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Invited Review

The radius of robust feasibility of uncertain mathematical programs: A Survey and recent developments[☆]

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ABSTRACT

The radius of robust feasibility provides a numerical value for the largest possible uncertainty set that guarantees feasibility of a robust counterpart of a mathematical program with uncertain constraints. The objective of this review of the state-of-the-art in this field is to present this useful tool of robust optimization to its potential users and to avoid undesirable overlapping of research works on the topic as those we have recently detected. In this paper we overview the existing literature on the radius of robust feasibility in continuous and mixed-integer linearly constrained programs, linearly constrained semi-infinite programs, convexly constrained programs, and conic linearly constrained programs. We also analyze the connection between the radius of robust feasibility and the distance to ill-posedness for different types of uncertain mathematical programs.

1. Introduction

The radius of robustness for certain desirable property of a given uncertain mathematical object is, roughly speaking, the greatest size of the uncertainty set such that this property is preserved for any possible value of the uncertain parameter. For instance, radii of robustness concepts have been introduced in the context of committee elections (Misra & Sonar, 2019), where the property to be preserved is the voting rule and the perturbations affect the input preference orders, and in Schur polynomials (Choo, 2014; Gao & Sun, 2002; Mastorakis, 2000), where the property to be preserved is the stability of the given (nominal) polynomial and the perturbations affect its coefficients.

In particular, two concepts of radii of robustness have been proposed in the emerging field of robust optimization (see, e.g., Ben-Tal, El Ghaoui, & Nemirovski, 2009, Ben-Tal & Nemirovski, 1999, Ben-Tal & Nemirovski, 2000, Ben-Tal & Nemirovski, 2001, Ben-Tal & Nemirovski, 2002, Ben-Tal & Nemirovski, 2008, Bertsimas & Brown, 2009, Bertsimas, Brown, & Caramanis, 2011, Bertsimas & Sim, 2004, etc.): that of radius of highly robustness introduced in Goberna, Jeyakumar, Li, and Vicente-Pérez (2018) for uncertain multi-objective convex programs, where the property to be preserved is the existence of highly robust weakly efficient solutions and the perturbations affect all the data, and that of radius of robust feasibility for optimization problems with uncertain constraints reviewed in this paper, which was introduced in Goberna, Jeyakumar, Li, and Vicente-Pérez (2014) as the largest size of the uncertainty sets so that the robust counterpart remains feasible.

We consider in this paper uncertain problems of the form

$$(P)\qquad \min_{x\in\mathbb{Z}^k\times\mathbb{R}^{n-k}} \; f(x) \quad \text{s.t.} \quad g(x)\in -K, \tag{1}$$

where x is the decision variable, $k\in\{0,\dots,n\}$, $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a deterministic function, $K\subset\mathbb{R}^I$ is a given convex cone, I is an arbitrary (possibly infinite) set, and $g=(g_i)_{i\in I}:\mathbb{R}^n\longrightarrow\mathbb{R}^I$ is an uncertain mapping. Following Goberna and López (1998, 2018), we associate with (P) its constraint system, posed in $\mathbb{Z}^k\times\mathbb{R}^{n-k}$, denoted by $\sigma_P=\{g(x)\in -K\}$, and its feasible set $F_P=\left\{x\in\mathbb{Z}^k\times\mathbb{R}^{n-k}: g(x)\in -K\right\}$.

According to Ben-Tal and Nemirovski (2002), the major questions associated with the application of the robust optimization methodology to (P) include the following:

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- When and how can we reformulate (P) as a (computational) tractable optimization problem, or at least approximate (P) by a tractable problem. This requires to chose a tractable uncertainty set, e.g., the solution set of a system of either linear inequalities, or conic quadratic inequalities, or linear matrix inequalities (see Ben-Tal & Nemirovski, 2002, Theorem 1). This objective is not viable for complex uncertain problems as the mixed-integer linear programs.
- How to specify reasonable uncertainty sets in specific applications.

Regarding the uncertain constraint mapping g in (P), we now assume that it not only depends on the decision variable x, but also on an uncertain parameter u, whose corresponding uncertainty set is some nonnegative multiple $\alpha\mathcal{U}$ of some *pattern-set* $\mathcal{U}$ that will be assumed to be a convex subset containing the null vector of certain linear space. To each $\alpha \geq 0$ determining the size of the uncertainty set $\alpha\mathcal{U}$, we associate the *parameterized robust counterparts* of σ_P, F_P, and (P), that is, the system, posed in $\mathbb{Z}^k \times \mathbb{R}^{n-k}$,

$$\sigma_P^\alpha := \{g(x,u) \in -K,\ u \in \alpha\mathcal{U}\},$$

the set

$$F_P^\alpha := \left\{x \in \mathbb{Z}^k \times \mathbb{R}^{n-k} : g(x,u) \in -K \ \text{ for all } u \in \alpha\mathcal{U}\right\}$$

and the optimization problem

$$(RP_\alpha) \quad \min_{x \in F_P^\alpha} \quad f(x),$$

respectively. Obviously, σ_P^α may be infeasible (i.e., F_P^α be empty) whenever α is too large, in which case the optimal value $v(RP_\alpha)$ of (RP_α) is $+\infty$ by convention. We assume that the *nominal problem* (RP_0) is feasible, i.e., $F_P^0 = F_P \neq \emptyset$. Thanks to the assumptions on $\mathcal{U}$,

$$0 \leq \alpha_1 \leq \alpha_2 \implies \alpha_1\mathcal{U} \subset \alpha_2\mathcal{U} \implies F_P^{\alpha_2} \subset F_P^{\alpha_1}. \tag{2}$$

Due to (2), $\{\alpha \in \mathbb{R}_+ : F_P^\alpha \neq \emptyset\}$ is an interval in $\mathbb{R}_+$ (with 0 as lower limit), maybe $\{0\}$ or the whole of $\mathbb{R}_+$.

The radius of robust feasibility can be defined as the largest size of the uncertainty set $\alpha\mathcal{U}$ so that (RP_α) is feasible. More precisely, the *radius of robust feasibility* (RRF in short) of the uncertain problem (P) is the extended real number

$$\rho_P := \sup\{\alpha \in \mathbb{R}_+ : F_P^\alpha \neq \emptyset\} \in \mathbb{R}_+ \cup \{+\infty\}. \tag{3}$$

The RRF ρ_P is said to be *attained* when the max in (3) exists and equals sup, that is, $F_P^{\rho_P} \neq \emptyset$.

We first note from (2) that the optimal value function $v(RP_\alpha)$ of (RP_α) is a non-decreasing function of α along the interval $[0,\rho_P]$. As pointed out in Bertsimas and Sim (2004), there is a price to pay for an increase of safety (understood as the degree of confidence in the feasibility of the computed optimal solution of the robust counterpart for any conceivable perturbation of the data).

Any robust optimizer should choose her/his suitable value of α, which determines her/his preferred uncertainty set $\alpha\mathcal{U}$, by balancing the price of robustness with the price of safeness according to her/his attitude towards risk, with $\alpha = \rho_P$ ($\alpha = 0$, respectively) only for extremely pessimistic (optimistic) decision makers. When $0 < \rho_P < +\infty$, sensible measures for the price of robustness and for the price of safeness, in scale 0–1, would be the ratios $\frac{v(RP_\alpha)-v(RP_0)}{v(RP_{\rho_P})-v(RP_0)}$ and $\frac{\alpha}{\rho_P}$, respectively.

Let us illustrate this decision-making situation regarding robust optimization modelling with a toy example.

Example 1. Consider the one-dimensional uncertain optimization problem

$$\begin{array}{lll}(P) & \min\limits_{x\in\mathbb{R}} & f(x) = x^2 \\ & \text{s.t.} & g(x) := (-x, x-2) \in -\mathbb{R}_+^2,\end{array}$$

where f is deterministic while the coefficients of each of the two constraints, $g_1(x) := -x \leq 0$ and $g_2(x) := x - 2 \leq 0$, are uncertain with pattern-set $\mathcal{U} = [-1,1]^2$. This means, regarding the first constraint $(-1)x \leq 0$, that it can be perturbed as $g_1(x,u) := (-1+u_1)x - u_2 \leq 0$, with $u = (u_1,u_2) \in [-1,1]^2$; analogously, the second constraint can be perturbed as $g_2(x,u) := (1+u_1)x - (2+u_2) \leq 0$, with $u_1,u_2 \in [-1,1]$. Then, the uncertain robust feasible set is given by $F_P^\alpha = \{x : g_i(x,u) \leq 0, i = 1,2, \text{ for all } u = (u_1,u_2) \in \alpha\mathcal{U}\}$. For $\alpha \geq 0$ sufficiently small, one has

$$\begin{aligned} F_P^\alpha &= \{x \in \mathbb{R} : (1 \pm \alpha)x \geq \pm\alpha, (1 \pm \alpha)x \leq 2 \pm \alpha\} \\ &= \left(\left[\frac{\alpha}{1-\alpha}, +\infty\right[\right) \cap \left(\left]-\infty, \frac{2-\alpha}{1+\alpha}\right]\right) = \left[\frac{\alpha}{1-\alpha}, \frac{2-\alpha}{1+\alpha}\right], \end{aligned}$$

so that $F_P^\alpha \neq \emptyset$ if and only if $\alpha \in \left[0, \frac{1}{2}\right]$, with $F_P^{0.5} = \{1\}$. Thus, the RRF $\rho_P = \frac{1}{2}$ is attained. Since $v(RP_\alpha) = \left(\frac{\alpha}{1-\alpha}\right)^2$, the maximum price of robustness is $v(RP_{\rho_P}) - v(RP_0) = 1 - 0 = 1$, and the prices of robustness and safeness for choosing a parameter α are, in scale 0–1, $\left(\frac{\alpha}{1-\alpha}\right)^2$ and 2α, respectively.

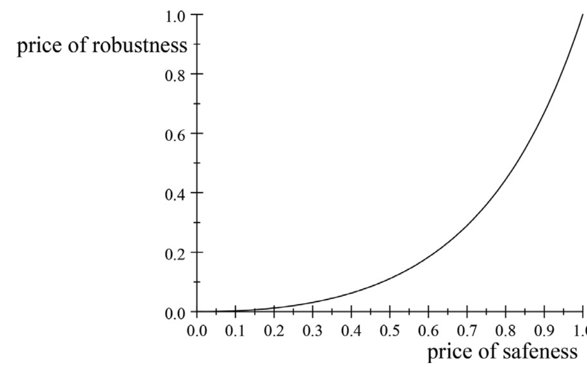

Fig. 1. Trade-off between prices.

Fig. 1 allows to compare both prices as α grows from 0 to 0.5, i.e., both prices between 0 and 1. It shows that small values of α have small impact on the optimal value of the robust counterpart, for instance, one gets the 25, 50 and 75% of the maximum price of robustness for $\alpha = \frac{1}{3}$, $\alpha = \sqrt{2} - 1 \simeq 0.4142$, and $\alpha = 2\sqrt{3} - 3 \simeq 0.4641$, respectively (equivalent to the 66.67%, the 82.84% and the 92.82% of the maximum price of safeness, respectively).

This bi-objective approach to robust modelling, consisting in the simultaneous maximization of the safety price and minimization of the robustness price has been used in the facility location setting (Carrizosa & Nickel, 2003), before the introduction of the RRF in the literature, i.e., without determining the interval of variation of α providing feasible robust counterparts.

The generic intention of this review of RRF is to present this useful tool of robust optimization to its potential users and to avoid undesirable overlapping of future research works on the topic. To do this, we comment known results on the RRF and prove new ones for the following five types of optimization problems:

- **Linear programming (LP) problems** (and other linearly constrained programs): $k = 0$, $I = \{1,\ldots,m\}$, $K = \mathbb{R}_+^m$ (where $\mathbb{R}_+ := [0,+\infty[$), and $g_1,\ldots,g_m$ are affine functions. The problem (P) in Example 1 belongs to this class of problems.
- **Mixed-integer linear programming (MILP) problems** (and other linearly constrained programs with integer constraints): $k > 0$, $I = \{1,\ldots,m\}$, $K = \mathbb{R}_+^m$, and $g_1,\ldots,g_m$ are affine functions.
- **Linear semi-infinite programming (LSIP) problems** (and other programs with infinitely many linear constraints): $k = 0$, I is an infinite set, $K = \mathbb{R}_+^I$, and g_i is an affine function for all $i \in I$.

- **Convex programming (CP) problems** (and other programs with a finite number of convex constraints): $k = 0$, $I = \{1, \ldots, m\}$, $K = \mathbb{R}^m_+$, and $g_1, \ldots, g_m$ are convex functions.
- **Conic linear programming (CLP) problems** (and other programs with linear conic constraint): $k = 0$, $I = \{1, \ldots, m\}$, and $g_1, \ldots, g_m$ are affine functions.

We are primarily interested in the available formulas and methods to compute the RRF ρ_P, or at least lower and upper bounds for ρ_P, always expressed in terms of the data, emphasizing those situations in which it is possible to obtain "numerically tractable" formulas and bounds for the RRF.

We are secondarily interested in necessary and sufficient conditions for $\rho_P > 0$ (positiveness of the RRF) and for the attainability of ρ_P, problems which have been recently considered in the framework of MILP (Liers, Schewe, & Thürauf, 2021) and we analyze here for LP, LSIP, CP, and CLP. We also provide scalarized versions of these results which are suitable for those uncertain optimization problems whose constraints, of the form $g_i(x) \le 0$, are expressed in different units.

Finally, we also consider the existing connection of the RRF with the well-studied concept of "distance to ill-posedness" in parametric LP, LSIP, and CLP, providing new numerically tractable formulas and bounds for the distance to ill-posedness with respect to feasibility in CLP. Formulas and bounds for the "distance to ill-posedness" in parametric LP and LSIP (respectively, CLP) can be found in Cánovas, Gómez-Senent, and Parra (2007), Cánovas, López, Parra, and Toledo (2005), Cánovas, López, Parra, and Toledo (2006), Cánovas, López, Parra, and Toledo (2011) (respectively, Renegar (1994), Freund and Vera (1999), Vera (2014)).

When (P) is an uncertain LP or CLP problem, or belongs to certain types of uncertain CP problems, the computation of $v(RP_\alpha)$ for a given $\alpha \in [0, \rho_P[$ requires to solve a tractable optimization problem provided that $\mathcal{U}$ is conveniently chosen (some tractable set). Otherwise, taking into account that rough estimations of the ratios $\frac{v(RP_\alpha)-v(RP_0)}{v(RP_{\rho_P})-v(RP_0)}$ are sufficient to decide the suitable α, the "computationally intractable" programs (RP_α) can be approximately solved by means of the available numerical methods, e.g., the linear SIP methods recently reviewed in Goberna and López (2018), for uncertain LP and LSIP, or convex SIP methods, as those proposed in Auslender, Ferrer, Goberna, and López (2015), Gao, Yiu, and Wu (2018), Guo and Sun (2020), Mehrotra and Papp (2014), Okuno, Hayashi, Yamashita, and Gomoto (2016) and Pang, Lv, and Wang (2016) (all of them published along the last six years), and references therein, for uncertain CP. The numerical Examples 3–5 in Liers et al. (2021) illustrate three different situations for the trade-off between robustness and minimum cost in MILP.

The paper is organized as follows. Section 2 deals with uncertain linearly constrained programs, Section 3 with uncertain mixed-integer linearly constrained programs, Section 4 with uncertain linearly constrained semi-infinite programs (including the connection between the RRF and the "distance to ill-posedness"), Section 5 with uncertain convexly constrained programs, and Section 6 with uncertain conic linearly constrained programs.

These methods are based on solving (preferably) tractable optimization problems whose objective function may be either the distance from the origin to certain subset of $\mathbb{R}^{n+1}$ (as it happens with the distance to ill-posedness in quantitative stability theory), the Minkowski gauge function or the support function of certain subset set of $\mathbb{R}^{n+1}$.

Section 7 analyzes for the first time the relationship between the RRF and the distance to ill-posedness for CLP uncertain programs. Finally, Section 8 summarizes the content of the paper and identifies the main open problems.

The main antecedents of the paper are as follows. The first two papers on RRF, (Goberna, Jeyakumar, Li, & Vicente Pérez, 2015; Goberna et al., 2014), dealt with uncertain LSIP and uncertain multi-objective LP, respectively, providing different proofs of the formula for the RRF, when the pattern-set is the Euclidean unit ball; the proof in Goberna et al. (2014) was based on formulas for the distance to ill-posedness in LSIP (Cánovas et al., 2005), where the notion of epigraphical set plays a crucial role, while the proof in Goberna et al. (2015) was based on an existence theorem for linear semi-infinite systems in Fan (1968); the unit ball was replaced, as pattern-set, in Chuong and Jeyakumar (2017) by an arbitrary convex body, using as main tools the gauge function and the epigraphical set; finally, (Liers et al., 2021) has revisited recently the RRF in LP, introducing safe and deterministic constraints and more flexible pattern-sets.

The unique antecedents for Sections 3 and 4 are (Goberna et al., 2014; Liers et al., 2021), respectively. The antecedents for Section 5, under affine perturbations, are (Chen, Li, Li, Lv, & Yao, 2020; Li & Wang, 2018), which mimic the methodology used in Chuong and Jeyakumar (2017) in the linear framework, i.e., they combine gauge functions and epigraphical sets; moreover, (Goberna, Jeyakumar, Li, & Linh, 2016) provided computable formulas for the RRF in CP under strong assumptions on the constraint functions, but allowing polynomial perturbations (instead of affine ones). Section 6 also has a unique antecedent, (Goberna, Jeyakumar, & Li, 2021), whose methodology is inspired in that of Goberna et al. (2014) via the linearization of the positive dual cone of K. Finally, the antecedents of Section 7 are a stream of works on ill-posedness in CLP, (Freund & Vera, 1999; Renegar, 1994; Vera, 2014), which is here tackled from the RRF perspective.

2. RRF of uncertain linearly constrained programs

We consider, as in uncertain LP, constraint systems posed in $\mathbb{R}^n$ of the form

$$\sigma_{LP} = \left\{ a_i^\top x \le b_i, i = 1, \ldots, m \right\},$$

where $(a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$ are uncertain vectors, with $a_i = \left(a_i^1, \ldots, a_i^n\right)^\top$ for $i \in I = \{1, \ldots, m\}$, and the symbol $^\top$ denotes transpose.

We also assume that the pattern-set $\mathcal{U} \subset \left(\mathbb{R}^{n+1}\right)^m$ is the cartesian product $\prod_{i \in I} \mathcal{U}_i$ of m convex sets $\mathcal{U}_i \subset \mathbb{R}^{n+1}$ such that $0_{n+1} \in \mathcal{U}_i$ for all $i \in I$. We denote by u_i the ith component of $u \in \mathcal{U}$, i.e., $u = (u_1, \ldots, u_m)$. Regarding g, whose ith component is $g_i(x) := a_i^\top x - b_i$, we assume the existence of a vector $(\overline{a}_i, \overline{b}_i) \in \mathbb{R}^{n+1}$ such that the uncertainty of g_i is captured by the expression

$$g_i(x, u) := \left((\overline{a}_i, \overline{b}_i) + u_i\right)^\top \begin{pmatrix} x \\ -1 \end{pmatrix},$$

for all $u_i \in \mathcal{U}_i$ and $x \in \mathbb{R}^n$. So, the parameterized *robust counterpart* of σ_{LP}, posed in $\mathbb{R}^n$, is

$$\sigma_{LP}^\alpha := \left\{ a_i^\top x \le b_i,\ (a_i, b_i) \in \left(\overline{a}_i, \overline{b}_i\right) + \alpha \mathcal{U}_i, i \in I \right\}$$

with solution set

$$F_{LP}^\alpha = \left\{ x \in \mathbb{R}^n : a_i^\top x \le b_i \text{ for all } (a_i, b_i) \in \left(\overline{a}_i, \overline{b}_i\right) + \alpha \mathcal{U}_i, i \in I \right\}.$$

Thus, the RRF of σ_{LP} reads

$$\rho_{LP} := \sup \{\alpha \in \mathbb{R}_+ : F_{LP}^\alpha \ne \emptyset\}.$$

From the existence theorem for linear systems (Fan, 1968, Theorem 1), denoting by cl cone X the closed convex hull of X,

$$F_{LP}^\alpha \ne \emptyset \Longleftrightarrow (0_n, -1) \notin \operatorname{cl\,cone} \left\{ \bigcup_{i \in I} \left[\left(\overline{a}_i, \overline{b}_i\right) + \alpha \mathcal{U}_i \right] \right\}.$$

We assume that $F_{LP}^0 \ne \emptyset$, that is $(0_n, -1) \notin \operatorname{cone}\left\{ \left(\overline{a}_i, \overline{b}_i\right), i \in I \right\}$ or, equivalently,

$$\sup_{x \in \mathbb{R}^n, y \in \mathbb{R}} \left\{ y : \overline{a}_i^\top x + y \le \overline{b}_i, i \in I \right\} \ge 0,$$

so that the fulfilment of $F_{LP}^0 \neq \emptyset$ can be checked by solving a feasible LP program.

Following Liers et al. (2021), the ith constraint is called *safe* (or *deterministic*) whenever $\mathcal{U}_i = \{0_{n+1}\}$. We denote by S the set of safe indices, i.e., $S = \{i \in I : \mathcal{U}_i = \{0_{n+1}\}\}$. Analogously, the jth variable x_j is called *safe* whenever the j-th projection of $\mathcal{U}_i$ is $\{0\}$ for all $i \in I$.

The following *interiority assumption* guarantees that $0 \leq \rho_{LP} < +\infty$ (cf. Goberna et al., 2015, Lemma 1), but obviously precludes the existence of safe constraints and variables.

(**A1**) There exists a compact convex set Z such that $0_{n+1} \in \operatorname{int} Z$ and $\mathcal{U}_i = Z$ for all $i \in I$.

From the existence theorem for linear systems (Fan, 1968, Theorem 1) (whose finite dimensional version is Gale's alternative theorem (Goberna and López, 1998, Corollary 3.1.1)), since $\mathcal{U}_i = Z$ for all $i \in I$, denoting by conv X the convex hull of X; ...

$$F_{LP}^{\alpha} \neq \emptyset \iff (0_n, -1) \notin \operatorname{cl} \mathbb{R}_+ \left(\operatorname{conv}\left\{(\overline{a}_i, \overline{b}_i), i \in I\right\} + \alpha Z\right). \tag{4}$$

Condition (4) can be checked by solving tractable programs in simple cases, e.g., an LP program whenever Z is a polytope.

We associate with σ_{LP} satisfying (**A1**) the *epigraphical set*

$$E(\overline{a}, \overline{b}) := \operatorname{conv}\left\{(\overline{a}_i, \overline{b}_i), i \in I\right\} + \mathbb{R}_+\{(0_n, 1)\}. \tag{5}$$

The first known formula for the RRF was given for the special case of (**A1**) in which $Z \subset \mathbb{R}^{n+1}$ is the unit Euclidean closed ball $\mathbb{B}_{n+1}$:

$$\begin{aligned} \rho_{LP} &= \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right) \\ &:= \inf\{\|(a,b)\| : (a,b) \in E(\overline{a}, \overline{b})\}, \end{aligned} \tag{6}$$

that is, the Euclidean distance from $E(\overline{a}, \overline{b})$ to the origin. The first proof of (6), in (Goberna et al., 2014, Theorem 2.5), used stability analysis tools introduced in Cánovas et al. (2005), while the second one, in Goberna et al. (2015, Theorem 4), was direct. In geometrical terms, computing ρ_{LP} by means of (6) consists in projecting the origin 0_{n+1} onto the epigraphical set. This geometrical problem can be reformulated as a tractable optimization one as follows:

$$\begin{aligned} (\rho_{LP})^2 &= \min_{(a,b) \in E(\overline{a}, \overline{b})} \|(a,b)\|^2 \\ &= \min_{(\delta,\mu) \in \Delta_m \times \mathbb{R}_+} \left\| \left(\sum_{i=1}^m \delta_i \overline{a}_i, \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right) \right\|^2, \end{aligned} \tag{7}$$

where $\Delta_m = \left\{\delta \in \mathbb{R}_+^m : \sum_{i=1}^m \delta_i = 1\right\}$ is the *unit simplex* in $\mathbb{R}^m$. The solvability of the linearly constrained convex quadratic program in (7) does not guarantee that ρ_{LP} is attained.

The unit ball $\mathbb{B}_{n+1}$ was replaced in Chuong and Jeyakumar (2017) by an arbitrary convex body (i.e., a full dimensional compact convex set) Z such that $0_{n+1} \in \operatorname{int} Z$. The assumptions on Z guarantee the continuity of its *Minkowski function* or *gauge* ϕ_Z, defined for every $x \in \mathbb{R}^n$ by

$$\phi_Z(x) = \inf\{t > 0 : x \in tZ\}.$$

The gauge ϕ_Z is a norm whenever Z is a symmetric compact convex set such that $0_{n+1} \in \operatorname{int} Z$.

Proposition 2 (Two formulas for ρ_{LP} *under* (**A1**)). *(Chuong & Jeyakumar, 2017, Theorem 2.1 and Corollary 2.1) Under* (**A1**), *one has*

$$\rho_{LP} = \inf_{(a,b) \in E(\overline{a}, \overline{b})} \phi_Z(-a, -b). \tag{8}$$

If, additionally, Z is symmetric, then

$$\rho_{LP} = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right). \tag{9}$$

where dist *is the distance associated to the norm ϕ_Z on $\mathbb{R}^{n+1}$ generated by Z.*

The analytic formula (8) is exactly (Chuong and Jeyakumar, 2017, Theorem 2.1) (and also the linear version of (Li & Wang, 2018, Theorem 3.1)), while the geometric formula (9) is (Chuong and Jeyakumar, 2017, Corollary 2.1) (and also the linear version of (Li & Wang, 2018, Corollary 3.2)).

From (8) it is possible to compute the RRF ρ_{LP} by solving tractable programs when Z is a *spectrahedron* in $\mathbb{R}^{n+1}$, i.e., there exist $n+2$ symmetric $q \times q$ matrices $A_0, \ldots, A_{n+1}$ such that

$$Z = \left\{ z \in \mathbb{R}^{n+1} : A_0 + \sum_{i=1}^{n+1} z_i A_i \succeq 0 \right\},$$

where $A_i \succeq 0$ means that A_i is a square positive semidefinite matrix. The uncertainty sets of many robust optimization problems arising in practice are spectrahedra, e.g., ellipsoids, balls, polytopes and boxes (Nie, 2013; Ramana & Goldman, 1995; Vinzant, 2014). Remarkable features of this large class of sets is that they are always closed and convex, as they can be written as intersections of closed half-spaces,

$$Z = \left\{ z \in \mathbb{R}^{n+1} : s^\top \left(A_0 + \sum_{i=1}^{n+1} z_i A_i \right) s \geq 0, s \in \mathbb{S}^{q-1} \right\},$$

where $\mathbb{S}^{q-1}$ denotes the unit sphere in $\mathbb{R}^q$, and that the other two conditions involved in (**A1**), boundedness of Z and $0_{n+1} \in \operatorname{int} Z$, can be checked in terms of the matrices $A_0, \ldots, A_{n+1}$ (see, e.g., Goberna & López, 1998, Theorems 5.9 and 9.3 and Nie, 2013, Page 252). In the following three cases, ρ_{LP} can be found by solving tractable programs obtained from (8):

- If Z is an ellipsoid centered at the origin, it can be written as $Z = \{z \in \mathbb{R}^{n+1} : z^\top M^{-1} z \leq 1\}$, with M being a positive definite symmetric $(n+1) \times (n+1)$ matrix. Then, by (Chuong and Jeyakumar, 2017, Corollary 3.1),

$$(\rho_{LP})^2 = \min_{(\delta,\mu) \in \Delta_m \times \mathbb{R}_+} f(\delta, \mu), \tag{10}$$

  where

$$f(\delta, \mu) = \left(\sum_{i=1}^m \delta_i \overline{a}_i, \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right)^\top M^{-1} \left(\sum_{i=1}^m \delta_i \overline{a}_i, \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right).$$

  So, ρ_{LP} can be found by solving a linearly constrained convex quadratic program. In particular, if $Z = \mathbb{B}_{n+1}$, (10) with $M = I_n$ collapses to (7).
- If Z is the ℓ_1 unit ball, i.e., $Z = \left\{z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} |z_i| \leq 1\right\}$, by (Chuong & Jeyakumar, 2017, Corollary 3.3),

$$\rho_{LP} = \min_{(\delta,\mu) \in \Delta_m \times \mathbb{R}_+} \left\{ \sum_{j=1}^n \left| \sum_{i=1}^m \delta_i \overline{a}_i^j \right| + \left| \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right| \right\}. \tag{11}$$

  Thus, ρ_{LP} can be computed by solving a linear program (see, e.g., (Aragón, Goberna, López, & Rodríguez, 2019, § 1.1.5.4)).
- If Z is the ℓ_∞ unit ball, i.e., $Z = \left\{z \in \mathbb{R}^{n+1} : |z_j| \leq 1, j = 1, \ldots, n+1\right\}$, by (Chuong & Jeyakumar, 2017, Corollary 3.4),

$$\begin{aligned} \rho_{LP} &= \min_{(\delta,\mu) \in \Delta_m \times \mathbb{R}_+} \max \left\{ \left| \sum_{i=1}^m \delta_i \overline{a}_i^j \right|, j = 1, \ldots, n; \left| \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right| \right\} \\ &= \min_{(\delta,\mu,\gamma) \in \Delta_m \times \mathbb{R}_+^2} \left\{ \gamma : \gamma \geq \pm \sum_{i=1}^m \delta_i \overline{a}_i^j, j = 1, \ldots, n; \gamma \geq \pm \left(\mu + \sum_{i=1}^m \delta_i \overline{b}_i \right) \right\}. \end{aligned} \tag{12}$$

  So, ρ_{LP} can be found by solving a linear program, too.

Now we show that, under (**A1**), it is possible to check the positivity and the attainability of ρ_{LP} by solving suitable LP programs.

Even though the next result is the linear version of Proposition 18 below, on uncertain CP, we include here a direct proof that will inspire the corresponding result for uncertain LSIP. It consists in proving that $\rho_{LP} > 0$ is equivalent to the fulfilment of the *Slater condition*, that is, the existence of some $\widehat{x} \in \mathbb{R}^n$ such that $\overline{a}_i^\top \widehat{x} < \overline{b}_i$ for all $i \in I$ (recall that, for finite linear systems, $\operatorname{int} F_{LP}^0 \neq \emptyset$ implies the Slater condition and the converse statement holds whenever $\left(\overline{a}_i, \overline{b}_i\right) \neq 0_{n+1}$ for all $i \in I$).

Proposition 3 (Positiveness of ρ_{LP} *under* (**A1**)*). If* (**A1**) *holds, then*

$$\rho_{LP} > 0 \iff \sup_{x \in \mathbb{R}^n, y \in \mathbb{R}} \left\{ y : \overline{a}_i^\top x + y \leq \overline{b}_i, i \in I \right\} > 0.$$

Proof. [$\Longrightarrow$] Let $\rho_{LP} > 0$. Then, by (9), $0_{n+1} \notin E(\overline{a}, \overline{b})$ and so, there exists a hyperplane in $\mathbb{R}^{n+1}$which separates strongly 0_{n+1} from the polyhedron $E(\overline{a}, \overline{b})$. Let $(u, v) \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ and $\gamma \in \mathbb{R}$ be such that

$$(u, v)^\top (a, b) < \gamma < (u, v)^\top 0_{n+1} = 0, \text{ for all } (a, b) \in E(\overline{a}, \overline{b}).$$

Since

$$0^+ E(\overline{a}, \overline{b}) = 0^+ \left[\operatorname{conv} \left\{ (\overline{a}_i, \overline{b}_i), i \in I \right\} + \mathbb{R}_+ \{(0_n, 1)\} \right] = \mathbb{R}_+ \{(0_n, 1)\},$$

where $0^+ E(\overline{a}, \overline{b})$ stands for the recession cone of $E(\overline{a}, \overline{b})$, and $(u, v)^\top (a, b)$ is bounded above on $E(\overline{a}, \overline{b})$, necessarily $v = (u, v)^\top (0_n, 1) \leq 0$. Two cases can arise:

I. If $v < 0$, given $i \in I$, one has $\left(-\frac{u}{v}, -1\right)^\top (\overline{a}_i, \overline{b}_i) < -\frac{\gamma}{v}$, i.e., $\overline{a}_i^\top \left(-\frac{u}{v}\right) + \frac{\gamma}{v} < \overline{b}_i$. Thus,

$$\sup_{x \in \mathbb{R}^n, y \in \mathbb{R}} \left\{ y : \overline{a}_i^\top x + y \leq \overline{b}_i, i \in I \right\} \geq \frac{\gamma}{v} > 0.$$

II. If $v = 0$, we have $\overline{a}_i^\top u < \gamma$ for all $i \in I$. For a sufficiently large positive scalar μ one has $\mu\left(\overline{a}_i^\top u - \gamma\right) \leq \overline{b}_i$, for all $i \in I$. Then, $\overline{a}_i^\top (\mu u) - \mu\gamma \leq \overline{b}_i$, for all $i \in I$. So,

$$\sup_{x \in \mathbb{R}^n, y \in \mathbb{R}} \left\{ y : \overline{a}_i^\top x + y \leq \overline{b}_i, i \in I \right\} \geq -\mu\gamma > 0.$$

[$\Longleftarrow$] Let $\overline{x} \in \mathbb{R}^n$ and $\overline{y} \in \mathbb{R}$ be such that $a_i^\top \overline{x} + \overline{y} \leq b_i$ for all $i \in I$ and $\overline{y} > 0$. We now show that $0_{n+1} \notin E(\overline{a}, \overline{b})$ by contradiction. Let $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}_+$ be such that $\sum_{i \in I} \lambda_i = 1$ and $\sum_{i \in I} \lambda_i (\overline{a}_i, \overline{b}_i) + \mu(0_n, 1) = 0_{n+1}$. We thus get the following contradiction:

$$0 = (\overline{x}, -1)^\top 0_{n+1} = \sum_{i \in I} \lambda_i \left(a_i^\top \overline{x} - b_i\right) - \mu$$

$$\leq -\left(\overline{y} \sum_{i \in I} \lambda_i + \mu \right) = -(\overline{y} + \mu) < 0.$$

This completes the proof. □

Section 3 shows why it is important in robust MILP to determine whether ρ_{LP} is attained or not. Recall that the pointed cone of a convex cone K in $\mathbb{R}^n$ is $K \cap (\operatorname{lin} K)^\perp$.

Proposition 4 (Attainment of ρ_{LP} *under* (**A1**)*). If the pointed cone of* $\operatorname{cone}\{\overline{a}_i, i \in I\} \times \mathbb{R}_+$ *is a half-line, then* ρ_{LP} *is attained. Moreover, if* Z *is symmetric,* ρ_{LP} *is attained if and only if*

$$(0_n, -1) \notin \mathbb{R}_+ \left(\operatorname{conv} \left\{ (\overline{a}_i, \overline{b}_i), i \in I \right\} + \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right) Z \right). \quad (13)$$

Proof. According to (Goberna et al., 2014, Proposition 2.3), a sufficient condition for ρ_{LP} being attained is that the recession cone $0^+ F_{LP}^0 = \{x \in \mathbb{R}^n : \overline{a}_i^\top x \leq 0, i \in I\}$ of the *nominal solution set* F_{LP}^0 of the nominal system σ_{LP}^0 is a linear subspace of $\mathbb{R}^n$. This fact follows by (Goberna & López, 1998, Theorem 5.13(ii)), since $0^+ F_{LP}^0$ is an affine manifold if and only if the pointed cone of $\operatorname{cone}\{\overline{a}_i, i \in I\} \times \mathbb{R}_+$ is a half-line.

Last statement immediately follows, under (**A1**), from Proposition 2 and (4). □

The second statement of Proposition 4 has the advantage that it only involves the data, $\{(\overline{a}_i, \overline{b}_i), i \in I\}$, and the disadvantage that it is a hardly checkable condition as it involves the exact value of $\rho_{LP} = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right)$. However, Proposition 4 can be useful in some particular cases.

- If $Z = \mathbb{B}_{n+1}$, (13) fails if and only if $(0_n, -1)$ can be expressed as $\beta\left(\sum_{i=1}^m \delta_i (\overline{a}_i, \overline{b}_i) + \gamma(u, v)\right)$, with $\beta > 0$, $\sum_{i=1}^m \delta_i = 1$, $\delta_i \geq 0$ for all $i = 1, \ldots, m$, $0 \leq \gamma \leq \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right) = \rho_{LP}$, and $(u, v) \in Z$. Denoting $\mu = \frac{1}{\beta}$, (13) holds if and only if there do not exist $\mu > 0$, $\delta \in \Delta_m$, and $(v, w) \in \rho_{LP}\mathbb{B}_{n+1}$ such that $(0_n, -\mu) = \sum_{i=1}^m \delta_i (\overline{a}_i, \overline{b}_i) + (v, w)$, i.e., $\sum_{i=1}^m \delta_i \overline{a}_i = -v$ and $\sum_{i=1}^m \delta_i \overline{b}_i + \mu = -w$, with $\|(v, w)\|^2 = \left\| \left(\sum_{i=1}^m \delta_i \overline{a}_i, \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right) \right\|^2 \leq (\rho_{LP})^2$. This is equivalent to assert that the optimal value of the following convex quadratic problem with linear objective function is not positive:

$$\sup_{\delta \in \Delta_m} \left\{ \mu : \left\| \left(\sum_{i=1}^m \delta_i \overline{a}_i, \mu + \sum_{i=1}^m \delta_i \overline{b}_i \right) \right\|^2 \leq (\rho_{LP})^2 \right\}.$$

- If $Z = \operatorname{conv}\left\{ \left(v_j, w_j\right), j \in J \right\} \subset \mathbb{R}^{n+1}$, with J finite, (13) holds if and only if there does not exist $\mu = \left(\mu_{i,j}\right)_{(i,j) \in I \times J} \in \mathbb{R}_+^{I \times J}$ such that

$$(0_n, -1) = \sum_{(i,j) \in I \times J} \mu_{i,j} \left[\left(\overline{a}_i, \overline{b}_i\right) + \rho_{LP}\left(v_j, w_j\right) \right],$$

  if and only if the following LP problem is not unbounded:

$$\inf_{\mu \in \mathbb{R}_+^{I \times J}} \left\{ \sum_{(i,j) \in I \times J} \mu_{i,j} \left(\overline{b}_i + \rho_{LP} w_j \right) : \sum_{(i,j) \in I \times J} \mu_{i,j} \left(\overline{a}_i + \rho_{LP} v_j \right) = 0_n \right\}.$$

The equal size condition on the uncertainty sets in (**A1**) may seem unrealistic in practical situations where the vector of coefficients of g_i represents a random vector in $\mathbb{R}^{n+1}$ with Gaussian distribution, with mean $\left(\overline{a}_i, \overline{b}_i\right)$ and variance-covariance matrix $\lambda_i I_n$, for some $\lambda_i > 0$ (e.g., the standard deviation of the scalars $\left\| (a_i, b_i) - \left(\overline{a}_i, \overline{b}_i\right) \right\|$) for all $i \in I$). Then, the natural choice of the uncertainty set for the i-th constraint is $\mathcal{U}_i = \lambda_i \mathbb{B}_{n+1}$, with $\lambda_i \in \mathbb{R}_{++} :=]0, +\infty[$, that is, the pattern set $\mathcal{U}$ is the cartesian product of m Euclidean balls of different radii, reflecting the fact that the vectors of coefficients of the different constraints have different degree of uncertainty. The next assumption allows to handle this type of situations, but not only for Euclidean balls.

(**A2**) There exists a vector $\lambda \in \mathbb{R}_{++}^m$ and a compact convex set Z such that $0_{n+1} \in \operatorname{int} Z$ and $\mathcal{U}_i = \lambda_i Z$ for all $i \in I$.

Obviously, the *scaled interiority assumption* (**A2**) means that the uncertainty sets of all constraints are identical up to scaling, i.e., that given $i \neq j$, $\mathcal{U}_i$ is a positive multiple of $\mathcal{U}_j$ and viceversa. Now, we associate with σ_{LP} satisfying (**A2**) the *scaled epigraphical set*

$$E(\overline{a}, \overline{b}, \lambda) := \operatorname{conv} \left\{ \lambda_i^{-1} (\overline{a}_i, \overline{b}_i), i \in I \right\} + \mathbb{R}_+ \{(0_n, 1)\}. \quad (14)$$

Obviously, (**A1**) is nothing else than (**A2**) with λ being the vector 1_m of all ones, and so $E(\overline{a}, \overline{b}) = E(\overline{a}, \overline{b}, 1_m)$.

Corollary 5 (Two formulas for ρ_{LP} *under* (**A2**)*). If* (**A2**) *holds, the RRF of* σ_{LP} *is*

$$\rho_{LP} = \inf_{(a,b) \in E(\overline{a}, \overline{b}, \lambda)} \phi_Z(-a, -b).$$

If, additionally, Z *is symmetric, then* $\rho_{LP} = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b}, \lambda)\right)$.

Proof. Since the parameterized scaled robust solution set is

$$\begin{aligned} \widetilde{F}_{LP}^\alpha &= \left\{ x \in \mathbb{R}^n : a_i^\top x \leq b_i \text{ for all } (a_i, b_i) \in (\overline{a}_i, \overline{b}_i) + \alpha \lambda_i Z, i \in I \right\} \\ &= \left\{ x \in \mathbb{R}^n : c_i^\top x \leq d_i \text{ for all } (c_i, d_i) \in \lambda_i^{-1} (\overline{a}_i, \overline{b}_i) + \alpha Z, i \in I \right\}, \end{aligned}$$

the scaled RRF $\rho_{LP} = \sup\left\{\alpha \in \mathbb{R}_+ : \widetilde{F}_{LP}^{\alpha} \neq \emptyset\right\}$ can be computed by replacing in Proposition 2 the vector $(\overline{a}_i, \overline{b}_i)$ by $\lambda_i^{-1}(\overline{a}_i, \overline{b}_i)$, for all $i \in I$. □

Formulas for ρ_{LP} when Z is the unit ball for the ℓ_2 (i.e., the Euclidean), the ℓ_1 and the ℓ_∞ norms, can be obtained just replacing $\overline{a}_i$ and $\overline{b}_i$ by $\lambda_i^{-1}\overline{a}_i$ and $\lambda_i^{-1}\overline{b}_i$, respectively, in (7), (11), and (12). For instance, if $Z = \mathbb{B}_{n+1}$, then

$$(\rho_{LP})^2 = \min_{(\delta,\mu)\in\Delta_m\times\mathbb{R}_+} \left\| \left(\sum_{i=1}^m \left(\frac{\delta_i}{\lambda_i}\right)\overline{a}_i, \mu + \sum_{i=1}^m \left(\frac{\delta_i}{\lambda_i}\right)\overline{b}_i \right) \right\|^2,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Since the Slater condition holds for the nominal system $\left\{\overline{a}_i^\top x \le \overline{b}_i, i \in I\right\}$ if and only if it holds for $\left\{\lambda_i^{-1}\overline{a}_i^\top x \le \lambda_i^{-1}\overline{b}_i, i \in I\right\}$, Proposition 3 remains valid under assumption (**A2**) (instead of (**A1**)).

Corollary 6 (Attainment of ρ_{LP} *under* (**A2**)). *If the pointed cone of* cone $\{\overline{a}_i, i \in I\} \times \mathbb{R}_+$ *is a half-line, then* ρ_{LP} *is attained. Moreover, if* Z *is symmetric,* ρ_{LP} *is attained if and only if*

$$(0_n, -1) \notin \mathbb{R}_+\left(\operatorname{conv}\left\{\lambda_i^{-1}(\overline{a}_i, \overline{b}_i), i \in I\right\} + \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b}, \lambda)\right)Z\right).$$

From now on in this section we get rid of interiority assumptions by introducing the following relaxation of (**A2**).

(**A3**) There exists a compact convex set Z and scalars $\lambda_i > 0$, $i \in I\backslash S$, such that $0_{n+1} \in Z \subset \mathbb{R}^{n+1}$ and $\mathcal{U}_i = \lambda_i Z$ for all $i \in I\backslash S$.

Obviously, (**A3**) means that the uncertainty sets of all non-safe constraints are identical up to scaling. For instance, in uncertain production planning problems, the right-hand side coefficients are the available amounts of different type of resources (raw materials, working force, etc.). Assuming that the left-hand side coefficients, depending on the available technology are deterministic, we could take $Z = \{0_n\} \times [-1, 1]$ and as λ_i an estimation of the mean of the ith demand.

In contrast with (**A1**) and (**A2**), (**A3**) is compatible with $\rho_{LP} = +\infty$ and with the existence of safe constraints and variables. This advantage is accompanied by two main disadvantages: under (**A3**) ρ_{LP} can seldom be obtained via a tractable optimization problem, even for the simplest instances of σ_{LP}, and do not exist counterparts of the previous results on positiveness and attainment of ρ_{LP} under (**A1**) and (**A2**).

We denote by δ_Z^* the *support function* of Z, that is, $\delta_Z^*(a, b) = \max\{z^\top(a, b) : z \in Z\}$. By the assumptions on Z, δ_Z^* is a continuous nonnegative and sublinear convex function. Let us associate with σ_{LP} the following auxiliary program:

$$\begin{array}{lll} (AP) & \displaystyle\min_{(y,w,t)\in\mathbb{R}^{n+m+1}} & \delta_Z^*(y, -t) \\ & \text{s.t.} & \overline{a}_i^\top y + w_i - t\overline{b}_i \le 0, i \in I\backslash S, \\ & & \overline{a}_i^\top y - t\overline{b}_i \le 0, i \in S, \\ & & w_i \ge \lambda_i, i \in I\backslash S, \\ & & t \ge 0. \end{array}$$

We denote by $v(AP) \in \mathbb{R}_+$ the optimal value of (AP).

Proposition 7 (A formula for ρ_{LP} *under* (**A3**)). *(Liers et al., 2021, Lemma 4.8) If assumption* (**A3**) *holds and there exists a feasible solution* (y, w, t) *of* (AP) *with* $t > 0$*, then*

$$\rho_{LP} = \begin{cases} \frac{1}{v(AP)}, & \text{if } v(AP) > 0, \\ +\infty, & \text{if } v(AP) = 0. \end{cases} \tag{15}$$

We now show the independence of Corollary 5 and Proposition 7 due to the fact that, even though (**A3**) is weaker than (**A1**), the additional assumption on (y, w, t) may fail under (**A1**). In fact, under (**A1**), $S = \emptyset$ and (AP) is equivalent to

$$\begin{array}{lll} (\widetilde{AP}) & \displaystyle\min_{(y,t)\in\mathbb{R}^{n+1}} & \delta_Z^*(y, -t) \\ & \text{s.t.} & \overline{a}_i^\top y + \lambda_i - t\overline{b}_i \le 0, i \in I, \\ & & t \ge 0, \end{array}$$

and there exists a feasible solution (y, w, t) of (AP) with $t > 0$ if and only if there exists a feasible solution (y, t) of $(\widetilde{AP})$ with $t > 0$. Then, if there exists a feasible solution (y, t) of $(\widetilde{AP})$ such that $t > 0$, one has

$$\overline{a}_i^\top\left(\frac{y}{t}\right) \le \overline{b}_i - \frac{\lambda_i}{t} < \overline{b}_i$$

for all $i \in I$, which entails $\frac{y}{t} \in \operatorname{int} F_{LP}^0$. Conversely, if $x \in \operatorname{int} F_{LP}^0$ there exists $\varepsilon > 0$ such that $\overline{a}_i^\top x + \varepsilon \le \overline{b}_i$, for all $i \in I$. Then, taking $t := \frac{1}{\varepsilon}\max\{\lambda_i, i \in I\} > 0$, (tx, t) is a feasible solution of $(\widetilde{AP})$ such that $t > 0$. So, under (**A1**), the existence of a feasible solution (y, w, t) of (AP) such that $t > 0$ means that $\dim F_{LP}^0 = n$, which is independent of (**A1**) and (**A3**).

The conceptual Algorithm 1 in Liers et al. (2021), based on Proposition 7, computes ρ_{LP} under the corresponding assumptions. Observe that (AP) is tractable in simple cases, e.g., it is an LP problem when Z is a polytope and a linearly constrained convex quadratic program when $Z = \mathbb{B}_{n+1}$.

3. RRF of uncertain mixed-integer linearly constrained programs

We now consider, as in uncertain MILP, constraint systems posed in $\mathbb{Z}^k \times \mathbb{R}^{n-k}$ (meaning that the first $k > 0$ decision variables $x_1, \ldots, x_k$ are integer) of the form

$$\sigma_{MILP} := \left\{a_i^\top x \le b_i, i = 1, \ldots, m\right\}.$$

We also assume that $\mathcal{U} = \prod_{i\in I}\mathcal{U}_i$ and consider the parameterized *robust counterpart* of σ_{MILP}, posed in $\mathbb{Z}^k \times \mathbb{R}^{n-k}$,

$$\sigma_{MILP}^{\alpha} := \left\{a_i^\top x \le b_i,\ (a_i, b_i) \in (\overline{a}_i, \overline{b}_i) + \alpha\mathcal{U}_i, i \in I\right\},$$

with solution set

$$F_{MILP}^{\alpha} = \left\{x \in \mathbb{Z}^k \times \mathbb{R}^{n-k} : a_i^\top x \le b_i \text{ for all } (a_i, b_i) \in (\overline{a}_i, \overline{b}_i) + \alpha\mathcal{U}_i, i \in I\right\}.$$

The RRF of σ_{MILP} is

$$\rho_{MILP} = \sup\{\alpha \in \mathbb{R}_+ : F_{MILP}^{\alpha} \neq \emptyset\},$$

where, to the best of our knowledge, no characterization of $F_{MILP}^{\alpha} \neq \emptyset$ in terms of the data is available. Observe that the relaxed system of σ_{MILP} and σ_{MILP}^{α} are respectively the systems σ_{LP} and σ_{LP}^{α} of Section 2, whose notation we maintain. In particular, ρ_{LP} denotes the RRF of the relaxed problem of σ_{MILP}. Of course, $0 \le \rho_{MILP} \le \rho_{LP} \le +\infty$. Assumptions (**A1**) and (**A3**) are also as in Section 2.

The next result summarizes the relationships between ρ_{MILP} and ρ_{LP} under assumption (**A1**), which implies $\rho_{LP} < +\infty$.

Proposition 8 (Attainment ofρ_{MILP} *and* ρ_{LP}). *(Liers et al., 2021, Theorem 2.6) Under* (**A1**)*, the following statements hold:*

(i) *If* ρ_{LP} *is not attained, then* $\rho_{MILP} = \rho_{LP}$.
(ii) *If* ρ_{MILP} *is attained, then* ρ_{LP} *is also attained.*
(iii) *The attainment of* ρ_{LP} *is compatible with* ρ_{MILP} *being attained or not.*

Statement (i) is particularly important, as it reduces, under the interiority assumption, the computation of the RRF of σ_{MILP} to that of its relaxed problem. The drawback is that applying Proposition 4 requires the exact computation of ρ_{LP}, while Proposition 2 cannot guarantee the identity $\rho_{MILP} = \rho_{LP}$.

The solution set of the nominal problem,

$$F^0_{MILP} = \left\{x \in \mathbb{Z}^k \times \mathbb{R}^{n-k} : \overline{a}_i^\top x \leq \overline{b}_i, i \in I\right\},$$

is closed (but not even connected), and it is related with $\rho_{MILP} = +\infty$ as follows.

Proposition 9 (Conditions for $\rho_{MILP} = +\infty$)*. (Liers et al., 2021, Lemmas 4.10 and 4.11) Assume that* (**A3**) *holds and F^0_{MILP} is bounded. The following statements hold:*

(i) *Either ρ_{MILP} is attained or $\rho_{MILP} = +\infty$.*
(ii) *$\rho_{MILP} = +\infty$ if and only if there exists $x \in \{0,1\}^n$ such that $\overline{a}_i^\top x \leq \overline{b}_i$ for all $i \in I$ and $\delta^*_{\mathcal{U}_i}(x,-1) \leq 0$ for all $i \in I \backslash S$.*

An algorithmic scheme (Liers et al., 2021, Algorithm 2) has been proposed by Liers, Schewe, and Thürauf to compute ρ_{MILP}, and the computational efficiency of several instances of that algorithm have been compared through numerical experiments which include 13 test problems which could not be solved in the time limit of 2 h by any method based on Algorithm 2 and implemented with an empirical stopping rule of absolute and relative tolerances 10^{-4}. Section 1 in Liers et al. (2021) briefly reviews applications of the RRF of uncertain MILP to facility location design (Carrizosa & Nickel, 2003), flexibility index problem (Zhang, Grossmann, & Lima, 2016), and design and control of gas networks (Aßmann, Liers, & Stingl, 2019; Koch, Hiller, Pfetsch, & Schewe, 2015; Schewe, Schmidt, & Thürauf, 2020).

4. RRF of uncertain linearly constrained semi-infinite programs

The seminal paper on the RRF (Goberna et al., 2014) dealt with uncertain linear semi-infinite systems posed in $\mathbb{R}^n$ of the form

$$\sigma_{LSIP} := \left\{a_i^\top x \leq b_i, i \in I\right\},$$

where I is an infinite index set. All symbols have the same meaning as in Section 2, with the unique difference of the cardinality of I. Here, we assume that $\mathcal{U} = \prod_{i\in I}\mathcal{U}_i$, with $\mathcal{U}_i = Z$, for all $i \in I$, and $Z \subset \mathbb{R}^{n+1}$ is a symmetric compact convex set such that $0_{n+1} \in \operatorname{int} Z$, that is, (**A1**) with Z symmetric. The results in (Goberna et al., 2014, Section 2) considered the particular case $Z = \mathbb{B}_{n+1}$. So, $\mathcal{U}$ is a convex subset of the infinite dimensional space $\left(\mathbb{R}^{n+1}\right)^I$ which contains the null function, and the parameterized *robust counterpart* of σ_{LSIP} of parameter $\alpha \geq 0$ is

$$\sigma^\alpha_{LSIP} := \left\{a_i^\top x \leq b_i,\ (a_i, b_i) \in \left(\overline{a}_i, \overline{b}_i\right) + \alpha\mathcal{U}_i, i \in I\right\},$$

whose solution set

$$F^\alpha_{LSIP} = \left\{x \in \mathbb{R}^n : a_i^\top x \leq b_i \text{ for all } (a_i, b_i) \in \left(\overline{a}_i, \overline{b}_i\right) + \alpha Z, i \in I\right\}$$

is closed and convex as it is the intersection of infinitely many closed half-spaces.

From the existence theorem for linear systems (Fan, 1968, Theorem 1),

$$F^\alpha_{LSIP} \neq \emptyset \iff (0_n, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\bigcup_{i\in I}\left[\left(\overline{a}_i, \overline{b}_i\right) + \alpha Z\right]\right\}.$$

We assume $F^0_{LSIP} \neq \emptyset$, that is,

$$(0_n, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\left(\overline{a}_i, \overline{b}_i\right), i \in I\right\}.$$

We denote the RRF of σ_{LSIP} by ρ^α_{LSIP}. Let $E(\overline{a}, \overline{b})$ be as in (5), i.e.,

$$E(\overline{a}, \overline{b}) := \operatorname{conv}\left\{(\overline{a}_i, \overline{b}_i), i \in I\right\} + \mathbb{R}_+\{(0_n, 1)\},$$

the difference being that, here, $E(\overline{a}, \overline{b})$ may be non-closed.

The following result, with $Z = \mathbb{B}_{n+1}$, has been recently used in computational geometry, in order to guarantee the existence of robust farthest Voronoi cells under perturbations of the sites preserving the generator (Goberna, Ridolfi, and Vera de Serio, 2020, Section 4.1).

Proposition 10 (A geometric formula for ρ_{LSIP} *under* (**A1**)*) Under* (**A1**) *with Z symmetric, the following formula for the RRF of σ_{LSIP} holds:*

$$\rho_{LSIP} = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right). \tag{16}$$

The proof of this proposition follows the lines of that of Goberna et al. (2014, Theorem 2.5) where $Z = \mathbb{B}_{n+1}$. Instead, now we assume (**A1**) with Z symmetric. Then, the proof follows analogously and gives us that

$$\rho_{LSIP} = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right) = \sup_{x\in\mathbb{R}^n}\inf_{i\in I}\frac{\overline{b}_i - \overline{a}_i^\top x}{\phi_Z(x,-1)_*},$$

the last value known as the consistency value of the linear system associated to F^0_{LSIP}, where $\phi_Z(\cdot)_*$ denotes the dual norm of $\phi_Z(\cdot)$, that is, $\phi_Z(v)_* := \max\{v^\top w : \|w\| \leq 1\}$.

In geometrical terms, computing ρ_{LSIP} consists in projecting the origin 0_{n+1} onto $\operatorname{cl} E(\overline{a}, \overline{b})$. In contrast with its LP counterpart, this geometrical problem can hardly be reformulated as a tractable optimization one. Actually, the proof of the above result consisted in showing that ρ_{LSIP} coincides with the distance from the nominal system $\overline{\sigma}_{LSIP} = \left\{\overline{a}_i^\top x \leq \overline{b}_i, i \in I\right\}$ to ill-posedness, a stability concept briefly introduced in the next remark.

Remark 11. The stability analysis of linear ordinary and semi-infinite systems posed in $\mathbb{R}^n$ is based on embedding the given *nominal constraint system* $\overline{\sigma} = \left\{\overline{a}_i^\top x \leq \overline{b}_i, i \in I\right\}$, identified with the couple $\overline{\sigma} = (\overline{a}, \overline{b}) \in (\mathbb{R}^n)^I \times \mathbb{R}^I$, into a suitable topological space of admissible perturbed systems, the so-called *space of parameters* Θ, which is formed by all linear systems having the same numbers of variables and constraints as $\overline{\sigma}$. So, the generic element of Θ is a couple $\sigma = (a, b)$ representing an admissible perturbation $\sigma = \left\{a_i^\top x \leq b_i, i \in I\right\}$ of $\overline{\sigma}$. We equip Θ with the pseudo metric (when I is infinite) or metric (when I is finite)

$$d(\sigma_1, \sigma_2) := \sup_{i\in I}\left\|\left(a_i^1, b_i^1\right) - \left(a_i^2, b_i^2\right)\right\|,$$

with $\sigma_1, \sigma_2 \in \Theta$. The *set of feasible parameters* Θ_c is formed by those feasible systems σ which result of perturbing $\overline{\sigma}$ while preserving the same numbers of variables and constraints as $\overline{\sigma}$. The *distance from $\overline{\sigma}$ to ill-posedness* (in the feasibility sense) is $\inf_{\sigma\in\Theta\backslash\Theta_c} d(\overline{\sigma}, \sigma)$. The equation

$$\inf_{\sigma\in\Theta\backslash\Theta_c} d(\overline{\sigma}_{LSIP}, \sigma) = \operatorname{dist}\left(0_{n+1}, E(\overline{a}, \overline{b})\right)$$

was proved in Cánovas et al. (2005), and the proof of Goberna et al. (2014, Theorem 2.5) consisted in showing that the RRF in LSIP, with the uncertainty pattern-set $\mathcal{U}$ described at the beginning of this section, is $\rho_{LSIP} = \inf_{\sigma\in\Theta\backslash\Theta_c} d(\overline{\sigma}_{LSIP}, \sigma)$. The argument is also valid when I is finite, but the proof of (6) in Goberna et al. (2015, Theorem 4) was direct.

The next two results, on the positiveness and the attainment of ρ_{LSIP}, extend Propositions 3 and 4 to the semi-infinite setting, with the inconvenient that they do not provide "computationally tractable" attainability tests. We first show that the *strong Slater condition* (existence of $\widehat{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $\overline{a}_i^\top \widehat{x} + \varepsilon \leq \overline{b}_i$ for all $i \in I$) implies the positiveness of ρ_{LSIP} while the converse statement is also true under a condition that is fulfilled in the main LSIP real applications.

Proposition 12 (Positiveness of ρ_{LSIP} *under* (**A1**)*) Under* (**A1**) *with Z symmetric, the following implication holds:*

$$\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}}\left\{y : \overline{a}_i^\top x + y \leq \overline{b}_i, i \in I\right\} > 0 \implies \rho_{LSIP} > 0.$$

The converse statement also holds whenever $\{(a_i, b_i), i \in I\}$ *is a compact subset of* $\mathbb{R}^{n+1}$.

Proof. The argument for the first statement is the same as the part $[\Longleftarrow]$ of Proposition 3, just replacing the positive cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$ by the positive cone $\mathbb{R}^{(I)}_+$ of the linear space $\mathbb{R}^{(I)}$ of generalized finite sequences (real-valued functions on I which vanish everywhere except on a finite subset of I). In fact, let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}$ be such that $a_i^\top \bar{x} + \bar{y} \leq b_i$ for all $i \in I$ and $\bar{y} > 0$. We now show that $0_{n+1} \notin E(\bar{a}, \bar{b})$ by contradiction. Let $\lambda \in \mathbb{R}^{(I)}_+$ and $\mu \in \mathbb{R}_+$ be such that $\sum_{i\in I} \lambda_i = 1$ and $\sum_{i\in I} \lambda_i(\bar{a}_i, \bar{b}_i) + \mu(0_n, 1) = 0_{n+1}$. We thus get the following contradiction:

$$0 = 0_{n+1}^\top \begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} = \sum_{i\in I} \lambda_i\left(a_i^\top \bar{x} - b_i\right) - \mu$$
$$\leq -\left(\bar{y}\sum_{i\in I} \lambda_i + \mu\right) = -(\bar{y} + \mu) < 0.$$

We now assume that $\{(a_i, b_i), i \in I\}$ is a compact subset of $\mathbb{R}^{n+1}$. *Then*, $E(\bar{a}, \bar{b})$ is a closed convex set. So, by (16), $\rho_{LSIP} > 0$ if and only if $0_{n+1} \notin E(\bar{a}, \bar{b})$. The rest of the proof is exactly the same as the part $[\Longrightarrow]$ of Proposition 3. □

Proposition 13 (Attainment of ρ_{LSIP} *under* (**A1**)*). Under* (**A1**) *with* Z *symmetric, the following statements hold:*

(i) ρ_{LSIP} *is attained if and only if*

$$(0_n, -1) \notin \operatorname{cl} \mathbb{R}_+\left(\operatorname{conv}\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\} + \operatorname{dist}\left(0_{n+1}, E(\bar{a}, \bar{b})\right)Z\right). \tag{17}$$

(ii) *If the pointed cone of* $\operatorname{cl}\operatorname{cone}\{\bar{a}_i, i \in I\} \times \mathbb{R}_+$ *is a half-line, then* ρ_{LSIP} *is attained.*

Proof. (*i*) From Gale's alternative theorem (see, e.g., Goberna & López, 1998, Corollary 3.1.1), $F_{\rho_{LSIP}} = \emptyset$ if and only if

$$(0_n, -1) \in \operatorname{cl}\operatorname{cone}\left(\bigcup_{i\in I}\left[(\bar{a}_i, \bar{b}_i) + \rho_{LSIP} Z\right]\right)$$
$$= \operatorname{cl} \mathbb{R}_+\left(\operatorname{conv}\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\} + \rho_{LSIP} Z\right).$$

(*ii*) As in Proposition 4, a sufficient condition for ρ_{LSIP} being attained is that the recession cone $0^+F^0_{LSIP} = \{x \in \mathbb{R}^n : \bar{a}_i^\top x \leq 0, i \in I\}$ of the *nominal solution set* F^0_{LSIP} of the nominal system $\bar{\sigma}_{LSIP}$ is a linear subspace of $\mathbb{R}^n$. By Goberna and López (1998, Theorem 5.13(ii)), this happens if and only if $\operatorname{cl}\operatorname{cone}\{\bar{a}_i, i \in I\} \times \mathbb{R}_+$ is a half-line.

Regarding Proposition 13(*i*), if $\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\}$ is compact and $\operatorname{conv}\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\} \cap \rho_{LSIP} Z = \emptyset$, then the closure operator can be removed from (17). In fact, under the additional assumption, $0_{n+1} \notin \operatorname{conv}\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\} + \rho_{LSIP} Z$, so that $\operatorname{conv}\left\{(\bar{a}_i, \bar{b}_i), i \in I\right\} + \rho_{LSIP} Z$ is a compact convex set that does not contain 0_{n+1}. Thus, $\operatorname{cone}(\bigcup_{i\in I}[(\bar{a}_i, \bar{b}_i) + \rho_{LSIP} Z])$ is closed. □

We finish this section by considering the scaled parameterized robust counterpart of σ_{LSIP} under the assumption that $\mathcal{U}_i = \lambda_i Z$, with $\lambda \in \mathbb{R}^I_{++}$. So, (**A2**) holds with Z symmetric. Let $E(\bar{a}, \bar{b}, \lambda)$ be as in (14), i.e.,

$$E(\bar{a}, \bar{b}, \lambda) := \operatorname{conv}\left\{\lambda_i^{-1}(\bar{a}_i, \bar{b}_i), i \in I\right\} + \mathbb{R}_+\{(0_n, 1)\}.$$

The proofs of the next three corollaries are similar to those of the corresponding propositions in Section 2. We only provide the first one.

Corollary 14 (A geometric formula for ρ_{LSIP} *under* (**A2**)*). Under* (**A2**) *with* Z *symmetric, the RRF of* σ_{LSIP} *is*

$$\rho_{LSIP} = \operatorname{dist}\left(0_{n+1}, E(\bar{a}, \bar{b}, \lambda)\right).$$

Proof. As in Corollary 5, one has

$$\widetilde{F}^\alpha_{LSIP} = \left\{x \in \mathbb{R}^n : c_i^\top x \leq d_i \text{ for all } (c_i, d_i) \in \lambda_i^{-1}(\bar{a}_i, \bar{b}_i) + \alpha Z, i \in I\right\}.$$

So, the scaled RRF $\rho_{LP} = \sup\left\{\alpha \in \mathbb{R}_+ : \widetilde{F}^\alpha_{LP} \neq \emptyset\right\}$ can be obtained by replacing in (16) the vector $(\bar{a}_i, \bar{b}_i)$ by $\lambda_i^{-1}(\bar{a}_i, \bar{b}_i)$, for all $i \in I$. □

Observe that, due to the infiniteness of I, the fulfilment of the Slater condition for the nominal system $\left\{\bar{a}_i^\top x \leq \bar{b}_i, i \in I\right\}$ is independent of its fulfilment by $\left\{\lambda_i^{-1}\bar{a}_i^\top x \leq \lambda_i^{-1}\bar{b}_i, i \in I\right\}$. Similarly, the boundedness of the set $\{(a_i, b_i), i \in I\}$ is independent of the boundedness of $\left\{\lambda_i^{-1}(a_i, b_i), i \in I\right\}$.

Corollary 15 (Positiveness of ρ_{LSIP} *under* (**A2**)*). Under* (**A2**) *with* Z *symmetric, the following implication holds:*

$$\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}}\left\{y : \lambda_i^{-1}\bar{a}_i^\top x + y \leq \lambda_i^{-1}\bar{b}_i, i \in I\right\} > 0 \implies \rho_{LSIP} > 0.$$

The converse statement also holds whenever $\left\{\lambda_i^{-1}(a_i, b_i), i \in I\right\}$ *is a compact subset of* $\mathbb{R}^{n+1}$.

Corollary 16 (Attainment of ρ_{LSIP} *under* (**A2**)*). Under* (**A2**) *with* Z *symmetric, the following statements hold:*

(i) ρ_{LSIP} *is attained if and only if*

$$(0_n, -1) \notin \operatorname{cl} \mathbb{R}_+\left(\operatorname{conv}\left\{\lambda_i^{-1}(\bar{a}_i, \bar{b}_i), i \in I\right\} + \operatorname{dist}\left(0_{n+1}, E(\bar{a}, \bar{b}, \lambda)\right)Z\right).$$

(ii) *If the pointed cone of* $\operatorname{cl}\operatorname{cone}\{\bar{a}_i, i \in I\} \times \mathbb{R}_+$ *is a half-line, then* ρ_{LSIP} *is attained.*

5. RRF of uncertain convexly constrained programs under affine perturbations

We now consider, as in uncertain CP, a convex constraint systems posed in $\mathbb{R}^n$ of the form

$$\sigma_{CP} = \{g_i(x) \leq 0, i = 1, \ldots, m\},$$

where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an uncertain convex function for $i \in I = \{1, \ldots, m\}$. We also assume that the pattern-set $\mathcal{U} \subset \left(\mathbb{R}^{n+1}\right)^m$ is the cartesian product $\prod_{i\in I}\mathcal{U}_i$ of m convex sets $\mathcal{U}_i \subset \mathbb{R}^{n+1}$ such that $\mathcal{U}_i \neq \emptyset$ for all $i \in I$. We denote by u_i the ith component of $u \in \mathcal{U}$, i.e., $u = (u_1, \ldots, u_m)$. Regarding g, whose ith component is g_i, we assume the existence of a convex function $\bar{g}_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ (the *nominal i-th constraint function*) such that the uncertainty of g_i is captured by the expression

$$g_i(x, u) := \bar{g}_i(x) + u_i^\top \begin{pmatrix} x \\ -1 \end{pmatrix}, \ \forall u_i \in \mathcal{U}_i, \ \forall x \in \mathbb{R}^n. \tag{18}$$

For any $i \in I$ we can pick a point $(\bar{a}_i, \bar{b}_i) \in \operatorname{int}\mathcal{U}_i$. Defining $\mathcal{V}_i = \mathcal{U}_i - (\bar{a}_i, \bar{b}_i)$, (18) becomes

$$g_i(x, u) := \bar{h}_i(x) + v_i^\top \begin{pmatrix} x \\ -1 \end{pmatrix}, \ \forall v_i \in \mathcal{V}_i, \ \forall x \in \mathbb{R}^n,$$

where the function $\bar{h}_i(x) := \bar{g}_i(x) + \bar{a}_i^\top x - \bar{b}_i$ is convex and $0_{n+1} \in \mathcal{V}_i$ for all $i \in I$. In the below formulas for the RRF appear the epigraphs of the conjugate functions of the constraints. Recall that $\bar{g}_i^*(x^*) := \sup_{x\in\mathbb{R}^n}\{(x^*)^\top x - \bar{g}_i(x)\}$ is the *conjugate* of $\bar{g}_i$ while $\operatorname{epi}\bar{g}_i^* = \left\{(x^*, r) \in \mathbb{R}^{n+1} : \bar{g}_i^*(x^*) \leq r\right\}$ is its *epigraph*. Accordingly, $\bar{h}_i^*(x^*) = \bar{b}_i + \bar{g}_i^*(x^*)$ and $\operatorname{epi}\bar{h}_i^* = \operatorname{epi}\bar{g}_i^* + (\bar{a}_i, \bar{b}_i)$.

Hence, we can assume without loss of generality that (18) holds with $0_{n+1} \in \mathcal{U}_i$ for all $i \in I$. In the same way, if $\operatorname{int}\mathcal{U}_i \neq \emptyset$ for all $i \in I$, we can assume without loss of generality that (18) holds with $0_{n+1} \in \operatorname{int}\mathcal{U}_i$ for all $i \in I$.

Thus, the parameterized *robust counterpart* of σ_{CP} is the convex system posed in $\mathbb{R}^n$

$$\sigma^\alpha_{CP} := \left\{\bar{g}_i(x) + a_i^\top x \leq b_i, \ (a_i, b_i) \in \alpha\mathcal{U}_i, i \in I\right\},$$

whose solution set is

$$F_{CP}^{\alpha} = \left\{x \in \mathbb{R}^n : \overline{g}_i(x) + a_i^\top x - b_i \le 0 \text{ for all } (a_i, b_i) \in \alpha\mathcal{U}_i, i \in I\right\}.$$

Since $\operatorname{epi}(\overline{g}_i + \langle a_i, \cdot\rangle - b_i)^* = \operatorname{epi}\overline{g}_i^* + (a_i, b_i)$, by Dinh, Goberna, and López (2006, Theorem 3.1),

$$F_{CP}^{\alpha} \neq \emptyset \iff (0_n, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\bigcup_{i\in I}\left(\operatorname{epi}\overline{g}_i^* + \alpha\mathcal{U}_i\right)\right\}. \tag{19}$$

We assume $F_{CP}^0 \neq \emptyset$, that is, $(0_n, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\bigcup_{i\in I}\operatorname{epi}\overline{g}_i^*\right\}$.

The RRF of σ_{CP} is

$$\rho_{CP} := \sup\{\alpha \in \mathbb{R}_+ : F_{CP}^{\alpha} \neq \emptyset\}.$$

We associate with σ_{CP} satisfying (**A1**) the *epigraphical set*

$$E(\overline{g}) := \operatorname{conv}\left(\bigcup_{i\in I}\operatorname{epi}\overline{g}_i^*\right),$$

where $\overline{g} := (\overline{g}_1, \dots, \overline{g}_m)$.

If $\overline{g}_i(x) = \overline{a}_i^\top x - \overline{b}_i$ for all $i \in I$, since $\overline{g}_i^* = \overline{b}_i + \delta_{\{\overline{a}_i\}}$, where $\delta_{\{\overline{a}_i\}}$ denotes the indicator function of $\{\overline{a}_i\}$ (i.e., $\delta_{\{\overline{a}_i\}}(x) = 0$ if $x = \overline{a}_i$ and $+\infty$ otherwise), one has

$$E(\overline{g}) := \operatorname{conv}\left(\bigcup_{i\in I}\{\overline{a}_i\} \times \left[\overline{b}_i, +\infty\right[\right) = E(\overline{a}, \overline{b}),$$

the epigraphical set defined in (5).

We first consider the RRF of σ_{CP} under the interiority assumption (**A1**).

Proposition 17 (Two exact formulas for ρ_{CP} *under* (**A1**)) *((Chen et al., 2020, Corollaries 3.1 and 3.2) and (Li and Wang, 2018, Theorem 3.1, Corollary 3.2)) Under* (**A1**),

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g})} \phi_Z(-a, -b).$$

If, additionally, Z is symmetric,

$$\begin{aligned}\rho_{CP} &= \operatorname{dist}(0_{n+1}, E(\overline{g})) \\ &= \inf_{\delta\in\Delta_{n+2},\ (a_k, b_k)\in\bigcup_{i\in I}\operatorname{epi}\overline{g}_i^*} \phi_Z\left(\sum_{k=1}^{n+2}\delta_k a_k, \sum_{k=1}^{n+2}\delta_k b_k\right).\end{aligned} \tag{20}$$

In some simple cases, (20) allows to compute ρ_{CP} by solving optimization problems:

- If $Z = \left\{z \in \mathbb{R}^{n+1} : z^\top M^{-1} z \le 1\right\}$, with M being a positive definite symmetric $(n+1)\times(n+1)$ matrix, then, by Chen et al. (2020, Corollary 3.3(i)),

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g})} \sqrt{(a,b)^\top M^{-1}(a,b)}. \tag{21}$$

- If $Z = \mathbb{B}_{n+1}$, by (21),

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g})} \sqrt{b^2 + \sum_{k=1}^{n} a_k^2}. \tag{22}$$

- If Z is the ℓ_1 unit ball $\left\{z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1}|z_i| \le 1\right\}$, by Chen et al. (2020, Corollary 3.3(iii)),

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g})} \left\{|b| + \sum_{k=1}^{n}|a_k|\right\}. \tag{23}$$

- If Z is the ℓ_∞ unit ball $\left\{z \in \mathbb{R}^{n+1} : |z_i| \le 1, i \in I\right\}$, by Chen et al. (2020, Corollary 3.3(iv)),

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g})} \max\{|b|, |a_k| : k = 1, \dots, n\}. \tag{24}$$

However, the above formulas (22)–(24) do not provide tractable optimization problems for ρ_{CP} when not all constraints are linear because $E(\overline{g})$ is seldom polyhedral.

In order to check the positivity of ρ_{CP} one has to decide whether the nominal constraint convex system satisfies, or not, the Slater condition. This can be done by maximizing a linear function under convex constraints, i.e., by solving certain CP problem.

Proposition 18 (Positiviness of ρ_{CP} *under* (**A1**)*). ((Chen et al., 2020, Proposition 3.1) and (Li & Wang, 2018, Theorem 3.5)) Assume that* (**A1**) *holds. Then,*

$$\begin{aligned}\rho_{CP} > 0 &\iff \sup_{x\in\mathbb{R}^n}\inf_{i\in I}\{y_i \in \mathbb{R} : \overline{g}_i(x) + y_i \le 0\} > 0 \\ &\iff \sup_{x\in\mathbb{R}^n, y\in\mathbb{R}}\{y : \overline{g}_i(x) + y \le 0, i \in I\} > 0.\end{aligned}$$

The next corollary is the result of combining (19) and (20).

Proposition 19 (Attainment of ρ_{CP} *under* (**A1**)*). Assume that* (**A1**) *holds with Z being symmetric. Then,* ρ_{CP} *is attained if and only if*

$$(0_n, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\left(\bigcup_{i\in I}\operatorname{epi}\overline{g}_i^*\right) + \operatorname{dist}(0_{n+1}, E(\overline{g}))Z\right\}.$$

As in previous sections, we consider scaled CP, i.e., the counterparts of the above propositions when (**A2**) holds instead of (**A1**). The parameterized scaled robust solution set is now

$$\begin{aligned}\widetilde{F}_{CP}^{\alpha} &= \left\{x \in \mathbb{R}^n : \overline{g}_i(x) + a_i^\top x - b_i \le 0 \text{ for all } (a_i, b_i) \in \alpha\lambda_i Z, i \in I\right\} \\ &= \left\{x \in \mathbb{R}^n : \lambda_i^{-1}\overline{g}_i(x) + c_i^\top x - d_i \le 0 \text{ for all } (c_i, d_i) \in \alpha Z, i \in I\right\}.\end{aligned}$$

We associate with σ_{CP} satisfying (**A2**) the *epigraphical set*

$$E(\overline{g}, \lambda) := \operatorname{conv}\left(\bigcup_{i\in I}\operatorname{epi}\left(\lambda_i^{-1}\overline{g}_i\right)^*\right).$$

It is easy to see that $\left(\lambda_i^{-1}\overline{g}_i\right)^*(x^*) = \overline{g}_i^*(\lambda_i x^*)$. Thus, defining $\Delta := \operatorname{diag}(\lambda_1, \dots, \lambda_m, 1)$, we can write

$$E(\overline{g}, \lambda) := \operatorname{conv}\left(\bigcup_{i\in I}\Delta\operatorname{epi}\overline{g}_i^*\right).$$

Corollary 20 (Two formulas for ρ_{CP} *under* (**A2**)*). If* (**A2**) *holds, the RRF of* σ_{CP} *is*

$$\rho_{CP} = \inf_{(a,b)\in E(\overline{g},\lambda)} \phi_Z(-a, -b).$$

If, additionally, Z is symmetric, then $\rho_{CP} = \operatorname{dist}(0_{n+1}, E(\overline{g}, \lambda))$.

Since the Slater condition holds for the nominal system $\left\{\overline{g}_i^*(x) \le 0, i \in I\right\}$ if and only if it holds for $\left\{\lambda_i^{-1}\overline{g}_i^*(x) \le 0, i \in I\right\}$, Proposition 18 remains valid under assumption (**A2**).

Corollary 21 (Attainment of ρ_{CP} *under* (**A2**)*). Assume that* (**A2**) *holds with Z being symmetric. Then,* ρ_{CP} *is attained if and only if*

$$(0, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\left(\bigcup_{i\in I}\Delta\operatorname{epi}\overline{g}_i^*\right) + \operatorname{dist}(0_{n+1}, E(\overline{g}, \lambda))Z\right\}.$$

The following assumption is an extension of the interiority assumption (**A2**) introduced in Section 2 as the uncertainty sets $\mathcal{U}_i$ are no longer required to be coincident up to scaling:

(**A4**) For each $i \in I$ there exists a compact convex set $Z_i \subset \mathbb{R}^{n+1}$ such that $0_{n+1} \in \operatorname{int} Z_i$ and $\mathcal{U}_i = Z_i$.

Assumption (**A4**) holds in convex programs with deterministic objective function and uncertain constraints whose uncertainty sets are closed balls for different norms, e.g., polyhedral and non-polyhedral balls, in which case they cannot be nonnegative multiples of a unique convex body.

Proposition 22 (Lower and upper bounds for ρ_{CP} *under* (**A4**)*). (Chen et al., 2020, Theorem 3.1) Under* (**A4**)*, the following inequalities hold:*

$$\inf_{(a,b)\in E(\overline{g})} \inf_{i\in I} \phi_{Z_i}(-a,-b) \le \rho_{CP} \le \inf_{(a,b)\in E(\overline{g})} \sup_{i\in I} \phi_{Z_i}(-a,-b).$$

Remark 23. The admissible perturbations of each constraint function $\overline{g}_i$ in σ_{CP} are not linear in Goberna et al. (2016), so that they are not covered by the above results. To be more precise, the admissible perturbations $\overline{g}_i$ are sums of nonnegative combinations of the m constraint functions with affine functions, so that each perturbed functions is convex. Even though the interior of the uncertainty pattern-set $\mathcal{U}$ does not contain the zero vector, in the same vein as (**A3**), (Goberna et al., 2016, Theorem 3.1) provides an exact formula for ρ_{CP} computable by solving a suitable tractable optimization problem

6. RRF of conic linearly constrained programs

This section deals with uncertain conic linear systems posed in $\mathbb{R}^n$ of the form

$$\sigma_{CLP} = \left\{ \begin{bmatrix} a_1^\top x - b_1 \\ \vdots \\ a_m^\top x - b_m \end{bmatrix} \in -K \right\},$$

where $\{0_m\} \neq K \subsetneq \mathbb{R}^m$ is a given closed pointed convex cone such that $\operatorname{int} K \neq \emptyset$ (implying that its positive dual cone $K^* = \left\{ y \in \mathbb{R}^m : z^\top y \ge 0, z \in K \right\}$ enjoys the same properties), and $(a_i, b_i) \in \mathbb{R}^{n+1}$, $i \in I = \{1, \dots, m\}$. Particular cases of (CLP) are:

- If $K = \mathbb{R}^m_+$, then σ_{CLP} coincides with the uncertain **linear system** σ_{LP} analyzed in Section 2.
- If f is linear and $K = S^q_+$ is the cone consisting of all $q \times q$ positive semi-definite symmetric matrices, then (P) in (1) is an uncertain **semi-definite programming (SDP)** problem, with constraint system σ_{DP}. Let $\operatorname{Tr}(M)$ be the trace of a matrix $M \in S^q$. As S^q and $\mathbb{R}^{q(q+1)/2}$ have the same dimensions, there exists an invertible linear map $L : S^q \to \mathbb{R}^{q(q+1)/2}$ such that
$$L(M_1)^\top L(M_2) = \operatorname{Tr}(M_1 M_2) \text{ for all } M_1, M_2 \in S^q. \tag{25}$$
By (25), L establishes an isomorphism between S^q and $\mathbb{R}^{q(q+1)/2}$, equipped with the trace and the Euclidean inner product, respectively which preserves inner products. So, one can identify the space S^q of all $(q \times q)$ symmetric matrices with the Euclidean space $\mathbb{R}^{q(q+1)/2}$.
- If f is linear and K is the second order cone $K^m_p = \{x \in \mathbb{R}^m : x_m \ge \|(x_1, \dots, x_{m-1})\|\}$, then (P) in (1) is an uncertain **second order cone programming (SOCP)** problem.

The uncertain constraint of σ_{CLP} can be written as $g(x) = [A \mid b]\begin{pmatrix} x \\ -1 \end{pmatrix} \in -K$, where $A := [a_1 \mid \dots \mid a_m]^\top \in \mathbb{R}^{m\times n}$ and $b = (b_1, \dots, b_m)^\top \in \mathbb{R}^m$. We assume the existence of a pattern-set $\mathcal{U}$ formed by $m \times (n+1)$ real matrices, $\mathcal{U}$ being a convex subset of $\mathbb{R}^{m\times(n+1)}$ containing the zero matrix, a matrix $\overline{A} := [\overline{a}_1 \mid \dots \mid \overline{a}_m]^\top \in \mathbb{R}^{m\times n}$, and a vector $\overline{b} = (\overline{b}_1, \dots, \overline{b}_m)^\top \in \mathbb{R}^m$ such that the uncertainty of g is captured by the expression

$$g(x,u) := \left(\left[\overline{A} \mid \overline{b}\right] + U\right)\begin{pmatrix} x \\ -1 \end{pmatrix}$$

for all $U \in \mathcal{U}$ and $x \in \mathbb{R}^n$. So, the parameterized *robust counterpart* of σ_{CLP}, depending on a parameter $\alpha \ge 0$, is the conic linear system posed in $\mathbb{R}^n$

$$\sigma^\alpha_{CLP} := \left\{ \left(\left[\overline{A} \mid \overline{b}\right] + U\right)\begin{pmatrix} x \\ -1 \end{pmatrix} \in -K,\ U \in \alpha\mathcal{U} \right\}.$$

It remains to choose a suitable pattern-set $\mathcal{U}$. The size of a matrix can be defined through any of the well-known matrix norms. Since our approach is based on the linearization of the solution set F^α_{CLP} of σ^α_{CLP}, we use a norm which is not popular (it does not appear in Horn and Johnson (1985, Chapter 5)) but allows to use the results on the RRF of uncertain LSIP problems. Given an $m \times n$ matrix $M = \left[m_{ij}\right]$, we define $\|M\|$ as the maximum of the Euclidean norms of the rows of M, that is,

$$\|M\| = \max_{i=1,\dots,m} \sqrt{\sum_{j=1}^n m_{ij}^2}. \tag{26}$$

Our pattern-set $\mathcal{U}$ will be the unit closed ball for the norm $\|\cdot\|$ defined by (26) in the linear space of $m \times (n+1)$ matrices, i.e., $\mathcal{U}$ is formed by the matrices U whose rows belong to $\mathbb{B}_{n+1}$. Interpreting the entries of the rows (a_i, b_i) as coefficients of linear inequalities $a_i^\top x \le b_i$, we could say that the interiority assumption (**A1**) holds with $Z = \mathbb{B}_{n+1}$.

So, the solution set of σ^α_{CLP} can be written as

$$F^\alpha_{CLP} = \left\{ x \in \mathbb{R}^n : [A \mid b]\begin{pmatrix} x \\ -1 \end{pmatrix} \in -K \text{ for all } (a_i, b_i) \in \left(\overline{a}_i, \overline{b}_i\right) + \alpha\mathbb{B}_{n+1}, i \in I \right\}, \tag{27}$$

and the RRF of σ_{CLP} is

$$\rho_{CLP} := \sup\{\alpha \in \mathbb{R}_+ : F^\alpha_{CLP} \neq \emptyset\} \in \mathbb{R} \cup \{+\infty\}.$$

The assumptions on K ensure the existence of a compact base $\mathcal{B}$ for K^*, that is, a compact and convex subset $\mathcal{B}$ of K^* such that $0_m \notin \mathcal{B}$ and $K^* = \mathbb{R}_+\mathcal{B}$ (see, e.g., Göpfert, Riahi, Tammer, & Zălinescu, 2003, Lemma 2.2.17). In what follows $\mathcal{B}$ is such a base for K^*. This allows us to represent F^α_{CLP} by a linear semi-infinite system which does not contain the trivial inequality $0_n^\top x \le 0$ from which Fan's existence theorem (Fan, 1968, Theorem 1) yields

$$F^\alpha_{CLP} \neq \emptyset \iff (0_n, 1) \notin \operatorname{cl}\operatorname{cone}\left(\bigcup_{y\in\mathcal{B}} \left\{ \sum_{i=1}^m y_i\left(\left(\overline{a}_i, \overline{b}_i\right) + \alpha\mathbb{B}_{n+1}\right) \right\} \right). \tag{28}$$

So, we assume $F^0_{CLP} \neq \emptyset$, i.e.,

$$(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\left\{ \sum_{i=1}^m y_i(\overline{a}_i, \overline{b}_i) : y \in \mathcal{B} \right\},$$

so that the fulfilment of $F^0_{CLP} \neq \emptyset$ can be checked by solving a feasible LP program whenever $\mathcal{B}$ is a polytope.

The *epigraphical set* of σ_{CLP} associated with a compact base $\mathcal{B}$ of K^* is the set

$$E(\overline{A}, \overline{b}, \mathcal{B}) := \left\{ y^\top\left[\overline{A} \mid \overline{b}\right] : y \in \mathcal{B} \right\} + (\{0_n\} \times \mathbb{R}_+).$$

Observe that $E(\overline{A}, \overline{b}, \mathcal{B})$ not only depends here on the nominal data ($\overline{A}$ and $\overline{b}$), as it happens in Cánovas et al. (2007), Cánovas et al. (2005), Cánovas et al. (2006), Cánovas et al. (2011), but also on the chosen compact basis $\mathcal{B}$ of K^*. As in uncertain LSIP, the epigraphical set $E(\overline{A}, \overline{b}, \mathcal{B})$ is the sum of a compact convex subset of $\mathbb{R}^{n+1}$ with the vertical ray emanating from 0_{n+1}, so it is a closed convex set too. If $K = \mathbb{R}^m_+$ and we define $\mathcal{B}$ as the convex hull of the canonical basis of $\mathbb{R}^m$, which is actually a base of $K^* = \mathbb{R}^m_+$, one gets the epigraphical set of σ_{LP} with constraints written in the form $a_i^\top x \le b_i$, $i \in I$.

The next result provides lower and upper bounds for ρ_{CLP} which are expressed in terms of $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$.

Proposition 24 (Lower and upper bounds for ρ_{CLP} *under* (**A1**)*). (Goberna et al., 2021, Theorem 3.1) Let $\mathcal{B}$ be a compact base of K^*. Then, the RRF of σ_{CLP} satisfies*

$$C_1(\mathcal{B}) \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) \le \rho_{CLP} \le C_2(\mathcal{B}) \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right),$$

(29)

where

$$C_1(\mathcal{B}) = 1/\max\left\{\|\sum_{i=1}^m y_i u_i\| : y \in \mathcal{B}, \|u_i\| \le 1\right\} \tag{30}$$

and

$$C_2(\mathcal{B}) = 1/\min\left\{\sum_{i=1}^m |y_i| : y \in \mathcal{B}\right\}. \tag{31}$$

Fortunately, it is possible to obtain tractable optimization problems for the computation of the term $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$ under a mild condition.

Proposition 25 (A computable formula for $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$ *under* (**A1**)*).* *(Goberna et al., 2021, Theorem 3.2) Let $\mathcal{B}$ be a compact base of K^*. Then,*

$$\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) = \inf_{(z,s,t,y)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{t : \begin{array}{c}\|(z,s)\| \le t, y \in \mathcal{B}, \\ z = \overline{A}^\top y,\ s \ge \overline{b}^\top y\end{array}\right\}.$$

In particular, if $\mathcal{B}$ is a spectrahedron with the form $\mathcal{B} = \{y \in \mathbb{R}^m : B_0 + \sum_{i=1}^m y_i B_i \succeq 0\}$ for some $s\times s$ symmetric matrices B_i, $i = 0, 1, \ldots, m$, then

$$\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)^2 = \inf_{(z,s,t,y)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{t : \begin{array}{l}\begin{bmatrix} tI_n & 0_n & z \\ 0_n^\top & t & s \\ z^\top & s & 1\end{bmatrix} \succeq 0, \\ z = \overline{A}^\top y,\ s \ge \overline{b}^\top y, \\ B_0 + \sum_{i=1}^m y_i B_i \succeq 0.\end{array}\right\}.$$

We now show how to obtain tractable lower and upper bounds for the RRF of uncertain SDP and SOC problems by exploiting the fact that K is self-dual for both types of problems, i.e., $K^* = K$.

- **SDP:** Taking $\mathcal{B} = \left\{M \in S_+^q : \operatorname{Tr}(M) = 1\right\}$ in Propositions 24 and 25, one gets
$$\frac{2}{q(q+1)}\sqrt{\upsilon_{SDP}} \le \rho_{SDP} \le \sqrt{q}\sqrt{\upsilon_{SDP}},$$
where ρ_{SDP} is the RRF of σ_{SDP} and υ_{SDP} is the optimal value of the following SDP problem:
$$\inf_{\substack{(z,s,t,y)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m \\ M\in S^q}}\left\{t : \begin{array}{l}\begin{bmatrix} tI_n & 0_n & z \\ 0_n^\top & t & s \\ z^\top & s & 1\end{bmatrix} \succeq 0, \\ z = \overline{A}^\top y,\ s \ge \overline{b}^\top y, \\ y = L(M) \\ M \in S_+^q,\ \operatorname{Tr}(M) = 1.\end{array}\right\}.$$
- **SOCP:** Taking $\mathcal{B} = \{y \in K_p^m : y_m = 1\}$ in Propositions 24 and 25, one gets
$$\frac{1}{\sqrt{m-1}+1}\upsilon_{SOC} \le \rho_{SOC} \le \upsilon_{SOC},$$
where υ_{SOC} is the optimal value of the following SOCP problem
$$\inf_{(z,s,t,y)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{t : \begin{array}{l}\|(z,s)\| \le t, \\ z = \overline{A}^\top y,\ s \ge \overline{b}^\top y, \\ \|(y_1,\ldots,y_{m-1})\| \le 1,\ y_m = 1.\end{array}\right\}.$$

To get an exact formula for ρ_{CLP} from the epigraphical set we need some qualification of the dual cone K^*, more exactly the set

$$\mathcal{B}_s := \left\{y \in K^* : \sum_{i=1}^m y_i = 1\right\} \tag{32}$$

be a compact base for K^*. According to Göpfert et al. (2003, Theorem 2.1.15) and (Aliprantis & Tourky, 2007, Theorem 1.47), a sufficient condition for the set $\mathcal{B}_s$ defined in (32) to be a compact base of K^* is that $\sum_{i=1}^m y_i > 0$ for all $y \in K^*\backslash\{0_m\}$ and $\mathcal{B}_s$ be bounded. In contrast with S_+^q and K_p^m, any K such that $\mathbb{R}_+^m \subset K$ satisfies this condition as $\sum_{i=1}^m y_i > 0$ for all $y \in K^*\backslash\{0_m\} \subset \mathbb{R}_+^m\backslash\{0_m\}$.

Proposition 26 (An exact formula for ρ_{CLP} *under* (**A1**)*). If $\mathcal{B}_s := \{y \in K^* : \sum_{i=1}^m y_i = 1\}$ is a compact base of K^*, then*

$$\rho_{CLP} \le \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right).$$

Moreover, the exact formula

$$\rho_{CLP} = \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right) \tag{33}$$

holds whenever the additional assumption $\mathbb{R}_+^m \subset K$ is satisfied.

Proof. Let C_1 and C_2 be the constants (30) and (31) in Proposition 24 in the particular case that $\mathcal{B} = \mathcal{B}_s = \{\lambda \in K^* : \sum_{i=1}^m \lambda_i = 1\}$. Obviously, $C_1 \le C_2$.

We first assume that $\mathcal{B}_s$ is a compact base. If $\lambda \in \mathcal{B}_s$, then $\sum_{i=1}^m |\lambda_i| \ge \sum_{i=1}^m \lambda_i = 1$, so that $C_2^{-1} = \min\left\{\sum_{i=1}^m |\lambda_i| : \lambda \in \mathcal{B}_s\right\} \ge 1$, which combined with Proposition 24 yields $\rho_{CLP} \le \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right)$.

We now assume that $\mathbb{R}_+^m \subset K$, so that $\mathcal{B}_s \subset K^* \subset \mathbb{R}_+^m$. Then we have

$$\begin{aligned} C_1^{-1} &= \max\left\{\|\sum_{i=1}^m \lambda_i u_i\| : \lambda \in \mathcal{B}_s, \|u_i\| \le 1\right\} \\ &\le \max\left\{\sum_{i=1}^m \lambda_i\|u_i\| : \lambda \in \mathcal{B}_s, \|u_i\| \le 1\right\} \le 1. \end{aligned}$$

This implies that $C_1 \ge 1$. This together with $C_1 \le C_2$ and $C_2^{-1} \ge 1$ gives us that $C_1 = C_2 = 1$. So, $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right) = \rho_{CLP}$, by applying again Proposition 24 with $\mathcal{B} = \mathcal{B}_s$. $\square$

- **LP:** Since the additional assumption in Proposition 26 trivially holds, (33) provides a new formula for ρ_{LP} by solving the following SOCP problem (compare with (7) and (15)):

$$\rho_{LP} = \inf_{(z,s,t,y)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{t : \begin{array}{l}\|(z,s)\| \le t, \\ z = \overline{A}^\top y,\ s \ge \overline{b}^\top y, \\ y \in \mathbb{R}_+^m,\ \sum_{i=1}^m y_i = 1\end{array}\right\}. \tag{34}$$

The next result is an immediate consequence of the double inequality (29) in Proposition 24.

Proposition 27 (Positiveness of ρ_{CLP} *under* (**A1**)*). Let $\mathcal{B}$ be a compact base of K^*. Then, $\rho_{CLP} > 0$ if and only if $0_{n+1} \notin E(\overline{A}, \overline{b}, \mathcal{B})$.*

The attainment of ρ_{CLP} can be characterized by putting $\alpha = \rho_{CLP}$ in (28).

Proposition 28 (Attainment of ρ_{CLP} *under* (**A1**)*). Assume that $\mathbb{R}_+^m \subset \Lambda^{-1}K$. Then, ρ_{CLP} is attainable if and only if*

$$(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\left(\bigcup_{y\in\mathcal{B}}\left\{\sum_{i=1}^m y_i\left((\overline{a}_i, \overline{b}_i) + \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right)\mathbb{B}_{n+1}\right)\right\}\right).$$

An application of Proposition 26 to uncertain support vector machine problems can be found in Goberna et al. (2021).

Assumption (**A2**), with $Z = \mathbb{B}_{n+1}$, in the robust CLP setting, consists in replacing $\mathcal{U}$ by $\Lambda\mathcal{U} := \{\Lambda U : U \in \mathcal{U}\}$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, with $\lambda_i > 0$, $i = 1, \ldots, m$, $\mathcal{U}$ being the pattern-set of σ^{α}_{CLP}. So, the parameterized scaled robust solution set is

$$\begin{aligned}\widetilde{F}^{\alpha}_{CLP} &= \left\{x \in \mathbb{R}^n : [A \mid b]\begin{pmatrix} x \\ -1\end{pmatrix} \in -K \text{ for all } (a_i, b_i) \in (\overline{a}_i, \overline{b}_i) + \alpha\lambda_i \mathbb{B}_{n+1}, i \in I\right\} \\ &= \left\{x \in \mathbb{R}^n : \left(\left[\overline{A} \mid \overline{b}\right] + U\right)\begin{pmatrix} x \\ -1\end{pmatrix} \in -K \text{ for all } U \in \alpha\Lambda\mathcal{U}\right\} \\ &= \left\{x \in \mathbb{R}^n : \left(\Lambda^{-1}\left[\overline{A} \mid \overline{b}\right] + V\right)\begin{pmatrix} x \\ -1\end{pmatrix} \in -\Lambda^{-1}K \text{ for all } V \in \alpha\mathcal{U}\right\},\end{aligned}$$

where the cone $\Lambda^{-1}K$ is a closed pointed convex cone such that $\operatorname{int}\left(\Lambda^{-1}K\right) \neq \emptyset$ and

$$\begin{aligned}\left(\Lambda^{-1}K\right)^{*} &= \left\{y \in \mathbb{R}^m : \left(\Lambda^{-1}y\right)^{\top} z \geq 0, z \in K\right\} \\ &= \left\{\Lambda v \in \mathbb{R}^m : v^{\top} z \geq 0, z \in K\right\} = \Lambda K^{*}.\end{aligned}$$

We associate with σ_{CLP} satisfying (**A2**) the *epigraphical set*

$$E(\overline{A}, \overline{b}, \mathcal{B}, \Lambda) := \left\{y^{\top}\left[\Lambda^{-1}\overline{A} \mid \Lambda^{-1}\overline{b}\right] : y \in \mathcal{B}\right\} + (\{0_n\} \times \mathbb{R}_+).$$

The following result is an immediate consequence of Propositions 25 and 26.

Corollary 29 (Bounds and an exact formula for ρ_{CLP} *under* (**A2**)). *Let $\mathcal{B}$ be a base of ΛK^*, and $C_1(\mathcal{B})$ and $C_2(\mathcal{B})$ be as in (30) and (31). Then,*

$$C_1(\mathcal{B}) \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}, \Lambda)\right) \leq \rho_{CLP} \leq C_2(\mathcal{B}) \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}, \Lambda)\right).$$

Moreover, if $\mathcal{B}_s := \left\{y \in \Lambda K^* : \sum_{i=1}^m y_i = 1\right\}$ is a compact base of ΛK^*, then

$$\rho_{CLP} \leq \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s, \Lambda)\right),$$

and the exact formula

$$\rho_{CLP} = \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s, \Lambda)\right)$$

holds whenever the additional assumption $\mathbb{R}^m_+ \subset \Lambda^{-1}K$ is satisfied.

Corollary 30 (Positiveness of ρ_{CLP} *under* (**A2**)). *Let $\mathcal{B}$ be a compact base of ΛK^*. Then, $\rho_{CLP} > 0$ if and only if $0_{n+1} \notin E(\overline{A}, \overline{b}, \mathcal{B}, \Lambda)$.*

Corollary 31 (Attainment of ρ_{CLP} *under* (**A2**)). *Assume that $\mathbb{R}^m_+ \subset \Lambda^{-1}K$. Then, ρ_{CLP} is attainable if and only if*

$$(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\left(\bigcup_{y \in \mathcal{B}}\left\{\sum_{i=1}^m y_i\left((\overline{a}_i, \overline{b}_i) + \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}_s)\right)\mathbb{B}_{n+1}\right)\right\}\right),$$

where $\mathcal{B}_s := \left\{y \in \Lambda K^* : \sum_{i=1}^m y_i = 1\right\}$.

7. Distance to ill-posedness

In this section, we show that our derived bounds and formulas for RRF can also provide new formulas for computing the distance to ill-posedness for uncertain conic systems.

Consider the conic linear system $Ax - b \in -K$ parameterized by the parameter $[A, b] \in \mathbb{R}^{m \times (n+1)}$. For two parameters $\left[A^1, b^1\right], \left[A^2, b^2\right] \in \mathbb{R}^{m \times (n+1)}$ with $A^i = \left[a^i_1 \mid \ldots \mid a^i_m\right]^{\top}$ and $b^i = \left(b^i_1, \ldots, b^i_m\right)^{\top}$, $i = 1, \ldots, m$, we define the distance between $\left[A^1, b^1\right]$ and $\left[A^2, b^2\right]$ as

$$d\left(\left[A^1, b^1\right], \left[A^2, b^2\right]\right) := \max_{i=1,\ldots,m}\left\{\left\|\left(a^1_i, b^1_i\right) - \left(a^2_i, b^2_i\right)\right\|\right\}.$$

Then, the so-called *space of parameters* Θ can be defined as the space of all $m \times (n+1)$ matrices $[A, b]$ equipped with the above metric d. The *set of feasible parameters* is defined as

$$\Theta_c := \{[A, b] \in \Theta : Ax - b \in -K \text{ for some } x \in \mathbb{R}^n\}.$$

Let $\left[\overline{A}, \overline{b}\right] \in \Theta_c$ be a given matrix parameter. We say the matrix parameter $\left[\overline{A}, \overline{b}\right] \in \Theta_c$ is *well-posed with respect to feasibility* when $\left[\overline{A}, \overline{b}\right] \in \operatorname{int}\Theta_c$. The *distance to ill-posedness* of the given matrix parameter $\left[\overline{A}, \overline{b}\right] \in \Theta_c$, denoted as $\delta\left(\overline{A}, \overline{b}\right)$, is defined as

$$\delta\left(\overline{A}, \overline{b}\right) = \operatorname{dist}\left(\left[\overline{A}, \overline{b}\right], \Theta \backslash \Theta_c\right).$$

Next, using results from the previous section, we provide computable bounds for the distance to ill-posedness of a given matrix parameter $\left[\overline{A}, \overline{b}\right]$. These bounds can be aggregated to those provided by Vera (2014, Section 3) under suitable assumptions.

Proposition 32 (Bounds for the distance to ill-posedness)**.** *Let $\mathcal{B}$ be a compact base for K^*. Then,*

$$\rho_{CLP} \leq \delta\left(\overline{A}, \overline{b}\right) \leq \frac{C_2(\mathcal{B})}{C_1(\mathcal{B})}\rho_{CLP}, \tag{35}$$

and

$$\begin{aligned}&C_1(\mathcal{B}) \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) \leq \delta\left(\overline{A}, \overline{b}\right) \\ &\quad \leq \frac{(C_2(\mathcal{B}))^2}{C_1(\mathcal{B})} \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right),\end{aligned} \tag{36}$$

where $C_1(\mathcal{B}), C_2(\mathcal{B})$ are given as in (30) and (31), that is,

$$C_1(\mathcal{B}) := 1/\max\left\{\left\|\sum_{i=1}^m \lambda_i u_i\right\| : \lambda \in \mathcal{B}, \|u_i\| \leq 1\right\}$$

and

$$\text{and } C_2(\mathcal{B}) := 1/\min\left\{\sum_{i=1}^m |\lambda_i| : \lambda \in \mathcal{B}\right\}.$$

Proof. For the sake of simplicity, we write C_1 and C_2 instead of $C_1(\mathcal{B})$ and $C_2(\mathcal{B})$, respectively, along the proof. We first establish (35). To do this, we observe that $\rho_{CLP} \leq \delta(\overline{A}, \overline{b})$ holds because, recalling (27),

$$F^{\alpha}_{CLP} \subset \{x \in \mathbb{R}^n : Ax - b \in -K\}$$

for all $[A, b] \in \Theta$ such that $d([A, b], [\overline{A}, \overline{b}]) \leq \alpha$.

Take an arbitrary $\rho > \rho_{CLP}$. By the same argument as in the first part of Proposition 24, we can write for any $\epsilon > 0$

$$\sum_{i=1}^m \gamma_i(\overline{a}_i, \overline{b}_i) + (0_n, \mu) = -(\rho C_1^{-1} + \epsilon)(u, s), \tag{37}$$

where $\gamma \in \mathcal{B}$, μ is a positive scalar, and $(u, s) \in \mathbb{B}_{n+1}$. Let $w_2 := \min\{\sum_{i=1}^m |\lambda_i| : \lambda \in \mathcal{B}\} > 0$, and, for each $i = 1, \ldots, m$,

$$(u_i, s_i) := \left(\frac{\operatorname{sign}\gamma_i}{\sum_{i=1}^m |\gamma_i|}u, \frac{\operatorname{sign}\gamma_i}{\sum_{i=1}^m |\gamma_i|}s\right) \in w_2^{-1}\mathbb{B}_{n+1}.$$

This together with the fact that $C_2 = w_2^{-1}$ implies that $\|(u_i, -s_i)\| \leq C_2$, $i = 1, \ldots, m$. Let $\epsilon > 0$. Defining $(a_i, b_i) := (\overline{a}_i, \overline{b}_i) + (\rho C_1^{-1} + \epsilon)(u_i, -s_i)$, $i = 1, \ldots, m$, $A := [a_1 \mid \ldots \mid a_m]^{\top}$ and $b := (b_1, \ldots, b_m)^{\top}$, we have

$$d\left([A, b], \left[\overline{A}, \overline{b}\right]\right) = (\rho C_1^{-1} + \epsilon)\max_{1 \leq i \leq m}\|(u_i, -s_i)\| \leq \left(\rho C_1^{-1} + \epsilon\right)C_2,$$

where the last inequality holds because $\|(u_i, -s_i)\| \leq C_2$. Moreover, noting the fact that $\sum_{i=1}^m \gamma_i(u_i, s_i) = (u, s)$, it follows from (37) that $(0_n, \mu) = \gamma^{\top}[A \mid b]$. So,

$$(0_n, 1) \in \operatorname{cone}\left\{\lambda^{\top}[A \mid b] : \lambda \in \mathcal{B}\right\},$$

which yields $[A, b] \notin \Theta_c$ by Proposition 28 for $\alpha = 0$. Hence, $\delta(\overline{A}, \overline{b}) \leq (\rho C_1^{-1} + \epsilon)C_2$. Letting $\epsilon \longrightarrow 0$ we have $\delta(\overline{A}, \overline{b}) \leq C_2(C_1)^{-1}\rho$, which shows that $\delta(\overline{A}, \overline{b}) \leq C_2(C_1)^{-1}\rho_{CLP}$. So, (37) holds.

Finally, combining (35) and Proposition 24, we see that (36) follows. Thus, the conclusion follows. □

As an immediate corollary, we obtain complete characterizations for well-posedness of a given data matrix $[\overline{A}, \overline{b}]$ with respect to feasibility of the linear conic system $Ax - b \in -K$.

Corollary 33 (Characterizing well-posedness)**.** *Let $\mathcal{B}$ be a compact base for K^*. Then $[\overline{A},\overline{b}] \in \Theta_c$ is well-posed if and only if $0_{n+1} \notin E(\overline{A},\overline{b},\mathcal{B})$.*

Proof. By Proposition 32, we see that

$$\left[\overline{A},\overline{b}\right] \in \operatorname{int}\Theta_c \Longleftrightarrow \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right) > 0 \Longleftrightarrow 0_{n+1} \notin E(\overline{A},\overline{b},\mathcal{B}).$$

Thus, the conclusion follows. □

The next exact formula, applied to the particular case of $K=\mathbb{R}^m_+$ and its natural base Δ_m, coincides with the formula in (Cánovas et al., 2005, Theorem 6) specialized to linear programming problems.

Proposition 34. *Let $\mathbb{R}^m_+ \subset K$ and $\mathcal{B}_s = \{y \in K^* : \sum_{i=1}^m y_i = 1\}$. Then,*

$$\delta\left(\overline{A},\overline{b}\right) = \rho_{CLP} = \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B}_s)\right).$$

Proof. Using same argument as in Proposition 26, we see that $C_1(\mathcal{B}_s) = C_2(\mathcal{B}_s) = 1$ and so, the conclusion follows from Proposition 32. □

In particular, one recovers the formula in (6) $\rho_{LP} = \operatorname{dist}(0_{n+1}, E(\overline{a},\overline{b}))$ by taking $K=\mathbb{R}^m_+$ in Proposition 34.

In concluding this Section, we briefly comment on the relationship between our results and previous works on the distance to ill-posedness. Firstly, note that our nominal conic linear system $\{\overline{A}x - \overline{b} \in -K\}$ satisfies the assumptions of Freund and Vera (1999, Theorems 6 and 7), where the norm of $[\overline{A},\overline{b}] \in \Theta$ is $\max\{\|\overline{A}\|, \|\overline{b}\|\}$, with $\|\overline{A}\| := \{\|\overline{A}x\| : \|x\| \le 1\}$ (instead of the norm used in Section 6, i.e., $\max_{j=1,\ldots,m}\{\|(\overline{a}_j,\overline{b}_j)\|\}$) and whose proofs are based on a result of Renegar (1994) and a suitable “linearization” of the involved cones.

Denote by $\delta_0(\overline{A},\overline{b})$ the distance from $[\overline{A},\overline{b}]$ to ill-posedness measured this way, which was proposed in Freund and Vera (1999) and by $\|u\|_* = \max\{y^\top x : \|y\| \le 1\}$ the *dual norm* of $u \in \mathbb{R}^m$. Theorem 7 in Freund and Vera (1999) asserts that

$$\beta_K v^{-1} \le \delta_0(\overline{A},\overline{b}) \le v^{-1}, \tag{38}$$

where $\beta_K := \sup_{u\in\mathbb{R}^m, \|u\|_*=1} \inf_{x\in\mathbb{R}^m, \|x\|=1} u^\top x$ is the so-called *coefficient of linearity* of the cone K (Freund and Vera, 1999, Definition 1) and

$$v := \inf\left\{\|x\| + \gamma : \gamma\overline{b} - \overline{A}x - \frac{1_m}{\sqrt{m}} \in K, \gamma \in \mathbb{R}_+\right\}.$$

Let $d := \max\{m,n\}$. From the known inequalities for pairs of vector and matrix norms in Horn and Johnson (1985) (see the tables in pp. 279 and 314) one gets

$$\frac{\delta(\overline{A},\overline{b})}{\sqrt{2d}} \le \delta_0(\overline{A},\overline{b}) \le d\,\delta(\overline{A},\overline{b}),$$

which combined with (29) yields

$$\left(\frac{C_1}{\sqrt{2d}}\right) \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right) \le \delta_0\left(\overline{A},\overline{b}\right) \le \left(\frac{dC_2^2}{C_1}\right) \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right). \tag{39}$$

Observe that (38) and (39) provide lower and upper bounds for $\delta_0(\overline{A},\overline{b})$ which are expressed as positive multiples of v^{-1} and $\operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right)$, respectively. In LP, $K=\mathbb{R}^m_+$ and one gets from either (38), with $\beta_K = \frac{1}{\sqrt{m}}$ (see the comment after Freund & Vera, 1999, Remark 4), or from (39), an interval for $\delta_0(\overline{A},\overline{b})$ while our Proposition 26 provides the exact formula $\delta(\overline{A},\overline{b}) = \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\Delta_m)\right)$, which has a nice geometric interpretation and can easily be computed by solving a second-order cone program, see (34). So, at least in LP, $\delta(\overline{A},\overline{b})$ seems preferable to $\delta_0(\overline{A},\overline{b})$ as a measure of the distance to ill-posedness.

Table 1
Classification of the main results on the RRF.

	RRF	(A1)	(A2)	(A3)	(A4)
Formula for	ρ_{LP}	Prop. 2	Cor. 5*	Prop. 7	
	ρ_{MILP}			Prop. 9	
	ρ_{LSIP}	Prop. 10*	Cor. 14*		
	ρ_{CP}	Prop. 17	Cor. 20*		Prop. 22
	ρ_{CLP}	Props. 24–26	Cor. 29*		
Positiveness of	ρ_{LP}	Prop. 3*			
	ρ_{MILP}				
	ρ_{LSIP}	Prop. 12*	Cor. 15*		
	ρ_{CP}	Prop. 18			
	ρ_{CLP}	Prop. 27	Cor. 30*		
Attainment of	ρ_{LP}	Prop. 4*	Cor. 6*		
	ρ_{MILP}	Prop. 8			
	ρ_{LSIP}	Prop. 13*	Cor. 16*		
	ρ_{CP}	Prop. 19*	Cor. 21*		
	ρ_{CLP}	Prop. 28	Cor. 31*		

8. Conclusions and further research

This paper provides formulas and methods to compute either the RRF, or at least lower and upper bounds for the RRF, for five types of uncertain optimization programs under different assumptions. It also provides conditions for the positiveness of the RRF and its attainment. Table 1 summarizes the content of Sections 2–6, classifying the given results according to two criteria: information provided on the RRF (either a formula for its exact or approximate computation, or a positiveness condition, or an attainment condition) and assumption (from (**A1**) to (**A4**)) under which the corresponding result is valid. Moreover, the new results (some of them are corollaries), are marked with an asterisk. The empty cells identify (not necessarily difficult) open problems on the RRF.

A good part of the above results involve formulas or conditions which are not checkable through tractable optimization problems. Specifying types of pattern-sets (likely spectrahedra) allowing to obtain checkable formulas or conditions is a challenging problem, together with the narrowing of the intervals for the RRF in those results providing lower and upper bounds, specially under (**A4**). There is no hope of characterizing the attainability of the RRF until the obtaining of an existence theorem for linear systems posed in $\mathbb{Z}^n$ (a hard theoretical open problem). Of course, the study of the RRF is still to be made for other types of optimization problems whose constraint system do not belong to the five type families of systems analyzed in this paper, e.g., uncertain convex semi-infinite programs, whose RRF will likely be characterized by combining the tools used in Sections 4 and 5.

A major challenging problem is to extend the RRF results to adjustable robust optimization (Ben-Tal et al., 2009), which offers less conservative decisions than the classical static (single-stage) robust optimization for multi-stage optimization problems involving both “here and now” and “wait and see” decision variables. In the simplest case of two-stage LP, according to Woolnough, Jeyakumar, and Li (2021), the parameterized *robust counterpart* of the nominal constraint system can be written as

$$\sigma^\alpha_{ALP} := \{A(u)x + By(u) \le d(u),\ u \in \alpha\mathcal{U}\},$$

where the pattern-set $\mathcal{U}$ is a convex subset containing the null vector of some linear space, $x \in \mathbb{R}^n$ is the first-stage “here and now” decision variable that is made before u is realized. The second-stage “wait and see” decision, $y(\cdot)$, that can be adjusted according to the actual data, is a mapping, rather than a vector. The coefficient matrix $A \in \mathbb{R}^{m\times n}$ and the right hand side vector $d \in \mathbb{R}^m$ depend on the uncertainty parameter u, while the (fixed recourse) coefficient matrix $B \in \mathbb{R}^{m\times k}$ does not depend on u. Then, the RRF of the nominal adjustable LP problem σ^0_{ALP} could be defined as

$\rho_{ALP} := \sup\{\alpha \in \mathbb{R}_+ : F^{\alpha}_{ALP} \neq \emptyset\}$,

where F^{α}_{ALP} represents the solution set of σ^{α}_{ALP}.

Formally, everything is apparently as in the definition of ρ_{LP}, but there is a substantial difference: the decision space is now infinite dimensional, making harder obtaining the exact value and even bounds for ρ_{ALP}, as well as conditions guaranteeing its positivity and attainability. An approach for obtaining numerically tractable results, based on decision rules, is to restrict $y(\cdot)$ to some specific class of functions such as affine or quadratic functions (see Ben-Tal et al., 2009, Woolnough et al., 2021). It is of particular interest to examine how the pattern-sets can be constructed using RRF for specific practical decision-making problems such as the lot-sizing problems and production planning problems (Ben-Tal et al., 2009).

The relationship between the RRF and the distance to ill-posedness in mathematical programming is mentioned in passing in Section 4, as it was used to obtain the RRF in linear semi-infinite programming in the first paper on RRF. Section 7 provides bounds for the distance to ill-posedness in parametric conic linear programming which are derived from the corresponding bounds for RRF obtained in Section 6. As a by-product, we characterize the well-posedness and obtain an exact formula for the distance to ill-posedness under the mentioned strong condition on K. All results in this section, Propositions 32 and 34, and Corollary 33, are original. Of course, new connections between both concepts can potentially be obtained by combining different norms in the space of parameters and different bases of K^*.

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References

Aliprantis, C. D., & Tourky, R. (2007). *Cones and duality*. Providence, Rhode Island: AMS.
Aragón, F. J., Goberna, M. A., López, M. A., & Rodríguez, M. M. L. (2019). *Nonlinear optimization*. Cham, Switzerland: Springer.
Aßmann, D., Liers, F., & Stingl, M. (2019). Decomposable robust two-stage optimization: An application to gas network operations under uncertainty. *Networks, 74*, 40–61.
Auslender, A., Ferrer, A., Goberna, M. A., & López, M. A. (2015). Comparative study of RPSALG algorithm for convex semi-infinite programming. *Computational Optimization and Applications, 60*, 59–87.
Ben-Tal, A., El Ghaoui, L., & Nemirovski, A. (2009). *Robust optimization*. Princeton: Princeton University Press.
Ben-Tal, A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research, 25*, 1–13.
Ben-Tal, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming, 88*, 411–424.
Ben-Tal, A., & Nemirovski, A. (2001). *Lectures on modern convex optimization: Analysis, algorithms, and engineering applications*. Philadelphia: SIAM.
Ben-Tal, A., & Nemirovski, A. (2002). Robust optimization – methodology and applications. *Mathematical Programming, 92*, 453–480.
Ben-Tal, A., & Nemirovski, A. (2008). Selected topics in robust convex optimization. *Mathematical Programming, 112*, 125–158.
Bertsimas, D., & Brown, D. B. (2009). Constructing uncertainty sets for robust linear optimization. *Operations Research, 57*, 1483–1495.
Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM Review, 53*, 464–501.
Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research, 52*, 35–53.
Cánovas, M. J., Gómez-Senent, F. J., & Parra, J. (2007). Stability of systems of linear equations and inequalities: Distance to ill-posedness and metric regularity. *Optimization, 56*, 1–24.
Cánovas, M. J., López, M. A., Parra, J., & Toledo, F. J. (2005). Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems. *Mathematical Programming, 103*, 95–126.
Cánovas, M. J., López, M. A., Parra, J., & Toledo, F. J. (2006). Distance to ill-posedness in linear optimization via the Fenchel-Legendre conjugate. *Journal of Optimization Theory and Applications, 130*, 173–183.
Cánovas, M. J., López, M. A., Parra, J., & Toledo, F. J. (2011). Distance to ill-posedness for linear inequality systems under block perturbations: Convex and infinite-dimensional cases. *Optimization, 60*, 925–946.
Carrizosa, E., & Nickel, S. (2003). Robust facility location. *Mathematical Methods of Operations Research, 58*, 331–349.
Chen, J., Li, J., Li, X., Lv, Y., & Yao, J. C. (2020). Radius of robust feasibility of system of convex inequalities with uncertain data. *Journal of Optimization Theory and Applications, 184*, 384–399.
Choo, Y. (2014). Improved optimum radius for robust stability of schur polynomials. *Journal of Optimization Theory and Applications, 161*, 553–556.
Chuong, T. D., & Jeyakumar, V. (2017). An exact formula for radius of robust feasibility of linear programs. *Journal of Optimization Theory and Applications, 173*, 203–226.
Dinh, N., Goberna, M. A., & López, M. A. (2006). From linear to convex systems: Consistency, farkas' lemma and applications. *Journal of Convex Analysis, 13*, 113–133.
Fan, K. (1968). On infinite systems of linear inequalities. *Journal of Mathematical Analysis and Applications, 21*, 475–478.
Freund, R. M., & Vera, J. R. (1999). Some characterizations and properties of the "distance to ill-posedness" and the condition measure of a conic linear system. *Mathematical Programming, 86*, 225–260.
Gao, L. X., & Sun, Y. X. (2002). On the optimum radius of robust stability for Schur polynomials. *Journal of Optimization Theory and Applications, 114*, 471–475.
Gao, M. J., Yiu, K.-F., & Wu, S. Y. (2018). A perturbation exchange algorithm for convex semi-infinite programming with applications in sparse beamformer design. *Pacific Journal of Optimization, 14*, 15–30.
Goberna, M. A., Jeyakumar, V., & Li, G. (2021). Calculating radius of robust feasibility of uncertain linear conic programs via semi-definite programs. *Journal of Optimization Theory and Applications, 189*, 597–622.
Goberna, M. A., Jeyakumar, V., Li, G., & Linh, N. (2016). Radius of robust feasibility formulas for classes of convex programs with uncertain polynomial constrains. *Operations Research Letters, 44*, 67–73.
Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2015). Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research, 242*, 730–743.
Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2018). Guaranteeing highly robust weakly efficient solutions for uncertain multi-objective convex programs. *European Journal of Operational Research, 70*, 40–50.
Goberna, M. A., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2014). Robust solutions of multi-objective linear semi-infinite programs under constraint data uncertainty. *SIAM Journal on Optimization, 24*, 1402–1419.
Goberna, M. A., & López, M. A. (1998). *Linear semi-infinite optimization*. Chichester: Wiley.
Goberna, M. A., & López, M. A. (2018). Recent contributions to linear semi-infinite optimization: An update. *Annals of Operations Research, 271*, 237–278.
Goberna, M. A., Ridolfi, A. B., & Vera de Serio, V. N. (2020). Selected applications of linear semi-infinite systems theory. *Vietnam Journal of Mathematics, 48*, 439–470.
Göpfert, A., Riahi, H., Tammer, C., & Zălinescu, C. (2003). *Variational methods in partially ordered spaces*. New York: Springer.
Guo, F., & Sun, X. (2020). On semi-infinite systems of convex polynomial inequalities and polynomial optimization problems. *Computational Optimization and Applications, 75*, 669–699.
Horn, R. A., & Johnson, C. R. (1985). *Matrix analysis*. New York: Cambridge University Press.
(2015). In T. Koch, B. Hiller, M. Pfetsch, & L. Schewe (Eds.), *Evaluating gas network capacities*. Philadelphia: SIAM.
Li, X.-B., & Wang, Q. L. (2018). A note on the radius of robust feasibility for uncertain convex programs. *Filomat, 32*, 6809–6818.
Liers, F., Schewe, L., & Thürauf, J. (2021). Radius of robust feasibility for mixed-integer problems. *INFORMS Journal on Computing*. https://doi.org/10.1287/ijoc.2020.1030.
Mastorakis, N. E. (2000). Optimum radius of robust stability for schur polynomials. *Journal of Optimization Theory and Applications, 104*, 165–174.
Mehrotra, S., & Papp, D. (2014). A cutting surface algorithm for semi-infinite convex programming with an application to moment robust optimization. *SIAM Journal on Optimization, 24*, 1670–1697.
Misra, N., & Sonar, C. (2019). Robustness radius for chamberlin-courant on restricted domains. In *SOFSEM 2019: theory and practice of computer science*. In *Lecture Notes in Comput. Sci., 11376. Cham* (pp. 341–353). Switzerland: Springer.
Nie, J. (2013). Semidefinite representability. In G. Blekherman, P. A. Parrilo, & R. R. Thomas (Eds.), *Semidefinite optimization and convex algebraic geometry. chapter 6* (pp. 251–291). Philadelphia: MOS-SIAM.
Okuno, T., Hayashi, S., Yamashita, N., & Gomoto, K. (2016). An exchange method with refined subproblems for convex semi-infinite programming problems. *Optimization Methods & Software, 31*, 1305–1324.
Pang, L.-P., Lv, J., & Wang, J. H. (2016). Constrained incremental bundle method with partial inexact oracle for nonsmooth convex semi-infinite programming problems. *Computational Optimization and Applications, 64*, 433–465.
Ramana, M., & Goldman, A. J. (1995). Some geometric results in semidefinite programming. *Journal of Global Optimization, 7*, 33–50.
Renegar, J. (1994). Some perturbation theory for linear programming. *Mathematical Programming, 65*, 73–91.
Schewe, L., Schmidt, M., & Thürauf, J. (2020). Structural properties of feasible bookings in the european entry-exit gas market system. *4OR, 18*, 197–218.

Vera, J. R. (2014). Geometric measures of convex sets and bounds on problem sensitivity and robustness for conic linear optimization. *Mathematical Programming, 147*, 47–79.
Vinzant, C. (2014). What is a spectrahedron? *Notices of the American Mathematical Society, 61*, 492–494.
Woolnough, D., Jeyakumar, V., & Li, G. (2021). Exact conic programming reformulations of two-stage adjustable robust linear programs with new quadratic decision rules. *Optimization Letters, 15*, 25–44.
Zhang, Q., Grossmann, I. E., & Lima, R. M. (2016). On the relation between flexibility analysis and robust optimization for linear systems. *AIChE Journal, 62*, 3109–3123.