

Persistence of attractors for one-step discretization of ordinary differential equations

Lars Grüne
Fachbereich Mathematik
J.W. Goethe-Universität
Postfach 111932
60054 Frankfurt a.M., Germany
gruene@math.uni-frankfurt.de

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Abstract: We consider numerical one-step approximations of ordinary differential equations and present two results on the persistence of attractors appearing in the numerical system. First, we show that the upper limit of a sequence of numerical attractors for a sequence of vanishing time step is an attractor for the approximated system if and only if for all these time steps the numerical one-step schemes admit attracting sets which approximate this upper limit set and attract with a uniform rate. Second, we show that if these numerical attractors themselves attract with a uniformly rate, then they converge to some set if and only if this set is an attractor for the approximated system. In this case, we can also give an estimate for the rate of convergence depending on the rate of attraction and on the order of the numerical scheme.

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1 Introduction

Attractors play an important role in the understanding of the behaviour of complex dynamical systems. It is therefore important to know about the effects of discretization errors on attractors in order to give a reasonable interpretation to numerical experiments and to justify numerical findings, which often are the only way to gather knowledge about complicated systems as analytic solutions are rarely available.

For one-step discretizations of ordinary differential equations (ODEs) the basic result in that direction has been obtained by Kloeden and Lorenz in 1986 [11]. In this paper it is shown that if the ODE possesses an attractor then the numerical approximations possess absorbing sets nearby, which converge to the attractor as the time step tends to 0. Later, this result was established also for multi-step schemes by the same authors [12].

One of the main contributions of these papers is the observation that the right approximating set for the attractor of an ODE in general is not an attractor but an absorbing set, which in turn contains an attractor of the numerical approximation. This fact, however, implies a major problem when looking at attractors of the numerical approximation: Even the upper limit set for vanishing time step of these attractors can be strictly smaller than the attractor of the approximated system, i.e. we only obtain semi-continuous convergence (see e.g. [4, Example (2.12)] for an illustration and also [6] for related results for finite-dimensional approximations of infinite dimensional systems). In other words, the existence of “numerical attractors” does not imply the existence of a nearby attractor for the approximated ODE; one can only conclude that if the unperturbed system has an attractor A then there exist attractors for the numerical approximations whose upper limit is contained in A . Unfortunately, also the knowledge about the existence of the absorbing sets close to the original attractor does not help in general, since in this case for each time step there exist infinitely many absorbing sets, and thus it is difficult to single out those sets approximating the attractor.

In Chapter 7 of the monograph [14] these results are discussed in detail, furthermore in Section 7.7 of this reference several conditions for convergence are given. For example, it is shown that convergence holds if both the continuous and the numerical attractors attract exponentially (in this case also an estimate for the rate of convergence is given), or if the continuous time attractor consists of the unstable manifolds of finitely many hyperbolic equilibrium points (which is shown to be true for gradient systems with a bounded set of hyperbolic equilibria). These conditions have in common that certain assumptions on the dynamics of the approximated ODE are made.

In the present paper we take the converse point of view. We ask whether it is possible to obtain criteria for convergence just by looking at the numerical approximations (this approach is also taken in [9] for attracting sets of Galerkin approximations to Navier-Stokes equations using, however, rather different techniques). And in fact we can give a positive answer, at least under the assumption that we have information about the behaviour of the numerical approximations for arbitrarily small time steps. More precisely we prove that the upper limit of the numerical attractors for vanishing time step is a “true” attractor if and only if for arbitrarily small time steps the numerical one-step schemes admit attracting sets with uniform attraction rate approximating this upper limit set (the suitable concept of attraction rate is defined precisely in Section 2). Furthermore, we show that if the numerical attractors themselves are attracting with uniform rate, then they converge to some set A if and only if this set A is an attractor for the approximated ODE. In this case, we can also give an estimate for the rate of convergence.

It is clear that an assumption “for arbitrarily small time steps” will be hard to check rigorously in practice. Nevertheless, the results suggest the following procedure for numerical simulations: When an attractor is observed in a numerical system then redo the computation with different time steps and compare the rates of attraction. If these rates vary for different time steps then the observed attractor is likely to be a numerical artefact and the numerical results should be interpreted with care.

Concerning a rigorous verification, we expect that the relation of the assumed rates of attraction to Lyapunov functions (see Remark 2.6) might lead to practicable ways of checking

the desired property. (The details are currently under investigation and will be addressed in a later paper.) Furthermore, the assumptions are always satisfied in the presence of hyperbolicity, with the additional nice outcome that in this case the order of convergence of the numerical attractors coincides with the order of convergence of the numerical scheme, see Remark 2.9. In any case, the results precisely show what is theoretically possible, and thus contribute to the understanding and interpretation of numerical results.

This paper is organized as follows: In Section 2 we fix the setup and notation and state the main results. Section 3 provides some facts on attractors and attracting sets. In Section 4 two basic results on the stability of uniformly attracting sets are proved, and finally in Section 5 we prove the main results.

2 Setup and Main Results

We consider the ordinary differential equation in \mathbb{R}^d

$$\dot{x} = f(x) \tag{2.1}$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is assumed to satisfy $\|f(x)\| \leq M$ for all $x \in \mathbb{R}^d$ and $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^d$ and constants $M, L > 0$. (These global assumptions can easily be weakened since we are only interested in the behaviour on compact subsets of the state space.) The solutions of (2.1) with initial value $x_0 \in \mathbb{R}^d$ for initial time $t_0 = 0$ will be denoted by $\varphi(t, x_0)$.

In order to provide a framework for numerical one-step approximations of (2.1) we fix some $h_0 > 0$ and consider difference equations for time steps $h \in (0, h_0]$

$$x(t+h) = \Psi_h(x(t)) \tag{2.2}$$

where $t \in h\mathbb{Z} := \{hk \mid k \in \mathbb{Z}\}$, and $\Psi_h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies $\|x - \Psi_h(x)\| \leq hM$ and $\|\Psi_h(x) - \Psi_h(y)\| \leq (1 + hL)\|x - y\|$.

A special case of (2.2) is the time- h map of (2.1) given by

$$x(t+h) = \Phi_h(x(t)) := \varphi(h, x(t)). \tag{2.3}$$

Note that L and M from (2.1) need to be slightly enlarged in order to meet the assumptions on (2.2).

Our main object of interest are the numerical one-step approximations of (2.1) (or, more precisely, of (2.3)) which we will denote by

$$x(t+h) = \tilde{\Phi}_h(x(t)). \tag{2.4}$$

Here we assume that (2.4) is of type (2.2) and, in addition,

$$\|\tilde{\Phi}_h(x) - \Phi_h(x)\| \leq Kh^{p+1}$$

for some $p \in \mathbb{N}$ and some $K > 0$. The value p is called the order of the scheme. Typical examples of these schemes are Runge-Kutta and Taylor schemes, which are described in any textbook on numerical methods for ordinary differential equations, see e.g. [2, 8, 13].

Each of these equations defines a (semi-)dynamical system either in continuous or discrete time, which we denote by $\varphi(t, \cdot)$, $\Psi_h(t, \cdot)$, $\Phi_h(t, \cdot)$ and $\tilde{\Phi}_h(t, \cdot)$. In what follows we will sometimes write $\Phi(t, \cdot)$ which can be either a continuous or a discrete time system, and set $\mathbb{T} = \mathbb{R}$ for continuous time and $\mathbb{T} = h\mathbb{Z}$ for discrete time systems, the proper meaning being clear from the context. Furthermore, we abbreviate $\mathbb{T}^+ := \{t \in \mathbb{T} \mid t > 0\}$, and for subsets $B \subset \mathbb{R}^d$ we use the convention $\Phi(t, B) = \bigcup_{x \in B} \{\Phi(t, x)\}$.

As we are going to derive estimates for distances between compact sets we briefly recall basic definitions for several of these distances.

Definition 2.1 Let $C, D \subset \mathbb{R}^d$ be nonempty compact sets, $x \in \mathbb{R}^d$, and let d be the Euclidian metric on \mathbb{R}^d . We define the distance from a point to a set by

$$d(x, D) := \min_{y \in D} d(x, y),$$

the nonsymmetric Hausdorff distance between two compact sets by

$$\text{dist}(C, D) := \max_{x \in C} \min_{y \in D} d(x, y),$$

the Hausdorff metric for compact sets by

$$d_H(C, D) := \max\{\text{dist}(C, D), \text{dist}(D, C)\},$$

and, if $C \subseteq D$, the minimal distance by

$$d_{\min}(C, D) := \inf_{x \notin D} \min_{y \in C} d(x, y).$$

For $\varepsilon > 0$ we denote the ε -ball around C by $\mathcal{B}(\varepsilon, C) := \{y \in \mathbb{R}^d \mid d(y, C) < \varepsilon\}$. If $C = \{x\}$ we also write $\mathcal{B}(\varepsilon, x)$.

Now we can define our objects of interest.

Definition 2.2 Let $\Phi = \varphi$ or $\Phi = \Psi_h$ for some $h > 0$.

A compact set $B \subset \mathbb{R}^d$ is called *forward invariant*, if $\Phi(t, B) \subseteq B$ for all $t \in \mathbb{T}^+$ and *invariant*, if $\Phi(t, B) = B$ for all $t \in \mathbb{T}^+$.

Given two compact forward invariant sets $A, B \subset \mathbb{R}^d$ with $A \subset \text{int}B$ we call A *attracting* with *attracted neighbourhood* B , if

$$\text{dist}(\Phi(t, B), A) \rightarrow 0$$

as $t \rightarrow \infty$.

A is called a *attractor* (with attracted neighbourhood B) if it is invariant and attracting with attracted neighbourhood B .

Note that A is a local attractor here, i.e. it is not assumed that each compact set $B \subset \mathbb{R}^d$ is an attracted neighbourhood.

The following definition, which is a slight variation of [5, Definition 2.8], is a tool to give some structure to the attraction property.

Definition 2.3 Let $\Phi = \varphi$ or $\Phi = \Psi_h$ for some $h > 0$, and consider two compact, forward invariant sets $A, B \subset \mathbb{R}^d$ with $A \subset \text{int}B$, where A is attracting with attracted neighbourhood B .

A family of compact, forward invariant sets B_ϑ , $\vartheta \in \mathbb{R}_0^+$, which depend continuously on ϑ (w.r.t. the Hausdorff metric d_H) and satisfy $B \subseteq B_0$ is called a *contracting family of neighbourhoods* if

- (i) $B_{\vartheta'} \subseteq B_\vartheta$ for all $\vartheta, \vartheta' \in \mathbb{R}_0^+$, $\vartheta' \geq \vartheta$
- (ii) $A = \bigcap_{\vartheta \in \mathbb{R}_0^+} B_\vartheta$
- (iii) $\Phi(t, B_\vartheta) \subseteq B_{\vartheta+t}$ for all $\vartheta \in \mathbb{R}_0^+$ and all $t \in \mathbb{T}^+$.

The next definition introduces a measure for the rate of convergence of B_ϑ to A as $\vartheta \rightarrow \infty$. As usual, we call a continuous function $\beta : [0, \infty) \rightarrow [0, \infty)$ of class \mathcal{K} , if it is monotone increasing and satisfies $\beta(0) = 0$.

Definition 2.4 Consider a contracting family of neighbourhoods B_ϑ and some class \mathcal{K} function β . Then B_ϑ is called *β -shrinking*, if

$$d_H(B_\vartheta, A) \leq \beta(1/\vartheta)$$

for all $\vartheta > 0$.

Throughout this paper the β -shrinking property will be used to give a uniformity condition for contracting families.

Next we give the property which will turn out to be crucial for estimating rates of convergence. Again, this is a slight variation of a concept from [5], namely of Definition 4.1 in this reference.

Definition 2.5 Let $\Phi = \varphi$ or $\Phi = \Psi_h$ for some $h > 0$, and let γ be a class \mathcal{K} function. We say that an attracting set A with attracted neighbourhood B is γ -attracting if it admits a contracting family of neighbourhoods B_ϑ , $\vartheta \geq 0$, which for all $\vartheta \geq 0$ and all $t \in [0, T_\vartheta] \cap \mathbb{T}$ for some $T_\vartheta \in \mathbb{T}^+$ satisfies the inequality

$$d_{\min}(\Phi(t, B_\vartheta), B_\vartheta) \geq t\gamma^{-1}(d_H(B_\vartheta, A)),$$

and we call it (γ, β) -attracting if this contracting family can be chosen to be β -shrinking. If, in addition, A is an attractor then we call A a γ -attractor or a (γ, β) -attractor, respectively.

Remark 2.6 (i) *Each attracting set is (γ, β) -contracting for suitable γ, β of class \mathcal{K} , which can be seen e.g. by taking the B_ϑ as suitably parameterized sublevel sets of Lyapunov functions V provided by [14, Theorem 1.7.6 and 2.7.6] (the construction*

in [14] actually goes back to [3] and [15, Theorem 22.1] and was also used in [11]): Since $V(\Phi(t, x)) \leq e^{-t}V(x)$ we can pick some $x_0 \in B$, $x_0 \notin A$ and set $B_\vartheta = \{x \in \mathbb{R}^d \mid V(x) \leq e^{-\vartheta}V(x_0)\}$. Then the bounds on V and the Lipschitz continuity allow an explicit computation of γ and β . It is, however, an open question whether we can always find a Lyapunov function whose sublevel sets realize the optimal rates γ and β .

- (ii) In the case of exponential attraction, i.e. when there exists constants $C, \lambda > 0$ with $d(\Phi(t, x), A) \leq Ce^{-\lambda t}d(x, A)$, we can choose $\gamma(s) = cs$ for some suitable $c > 0$ which follows from the fact that for each $\delta > 0$ with $e^{\delta t_0}Ce^{-\lambda t_0} < 1$ for some $t_0 > 0$ the function

$$V(x) := \sup_{t \geq 0} e^{\delta t}d(\Phi(t, x), A) = \sup_{t \in [0, t_0]} e^{\delta t}d(\Phi(t, x), A)$$

is easily verified as a Lipschitz continuous Lyapunov function (cf. [15, Theorem 19.2]) satisfying $V(\Phi(t, x)) \leq e^{-\delta t}V(x)$. The construction from (i) then gives the appropriate sublevel sets. (For a direct construction of a contracting family of neighbourhoods in the exponential case we refer to [5, Section 7].)

As an illustration of this concept consider the simple two-dimensional system

$$\dot{x} = -x, \quad \dot{y} = -y$$

possessing $A = \{(0, 0)^T\}$ as exponential attractor. Then for each $\delta \in (0, 1]$ the family of neighbourhoods

$$B_\vartheta = [-e^{-\vartheta}, e^{-\vartheta}] \times [-e^{-\delta\vartheta}, e^{-\delta\vartheta}]$$

form a contracting family with $\beta(1/\vartheta) = e^{-\delta\vartheta}$. The function γ , however, has to be chosen as $\gamma(s) \geq s^\delta$. Thus exponential shrinking of the B_ϑ does not necessarily imply $\gamma(s) = cs$, however, a suitable choice of this family (i.e. $\delta = 1$) guarantees linearity of γ . An interesting (but to the author's knowledge unsolved) question in this context is whether for a given system one can always obtain a relation between β and γ by choosing a suitable family B_ϑ .

Using the concept of (γ, β) -attraction we can now formulate our main results.

Theorem 2.7 Consider a positive sequence of time steps $h_n \rightarrow 0$ as $n \rightarrow \infty$. Let A_{h_n} be attractors for the numerical scheme $\tilde{\Phi}_{h_n}$ with attracted neighbourhood B and consider the upper limit set

$$A := \bigcap_{N > 0} \overline{\bigcup_{n \geq N} A_{h_n}}.$$

Then A is an attractor for the continuous time system (2.1) if and only if there exist functions γ, β of class \mathcal{K} and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exist (γ, β) -attracting sets $\tilde{A}_{h_n} \supseteq A_{h_n}$ for $\tilde{\Phi}_{h_n}$ with $d_H(\tilde{A}_{h_n}, \overline{\bigcup_{n \geq N} A_{h_n}}) \rightarrow 0$ as $N \rightarrow \infty$.

In this case A is a $(2\gamma(5 \cdot), 2\beta(4 \cdot))$ -attractor.

The proof is given in Section 5.

In other words, the upper limit A is an attractor if and only if we find uniformly attracting sets (i.e. all with the same rates γ and β) for vanishing time step which approximate A .

Note that it is not necessary that the attractors A_{h_n} themselves are (γ, β) -attracting. In this case, however, further implications are possible, as formulated in the following theorem.

Theorem 2.8 Consider a positive sequence $h_n \rightarrow 0$ as $n \rightarrow \infty$ and two functions γ, β of class \mathcal{K} . Assume there exist (γ, β) -attractors A_{h_n} with attracted neighbourhood B for the numerical scheme $\tilde{\Phi}_{h_n}$ and let $A \subset \text{int}B$ be a compact set. Then the following four statements are equivalent.

- i) A is an attractor with attracted neighbourhood B for the continuous time system (2.1).
- ii) A is a $(2\gamma(5\cdot), 2\beta(4\cdot))$ -attractor with attracted neighbourhood B for the continuous time system (2.1).
- iii) $d_H(A, A_{h_n}) \rightarrow 0$ as $n \rightarrow \infty$.
- iv) $d_H(A, A_{h_n}) \leq 2\gamma(10Kh_n^p)$ for all h_n sufficiently small.

The proof can also be found in Section 5.

Remark 2.9 (i) *This theorem generalizes [14, Theorem 7.7.1] to arbitrary rates of attraction and gives a better rate of convergence in the case of exponential attraction, where we have $\gamma(s) = cs$ for some $c > 1$, cf. Remark 2.6 (ii).*

(ii) *If the system (2.1) is uniformly hyperbolic in a neighbourhood of A , then also the one-step approximations are uniformly hyperbolic for $h > 0$ sufficiently small. In this case the attractors attract exponentially (see [7]), i.e. we obtain $\gamma(s) = cs$ from Remark 2.6 (ii). Thus hyperbolicity implies*

$$d_H(A_h, A) \leq CKh^p$$

for some suitable $C > 0$ without any additional assumptions on A and A_h .

3 Some facts on attractors and attracting sets

For the convenience of the reader in this section we provide some facts about attractors and attracting sets for the system and its time- h map.

It is well known that each forward invariant attracting set contains an attractor (see e.g. [14, Theorem 2.7.4(iii)], observing that our attracting sets are uniformly asymptotically stable in the sense of [14, Definition 2.7.3]). In the next Lemma we closer investigate the relation between these two concepts.

Lemma 3.1 Let $\Phi = \varphi$ or $\Phi = \Psi_h$ for some $h > 0$. Then a compact forward invariant attracting set A for Φ with attracted neighbourhood B is an attractor with attracted neighbourhood B if and only if it is the minimal compact forward invariant attracting set (w.r.t. set inclusion) with attracted neighbourhood B . In particular for each compact set $B \subset \mathbb{R}^d$ there exists at most one attractor with attracted neighbourhood B .

Proof: Let A be an attractor with attracted neighbourhood B . Then in particular A is invariant. Now assume that $\tilde{A} \subset A$, $\tilde{A} \neq A$, is a forward invariant attracting set. Then there exists a neighbourhood $\mathcal{N} \supset \tilde{A}$ with $A \not\subseteq \mathcal{N}$, such that $\Phi(t, B) \subset \mathcal{N}$ for some $t \in \mathbb{T}^+$, i.e. in particular $\Phi(t, A) \neq A$ which contradicts the invariance of A .

Let conversely A be a minimal forward invariant attracting set. Then A contains an attractor which again is a forward invariant attracting set. Hence by minimality it coincides with A . \square

The next Lemma shows that the attractor is also the maximal compact invariant set contained in $\text{int}B$.

Lemma 3.2 Let $\Phi = \varphi$ or $\Phi = \Psi_h$ for some $h > 0$ and let A be an attractor with attracted neighbourhood B for Φ . Then each compact invariant set $D \subset \text{int}B$ is contained in A .

Proof: Let $D \subset \text{int}B$ be an invariant set. Then $D = \Phi(t, D) \subset \Phi(t, B)$ for all $t \in \mathbb{T}^+$. On the other hand, for each neighbourhood $\mathcal{N} \supset A$ we know that $\Phi(t, B) \subset \mathcal{N}$ for all $t \in \mathbb{T}^+$ sufficiently large. Hence $D \subset \mathcal{N}$ which implies the assertion. \square

In the next two lemmas we investigate the relation between attracting sets and attractors for the continuous time system and its time- h map.

Lemma 3.3 Consider system (2.1). Then a forward invariant set A is an attracting set with attracted neighbourhood B if and only if there exists $T > 0$ such that

$$\lim_{i \rightarrow \infty, i \in \mathbb{N}} \text{dist}(\varphi(iT, B), A) = 0. \quad (3.1)$$

Proof: Obviously, if A is attracting then (3.1) holds for all $T > 0$. Now let conversely (3.1) hold for some $T > 0$. Then forward invariance of A and continuous dependence on the initial value imply that for each $\delta > 0$ there exists $\varepsilon > 0$ with

$$d_H(D, A) < \varepsilon \quad \Rightarrow \quad d_H(\varphi(t, D), A) < \delta$$

for all $t \in [0, T]$. Thus the assumption implies $\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, B), A) = 0$, hence A is an attracting set. \square

Lemma 3.4 Let $h > 0$ and A_h be an attractor with attracted neighbourhood B for the time- h map Φ_h of the continuous time system (2.1). Then A_h is also an attractor with attracted neighbourhood B for system (2.1).

Proof: We first show that $\varphi(t, A_h) \subseteq A_h$ for each $t \in \mathbb{R}$. By invariance of A_h for Φ_h we know $\Phi_h(\varphi(t, A_h)) = \varphi(t, \Phi_h(A_h)) = \varphi(t, A_h)$, hence $\varphi(t, A_h)$ is invariant for Φ_h , and by Lemma 3.2 it is contained in A_h .

Thus we can conclude $\varphi(t, A_h) \subseteq A_h$ for each $t \in \mathbb{R}$, hence also $A_h = \varphi(-t, \varphi(t, A_h)) \subseteq \varphi(-t, A_h)$ for each $t \in \mathbb{R}$ and consequently A_h is invariant for φ .

Finally, since A_h is invariant for φ and an attracting set for Φ_h , by Lemma 3.3 it is also an attracting set for φ with attracted neighbourhood B , thus an attractor. \square

4 Stability of attracting sets

In this section we provide two stability results for (γ, β) -attracting sets in the Propositions 4.2 and 4.6. The first considers stability under perturbation of a difference equation, the second stability when passing to a limit of time- h maps as $h \rightarrow 0$.

Before formulating these result we introduce two auxiliary systems which will be useful for the proofs. Consider the differential equation (2.1) and the difference equation (2.2). For these equations we consider the α -perturbed systems

$$\dot{x}(t) = f(x(t)) + \alpha u(t), \quad x(t+h) = \Psi_h(x(t)) + \alpha h u_h(t)$$

with solutions $\varphi^\alpha(t, x, u(\cdot))$ and $\Psi_h^\alpha(t, x, u_h(\cdot))$, where $u(\cdot) \in \mathcal{U}$ and $u_h(\cdot) \in \mathcal{U}_h$ with $\mathcal{U} := \{u : \mathbb{R} \rightarrow \mathcal{B}(1, 0) \mid u \text{ measurable}\}$ and $\mathcal{U}_h := \{u_h : h\mathbb{Z} \rightarrow \mathcal{B}(1, 0)\}$. (Recall that $\mathcal{B}(1, 0)$ is the ball with radius 1 around the origin in \mathbb{R}^d). The set valued maps

$$\varphi^{\text{infl}}(t, x, \alpha) := \bigcup_{u(\cdot) \in \mathcal{U}} \varphi^\alpha(t, x, u(\cdot)), \quad \Psi_h^{\text{infl}}(t, x, \alpha) := \bigcup_{u_h(\cdot) \in \mathcal{U}_h} \Psi_h^\alpha(t, x, u_h(\cdot))$$

are called the α -inflated dynamics (they can alternatively be defined via differential or difference inclusions, cf. [10]). Again for $B \subset \mathbb{R}^d$ we define $\varphi^{\text{infl}}(t, B, \alpha) := \bigcup_{x \in B} \varphi^{\text{infl}}(t, x, \alpha)$, and analogously for Ψ_h^{infl} . The following elementary Lemma follows e.g. from [5, Lemma 10.1 and 10.2] for the continuous time case and is easily extended also to the discrete time case.

Lemma 4.1 Let $B \subset \mathbb{R}^d$ be a compact set and $t \in \mathbb{T}^+$. Then the inclusions

$$\mathcal{B}(\varphi(t, B), \alpha t / (Lt + 1)) \subseteq \varphi^{\text{infl}}(t, B, \alpha) \subseteq \mathcal{B}(\varphi(t, B), \alpha t e^{Lt})$$

and

$$\mathcal{B}(\Psi_h(t, B), \alpha t / (Lt + 1)) \subseteq \Psi_h^{\text{infl}}(t, B, \alpha) \subseteq \mathcal{B}(\Psi_h(t, B), \alpha t e^{Lt})$$

hold for the constant L from the Lipschitz estimates for (2.1) and (2.2), respectively.

We now turn to a more specific perturbation of difference equations which covers our numerical one-step approximations. For the difference equation (2.2) given by Ψ_h we consider the perturbed system $\tilde{\Psi}_{2h}$ with doubled time step $2h$

$$x(t+2h) = \tilde{\Psi}_{2h}(x(t)) \tag{4.1}$$

In order to estimate the difference between Ψ_h and $\tilde{\Psi}_{2h}$ we make the following assumption on the perturbed system

$$\sup_{x \in \mathbb{R}^d} \|\Psi_h(2h, x) - \tilde{\Psi}_{2h}(2h, x)\| \leq h\alpha \tag{4.2}$$

for some $\alpha > 0$.

Note that we do not make any regularity assumptions on $\tilde{\Psi}$. The doubled time step $2h$ is necessary in order to make a connection between attracting sets of Ψ_h and $\tilde{\Psi}_{2h}$, which will become clear in the proof of the following proposition. This provides a stability result for (γ, β) -attracting sets under perturbations of type (4.1).

Proposition 4.2 Suppose for some $h \in (0, h_0]$ that A_h is a (γ, β) -attracting set with attracted neighbourhood B for system (2.2). Then for each system $\tilde{\Psi}_{2h}$ satisfying (4.2) for some $\alpha \in (0, \alpha_0]$, $\alpha_0 = \gamma^{-1}(d_H(B, A_h))/2$, there exists a $(\gamma(4\cdot), \beta(2\cdot))$ -attracting set A_{2h}^α with attracted neighbourhood B and

$$d_H(A_{2h}^\alpha, A_h) \leq \gamma(2\alpha).$$

In particular the constant α_0 only depends on γ and $d_H(B, A_h)$ but not on h .

Proof: Consider the contracting family of neighbourhoods B_ϑ realizing the assumed attraction rate and fix some $\alpha \in (0, \alpha_0]$. We choose ϑ_α minimal such that $d_H(B_{\vartheta_\alpha}, A_h) = \gamma(2\alpha)$ which implies $d_H(B_\vartheta, A_h) \geq \gamma(2\alpha)$ for all $\vartheta \in [0, \vartheta_\alpha]$. Now we set

$$A_{2h}^\alpha = B_{\vartheta_\alpha}$$

and define a family of neighbourhoods B_ϑ^{2h} of A_{2h}^α by

$$\begin{aligned} B_\vartheta^\alpha &= B_{\vartheta/2} & \vartheta &\leq 2\vartheta_\alpha \\ B_\vartheta^\alpha &= B_{\vartheta_\alpha} & \vartheta &\geq 2\vartheta_\alpha \end{aligned}$$

The continuity and $\beta(2\cdot)$ -shrinking w.r.t. A_{2h}^α of this family follow immediately from the construction. Furthermore,

$$\begin{aligned} d_{\min}(\tilde{\Psi}_{2h}(B_\vartheta^\alpha), B_\vartheta^\alpha) &\geq h\gamma^{-1}(d_H(B_\vartheta^\alpha, A_h)) - h\alpha \\ &\geq h\gamma^{-1}(d_H(B_\vartheta^\alpha, A_h))/2 \geq h\gamma^{-1}(d_H(B_\vartheta^\alpha, A_h^\alpha))/2 \end{aligned}$$

which yields the desired estimate for the γ -contraction the B_ϑ^α . In addition, for each $\vartheta \geq 0$ with $\vartheta \leq 2\vartheta_\alpha - 2h$ the construction implies

$$\begin{aligned} d_{\min}(\tilde{\Psi}_{2h}(B_\vartheta^\alpha), B_{\vartheta+2h}^\alpha) &\geq d_{\min}(\Psi_h(2h, B_{\vartheta/2}), B_{\vartheta/2+h}) - h\alpha \\ &\geq h\gamma^{-1}(d_H(B_{\vartheta/2+h}, A_h)) - h\alpha > 0 \end{aligned}$$

which implies $\tilde{\Psi}_{2h}(B_\vartheta^\alpha) \subset B_{\vartheta+2h}^\alpha$.

For $\vartheta \geq 2\vartheta_\alpha - 2h$ we have $B_{\vartheta+2h}^\alpha = B_{2\vartheta_\alpha}^\alpha$, hence we obtain

$$\tilde{\Psi}_{2h}(B_\vartheta^\alpha) \subseteq \tilde{\Psi}_{2h}(B_{2\vartheta_\alpha-2h}^\alpha) \subseteq B_{2\vartheta_\alpha}^\alpha = B_{\vartheta+2h}^\alpha$$

which shows that the B_ϑ^α form a contracting family and thus finishes the proof. \square

We now turn to the question of (α, β) -attraction for the continuous time system (2.1) for limits of (α, β) -attracting sets for its time- h map (2.3) as $h \rightarrow 0$. Before proving the desired result in Proposition 4.6 we provide a series of lemmas necessary for the proof.

Lemma 4.3 Consider a compact set B satisfying $d_{\min}(\varphi(t, B), B) \geq Ct$ for all $t \in [0, T]$ and some $T > 0$. Then for each $t \in \mathbb{R}^+$ the inequality $d_{\min}(\varphi(t, B), B) \geq tC/(Lt + 1)$ holds for the Lipschitz constant L of (2.1).

Proof: From the assumption and Lemma 4.1 for each $\varepsilon > 0$ we can conclude $\varphi^{\text{infl}}(t, B, C - \varepsilon) \subset B$ for all $t \in [0, T_\varepsilon]$ for some $T_\varepsilon > 0$ sufficiently small. Since $\varphi^{\text{infl}}(t + t_1, B, C - \varepsilon) = \varphi^{\text{infl}}(t, \varphi^{\text{infl}}(t_1, B, C - \varepsilon), C - \varepsilon)$ this inclusion in fact holds for all $t > 0$. Thus again by Lemma 4.1 this yields $d_{\min}(\varphi(t, B), B) \geq (C - \varepsilon)t/(Lt + 1)$ implying the assertion since $\varepsilon > 0$ was arbitrary. \square

Lemma 4.4 Let A_h be a (γ, β) -attracting set for the time- h map (2.3) for some $h > 0$ and attracted neighbourhood B . Then the set A_h admits a $(e^{Lh}\gamma(e^{Lh}\cdot), e^{Lh}\beta(2\cdot))$ -contracting family of neighbourhoods B_ϑ for the time- $2h$ map satisfying $B \subset B_0$ and

$$\varphi^{\text{infl}}(t, B_\vartheta, e^{-Lh}\gamma^{-1}(e^{-Lh}d_H(B_\vartheta, A))) \subseteq \tilde{B}_\vartheta$$

for all $\vartheta, t \in \mathbb{R}_0^+$ where L is the Lipschitz constant for system (2.1).

Proof: Consider the contracting family \tilde{B}_ϑ for A_h realizing the assumed rate of contraction. Then Lemma 4.1 implies $\varphi^{\text{infl}}(h, \tilde{B}_\vartheta, e^{-Lh}\gamma^{-1}(d_H(\tilde{B}_\vartheta, A))) \subseteq \tilde{B}_\vartheta$. Hence defining

$$B_{2\vartheta} := \bigcup_{t \geq 0} \varphi^{\text{infl}}(t, \tilde{B}_\vartheta, e^{-Lh}\gamma^{-1}(d_H(\tilde{B}_\vartheta, A)))$$

we obtain

$$B_{2\vartheta} = \bigcup_{t \in [0, h]} \varphi^{\text{infl}}(t, \tilde{B}_\vartheta, e^{-Lh}\gamma^{-1}(d_H(\tilde{B}_\vartheta, A))).$$

By construction this yields

$$\varphi^{\text{infl}}(t, B_{2\vartheta}, e^{-Lh}\gamma^{-1}(d_H(\tilde{B}_\vartheta, A))) \subseteq B_{2\vartheta} \text{ and } \varphi(h, B_{2\vartheta}) \subseteq \tilde{B}_\vartheta,$$

furthermore Lemma 4.1 implies $d_H(B_{2\vartheta}, A) \leq e^{Lh}d_H(\tilde{B}_\vartheta, A)$. Finally, since

$$\varphi(2h, B_\vartheta) = \varphi(h, \underbrace{\varphi(h, B_\vartheta)}_{\subseteq \tilde{B}_{\vartheta/2}}) \subseteq \tilde{B}_{\vartheta/2+h} \subseteq B_{\vartheta+2h},$$

these sets indeed form a contracting family of neighbourhoods for the time- $2h$ map. \square

We will make use of the following concept: Consider a family of compact sets $B_i \subset \mathbb{R}^d$. Then we define the limes inferior of these sets by

$$\text{Liminf}_{i \rightarrow \infty} B_i := \{x \in \mathbb{R}^d \mid \lim_{i \rightarrow \infty} d(x, B_i) = 0\}.$$

We refer to [1] for more information on this concept. The following Lemma summarizes those properties of this set which we will need in what follows.

Lemma 4.5 Consider two families of compact sets $B_i, C_i \subset \mathbb{R}^d$ and let $B = \text{Liminf}_{i \rightarrow \infty} B_i$ and $C = \text{Liminf}_{i \rightarrow \infty} C_i$. Then the following assertions hold

- (i) If there exists $N > 0$ with $B_i \subseteq C_i$ for all $i \geq N$, then $B \subseteq C$.
- (ii) For all $\alpha \geq 0$ and all $u(\cdot) \in \mathcal{U}$ we have $\varphi^\alpha(t, B, u(\cdot)) \subseteq \text{Liminf}_{i \rightarrow \infty} \varphi^\alpha(t, B_i, u(\cdot))$.

- (iii) If for some $\alpha \geq 0$ there exists $N > 0$ with $\varphi^{\text{infl}}(t, B_i, \alpha) \subseteq C_i$ for all $i \geq N$, then $\varphi^{\text{infl}}(t, B, \alpha) \subseteq C$.
- (iv) If there exists $N > 0$ with $B_i \subseteq C_i$ for all $i \geq N$, and furthermore $d_H(B_i, B) \rightarrow 0$ as $i \rightarrow \infty$, then $d_H(B, C) \leq \liminf_{i \rightarrow \infty} d_H(B_i, C_i)$.

Proof: Assertion (i) follows directly from the definition, (ii) follows from [1, Proposition 1.2.2(ii)] since $\varphi^\alpha(t, \cdot, u(\cdot))$ is continuous.

In order to prove (iii) let $x \in \varphi^{\text{infl}}(t, B, \alpha)$. Then there exist $y \in B$ and $u(\cdot) \in \mathcal{U}$ such that $x = \varphi^\alpha(t, y, u(\cdot))$. Hence using first (ii) and then the assumption and (i) we obtain

$$x \in \varphi^\alpha(t, B, u(\cdot)) \subseteq \text{Liminf}_{i \rightarrow \infty} \underbrace{\varphi^\alpha(t, B_i, u(\cdot))}_{\subseteq C_i} \subseteq C$$

and hence the assertion.

For the proof of (iv) observe that by (i) the assumption implies $B \subseteq C$. Hence we obtain the equalities

$$d_H(B, C) = \sup_{x \in C} d(x, B), \quad d_H(B_i, C_i) = \sup_{x \in C_i} d(x, B_i)$$

for all $i \in \mathbb{N}$ sufficiently large and

$$d(x, B) \leq \liminf_{i \rightarrow \infty} \left(d(x, x_i) + d(x_i, B_i) + d_H(B, B_i) \right) = \liminf_{i \rightarrow \infty} d(x_i, B_i)$$

for each point $x \in \mathbb{R}^d$ and any sequence x_i with $x_i \rightarrow x$ as $i \rightarrow \infty$. Now let $x \in C$. Then by the definition of Liminf there exists a sequence $x_i \in C_i$ with $x_i \rightarrow x$ as $i \rightarrow \infty$. Thus

$$d(x, B) \leq \liminf_{i \rightarrow \infty} d(x_i, B_i) \leq \liminf_{i \rightarrow \infty} d_H(B_i, C_i).$$

Since x was arbitrary in C this implies the assertion. \square

Now we have all necessary tools for proving the second stability result.

Proposition 4.6 Consider a sequence $h_n \rightarrow 0$, and assume that there exist functions γ, β of class \mathcal{K} and (γ, β) -attracting sets with attracted neighbourhood B for the time- h_n map Φ_{h_n} of the continuous time system (2.1). Assume, furthermore, that there exists a set A with $d_H(A_{h_n}, A) \rightarrow 0$. Then for each $C > 1$ the set A is $(C\gamma(C \cdot), C\beta(2 \cdot))$ -attracting for the continuous time system (2.1).

Proof: Fix some $C > 1$ and let $\tilde{C} \in (1, C)$. Assuming all h_n to be sufficiently small and to be monotone decreasing (by taking an appropriate subsequence) by Lemma 4.4 we know that the A_{h_n} are $(\tilde{C}\gamma(\tilde{C} \cdot), \tilde{C}\beta(2 \cdot))$ attracting for Φ_{2h_n} , with contracting families B_ϑ^n satisfying

$$\varphi^{\text{infl}}(t, B_\vartheta^n, \gamma^{-1}(d_H(B_\vartheta^n, A_{h_n})/\tilde{C})/\tilde{C}) \subseteq B_\vartheta^n. \quad (4.3)$$

We claim that the family of sets

$$B_\vartheta := \text{Liminf}_{i \rightarrow \infty} B_\vartheta^i$$

gives the desired contracting family of neighbourhoods.

Property (i) and (ii) of Definition 2.3 are immediate from Lemma 4.5 (i). We now prove continuity of the family, i.e. $d_H(B_\vartheta, B_{\vartheta'}) \rightarrow 0$ as $\vartheta' \rightarrow \vartheta$: Consider two values $0 \leq \vartheta_1 \leq \vartheta_2$. Then the assumption implies $\Phi_{2h_n}(t, B_{\vartheta_1}^n) \subseteq B_{\vartheta_2}^n$ for each $t \in 2h_n\mathbb{Z}$, $t \geq \vartheta_2 - \vartheta_1$, and consequently for each $\varepsilon > 0$ and each n with $2h_n \leq \varepsilon$ the inclusion $\varphi(\vartheta_2 - \vartheta_1 + \varepsilon, B_{\vartheta_1}^n) \subseteq B_{\vartheta_2}^n$ holds. Thus by Lemma 4.5 (ii) and (i)

$$\varphi(\vartheta_2 - \vartheta_1 + \varepsilon, B_{\vartheta_1}) \subseteq \text{Liminf}_{n \rightarrow \infty} \varphi(\vartheta_2 - \vartheta_1 + \varepsilon, B_{\vartheta_1}^n) \subseteq \text{Liminf}_{n \rightarrow \infty} B_{\vartheta_2}^n = B_{\vartheta_2}.$$

Since $\varepsilon > 0$ was arbitrary this yields $d_H(B_{\vartheta_1}, B_{\vartheta_2}) \leq M(\vartheta_2 - \vartheta_1)$ implying continuity.

The same construction shows Property (iii) of Definition 2.3, and the desired distance from A follows from Lemma 4.5 (iv).

Finally, from (4.3) and Lemma 4.5 (iii) we can conclude

$$\varphi^{\text{infl}}(t, B_\vartheta, \gamma^{-1}(d_H(B_\vartheta^n, A)/\tilde{C})/\tilde{C}) \subseteq B_\vartheta$$

which by Lemma 4.1 gives the desired contraction for any $T \leq (C - \tilde{C})/(L\tilde{C})$. \square

5 Proofs of Theorem 2.7 and Theorem 2.8

Proof of Theorem 2.7: Throughout the proof we fix the attracted neighbourhood B , i.e. “attracting” is to be understood as “attracting with attracted neighbourhood B ”.

Let A be an attractor. Then by Remark 2.6 there exist functions $\tilde{\gamma}, \tilde{\beta}$ such that A is $(\tilde{\gamma}, \tilde{\beta})$ -attracting for the continuous time system, thus also for each time- $h_n/2$ map. Hence by Proposition 4.2 with $h = h_n/2$, $A_h = A$, $\Psi_h = \Phi_h$, $\tilde{\Psi}_{2h} = \tilde{\Phi}_{2h}$ and $\alpha = Kh^p$, for $\gamma(r) = \tilde{\gamma}(4r)$, $\beta(r) = \tilde{\beta}(2r)$ and all sufficiently small $h_n > 0$ there exist (γ, β) -attracting sets \tilde{A}_{h_n} for the one-step scheme converging to A as $h_n \rightarrow 0$. By Lemma 3.1 these sets contain the attractors A_{h_n} .

Let conversely \tilde{A}_{h_n} be sets as in the assertion of the theorem. Then the assumption on (2.4) implies

$$\|\Phi_{h_n}(2h_n, x) - \tilde{\Phi}_{h_n}(2h_n, x)\| \leq (1 + e^{Lh_n})Kh_n^{p+1}.$$

Thus by Proposition 4.2 with $h = h_n$, $A_h = \tilde{A}_h$, $\Psi_h = \tilde{\Phi}_h$, $\tilde{\Psi}_{2h} = \Phi_{2h}$ and $\alpha = (1 + e^{Lh_n})Kh_n^p$, for each h_n sufficiently small there exist $(\gamma(4\cdot), \beta(2\cdot))$ -attracting sets A'_{2h_n} for the time- $2h_n$ maps Φ_{2h_n} converging to A . By Proposition 4.6 (applied with $C = 5/4$) this implies that A is a $(2\gamma(5\cdot), 2\beta(4\cdot))$ -attracting set. It remains to show that A is an attractor.

By Lemma 3.1 there exists an attractor $\bar{A} \subseteq A$. Assume $\bar{A} \neq A$, then there exists a compact neighbourhood $\mathcal{N} \subset \mathbb{R}^d$ with $\bar{A} \subset \text{int}\mathcal{N}$ and $A \not\subset \mathcal{N}$. As in the first part of the proof, by Remark 2.6 and Proposition 4.2 we can conclude the existence of attracting sets \bar{A}_h for $\tilde{\Phi}_h$ converging to \bar{A} . In particular this yields $\bar{A}_h \subset \mathcal{N}$ for $h > 0$ sufficiently small. Since the A_{h_n} are attractors for $\tilde{\Phi}_{h_n}$, by Lemma 3.1 they must be contained in the \bar{A}_{h_n} , which implies $A_{h_n} \subset \mathcal{N}$, and hence

$$\overline{\bigcup_{n \geq N} A_{h_n}} \subset \mathcal{N}$$

for all $N \in \mathbb{N}$ sufficiently large implying

$$A \not\subseteq \overline{\bigcup_{n \geq N} A_{h_n}}$$

which contradicts the definition of A . \square

Proof of Theorem 2.8: As in the proof of Theorem 2.7 we fix the attracted neighbourhood B . Now “(iv) \Rightarrow (iii)” and “(ii) \Rightarrow (i)” are obvious and “(iii) \Rightarrow (ii)” follows immediately from Theorem 2.7.

We now show “(ii) \Rightarrow (iv)”:

Since A is also a $(2\gamma(5 \cdot), 2\beta(4 \cdot))$ -attractor for the time $h_n/2$ -map, by Proposition 4.2 with $h = h_n/2$, $A_h = A$, $\Psi_h = \Phi_h$, $\tilde{\Psi}_{2h} = \tilde{\Phi}_{2h}$ and $\alpha = Kh_n^p$, we know the existence of $(2\gamma(20 \cdot), 2\beta(8 \cdot))$ -attracting sets \tilde{A}_{h_n} for the one-step scheme with

$$d_H(A, \tilde{A}_{h_n}) \leq 2\gamma(10Kh_n^p),$$

and by Lemma 3.1 we know $A_{h_n} \subseteq \tilde{A}_{h_n}$, hence

$$\text{dist}(A_{h_n}, A) \leq 2\gamma(10Kh_n^p).$$

Conversely, the assumption on (2.4) implies

$$\|\Phi_{h_n}(2h_n, x) - \tilde{\Phi}_{h_n}(2h_n, x)\| \leq (1 + e^{Lh_n})Kh_n^{p+1}.$$

Hence by the assumption on the A_{h_n} and by Proposition 4.2 with $h = h_n$, $\Psi_h = \tilde{\Phi}_h$, $\tilde{\Psi}_{2h} = \Phi_{2h}$ and $\alpha = (1 + e^{Lh_n})Kh_n^p$ we can conclude the existence of attracting sets \bar{A}_{2h_n} for the time- $2h_n$ map satisfying

$$d_h(\bar{A}_{2h_n}, A_{h_n}) \leq \gamma(10Kh_n^p)$$

for all h_n sufficiently small. By Lemma 3.1 and Lemma 3.4 we know $A \subset \bar{A}_{2h_n}$, hence

$$\text{dist}(A, A_{h_n}) \leq \gamma(10Kh_n^p)$$

which shows (iv).

Finally, we show “(i) \Rightarrow (iii)”, which finishes the proof:

Observe by Remark 2.6 that there exist class \mathcal{K} functions $\tilde{\gamma}, \tilde{\beta}$ such that A is a $(\tilde{\gamma}, \tilde{\beta})$ -attractor. Without loss of generality we may assume $\tilde{\gamma} \geq \gamma$ and $\tilde{\beta} \geq \beta$. Hence also the A_{h_n} are $(\tilde{\gamma}, \tilde{\beta})$ -attractors, and by the same arguments as for “(ii) \Rightarrow (iv)”, above, we obtain

$$d_H(A, A_{h_n}) \leq \tilde{\gamma}(Ch_n^p)$$

for some suitable $C > 0$ implying (iii). \square

References

- [1] J.-P. AUBIN AND H. FRANKOWSKA, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
- [2] P. DEUFLHARD AND F. BORNEMANN, *Numerische Mathematik. II: Integration gewöhnlicher Differentialgleichungen*, de Gruyter, Berlin, 1994.
- [3] P. DIAMOND AND P. E. KLOEDEN, *Spatial discretization of mappings*, *Comput. Math. Appl.*, 25 (1993), pp. 85–94.
- [4] B. G. GARAY AND P. E. KLOEDEN, *Discretization near compact invariant sets*, *Random Comput. Dyn.*, (1997), pp. 93–12.
- [5] L. GRÜNE, *Convergence rates of perturbed attracting sets with vanishing perturbation*. DFG-Schwerpunktprogramm "Dynamik: Analysis, effiziente Simulation und Ergodentheorie", Preprint 21/99, 1999. Submitted.
- [6] J. K. HALE, X.-B. LIN, AND G. RAUGEL, *Upper semicontinuity of attractors for approximations of semigroups and partial differential equations*, *Math. Comput.*, 50 (1988), pp. 89–123.
- [7] M. W. HIRSCH, J. PALIS, C. C. PUGH, AND J. SHUB, *Neighborhoods of hyperbolic sets*, *Invent. Math.*, 9 (1970), pp. 121–134.
- [8] A. ISERLES, *A first course in the numerical analysis of differential equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1995.
- [9] P. E. KLOEDEN, *Asymptotically stable attracting sets in the Navier–Stokes equations*, *Bulletin Austral. Math. Soc.*, (1986), pp. 37–52.
- [10] P. E. KLOEDEN AND V. S. KOZYAKIN, *The inflation of attractors and their discretization*. Preprint 22/99, DFG-Schwerpunktprogramm "Dynamik: Analysis, effiziente Simulation und Ergodentheorie", 1999.
- [11] P. E. KLOEDEN AND J. LORENZ, *Stable attracting sets in dynamical systems and their one-step discretizations*, *SIAM J. Numer. Anal.*, 23 (1986), pp. 986–995.
- [12] ———, *A note on multistep methods and attracting sets of dynamical systems*, *Numer. Math.*, 56 (1990), pp. 667–673.
- [13] J. STOER AND R. BULIRSCH, *Introduction to Numerical Analysis*, Springer Verlag, New York, 1980.
- [14] A. M. STUART AND A. R. HUMPHRIES, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, 1996.
- [15] T. YOSHIKAWA, *Stability Theory by Lyapunov's Second Method*, The Mathematical Society of Japan, Tokyo, 1966.