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Sharing the Cost of Maximum Quality Optimal Spanning Trees

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Abstract

Minimum cost spanning tree problems have been widely studied in operation research and economic literature. Multi-criteria optimal spanning trees provide a more realistic representation of different actual problems. Once an optimal tree is obtained, how to allocate its cost among the agents defines a situation quite different from what we have in the minimum cost spanning tree problems. In this paper, we analyze a multi-criteria problem where the objective is to connect a group of agents to a source with the highest possible quality at the cheapest cost. We compute optimal networks and propose cost allocations for the total cost of the project. We analyze properties of the proposed solution; in particular, we focus on coalitional stability (*core selection*), a central concern in the literature on minimum cost spanning tree problems.

Keywords: Minimum cost spanning tree, Multi-criteria decision making, Quality, Cost sharing
JEL classification: C71, D63, D71.

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1. Introduction

The minimum cost spanning tree problem (*mcstp*) analyzes situations in which some individuals, who are located at different places, want to be connected to a source in order to obtain a good or a service. There are some known fixed costs of linking any two individuals, and of linking each individual to the source. These problems are used to analyze different real-life issues, from telephone and cable TV, to water supply networks. There are several methods to obtain a way of connecting all agents to the source so that the total cost of the selected network is minimum (Boruvka (1926), Kruskal (1956), Prim (1957), for instance). The allocation of this cost among the individuals participating on the network, once the minimum cost spanning tree is obtained, is an issue deeply studied in the literature, where different solutions have been proposed: for instance, *Bird rule* (Bird, 1976), *Kar* (Kar, 2002), *Folk* (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007), *Cycle-complete* (Trudeau, 2012), etc.

A more realistic situation appears when considering that there are several attributes defining the connection network (cost for connection, maintenance cost, quality of connection...) that can be represented by multi-criteria spanning tree problems. These multiple objectives usually conflict with each other, so the solutions to this kind of problems are selected from the Pareto (non-dominated) alternatives. The calculation of this Pareto set is not easy because, in general, it is an NP-hard problem. Usual solutions consist of simplifying the multiple objectives into a single objective, or prioritizing the objectives in a specific order.

The present paper deals with a particular multi-criteria spanning tree problem with two objectives. The model is inspired in the following example about a water supply network. Water must be supplied to all the individuals in a society from a source, and due to the physical characteristics of the possible connections between individuals, these connections may have different costs and different qualities. We consider only two levels of quality (low and high). Low quality connection entails that the water passing through this connection gets some impurities (as dust, sand, clay ...). An agent obtains impure water whenever some connection in his path to the source has low quality. On the contrary, an agent will obtain pure water only if all the connections from the source to him are of high quality. In this situation, we consider that the decision maker's objective is to connect all the individuals to the source providing high quality connection to the greatest number of agents at the minimum possible cost.

Then, the first question to solve is which network should be implemented. We show that we can obtain an optimal spanning tree for this problem by using an adaptation of Prim's algorithm. Once the collective optimal spanning tree is obtained, an important question is how to allocate the total cost among the agents in the network, because this cost sharing should depend not only on the connection costs, but also on the quality that each individual obtains.

To find a fair and in some sense stable division of the optimal cost among the agents involved in a minimum cost spanning tree problem has been extensively studied by using cooperative games. Since cooperation is necessary, the literature singles out stand alone core stability as the key property of any allocation rule: in order for the agents to be willing to participate, no coalition of agents should be charged more than the cheapest cost of connecting all of them to the source, independently of agents outside the coalition. In our context, agents may be willing to pay more if they can improve in quality, so the stand alone core stability needs to be redefined. We propose a cost allocation of the optimal cost extending the *Folk* solution from *mcstp* to our more general context. We prove that our extension is a core selection that fulfills appealing properties on responsiveness. In addition, we prove that the agents not obtaining high quality in the implemented network are not harmed by the additional cost due to other agents getting high quality.

2. Problem description

Consider a complete undirected graph $G = (N_\omega, E)$, where $N = \{1, 2, \ldots, n\}$ is a finite set of nodes representing individuals who want to be connected to a *source* ω. For any subset of agents $S \subseteq N$, S_ω denotes the set of agents in S and the source, i.e., $S_\omega = S \cup \{\omega\}$. On the other hand, $E = \{e_{ij} = (i, j) : i, j = \omega, 1, ..., n; i \neq j\}$ is a finite set of edges that connect the individuals and the source; if $i \neq \omega, j \neq \omega$, $e_{ij} = e_{ji}$ is the edge joining individuals i and j, and $e_{i\omega} = e_{\omega i}$ is the edge joining individual i to the source ω.

A *spanning tree* over N_ω is an undirected graph with no cycles that connects all elements of N_ω. We denote by $\mathcal{S}(N_\omega)$ the set of all the spanning trees over N_ω. In a spanning tree each agent is connected to the source ω. Moreover, for each spanning tree p, there is a unique path from any agent i to the source. We can identify a spanning tree with a *predecessor map* $p : N \to N_\omega$ so that $j = p(i)$ is the agent (or the source) to whom i connects in his way toward the source. This map p defines the edges $e_i^p = (i, p(i))$ in the spanning tree. The path from any agent i to the source is given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t(i)-1}(i), p^{t(i)}(i) = \omega)$, for some $t(i) \in \mathbb{N}$. We use the notation $p^0(i) = i$.

Each edge in the model has a non-negative cost and a quality level. We consider each edge has only two possible quality levels: low quality represented by 0, and high quality represented by 1. For each pair $(i, j) \in N_\omega$, $i \neq j$, $c_{ij} \in \mathbb{R}_+$ and $q_{ij} \in \{0, 1\}$ represent, respectively, the cost and the quality of the edge e_{ij}. For each $i \in N_\omega$ we define $c_{ii} = q_{ii} = 0$ and then $C = [c_{ij}]_{(n+1)\times(n+1)}$ and $Q = [q_{ij}]_{(n+1)\times(n+1)}$ are symmetric matrices representing, respectively, the cost and quality of each connection in the graph. A ***quality minimum cost spanning tree*** problem (*qmcstp*) is represented by the triplet (N_ω, C, Q). We denote by $\mathcal{N}_n^q$ the set of all the quality minimum cost spanning tree problems with n individuals and the described characteristics.

The cost of building a spanning tree p is the sum of the cost of the edges in this tree. The first algorithm to find the spanning tree of minimum cost was proposed by Boruvka (1926) as a method of building an efficient electricity network for Moravia (Hungary). Other popular algorithms to find such a tree are those of Kruskal (1956) and Prim (1957). In this work we will use Prim's algorithm. The achieved solution, the *minimum cost spanning tree*, may not be unique. We denote by m ($m \in \mathcal{S}(N_\omega)$) a tree with minimum cost and by $C(m)$ its cost. That is,

$$C(m) = \sum_{i=1}^{n} c_{im(i)} \leqslant C(p) = \sum_{i=1}^{n} c_{ip(i)} \quad \text{for all spanning tree } p \in \mathcal{S}(N_\omega)$$

When necessary, we will make explicit the dependence of this cost function on the cost matrix.

For any spanning tree, the quality each agent obtains depends on all the connections between the individual and the source; i.e., on the unique path from agent i to the source. We suppose that if in this path there is an edge of low quality then the individual has a connection with the source of low quality (if water acquires impurities somewhere along the path, the agent will get impure water). In this way, for each individual i, and each spanning tree $p \in \mathcal{S}(N_\omega)$

$$Q^i(p) = \prod_{r=0}^{t(i)} q_{p^r(i)p^{r+1}(i)}$$

denotes the quality that individual i gets, $Q^i(p) \in \{0,1\}$. Then, $Q(p) = \sum_{i=1}^{n} Q_p^i$ represents the number of agents obtaining high quality with the spanning tree p, and we use this value as a measure of the quality of the spanning tree. When necessary, we will make explicit the dependence of these quality functions on the quality matrix.

The objective is to implement a spanning tree with the highest possible quality at the lowest cost. We call such tree a max-quality optimal tree. Note that a traditional *mcstp* can be considered a particular case of our problem in which the quality matrix has all the entries of the same value; for instance, (N_ω, C) is the same problem as $(N_\omega, C, [0])$.

Definition 1. *A **max-quality optimal tree** in a qmcstp (N_ω, C, Q) is a spanning tree $M_q \in \mathcal{S}(N_\omega)$ satisfying:*

a) $Q(M_q) \geqslant Q(p)$ for all spanning tree $p \in \mathcal{S}(N_\omega)$.

b) $C(M_q) \leqslant C(r)$ for all spanning tree $r \in \mathcal{S}(N_\omega)$ such that $Q(r) = Q(M_q)$.

3. Max-quality optimal trees

In order to build a max-quality optimal tree, first note that quality requirements originates a partition into the set of agents: those who can achieve high quality, $Q_p^i = 1$, for some spanning tree $p \in \mathcal{S}(N_\omega)$, and the rest of the agents that can only receive low quality independently of the implemented spanning tree. We will denote these sets by N^1 and N^0, respectively

$$N^1 = \left\{i \in N : Q_p^i = 1 \text{ for some } p \in \mathcal{S}(N_\omega)\right\} \quad N^0 = \left\{i \in N : Q_p^i = 0 \text{ for all } p \in \mathcal{S}(N_\omega)\right\}$$

$N = N^1 \cup N^0$, $N^1 \cap N^0 = \varnothing$. Obviously, $i \in N^1$ does not imply $Q_p^i = 1$ for any spanning tree $p \in \mathcal{S}(N_\omega)$. The sets N^0 and N^1 depend on the quality matrix Q although we do not specify this dependence unless it is necessary. We will use the notation $S^0 = S \cap N^0$ and $S^1 = S \cap N^1$, for any subset of individuals $S \subseteq N$.

In order to share the cost of a selected spanning tree, each agent's allocation should depend on the quality he receives. In particular, an important problem will arise if an agent who is able to receive high quality, $i \in N^1$, finally receives low quality: how could this agent be compensated for this lack of quality? The following result proves that this problem can not happen in our model because for any max-quality optimal tree all the individuals who can obtain high quality will obtain it.

Proposition 1. *Let $M_q \in \mathcal{S}(N_\omega)$ be a max-quality optimal tree in (N_ω, C, Q). Then, $Q(M_q) = |N^1|$.*

Proof. Let $i \in N^1$ and let $p \in \mathcal{S}(N_\omega)$ be a spanning tree such that $Q^i(p) = 1$, and suppose $Q^i(M_q) = 0$. Then $Q^{p^k(i)}(p) = 1$, for $k = 0, 1, \ldots, t_p(i)$, $p^{t_p(i)}(i) = \omega$. Let us consider the set of agents $T = \left\{i, p(i), p^2(i), \ldots, p^{t_p(i)-1}(i)\right\}$, the chain which connects i to the source with the tree p, and let us define the new tree $s \in \mathcal{S}(N_\omega)$ as

$$s(j) = \begin{cases} p(j) & \text{if } j \in T \\ \\ M_q(j) & \text{if } j \notin T \end{cases}$$

First note that s is a spanning tree, since agents in T are connected to the source *via* p, and agents outside T either connect to the source *via* M_q, or connect to some agent in T already connected to the source.

Moreover, for all k and j, $q_{s^k(j)\, s^{k+1}(j)} \geqslant q_{M_q^k(j)\, M_q^{k+1}(j)}$ so

1. For all $j \neq i$, $Q^j(s) = \displaystyle\prod_{k=0}^{t_s(j)-1} q_{s^k(j)\, s^{k+1}(j)} \geqslant \prod_{k=0}^{t_{M_q}(j)-1} q_{M_q^k(j)\, M_q^{k+1}(j)} = Q^j(M_q)$
2. $Q^i(s) = 1 > Q^i(M_q) = 0$.

Then, $Q(s) = \displaystyle\sum_{j \in N} Q_s^j > \sum_{j \in N} Q_{M_q}^j = Q(M_q)$, a contradiction. Then $Q^i(M_q) = 1$. ■

Obviously, the existence of high quality connections between individuals in N^0 and N^1 is not possible because in case they existed, the individual in N^0 could connect to an individual in N^1 obtaining a high quality connection throughout him.

Prim's algorithm (Prim, 1957) is a well-known method to find a minimum cost spanning tree in a *mcstp* (N_ω, C). This method has n steps, as much as the number of individuals in the network. First, it connects to the source the agent i with smallest cost to the source, $c_{i\omega} \leqslant c_{j\omega}$, for all $j \in N$. In case that more than one agent fulfills this condition, any of them can be selected. In the second step, an agent in $N \setminus \{i\}$ with the smallest cost either to the source or to agent i, who is already connected, is selected and this connection is used. The process continues until all agents are connected, at each step connecting an agent still not connected to a connected agent or to the source. In general, several spanning trees with the same minimum cost may exist but, in case all the connection costs of the problem are different, there is only one minimum cost spanning tree.

Now, we adapt this algorithm for quality minimum spanning tree problems. We follow the next process:

Step 1 Identify the agents in N^1 (and then agents in $N^0 = N \setminus N^1$ are also identified).

Step 2 Modify the connection costs among individuals in N_ω in the following way:

$$c^+_{ij} = \begin{cases} +\infty & \text{if } i,j \in N^1_\omega \text{ and } q_{ij} = 0 \\ c_{ij} & \text{otherwise} \end{cases}$$

We call C^+ the *effective* cost matrix.

Step 3 Apply Prim's algorithm to find a minimum cost spanning tree in the problem $\left(N^1_\omega, C^+|_{N^1_\omega}\right)$. In this way, all individuals in N^1 are already connected.

Step 4 Choose the agent in N^0 with the minimum cost connection to N^1_ω and connect this agent with this cheapest connection.

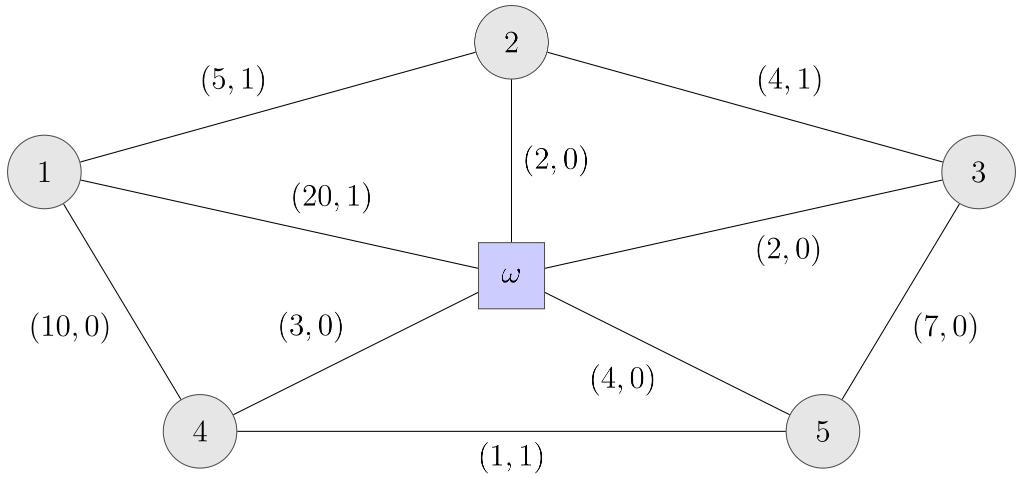
Step 5 Continue using Prim's algorithm in order to connect the remaining agents in N^0.

Proposition 2. *The described algorithm provides a max-quality optimal spanning tree for any qmstp* (N_ω, C, Q).

Proof. Let (N_ω, C, Q) be a *qmcstp* and apply the modified Prim's algorithm to obtain a tree p^*. From the way p^* is obtained (by applying Prim's algorithm in two *mcst* subproblems), it is clear that the result is a spanning tree and that all agents are connected to the source. Moreover, since the costs of the low quality edges among individuals in N^1 have been modified to $+\infty$, all the agents in N^1 are connected with high quality connections in p^*, so this is a maximum quality spanning tree. In order to prove the optimality in costs (there is not a maximum quality tree with a strictly lower cost), suppose a spanning tree r such that $Q(r) = Q(p^*)$. Since all agents in N^1 (and only these agents) are connected with high quality to the source, the low quality edges for agents in N^1 are not used, so $r|_{N^1}$ is a spanning tree in the problem $(N^1, C^+|_{N^1_\omega})$, so $C(r|_{N^1}) \geqslant C(p^*|_{N^1})$. Now, the connections used in r by agents $i \in N^0$ are available when applying the modified Prim's algorithm, so $C(r) \geqslant C(p^*)$ and p^* is a max-quality optimal tree. ■

The first step in the previous algorithm is the identification of the agents in N^1 (and consequently in N^0). It is not difficult to identify agents in those sets by considering the quality matrix Q as the adjacent matrix of a binary relation using any efficient algorithm for computing its transitive closure (Q_∞, the smallest transitive binary relation containing Q). Existing algorithms to compute Q_∞ can be classified into two categories: matrix multiplication based algorithms (see for instance Sankowski and Mucha (2008)) and graph traversal based algorithms (see for instance Simon (1988)). The agents in N^1 are those which can be reached from the source with the transitive closure. Then, if we know the matrix Q_∞, its first row will have entries with value 1 for the agents in N^1 and entries with value 0 for those in N^0, so Q_∞ characterizes the individuals that can obtain high quality.[1] The following example with five agents illustrates how the modified Prim's algorithm works.

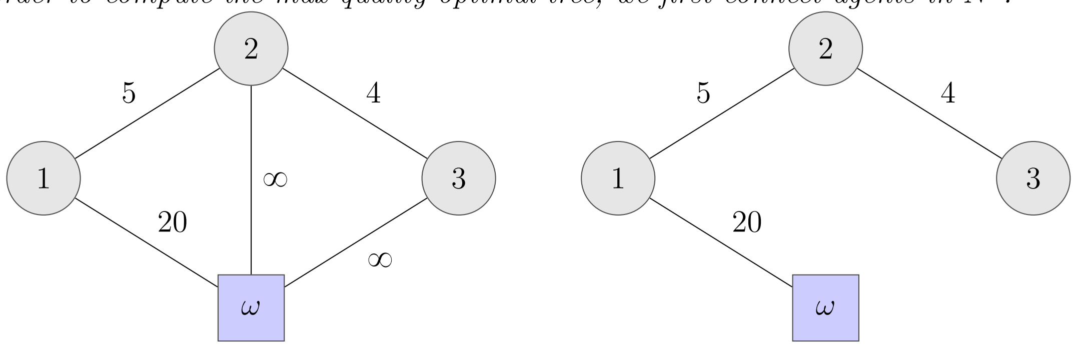
Example 1. *Let us consider the qmcstp with* $n = 5$ *individuals depicted in the following figure: the vector in each edge denotes, respectively, the cost and quality of such edge (the non depicted edges have a cost* $c_{ij} = 30$ *units, and low quality,* $q_{ij} = 0$*).*



It is easy to observe that only agents 3, 4 *and* 5 *are able to receive high quality and* $N^1 = \{1, 2, 3\}$, $N^0 = \{4, 5\}$. *Note that the matrix of the transitive closure of* Q *is*

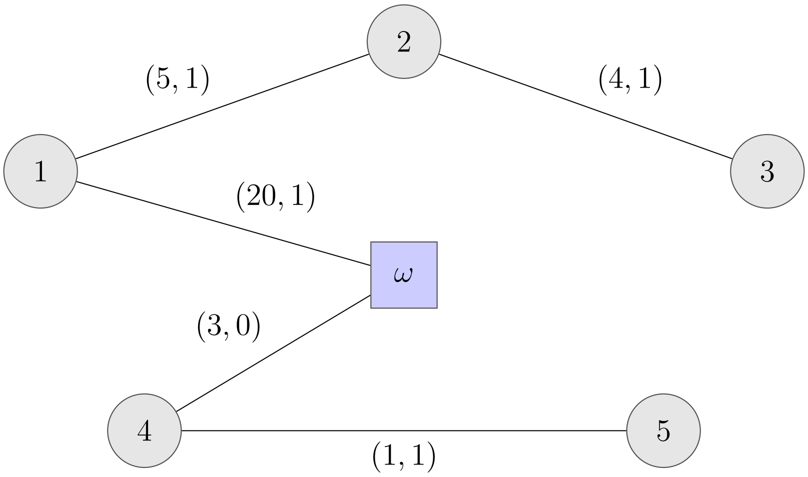
$$Q_\infty = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In order to compute the max-quality optimal tree, we first connect agents in N^1*:*



[1]Note that we consider N_ω ordered as $\{\omega, 1, 2, \ldots, n\}$, so the first row in Q corresponds to the quality levels of the connections of agents in N with the source, $q_{\omega i}$.

Now, we connect the individual in N^0 with minimum cost to those in N^1_ω and follow Prim's algorithm with the original costs. The final configuration of the (unique) max-quality optimal tree is



Therefore, the max-quality optimal tree has a cost $C(M_q) = 33$ units and agents 1, 2 and 3 receive high quality, $Q(M_q) = 3$.

Without considering quality, the unique minimum cost spanning tree m of the problem is: $m(2) = \omega$, $m(3) = \omega$, $m(4) = \omega$, $m(5) = 4$, $m(1) = 2$. The minimal cost is $C(m) = 13$ and the quality of this tree is $Q(m) = 0$. Therefore, obtaining high quality for the agents in N^1 supposes an extra cost of 20 units.

4. Sharing the cost of the max-quality optimal spanning trees

Once a particular spanning tree is selected, an important issue is how to allocate its cost among the agents. A *sharing rule* (or simply, a *solution*) is a function $\alpha : \mathcal{N}^q_n \to \mathbb{R}^n_+$ that proposes for any problem (N_ω, C, Q) an allocation for the cost of any max-quality optimal tree $M_p \in \mathcal{S}(N_\omega)$

$$\alpha(N_\omega, C, Q) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+, \quad \text{such that} \quad \sum_{i=1}^{n} \alpha_i = C(M_p).$$

Remark 1. *We assume that a sharing rule is non-negative, although this is not a general assumption in the literature. For instance, Kar or Cycle-complete solutions may propose negative allocations for some agents. Nevertheless, as pointed out in Bogomolnaia and Moulin (2010), non-negativity is considered "virtually ubiquitous in the literature on fair cost sharing."*

In this kind of situations cooperation is necessary in order to implement the efficient network. Then, a usual requirement is that the sharing of the cost of a tree is both individually rational (*no one pays more than his own connection to the source*) and collectively stable (*no coalition of agents should be charged more than the cost of connecting all of them to the source*). In order to define these concepts, a monotonic cooperative game associated to a *mcstp* (N_ω, C) is introduced by defining the characteristic function for any $S \subseteq N$:

$$v(S, C) = \min_{T \subseteq N} \{C(m(T_\omega, C|_T)) : S \subseteq T \subseteq N\}$$

where $C(m(T_\omega, C|_T))$ denotes the cost of the minimum cost spanning tree m in the problem $(T_\omega, C|_T)$. The above definition allows members of a coalition to freely connect throughout individuals outside their coalition in order to obtain the cheapest way of connecting the source. In this case, the cooperative game is monotonic. This situation is known as *non-property rights* approach and it is related with the non-negativity of the cost allocation (see, for instance, Bogomolnaia and Moulin (2010) and Trudeau (2013)).We follow this non-property rights approach.

In our context, however, individuals may be willing to pay more than their minimum cost, if the quality they receive is improved. That is, the distribution of the cost should depend not only on the cost of the connections, but also on the quality that each agent could obtain and the quality that the individual actually obtains with the implemented spanning tree. As we are maximizing the quality that the spanning tree provides, a modification of the characteristic function defining the cooperative game is needed by only considering spanning trees ensuring maximum quality. Then, given a *qmcstp* (N_ω, C, Q), for each *coalition* $S \subseteq N$, the quality characteristic function v^Q is defined by

$$v^Q(S, C, Q) = \min_{T \subseteq N} \{C(M_q(T_\omega, C|_T, Q|_T)) : S \subseteq T \subseteq N\}$$

where $C(M_q(T_\omega, C|_T, Q|_T))$ denotes the cost of the max-quality optimal tree in the problem $(T_\omega, C|_T, Q|_T)$. Unless necessary, we will simply use $v^Q(S)$. Now, the *quality-core* of a *qmcstp* (N_ω, C, Q) is defined in the usual way: those allocations of the total cost such that no individual, nor group of individuals, has incentives to quit and take on the project alone, since they can not improve in quality or reduce their cost.

Definition 2. *The Q-core associated to a quality minimum spanning tree problem* (N_ω, C, Q) *is defined by:*

$$co(N_\omega, C, Q) = \left\{ \alpha \in \mathbb{R}^n_+ : \sum_{i \in S} \alpha_i \leqslant v^Q(S), \ \forall S \subseteq N, \quad \sum_{i \in N} \alpha_i = v^Q(N) = C(M_q) \right\}$$

where $C(M_q)$ *denotes the cost of a max-quality optimal tree in the problem* (N_ω, C, Q).

Other compelling properties of a solution, besides to be in the core, are: Cost Monotonicity (the cost share should weakly increase if some of the direct connecting costs to other users or to the source increase), Continuity (cost shares should be continuous functions of the connecting costs) and Ranking (an agent more expensive to connect than another agent must not pay less). Next we define these properties, in the context of quality minimum cost spanning trees. On the other hand, we introduce properties concerning quality requirements: Quality Consistency, Quality Monotonicity and Quality Ranking.

- **Continuity (CO)**: The sharing rule $\alpha(N_\omega, C, Q)$ is a continuous function of C.

- **Strong Cost Monotonicity (SCM)**: For any pair of *qmcstp* with the same quality matrix, (N_ω, C, Q), (N_ω, C', Q), $C \leqslant C'$ implies $\alpha_i(N_\omega, C, Q) \leqslant \alpha_i(N_\omega, C', Q)$, for all $i \in N$.[2]

- **Cost Ranking (CRKG)**: For any *qmcstp* (N_ω, C, Q)

  If $i, j \in N^0$, $c_{ik} \leqslant c_{jk}$ for all $k \in N_\omega$ implies $\alpha_i(N_\omega, C, Q) \leqslant \alpha_j(N_\omega, C, Q)$.

  If $i, j \in N^1$, $c_{ik} \leqslant c_{jk}$ for all $k \in N_\omega$ implies $\alpha_i(N_\omega, C, Q) \leqslant \alpha_j(N_\omega, C, Q)$.

- **Quality Consistency (QC)**: For any *qmcstp* (N_ω, C, Q), if $i \in N^0(Q)$, then $\alpha_i(N_\omega, C, Q) \leqslant \alpha_i(N_\omega, C, [0])$.

  QC asks that the additional cost for obtaining maximum quality will not be charged on the agents that can not get high quality.

- **Quality Mononotonicity (QM)**: For any pair of *qmcstp* with the same cost matrix, (N_ω, C, Q), (N_ω, C, Q'), such that matrices Q and Q' coincide except in $q_{ij} = 0 < q'_{ij} = 1$ and $i \in N^1(Q')$ but $i \notin N^1(Q)$, then $\alpha_i(N_\omega, C, Q) \leqslant \alpha_i(N_\omega, C, Q')$.

  QM asks that whenever the quality of a connection is improved and an agent switch to receive high quality, this agent will not pay less in the new optimal spanning tree.

- **Quality Ranking (QRKG)**: For any *qmcstp* (N_ω, C, Q)

  If $i \in N^0$, $j \in N^1$, $c_{ik} = c_{jk}$ for all $k \in N_\omega$ implies $\alpha_i(N_\omega, C, Q) \leqslant \alpha_j(N_\omega, C, Q)$.

  QRKG asks that when two agents have identical connection costs but different possible quality levels, then the agent receiving high quality will not pay less than the one with low quality.

[2]This property implies the usual *cost monotonicity* condition (Bergantiños and Vidal-Puga, 2007): For any pair of *qmcstp* with the same quality matrix, (N_ω, C, Q) and (N_ω, C', Q) such that matrices C and C' coincide except in $c_{ik} < c'_{ik}$ for some $i \in N, k \in N_\omega$, then $\alpha_i(N_\omega, C, Q) \leqslant \alpha_i(N_\omega, C', Q)$.

As we have mentioned, in order to allocate the optimal cost of the network among the agents, we are interested in allocations within the Q-core, so that no subset of agents has incentives to give up the grand coalition in order to do the project independently since they can not improve in quality or reduce their cost. Then the non-emptiness of the Q-core is a crucial issue. For minimum cost spanning tree problems, Bird (1976) proposed an intuitive solution concept that always belongs to the core and, moreover, is easy to compute whenever the minimum cost spanning tree is unique. This solution essentially proposes that each agent pays the connection he directly uses in the optimal tree and can be extended to quality minimum spanning tree problems in a straightforward way.

Definition 3. *Given a qmcstp* (N_ω, C, Q)

1) *If there is just one max-quality optimal tree,* $M_q \in \mathcal{S}(N_\omega)$*, the* ***Bird solution*** *proposes the allocation:*

$$B_i(N_\omega, C, Q) = c_{i\, M_q(i)}$$

that is, each agent pays for the connection he directly uses.

2) *If there are several max-quality optimal trees,* $M_q^1,\ M_q^2,\ \ldots,\ M_q^k$*, then the* ***Bird solution*** *proposes the allocation:*

$$B_i(N_\omega, C, Q) = \frac{1}{k}\sum_{t=1}^{k} c_{i\, M_q^t(i)}$$

that is, the arithmetic average of the allocations corresponding to each max-quality optimal tree.

Next result shows that, as in the classical minimum cost spanning tree problem, the adaptation of *Bird* solution always belongs to the Q-core, showing that the Q-core is always a non-empty set.

Proposition 3. *The Bird solution of a quality minimum cost spanning tree problem is in the Q-core of the associated monotonic cooperative game. Then, for any qmcstp* (N_ω, C, Q) *its Q-core is a non-empty set.*

Proof. First consider a max-quality optimal tree $M_q^*(N_\omega, C, Q)$, and we allocate to each agent $i \in N$ the cost of the connection he uses directly in his way to the source, $B_i^* = c_{i\, M_q^*(i)}$. If we prove that this allocation is in the *Q-core*, as it is a convex set, the arithmetic average defining Bird solution (a particular convex combination) will lie in the *Q-core*.

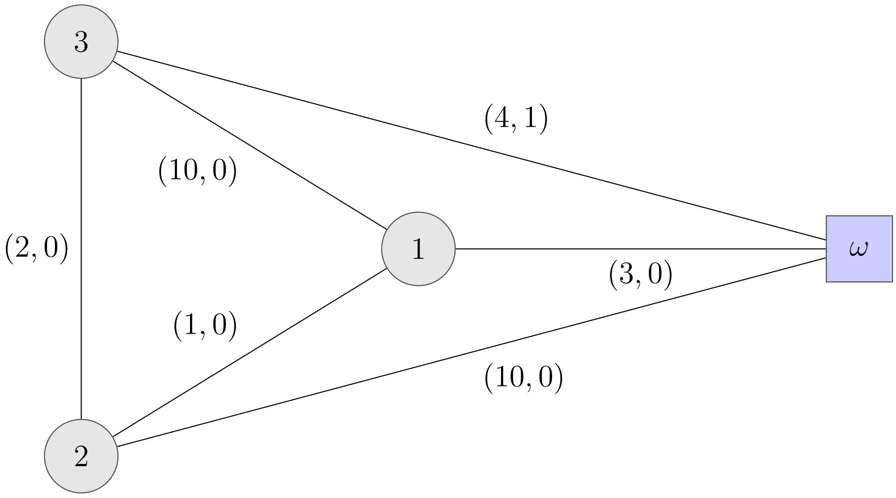
Suppose the result is not true; then, there is $S \subseteq N$, and $T \supseteq S$ such that

$$\sum_{i \in S} B_i^* > v^Q(S, C, Q) = C\left(M_q(T_\omega, C|_T, Q|_T)\right)$$

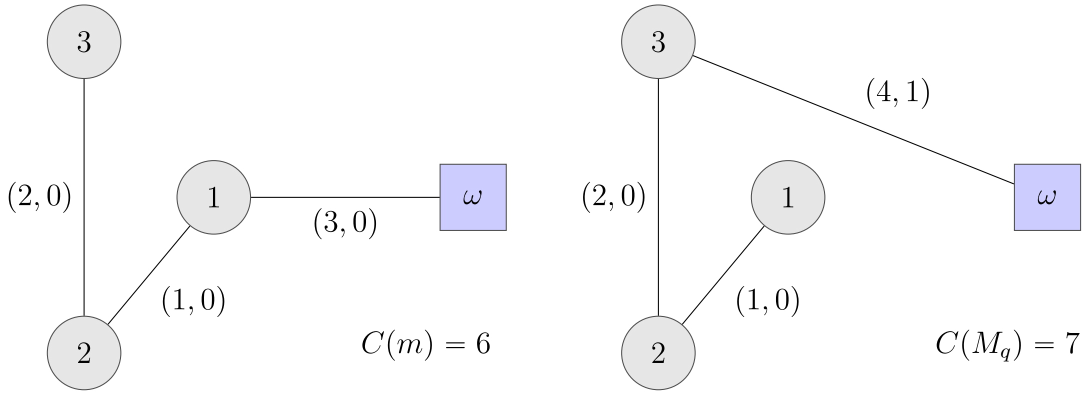
then,
$$\sum_{i\in T} B_i^* = \sum_{i\in T} c_{i\,M_q^*(i)} > v^Q(T,C,Q) = C\left(M_q(T_\omega, C|_T, Q|_T)\right).$$
By following the argument in Granot and Huberman (1984), departing from the max-quality optimal tree in T, $M_q(T_\omega, C|_T, Q|_T)$, we can add edges from $M_q^*(N_\omega, C, Q)$ joining agents outside T so that the cost of the max-quality tree so obtained is strictly lower than the one in the max-quality optimal tree $M_q^*(N_\omega, C, Q)$, and the quality remains unchanged. But this fact contradicts the optimality of $M_q^*(N_\omega, C, Q)$. ■

As Bird solution for *mcstp* is a particular case of our extension, $B(N_\omega, C) = B(N_\omega, C, [0])$, we know that this solution is little responsive to changes in costs, proposes extreme points in the core and fails to fulfill continuity and monotonicity properties. In addition, the Bird extension may have unwanted behavior in this context. Specifically, as shown in the following example, this solution can allocate to an agent in N^0 a higher payment than he would pay in the *mcstp* even though he does not improve in quality; that is, it violates Quality Consistency.

Example 2. *Let $N = \{1,2,3\}$ the set of agents in the following qmcstp*



The minimum cost spanning tree and the max-quality optimal tree in this problem are, respectively,



The Bird solution in the mcstp proposes the allocation $B = (3, 1, 2)$, whereas in the quality problem proposes $B^q = (1, 2, 4)$. Although agent 3, the one that obtains high quality, pays for the extra cost, agent 2 is allocated a greater amount in the quality problem even though he does not get high quality.

4.1. ***Extending the Folk solution***

In order to obtain a Q-core allocation with better properties, we are going to consider the extension of a solution that was first suggested for *mcstp* by Feltkamp et al. (1994) and independently rediscovered by Bergantiños and Vidal-Puga (2007) (we follow Bogomolnaia and Moulin (2010) and call it the *Folk* solution). This solution is based in the concept of the *irreducible* cost matrix, introduced by Bird (1976), that reduces the cost of each edge as much as possible, with the constraint to leave the total optimal cost of the project unchanged.

To extend the concept of irreducible cost matrix to quality minimum cost spanning tree problems, we use the effective cost matrix C^+ introduced in the modified Prim's algorithm. It must be noticed that the optimal cost of the max-quality optimal tree in the problems (N, C^+, Q) and (N, C, Q) is the same and then $v^Q(N, C^+, Q) = v^Q(N, C, Q)$.

Definition 4. *Given a qmcstp (N_ω, C, Q), the* ***q-irreducible*** *cost matrix is defined as the smallest[3] matrix C^* such that $C^* \leqslant C^+$ and $v^Q(N, C^*, Q_\infty) = v^Q(N, C, Q)$.*

The following method provides a way to calculate the q-irreducible cost matrix:

Step 1 Identify the agents in N^1 (and then agents in $N^0 = N \setminus N^1$ are also identified).
Step 2 Find a minimum cost spanning tree m_1 in the problem $\left(N^1_\omega, C^+|_{N^1_\omega}\right)$.
Step 3 For all $i, j \in N^1_\omega$ define the q-irreducible cost as:

$$c^*_{ij} = \max_{(k,l)\in m_1(i,j)} \left\{c^+_{kl}\right\} \qquad m_1(i,j): \text{ unique path in } m_1 \text{ from } i \text{ to } j$$

[3] The smallest in the sense that for any C' such that $C' \leqslant C^+$ and $v^Q(N, C', Q_\infty) = v^Q(N, C, Q)$ we have $C^* \leqslant C'$.

Step 4 Define the connection cost among any $i \in N^0$ and ω in the following way:

$$c^{++}_{i\omega} = \min\left\{c_{ik} : k \in N^1_\omega\right\}$$

and maintain the cost among individuals in N^0, $c^{++}_{ij} = c_{ij}$, for $i, j \in N^0$.

Step 5 Find a minimum cost spanning tree m_2 in the problem (N^0_ω, C^{++}).

Step 6 For all $i, j \in N^0_\omega$ define the q-irreducible cost as:

$$c^*_{ij} = \max_{(k,l)\in m_2(i,j)} \{c_{kl}\} \qquad m_2(i,j) : \text{ unique path in } m_2 \text{ from } i \text{ to } j$$

Step 7 For all $i \in N^0$ and $j \in N^1$ define the q-irreducible cost as:

$$c^*_{ij} = c^*_{i\omega}$$

Proposition 4. *The matrix C^* constructed with the above process is the q-irreducible cost matrix for the qmcstp (N_ω, C, Q).*

Proof. Given a *mcstp* (N_ω, C) the irreducible cost matrix (see Bird (1976); Bergantiños and Vidal-Puga (2007)) $\widetilde{C}$ is:

$$\widetilde{c}_{ij} = \max_{(k,l)\in m(i,j)} \{c_{kl}\} \qquad m_{(i,j)} : \text{ unique path in } m \text{ from } i \text{ to } j$$

where m is any *mcst* of the problem (N_ω, C)[4].

First, let us see that $C^* \leqslant C^+$. By construction, for all $i, j \in N^1_\omega$, the matrix $C^*|_{N^1_\omega} = \widetilde{C^+}|_{N^1_\omega}$ and then $C^*|_{N^1_\omega} \leqslant C^+|_{N^1_\omega}$. For all $i, j \in N^0$, the matrix $C^*|_{N^0} = \widetilde{C^{++}}|_{N^0}$ and then $C^*|_{N^0} \leqslant C^{++}|_{N^0} = C^+|_{N^0}$. Finally, for all $j \in N^1_\omega$, $i \in N^0$, $c^*_{ij} = c^*_{i\omega} \leqslant c^{++}_{i\omega} = \min\{c_{ik} : k \in N^1_\omega\} \leqslant c_{ij} = c^+_{ij}$.

Obviously, $v^Q(N, C^*, Q_\infty) = v^Q(N, C^*, Q) \leqslant v^Q(N, C^+, Q) = v^Q(N, C, Q)$. In order to see that the equality holds, suppose a max-quality optimal tree in the *qmcstp* (N, C^*, Q), M^*_q and a max-quality optimal tree for (N, C, Q), M_q.

Then $v^Q(N, C^*, Q) = C(M^*_q, C^*)$ and $v^Q(N, C, Q) = C(M_q, C)$. As for any connection in M_q, $c_{ij} = c^*_{ij}$ we have that $C(M^*_q, C^*) \leqslant C(M_q, C^*) = C(M_q, C)$. In case that $C(M^*_q, C^*) < C(M_q, C)$, there is $i \in N$ such that $c^*_{iM^*_q(i)} < c_{iM_q(i)}$, but by the way C^* has been constructed this is not possible. Then $C(M_q, C) = C(M^*_q, C^*)$.

Finally, let us see that if we consider a cost matrix C' such that $C' \leqslant C^+$ and $v^Q(N, C', Q) = v^Q(N, C, Q)$ we have $C^* \leqslant C'$. Suppose that $c^*_{ij} > c'_{ij}$ for some $i, j \in$

[4]This notion of irreducible cost is independent of the selection of the minimum cost spanning tree m. Matrix $\widetilde{C}$ has the property that the cost of the *mcst* is the same in the problems (N_ω, C) and $(N_\omega, \widetilde{C})$.

N_ω. If $i,j \in N^1_\omega$ this fact contradicts $C^*|_{N^1_\omega} = \widetilde{C^+}|_{N^1_\omega}$. If $i,j \in N^0$ this contradicts $C^*|_{N^0} = \widetilde{C^{++}}|_{N^0}$. If $j \in N^1_\omega$, $i \in N^0$, and $c'_{ij} < c^*_{ij} = \min\{c_{ik} : j \in N^1_\omega\} \leqslant c_{ij} = c^+_{ij}$, then we can consider the max-quality optimal tree for the problem (N,C,Q), M_q, that fulfills $v^Q(N,C^+,Q) = v^Q(N,C,Q) = C(M_q,C^+)$. By changing in M_q the first connection of individuals from N^0 to individuals in N^1 by (i,j), we obtain that the new spanning tree M'_q has the same quality level and fulfills $v^Q(N,C',Q) \leqslant C(M'_q,C') < C(M_q,C) = v^Q(N,C,Q)$, a contradiction. ■

Once the q-irreducible matrix is obtained, an irreducible quality minimum cost spanning tree problem can be defined by considering the transitive closure of Q. That is, to any *qmcst* problem (N_ω,C,Q) we associate the q-irreducible problem (N_ω,C^*,Q_∞). Now, we define the *q-Folk* solution by using the Shapley value of the monotonic cooperative game defined by (N_ω,C^*,Q_∞).

Definition 5. *Given a qmcstp (N_ω,C,Q) the* ***q-Folk*** *solution allocates to each agent $i \in N$ the amount*

$$F^q_i(N_\omega,C,Q) = Sh_i(N,v^{*Q}) \qquad i = 1,2,\ldots,n$$

where v^{*Q} is the quality cooperative game defined by the problem (N_ω,C^*,Q_∞).

For short, sometimes we use the notation $v^{*Q}(S) = v^Q(S,C^*,Q_\infty)$. The following result shows that the *q-Folk* proposal is in the Q-core. The proof is based on the fact that v^{*Q} is a concave game and then the Shapley value is in the core of v^{*Q}, which is included in the Q-core associated to the problem (N_ω,C,Q) because all the max-quality optimal trees in (N_ω,C,Q) are spanning trees with the same quality for (N_ω,C^*,Q_∞).

Proposition 5. *The q-Folk solution of a qmcstp (N_ω,C,Q) is in the Q-core of the associated monotonic cooperative game.*

Proof. Let us see that the cooperative game v^{*Q} is concave.

By the way in which C^* is computed, without lost of generality, we can consider the individuals in N ordered in the following way (see, for instance, Bergantiños and Vidal-Puga (2007)): $N^1 = \{i^1_1, i^1_2, ..., i^1_{n_1}\}$, $N^0 = \{i^0_1, i^0_2, ..., i^0_{n_0}\}$ such that

$$M^*_q(i^1_1) = \omega, M^*_q(i^1_k) = i^1_{k-1} \text{ for all } k, 1 < k \leqslant n_1$$

$$M^*_q(i^0_1) \in N^1_\omega, M^*_q(i^0_k) = i^0_{k-1} \text{ for all } k, 1 < k \leqslant n_0$$

Then it is easy to check that if we consider $S \subseteq N$ and $i \notin S$:

- if $i \in N^1$ then $v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min\limits_{j \in S^1_\omega} c^*_{ij}$.
- if $i \in N^0$ then $v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min\limits_{j \in S^0_\omega} c^*_{ij}$.

Then if $S \subseteq T \subseteq N$ and $i \notin T$, for $i \in N^1$,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{j \in S_\omega^1} c_{ij}^* \geqslant \min_{j \in T_\omega^1} c_{ij}^* = v^{*Q}(T \cup \{i\}) - v^{*Q}(T)$$

In the case $i \in N^0$,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{j \in S_\omega^0} c_{ij}^* \geqslant \min_{j \in T_\omega^0} c_{ij}^* = v^{*Q}(T \cup \{i\}) - v^{*Q}(T)$$

So the cooperative game is concave and the Shapley value is in the core of v^{*Q}, which is included in the Q-core. ■

Example 3. *For the qmcstp in Example 1, the q-irreducible cost matrix is*

$$C^* = \begin{pmatrix} 0 & 20 & 20 & 20 & 3 & 3 \\ 20 & 0 & 5 & 5 & 3 & 3 \\ 20 & 5 & 0 & 4 & 3 & 3 \\ 20 & 5 & 4 & 0 & 3 & 3 \\ 3 & 3 & 3 & 3 & 0 & 1 \\ 3 & 3 & 3 & 3 & 1 & 0 \end{pmatrix}$$

and the q-Folk solution is $F^q = (10, 9.5, 9.5, 2, 2)$. *Folk allocation for the mcstp (without considering quality requirements) is* $F = (5, 2, 2, 2, 2)$. *The three agents obtaining high quality share the extra-cost.*

5. Responsiveness of the *q-Folk* solution

In this section we show that the *q-Folk* solution keeps the properties of the *Folk* solution and also satisfies the introduced quality properties.

Proposition 6. *The q-Folk solution satisfies CO, SCM and CRKG.*

Proof. The proof follows similar arguments as in the case of the classical *mcstp* (see Bergantiños and Vidal-Puga (2007)).

1. The *q-Folk* solution satisfies CO:

For a fixed quality matrix Q, the *q-Folk* solution can be seen as the composition of the following functions that are obviously continuous: $g_1(C) = C^+$, $g_2(C) = C^*$ and $g_3(C) = v^{*Q}$, $g_4(v) = Sh(N, v)$. So the *q-Folk* solution is continuous with respect to the cost matrix.

2. The *q-Folk* solution satisfies SCM: Let $S \subseteq N$ and $i \notin S$. If $i \in N^1$, we know that

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{j \in S_\omega^1} c_{ij}^*$$

On the other hand, $C \leqslant C'$ implies $C^* \leqslant C'^*$ and, as the quality values are the same for both problems,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{j \in S^1_\omega} c^*_{ij} \leqslant \min_{j \in S^1_\omega} c'^*_{ij} = v'^{*Q}(S \cup \{i\}) - v'^{*Q}(S)$$

The same reasoning applies if $i \in N^0$. Then,

$$F^q(N_\omega, C, Q) = Sh(N, v^{*Q}) \leqslant Sh(N, v'^{*Q}) = F^q(N_\omega, C', Q).$$

3. The *q-Folk* solution satisfies CRKG:

We prove the property for agents in N^1. The case for agents in N^0 follows an identical argument. Let $i, j \in N^1$ such that $c_{ik} \leqslant c_{jk}$ for all $k \in N_\omega$. First, we note that $c^*_{ik} \leqslant c^*_{jk}$ for all $k \in N_\omega$, since

a) If $k \in N^0$, $c^*_{ik} = c^*_{jk}$.

b) If $k \in N^1$ and we suppose $c^*_{ik} > c^*_{jk}$, the edge (i, j'), $j' = M_q(j)$ (or $j = M_q(j')$, depending on the position of k) fulfills

$$c_{ij'} \leqslant c_{jj'} \leqslant c^*_{jk} < c^*_{ik}$$

so adding this edge and removing the expensive one we can obtain a cheaper spanning tree, contradicting the optimality of M_q.

Therefore, for any subset of individuals $S \subseteq N$, $i \notin S$ we obtain:

1) If $j \notin S$,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{k \in S^1_\omega} c^*_{ik} \leqslant \min_{k \in S^1_\omega} c^*_{jk} = v^{*Q}(S \cup \{j\}) - v^{*Q}(S)$$

2) If $j \in S$, then $S = T \cup \{j\}$, $i, j \notin T$, and we need to prove

$$v^{*Q}(T \cup \{i, j\}) - v^{*Q}(T \cup \{j\}) \leqslant v^{*Q}(T \cup \{i, j\}) - v^{*Q}(T \cup \{i\})$$

or, equivalently,

$$v^{*Q}(T \cup \{i\}) - v^{*Q}(T) \leqslant v^{*Q}(T \cup \{j\}) - v^{*Q}(T)$$

that reduces to the previous case.

So,

$$F^q_i(N_\omega, C, Q) = Sh_i(N, v^{*Q}) \leqslant Sh_j(N, v^{*Q}) = F^q_j(N_\omega, C, Q). \quad \blacksquare$$

Proposition 7. *The q-Folk solution satisfies QC, QM and QRKG.*

Proof.
1. The *q-Folk* solution satisfies QC:

Let (N_ω, C, Q) be a *qmcstp* and let $i \in N^0$ an individual receiving low quality. Let us see that $F_i^q(N_\omega, C, Q) \leqslant F_i(N_\omega, C, [0])$. Note that $F(N_\omega, C, [0])$ is the Folk solution for the *mcstp* (N_ω, C). Then, it is enough to prove that for all $S \subseteq N$, $i \notin S$, if $\widetilde{v}$ is the monotonic game for the problem $(N_\omega, \widetilde{C})$

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) \leqslant \widetilde{v}(S \cup \{i\}) - \widetilde{v}(S)$$

or, equivalently, see for instance Bogomolnaia and Moulin (2010),

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{j \in S^0_\omega} c^*_{ij} = \min_{j \in S_\omega} c^*_{ij} \leqslant \widetilde{v}(S \cup \{i\}) - \widetilde{v}(S) = \min_{j \in S_\omega} \widetilde{c}_{ij}$$

Then, if $c^*_{ij} \leqslant \widetilde{c}_{ij}$, for all $j \in S_\omega$ we obtain the result. But,

$$C^*|_{N_0} = \widetilde{C^{++}}|_{N_0} \leqslant \widetilde{C}|_{N_0}$$

since $C^{++}|_{N_0} \leqslant C|_{N_0}$. Then, $c^*_{ij} \leqslant \widetilde{c}_{ij}$, for $j \in S^0$. On the other hand, if $j \in S^1$,

$$c^*_{ij} = c^*_{i\omega} = \widetilde{c^{++}}_{i\omega} \leqslant \widetilde{c}_{ij}$$

and $c^*_{ij} \leqslant \widetilde{c}_{ij}$, for all $j \in S_\omega$.

2. The *q-Folk* solution satisfies QM:

Let (N_ω, C, Q) and (N_ω, C, Q'), two *qmcst* problems with the same cost matrix, such that matrices Q and Q' coincide except in $q_{ij} = 0 < q'_{ij} = 1$ with $i \in N^1(Q')$ but $i \notin N^1(Q)$. Then, $j \in N^1(Q)$ and $N^1(Q) \cup \{i\} \subseteq N^1(Q')$.

Let us consider a subset of individuals, $S \subseteq N$, such that $i \notin S$. In order to prove that

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) \leqslant v^{*Q'}(S \cup \{i\}) - v^{*Q'}(S)$$

as the entries of the q-irreducible matrix C^* depend on the quality matrix Q we write $c^*(Q)_{ij}$ and $c^*(Q')_{ij}$ to denote this dependence. The individuals in S that can obtain high quality also depend on quality matrices, so we denote them by $S^0(Q) = S \cap N^0(Q)$ and $S^1(Q) = S \cap N^1(Q)$. Then,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{k \in S^0(Q)_\omega} c^*(Q)_{ik} \qquad \text{since } i \in N^0(Q) \tag{1}$$

$$v^{*Q'}(S \cup \{i\}) - v^{*Q'}(S) = \min_{k \in S^1(Q')_\omega} c^*(Q')_{ik} \qquad \text{since } i \in N^1(Q') \tag{2}$$

Note that the expression in Equation (1) is lower or equal to c_{ij}, since

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{k \in S^0(Q)_\omega} c^*(Q)_{ik} \leqslant c^*(Q)_{i\omega} = c^*(Q)_{ij} \leqslant c_{ij}$$

In order to prove that the expression in Equation (1) is not greater than the expression in Equation (2) we distinguish two cases:

a) If $S^0(Q) = S^0(Q')$, then for any max-quality optimal spanning tree M'_q in the problem (N_ω, C, Q'), it is clear that the edge (i,j) must belong to M'_q, since it is the only possibility for agent i to obtain high quality. Then, $c_{ij} = c^*(Q')_{ij}$ and $(i,j) \in M'_q(i,k)$ for any $k \in N^1(Q')$. If $k \in S^1(Q') = S^1(Q)$, then

$$c^*(Q')_{ik} \geqslant c^*(Q')_{ij} = c_{ij}$$

and the required inequality holds.

b) If $S^0(Q) \neq S^0(Q')$, then $S^1(Q') \cap S^0(Q) \neq \varnothing$. For all $k_1 \in S^1(Q') \cap S^0(Q)$

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) \leqslant c^*(Q)_{k_1 i}$$

On the other hand,

$$v^{*Q'}(S \cup \{i\}) - v^{*Q'}(S) = \min_{k \in S^1(Q')_\omega} c^*(Q')_{ik} = c^*(Q')_{ik_2}$$

If $k_2 \in S^1(Q)_\omega$, as in the previous case $c^*(Q')_{ik_2} \geqslant c_{ij}$. Finally, if $k_2 \notin S^1(Q)_\omega$ then $k_2 \in S^1(Q') \cap S^0(Q)$ and

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) \leqslant c^*(Q)_{k_2 i} \leqslant c^*(Q')_{k_2 i}$$

proving the inequality.

3. The *q-Folk* solution satisfies QRKG:

Let (N_ω, C, Q) be a *qmcstp* such that there are two individuals $i \in N^0$ and $j \in N^1$ with equal connecting costs: $c_{ik} = c_{jk}$, for all $k \in N_\omega$. Let M_q be a max-quality optimal tree in the problem, $M_q = m_1 \cup m_2$ (see the construction of the optimal tree). For $r, l \in N_\omega$ let us denote by $N_{M_q}(r,l)$ the set of all the nodes in the path in M_q connecting r and l.

a) Let $S \subseteq N$, such that $i \notin S$, $j \notin S$. Then,

$$v^{*Q}(S \cup \{i\}) - v^{*Q}(S) = \min_{k \in S^0_\omega} c^*_{ik} = c^*_{ik^0}$$

$$v^{*Q}(S \cup \{j\}) - v^{*Q}(S) = \min_{k \in S^1_\omega} c^*_{jk} = c^*_{jk^1}$$

Let us prove first that $c^*_{ik^0} = c^*_{i\omega} = c_{iM_q(i)}$.

For any $r \in N_{M_q}(i,\omega)$, we know that $c^*_{ir} \geqslant c_{iM_q(i)}$. In the case $c^*_{ir} > c_{iM_q(i)}$, we could remove in the spanning tree M_q the edge with cost, c^*_{ir} adding the edge $(M_q(i), j)$ and obtaining a new tree with maximum quality and lower cost, since $c_{jM_q(i)} = c_{iM_q(i)} < c^*_{ir}$, which contradicts the optimality of the tree M_q. Consequently, we obtain $c^*_{ir} = c_{iM_q(i)}$. In particular, by applying this reasoning for $r = \omega$, then $c^*_{i\omega} = c_{iM_q(i)}$. We now distinguish three possibilities depending on the positions of the nodes i and k^0:

a1) If $k^0 \in N_{Mq}(i,\omega)$, we have shown $c^*_{ik^0} = c^*_{i\omega} = c_{iM_q(i)}$.

a2) If $i \in N_{Mq}(k^0,\omega)$, let $i' \in N_{Mq}(k^0,\omega)$ such that $M_q(i') = i$. Then $c_{i'i} \leqslant c^*_{ik^0}$. If $c^*_{ik^0} < c^*_{i\omega} = c_{iM_q(i)}$, we can change the edge $(i, M_q(i))$ by the edge (i', j) with the same quality and lower cost, contradicting the optimality of M_q. So, $c^*_{ik^0} \geqslant c^*_{i\omega}$ and as we are minimizing this cost, $c^*_{ik^0} = c^*_{i\omega}$.

a3) In other case, $k^0 \notin N_{Mq}(i,\omega)$, $i \notin N_{Mq}(k^0,\omega)$, then $M_q(i) \in N_{Mq}(i, k^0)$, so $c^*_{ik^0} \geqslant c^*_{i\omega} = c_{iM_q(i)}$ and as we are minimizing this cost, $c^*_{ik^0} = c^*_{i\omega}$.

Then, the equality $c^*_{ik^0} = c^*_{i\omega} = c_{iM_q(i)}$ is proved.

Consider now the following possibilities for $c^*_{jk^1}$:

i) If $k^1 \in N_{Mq}(j,\omega)$ and $c^*_{jk^1} < c^*_{i\omega} = c_{iM_q(i)}$, we can change in M_q the edge $(i, M_q(i))$ by $(i, M_q(j))$, obtaining a new spanning tree with the same quality and lower cost, since $c_{iM_q(j)} = c_{jM_q(j)} \leqslant c^*_{jk^1} < c^*_{i\omega}$, which contradicts the optimality of M_q. Then, $c^*_{jk^1} \geqslant c^*_{i\omega} = c^*_{ik^0}$.

ii) If $j \in N_{Mq}(k^1,\omega)$, let $j' \in N_{Mq}(k^1,\omega)$ such that $M_q(j') = j$. Then, $c_{j'j} \leqslant c^*_{jk^1}$. If $c^*_{jk^1} < c^*_{i\omega}$, we can change the edge $(i, M_q(i))$ by the edge (i, j') obtaining the same quality and lower cost, contradicting the optimality of M_q.

iii) In other case, $k^1 \notin N_{Mq}(j,\omega)$, $j \notin N_{Mq}(k^1,\omega)$, we can change the edge $(i, M_q(i))$ by the edge $(i, M_q(j))$ obtaining a tree with the same quality and lower cost, contradicting the optimality of M_q.

Then, in all cases $c^*_{ik^0} \leqslant c^*_{jk^1}$.

b) Let $S \subseteq N$, such that $i \notin S$, $j \in S$. Then, $S = T \cup \{j\}$, such that $i, j \notin T$. We want to prove that

$$v^{*Q}(T \cup \{i,j\}) - v^{*Q}(T \cup \{j\}) \leqslant v^{*Q}(T \cup \{i,j\}) - v^{*Q}(T \cup \{i\})$$

or, equivalently, for each $T \subseteq N$, $i, j \notin T$,

$$v^{*Q}(T \cup \{i\}) - v^{*Q}(T) \leqslant v^{*Q}(T \cup \{j\}) - v^{*Q}(T)$$

that coincides with the property proved in a).

Then, $F^q_i(N_\omega, C, Q) \leqslant F^q_j(N_\omega, C, Q)$. ■

6. Final comments

We have presented an extension of the well-known minimum cost spanning tree problem, in which agents care about the quality they receive with the optimal network. In fact, we prioritize the quality side of the problem, selecting the cheapest tree among those with highest quality. Our approach has two remarkable points:

a) In the optimal tree, all the individuals who can reach high quality will have it.
b) Individuals not receiving high quality are not harmed by the greater cost of the implemented quality optimal network. The additional cost (if any) is shared by the agents obtaining high quality.

We have extended to our context of quality spanning trees one of the most relevant solution concept in the *mcstp* literature: the *Folk* solution. And we have showed that our extension maintains the main properties of the original solution, specially the core selection: no individual, nor coalition of individuals, wants to leave the great coalition. Properties analyzing the behavior of our extension regarding quality aspects have also been introduced.

There are some aspects that could be more generals in our model. One of them is the possible quality levels: we suppose that quality may only take two levels (high and low). This allows us to determine the objective in quality easily (maximize the number of agents with high quality). If we had more quality levels, for instance high, medium and low, there would not be a trivial way to derive the quality that an agent would receive and the maximization objective would not be clear either.

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