X_* \subsetneq V$, Algorithms 5 and 6 output a 1-stage SO containing the solution to the global omniscience problem: $X_*^{(1)} = V$ and an optimal rate vector $\mathbf{r}_V^{(1)} \in R_{\text{ACO}}(V)$ and $\mathbf{r}_V^{(1)} \in R_{\text{NCO}}(V)$ for the asymptotic and non-asymptotic models, respectively.

V. Conclusion

This paper proposed a PAR algorithm that reduces the complexity of solving the minimum sum-rate problem in CO by a factor of $|V|$. We observed the structural properties of the existing CoordSatCap algorithm that determines the Dilworth truncation $\hat{f}_\alpha(V)$ for a minimum sum-rate estimate α and proved that the objective function in a nesting SFM problem exhibits the strict strong map property in α. We proposed a StrMap algorithm that searches the minimizer for all α by $O(1)$ calls of the PSFM algorithm that completes in $O(\text{SFM}(|V|))$ time. Based on this fact, we proposed a PAR algorithm that solves the minimum sum-rate problem in $O(|V| \cdot \text{SFM}(|V|))$ time. We showed a distributed implementation of PAR, which incurs computation complexity $O(\text{SFM}(|V|))$ at each user.

We also utilized the PAR algorithm to efficiently solve the SO problem. For the two-stage SO, we showed that by applying a lower bound $\underline{\alpha}$ on the minimum sum-rate to the results at the end of each iteration of the PAR algorithm, a complimentary subset and a local omniscience achievable rate vector can be found. For the multi-stage SO, we propose sufficient conditions for a K-stage SO to be achievable. We used these conditions to propose algorithms for searching for an achievable multi-stage SO strategy for asymptotic and non-asymptotic models. It shows that both two-stage and multi-stage SO can be solved in $O(|V| \cdot \text{SFM}(|V|))$ time.

In addition to the brief discussion on the related problems in Section III-D, it is worth studying how the PAR algorithm contributes to the recent developments in secret key agreement problem in [52], [53] and the agglomerative approach for the information-theoretic clustering problem in [54]. It is also of interest to see how the results on the non-asymptotic model derived in this paper can be applied to a practical CCDE system. While these results can be directly implemented by the random linear network coding [51] where the coefficient is chosen from a sufficiently large Galois field (See examples in [1]), it is worth understanding how to determine the content in each transmission for other network coding schemes.

Appendix A
Properties of PSP in α and the Decomposition Algorithm

For f being a submodular function, e.g., the entropy function H or the cut κ function. The solution to the minimization $\min_{\mathcal{P} \in \Pi(V)} f_\alpha[\mathcal{P}]$, where $f_\alpha[\mathcal{P}] = \sum_{C \in \mathcal{P}} f_\alpha(C)$, is segmented in α by critical points $\alpha^{(j)}$ and the partitions $\mathcal{P}^{(j)}$ for all $j \in \{0, \dots, p\}$ as described in Section II-B2. The $\alpha^{(j)}$'s and $\mathcal{P}^{(j)}$ satisfy the following lemma.

Lemma 17 ([21, Sections 2.2 and 3] [41, Definition 3.8]). *For any two $\mathcal{P}^{(j)}$ and $\mathcal{P}^{(j')}$ such that $j < j'$ (or $\mathcal{P}^{(j')} \prec \mathcal{P}^{(j)}$), let*

$$\alpha = f(V) - \frac{f[\mathcal{P}^{(j')}] - f[\mathcal{P}^{(j)}]}{|\mathcal{P}^{(j')}| - |\mathcal{P}^{(j)}|}.$$

The following statements hold.

(a) If $j + 1 = j'$, $\alpha = \alpha^{(j')}$;
(b) if $j + 1 < j'$, $\alpha^{(j')} < \alpha \leq \alpha^{(j)}$. ∎

Based on Lemma 17, the call $\text{DA}(\{\{i\} : i \in V\}, \{V\})$ of the decomposition algorithm (DA) in Algorithm 7 returns all partitions in $\{\mathcal{P}^{(j)} : j \in \{0, \dots, p\}\}$ of the PSP. The corresponding critical points $\alpha^{(j)}$ can be determined by Lemma 17(a). The MDA algorithm in [18, Algorithm 1] is a revised version of the DA algorithm for the purpose of determining only the first partition $\mathcal{P}^{(1)}$, which determines the solution to the minimum sum-rate problem in CO. Lemma 17 also ensures the validity of StrMap algorithm in Algorithm 2.

Algorithm 7: Decomposition Algorithm (DA) [21, Algorithm SPLIT] [41, Algorithm II]

input : $\mathcal{P}^{(j)}, \mathcal{P}^{(j')}$ in the PSP of V such that $\mathcal{P}^{(j')} \prec \mathcal{P}^{(j)}$.
output: $\{\mathcal{P}^{(j)}, \mathcal{P}^{(j+1)}, \dots, \mathcal{P}^{(j')}\}$.

$\alpha := H(V) - \frac{f[\mathcal{P}^{(j')}] - f[\mathcal{P}^{(j)}]}{|\mathcal{P}^{(j')}| - |\mathcal{P}^{(j)}|}$;
$(\mathbf{r}_{\alpha,V}, \mathcal{Q}_{\alpha,V}) := \text{CoordSatCap}(\alpha, f, V, \Phi)$ where Φ is an arbitrarily chosen linear ordering of V;
if $\mathcal{Q}_{\alpha,V} = \mathcal{P}^{(j')}$ **then return** $\{\mathcal{P}^{(j)}, \mathcal{P}^{(j')}\}$;
else return $\text{DA}(\mathcal{P}^{(j)}, \mathcal{Q}_{\alpha,V}) \cup \text{DA}(\mathcal{Q}_{\alpha,V}, \mathcal{P}^{(j')})$;

Appendix B
Proof of Lemma 4

The fact that $\mathbf{r}_{\alpha,V} \in P(f_\alpha)$ holds throughout Algorithm 1 is shown in [21, Section 4.2] [18, Lemma 19] and the equality of two polyhedra $P(f_\alpha) = P(\hat{f}_\alpha)$ is proved in [19, Theorem 25]. (a) is the result in [21, Theorem 8 and Lemma 9].

We prove (b) and (c) as follows. All $C \in \mathcal{Q}_{\alpha,V_i}$ are *tight sets* [21, Section 4.2], i.e., $r_\alpha(C) = f_\alpha(C), \forall C \in \mathcal{Q}_{\alpha,V_i}$. In addition, for each $C \in \mathcal{Q}_{\alpha,V_i}$, $r_\alpha(C) = f_\alpha(C) \leq \hat{f}_\alpha(C)$ since $\mathbf{r}_{\alpha,V} \in P(f_\alpha) = P(\hat{f}_\alpha)$. But, $\hat{f}_\alpha(C) \leq f_\alpha(C)$, too, based on the definition of Dilworth truncation (3). So, $r_\alpha(C) = f_\alpha(C) = \hat{f}_\alpha(C)$ for all $C \in \mathcal{Q}_{\alpha,V_i}$ and therefore $r_\alpha(\tilde{\mathcal{X}}) = r_\alpha[\mathcal{X}] = f_\alpha[\mathcal{X}] = \hat{f}_\alpha[\mathcal{X}]$ for all $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$. We also have $\mathcal{X} = \bigwedge \text{argmin}_{\mathcal{P} \in \Pi(\tilde{\mathcal{X}})} f_\alpha[\mathcal{P}], \forall \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$ because, otherwise, either $\mathcal{Q}_{\alpha,V_i} \notin \text{argmin}_{\mathcal{P} \in \Pi(V_i)} f_\alpha[\mathcal{P}]$ or $\mathcal{Q}_{\alpha,V_i}$ is not the finest minimizer. Therefore, (b) holds. (c) also holds because of the properties of the PSP in Section II-B2. ∎

Appendix C
Proof of Lemma 8

Lemma 8(a) is a result in [42, Theorem 31] of the strict strong map: for $\mathcal{S}_j = \bigcap \text{argmin}\{g_{\alpha_j}(\tilde{\mathcal{X}}) : \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha_j,V_{i-1}} \sqcup \{\{\phi_i\}\}\}$ and $\mathcal{S}_{j-1} = \bigcup \text{argmin}\{g_{\alpha_j}(\tilde{\mathcal{X}}) : \{\phi_i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha_j,V_{i-1}} \sqcup \{\{\phi_i\}\}\}$ for all $j \in \{1, \dots, q\}$. Here, $g_{\alpha_j}(\tilde{\mathcal{S}}_{j-1}) = g_{\alpha_j}(\tilde{\mathcal{S}}_j)$ is equivalent to $r_{\alpha_j}(\tilde{\mathcal{S}}_{j-1} \setminus \tilde{\mathcal{S}}_j) = H(\tilde{\mathcal{S}}_{j-1}) - H(\tilde{\mathcal{S}}_j)$. This proves (a).

For $j < j'$, convert (11) to $H(\tilde{\mathcal{S}}_j) - H(\tilde{\mathcal{S}}_{j'}) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}]$. Since

$$\begin{aligned} H(\tilde{\mathcal{S}}_j) - H(\tilde{\mathcal{S}}_{j'}) &= \sum_{m=j+1}^{j'} \big(H(\tilde{\mathcal{S}}_{m-1}) - H(\tilde{\mathcal{S}}_m) \big) \\ &= \sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m), \end{aligned}$$

we have $f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] = \sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m)$. We prove (b) by contradiction as follows.

For the case $\mathcal{P}_d \prec \mathcal{Q}_{\alpha_{j'},V_i}$, let $\mathcal{P}^{(l)}$ be one of the partitions in the PSP of V_{i-1}, the segmented $\mathcal{Q}_{\alpha,V_{i-1}}$, such that $\mathcal{P}^{(l)} \sqcup \{\{\phi_i\}\} = \mathcal{P}_d$, then we must have $\alpha^{(l)} < \alpha_{j'}$ based on

Lemma 1. Assume that $\alpha > \alpha_{j+1}$. Based on Lemma 4(c), we have

$$\sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] >$$
$$f_{\alpha_{j+1}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] \geq \sum_{m=j+1}^{j'} f_{\alpha_m}[\langle \tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m \rangle_{\mathcal{P}_d}]$$

contradicting $\mathbf{r}_{\alpha,V} \in P(f_\alpha), \forall \alpha$ in Lemma 4. Assume that $\alpha \leq \alpha^{(l)}$. Based on Lemma 4(b) and (c) and the Dilworth truncation (3), we have

$$\sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] \leq$$
$$f_{\alpha^{(l)}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_d}] = \hat{f}_{\alpha^{(l)}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'}) < \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'}) \leq$$
$$\sum_{m=j+1}^{j'} \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) \leq \sum_{m=j+1}^{j'} \hat{f}_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m)$$

contradicting $r_\alpha(\tilde{\mathcal{X}}) = \hat{f}_\alpha(\tilde{\mathcal{X}})$ for all $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}$ and α in Lemma 4(b). So, we must have $\alpha_{j+1} \geq \alpha > \alpha^{(l)}$, i.e., (b)-(i) holds.

For $\mathcal{P}_d = \mathcal{Q}_{\alpha_{j'},V_i}$, consider the case when $j+1 < j'$. Assume that $\alpha > \alpha_{j+1}$. We have $\sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] > f_{\alpha_{j+1}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] \geq \sum_{m=j+1}^{j'} f_{\alpha_m}[\langle \tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m \rangle_{\mathcal{P}_{\alpha_{j'}}}]$ contradicting $\mathbf{r}_{\alpha,V} \in P(f_\alpha), \forall \alpha$ in Lemma 4. Assume that $\alpha \leq \alpha_{j'}$. We have $\sum_{m=j+1}^{j'} r_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] \leq f_{\alpha_{j'}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] = \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'}) \leq \sum_{m=j+1}^{j'} \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m) < \sum_{m=j+1}^{j'} \hat{f}_{\alpha_m}(\tilde{\mathcal{S}}_{m-1} \setminus \tilde{\mathcal{S}}_m)$ contradicting $r_\alpha(\tilde{\mathcal{X}}) = \hat{f}_\alpha(\tilde{\mathcal{X}})$ for all $\mathcal{X} \subseteq \mathcal{Q}_{\alpha,V}$ and α in Lemma 4(b). Therefore, we must have $\alpha_{j+1} \geq \alpha > \alpha_{j'}$.

Consider the case when $j+1 = j'$. Assume that $\alpha > \alpha_{j'}$. Then, we have $r_{\alpha_{j'}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'}) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] > f_{\alpha_{j'}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}]$ contradicting $\mathbf{r}_{\alpha_{j'},V} \in P(f_{\alpha_{j'}})$ in Lemma 4. Assume $\alpha < \alpha_{j'}$. We have $r_{\alpha_{j'}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'}) = f_\alpha[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] < f_{\alpha_{j'}}[\langle \tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'} \rangle_{\mathcal{P}_{\alpha_{j'}}}] = \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{S}}_j \setminus \tilde{\mathcal{S}}_{j'})$ contradicting $r_{\alpha_{j'}}(\tilde{\mathcal{X}}) = \hat{f}_{\alpha_{j'}}(\tilde{\mathcal{X}}), \forall \mathcal{X} \subseteq \mathcal{P}_{\alpha_{j'}}$ in Lemma 4(b). Therefore, we must have $\alpha = \alpha_{j'}$. This proves (b)-(ii). ■

APPENDIX D
SFM AND PSFM

For submodular set function $f\colon 2^V \mapsto \mathbb{R}$, $\min\{f(X)\colon X \subseteq V\}$ is a submodular function minimization (SFM) problem. There exist various polynomial-time algorithms for solving this SFM problem [19]. We denote the complexity of solving this problem by $O(\text{SFM}(|V|))$, which is polynomial in $|V|$. Let δ be the upper bound on the complexity of evaluating the value of $f(X)$ for $X \subseteq V$. The complexity $O(\text{SFM}(|V|))$ of the SFM algorithms in [55]–[58] is in the order of $|V|^5$ to $|V|^8$, e.g., the SFM algorithms in [58] and [59] have the complexity $O(|V|^8 \cdot \delta)$ and $O(|V|^5 \cdot \delta + |V|^6)$, respectively. See the summary of these SFM algorithms in Appendix D of the arXiv version of [18]. The most recent SFM algorithm is the Fujishige-Wolfe algorithm proposed in [60] based on the minimum-norm point method in [61], which is implemented in the SFM toolbox [62]. The complexity of the Fujishige-Wolfe algorithm is proved in [63, Theorem 1] to be $O((|V|^3\delta + |V|^4)M^2)$, where $M = \max_{i \in V}\{|f(\{i\})|, |f(V) - f(V \setminus \{i\})|\}$.[23]

In the computation complexity of all algorithms proposed in this paper, we neglect the computations other than the call of the SFM algorithm, e.g., the union, summation and subtraction operations in Algorithm 6, because they are much less complex than the SFM algorithm. For example, the value $\hat{\alpha}$ in Remark 11 can be searched over no more than i regions: $[\alpha^{(p)}, \alpha^{(p-1)}]$, $(\alpha^{(p-1)}, \alpha^{(p-2)}]$, ..., $(\alpha^{(1)}, \alpha^{(0)}]$, where $p \leq |V_i| = i$. The run time is $O(i)$, where $i \leq |V|$. This operation does not depend on δ and therefore is much simpler than the SFM algorithm. In CO and SO, δ depends on the run time of evaluating the entropy function $H(X)$. For example, in CCDE, δ refers to the complexity of the matrix rank function, which is polynomial in $|V|$, e.g., $O(|V|^{2.38})$ [64].

The authors in [46] studied a specific type of SFM for the graph model, the push-relabel max-flow/Min-cut algorithm in [65]. It was shown that if the capacities of edges from the source node and to the sink node are monotonically changing in a real-valued parameter α, the max-flows/min-cuts for each α can be determined in order. The max-flow algorithm in [65] was then extended to a parameterized max-flow algorithm, which solves a finite sequence of min-cut problems parameterized by α at the same asymptotic time as the push-relabel MaxFlow algorithm.[24]

The same technique was further applied to extend the SFM algorithms to the PSFM algorithms in [33]–[35] that can solve a sequence of SFM problems having the strong map property in α. However, all these algorithms require a finite number of monotonic values of α, e.g., the critical values $\alpha_0, \ldots, \alpha_q$ in Lemma 7, as input. Since we do not know the critical values α_j's in advance, these PSFM algorithms cannot be directly applied to solve the problem $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ in the PAR algorithm. This is the reason why we derive Lemma 8 and propose the StrMap algorithm (Algorithm 2). The StrMap algorithm invokes at most $2q$ recursions to search for all α_j's and $\tilde{\mathcal{S}}_j$'s that determine the minimum and the minimizer of $\min\{g_\alpha(\tilde{\mathcal{X}})\colon \{i\} \in \mathcal{X} \subseteq \mathcal{Q}_{\alpha,V_i}\}$ for all α. The StrMap algorithm can be implemented by the PSFM algorithms in [33]–[35]. See the Slicing algorithm in [33, Section 4.2], where it is shown that the StrMap algorithm completes at the same time as the PSFM algorithms and therefore its complexity is $O(\text{SFM}(|V|))$. There is an example in [43, Algorithm 7] showing how to run StrMap by a parametric version of Schrijvers SFM algorithm in [58]. The idea is the same as [33, Section 4.2]: running the push-relabel SFM [33, Section 4.2] in descending α and the reverse-push-

[23]For the CO problem, $M \leq H(V)$.

[24]Here, 'asymptotic' refers to the asymptotic limits of the complexity notation $O(\cdot)$: for the actual running time $a(|V|)$, the asymptotic complexity is $O(b(|V|))$ if $\lim_{|V| \to \infty} \frac{a(|V|)}{b(|V|)} = c$ for some constant c.

relabel SFM of [33, Section 4.2] in ascending α. They incur the same label and push operations (in terms of asymptotic complexity) as the SFM algorithm [33].

For determining the PSP of a graph, Kolmogorov proposed an algorithm [66, Fig. 3] with the parametric MaxFlow [46] being the subroutine, the contribution of which is similar to the PAR algorithm: it reduces the previous complexity $O(|V|^2 \cdot \text{MaxFlow}(|V|))$ for determining the network strength and the maximum number of edge-disjoint spanning trees [22], [23] to $O(|V| \cdot \text{MaxFlow}(|V|))$. Here, $\text{MaxFlow}(|V|)$ denotes the complexity of determining the max-flow in a $|V|$-vertex graph. The method in [66, Fig. 3] is to keep updating the rate vector $\mathbf{r}_{\alpha,V}$ at the end of each iteration to maintain each dimension $r_{\alpha,i}$ monotonically increasing in α. We show in [43, Theorem E.1] that the purpose of this rate update is in fact to preserve a non-strict strong map property of the corresponding SFM such that the minimizer forms a 'nesting' set sequence in α as stated in [66, Lemmas 4 and 5]. However, this method cannot be applied to the CO and SO problem. In [43, Theorem E.1], we follow the rate adaptation method in [66, Fig. 3] and the non-strict strong map to derive the properties of the critical points (a result similar to Lemma 8). It is shown that a small positive ϵ needs to be applied to check if there is exists any critical point in the region $(\alpha_{j'}, \alpha_j]$. This means that if this ϵ is not set to be small enough, we might not be able to find all critical points. See the explanation in [43, Appendix E].

Appendix E
Proof of Theorem 10

We start with the necessary and sufficient condition for a user subset to be complimentary in [6], [27]. We rewrite this condition in terms of the Dilworth truncation $\hat{f}_\alpha$ below and relax it to a sufficient condition that only requires a lower bound on the minimum sum-rate $R_{\text{ACO}}(V)$ or $R_{\text{NCO}}(V)$. Since V is always a complimentary subset, we restrict our attention to proper subsets of V.

Corollary 18. *A user subset $X_* \subsetneq V$ such that $|X_*| > 1$ is complimentary if and only if $f_{R_{ACO}(V)}(X_*) = \hat{f}_{R_{ACO}(V)}(X_*)$ for the asymptotic model and $f_{R_{NCO}(V)}(X_*) = \hat{f}_{R_{NCO}(V)}(X_*)$ for the non-asymptotic model.*

Proof: Consider the asymptotic model first. Based on [6, Theorem 4.2], $R_{\text{ACO}}(X_*) \leq R_{\text{ACO}}(V) - H(V) + H(X_*) = f_{R_{\text{ACO}}(V)}(X_*)$ is the necessary and sufficient condition for X_* to be complimentary. We also have $R_{\text{ACO}}(X_*) \geq \sum_{C \in \mathcal{P}} \frac{H(X_*)-H(C)}{|\mathcal{P}|-1}, \forall \mathcal{P} \in \Pi(X_*) \colon |\mathcal{P}| > 1$. So, $\sum_{C \in \mathcal{P}} \frac{H(X_*)-H(C)}{|\mathcal{P}|-1} \leq R_{\text{ACO}}(V) - H(V) + H(X_*), \forall \mathcal{P} \in \Pi(X_*) \colon |\mathcal{P}| > 1$, which is equivalent to $f_{R_{\text{ACO}}(V)}(X_*) \leq \sum_{C \in \mathcal{P}} f_{R_{\text{ACO}}(V)}(C), \forall \mathcal{P} \in \Pi(X_*)$, i.e., $f_{R_{\text{ACO}}(V)}(X_*) = \hat{f}_{R_{\text{ACO}}(V)}(X_*)$. In the same way, one can prove that the necessary and sufficient condition for a subset X_* to be complimentary in [6, Theorem 5.2], $H(V) - H(X_*) + R_{\text{NCO}}(X_*) \leq R_{\text{NCO}}(V)$, is equivalent to $f_{R_{\text{NCO}}(V)}(X_*) = \hat{f}_{R_{\text{NCO}}(V)}(X_*)$. ■

In [6, Theorems 4.2 and 5.2], the necessary and sufficient condition in Corollary 18 is written as $I(X_*) \geq I(V)$ and $\lfloor I(X_*) \rfloor \geq \lfloor I(V) \rfloor$ for the asymptotic and non-asymptotic models, respectively, via the dual relationships: $R_{\text{ACO}}(V) = H(V) - I(V)$ and $R_{\text{NCO}}(V) = H(V) - \lfloor I(V) \rfloor$ [13], [16]. Here, $I(V)$ is the amount of information shared by users in V [43, Section IV]. The interpretation of the necessary and sufficient condition in Corollary 18 is: the sources in a complimentary subset X_* are more correlated to each other than with the remaining sources in $V \setminus X_*$. This is a common situation in real world applications. For example, the sensors that are geographically close to each other usually record data that are statistically similar.

Lemma 19 (sufficient condition)**.** *A user subset $X_* \subsetneq V$ such that $|X_*| > 1$ is complimentary if $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$ for $\underline{\alpha} \leq R_{ACO}(V)$ for the asymptotic model and an integer-valued $\underline{\alpha} \leq R_{NCO}(V)$ for the non-asymptotic model.*

Proof: For any $\underline{\alpha}, \underline{\alpha}' \leq R_{\text{ACO}}(V)$ such that $\underline{\alpha} < \underline{\alpha}'$, if $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$, then

$$\begin{aligned} & f_{\underline{\alpha}'}[\mathcal{P}] - f_{\underline{\alpha}'}(X_*) \\ & \quad = H[\mathcal{P}] - H(X_*) - (|\mathcal{P}|-1)\Big(H(V) - \underline{\alpha}'\Big) \\ & \quad > H[\mathcal{P}] - H(X_*) - (|\mathcal{P}|-1)\big(H(V) - \underline{\alpha}\big) \\ & \quad = f_{\underline{\alpha}}[\mathcal{P}] - f_{\underline{\alpha}}(X_*) \geq 0 \end{aligned} \tag{20}$$

for all $\mathcal{P} \in \Pi(X_*)$ such that $|\mathcal{P}| > 1$, where $H[\mathcal{P}] = \sum_{C \in \mathcal{P}} H(C)$. So, $f_{R_{\text{ACO}}(V)}[\mathcal{P}] - f_{R_{\text{ACO}}(V)}(X_*) \geq f_{\underline{\alpha}}[\mathcal{P}] - f_{\underline{\alpha}}(X_*) \geq 0, \forall \mathcal{P} \in \Pi(X_*) \colon |\mathcal{P}| > 1$. The condition $f_{R_{\text{ACO}}(V)}(X_*) = \hat{f}_{R_{\text{ACO}}(V)}(X_*)$ in Corollary 18 holds. Therefore, X_* is complimentary in the asymptotic model. In the same way, one can prove the sufficient condition $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$ for the non-asymptotic model. ■

We derive a lower bound $\underline{\alpha}$ that tells the existence of a complimentary subset as follows.

Corollary 20. *For any two lower bounds $\underline{\alpha}$ and $\underline{\alpha}'$ on the minimum sum-rate $R_{ACO}(V)$ for the asymptotic model, or on $R_{NCO}(V)$ for the non-asymptotic model, such that $\underline{\alpha} < \underline{\alpha}'$,*

$$\begin{aligned} & \{X_* \subsetneq V \colon |X_*| > 1, f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)\} \\ & \qquad \subseteq \{X_* \subsetneq V \colon |X_*| > 1, f_{\underline{\alpha}'}(X_*) = \hat{f}_{\underline{\alpha}'}(X_*)\}. \end{aligned}$$

Proof: As shown in the proof of Lemma 19, for any $X_* \subsetneq V$ such that $|X_*| > 1$, if $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$, then $f_{\underline{\alpha}}[\mathcal{P}] - f_{\underline{\alpha}}(X_*) \geq 0, \forall \mathcal{P} \in \Pi(X_*) \colon |\mathcal{P}| > 1$ and inequality (20) holds. We necessarily have $f_{\underline{\alpha}'}(X_*) = \hat{f}_{\underline{\alpha}'}(X_*)$ so that $X_* \in \{X_* \subsetneq V \colon |X_*| > 1, f_{\underline{\alpha}'}(X_*) = \hat{f}_{\underline{\alpha}'}(X_*)\}$. Corollary holds. ■

Lemma 19 does not tell the non-existence of a complimentary subset. Corollary 20 states that the number of complimentary subsets searched by Lemma 19 shrinks to zero when the value of lower bound $\underline{\alpha}$ decreases. Therefore, we need to choose a $\underline{\alpha}$ large enough to capture at least one of the complimentary subsets (if there exists one).

Lemma 21. *There does not exist any complimentary subset in the asymptotic model if no $X_* \subsetneq V$ such that $|X_*| > 1$ satisfies $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$ for $\underline{\alpha} = \sum_{i \in V} \frac{H(V)-H(\{i\})}{|V|-1}$ for the asymptotic model and $\underline{\alpha}_{NCO} = \lceil \sum_{i \in V} \frac{H(V)-H(\{i\})}{|V|-1} \rceil$ for the non-asymptotic model.*

Proof: The proof is based on Lemma 17. The method is to show that if no X_* holds $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$, there is only singleton partition and ground set partition in the PSP, which indicates the omniscience can only be attainted in a one-off manner. If $\nexists X_* \subsetneq V \colon |X_*| > 1, f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$, we must have $f_{\underline{\alpha}}(X_*) > f_{\underline{\alpha}}[\mathcal{P}]$ for some $\mathcal{P} \in \Pi(X_*) \colon |\mathcal{P}| > 1$ for all $X_* \subsetneq V$ such that $|X_*| > 1$. This necessarily means that $\mathcal{Q}_{\underline{\alpha},V} = \bigwedge \operatorname{argmin}_{\mathcal{P} \in \Pi(V)} f_{\underline{\alpha}}[\mathcal{P}] = \{\{i\} \colon i \in V\}$. In the case when $\underline{\alpha} = \sum_{i \in V} \frac{H(V) - H(\{i\})}{|V|-1}$, Lemma 17(a) holds. That is, the PSP of V only contains one critical point $\alpha^{(1)} = \underline{\alpha}$ with the partition chain $\{\{i\} \colon i \in V\} = \mathcal{P}^{(1)} \prec \mathcal{P}^{(0)} = \{V\}$. In this case, the value of $\underline{\alpha}$ in the lemma is in fact the minimum sum-rate for the asymptotic model, i.e., $\alpha^{(1)} = R_{\text{ACO}}(V) = \underline{\alpha}$, where the necessary and sufficient condition in Corollary 18 does not hold, i.e., $\nexists X_* \subsetneq V \colon |X_*| > 1, f_{R_{\text{ACO}}(V)}(X_*) = \hat{f}_{R_{\text{ACO}}(V)}(X_*)$. Therefore, there is no complimentary subset for SO in the asymptotic model. Similarly, for $\underline{\alpha}' = \lceil \sum_{i \in V} \frac{H(V) - H(\{i\})}{|V|-1} \rceil$ for the non-asymptotic model, we have $\mathcal{Q}_{\underline{\alpha}',V} = \{\{i\} \colon i \in V\} = \mathcal{Q}_{\underline{\alpha},V}$ due to the property of the PSP $\mathcal{Q}_{\underline{\alpha},V} \preceq \mathcal{Q}_{\underline{\alpha}',V}$ for $\underline{\alpha} \leq \underline{\alpha}'$. This necessarily means $\underline{\alpha}' = R_{\text{NCO}}(V)$. Corollary 18 holds and there is no complimentary subset in the non-asymptotic model. ■

The lower bounds $\underline{\alpha}$ can be determined by $O(|V|)$ calls of the entropy function. An example to demonstrate Lemma 21 for the asymptotic model is the independent source model with the terminals Z_i being independent of each other, where we have only two trivial partitions $\mathcal{P}^{(1)} = \{\{i\} \colon i \in V\}$ and $\mathcal{P}^{(0)} = \{V\}$ in the PSP and $\alpha^{(1)} = R_{\text{ACO}}(V) = H(V)$. The lower bound in Lemma 21 is $\underline{\alpha} = \sum_{i \in V} \frac{H(V) - H(\{i\})}{|V|-1} = \alpha^{(1)}$ and the partition $\mathcal{P}^{(1)} = \{\{i\} \colon i \in V\}$ indicates that no $X_* \subsetneq V$ such that $|X_*| > 1$ satisfies $f_{\underline{\alpha}}(X_*) = \hat{f}_{\underline{\alpha}}(X_*)$. In this case, there does not exist any complimentary subset for the asymptotic model in the independent source. Lemmas 21 and 19 prove Theorem 10.

APPENDIX F
PROOF OF COROLLARY 13

We prove this corollary by showing that $\{(\tilde{\mathcal{X}}_*^{(k)}, \mathbf{r}_V^{(k)}) \colon k \in \{1, \ldots, p\}\}$ returned by Algorithm 5 satisfies the achievability of the multi-stage SO in Proposition 12. Based on Theorem 10, $\mathcal{X}_*^{(k)}$ obtained in step 5 contains all nonsingleton subsets of $\mathcal{P}^{(p-k)}$ that are complimentary. For all $k \in \{1, \ldots, p\}$ and each $C \in \mathcal{X}_*^{(k)}$, $\hat{\alpha} = \min\{\alpha \in \mathbb{R} \colon f_\alpha(C) = \hat{f}_\alpha(C)\} = \alpha^{(p-k+1)}$ so that $\mathbf{r}_{\alpha^{(p-k+1)},C} \in \mathscr{R}^*_{\text{ACO}}(C)$ with sum-rate $r_{\alpha^{(p-k+1)}}(C) = R_{\text{ACO}}(C)$ according to Theorem 10 and Remark 11. Here, we only know the sum-rate $r_{\alpha^{(p-k+1)}}(C)$ is optimal for each k, but are unclear how to allocate it to individual users that achieves the local omniscience in Proposition 12(a) and the monotonicity of the rates in Proposition 12(c). Below, we prove by recursion that the rate allocation procedure in Algorithm 5 satisfies Proposition 12(a) and (c).

For $k = 1$, since $\mathcal{P}^{(p)} = \{\{i\} \colon i \in V\}$, we have $\langle C \rangle_{\mathcal{P}^{(p)}}$ contains only singletons for all $C \in \mathcal{X}_*^{(1)} = \{C \in \mathcal{P}^{(p-1)} \colon |C| > 1\}$. In this case, $\mathbf{r}_{\alpha^{(p)},V} \in \mathscr{R}^*_{\text{ACO}}(C)$ with the dimensions $r_{\alpha^{(p)},i} \geq 0$ for all $i \in C$ [19, Lemma 3.23] [18, Theorem 9], i.e., the monotonicity in Proposition 12(b) holds. After step 7, the local omniscience is attained in each $C \in \mathcal{X}_*^{(1)}$ with $r_i^{(1)} = r_{\alpha^{(p)},i}$ for all $i \in C$, i.e., the sum-rate $r^{(1)}(C) = r_{\alpha^{(p)}}(C)$ is assigned to the users in C.

By recursion, before step 7 of Algorithm 5 in iteration k, we have all nonsingletons $C' \in \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$ attain local omniscience by an optimal rate vector $\mathbf{r}_{C'}^{(k')} = \mathbf{r}_{\alpha^{(p-k'+1)},C'} \in \mathscr{R}^*_{\text{ACO}}(C')$ with sum-rate $r^{(k')}(C') = r_{\alpha^{(p-k'+1)}}(C')$ at some previous stage $k' < k$. So, all C' can be treated as super-users with the index $\tilde{C}'$ and, for the problem of attaining the local omniscience in C, it suffices to consider the super-user system $\tilde{C} = \{\tilde{C}' \colon C' \in \langle C \rangle_{\mathcal{P}^{(p-k+1)}}\}$. For $r_{\alpha^{(p-k+1)},\tilde{C}'} = r_{\alpha^{(p-k+1)}}(C') = \sum_{i \in C'} r_{\alpha^{(p-k+1)},i}$, we have $\mathbf{r}_{\alpha^{(p-k+1)},C} \in \mathscr{R}^*_{\text{ACO}}(C)$ reduce to $\mathbf{r}_{\alpha^{(p-k+1)},\tilde{C}} \in \mathscr{R}^*_{\text{ACO}}(\tilde{C})$ with $r_{\alpha^{(p-k+1)}}(C) = R_{\text{ACO}}(C) = R_{\text{ACO}}(\tilde{C}) = r_{\alpha^{(p-k+1)}}(\tilde{C})$. Therefore, we just need to assign the rates $r_{\alpha^{(p-k+1)}}(C')$ to the users in C', where the monotonicity in Proposition 12(b) also holds for all C', based on Lemma 4(c):

$$\begin{aligned} \Delta r &= r_{\alpha^{(p-k+1)}}(C') - r^{(k')}(C') \\ &= r_{\alpha^{(p-k+1)}}(C') - r_{\alpha^{(p-k'+1)}}(C') > 0 \end{aligned} \tag{21}$$

since $\alpha^{(p-k+1)} > \alpha^{(p-k'+1)}$ and $C' \in \mathcal{P}^{(p-k+1)}$ so that $\langle C' \rangle_{\mathcal{P}^{(p-k'+1)}} \subseteq \mathcal{P}^{(p-k'+1)}$ for all $k' < k$; for all singleton $C' \in \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$, we have $r^{(k-1)}(C') = 0$ so that $\Delta r = r_{\alpha^{(p-k+1)}}(C') - r^{(k-1)}(C') \geq 0$ [19, Lemma 3.23] [18, Theorem 9], i.e., Proposition 12(c) holds. Note, the above rate updates only need to be considered for all $C \in \mathcal{X}_*^{(k)}$ such that $C \neq \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$. This is because, if $C = \langle C \rangle_{\mathcal{P}^{(p-k+1)}}$, the local omniscience in C has already been attained in the previous stages.

We then prove the global omniscience is attained in the final recursion. At the end of the last stage $k = p$, $\mathbf{r}_{\alpha^{(p-k+1)},C} \in \mathscr{R}^*_{\text{ACO}}(C)$ ensures $\mathbf{r}_V^{(p)} \in R_{\text{ACO}}(V)$ since, in $\mathcal{X}_*^{(p)} = \{C \in \mathcal{P}^{(0)} \colon |C| > 1\} = \{V\}$, $C = V$ is the only nonsingleton subset and $\mathbf{r}_{\alpha^{(1)},V} \in \mathscr{R}^*_{\text{ACO}}(V)$. Also, because $\mathcal{P}^{(p-k+1)} \prec \mathcal{P}^{(p-k)}$, we have $\tilde{\mathcal{X}}_*^{(k-1)} \subsetneq \tilde{\mathcal{X}}_*^{(k)}$ for all $k \in \{2, \ldots, p\}$. It is clear that $r_i^{(k)} = 0$ for all $i \in V \setminus X_*^{(k)}$ in each $k \in \{1, \ldots, p\}$, which satisfies Proposition 12(b). Therefore, $\{(\tilde{\mathcal{X}}_*^{(k)}, \mathbf{r}_V^{(k)}) \colon k \in \{1, \ldots, p\}\}$ is achievable. ■

APPENDIX G
PROOF OF COROLLARY 15

It is clear that all $X_*^{(k)}$'s are complimentary according to Lemma 19 and form the chain $X_*^{(1)} \subsetneq \ldots \subsetneq X_*^{(K)} = V$. Then, to prove the achievability of the returned K-stage SO, we just need to prove the monotonicity of the rate vector in Proposition 12(c), because (b) holds by the assignment $r_i^{(k)} = 0$ for all $i \in V \setminus X_*^{(k)}$ at each stage k (as in the second for-loop in Algorithm 6).

Consider the rate vector $\bar{\mathbf{r}}_{\alpha,V}$ returned by the call $\text{PAR}(H, V, \bar{\Phi})$. Since $g_\alpha(\tilde{\mathcal{U}}_{\alpha,V_i})$ taking integer values for the integer-valued entropy function H and α in the non-asymptotic model, we have $\bar{\mathbf{r}}_{\alpha,V} \in \mathbb{Z}^{|V|}$ for all integer-valued α, i.e., $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)},V} \in \mathbb{Z}^{|V|}$ for all k. If $\bar{\alpha}^{(k)} = \alpha^{(j)}$, we have shown in the proof of Corollary 13 that $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)},X_*^{(k)}}$ is an optimal rate vector that attains local omniscience in $X_*^{(k)}$ with the minimum sum-rate $R_{\text{NCO}}(X_*^{(k)}) = R_{\text{ACO}}(X_*^{(k)})$;

when $\bar{\alpha}^{(k)} \in (\alpha^{(j)}, \alpha^{(j-1)}) \cap \mathbb{Z}$, while Theorem 10(b) states that $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)}, X_*^{(k)}} \in \mathscr{R}^*_{\text{NCO}}(X_*^{(k)})$ if $\bar{\alpha}^{(k)} = \min\{\alpha \colon \alpha \in [\alpha^{(j)}, \alpha^{(j-1)}) \cap \mathbb{Z}\}$, it can be proven in the same way that $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)}, X_*^{(k)}} \in \mathscr{R}_{\text{CO}}(X_*^{(k)}) \cap \mathbb{Z}^{|X_*^{(k)}|}$ for any $\alpha \in [\alpha^{(j)}, \alpha^{(j-1)})$, i.e., $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)}, X_*^{(k)}}$ is achievable, but may not be optimal. Thus, $\bar{\mathbf{r}}_{\bar{\alpha}^{(k)}, X_*^{(k)}}$ attains the local omniscience in $X_*^{(k)}$.

For the linear ordering $\bar{\Phi}$ satisfying (19), we have $V_{|X_*^{(k)}|} = X_*^{(k)}$ for all $k \in \{1, \ldots, K\}$. Based on Lemma 4(a), the call $\text{PAR}(V, H, \bar{\Phi})$ outputs a rate vector $\bar{\mathbf{r}}_{\alpha,V}$ such that $\bar{r}_\alpha(X_*^{(k)}) = \bar{r}_\alpha\big(V_{|X_*^{(k)}|}\big) = \hat{f}_\alpha(V_{|X_*^{(k)}|}) = \hat{f}_\alpha(X_*^{(k)})$ for all α. Then, according to Lemma 4(c), for all $k \in \{1, \ldots, K-1\}$, since $\bar{\alpha}^{(k)} < \bar{\alpha}^{(k+1)}$, $\bar{r}_{\bar{\alpha}^{(k+1)}}(X_*^{(k)}) - \bar{r}_{\bar{\alpha}^{(k)}}(X_*^{(k)}) > 0$. This proves Proposition 12(c). ■

References

[1] N. Ding, R. A. Kennedy, and P. Sadeghi, "A practical approach for successive omniscience," in *Proc. IEEE Int. Symp. Inf. Theory*, Aachen, 2017, pp. 2578–2582.

[2] N. Ding, P. Sadeghi, and T. Rakotoarivelo, "Improving computational efficiency of communication for omniscience and successive omniscience," in *Proc. Annu. Allerton Conf. Commun., Control, and Comput.*, Monticello, IL, 2018, pp. 1073–1080.

[3] I. Csiszár and P. Narayan, "Secrecy capacities for multiple terminals," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3047–3061, Dec. 2004.

[4] S. Nitinawarat and P. Narayan, "Perfect omniscience, perfect secrecy, and steiner tree packing," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6490–6500, Dec. 2010.

[5] C. Chan, "Linear perfect secret key agreement," in *Proc. IEEE Inf. Theory Workshop*, Paraty, Brazil, 2011, pp. 723–726.

[6] C. Chan, Al-Bashabsheh, Q. Zhou, N. Ding, T. Liu, and A. Sprintson, "Successive omniscience," *IEEE Trans. Inf. Theory*, vol. 62, no. 6, pp. 3270–3289, Apr. 2016.

[7] S. El Rouayheb, A. Sprintson, and P. Sadeghi, "On coding for cooperative data exchange," in *Proc. IEEE Inf. Theory Workshop*, Cairo, Egypt, 2010, pp. 1–5.

[8] T. Courtade and R. Wesel, "Efficient universal recovery in broadcast networks," in *Proc. Annu. Allerton Conf. Commun., Control, and Comput.*, Monticello, IL, 2010, pp. 1542–1549.

[9] T. Courtade, B. Xie, and R. Wesel, "Optimal exchange of packets for universal recovery in broadcast networks," in *Proc. Military Commun. Conf.*, San Jose, CA, 2010, pp. 2250–2255.

[10] A. Sprintson, P. Sadeghi, G. Booker, and S. El Rouayheb, "A randomized algorithm and performance bounds for coded cooperative data exchange," in *Proc. IEEE Int. Symp. Inf. Theory*, Austin, TX, 2010, pp. 1888–1892.

[11] D. Ozgul and A. Sprintson, "An algorithm for cooperative data exchange with cost criterion," in *Proc. Inf. Theory Appl. Workshop*, La Jolla, CA, 2011, pp. 1–4.

[12] N. Abedini, M. Manjrekar, S. Shakkottai, and L. Jiang, "Harnessing multiple wireless interfaces for guaranteed QoS in proximate P2P networks," in *Proc. IEEE Int. Conf. Commun. China*, Beijing, China, 2012, pp. 18–23.

[13] T. Courtade and R. Wesel, "Weighted universal recovery, practical secrecy, and an efficient algorithm for solving both," in *Proc. Annu. Allerton Conf. Commun., Control, and Comput.*, Monticello, IL, 2011, pp. 1349–1357.

[14] ——, "Coded cooperative data exchange in multihop networks," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 1136–1158, Feb. 2014.

[15] N. Milosavljevic, S. Pawar, S. E. Rouayheb, M. Gastpar, and K. Ramchandran, "Efficient algorithms for the data exchange problem," *IEEE Trans. Inf. Theory*, vol. 62, no. 4, pp. 1878 – 1896, Apr. 2016.

[16] C. Chan, A. Al-Bashabsheh, J. Ebrahimi, T. Kaced, and T. Liu, "Multivariate mutual information inspired by secret-key agreement," *Proc. IEEE*, vol. 103, no. 10, pp. 1883–1913, Oct. 2015.

[17] N. Ding, C. Chan, Q. Zhou, R. A. Kennedy, and P. Sadeghi, "A faster algorithm for asymptotic communication for omniscience," in *Proc. IEEE Globecom Workshop Network Coding Appl.*, Washington, D.C., 2016, pp. 1–6.

[18] ——, "Determining optimal rates for communication for omniscience," *IEEE Trans. Inf. Theory*, vol. 64, no. 3, pp. 1919–1944, Mar. 2018.

[19] S. Fujishige, *Submodular functions and optimization*, 2nd ed. Amsterdam, The Netherlands: Elsevier, 2005.

[20] C. Chan, A. Al-Bashabsheh, Q. Zhou, T. Kaced, and T. Liu, "Info-clustering: A mathematical theory for data clustering," *IEEE Trans. Mol. Biol. Multi-Scale Commun.*, vol. 2, no. 1, pp. 64–91, Jun. 2016.

[21] K. Nagano, Y. Kawahara, and S. Iwata, "Minimum average cost clustering," in *Proc. Advances in Neural Inf. Process. Syst.*, Vancouver, Candada, 2010, pp. 1759–1767.

[22] W. H. Cunningham, "Optimal attack and reinforcement of a network," *J. ACM*, vol. 32, no. 3, pp. 549–561, Jul. 1985.

[23] M. Baïou, F. Barahona, and A. R. Mahjoub, "Partition inequalities: Separation, extensions and network design," IBM, Tech. Rep. RC25233, 2011.

[24] C. Ye and A. Reznik, "Group secret key generation algorithms," in *Proc. IEEE Int. Symp. Inf. Theory*, Nice, France, 2007, pp. 2596–2600.

[25] S. Nitinawarat, C. Ye, A. Barg, P. Narayan, and A. Reznik, "Secret key generation for a pairwise independent network model," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6482–6489, Dec. 2010.

[26] S. Nitinawarat and P. Narayan, "Perfect omniscience, perfect secrecy, and steiner tree packing," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6490–6500, Dec 2010.

[27] C. Chan, A. Al-Bashabsheh, J. B. Ebrahimi, T. Kaced, S. Kadhe, T. Liu, A. Sprintson, M. Yan, and Q. Zhou, "Successive omniscience," in *Proc. Int. Symp. Network Coding*, Sydney, Australia, 2015, pp. 21–25.

[28] N. Ding, R. A. Kennedy, and P. Sadeghi, "Iterative merging algorithm for cooperative data exchange," in *Proc. Int. Symp. Network Coding*, Sydney, Australia, 2015, pp. 41–45.

[29] A. Heidarzadeh and A. Sprintson, "Successive local and successive global omniscience," in *Proc. IEEE Int. Symp. Inf. Theory*, Aachen, Germany, 2017, pp. 2313–2317.

[30] H. Tyagi and S. Watanabe, "Universal multiparty data exchange," in *Proc. IEEE Int. Symp. Inf. Theory*, Barcelona, Spain, 2016, pp. 2329–2333.

[31] ——, "Universal multiparty data exchange and secret key agreement," *IEEE Trans. Inf. Theory*, vol. 63, no. 7, pp. 4057–4074, Jul. 2017.

[32] ——, "Optimality of the recursive data exchange protocol," in *Proc. IEEE Int. Symp. Inf. Theory*, Aachen, Germany, 2017, pp. 2855–2859.

[33] L. Fleischer and S. Iwata, "A push-relabel framework for submodular function minimization and applications to parametric optimization," *Discrete Applied Math.*, vol. 131, no. 2, pp. 311 – 322, 2003.

[34] K. Nagano, "A faster parametric submodular function minimization algorithm and applications," University of Tokyo, Tech. Rep. 92 (2007-AL-114), 2007.

[35] S. Iwata, K. Murota, and M. Shigeno, "A fast parametric submodular intersection algorithm for strong map sequences," *Math. Oper. Res.*, vol. 22, no. 4, pp. 803–813, Nov. 1997.

[36] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. 19, no. 4, pp. 471–480, Jul. 1973.

[37] T. Cover, "A proof of the data compression theorem of Slepian and Wolf for ergodic sources," *IEEE Trans. Inf. Theory*, vol. 21, no. 2, pp. 226–228, Mar. 1975.

[38] T. M. Cover and J. A. Thomas, *Elements of information theory*. Hoboken, NJ: John Wiley & Sons, 2012.

[39] C. Chan, "On tightness of mutual dependence upperbound for secret-key capacity of multiple terminals," *arXiv preprint arXiv:0805.3200*, 2008.

[40] R. P. Dilworth, "Dependence relations in a semi-modular lattice," *Duke Math. J.*, vol. 11, no. 3, pp. 575–587, Feb. 1944.

[41] H. Narayanan, "The principal lattice of partitions of a submodular function," *Linear Algebra its Appl.*, vol. 144, pp. 179 – 216, Jan. 1991.

[42] S. Fujishige, "Theory of principal partitions revisited," in *Research Trends in Combinatorial Optimization*. Berlin, Germany: Springer, 2009, pp. 127–162.

[43] N. Ding, P. Sadeghi, and T. Rakotoarivelo, "Part I: Improving computational efficiency of communication for omniscience," *arXiv preprint arXiv:1912.11808*, 2019.

[44] W. T. Tutte, "On the problem of decomposing a graph into n connected factors," *J. Lond. Math. Soc.*, vol. s1-36, no. 1, pp. 221–230, 1961.

[45] C. S. A. Nash-Williams, "Edge-disjoint spanning trees of finite graphs," *J. Lond. Math. Soc.*, vol. s1-36, no. 1, pp. 445–450, 1961.

[46] G. Gallo, M. D. Grigoriadis, and R. E. Tarjan, "A fast parametric maximum flow algorithm and applications," *SIAM J. Comput.*, vol. 18, no. 1, pp. 30–55, 1989.

[47] N. Ding, D. Smith, T. Rakotoarivelo, and P. Sadeghi, "Fairness in multiterminal data compression: Decomposition of shapley value," in *Proc. IEEE Int. Symp. Inf. Theory*, Vail, CO, 2018, pp. 886–890.

[48] C. Chan, A. Al-Bashabsheh, and Q. Zhou, "Change of multivariate mutual information: From local to global," *IEEE Trans. Inf. Theory*, vol. 64, no. 1, pp. 57–76, Jan. 2018.

[49] N. Milosavljevic, S. Pawar, S. El Rouayheb, M. Gastpar, and K. Ramchandran, "Deterministic algorithm for the cooperative data exchange problem," in *Proc. IEEE Int. Symp. Inf. Theory*, St. Petersburg, Russia, 2011, pp. 410–414.

[50] N. Ding, P. Sadeghi, D. Smith, and T. Rakotoarivelo, "Attaining fairness in communication for omniscience," *arXiv*, 2019. [Online]. Available: http://arxiv.org/abs/1902.03706

[51] T. Ho, M. Medard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, "A random linear network coding approach to multicast," *IEEE Trans. Inf. Theory*, vol. 52, no. 10, pp. 4413–4430, Oct. 2006.

[52] M. Mukherjee, N. Kashyap, and Y. Sankarasubramaniam, "On the public communication needed to achieve SK capacity in the multiterminal source model," *IEEE Trans. Inf. Theory*, vol. 62, no. 7, pp. 3811–3830, Jul. 2016.

[53] C. Chan, M. Mukherjee, N. Kashyap, and Q. Zhou, "Secret key agreement under discussion rate constraints," in *Proc. IEEE Int. Symp. Inf. Theory*, Aachen, Germany, 2017, pp. 1519–1523.

[54] C. Chan, A. Al-Bashabsheh, and Q. Zhou, "Agglomerative info-clustering," in *Proc. IEEE Int. Symp. Inf. Theory*, Vail, CO, 2018, pp. 1725–1729.

[55] M. Grötschel, L. Lovász, and A. Schrijver, "The ellipsoid method and its consequences in combinatorial optimization," *Combinatorica*, vol. 1, no. 2, pp. 169–197, Jun. 1981.

[56] L. Khachiyan, "Polynomial algorithms in linear programming," *USSR Comput. Math. Math. Physics*, vol. 20, no. 1, pp. 53 – 72, 1980.

[57] S. Iwata, "Submodular function minimization," *Math. Programming*, vol. 112, no. 1, pp. 45–64, Mar. 2007.

[58] A. Schrijver, "A combinatorial algorithm minimizing submodular functions in strongly polynomial time," *J. Combinatorial Theory Series B*, vol. 80, no. 2, pp. 346 – 355, Nov. 2000.

[59] J. B. Orlin, "A faster strongly polynomial time algorithm for submodular function minimization," *Math. Programming*, vol. 118, no. 2, pp. 237–251, May 2009.

[60] S. Fujishige and S. Isotani, "A submodular function minimization algorithm based on the minimum-norm base," *Pacific J. Optimization*, vol. 7, no. 1, pp. 3–17, 2011.

[61] P. Wolfe, "Finding the nearest point in a polytope," *Math. Programming*, vol. 11, no. 1, pp. 128–149, Dec. 1976.

[62] A. Krause, "SFO: A toolbox for submodular function optimization," *J. Mach. Learning Res.*, vol. 11, pp. 1141–1144, Jan. 2010.

[63] D. Chakrabarty, P. Jain, and P. Kothari, "Provable submodular minimization via Fujishige-Wolfe algorithm," *Proc. Advances in Neural Inf. Process. Syst.*, 2014.

[64] H. Y. Cheung, T. C. Kwok, and L. C. Lau, "Fast matrix rank algorithms and applications," *J. ACM*, vol. 60, no. 5, pp. 1–25, 2013.

[65] A. V. Goldberg and R. E. Tarjan, "A new approach to the maximum-flow problem," *J. ACM*, vol. 35, no. 4, pp. 921–940, Oct. 1988.

[66] V. Kolmogorov, "A faster algorithm for computing the principal sequence of partitions of a graph," *Algorithmica*, vol. 56, no. 4, pp. 394–412, Apr. 2010.