SET-VALUED INTERPOLATION, DIFFERENTIAL INCLUSIONS, AND SENSITIVITY IN OPTIMIZATION

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Abstract. Set-valued interpolation and integration methods are introduced with special emphasis on error representations and error estimates with respect to Hausdorff distance. The connection between order of convergence results and sensitivity properties of finite-dimensional convex optimization problems is discussed. The results are applied to the numerical approximation of reachable sets of linear control problems by quadrature formulae and interpolation techniques for set-valued mappings.
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1. Introduction

Numerical methods for the solution of differential inclusions follow three directions.

(i) Compute special solution trajectories with additional qualitative or quantitative properties:

Such trajectories have to be computed by difference methods with additional selection procedures choosing points from the set-valued right-hand side in an appropriate way. Common strategies result, e.g., in the discrete analogue of heavy solutions, or slow solutions, or selections with a certain kind of discrete minimal variation. These selection procedures require the solution of finite-dimensional optimization problems at every gridpoint. Qualitative and quantitative sensitivity properties of this family of optimization problems determine qualitative and quantitative properties of the approximated solution, cp. in this connection [23] and the survey [19].

(ii) Compute all solution trajectories, or at least all belonging to a given class of functions:

This is theoretically and computationally an extremely difficult task. Applying the abstract framework of general discretization theory requires correct notions of stability and consistency. Conditions assuring order of convergence higher than 2 are not available until now. In principle, a proper calculus of higher order derivatives is required for set-valued mappings, guaranteeing Taylor expansions with valid error estimates with respect to Hausdorff distance. Some results concerning Euler's method resp. Euler-Cauchy method and order of convergence equal to 1 resp. equal to 2 are available, cp. [33], [34]. Every solution belonging to an appropriate Sobolev space can be approximated in a theoretical sense by a higher order linear multistep method, where the relevant notion of consistency is related to stability properties of a family of perturbed optimization problems, cp. Definition 3.2 in [23].

(iii) Compute the reachable set of all solution trajectories at a prescribed point in time:

The techniques mentioned in (ii) like Euler's method resp. Euler-Cauchy method yield, as a by-product, first resp. second order discrete approximations of reachable sets of special classes of differential inclusions. In [16] even higher order of convergence is proven for a method exploiting fully the structure of special linear differential inclusions with polyhedral control region. In the sequel of papers [7], [6], [4], and in the thesis [5], the discrete approximation of reachable sets of linear differential inclusions is totally reduced to the numerical integration of set-valued mappings. The basis of this approach consists in adaptations of quadrature formulae and extrapolation methods to the calculation of Aumann's integral for set-valued mappings. In principle, classical quadrature methods are applied to the support functional of the set-valued integrand. For every point in the integration interval and every unit vector in state space, the value of the support functional is determined by a convex optimization problem. Smoothness properties of this support functional as a function on the integration interval uniformly with respect to the unit ball in state space, thus strong stability and sensitivity properties of an infinite family of convex optimization problems, determine the order of the integration method and, consequently, the order of suitably defined discrete approximations of reachable sets. In this framework, higher order discrete approximations to reachable sets can be defined at least for special classes of linear differential inclusions. Originally, only the use of quadrature formulae with nonnegative weights seemed to be reasonable, like some open or closed Newton-Cotes formulae, Gauss quadrature, or Romberg's extrapolation method with Romberg's stepsize sequence. But exploiting some ideas in [8], compare also [9], depending on the geometry of the set-valued integrand, even quadrature fomulae with negative weights could be applied, thus opening the way to all kinds of extrapolation methods, error estimates by inclusion, and stepsize control for set-valued integration.

As outlined above, there exists an intrinsic relationship between numerical methods for differential inclusions and questions of sensitivity and stability analysis of finite dimensional optimization problems. The main objective of this paper is to describe this relationship. Hoping, that a numerical treatment of linear differential inclusions in the very spirit of set-valued numerical analysis will also be of value for a more satisfactory numerical treatment of nonlinear differential inclusions, we will concentrate on aspect (iii). Contrary to the thesis [5], where set-valued integration is the exclusive mathematical tool, we try to broaden the mathematical background to set-valued interpolation. The reader will easily recognize, that the techniques apply to set-valued mappings of several variables as well, thus opening the access to finite element methods for the discrete approximation of nonlinear differential inclusions in the, hopefully, near future.

2. Set-Valued Interpolation

In the following, we introduce set-valued interpolation as a mathematical tool to approximate set-valued mappings by simpler set-valued mappings. Deliberately, we avoid the technique of embedding spaces of convex sets into normed linear spaces, cp. the papers [28], [21], [30], [10], and [18]. This technique leaves the question unanswered how to interpret the results in the original spaces. Instead, we stay completely in the framework of set-valued mappings. Naturally, the problem arises how to define differences of sets in an appropriate way. This is done by a method already used in [8] for the proof of error estimates for set-valued quadrature formulae with negative weights, and in [5] for the derivation of inclusions of set-valued integrals by extrapolation methods. Only for simplicity we restrict ourselves to interpolation by set-valued polynomials, extensions to other function classes and even to interpolation of set-valued mappings of several variables by set-valued finite elements being rather obvious.

2.1. Interpolation Problem. *Let $I = [a, b]$ with $a < b$ and*

$$F : I \Longrightarrow \mathbb{R}^n$$

be a set-valued mapping with non-empty, convex and compact values.

Choose $N \in \mathbb{N}$ and a grid

$$a \le t_0 < t_1 < \ldots < t_N \le b \ ,$$

and compute for every $l \in \mathbb{R}^n$ the polynomial

$$p_N(l, \cdot)$$

of degree $\le N$ with

$$p_N(l, t_j) = \delta^\star(l, F(t_j)) \qquad (j = 0, \ldots, N) \ .$$

■

Here, we denote by

$$\delta^\star(l, A) = \sup_{z \in A} l^\star z \qquad (l \in \mathbb{R}^n)$$

the so-called support functional of the set $A \subset \mathbb{R}^n$. It is well-known, that $\delta^\star(\cdot, A)$ is a real-valued, positively homogenous continuous and convex functional on the whole of $\mathbb{R}^n$ for every non-empty convex and compact set A. Moreover, the polynomial $p_N(l, \cdot)$ exists and is uniquely determined for every $l \in \mathbb{R}^n$. Naturally, except constant or linear interpolation or use of interpolation techniques with non-negative basis functions, cp. e.g. [24], [25], [35], the polynomial $p_N(l, t)$ is not for all $t \in I$ the support functional of a convex set. This can easily be seen by inspection of Lagrange's interpolation formula,

$$p_N(l, t) = \sum_{j=0}^{N} \delta^\star(l, F(t_j)) \prod_{\substack{\mu=0 \\ \mu \ne j}}^{N} \frac{(t - t_\mu)}{(t_j - t_\mu)} \ , \tag{2.1}$$

which, for fixed $t \in I$, is a linear combination of support functionals with, unfortunately, some negative weights in general. Hence, $p_N(\cdot, t)$ is real-valued, positively homogeneous and continuous for every $t \in I$, but in general not convex.

A way out of this difficulty consists in the replacement of $p_N(\cdot, t)$ by its convexification resp. double conjugate

$$p_N^{\star\star}(\cdot, t) \qquad (t \in I) \ ,$$

which can be computed as follows.

By definition, cp. e.g. [29], we have

$$\begin{aligned} p_N^\star(z,t) &= \sup_{l\in\mathbb{R}^n} [z^\star l - p_N(l,t)] \\ &= \begin{cases} 0, & \text{if } z^\star l \le p_N(l,t) \text{ for all } l \in \mathbb{R}^n \ , \\ \infty, & \text{if } z^\star l > p_N(l,t) \text{ for at least one } l \in \mathbb{R}^n \ . \end{cases} \end{aligned}$$

Hence, $p_N^\star(z,t)$ is the indicator function of the set

$$P_N(t) = \{z \in \mathbb{R}^n : z^\star l \le p_N(l,t) \text{ for all } l \in \mathbb{R}^n\} \ , \tag{2.2}$$

and therefore

$$p_N^{\star\star}(\cdot,t) = \delta^\star(\cdot, P_N(t))$$

is the support functional of $P_N(t)$ for every $t \in I$.

2.2. Lemma. *On the standard assumptions of Interpolation Problem 2.1, the set $P_N(t)$ is closed, convex and bounded for every $t \in I$.*

Proof. According to (2.2) the set $P_N(t)$ is the intersection of closed half spaces in $\mathbb{R}^n$, therefore $P_N(t)$ is convex and closed. Moreover, (2.1) shows that $p_N(l,t)$ is bounded uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$,

$$p_N(l,t) \le c(t) \qquad (\|l\|_2 = 1) \ , \tag{2.3}$$

this implies for $z \in P_N(t)$

$$\|z\|_2^2 \le p_N(z,t)$$

and hence, for $\|z\|_2 \ne 0$,

$$\begin{aligned} \|z\|_2 &\le \frac{1}{\|z\|_2} p_N(z,t) \\ &= p_N\left(\frac{z}{\|z\|_2}, t\right) \\ &\le c(t) \qquad\qquad (t \in I) \ . \end{aligned}$$

■

Since

$$p_N^{\star\star}(l,t_j) = \delta^{\star\star\star}(l, F(t_j)) = \delta^\star(l, F(t_j))$$

and $F(t_j)$ is closed and convex,

$$P_N(t_j) = F(t_j) \qquad (j = 0, \ldots, N) \ .$$

Therefore, in a very natural way, we can define the set-valued interpolation "polynomial" which solves Interpolation Problem 2.1.

2.3. Definition. *For every $l \in \mathbb{R}^n$ let*

$$p_N(l,\cdot)$$

be the interpolation polynomial which solves Interpolation Problem 2.1.

Then the set-valued mapping

$$P_N : I \Longrightarrow \mathbb{R}^n \ ,$$

defined by

$$P_N(t) = \{z \in \mathbb{R}^n : z^\star l \le p_N(l,t) \text{ for all } l \in \mathbb{R}^n\} \qquad (t \in I) \ ,$$

is called the set-valued solution of Interpolation Problem 2.1. ■

At this point, we should add a warning: Neither is $P_N(t)$ in general polynomial with respect to t, nor is $P_N(t)$ necessarily non-empty for all $t \in I$. Hence, it is crucial to give conditions which guarantee $P_N(t) \neq \emptyset$ for all $t \in I$. In addition, these conditions should allow the proof of error estimates with respect to Hausdorff distance between $F(t)$ and $P_N(t)$ which are analogous to error estimates between the scalar functions $\delta^\star(l, F(t))$ and $p_N(l,t)$. For this purpose, we use the following result which was already exploited in [8] for the proof of error estimates for set-valued quadrature formulae with negative weights.

2.4. Lemma. *Consider a fixed $t \in I$ where $p_N(\cdot,t)$ is not itself a support functional. Assume moreover, that there exists a ball*

$$B(m(t),r(t)) = \{z \in \mathbb{R}^n : \|z - m(t)\|_2 \le r(t)\}$$

with center $m(t) \in \mathbb{R}^n$ and radius $r(t) > 0$, which is contained entirely in $P_N(t)$,

$$B(m(t),r(t)) \subset P_N(t) \ .$$

Define, as in (2.3),

$$c(t) = \sup_{\|l\|_2=1} p_N(l,t) \ .$$

Then the following error estimate holds

$$\text{haus}\,(F(t), P_N(t)) \le \frac{2c(t)}{r(t)} \sup_{\|l\|_2=1} |\delta^\star(l,F(t)) - p_N(l,t)| \ .$$

■

Here, $\text{haus}(\cdot,\cdot)$ denotes Hausdorff distance with respect to Euclidean norm $\|\cdot\|_2$. The proof is contained in [8] and [5]. More convenient in applications is the following condition on $F(t)$ itself.

2.5. Corollary. *Consider again a fixed* $t \in I$ *where* $p_N(\cdot, t)$ *is not itself a support functional. Assume moreover that the ball* $B(m(t), r(t))$ *with center* $m(t) \in \mathbb{R}^n$ *and radius* $r(t) > 0$ *is contained entirely in* $F(t)$.

Then for every

$$\epsilon(t) = \sup_{\|l\|_2=1} |\delta^\star(l, F(t)) - p_N(l, t)|$$

with $0 < \epsilon(t) < r(t)$ *the following error estimate holds*

$$\text{haus}\,(F(t), P_N(t)) \leq \frac{2c(t)}{r(t) - \epsilon(t)}\,\epsilon(t)\ .$$

Proof. Since $B(m(t), r(t)) \subset F(t)$, it follows

$$\begin{aligned} & \delta^\star(l, B(m(t), r(t)) \\ = \;& l^\star m(t) + r(t)\|l\|_2 \\ \leq \;& \delta^\star(l, F(t))\ , \end{aligned}$$

hence

$$\begin{aligned} & l^\star m(t) + r(t)\|l\|_2 \\ \leq \;& p_N(l, t) + \epsilon(t)\|l\|_2\ , \end{aligned}$$

whence it follows

$$l^\star m(t) + (r(t) - \epsilon(t))\|l\|_2 \leq p_N(l, t) \qquad (l \in \mathbb{R}^n)\ .$$

This means that the ball $B(m(t), r(t) - \epsilon(t))$ is contained in $P_N(t)$, and the estimate follows from Lemma 2.4. ■

If for a fixed $t \in I$ the interpolating function is itself a support functional, which is clear for all grid points, and for linear interpolation or other interpolation techniques with non-negative basis functions, then the error estimate does not depend any longer on the geometry of the set-valued mapping $F(\cdot)$. Then the following estimate, cp. [11], [21], holds.

2.6. Lemma. *Consider a fixed* $t \in I$ *where* $p_N(\cdot, t)$ *is itself a support functional of a non-empty convex and compact set* $P_N(t)$. *Then*

$$\text{haus}\,(F(t), P_N(t)) = \sup_{\|l\|_2=1} |\delta^\star(l, F(t)) - p_N(l, t)|\ .$$

■

The last representation of Hausdorff distance is extremely useful for the direct proof of error estimates for set-valued quadrature formulae with non-negative weights without recourse to set-valued interpolation, cp. [7], [6], [4], [5], and Section 5.

By Lemma 2.4, Corollary 2.5 and Lemma 2.6, the error between $F(t)$ and $P_N(t)$ with respect to Hausdorff distance is totally reduced to the classical error between $\delta^\star(l, F(t))$ and $p_N(l,t)$ and, eventually, some upper bounds for $c(t)$ and positive lower bounds for $r(t)$ which depend on the geometry of $P_N(t)$ resp. $F(t)$. As we will see in Section 3, continuity and differentiability properties of $\delta^\star(l, F(t))$ with respect to $t \in I$ uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$ play a crucial role for the classical error. But, we want to stress that such regularity properties of $\delta^\star(l, F(\cdot))$ can only be expected to hold for special classes of set-valued mappings $F(\cdot)$, cp. Section 4. In any case, all subsequent error representations and error estimates have to be done very cautiously to exploit at least some absolute continuity properties for reasonably large classes of problems.

3. Representation of the Interpolation Error

There are several methods, to prove estimates for the interpolation error

$$R_N(l,t) = \delta^\star(l, F(t)) - p_N(l,t) \ .$$

One could follow classical lines, cp. e.g. [32], which usually requires a little bit too strong smoothness assumptions on $\delta^\star(l, F(\cdot))$. One could also follow an approach exploiting systematically moduli of smoothness of $\delta^\star(l, F(\cdot))$, cp. [31]. This approach yields the weakest estimates for problems in one variable. Instead, we present an approach which leads to weak error estimates for an especially important class of problems, and which can easily be extended to interpolation problems in several variables, cp. [15] and [14].

For simplicity, in this section we use the abbreviation

$$f(t) = \delta^\star(l, F(t))$$

and suppress the explicit indication of l whenever possible.

Hence, $f(t)$ satisfies

$$f(t) = p_N(t) + R_N(t) \qquad (t \in I) \ . \tag{3.1}$$

We follow the idea in [15], cp. also [14], pp. 127–130. Taylor's theorem in [17] yields, for $N \geq 2$,

$$\begin{aligned} f(t_j) \quad &= \quad f(t) + f'(t)(t_j - t) + \ldots + \frac{1}{(N-2)!} f^{(N-2)}(t)(t_j - t)^{N-2} \\ &+ \int_0^1 \frac{(1-\zeta)^{N-2}}{(N-2)!} f^{(N-1)}(t + \zeta(t_j - t))(t_j - t)^{N-1} d\zeta \end{aligned}$$

for $j = 0, \ldots, N$ and all $t \in I$ as long as at least $f^{(N-1)}(\cdot)$ is continuous. If in addition $f^{(N-1)}(\cdot)$ is absolutely continuous, then $f^{(N)}(\cdot)$ exists almost everywhere and is integrable on I, partial integration is justified and gives

$$\begin{aligned} f(t_j) &= f(t) + f'(t)(t_j - t) + \ldots + \frac{1}{(N-2)!} f^{(N-2)}(t)(t_j - t)^{N-2} \\ &\quad + \left[\frac{-(1-\zeta)^{N-1}}{(N-2)!(N-1)} f^{(N-1)}(t + \zeta(t_j - t))(t_j - t)^{N-1} \right]_{\zeta=0}^{\zeta=1} \\ &\quad - \int_0^1 \frac{-(1-\zeta)^{N-1}}{(N-1)!} f^{(N)}(t + \zeta(t_j - t))(t_j - t)^N d\zeta \\ &= f(t) + f'(t)(t_j - t) + \ldots + \frac{1}{(N-1)!} f^{(N-1)}(t)(t_j - t)^{N-1} \\ &\quad + \int_0^1 \frac{(1-\zeta)^{N-1}}{(N-1)!} f^{(N)}(t + \zeta(t_j - t))(t_j - t)^N d\zeta \ . \end{aligned}$$

Hence, for almost all $t \in I$, we have the representation, which holds for $N = 1$ as well,

$$\begin{aligned} f(t_j) &= f(t) + f'(t)(t_j - t) + \ldots + \frac{1}{N!} f^{(N)}(t)(t_j - t)^N \\ &\quad + \int_0^1 \frac{(1-\zeta)^{N-1}}{(N-1)!} \left[f^{(N)}(t + \zeta(t_j - t)) - f^{(N)}(t) \right] (t_j - t)^N d\zeta \ . \end{aligned}$$

Consider t as a fixed parameter, then the polynomial of degree at most equal to N

$$p(z) = f(t) + f'(t)(z - t) + \ldots + \frac{1}{N!} f^{(N)}(t)(z - t)^N$$

satisfies

$$\frac{d^\nu}{dz^\nu} p(z) \,|_{z=t} = f^{(\nu)}(t) \qquad (\nu = 0, \ldots, N) \tag{3.2}$$

and, for $j = 0, \ldots, N$,

$$p(t_j) = f(t_j) - \int_0^1 \frac{(1-\zeta)^{N-1}}{(N-1)!} \left[f^{(N)}(t + \zeta(t_j - t)) - f^N(t) \right] (t_j - t)^N d\zeta \ .$$

Therefore, it coincides with the Lagrange interpolation polynomial of degree at most equal to N which attains the same values at the nodes t_j,

$$\begin{aligned} p(z) \equiv \sum_{j=0}^{N} &\left[f(t_j) - \int_0^1 \frac{(1-\zeta)^{N-1}}{(N-1)!} \left[f^{(N)}(t + \zeta(t_j - t)) - f^{(N)}(t) \right] (t_j - t)^N d\zeta \right] \\ &\cdot \prod_{\substack{\mu=0 \\ \mu \neq j}}^{N} \frac{z - t_\mu}{t_j - t_\mu} \ . \end{aligned}$$

Remembering that

$$p_N(z) \equiv \sum_{j=0}^{N} f(t_j) \prod_{\substack{\mu=0\\ \mu\neq j}}^{N} \frac{z-t_\mu}{t_j-t_\mu}\ ,$$

we get from (3.2) the following

3.1. Error Representation. *Let $f^{(N-1)}(\cdot)$ be absolutely continuous on I. Then for $\nu = 0,\ldots,N$ and almost all $t \in I$ the following representation holds*

$$\begin{aligned} f^{(\nu)}(t) &= p_N^{(\nu)}(t) \\ &\quad - \sum_{j=0}^{N} \left[\int_0^1 \frac{(1-\zeta)^{N-1}}{(N-1)!} \left[f^{(N)}(t+\zeta(t_j-t)) - f^{(N)}(t) \right] (t_j-t)^N d\zeta \right] \\ &\quad \cdot \left(\frac{d^\nu}{dz^\nu} \prod_{\substack{\mu=0\\ \mu\neq j}}^{N} \frac{z-t_\mu}{t_j-t_\mu} \right) \Bigg|_{z=t} \ . \end{aligned}$$

■

This error representation clearly shows that the variation of $f^{(N)}(\cdot)$ on I plays a crucial rôle, where this variation has to be defined in an appropriate way, since $f^{(N)}(\cdot)$ is only integrable. Fortunately, $f^{(N)}(\cdot)$ appears only in integrated form. Hence, the following definition is sufficient for our purposes, cp. [5], p. 15,

$$\begin{aligned} \operatorname*{Var}_I\ f^{(N)}(\cdot) = \inf \ \Big\{ & \operatorname*{var}_I\ g(\cdot) : \\ & g(\cdot) : I \longrightarrow \mathbb{R}^n \text{ is integrable and} \\ & g(t) = f^{(N)}(t) \text{ for almost all } t \in I \Big\}\ , \end{aligned} \tag{3.3}$$

where var$(\cdot)$ denotes the usual variation of a vector valued function with respect to Euclidean norm. In the rest of this paper, the variation of integrable functions is to be understood in the sense of (3.3).

Assuming $f(\cdot)$ to be absolutely continuous and $N = 1,\ \nu = 0$, we get

$$\begin{aligned} f(t) &= \frac{t-t_1}{t_0-t_1} f(t_0) + \frac{t-t_0}{t_1-t_0} f(t_1) \\ &\quad - \int_0^1 \left[f'(t+\zeta(t_0-t)) - f'(t) \right] (t_0-t) d\zeta \cdot \frac{t-t_1}{t_0-t_1} \\ &\quad - \int_0^1 \left[f'(t+\zeta(t_1-t)) - f'(t) \right] (t_1-t) d\zeta \cdot \frac{t-t_0}{t_1-t_0} \end{aligned} \tag{3.4}$$

for almost all $t \in I$ as a very special case. This example is special in another sense as well. Since the basis functions

$$\frac{t-t_1}{t_0-t_1}, \ \frac{t-t_0}{t_1-t_0}$$

are nonnegative on $I = [t_0, t_1]$,

$$p_1(l,t) = \frac{t-t_1}{t_0-t_1}\delta^\star(l,F(t_0)) + \frac{t-t_0}{t_1-t_0}\delta^\star(l,F(t_1))$$

is itself a support functional of the non-empty convex and compact set

$$P_1(t) = \frac{t-t_1}{t_0-t_1}F(t_0) + \frac{t-t_0}{t_1-t_0}F(t_1) \ ,$$

hence Lemma 2.6 applies, and (3.4) results directly in the following error estimate for the Hausdorff distance between $F(t)$ and $P_1(t)$.

3.2. Linear Interpolation. *Let $\delta^\star(l,F(\cdot))$ be absolutely continuous, and let $\frac{d}{dt}\delta^\star(l,F(\cdot))$ be of bounded variation in I uniformly for all $l \in \mathbb{R}^n$ with* $\|l\|_2 = 1$.

Then, for linear set-valued interpolation, the following error estimate holds

$$\begin{aligned} & \operatorname{haus}(F(t),P_1(t)) \\ = \ & \sup_{\|l\|_2=1} |\delta^\star(l,F(t)) - p_1(l,t)| \\ \leq \ & \sup_{\|l\|_2=1} \operatorname*{Var}_I \left(\frac{d}{dt}\delta^\star(l,F(\cdot))\right) \cdot \frac{(t-t_0)(t_1-t)}{t_1-t_0} \ . \end{aligned}$$

∎

Naturally, such error representations suggest the use of piecewise polynomial interpolation of set-valued mappings to get error estimates in terms of stepsize. Piecewise linear interpolation leads to corresponding error estimates for the composite trapezoidal rule for set-valued mappings which is the basis for extrapolation methods for set-valued integration, cp. Section 5 and [7], [4], [5].

For later use, we add another special case, set-valued interpolation by polynomials of second degree. Assuming now $\frac{d}{dt}f(\cdot)$ to be absolutely continuous and $N = 2$, $\nu = 0$, we get

$$
\begin{aligned}
f(t) &= \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} f(t_0) + \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)} f(t_1) + \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)} f(t_2) \\
&- \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} \int_0^1 (1-\zeta) \left[f^{(2)}(t+\zeta(t_0-t)) - f^{(2)}(t) \right] (t_0-t)^2 d\zeta \\
&- \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)} \int_0^1 (1-\zeta) \left[f^{(2)}(t+\zeta(t_1-t)) - f^{(2)}(t) \right] (t_1-t)^2 d\zeta \\
&- \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)} \int_0^1 (1-\zeta) \left[f^{(2)}(t+\zeta(t_2-t)) - f^{(2)}(t) \right] (t_2-t)^2 d\zeta \ .
\end{aligned}
$$

Now, clearly, for fixed $t \in I$

$$
p_2(l,t) = \sum_{j=0}^{2} \delta^\star(l, F(t_j)) \prod_{\substack{\mu=0 \\ \mu \neq j}}^{2} \frac{(t-t_\mu)}{(t_j-t_\mu)}
$$

is positively homogeneous with respect to l, but not any longer necessarily convex, since the Lagrangean elementary polynomials generally have different signs. Therefore, applying Corollary 2.5, we get the following error representation for set-valued quadratic interpolation.

3.3. Quadratic Interpolation. *Let* $\frac{d}{dt}\delta^\star(l, F(\cdot))$ *be absolutely continuous and* $\frac{d^2}{dt^2}\delta^\star(l, F(\cdot))$ *of bounded variation in* I *uniformly for all* $l \in \mathbb{R}^n$ *with* $\|l\|_2 = 1$. *Assume moreover that for* $t \in I$ *the ball* $B(m(t), r(t))$ *with center* $m(t) \in \mathbb{R}^n$ *and radius* $r(t) > 0$ *is contained in* $F(t)$, *and that*

$$
\epsilon(t) = \sup_{\|l\|_2=1} |\delta^\star(l, F(t)) - p_2(l,t)|
$$

is small enough, i.e. $0 \leq \epsilon(t) < r(t)$. *Let* $c(t) = \sup_{\|l\|_2=1} p_2(l,t)$.

Then the following error estimate holds

$$
\begin{aligned}
&\operatorname{haus}(F(t), P_2(t)) \\
\leq\ & \frac{2c(t)}{r(t)-\epsilon(t)} \sup_{\|l\|_2=1} \operatorname*{Var}_I \left(\frac{d^2}{dt^2}\delta^\star(l, F(\cdot)) \right) \cdot \frac{1}{2} \\
&\cdot \left[\left| \frac{(t-t_0)^2(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} \right| + \left| \frac{(t-t_0)(t-t_1)^2(t-t_2)}{(t_1-t_0)(t_1-t_2)} \right| \right. \\
&\left. + \left| \frac{(t-t_0)(t-t_1)(t-t_2)^2}{(t_2-t_0)(t_2-t_1)} \right| \right] .
\end{aligned}
$$

■

Again, in concrete applications, one should use piecewise quadratic set-valued interpolation to get reasonable error estimates in terms of stepsize, compare in this connection Section 6.

All these representations of the interpolation error clearly show that in the case of interpolation by polynomials of degree at most equal to N the absolute continuity of

$$\frac{d^{N-1}}{dt^{N-1}}\delta^\star(l, F(\cdot))$$

and the variation of

$$\frac{d^{N}}{dt^{N}}\delta^\star(l, F(\cdot))$$

on I are essential for the error. For the special case $N = 1$, we only need that $\delta^\star(l, F(\cdot))$ itself is absolutely continuous and $\frac{d}{dt}\delta^\star(l, F(\cdot))$ of bounded variation. Surprisingly enough, this property is satisfied automatically for set-valued mappings defined by a broad class of linear differential inclusions, cp. [16] and Section 4.

4. The Rôle of Sensitivity

As outlined in Section 2, cp. especially Lemma 2.4, Corollary 2.5, and Lemma 2.6, and exploited in Section 3, cp. Error Representation 3.1 and the special cases 3.2 and 3.3, the error between $F(t)$ and $P_N(t)$ with respect to Hausdorff distance is reduced to the classical error between $\delta^\star(l, F(t))$ and $p_N(l, t)$ uniformly with respect to all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.

This is the point where sensitivity enters the scene, since the classical error

$$\delta^\star(l, F(t)) - p_N(l, t)$$

is determined, for every fixed $l \in \mathbb{R}^n$, by regularity properties of the value function $\delta^\star(l, F(\cdot))$ of the following family of convex optimization problems.

4.1. Perturbed Optimization Problems. *For every fixed $t \in I$, maximize*

$$l^\star z$$

subject to

$$z \in F(t)\,.$$

■

Here, the perturbation parameter is $t \in I$, the vector $l \in \mathbb{R}^n$ is considered to be fixed, and continuity and differentiability properties of the corresponding value function $\delta^\star(l, F(t))$ with respect to $t \in I$ uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$ play a crucial rôle. Naturally, such additional regularity properties of $\delta^\star(l, F(\cdot))$ can only be expected to hold for special classes of set-valued mappings $F(\cdot)$ or for some concrete problems. A relatively complete overview of such classes is contained in [5], pp. 81–106, which is based on the results of [26], [27], [1], [2], [16]. In the following, we cite only the most important cases.

4.2. Theorem. *(a) Let $U \subset \mathbb{R}^m$ be compact and non-empty, and let the single-valued mapping*

$$f : I \times U \longrightarrow \mathbb{R}^n$$

be a parametrization of F,

$$F(t) = f(t, U) \qquad (t \in I)\ ,$$

with compact values.

Let $f(t, \cdot)$ be upper semicontinuous on U for all $t \in I$, and let there exist a Lipschitz constant L with

$$\|f(t_1, u) - f(t_2, u)\|_2 \le L|t_1 - t_2| \qquad (t_1, t_2 \in I, u \in U)\ .$$

Then, for every $u \in U$, $f(\cdot, u)$ is absolutely continuous, and the family

$$\left(\frac{\partial f}{\partial t}(\cdot, u)\right)_{u \in U}$$

is integrable. Assume moreover, that this family is jointly of bounded variation in the following sense:

There exist integrable functions

$$g(\cdot, u) : I \to \mathbb{R}^n \qquad (u \in U)$$

with

$$g(t, u) = \frac{\partial f}{\partial t}(t, u)$$

for almost all $t \in I$, such that all the numbers

$$\sum_{i=0}^{m-1} \|g(t_{i+1}, u_i) - g(t_i, u_i)\|_2$$

are bounded uniformly for all subdivisions

$$a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b\ ,$$

all $u_i \in U$, and all $m \in \mathbb{N}$.

Then, $\delta^\star(l, F(\cdot))$ *is Lipschitz continuous, and* $\frac{d}{dt}\delta^\star(l, F(\cdot))$ *of bounded variation uniformly for all* $l \in \mathbb{R}^n$ *with* $\|l\|_2 = 1$.

(b) Let $U \subset \mathbb{R}^m$ *be compact and non-empty, and let the* $n \times m$*-matrix function* $A(\cdot)$ *describe the following parametrization of* F,

$$F(t) = A(t)U \qquad (t \in I) \ .$$

Let $A(\cdot)$ *be absolutely continuous, and* $\frac{d}{dt}A(\cdot)$ *of bounded variation.*

Then, $\delta^\star(l, F(\cdot))$ *is Lipschitz continuous, and* $\frac{d}{dt}\delta^\star(l, F(\cdot))$ *of bounded variation uniformly for all* $l \in \mathbb{R}^n$ *with* $\|l\|_2 = 1$. ■

Sometimes, it is possible to compute the support functional exactly for all $t \in I$, and to examine its regularity properties directly.

4.3. Examples. *(i) Let*

$$F(t) = \{z \in \mathbb{R}^n : \|z - m(t)\|_p \le r(t)\}$$

be a varying ball in $\mathbb{R}^n$ *with center* $m(t) \in \mathbb{R}^n$ *and radius* $r(t) \ge 0$, *where* $1 \le p \le \infty$. *Then*

$$\delta^\star(l, F(t)) = l^\star m(t) + r(t)\|l\|_q \qquad (t \in I) \ ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let

$$a_i(t) \le b_i(t) \qquad (i = 1, \dots, n)$$

and

$$F(t) = \prod_{i=1}^n [a_i(t), b_i(t)] \ .$$

Then

$$\delta^\star(l, F(t)) = \sum_{i=1}^n l_i \left[\frac{1 - \operatorname{sign}(l_i)}{2} a_i(t) + \frac{1 + \operatorname{sign}(l_i)}{2} b_i(t)\right] \qquad (t \in I)$$

for all $l = (l_1, \dots, l_n)^\star \in \mathbb{R}^n$.

(iii) Let

$$F(t) = \operatorname{co}\{p_1(t), \dots, p_r(t)\} \qquad (t \in I)$$

be a convex polyhedron with corners $p_1(t), \dots, p_r(t) \in \mathbb{R}^n$. *Then*

$$\delta^\star(l, F(t)) = \max_{j=1,\dots,r} l^\star p_j(t) \qquad (t \in I) \ .$$

If, for every $l \in \mathbb{R}^n$, there exists a corner

$$p_{j_l}(t) \in \{p_1(t), \ldots, p_r(t)\}$$

with

$$l^\star p_{j_l}(t) = \max_{j=1,\ldots,r} l^\star p_j(t) \qquad (t \in I) \ ,$$

then differentiability properties of $p_{j_l}(\cdot)$ on I are inherited by $\delta^\star(l, F(\cdot))$.

(iv) Consider a real function $\phi : I \to \mathbb{R}$ and a nonempty subset $U \subset \mathbb{R}^n$. Then

$$\delta^\star(l, \phi(t)U) = \begin{cases} \phi(t)\delta^\star(l, U) & (\phi(t) \geq 0) \\ -\phi(t)\delta^\star(l, -U) & (\phi(t) < 0) \end{cases} .$$

Hence, as long as $\phi(\cdot)$ does not change sign, differentiability properties of $\phi(\cdot)$ are inherited by $\delta^\star(l, \phi(\cdot)U)$.

If, moreover,

$$U = -U \ ,$$

then

$$\delta^\star(l, \phi(t)U) = |\phi(t)|\delta^\star(l, U) \qquad (t \in I) \ .$$

Hence, differentiability properties of $|\phi(\cdot)|$, especially those at zeros of $\phi(\cdot)$, determine the differentiability properties of $\delta^\star(l, \phi(\cdot)U)$.

(v) Let $\Phi(\cdot)$ be an $n \times m$-matrix function and

$$B(m(t), r(t)) = \{z \in \mathbb{R}^m : \|z - m(t)\|_2 \leq r(t)\} \qquad (t \in I)$$

a varying ball in $\mathbb{R}^m$ with center $m(t) \in \mathbb{R}^m$ and radius $r(t) \geq 0$. Define

$$F(t) = \Phi(t)B(m(t), r(t)) \qquad (t \in I) \ .$$

Then

$$\delta^\star(l, F(t)) = l^\star\Phi(t)m(t) + r(t)\|\Phi^\star(t)l\|_2 \qquad (t \in I) \ .$$

Hence, as long as $\Phi^\star(t)l \neq 0_{\mathbb{R}^m}$ (which is, e.g., the case for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$ if the rows of $\Phi(t)$ are linearly independent for all $t \in I$), differentiability properties of $m(\cdot), r(\cdot)$, and $\phi(\cdot)$ are inherited by $\delta^\star(l, F(\cdot))$. ∎

The situation is much worse with a varying ball in $\mathbb{R}^m$ with respect to infinity norm,

$$B_\infty(m(t), r(t)) = \{z \in \mathbb{R}^m : \|z - m(t)\|_\infty \leq r(t)\} \qquad (t \in I) \ .$$

Again, let $\Phi(\cdot)$ be an $n \times m$-matrix function on I and

$$F(t) = \Phi(t) B_\infty(m(t), r(t)) \qquad (t \in I) \ .$$

Now, according to Example 4.3 (i), we have

$$\delta^\star(l, F(t)) = l^\star \Phi(t) m(t) + r(t) \|\Phi^\star(t) l\|_1 \qquad (t \in I) \ ,$$

and only under rather special circumstances differentiability properties of $m(\cdot), r(\cdot)$, and $\Phi(\cdot)$ lead to the corresponding differentiability properties of $\delta^\star(l, F(\cdot))$, cp. [6], Example 2.

Summarizing, we want to stress that Theorem 4.2 just suffices to justify the error estimate for (piecewise) linear set-valued interpolation for broader classes of parametrized set-valued mappings, whereas Examples 4.3 justify even higher order set-valued interpolation by (piecewise) polynomials for more restricted classes of set-valued mappings. In the following sections, these results are used for the derivation of error estimates for set-valued integration and discrete approximations of attainable sets.

5. Set-Valued Integration

Set-valued integration can be introduced in different ways either following [10] and [18] exploiting abstract embedding theorems for spaces of convex sets [28], [21], or in a direct way for quadrature formulae with non-negative weights, cp. [7], [4], [6], [5], resp. for quadrature formulae with negative weights, cp. [8] and [9]. First we give a motivation by a third approach following the classical introduction of interpolatory quadrature formulae: Interpolate the set-valued integrand by a set-valued mapping in the sense of Definition 2.3 and integrate this set-valued mapping in the sense of Aumann [3].

5.1. Definition. *Let $I = [a, b]$ with $a < b$ and*

$$F : I \Longrightarrow \mathbb{R}^n$$

be a set-valued mapping. Then

$$\int_I F(\tau)\, d\tau = \{ z \in \mathbb{R}^n : \ \text{there exists an integrable selection } f(\cdot) \text{ of } F(\cdot) \text{ on } I \text{ with } z = \int_I f(\tau)\, d\tau \}$$

is called Aumann's integral of $F(\cdot)$ over I. ■

The following theorem is fundamental, for proofs cp. e.g. [2] and [22].

5.2. Theorem. *Let $F : I \Longrightarrow \mathbb{R}^n$ be a measurable set-valued mapping with non-empty and closed images. Then*

$$\int_I F(\tau)\, d\tau$$

is convex.

If, moreover, $F(\cdot)$ is integrably bounded, i.e., if there exists a function $k(\cdot) \in L_1(I)$ with

$$\sup_{f(t) \in F(t)} \|f(t)\|_2 \leq k(t) \tag{5.1}$$

for almost all $t \in I$, then

$$\int_I F(\tau)\, d\tau = \int_I \mathrm{co}(F(\tau))\, d\tau \tag{5.2}$$

is non-empty, compact, and convex. ■

Here, $\mathrm{co}(\cdot)$ denotes convex hull operation. The representation (5.2) suggests the interpolation of the set-valued mapping $\mathrm{co}(F(\cdot))$ in the sense of Interpolation Problem 2.1 and Definition 2.3. Just for simplicity, we assume in the following that $F(\cdot)$ itself is a measurable, integrably bounded set-valued mapping with non-empty compact convex values. Then we need not distinguish between

$$F(\tau),\ \mathrm{co}(F(\tau)),\ \overline{\mathrm{co}}(F(\tau))\ ,$$

where $\overline{\mathrm{co}}(F(\tau))$ denotes the closed convex hull of $F(\tau)$.

According to Interpolation Problem 2.1, having set-valued interpolatory quadrature formulae in mind, choose $N \in \mathbb{N}$ and a grid

$$a \leq t_0 < t_1 < \ldots < t_N \leq b\ ,$$

and compute for every $l \in \mathbb{R}^n$ the polynomial

$$p_N(l, \cdot)$$

of degree $\leq N$ with

$$p_N(l, t_j) = \delta^\star(l, F(t_j)) \qquad (j = 0, \ldots, N)\ ,$$

respectively the set-valued mapping

$$P_N(\cdot) = \{z \in \mathbb{R}^n : z^\star l \leq p_N(l, \cdot) \quad \text{for all } l \in \mathbb{R}^n\}$$

with support functional

$$p_N^{\star\star}(l, t)$$

for every $t \in I$.

According to the proof of Lemma 2.2, $P_N(t)$ is closed, convex, and even uniformly bounded for all $t \in I$ with $P_N(t) \neq \emptyset$. Moreover, the representation (2.1) shows that $p_N(l,t)$ is continuous with respect to $t \in I$. Hence, following [2], if $P_N(t) \neq \emptyset$ for all $t \in I$, then Aumann's integral

$$\int_I P_N(\tau)\, d\tau$$

exists, is non-empty, convex, and compact, and satisfies

$$\begin{aligned} \delta^\star(\cdot, \int_I P_N(\tau)\, d\tau) &= \int_I \delta^\star(\cdot, P_N(\tau))\, d\tau \\ &= \int_I p_N^{\star\star}(\cdot, \tau)\, d\tau \ . \end{aligned}$$

Remembering the classical interpolatory quadrature formula defined by $p_N(l,\cdot)$, i.e.

$$\int_I p_N(l,\tau)\, d\tau = \sum_{j=0}^{N} \delta^\star(l, F(t_j)) \int_I \prod_{\substack{\mu=0 \\ \mu\neq j}}^{N} \frac{(\tau - t_\mu)}{(t_j - t_\mu)}\, d\tau \ ,$$

this suggests the use of this quadrature formula after convexification with respect to $l \in \mathbb{R}^n$:

$$\left[\sum_{j=0}^{N} \delta^\star(\cdot, F(t_j)) \int_I \prod_{\substack{\mu=0 \\ \mu\neq j}}^{N} \frac{(\tau - t_\mu)}{(t_j - t_\mu)} d\tau \right]^{\star\star} . \tag{5.3}$$

For quadrature formulae with nonnegative weights, e.g. for closed Newton-Cotes formulae with N nodes ($N = 2, \ldots, 8, 10$), this representation simplyfies, since a linear combination of support functionals with nonnegative coefficients is again a support functional. Hence, if all the weights

$$c_j = \int_I \prod_{\substack{\mu=0 \\ \mu\neq j}}^{N} \frac{(\tau - t_\mu)}{(t_j - t_\mu)} d\tau \qquad (j = 0, \ldots, N)$$

are nonnegative, then (5.3) simplifies to the support functional

$$\sum_{j=0}^{N} c_j \delta^\star(\cdot, F(t_j))$$

of the set

$$\sum_{j=0}^{N} c_j F(t_j) \ .$$

One drawback of the above approach is, that one needs additional geometric assumptions to guarantee that the interpolatory set-valued polynomial has non-empty values. For quadrature formulae with nonnegative weights, there is a direct approach avoiding this drawback and with a wider range of applicability even to set-valued Gauß quadrature formulae and extrapolation methods, cp. [7], [4], [6], [5]. In the following, we give a brief sketch of this direct approach.

It is well-known that, under all assumptions of Theorem 5.2, we have

$$\begin{aligned}\int_I F(\tau)\,d\tau = \{z \in \mathbb{R}^n : l^\star z \quad &\leq \quad \delta^\star(l, \int_I F(\tau)\,d\tau) \\ &= \quad \int_I \delta^\star(l, F(\tau))\,d\tau \quad \text{for all} \ \ l \in \mathbb{R}^n\} \ .\end{aligned}$$

This suggests the approximation of

$$\int_I \delta^\star(l, F(\tau))\,d\tau$$

by a quadrature formulae $J(l, F)$

$$\int_I \delta^\star(l, F(\tau))\,d\tau = J(l, F) + R(l, F)$$

with remainder term $R(l, F)$ depending on $l \in \mathbb{R}^n$ and $F(\cdot)$.

E.g. for composite closed Newton-Cotes fomulae, Gauß quadrature or Romberg integration, $J(l, F)$ has the representation

$$J(l, F) = \sum_{j=0}^{N} c_j \delta^\star(l, F(t_j)) \tag{5.4}$$

with a grid of nodes

$$a \leq t_0 < t_1 < \ldots < t_N \leq b$$

and suitable weights

$$c_j \in \mathbb{R} \qquad (j = 0, \ldots, N).$$

If all these weights are nonnegative, then the quadrature formula (5.4) can be interpreted as a support functional

$$J(l,F) = \delta^\star\left(l, \sum_{j=0}^{N} c_j F(t_j)\right)$$

of the non-empty, compact, and convex set

$$\sum_{j=0}^{N} c_j F(t_j)\ .$$

Then, without any additional geometric assumption, Lemma 2.6, which is true for an arbitrary pair of non-empty, compact, convex sets and their corresponding support functionals, can be applied directly to the sets

$$\int_I F(\tau)\,d\tau, \quad \sum_{j=0}^{N} c_j F(t_j)\ ,$$

resulting in the following

5.3. Theorem. *Let $F : I \Rightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with non-empty, convex and compact values, and let the quadrature formula $J(\cdot,F)$ have nodes t_j, nonnegative weights c_j ($j = 0,\ldots,N$), and remainder term $R(\cdot,F)$.*

Then the error estimate holds

$$\text{haus}\left(\int_I F(\tau)d\tau,\ \sum_{j=0}^{N} c_j F(t_j)\right) = \sup_{\|l\|_2=1} |R(l,F)|\ .$$

■

Obviously, the order of the error is determined again by regularity properties of the scalarized integrand

$$\delta^\star(l,F(t)) \qquad (t \in I)$$

with respect to t uniformly with respect to all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.

To be more specific, we cite only two special cases, the set-valued analogues of composite trapezoidal rule and composite Simpson's rule. In fact, these two methods form the first two stages of a set-valued analogue of Romberg's extrapolation method, cp. [4], [6], and [5] for more details and complete proofs.

5.4. Theorem. *Let $F : I \Rightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with non-empty, convex, and compact values, and let*

$$\delta^\star(l, F(\cdot))$$

be absolutely continuous with first derivative with respect to t of bounded variation uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$.

Then the error between Aumann's integral

$$\int_I F(\tau)\, d\tau$$

and composite trapezoidal rule

$$h\left[\frac{1}{2}F(t_0) + F(t_1) + \ldots + F(t_{N-1}) + \frac{1}{2}F(t_N)\right]$$

on the grid

$$t_j = a + jh \qquad (j = 0, \ldots, N)$$

with meshsize

$$h = \frac{b-a}{N}$$

is of order 2 in h with respect to Hausdorff distance.

If, in addition, $\delta^\star(l, F(\cdot))$ has an absolutely continuous second derivative and if its third derivative is of bounded variation uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$, then the error of composite Simpson's rule

$$\frac{h}{3}\left[F(t_0)+4F(t_1)+2F(t_2)+4F(t_3) + \ldots +2F(t_{N-2})+4F(t_{N-1})+F(t_N)\right]$$

on the same grid for even N is of order 4 in h with respect to Hausdorff distance. ■

Under additional smoothness assumptions Romberg extrapolation gives approximations of even higher order with respect to Hausdorff distance, compare the examples and numerical tests in [4], [6], [5].

We want to emphasize that, due to Theorem 4.2, we get second order convergence for a remarkable big class of parametrized set-valued mappings. Due to Examples 4.3, we get even higher order of convergence for more specific problem classes. Exploiting inclusion techniques, based on the asymptotic expansion of composite trapezoidal rule, for such problem classes, one can even get inner and outer approximations to the exact Aumann integral which converge of the same order with respect to Hausdorff distance as the underlying Romberg extrapolation method, cp. [5].

6. Approximating Reachable Sets by Set-Valued Integration and Interpolation

We want to outline in the following how set-valued integration and interpolation could be applied in a combined form to the discrete approximation of reachable sets of linear differential inclusions. To be more specific, we analyse the following standard

6.1. Linear Control Problem. *Let the $n \times n$-matrix function $A(\cdot)$ and the $n \times m$-matrix function $B(\cdot)$ be (at least) integrable on $I = [a, b]$, and the control region $U \subset \mathbb{R}^m$ be non-empty, compact, and convex. Let $Y_0 \subset \mathbb{R}^n$ be a non-empty, compact, and convex starting set.*

Find an absolutely continuous function $y : I \longrightarrow \mathbb{R}^n$ with

$$\begin{aligned} y'(t) &\in A(t)y(t) + B(t)U \quad \text{for almost all } t \in I \ , \\ y(a) &\in Y_0 \ . \end{aligned}$$

■

The reachable set of this linear control problem at time $t \in [a, b]$ can easily be represented as Aumann's integral

$$Y(t) = \Phi(t, a)Y_0 + \int_a^t \Phi(t, \tau)B(\tau)U \, d\tau \ , \tag{6.1}$$

where $\Phi(t, \tau)$ is a fundamental solution of the homogeneous system

$$\frac{d}{dt}\Phi(t, \tau) = A(t)\Phi(t, \tau)$$

for almost all $t \in I$, satisfying the initial condition

$$\Phi(\tau, \tau) = E_n \ .$$

Now, approximating $Y(\cdot)$ numerically for all $t \in I$ requires an appropriate combination of the following four procedures a), b), c), and d).

a) Apply a set-valued quadrature formula

$$\sum_{j=0}^{N} c_j \Phi(t, t_j) B(t_j) U$$

with nonnegative weights on, say, an equidistant grid

$$t_j = a + jh \qquad (j = 0, \ldots, N), \quad h = \frac{t - a}{N} \ ,$$

which is of order q with respect to h, i.e.

$$\begin{aligned}&\text{haus}\left(\int_a^t \Phi(t,\tau)B(\tau)U\,d\tau,\ \sum_{j=0}^N c_j\Phi(t,t_j)B(t_j)U\right)\\ \le\ &\alpha_1 h^q\ .\end{aligned}$$

Using the methods outlined in Section 5, such formulae exist, if

$$\delta^\star(l,\Phi(t,\cdot)B(\cdot)U)$$

has an absolutely continuous $(q-2)$-nd derivative and if the $(q-1)$-st derivative is of bounded variation uniformly with respect to all $l\in\mathbb{R}^n$ with $\|l\|_2=1$. E.g., one could use the set-valued analogue of Romberg's method, cp. [4], [5]. For $q=2$ resp. $q=4$ this amounts to the use of set-valued composite trapezoidal rule resp. set-valued Simpson's rule according to Theorem 5.4.

b) Naturally, the actual computation in a) is done with approximations $\tilde\Phi(t,t_j)\quad (j=0,\ldots,N)$ of the exact values of the fundamental solution which should be of order q as well,

$$\sup_{0\le j\le N}\|\tilde\Phi(t,t_j)-\Phi(t,t_j)\|_2\le\alpha_2h^q\ .$$

This is possible if $A(\cdot)$ has an absolutely continuous $(q-2)$-nd derivative and if its $(q-1)$-st derivative is of bounded variation, cp. [31]. For similar results cp. [12], [13]. Normally, $A(\cdot)$ is even smoother, and the choice of an algorithm of order q does not cause any problem. Since the weights c_j are nonnegative, it follows

$$\begin{aligned}&\text{haus}\left(\sum_{j=0}^N c_j\Phi(t,t_j)B(t_j)U,\ \sum_{j=0}^N c_j\tilde\Phi(t,t_j)B(t_j)U\right)\\ \le\ &\sum_{j=0}^N c_j\,\text{haus}\Big(\Phi(t,t_j)B(t_j)U,\ \tilde\Phi(t,t_j)B(t_j)U\Big)\\ \le\ &\sum_{j=0}^N c_j\|\Phi(t,t_j)-\tilde\Phi(t,t_j)\|_2\|B(t_j)U\|_2\ ,\end{aligned}$$

where $\|\cdot\|_2$ denotes simultaneously spectral norm and the norm of a set,

$$\|B(t_j)U\|_2=\sup_{u\in U}\|B(t_j)u\|_2\ .$$

Hence, the combined procedures a) and b) maintain the overall order of convergence q, if

$$\sum_{j=0}^{N} c_j \|B(t_j)U\|_2$$

is uniformly bounded for $N \in \mathbb{N}$. Keeping in mind that c_j for $j = 0, \dots, N$ are the weights of a convergent quadrature formula, this is the case, if, e.g., $B(\cdot)$ is bounded on I.

Summarizing the procedures a) and b), we get

6.2. Theorem. *Assume for Linear Control Problem 6.1 that the $n \times m$-matrix function $B(\cdot)$ is bounded on I and that $A(\cdot)$, $\delta^\star(l, \Phi(t,\cdot)B(\cdot)U)$ have an absolutely continuous $(q$–$2)$-nd derivative and that the $(q$–$1)$-st derivative is of bounded variation uniformly with respect to all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$. Then, combining a set-valued quadrature formula with nonnegative weights of order q with a difference method of order q for the computation of the fundamental solution yields a method of order q for the discrete approximation of the reachable set at time t.* ■

Numerical tests, based on this result, can be found for various examples in [7], [4], [6], [5]. For linear systems with polyhedral control regions cp. [16] and [20], where higher order methods are presented which are based on Pontryagin's maximum principle.

c) A method, approximating the reachable set $Y(\cdot)$ on the whole time interval I, necessarily should comprise a third procedure:
Having computed

$$Y(s_j) \qquad (j = 0, \dots, q)$$

for some points

$$s_j = s_0 + j\hat{h} \in I, \quad \hat{h} > 0\ ,$$

approximate $Y(\cdot)$ on $[s_0, s_q]$ by means of set-valued polynomial interpolation in the sense of Section 2. According to Error Representation 3.1, cp. also the special cases 3.2 and 3.3, direct interpolation of $Y(\cdot)$ would require regularity properties of

$$\delta^\star\left(l, \Phi(t,a)Y_0 + \int_a^t \Phi(t,\tau)B(\tau)U\, d\tau\right)$$

with respect to t uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$. By no means, it is easy to derive such properties from regularity properties of

$$\delta^\star(l, \Phi(t,\tau)B(\tau)U)$$

with respect to τ, which are needed anyway for set-valued integration. Therefore, we choose another representation of $Y(t)$,

$$\begin{aligned} Y(t) &= \Phi(t,a)Y_0 + \int_a^t \Phi(t,\tau)B(\tau)U\,d\tau \\ &= \Phi(t,a)\left[Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\,d\tau\right] , \end{aligned}$$

and treat the single-valued factor

$$\Phi(t,a)$$

and the set-valued factor

$$Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\,d\tau$$

separately. Then, only regularity properties of

$$\begin{aligned} &\delta^\star\left(l, Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\,d\tau\right) \\ = \; &\delta^\star(l,Y_0) + \int_a^t \delta^\star(l,\Phi(a,\tau)B(\tau)U)\,d\tau \end{aligned}$$

with respect to t uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$ are needed. If the $(q-2)$-nd derivative of $\delta^\star(l,\Phi(a,\cdot)B(\cdot)U)$ is absolutely continuous and the $(q-1)$-st derivative of bounded variation uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$, then

$$\int_a^t \delta^\star(l,\Phi(a,\tau)B(\tau)U)\,d\tau$$

has absolutely continuous $(q–1)$-st derivative with q-th derivative of bounded variation. This fits favourably to Error Representation 3.1, leading to error estimates of order q in $\hat{h}$ with respect to Hausdorff distance. Naturally, for $q > 1$, additional conditions on the geometry of the sets

$$Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\,d\tau \qquad (s_0 \le t \le s_q) ,$$

cp. Corollary 2.5, have to be satisfied. Due to Theorem 4.2 (b), the case $q = 2$ is especially important. If $\Phi(a,\cdot)B(\cdot)$ is absolutely continuous and its first derivative of bounded variation, then piecewise quadratic interpolation, according to the special case 3.3, can be used for second order approximation on the whole interval I.

Naturally, since the data to be interpolated,

$$Y_0 + \int_a^{s_j} \Phi(a,\tau)B(\tau)U\, d\tau \qquad (j = 0, \dots, q) \ ,$$

can only be computed up to an error of order q with respect to some chosen stepsize h according to procedures a) and b), one never will get the exact set-valued interpolatory polynomial $P_q(\cdot)$, but only some set-valued approximation $\hat{P}_q(\cdot)$. Assuming, that the error

$$\text{haus}\left(P_q(t), \hat{P}_q(t)\right) \leq \alpha_3 \hat{h}^q \qquad (s_0 \leq t \leq s_q)$$

and, similarily as in b),

$$\|\hat{\Phi}(t,a) - \Phi(t,a)\|_2 \leq \alpha_4 \hat{h}^q \qquad (s_0 \leq t \leq s_q)$$

for the approximation $\hat{\Phi}(t,a)$ of the exact fundamental system $\Phi(t,a)$, we get the following estimate

$$\begin{aligned}
& \text{haus}\left(Y(t), \hat{\Phi}(t,a)\hat{P}_q(t)\right) \\
= \ & \text{haus}\left(\Phi(t,a)\left[Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\, d\tau\right],\ \hat{\Phi}(t,a)\hat{P}_q(t)\right) \\
\leq \ & \text{haus}\left(\Phi(t,a)\left[Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\, d\tau\right],\ \Phi(t,a)P_q(t)\right) \\
& + \text{haus}\left(\Phi(t,a)P_q(t),\ \Phi(t,a)\hat{P}_q(t)\right) \\
& + \text{haus}\left(\Phi(t,a)\hat{P}_q(t),\ \hat{\Phi}(t,a)\hat{P}_q(t)\right) \\
\leq \ & \|\Phi(t,a)\|_2 \ \text{haus}\left(Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\, d\tau, P_q(t)\right) \\
& + \|\Phi(t,a)\|_2 \ \alpha_3 \hat{h}^q \\
& + \alpha_4 \hat{h}^q \ \|\hat{P}_q(t)\|_2
\end{aligned}$$

for $s_0 \leq t \leq s_q$. Hence, in fact, the order of the "theoretical error"

$$\text{haus}\left(Y_0 + \int_a^t \Phi(a,\tau)B(\tau)U\, d\tau, P_q(t)\right)$$

determines the order of the global error.

d) The last part of this combined method consists in the choice of an appropriate computer aided graphical visualization method of the sets

$$\hat{\Phi}(t,a)\hat{P}_q(t) \qquad (s_0 \leq t \leq s_q) \ .$$

The grid data for interpolation result from set-valued numerical integration with nonnegative weights, and we recommend the dual approach by supporting hyperplanes discussed thoroughly in [5] including an analysis of the additional error due to the fact that only finitely many hyperplanes can be computed. For higher dimensional problems, actually support points of these hyperplanes should be plotted, not the hyperplanes themselves. The dual representation of $\hat{\Phi}(t,a)\hat{P}_q(t)$ between gridpoints is more difficult, since in the representation (2.2) of $\hat{P}_q(t)$ even non-supporting hyperplanes will occur, i.e. one has to develop an appropriate device to get rid of such hyperplanes and to restrict this representation to actually supporting hyperplanes resp. corresponding support points. The complexity of this task is dimension dependent. In the following numerical tests, we restrict ourselves to state space dimension 2.

The combined procedures a), b), c), and d) are visualized for the following

6.3. Example. *Let* $I = [1,2]$, *and for all* $t \in I$

$$\begin{aligned} A(t) &= \begin{pmatrix} 0 & 1 \\ -\dfrac{2}{t^2} & \dfrac{2}{t} \end{pmatrix} , \\ B(t) &= \begin{pmatrix} \sin(2\pi t) \\ \cos(2\pi t) \end{pmatrix} , \\ U &= [-1,\ 1] , \\ Y_0 &= [-0.1,\ 0.1]^2 . \end{aligned}$$

With these data, approximate the reachable set

$$Y(t) = \Phi(t,1)Y_0 + \int_1^t \Phi(t,\tau)B(\tau)U\,d\tau$$

on the whole internal $[1,2]$. ∎

Due to Theorems 4.2, 5.4, and 6.2, the sets

$$Y(1),\ Y(1.5),\ \text{resp. } Y(2)$$

can be computed by a combined method of order 2 very precisely. The results are plotted in Figures 1, 2, resp. 3. Since the control region is polyhedral, the methods in [16], [20] could have been used equally well.

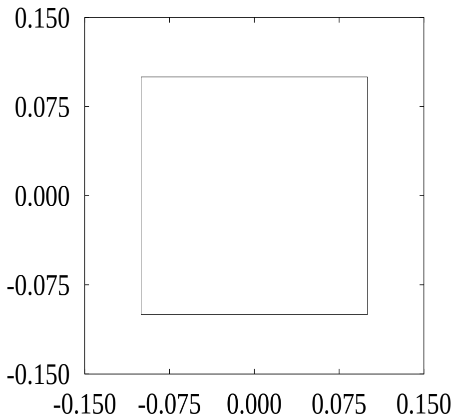

Figure 1. $Y(1)$

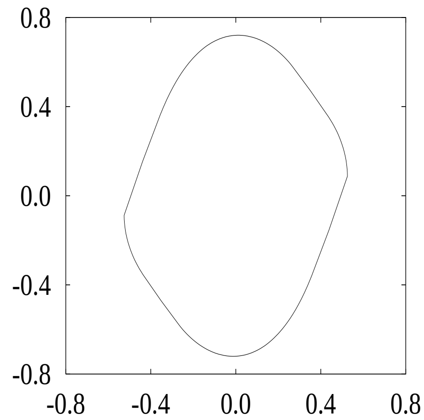

Figure 2. $Y(1.5)$

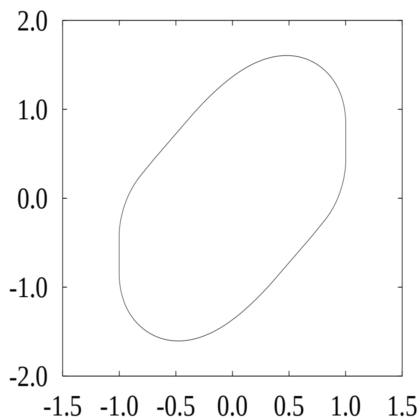

Figure 3. $Y(2)$

Consider now the set-valued function $\hat{P}_2(\cdot)$, which interpolates the sets

$$Y_0 + \int_1^{s_j} \Phi(1,\tau)B(\tau)U \, d\tau$$

on the grid

$$s_0 = 1, \; s_1 = 1.5, \; s_2 = 2 \; .$$

According to Definition 2.3, this set-valued mapping is related to a scalar quadratic interpolation polynomial $\hat{p}_2(l,\cdot)$ in the following way,

$$\hat{P}_2(t) = \left\{ z \in \mathbb{R}^2 : l^\star z \leq \hat{p}_2(l,t) \quad \text{for all } l \in \mathbb{R}^2 \right\} \; . \tag{6.2}$$

This opens the way, at least in $\mathbb{R}^2$, for a dual representation of $\hat{P}_2(t)$ as intersection of halfspaces. In Figure 4, the image of this representation under the linear transformation $\hat{\Phi}(t,1)$ is visualized for $t = 1.1$. Observe, that not all of these halfspaces are supporting ones due to the non-convexity of $\hat{p}_2(l, 1.1)$ with respect to l, cp. the magnification around the upper left corner in Figure 5.

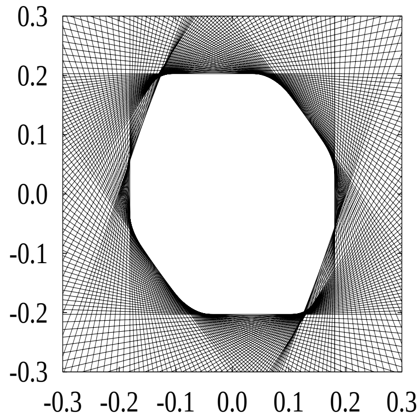

Figure 4. $\hat{\Phi}(1.1, 1)\hat{P}_2(1.1)$

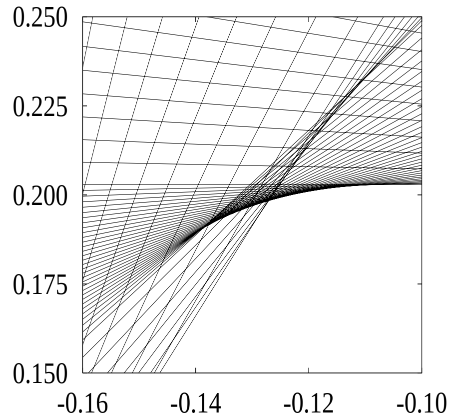

Figure 5. Magnification

Figures 6, 7, 8, resp. 9 show the approximations $\hat{\Phi}(t,1)\hat{P}_2(t)$ of the reachable set $Y(t)$, based on the evaluation of the set-valued polynomial $\hat{P}_2(t)$ and the linear transformation $\hat{\Phi}(t,1)$ for the successive time points $1.3, 1.4, 1.7$, resp. 1.8, compared with the exact reachable set (dotted line).

Observe, that, in principle, it would have been sufficient to compute the data sets for set-valued interpolation, cp. Figures 1, 2, 3, within an error of order 2 with respect to the stepsize for interpolation to get an overall error of order 2. In practical computations, one has to restrict oneself in (6.2) to a finite collection of vectors $l \in \mathbb{R}^2$ with $\|l\|_2 = 1$. Therefore, to retain order of convergence equal to 2 for the actually computed discrete approximations in Figures 6, 7, 8, 9, one has to choose the vectors l from an appropriately adjusted grid on the unit sphere in $\mathbb{R}^2$.

Acknowledgement. I appreciate the help of Robert Baier in preparing the plots. They are essentially based on his program package on the approximation of reachable sets of linear differential inclusions by set-valued integration methods.

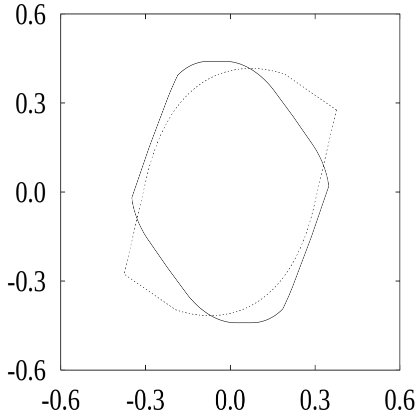

Figure 6. $\hat{\Phi}(1.3, 1)\hat{P}_2(1.3)$

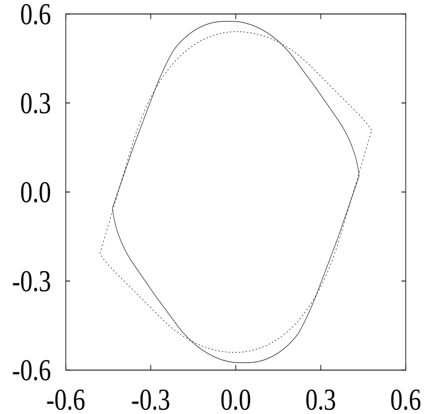

Figure 7. $\hat{\Phi}(1.4, 1)\hat{P}_2(1.4)$

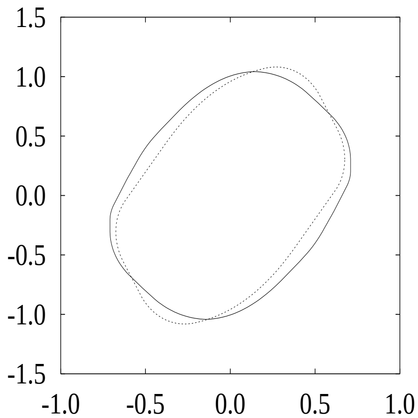

Figure 8. $\hat{\Phi}(1.7, 1)\hat{P}_2(1.7)$

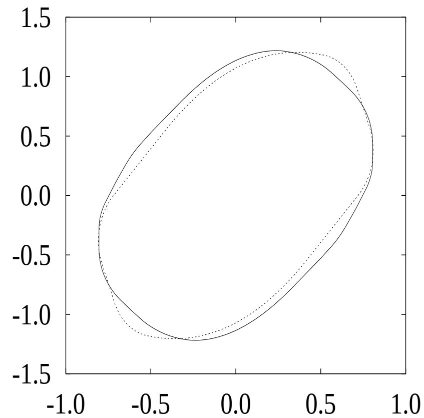

Figure 9. $\hat{\Phi}(1.8, 1)\hat{P}_2(1.8)$

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