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Calculating Radius of Robust Feasibility of Uncertain Linear Conic Programs via Semidefinite Programs

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Abstract The radius of robust feasibility provides a numerical value for the largest possible uncertainty set that guarantees robust feasibility of an uncertain linear conic program. This determines when the robust feasible set is non-empty. Otherwise the robust counterpart of an uncertain program is not well-defined as a robust optimization problem. In this paper, we address a key fundamental question of robust optimization: How to compute the radius of robust feasibility of uncertain linear conic programs, including linear programs? We first provide computable lower and upper bounds for the radius of robust feasibility for general uncertain linear conic programs under the commonly used ball uncertainty set. We then provide important classes of linear conic programs where the bounds are calculated by finding the optimal values of related semidefinite linear programs, among them uncertain semidefinite programs, uncertain second-order cone programs and uncertain support vector machine problems. In the case of an uncertain linear program, the exact formula allows us to calculate the radius by finding the optimal value of an associated second-order cone program.

Keywords Linear conic programs · Semidefinite programs · Parametric optimization · Robust feasibility

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1 Introduction

Consider the following linear conic programming problem

$$(P) \quad \min_{x\in\mathbb{R}^n} c^T x \\ \text{s.t.} \begin{bmatrix} a_1^T x + b_1 \\ \vdots \\ a_m^T x + b_m \end{bmatrix} \in -K, \tag{1}$$

where $\{0_m\} \neq K \subsetneqq \mathbb{R}^m$ is a given closed pointed convex cone with nonempty interior (which implies that its positive dual cone K^* enjoys the same properties), $c \in \mathbb{R}^n$, and the vectors $(a_i, b_i) \in \mathbb{R}^{n+1}, 1 \leq i \leq m$, or, equivalently, the matrix $A := [a_1 \mid ... \mid a_m]^T \in \mathbb{R}^{m\times n}$ and the vector $b := (b_1, ..., b_m)^T \in \mathbb{R}^m$, are the data associated to the conic programming problem (P). This model problem has found numerous applications in engineering, statistics and finance ([1], [2]), and covers many important optimization problems such as

(SDPs) Semi-definite programming (SDP in brief) problems, where $K = S_+^q$ is the cone consisting of all $(q \times q)$ positive semi-definite symmetric matrices.
(SOCPs) Second order cone programming problems, where

$$K = \{x \in \mathbb{R}^m : x_m \geq \|(x_1, ..., x_{m-1})\|\}$$

is the so-called second order cone (SOC in short), usually denoted by K_p^m.
(LPs) Linear programming (LP) problems, where $K = \mathbb{R}_+^m$. We must emphasize that most results in this paper are new even in this particular setting.

In practice, the data associated to the optimization problem (P) are uncertain due to measurement errors or prediction errors. One of the prominent ways of dealing with optimization under data uncertainty is the robust optimization approach. Following this approach, one assumes that the constraint data (a_i, b_i) ranges in some *uncertainty set*. We note that it is known in the robust optimization literature [3] that the general case where the linear objective data c is also uncertain can be easily converted to the current setting by introducing an auxiliary variable.

(**Uncertainty sets**) Given $i \in \{1, ..., m\}$, the uncertainty set for the constraint data (a_i, b_i) is the ball $\mathcal{U}_{r_i} := (\overline{a}_i, \overline{b}_i) + r_i\mathbb{B}_{n+1}$, where $r_i \in \mathbb{R}_+$ (with $r_i = 0$ when (a_i, b_i) is deterministic) and $\mathbb{B}_{n+1}$ is the Euclidean closed unit ball in $\mathbb{R}^{n+1}$.

Denote $\overline{A} = [\overline{a}_1 \mid ... \mid \overline{a}_m]^T$ and $\overline{b} = (\overline{b}_1, ..., \overline{b}_m)^T$. In practical situations, the matrix $[\overline{A}, \overline{b}]$ is called the nominal data which may correspond to a central value of a sample of observed matrices $[A, b]$. For instance, $(\overline{a}_i, \overline{b}_i)$ could be the mean vector of a sample of (a_i, b_i) −vectors. Denoting $r = (r_1, \ldots, r_m) \in \mathbb{R}_+^m$, the *robust counterpart* of (P), depending on r, can be formulated as

$$(P_r) \min_{x\in\mathbb{R}^n} c^T x$$
$$\text{s.t.} \begin{bmatrix} a_1^T x + b_1 \\ \vdots \\ a_m^T x + b_m \end{bmatrix} \in -K, \forall (a_i, b_i) \in \mathcal{U}_{r_i}, 1 \leq i \leq m,$$

which gives the worst-case solution for all possible realization of the scenarios in the constraint uncertainty set. The feasible set of (P_r)

$$F_r(\overline{A}, \overline{b}) := \{x \in \mathbb{R}^n : Ax + b \in -K, \ \forall (a_i, b_i) \in \mathcal{U}_{r_i}, i = 1, ..., m\}$$

is referred as the robust feasible set, where $A := [a_1 \mid ... \mid a_m]^T \in \mathbb{R}^{m\times n}$ and $b := (b_1, ..., b_m)^T \in \mathbb{R}^m$.

A great deal of work has been published on the efficient computation of optimal solutions of the robust counterpart of a variety of uncertain optimization problems. These problems are inherently semi-infinite, but they can be reduced to tractable semi-definite programs under suitable assumptions (see, e.g., [3], [4], [5], [6], and references therein).

One of the key questions in robust optimization is to determine when the robust feasible set is indeed nonempty (otherwise, the robust counterpart problem is not well-defined). An important concept quantifying the robust feasibility issue is the so-called *radius of robust feasibility* (RRF in short) of the nominal data, that can be roughly defined as the largest size of the uncertainty set so that the robust feasible set is nonempty. Formulas for the RRF have been given in [7], [8] and [9] for robust LP problems, in [10] for robust linear semi-infinite programming (LSIP) problems, in [11], [12] and [13] for robust convex programs, and in [9] for robust mixed-integer LP problems. The uncertainty sets are balls in [10] and [8], spectrahedra (e.g., ellipsoids, polytopes, and boxes) in [7], and more general compact convex sets in [11], [12], [13] and [9]. Most of the mentioned works discuss tractability issues regarding the proposed formulas for the computation of the RRF. In particular, in [7] and [12] the RRF of linear and convex polynomial problems is computed by solving associated semi-definite programs while [9] proposes to compute the RRF of linear and mixed linear programs via fractional programming and effective binary search algorithms, respectively. The introduction of the latter paper briefly reviews applications of the RRF to facility location design [14], flexibility index problem [15], and design and control of gas networks [16].

The uncertainty sets are also Euclidean balls in [8], devoted to certify the existence of highly robust solutions (i.e., robust feasible solutions which are optimal for any scenario) in robust multi-objective linear and convex programming with uncertain objectives through the estimation of the corresponding radius of highly robust efficiency. The formulas provided in this paper can also be used in robust scalar (resp., multi-objective) linear conic programming with deterministic constraints and uncertain objective (resp., objectives).

Although any conic LP problem can be reformulated as an LSIP problem through the "natural" linearization of the conic linear system $\left\{\overline{A}x+\overline{b}\in -K\right\}$ as $\sigma_{K^*}:=\left\{a^Tx\leq -b,(a,b)\in T_{K^*}\right\}$, where the index set is

$$T_{K^*}:=\left\{\sum_{i=1}^m \lambda_i(\overline{a}_i,\overline{b}_i):\lambda\in K^*\right\},$$

with K^* denoting the dual cone of K, this approach is seldom useful in examining uncertain conic LP, in particular from the stability and robustness perspectives, as σ_{K^*} is not even stable in the sense of LSIP because the latter system contains the trivial inequality $0_m^Tx\leq 0$ (recall [17, Theorem 6.1]). Therefore, none of the approaches used in the above mentioned papers [7,8,9, 10,11,12,15] can be directly adapted to deal with problems with conical constraints which calls for further research on the study of RRF for uncertain conic programming problems. Moreover, the mathematical formulae for estimating the RRF are often very difficult to validate numerically. In this paper we make the following contributions to robust linear conic programming:

(i) We first establish computable lower and upper bounds for the radius of robust feasibility for general uncertain linear conic programs under the commonly used ball uncertainty set.
(ii) We then show how the bounds can be calculated for important classes of linear conic programs by finding the optimal values of related semidefinite linear programs (SDPs), among them uncertain SDPs, uncertain second-order cone programs and uncertain support vector machine problems. In the case of an uncertain linear program, the exact formula allows us to calculate the radius by finding the optimal value of an associated second-order cone program.

The paper is organized as follows. Section 2 introduces the important concepts of radius of robust feasibility as well as admissible set of parameters formed by the parameters $r\in\mathbb{R}_+^m$ such that the robust feasible set $F_r(\overline{A},\overline{b})$ is nonempty. Section 3 provides lower and upper bounds for the RRF of uncertain conic programs. Section 4 establishes computational tractable bounds for uncertain semi-definite programs and second order cone programs. Section 5 provides conclusions and some future research directions. Finally, the appendix presents proofs of certain technical results.

2 Preliminaries and Radius of Robust Feasibility

Let us start introducing the necessary notation. We denote by 0_n, 1_n, $\|\cdot\|$, $\mathbb{B}_n$, and d the vector of all zeros, the vector of all ones, the Euclidean norm, the corresponding closed unit ball, and the Euclidean distance in $\mathbb{R}^n$, respectively. We also denote by $\{e_1,...,e_n\}$ the canonical basis and by $\Delta_n=\left\{x\in\mathbb{R}_+^n:1_n^Tx=1\right\}$ the unit simplex in $\mathbb{R}^n$. Given $\emptyset\neq X\subseteq\mathbb{R}^n$, $\operatorname{int}X$, $\operatorname{bd}X$, $\overline{X}$, $\operatorname{aff}X$, $\operatorname{conv}X$, denote the interior, the boundary, the closure, the affine hull, and the convex hull of X, respectively, whereas $\operatorname{cone} X := \mathbb{R}_+ \operatorname{conv} X$ denotes the convex conical hull of $X \cup \{0_n\}$. We represent by $\operatorname{dist}(\overline{x}, X) = \inf_{x\in X} d(\overline{x}, x)$ the distance from $\overline{x}$ to a set $X \subseteq \mathbb{R}^n$, with $\operatorname{dist}(\overline{x}, \emptyset) = +\infty$ by convention. A set K is called a cone if and only if $\lambda x \in K$ for any $\lambda \geq 0$ and $x \in K$. The (positive) dual cone of a cone $K \subseteq \mathbb{R}^m$ is defined as

$$K^* := \{a \in \mathbb{R}^m : a^T x \geq 0 \text{ for all } x \in K\}.$$

Throughout the paper, we assume that the cone $\{0_m\} \neq K \subsetneqq \mathbb{R}^m$ is a given closed pointed convex cone with nonempty interior. Moreover, we also assume that the feasible set of the nominal problem P_{0_m} is nonempty, that is, $\left\{x \in \mathbb{R}^n : \overline{A}x + \overline{b} \in -K\right\} \neq \emptyset$, where $\overline{A} = [\overline{a}_1 \mid ... \mid \overline{a}_m]^T$ and $\overline{b} = \left(\overline{b}_1, ..., \overline{b}_m\right)^T$. We note that these assumptions on the cone K in (P) are standard assumptions in the linear conic programming literature. In particular, the assumption $\{0_m\} \neq K \subsetneqq \mathbb{R}^m$ eliminates uninteresting cases (the feasible sets of the involved problems being either affine manifolds or the whole space); the condition that $\operatorname{int} K \neq \emptyset$ ensures [18,19] the existence of a compact base $\mathcal{B}$ for K^* (i.e., a compact and convex subset $\mathcal{B}$ of K^* such that $0_m \notin \mathcal{B}$ and $K^* = \mathbb{R}_+\mathcal{B}$).

Next, we introduce the important definition of the admissible set of parameters, which is formed by those parameters $r \in \mathbb{R}^m_+$ so that the robust feasible set is nonempty. This definition plays an important role in defining the concept of RRF considered later. We also emphasize that this concept appears to be new and was not examined in the previous study of the literature of RRF.

Definition 2.1 (Admissible set) Let $\overline{A} = [\overline{a}_1 \mid ... \mid \overline{a}_m]^T \in \mathbb{R}^{m\times n}$ and let $\overline{b} = \left(\overline{b}_1, ..., \overline{b}_m\right)^T \in \mathbb{R}^m$. The set

$$C(\overline{A}, \overline{b}) := \{r = (r_1, \ldots, r_m) \in \mathbb{R}^m_+ : F_r(\overline{A}, \overline{b}) \neq \emptyset\} \subseteq \mathbb{R}^m$$

is called the *admissible set of parameters* of the uncertain problem (P), where $F_r(\overline{A}, \overline{b})$ is the robust feasible set given by

$$F_r(\overline{A}, \overline{b}) := \{x \in \mathbb{R}^n : Ax + b \in -K,\ \forall\, (a_i, b_i) \in \mathcal{U}_{r_i}, i = 1, ..., m\},$$

$A := [a_1 \mid ... \mid a_m]^T \in \mathbb{R}^{m\times n}$, $b := (b_1, ..., b_m)^T \in \mathbb{R}^m$ and $\mathcal{U}_{r_i}$ is the ball uncertainty set defined as $\mathcal{U}_{r_i} := \left(\overline{a}_i, \overline{b}_i\right) + r_i\mathbb{B}_{n+1}$,

From its definition, $C(\overline{A}, \overline{b})$ is radiant (in the sense that $\mu C(\overline{A}, \overline{b}) \subseteq C(\overline{A}, \overline{b})$ for all $\mu \in [0, 1]$). However, we observe that

- The set $C(\overline{A}, \overline{b})$ may be reduced to $\{0_m\}$. For example, $C(\overline{A}, \overline{b}) = \{0_2\}$ when $\left[\overline{A} \mid \overline{b}\right] = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, with $K = -\mathbb{R}^2_+$.
- The set $C(\overline{A}, \overline{b})$ may contain nonzero points despite being contained in $\operatorname{bd}\mathbb{R}^m_+$. For instance, $C(\overline{A}, \overline{b}) = \operatorname{conv}\{(0, 0, 0), (0, 0, 1)\}$ when $\left[\overline{A} \mid \overline{b}\right] = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $K = -\mathbb{R}^3_+$.

– The set $C(\overline{A},\overline{b})$ can be a non-closed set with nonempty interior. For example, $C(\overline{A},\overline{b})=[0,1)^2$ when $\left[\overline{A}\mid\overline{b}\right]=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ and $K=-\mathbb{R}^2_+$.

Next, we summarize some basic properties of the admissible set of parameters below. These interesting mathematical properties could be of some independent interest. Its proof is included in the appendix for the purpose of self-containment.

Proposition 2.1 (Basic properties of $C(\overline{A},\overline{b})$) *Consider the admissible set $C(\overline{A},\overline{b})$.*

(a) (Dual characterization)

$$C(\overline{A},\overline{b})=\left\{r\in\mathbb{R}^m_+ : (0_n,1)\notin\overline{\operatorname{cone}\bigcup_{\lambda\in\mathcal{B}}\left\{\sum_{i=1}^m\lambda_i\left((\overline{a}_i,\overline{b}_i)+r_i\mathbb{B}_{n+1}\right)\right\}}\right\}.$$

where $\mathcal{B}$ is a compact base of K^*.

(b) (Boundedness) The set $C(\overline{A},\overline{b})$ is a bounded set which can be expressed as a union of segments emanating from 0_m.

(c) (Characterization of the interior) Let $\sigma_r^{\mathcal{B}}$ be the linear system describing $F_r(\overline{A},\overline{b})$ in (17). If $r\in\mathbb{R}^m_{++}$ and $\sigma_r^{\mathcal{B}}$ satisfies the Slater condition, then $r\in\operatorname{int}C(\overline{A},\overline{b})$. Conversely, if $r\in\operatorname{int}C(\overline{A},\overline{b})$ and $\sum_{i=1}^m r_i\lambda_i>0$ for all $\lambda\in\mathcal{B}$, then $\sigma_r^{\mathcal{B}}$ satisfies the Slater condition.

We now introduce the concept of radius of robust feasibility (RRF) for (P), as the supremum of those $\alpha\geq 0$ such that $F_{(\alpha,\ldots,\alpha)}(\overline{A},\overline{b})\neq\emptyset$.

Definition 2.2 (Radius of robust feasibility) The radius of robust feasibility (RRF) for (P) is defined as

$$\rho(\overline{A},\overline{b}):=\sup\{\alpha\in\mathbb{R}_+ : \alpha 1_m\in C(\overline{A},\overline{b})\},$$

where 1_m is the vector in $\mathbb{R}^m$ whose components are all equal to one.

The RRF $\rho(\overline{A},\overline{b})$ (Definition 2.2) is, roughly speaking, a measure of the maximal perturbations of a robust optimization problem which result in the problem still being feasible. The precise definition is a natural extension to robust conic linear programs of the homonymous concept introduced in [8] and [10] in the robust LP and LSIP settings, respectively.

Proposition 2.2 (Basic properties of radius of robust feasibility) *The following properties of the RRF of (P) hold:*

(i) $\rho(\overline{A},\overline{b})=\sup_{r\in C(\overline{A},\overline{b})}\min_{1\leq i\leq m}r_i$.

(ii) If $\rho(\overline{A},\overline{b})>0$, $\left[0,\rho(\overline{A},\overline{b})\right)^m\subseteq C(\overline{A},\overline{b})$.

(iii) $\rho(\overline{A},\overline{b})>0$ *if and only if* $\operatorname{int}C(\overline{A},\overline{b})\neq\emptyset$.

(iv) If there exists $\bar{x}\in\mathbb{R}^n$ such that $\overline{A}\overline{x}+\overline{b}\in-\operatorname{int}K$, then $\rho(\overline{A},\overline{b})>0$.

Proof We first see that the admissible set of parameters $C(\overline{A},\overline{b})$ satisfies

$$C(\overline{A},\overline{b}) = \left(C(\overline{A},\overline{b}) - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m. \tag{2}$$

If $r \in C(\overline{A},\overline{b})$, by definition of $C(\overline{A},\overline{b})$, any vector of $\left(r - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m$ belongs to $C(\overline{A},\overline{b})$. So, $\left(C(\overline{A},\overline{b}) - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m \subseteq C(\overline{A},\overline{b})$. Conversely, since

$$C(\overline{A},\overline{b}) = \left(C(\overline{A},\overline{b}) - 0_m\right) \cap \mathbb{R}_+^m \subseteq \left(C(\overline{A},\overline{b}) - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m,$$

we have $C(\overline{A},\overline{b}) = \left(C(\overline{A},\overline{b}) - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m$.

(i) Let $\alpha \geq 0$ be such that $\alpha 1_m \in C(\overline{A},\overline{b})$. Since $\alpha = \min\{\alpha,...,\alpha\} \leq \sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i$, we have $\rho(\overline{A},\overline{b}) \leq \sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i$.

Conversely, given $\epsilon > 0$, there exists $r \in C(\overline{A},\overline{b})$ and $j \in \{1,...,m\}$ such that $r_j \leq r_i$ for all $1 \leq i \leq m$ and $\sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i - \epsilon < r_j$. Since $r - r_j 1_m \in \mathbb{R}_+^m$, (2) yields

$$r_j 1_m \in \left(r - \mathbb{R}_+^m\right) \cap \mathbb{R}_+^m \subseteq C(\overline{A},\overline{b}).$$

Then, $r_j \leq \rho(\overline{A},\overline{b})$, so that $\sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i - \epsilon < \rho(\overline{A},\overline{b})$ for all $\epsilon > 0$. So, we see that $\sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i \leq \rho(\overline{A},\overline{b})$.

(ii) It follows from (2).

(iii) The direct statement follows from (ii) and the converse from (i).

(iv) The mapping $\Phi(A,b) := A\overline{x} + b$ is continuous on $\mathbb{R}^{m\times n} \times \mathbb{R}^m$ and satisfies $\Phi\left(\overline{A},\overline{b}\right) = \overline{A}\overline{x} + \overline{b} \in -\operatorname{int} K$, so that $\Phi(A,b) \in -K$ for (A,b) close enough to $\left(\overline{A},\overline{b}\right)$. Thus, $\overline{x} \in F_r(\overline{A},\overline{b})$ for any $r \in \mathbb{R}_+^m$ sufficiently close to 0_m and, so, $\rho(\overline{A},\overline{b}) = \sup_{r\in C(\overline{A},\overline{b})} \min_{1\leq i\leq m} r_i > 0$.

The characterization of $\rho(\overline{A},\overline{b})$ in Proposition 2.2(i) was used in [12] as definition of RRF for a class of convex problems with polynomial constraints without proving the equivalence between both concepts. Statement (ii) shows the identifiable part of $C(\overline{A},\overline{b})$ when $\rho(\overline{A},\overline{b})$ can be computed (see Theorem 3.2 below): the (non-closed) hypercube $\left[0,\rho(\overline{A},\overline{b})\right)^m$. The interior point condition $\overline{A}\overline{x} + \overline{b} \in -\operatorname{int} K$ in statement (iv) is called *Slater condition* for $\left(\overline{P}\right)$.

Next, we present a simple linear program example illustrating the admissible set of parameters and the radius of robust feasibility. This example is mainly used for illustrative purposes. Examples of conic program will also be proposed based on this illustrative example and discussed later on.

Example 2.1 ***(Illustrative example)*** Consider the simple linear program

$$\min_{x\in\mathbb{R}} \; x, \text{ s.t. } \begin{bmatrix} 2x \\ -x-3 \end{bmatrix} \in -\mathbb{R}_+^2. \tag{3}$$

Proposition 2.1 part (a) will allow us to obtain a tomographic description of the plane set

$$C(\overline{A},\overline{b})=\left\{r\in\mathbb{R}^2_+ : (0,1)\notin\overline{\operatorname{cone}\{((2,0)+r_1\mathbb{B}_2)\cup((-1,-3)+r_2\mathbb{B}_2)\}}\right\}$$

as the union of its intersections with all vertical lines.
Given $r=(r_1,r_2)\in\mathbb{R}^2$ such that $r_1\geq 2$, $F_r(\overline{A},\overline{b})=\emptyset$. So, we fix $0\leq r_1<2$ and calculate those r_2 such that $F_r(\overline{A},\overline{b})\neq\emptyset$. For $r_1=0$, $F_r(\overline{A},\overline{b})\neq\emptyset$ if and only if $0\leq r_2\leq d\left((-1,3),y=0\right)=3$. We now take $0<r_1<2$. The tangent lines from 0_2 to $\operatorname{bd}\left((2,0)+r_1\mathbb{B}_2\right)$ are $\pm r_1x-\sqrt{4-r_1^2}y=0$, denoted by L_+ and L_-, respectively. So, $F_r(\overline{A},\overline{b})\neq\emptyset$ if and only if $(-1,3)$ is above L_+ and L_-, i.e., $r_1\leq\frac{6}{\sqrt{10}}$ and

$$r_2\leq\min\left\{d\left((-1,3),L_+\right),d\left((-1,3),L_-\right)\right\}=d\left((-1,3),L_-\right)=\frac{3\sqrt{4-r_1^2}-r_1}{2}.$$

This amounts to saying that

$$\begin{aligned}C(\overline{A},\overline{b})&=\left\{r\in\mathbb{R}^2_+ : 0\leq r_1\leq 3\sqrt{\tfrac{2}{5}},0\leq r_2\leq\tfrac{3\sqrt{4-r_1^2}-r_1}{2}\right\}\\&=\left\{r\in\mathbb{R}^2_+ : 5r_1^2+2r_1r_2+2r_2^2\leq 18\right\}.\end{aligned}$$

For problem given in (3), we illustrate how Proposition 2.2 can help to find the greatest square supported by the coordinate axes contained in $C(\overline{A},\overline{b})$. As the line $r_2=r_1$ intersects $\operatorname{bd}C(\overline{A},\overline{b})$ at $(0,0)$ and $r^1:=\left(\sqrt{2},\sqrt{2}\right)$, the greatest α such that $F_{(\alpha,\alpha)}\neq\emptyset$ is $\rho(\overline{A},\overline{b})=\sqrt{2}$, with $F_{r^1}(\overline{A},\overline{b})=\{-1\}$ (a singleton set). Moreover, according to Proposition 2.2(ii), the feasibility of the robust counterpart is guaranteed for any r in the square $\left[0,\rho(\overline{A},\overline{b})\right)^2$ (possibly with $r_2\neq r_1$).

3 Bounds for Radius of Robust Feasibility

In this section, we obtain lower and upper bounds for $\rho(\overline{A},\overline{b})$, for an arbitrary compact base $\mathcal{B}$ of K^*, by using the following lemma. We note that this lemma is an auxiliary result using convex analysis, which allows to replace the finite set C (or, equivalently, the polytope $\operatorname{conv}C$) in [8, Lemma 3] by an arbitrary compact convex set; our proof here emphasizes the role played by the limit superior of some sequence of scalars which arises in the argument.

Lemma 3.1 *Let $C\subseteq\mathbb{R}^{n+1}$ be a nonempty compact convex set and $\alpha\geq 0$. Suppose that*

$$(0_n,1)\in\overline{\operatorname{cone}\left(C+\alpha\mathbb{B}_{n+1}\right)}. \tag{4}$$

Then, for all $\epsilon>0$, we have

$$(0_n,1)\in\operatorname{cone}\left(C+(\alpha+\epsilon)\mathbb{B}_{n+1}\right).$$

Proof Let $\epsilon > 0$. Since $(0_n, 1) \in \overline{\text{cone}\,(C + \alpha\mathbb{B}_{n+1})} = \overline{\mathbb{R}_+ (C + \alpha\mathbb{B}_{n+1})}$, there exist sequences $\{(y_k, s_k)\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^{n+1}$, $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$, $\{(x_k, t_k)\}_{k\in\mathbb{N}} \subseteq C$ and $\{(z_k, w_k)\}_{k\in\mathbb{N}} \subseteq \mathbb{B}_{n+1}$ such that

$$(y_k, s_k) = \mu_k \left((x_k, t_k) + \alpha(z_k, w_k)\right) \to (0_n, 1). \tag{5}$$

Two cases are possible for $\{\mu_k\}_{k\in\mathbb{N}}$.

Case 1: $\limsup \mu_k < +\infty$. We can assume that $\mu_k \longrightarrow \mu \in \mathbb{R}_+$ as $k \to \infty$. From (5), $\mu > 0$. Then, for sufficiently large k, one has $\frac{\mu}{2\mu_k} < 1$ and

$$\mu_k \left((x_k, t_k) + \alpha(z_k, w_k)\right) \in (0_n, 1) + \frac{\epsilon\mu}{2}\mathbb{B}_{n+1}.$$

Let $(u_k, v_k) \in \mathbb{B}_{n+1}$ be such that $\mu_k \left((x_k, t_k) + \alpha(z_k, w_k)\right) = (0_n, 1) + \frac{\epsilon\mu}{2}(u_k, v_k)$. Then,

$$(0_n, 1) = \mu_k \left((x_k, t_k) + \alpha(z_k, w_k) - \frac{\epsilon\mu}{2\mu_k}(u_k, v_k)\right) \in \text{cone}\,(C + (\alpha + \epsilon)\mathbb{B}_{n+1}).$$

Case 2: $\limsup \mu_k = +\infty$. We may assume that $\mu_k \to +\infty$ as $k \to \infty$. We also assume by contradiction that

$$(0_n, 1) \notin \text{cone}\,(C + (\alpha + \epsilon)\mathbb{B}_{n+1}).$$

Then, by the separation theorem, there exists $(\xi, r) \in \mathbb{R}^{n+1}\backslash\{0_{n+1}\}$ such that

$$r = \langle(\xi, r), (0_n, 1)\rangle \leq 0 \leq \langle(\xi, r), (y, s)\rangle, \tag{6}$$

for all $(y, s) \in \text{cone}\,(C + (\alpha + \epsilon)\mathbb{B}_{n+1})$. Let $(y, s) := \frac{(\xi,r)}{\|(\xi,r)\|} \in \mathbb{B}_{n+1}$. Note that

$$\mu_k \left((x_k, t_k) + \alpha(z_k, w_k) - \epsilon(y, s)\right) \in \mathbb{R}_+ (C + (\alpha + \epsilon)\mathbb{B}_{n+1}).$$

Then, (5) and (6) imply that

$$\begin{aligned} 0 &\leq \langle(\xi, r), \mu_k \left((x_k, t_k) + \alpha(z_k, w_k)\right)\rangle - \mu_k\, \epsilon\|(\xi, r)\| \\ &= \langle(\xi, r), (y_k, s_k)\rangle - \mu_k\epsilon\|(\xi, r)\|, \end{aligned}$$

with $\langle(\xi, r), (y_k, s_k)\rangle \to r$ and $\mu_k\epsilon\|(\xi, r)\| \to +\infty$. We got a contradiction.

As an illustration of the above discussion on the value of $\limsup \mu_k$, if we take $C = \{(1, 0_n)\}$, (4) holds if and only if $\alpha \geq 1$, we are necessarily in Case 2 when $\alpha = 1$, and both cases are possible when $\alpha > 1$.

To obtain the lower and upper bound for the RRF, we need the following definition of epigraphical set.

Definition 3.1 (Epigraphical set of (P)**)** The *epigraphical set* of (P) associated with a compact base $\mathcal{B}$ of K^* is the set

$$E(\overline{A}, \overline{b}, \mathcal{B}) := \left\{\lambda^T \left[\,\overline{A} \mid -\overline{b}\,\right] : \lambda \in \mathcal{B}\right\} + \{0_n\} \times \mathbb{R}_+.$$

The concept of epigraphical set is the adaptation to robust conic LP of the hypographical set introduced in [20] to measure the distance to ill-posedness in the framework of quantitative stability in LSIP. In contrast with the LP and LSIP adaptations in [10] and [8], the epigraphical set $E(\overline{A}, \overline{b}, \mathcal{B})$ not only depends here on the nominal data ($\overline{A}$ and $\overline{b}$), but also on the chosen compact basis $\mathcal{B}$ of K^*.

Obviously, $E(\overline{A}, \overline{b}, \mathcal{B})$ is a closed convex set. We observe that $0_{n+1} \notin \operatorname{int} E(\overline{A}, \overline{b}, \mathcal{B})$. To show this, we argue by contradiction and assume that $\epsilon \mathbb{B}_{n+1} \subseteq E(\overline{A}, \overline{b}, \mathcal{B})$ for some $\epsilon > 0$. Then there exist $\widetilde{\lambda} \in \mathcal{B}$ and $\mu \geq 0$ such that $(0_n, -\epsilon) = \widetilde{\lambda}^T \left[\overline{A} \mid -\overline{b}\right] + (0_n, \mu)$ and we have

$$(0_n, 1) = \frac{1}{\epsilon + \mu} \widetilde{\lambda}^T \left[\overline{A} \mid \overline{b}\right] \in \left\{\lambda^T \left[\overline{A} \mid \overline{b}\right] : \lambda \in K^*\right\},$$

so that $F_{0_m}(\overline{A}, \overline{b}) = \emptyset$ (contradiction).

Theorem 3.1 (Lower/Upper bounds for the radius of robust feasibility) *Let $\mathcal{B}$ be a compact base of K^*. Then, the RRF satisfies*

$$C_1 \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) \leq \rho(\overline{A}, \overline{b}) \leq C_2 \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right), \tag{7}$$

where

$$C_1 = C_1(\mathcal{B}) = 1/\max\left\{\|\sum_{i=1}^m \lambda_i u_i\| : \lambda \in \mathcal{B}, \|u_i\| \leq 1\right\},$$

$$C_2 = C_2(\mathcal{B}) = 1/\min\left\{\sum_{i=1}^m |\lambda_i| : \lambda \in \mathcal{B}\right\}.$$

Remark 3.1 Before get into the proof of Theorem 3.1, we first make the following remark:

- (Invariance of the bounds in scaling the compact base) We first observe that, if $\mathcal{B}$ is a compact base for K^*, then $\mu\mathcal{B}$ is also a compact base for K^* for any $\mu > 0$. On the other hand, note that for any $\mu > 0$, $C_i(\mu\mathcal{B}) = \frac{1}{\mu} C_i(\mathcal{B})$, $i = 1, 2$ and $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mu\mathcal{B})\right) = \mu \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$ (see (9)). So, we see that the lower/upper bounds remain the same if we replace $\mathcal{B}$ by $\mu\mathcal{B}$ with any $\mu > 0$.
- (Tightness of the bounds) As we will see in Example 3.1 and Example 3.2, the obtained bounds can be tight.
- (Gaps between the lower and upper bounds) Denote the ratio between the lower bound and upper bound by τ. Then, $\tau = C_1/C_2 \in (0, 1]$. In general, this ratio can depend on the dimension of the cone K^*. For example, if K is the second-order cone in $\mathbb{R}^m$, then, $\tau = \frac{1}{\sqrt{m-1}+1}$ (Corollary 4.2). If K is the positive semi-definite cone S^q_+, then, $\tau \in [\frac{2}{q^{3/2}(q+1)}, 1]$ (Corollary 4.1).

Proof Let $\rho > \rho(\overline{A}, \overline{b})$. Then, $\rho 1_m \notin C(\overline{A}, \overline{b})$. Recall that

$$C(\overline{A}, \overline{b}) = \left\{r \in \mathbb{R}^m_+ : (0_n, 1) \notin \overline{\operatorname{cone} \bigcup_{\lambda \in \mathcal{B}} \left\{\sum_{i=1}^m \lambda_i \left((\overline{a}_i, \overline{b}_i) + r_i \mathbb{B}_{n+1}\right)\right\}}\right\}.$$

This shows that

$$(0_n, 1) \in \overline{\text{cone}\left[\left\{\lambda^T \left[\overline{A} \mid \overline{b}\right] + \rho \sum_{i=1}^{m} \lambda_i u_i : \lambda \in \mathcal{B}, u_i \in \mathbb{B}_{n+1}\right\}\right]},$$

Thus, letting $w_1 := \max\{\| \sum_{i=1}^{m} \lambda_i u_i \| : \lambda \in \mathcal{B}, \|u_i\| \leq 1\}$, we have

$$(0_n, 1) \in \overline{\text{cone}\left[\left\{\lambda^T \left[\overline{A} \mid \overline{b}\right] + \rho w_1 \mathbb{B}_{n+1} : \lambda \in \mathcal{B}\right\}\right]},$$

where $\left\{\lambda^T \left[\overline{A} \mid \overline{b}\right] : \lambda \in \mathcal{B}\right\}$ is a compact convex set. Thus, according to Lemma 3.1,

$$(0_n, 1) \in \text{cone}\left[\left\{\lambda^T \left[\overline{A} \mid \overline{b}\right] : \lambda \in \mathcal{B}\right\} + (\rho w_1 + \epsilon)\mathbb{B}_{n+1}\right], \ \forall \epsilon > 0,$$

or, equivalently,

$$(0_n, -1) \in \text{cone}\left[\left\{\lambda^T \left[\overline{A} \mid -\overline{b}\right] : \lambda \in \mathcal{B}\right\} + (\rho w_1 + \epsilon)\mathbb{B}_{n+1}\right], \ \forall \epsilon > 0.$$

Hence, there exist $\bar{\lambda}^j \in \mathcal{B}$, $j = 1, ..., n$, $\mu_j \geq 0$, and vectors (u_j, s_j), $j = 1, \ldots, n+1$, such that $\| (u_j, s_j) \| \leq 1$ and

$$\sum_{j=1}^{n+1} \mu_j \sum_{i=1}^{m} \bar{\lambda}_i^j (\overline{a}_i, -\overline{b}_i) + (0_n, 1) = -(\rho w_1 + \epsilon) \sum_{j=1}^{n+1} \mu_j \left(u_j, s_j\right). \tag{8}$$

Clearly, we observe that $\sum_{j=1}^{n+1} \mu_j > 0$. Dividing both sides by $\sum_{j=1}^{n+1} \mu_j$, we have

$$\sum_{i=1}^{m} \gamma_i (\overline{a}_i, -\overline{b}_i) + \left(0_n, \frac{1}{\sum_{j=1}^{n+1} \mu_j}\right) = -(\rho w_1 + \epsilon)\left(u, s\right),$$

where $(u, s) = \frac{\sum_{j=1}^{n+1} \mu_j (u_j, s_j)}{\sum_{j=1}^{n+1} \mu_j} \in \mathbb{B}_{n+1}$ and $\gamma_i = \frac{\sum_{j=1}^{n+1} \mu_j \bar{\lambda}_i^j}{\sum_{j=1}^{\mu} \mu_j}$, with $(\gamma_1, \ldots, \gamma_m) \in \text{conv}\, \mathcal{B} = \mathcal{B}$. So, we see that

$$E(\overline{A}, \overline{b}, \mathcal{B}) \cap (\rho w_1 + \epsilon)\mathbb{B}_{n+1} \neq \emptyset$$

for all $\epsilon > 0$, which implies that $\text{dist}(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})) \leq \rho w_1 + \epsilon$. Letting $\epsilon \to 0$, we see that $\rho \geq \frac{1}{w_1} \text{dist}(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B}))$. Hence,

$$\rho(\overline{A}, \overline{b}) \geq \frac{1}{w_1} \ \text{dist}(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})).$$

To see the second inequality, let $\rho > 0$ be such that

$$\text{dist}(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})) < \rho.$$

Then, there exists $(z, s) \in E(\overline{A}, \overline{b}, \mathcal{B})$ such that $\|(z, s)\| \leq \rho$. So, there exist $\widetilde{\lambda} \in \mathcal{B}$, and $\mu \geq 0$ such that $(z, s) = \sum_{i=1}^{m} \widetilde{\lambda}_i (\overline{a}_i, -\overline{b}_i) + (0_n, \mu)$. Let $\epsilon > 0$ be an arbitrary positive number. Then we have

$$(0_n, -(\mu+\epsilon)) \in \sum_{i=1}^{m} \widetilde{\lambda}_i(\overline{a}_i, -\overline{b}_i) + (\rho+\epsilon)\mathbb{B}_{n+1},$$

Dividing both sides by $\mu+\epsilon$, one has

$$(0_n, -1) \in \left(\tfrac{1}{\mu+\epsilon}\right)\left(\sum_{i=1}^{m} \widetilde{\lambda}_i(\overline{a}_i, -\overline{b}_i) + (\rho+\epsilon)\,\mathbb{B}_{n+1}\right),$$

Thus, there exists $(u,r) \in \mathbb{B}_{n+1}$ such that

$$(0_n, 1) \in \left(\tfrac{1}{\mu+\epsilon}\right)\left(\sum_{i=1}^{m} \widetilde{\lambda}_i(\overline{a}_i, -\overline{b}_i) + (\rho+\epsilon)\,(u,r)\right).$$

Let $w_2 := \min\{\sum_{i=1}^{m} |\lambda_i| : \lambda \in \mathcal{B}\} > 0$ and

$$(u_i, r_i) = \left(\tfrac{\operatorname{sign}\widetilde{\lambda}_i}{\sum_{i=1}^{m}|\widetilde{\lambda}_i|}u, \tfrac{\operatorname{sign}\widetilde{\lambda}_i}{\sum_{i=1}^{m}|\widetilde{\lambda}_i|}r\right) \in w_2^{-1}\mathbb{B}_{n+1}.$$

Then, we have $\sum_{i=1}^{m} \widetilde{\lambda}_i(u_i, r_i) = (u,r)$, and so,

$$\begin{aligned}(0_n, 1) &\in \left(\tfrac{1}{\mu+\epsilon}\right)\left(\sum_{i=1}^{m} \widetilde{\lambda}_i(\overline{a}_i, -\overline{b}_i) + \sum_{i=1}^{m} \widetilde{\lambda}_i\,(\rho+\epsilon)\,(u_i, r_i)\right)\\ &\subseteq \operatorname{cone}\left\{\sum_{i=1}^{m} \lambda_i\left((\overline{a}_i, -\overline{b}_i) + \left(\tfrac{\rho+\epsilon}{w_2}\right)\mathbb{B}_{n+1}\right) : \lambda \in \mathcal{B}\right\}.\end{aligned}$$

This shows that $\frac{(\rho+\epsilon)}{w_2}1_m \notin C(\overline{A}, \overline{b})$, and hence, $\rho(\overline{A}, \overline{b}) < \dfrac{(\rho+\epsilon)}{w_2}$. Letting $\epsilon \to 0$, we have $\rho \geq w_2\,\rho(\overline{A}, \overline{b})$. So, $\operatorname{dist}(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})) \geq w_2\,\rho(\overline{A}, \overline{b})$, and hence, the conclusion holds.

Example 3.1 Consider the problem obtained by replacing $\mathbb{R}^2_+$, in Example 2.1, by the second-order cone (SOC), $K_p^2 = \{x \in \mathbb{R}^2 : x_2 \geq |x_1|\}$. We first compute its RRF, say $\rho_p(\overline{A}, \overline{b})$, by means of Proposition 2.1 part (a) .
Let $\alpha > 0$. Taking the compact base $\mathcal{B} = [-1,1] \times \{1\}$, one has

$$\begin{aligned}(\alpha,\alpha) \in C(\overline{A}, \overline{b}) &\Longleftrightarrow (0,1) \notin \overline{\operatorname{cone} \bigcup_{|\lambda_1|\leq 1} \{\lambda_1\,((2,0)+\alpha\mathbb{B}_2) + (-1,-3) + \alpha\mathbb{B}_2\}}\\ &\Longleftrightarrow (0,1) \notin \overline{\operatorname{cone}\left\{(-1,-3) + \alpha\mathbb{B}_2 + \bigcup_{|\lambda_1|\leq 1}\{\lambda_1\,((2,0)+\alpha\mathbb{B}_2)\}\right\}}.\\ &\Longleftrightarrow (0,1) \notin \overline{\operatorname{cone}\{A + (-1,-3) + \alpha\mathbb{B}_2\}},\end{aligned}$$

where $A := \bigcup_{|\lambda_1|\leq 1} \{\lambda_1\,((2,0)+\alpha\mathbb{B}_2)\}$ is the union of $\operatorname{conv}\{((2,0)+\alpha\mathbb{B}_2) \cup \{0,0\}\}$ with its symmetric set w.r.t. $(0,0)$. $C(\overline{A}, \overline{b})$, the points of D closest to the line

$x_2 = 0$ are $(-3, -3 + 2\alpha)$ and $(1, -3 + 2\alpha)$. So, $(\alpha, \alpha) \in C(\overline{A}, \overline{b})$ if and only if $2\alpha \leq 3$, i.e., $\rho_p(\overline{A}, \overline{b}) = \frac{3}{2}$.

Let us compare this exact value of the radius with the result of applying Theorem 3.1. Let $\mathcal{B}$ be the "natural" base $[-1, 1] \times \{1\}$ of K_p^2. Then,

$$\begin{aligned} E_p(\overline{A}, \overline{b}, \mathcal{B}) &:= \left\{\lambda^T \left[\overline{A} \mid -\overline{b}\right] : \lambda \in \mathcal{B}\right\} + \{0\} \times \mathbb{R}_+ \\ &= \operatorname{conv}\left\{\lambda^T \left[\overline{A} \mid -\overline{b}\right] : \lambda = (\pm 1, 1)\right\} + \{0\} \times \mathbb{R}_+ \\ &= ([-3, 1] \times 3) + \{0\} \times \mathbb{R}_+ \\ &= [-3, 1] \times [3, +\infty) , \end{aligned}$$

with $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) = 3$. In this case, direct calculation shows that

$$C_1 = 1/\left\{\max \|\lambda_1 u_1 + u_2\| : \lambda_1 \in [-1, 1], \|u_1\| \leq 1, \|u_2\| \leq 1\right\} = \frac{1}{2}$$

and $C_2 = 1/\min\{|\lambda_1| + 1 : \lambda_1 \in [-1, 1]\} = 1$. Thus, our previous result shows that

$$\rho_p(\overline{A}, \overline{b}) \in \left[\frac{3}{2}, 3\right],$$

i.e., the lower bound $C_1 \operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$ is exact here.

The lower and the upper bounds for the RRF provided by Theorem 3.1 involve two constants depending on the chosen base $\mathcal{B}$ of K, C_1 and C_2, and a non-negative number, $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$, which also depends on the nominal matrix $\left[\overline{A}, \overline{b}\right]$. We now provide a computable formula for the distance from the epigraphical set to the origin.

Theorem 3.2 (A computable formula for $\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right)$**)** *Let $\mathcal{B}$ be a compact base of K^*. Then,*

$$\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) = \inf_{(z,s,t,\lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m} \left\{ t \;\middle|\; \begin{array}{l} \|(z, s)\| \leq t, \\ z = \overline{A}^T \lambda, \ s \geq -\overline{b}^T \lambda, , \\ \lambda \in \mathcal{B}. \end{array} \right\}. \tag{9}$$

In particular, let $\mathcal{B}$ is a spectrahedron with the form

$$\mathcal{B} = \{\lambda \in \mathbb{R}^m : B_0 + \sum_{i=1}^{m} \lambda_i B_i \succeq 0\}$$

for some $(s \times s)$ symmetric matrices B_i, $i = 0, 1, \ldots, m$. Then one has

$$\operatorname{dist}\left(0_{n+1}, E(\overline{A}, \overline{b}, \mathcal{B})\right) = \sqrt{f^*},$$

where f^* is the optimal value of the following semi-definite program:

$$\inf_{(z,s,t,\lambda)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{ t \;\middle|\; \begin{array}{l} \begin{bmatrix} tI_n & 0_n & z \\ 0_n^T & t & s \\ z^T & s & 1 \end{bmatrix} \succeq 0 \\ z = \overline{A}^T\lambda,\; s \geq -\overline{b}^T\lambda, \\ B_0 + \sum_{i=1}^m \lambda_i B_i \succeq 0. \end{array} \right\}. \tag{10}$$

Proof By definition,

$$\begin{aligned} \operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right) &= \inf_{(z,s)\in\mathbb{R}^n\times\mathbb{R}}\{\|(z,s)\| : (z,s)\in E(\overline{A},\overline{b},\mathcal{B})\} \qquad (11)\\ &= \inf_{(z,s,t)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}}\{t : t\geq \|(z,s)\|, (z,s)\in E(\overline{A},\overline{b},\mathcal{B})\}. \end{aligned}$$

From the definition of $E(\overline{A},\overline{b},\mathcal{B})$, one has

$$(z,s)\in E(\overline{A},\overline{b},\mathcal{B}) \Leftrightarrow \exists\, w\geq 0,\, \lambda\in\mathcal{B}:\; (z,s)=\sum_{i=1}^m \lambda_i(\overline{a}_i,-\overline{b}_i)+(0_n,w). \tag{12}$$

So, (9) holds.

To see the second assertion, we note that

$$\begin{aligned} &\operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right)^2 \\ &= \inf_{(z,s)\in\mathbb{R}^n\times\mathbb{R}}\{\|(z,s)\|^2 : (z,s)\in E(\overline{A},\overline{b},\mathcal{B})\} \\ &= \inf_{(z,s,t)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}}\{t : t\geq \|(z,s)\|^2, (z,s)\in E(\overline{A},\overline{b},\mathcal{B})\}, \end{aligned} \tag{13}$$

where $t\geq\|(z,s)\|^2$ can be replaced by

$$\begin{bmatrix} tI_n & 0_n & z \\ 0_n^T & t & s \\ z^T & s & 1 \end{bmatrix} \succeq 0$$

thanks to the Schur complement (see, e.g., [5, Lemma 4.2.1]). So, one gets the desired conclusion $\operatorname{dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right)=\sqrt{f^*}$ from (13) and (12).

As a corollary, we obtain the radius formula of robust feasibility for an uncertain linear program which was established in [10]. Here, we further show that this formula leads to efficient computation of tight upper bounds for the RRF via second order cone programs.

Corollary 3.1 (Computable exact formula for $\rho(\overline{A},\overline{b})$ of uncertain LPs) *Let $K=\mathbb{R}^m_+$. Then, the RRF satisfies*

$$\rho(\overline{A},\overline{b}) = \operatorname{dist}(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})) = f^*_{LP},$$

where f^*_{LP} is the optimal value of the following second order programming problem:

$$\inf_{(z,s,t,\lambda)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m}\left\{t \;\middle|\; \begin{array}{l} \|(z,s)\|\le t, \\ z=\overline{A}^T\lambda,\ s\ge -\overline{b}^T\lambda, \\ \lambda\in\mathbb{R}^m_+,\ \sum_{i=1}^m\lambda_i=1 \end{array}\right\}. \tag{14}$$

Proof Let $K=\mathbb{R}^m_+$. Then, $K^*=K=\mathbb{R}^m_+$. Then, the simplex $\Delta=\{\lambda:\lambda\in\mathbb{R}^m_+,\sum_{i=1}^m\lambda_i=1\}$ is a natural compact base for $\mathbb{R}^m_+$. Thus, the conclusion follows by letting $\mathcal{B}=\Delta$ in (9) in Theorem 3.2.

We now provide a simple example illustrating Corollary 3.1.

Example 3.2 Consider the same problem examined in Example 2.1. Then, the second order problem in (14) becomes here

$$\inf_{(z,s,t,\lambda)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2}\left\{t \;\middle|\; \begin{array}{l} \|(z,s)\|\le t, \\ z=2\lambda_1-\lambda_2,\ s\ge 3\lambda_2, \\ \lambda_1+\lambda_2=1,\lambda_1\ge 0,\lambda_2\ge 0, \end{array}\right\},$$

whose optimal set is $\{1\}\times\{1\}\times\left\{\sqrt{2}\right\}\times\left\{\left(\frac{2}{3},\frac{1}{3}\right)\right\}$. Thus, we get again $\rho(\overline{A},\overline{b})=\sqrt{2}$ which coincides with the computation in Example 2.1.

4 Bounds for Radius of Robust Feasibility of SDPs & SOCPs

We now consider uncertain linear semi-definite programming problems and provide computable bounds for the RRF. To do this, recall that S^q_+ is the cone which consists of all $(q\times q)$ positive semi-definite matrices. Denote the set of all $(q\times q)$ symmetric matrices by S^q and let $\mathrm{Tr}(M)$ be the trace of a matrix $M\in S^q$. As S^q and $\mathbb{R}^{q(q+1)/2}$ have the same dimensions, there exists an invertible linear map $L:S^q\to\mathbb{R}^{q(q+1)/2}$ such that

$$L(M_1)^TL(M_2)=\mathrm{Tr}(M_1M_2) \text{ for all } M_1,M_2\in S^q.$$

We now identify the space of $(q\times q)$ symmetric matrices S^q, equipped with the trace inner product, as $\mathbb{R}^{q(q+1)/2}$ with the usual Euclidean inner product by associating each symmetric matrix M to $L(M)$.

Corollary 4.1 (Numerically tractable bounds for $\rho(\overline{A},\overline{b})$ of uncertain SDPs) *Identify S^q with $\mathbb{R}^{q(q+1)/2}$ via the mapping L given as above and let K be the positive semi-definite cone S^q_+. Let $\mathcal{B}=\{\Lambda\in S^q:\Lambda\in S^q_+,\mathrm{Tr}(\Lambda)=1\}$ be the natural compact base for K. Then, the RRF satisfies*

$$\frac{2}{q(q+1)}\sqrt{f^*_{SDP}}\le\rho(\overline{A},\overline{b})\le\sqrt{q}\sqrt{f^*_{SDP}}\ ,$$

where f^*_{SDP} is the optimal value of the following semi-definite program:

$$\inf_{\substack{(z,s,t,\lambda)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m\\ \Lambda\in S^q}}\left\{ t \;\middle|\; \begin{array}{l} \begin{bmatrix} tI_n & 0_n & z\\ 0_n^T & t & s\\ z^T & s & 1\end{bmatrix}\succeq 0,\\ \\ z=\overline{A}^T\lambda,\ s\ge -\overline{b}^T\lambda,\\ \lambda=L(\Lambda)\\ \Lambda\in S^q_+,\ \mathrm{Tr}(\Lambda)=1.\end{array}\right\}.$$

Proof Let K be the positive semi-definite cone S^q_+. Then, $K^*=K=S^q_+$. Let $\mathcal{B}=\{\Lambda\in S^q:\Lambda\in S^q_+,\mathrm{Tr}(\Lambda)=1\}$ be the natural compact base for S_q^+. Then, Theorem 3.1 and Theorem 3.2 imply that

$$C_1\sqrt{f^*_{SDP}}\le\rho(\overline{A},\overline{b})\le C_2\sqrt{f^*_{SDP}}\,,$$

where $m=\frac{q(q+1)}{2}$,

$$C_1=1/\max\left\{\|\sum_{i=1}^m\lambda_i u_i\|:\lambda=L(\Lambda),\ \Lambda\in\mathcal{B},\|u_i\|\le 1\right\}$$

and

$$C_2=1/\min\left\{\sum_{i=1}^m|\lambda_i|:\lambda=L(\Lambda),\ \Lambda\in\mathcal{B}\right\}.$$

To see the conclusion, it suffices to show that $C_1\ge\frac{2}{q(q+1)}$ and $C_2\le\sqrt{q}$. To see this, from the definition of L, we have $\|\lambda\|^2=\|L(\Lambda)\|^2=\mathrm{Tr}(\Lambda^2)$. Let $\Lambda=U\Sigma U^T$ be the singular value decomposition of $\Lambda\in S^q$ where U is an orthonormal matrix and Σ is a diagonal matrix whose diagonal elements are the eigenvalues of Λ. Then, $\Lambda^2=U\Sigma^2U^T$. It follows from the arithmetic-quadratic inequality that for all $\Lambda\in\mathcal{B}$,

$$\frac{1}{q}=\frac{1}{q}\mathrm{Tr}(\Lambda)^2\le\mathrm{Tr}(\Lambda^2)\le[\mathrm{Tr}(\Lambda)]^2=1.$$

This implies that $\frac{1}{\sqrt{q}}\le\|\lambda\|\le 1$ for all $\Lambda\in\mathcal{B}$. Thus, one has

$$C_1\ge 1/\max\left\{\sum_{i=1}^m|\lambda_i|:\lambda=L(\Lambda),\ \Lambda\in\mathcal{B}\right\}\ge\frac{1}{m}=\frac{2}{q(q+1)}.$$

Moreover, as $\sum_{i=1}^m|\lambda_i|\ge\|\lambda\|\ge\frac{1}{\sqrt{q}}$, one has

$$C_2=1/\min\left\{\sum_{i=1}^m|\lambda_i|:\lambda=L(\Lambda),\ \Lambda\in\mathcal{B}\right\}\le\sqrt{q}.$$

So, the conclusion follows.

Example 4.1 Consider the following semi-definite program (SDP)

$$(SDP) \quad \min\left\{-x_1 : \begin{bmatrix} -1 & x_2 \\ x_2 & -1+x_1 \end{bmatrix} \in -S^2_+\right\}.$$

We now investigate the RRF for this SDP.

Let $L : S^2 \to \mathbb{R}^3$ be defined as $L\left(\begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}\right) = (z_1, \sqrt{2}z_2, z_3)^T$. Clearly, L is a one-to-one mapping with $L(M_1)^T L(M_2) = \mathrm{Tr}(M_1 M_2)$. So, the feasible set of (SDP) can be written as $\{x \in \mathbb{R}^2 : \overline{A}x + \overline{b} \in -L\left(S^2_+\right)\}$, with $\overline{A} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \\ 1 & 0 \end{bmatrix}$, $\overline{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $L\left(S^2_+\right) = \left\{y \in \mathbb{R}^3 : y_2^2 \le 2y_1y_3, y_1 \ge 0, y_3 \ge 0\right\}$. Direct computation shows that $f^*_{\mathrm{SDP}} = 1$ in this case with an optimal solution $(z, s, t, \lambda) = (0_2, 1, 1, (1, 0, 0)^T)$. Then, the preceding corollary implies that the RRF satisfies $\frac{1}{3} \le \rho(\overline{A}, \overline{b}) \le \sqrt{2}$.

On the other hand, it can be directly verified that the true RRF for this example satisfies $\rho(\overline{A}, \overline{b}) \in [\frac{1}{2}, 1]$. Indeed, let $\Delta b(\epsilon) = (1+\epsilon, 0, 0)^T$ for any $\epsilon > 0$. note that $\{x : \overline{A}x + \overline{b} + \Delta b(\epsilon) \in -S^2_+\} = \emptyset$. This shows that $\rho(\overline{A}, \overline{b}) \le 1 + \epsilon$ for all $\epsilon > 0$, and hence $\rho(\overline{A}, \overline{b}) \le 1$. Moreover, let $\overline{A} = [\overline{a}_1 \mid \overline{a}_2 \mid \overline{a}_3]^T$ and $A = [a_1 \mid a_2 \mid a_3]^T$. Then, for all $(a_i, b_i) \in (\overline{a}_i, \overline{b}_i) + \frac{1}{2}\mathbb{B}_3$, one sees that $b_1 \in [-\frac{3}{2}, -\frac{1}{2}]$, $b_2 \in [-\frac{1}{2}, \frac{1}{2}]$ and $b_2 \in [-\frac{3}{2}, -\frac{1}{2}]$. So, $0_2 \in \{x : Ax + b \in -L\left(S^2_+\right)\}$. This shows that $\rho(\overline{A}, \overline{b}) \ge \frac{1}{2}$.

Next, we see that, in the case for an uncertain second-order cone program (that is, $K = K^m_p := \{x \in \mathbb{R}^m : x_m \ge \|(x_1, ..., x_{m-1})\|\}$), the lower and upper bound of the RRF can be computed by solving a second-order cone programming problem.

Corollary 4.2 (Numerically tractable bounds for $\rho(\overline{A}, \overline{b})$ of uncertain SOCPs) *Let K be the second-order cone K^m_p in $\mathbb{R}^m$, and let $\mathcal{B} = \{\lambda \in \mathbb{R}^m : \|(\lambda_1, \ldots, \lambda_{m-1})\| \le 1$ and $\lambda_m = 1\}$ be the natural compact base for K^m_p. Then, the RRF satisfies*

$$\frac{1}{\sqrt{m-1}+1} f^*_{SOC} \le \rho(\overline{A}, \overline{b}) \le f^*_{SOC},$$

where f^*_{SOC} is the optimal value of the following second order cone program:

$$\inf_{(z,s,t,\lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m} \left\{ t \;\middle|\; \begin{array}{l} \|(z,s)\| \le t, \\ z = \overline{A}^T\lambda,\ s \ge -\overline{b}^T\lambda, \\ \|(\lambda_1, \ldots, \lambda_{m-1})\| \le 1,\ \lambda_m = 1. \end{array} \right\}.$$

Proof Let K be the second-order cone K^m_p in $\mathbb{R}^m$. Then, $K^* = K = K^m_p$. Let $\mathcal{B} = \{\lambda \in \mathbb{R}^m : \|(\lambda_1, \ldots, \lambda_{m-1})\| \le 1$ and $\lambda_m = 1\}$ be the natural compact base for K_p^m. We claim that

$$C_1 = 1/\max\left\{\|\sum_{i=1}^{m}\lambda_i u_i\| : \lambda \in \mathcal{B}, \|u_i\| \leq 1\right\} = \frac{1}{\sqrt{m-1}+1}. \tag{15}$$

Indeed, by the triangle inequality, one has

$$\begin{aligned}
&\max\left\{\|\sum_{i=1}^{m}\lambda_i u_i\| : \lambda \in \mathcal{B}, \|u_i\| \leq 1\right\}\\
&= \max\left\{\|\sum_{i=1}^{m-1}\lambda_i u_i + u_m\| : \|(\lambda_1,\dots,\lambda_{m-1})\| \leq 1, \|u_i\| \leq 1\right\}\\
&\leq \max\left\{\sum_{i=1}^{m-1}|\lambda_i| + 1 : \|(\lambda_1,\dots,\lambda_{m-1})\| \leq 1\right\}\\
&= \sqrt{m-1}+1.
\end{aligned}$$

Moreover, for $u_i = u$ with $\|u\| = 1$ and $\lambda = (\underbrace{\frac{1}{\sqrt{m-1}},\dots,\frac{1}{\sqrt{m-1}}}_{m-1}, 1) \in \mathcal{B}$,

$$\|\sum_{i=1}^{m}\lambda_i u_i\| = (\sqrt{m-1}+1)\|u\| = \sqrt{m-1}+1.$$

Thus, (15) holds. Direct verification also shows that

$$C_2 = 1/\min\left\{\sum_{i=1}^{m}|\lambda_i| : \lambda \in \mathcal{B}\right\} = 1.$$

Therefore, the conclusion follows from Theorem 3.1 and Theorem 3.2 (equation (9)).

Robust Separability in Uncertain SVMs

The support vector machine for binary classification is a useful technique in generating an optimal classifier (hyperplane) which separates the training data into two classes. It has found numerous applications in engineering, medical imaging and computer science. Let $(u_i, \alpha_i) \in \mathbb{R}^s \times \{-1, 1\}$ be the given training data where α_i is the class label for each data u_i. An optimization model problem of support vector machine for binary classification can be stated as follows [3, Section 12.1.1]:

$$\begin{aligned}
\min_{(w,\gamma)\in\mathbb{R}^s\times\mathbb{R}} \quad & \|w\|\\
\text{s.t.} \quad & \alpha_i(u_i^T w + \gamma) \geq 1, i = 1,\dots,m.
\end{aligned}$$

In practice, the given data u_i, $i = 1, \ldots, m$, are often uncertain. We assume that these data are subject to the following norm data uncertainty:

$$u_i \in \mathcal{V}_i(r) = \overline{u}_i + r_i \mathbb{B}_s.$$

Let $r = (r_1, \ldots, r_m) \in \mathbb{R}^m_+$. Then, the robust support vector machine can be stated as

$$\begin{aligned}(RSVM_r) \quad &\min_{(w,\gamma)\in\mathbb{R}^s\times\mathbb{R}} \quad \|w\| \\ &\text{s.t.} \qquad \alpha_i(u_i^T w + \gamma) \geq 1, \ \forall\, u_i \in \mathcal{V}_i(r), \ i = 1, \ldots, m.\end{aligned}$$

If the feasible set F_r of $(RSVM_r)$ is nonempty, then a linear binary classification is possible even if the training data is subject to measurement or prediction error and the error level is controlled by r. Thus, the range of values of r, guaranteeing the non-emptiness of the feasible set F_r of $(RSVM_r)$, quantifies the robustness of the linear separability of the training data in uncertain support vector machine problems. So, we now investigate the question: for what values of $r > 0$ so that the feasible set F_r of $(RSVM_r)$ is nonempty.

The robust SVM problem can be further rewritten into a robust conic programming problem as follows

$$\begin{aligned}(RSVM_r) \quad &\min_{(w,\gamma,t)\in\mathbb{R}^s\times\mathbb{R}\times\mathbb{R}} \quad t \\ &\text{s.t.} \qquad \|w\| \leq t \\ &\qquad\qquad (-\alpha_i u_i)^T w + (-\alpha_i)\gamma + 1 \leq 0, \ \forall\, u_i \in \mathcal{V}_i(r), \ i = 1, \ldots, m.\end{aligned}$$

Let $K = K_p^{s+1} \times \mathbb{R}^m_+$, where K_p^{s+1} is the second-order cone in $\mathbb{R}^{s+1}$. Let $\overline{a}_i = (-e_i^T, 0, 0)^T$ and $\overline{b}_i = 0$ for $i = 1, \ldots, s$; $\overline{a}_{s+1} = (0_s^T, 0, -1)^T$ and $\overline{b}_{s+1} = 0$; $\overline{a}_i = (-\alpha_{i-s-1} u_{i-s-1}^T, -\alpha_{i-s-1}, 0)^T$ and $\overline{b}_i = 1$ for $i = s+2, \ldots, m+s+1$. Define

$$\mathcal{U}_i(\mu_i) = (\overline{a}_i, \overline{b}_i) + \mu_i \mathbb{B}_{s+3}, \ i = 1, \ldots, m+s+1,$$

where $\mu_i \geq 0$. We now consider a closely related robust SOCP problem

$$(SOCP_\mu) \ \min_{(w,\gamma,t)\in\mathbb{R}^s\times\mathbb{R}\times\mathbb{R}} t$$

$$\text{s.t.} \qquad \begin{bmatrix} a_1^T x + b_1 \\ \vdots \\ a_{m+s+1}^T x + b_{m+s+1} \end{bmatrix} \in -K, \forall\ (a_i, b_i) \in \mathcal{U}_i(\mu_i).$$

Denote the feasible set of $(SOCP_\mu)$ as F'_μ with $\mu = (\mu_1, \ldots, \mu_{m+s+1})$. Then, for any $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ and $\overline{r} = (\underbrace{0, \ldots, 0}_{s+1}, r_1, \ldots, r_m) \in \mathbb{R}^{m+s+1}$

$$F'_{\overline{r}} \neq \emptyset \Longrightarrow F_r \neq \emptyset. \tag{16}$$

Define $\overline{A} = [\overline{a}_1 \mid \ldots \mid \overline{a}_{m+s+1}]^T$, $\overline{b} = \left(\overline{b}_1, \ldots, \overline{b}_{m+s+1}\right)^T$ and let $\rho(\overline{A}, \overline{b})$ be the robust feasibility of $(SOCP_\mu)$. Then for all $r \in \left[0, \rho(\overline{A}, \overline{b})\right)^m$, the feasible set

F_r of the robust support vector machine problem $(RSVM_r)$ will be nonempty. In particular, we have the following result:

Corollary 4.3 *Let $r=(r_1,\ldots,r_m)$ and denote the feasible set of the robust support vector machine problem $(RSVM_r)$ by F_r. Then, $F_r \neq \emptyset$ for all $r \in \mathbb{R}^m_+$ with $r_i \leq \frac{1}{\sqrt{s+1}} f^*$, where f^* is the optimal value of the following second-order cone program*

$$\inf_{(z,s,t,\lambda)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{m+s+1}} \left\{ t \;\middle|\; \begin{array}{l} \|(z,s)\| \leq t, \\ z = \overline{A}^T\lambda, \;\; s \geq -\overline{b}^T\lambda, \\ \|(\lambda_1,\ldots,\lambda_s)\| \leq \lambda_{s+1}, \\ \sum_{i=1}^{m+1}\lambda_{s+i} = 1, \lambda_{s+1} \geq 0,\ldots,\lambda_{m+s+1} \geq 0. \end{array} \right\}.$$

Proof Note that $K^* = K = K_p^{s+1} \times \mathbb{R}^m_+$. So, a compact base $\mathcal{B}$ for K^* is

$$\{\lambda : \|(\lambda_1,\ldots,\lambda_s)\| \leq \lambda_{s+1}, \sum_{i=1}^{m+1}\lambda_{s+i} = 1, \lambda_{s+1} \geq 0,\ldots,\lambda_{m+s+1} \geq 0\}.$$

Let $\rho(\overline{A},\overline{b})$ be the RRF of $(SOCP_\mu)$. From Theorem 3.1, one sees that

$$\rho(\overline{A},\overline{b}) \geq C_1 \text{ dist}\left(0_{n+1}, E(\overline{A},\overline{b},\mathcal{B})\right) = C_1 f^*,$$

where the equality follows from Theorem 3.2. So, for all r_i with $0 \leq r_i \leq C_1 f^*$, $i=1,\ldots,m$, $F'_{\overline{r}} \neq \emptyset$ where $\overline{r} = (0,\ldots,0,r_1,\ldots,r_m) \in \mathbb{R}^{m+s+1}$. This together with (16) implies that F_r is nonempty if $0 \leq r_i \leq C_1 f^*$, $i=1,\ldots,m$. Now the conclusion follows by noting that

$$C_1 = 1/\max\left\{ \|\sum_{i=1}^{m+s+1}\lambda_i u_i\| : \lambda \in \mathcal{B}, \|u_i\| \leq 1 \right\} = \frac{1}{\sqrt{s}+1},$$

where the last equality holds by using the same reasoning as in the proof of Corollary 4.2.

5 Conclusions

In this paper, we introduced the notion of radius of robust feasibility for an uncertain linear conic program, which provides a numerical value for the largest size of a ball uncertainty set that guarantees non-emptiness of the robust feasible set. We then provided formulas for estimating the radius of robust feasibility of uncertain conic programs, using the tools of convex analysis and parametric optimization. We also established computationally tractable bounds for two important uncertain conic programs: semi-definite programs and second-order cone programs. In the special case of uncertain linear programs, the formula allows us to calculate the radius by finding the optimal value of an associated second-order cone program.

Our results suggest some interesting further work. For example, the radius of robust feasibility formula was achieved for commonly used ball uncertainty set. It would be of interest to examine how our approach can be extended to cover other commonly used uncertainty sets such as the polytope uncertainty sets or intersection of polytope and norm uncertainty sets. Another interesting topic of study would be to extend our approach to calculate the radius of robust feasibility for uncertain discrete nonlinear optimization problems (see a promising computational approach initialized in [9] for the linear cases).

Appendix: Proofs of Basic Properties of Admissible Sets
In this appendix, we provide the proof of the basic properties of the admissible set of parameters (Proposition 2.1).

Proof of Proposition 2.1:

Proof [Proof of (a)] Direct verification gives us that

$$F_r(\overline{A},\overline{b}) = \{x \in \mathbb{R}^n : \sum_{i=1}^m \lambda_i(\overline{a}_i + \Delta a_i)^T x + \sum_{i=1}^m \lambda_i(\overline{b}_i + \Delta b_i) \leq 0,$$
$$\forall \lambda \in K^*, \|(\Delta a_i, \Delta b_i)\| \leq r_i,\ i = 1,\dots,m\}.$$

It now follows from the well-known existence theorem for linear systems [17, Corollary 3.1.1] that $F_r(\overline{A},\overline{b}) \neq \emptyset$ if and only if

$$(0_n,-1) \notin \overline{\text{cone}\left\{\sum_{i=1}^m \lambda_i\left(\overline{a}_i + \Delta a_i, -\overline{b}_i - \Delta b_i\right) : \lambda \in \mathcal{B}, \|(\Delta a_i, \Delta b_i)\| \leq r_i, \forall i\right\}},$$

which, in turn, is equivalent to the statement

$$(0_n,1) \notin \overline{\text{cone}\left\{\sum_{i=1}^m \lambda_i\left(\overline{a}_i + \Delta a_i, \overline{b}_i + \Delta b_i\right) : \lambda \in \mathcal{B}, \|(\Delta a_i, \Delta b_i)\| \leq r_i, \forall i\right\}}$$
$$= \overline{\text{cone}\left\{\sum_{i=1}^m \lambda_i\left((\overline{a}_i,\overline{b}_i) + r_i\mathbb{B}_{n+1}\right) : \lambda \in \mathcal{B}\right\}}.$$

Thus, the conclusion follows.

[Proof of (b)] Let $\mathcal{B}$ be the compact base of K^*. Then, $0_m \notin \mathcal{B}$, and so,

$$\mu := \min\{\min_{1\leq i\leq m} \left|\lambda_i\right| : \lambda \in \mathcal{B}\} > 0.$$

Define $M = \max\{\|\sum_{i=1}^m \lambda_i(\overline{a}_i,\overline{b}_i)\| : \lambda \in \mathcal{B}\} < +\infty$. We shall prove by contradiction that $C(\overline{A},\overline{b}) \subseteq \frac{M}{\mu}\mathbb{B}_m$. If, in the contrary, $C(\overline{A},\overline{b}) \nsubseteq \frac{M}{\mu}\mathbb{B}_m$, then we can take $r \in C(\overline{A},\overline{b})$ such that $\|r\| \geq \frac{M+\epsilon}{\mu}$ for some $\epsilon > 0$. Now, fix any $\widetilde{\lambda} \in \mathcal{B}$. Note that

$$\sum_{i=1}^m \left|\widetilde{\lambda}_i\right| r_i \geq \mu \sum_{i=1}^m r_i \geq \mu\sqrt{\sum_{i=1}^m r_i^2} \geq M + \epsilon \geq \|\sum_{i=1}^m \widetilde{\lambda}_i(\overline{a}_i,\overline{b}_i)\| + \epsilon.$$

It follows that

$$\epsilon\mathbb{B}_{n+1} \subseteq \sum_{i=1}^m \widetilde{\lambda}_i(\overline{a}_i,\overline{b}_i) + \sum_{i=1}^m \left|\widetilde{\lambda}_i\right| r_i\mathbb{B}_{n+1} = \sum_{i=1}^m \widetilde{\lambda}_i\left(\left(\overline{a}_i,\overline{b}_i\right) + r_i\mathbb{B}_{n+1}\right)$$
$$\subseteq \bigcup_{\lambda\in\mathcal{B}} \left\{\sum_{i=1}^m \lambda_i\left(\left(\overline{a}_i,\overline{b}_i\right) + r_i\mathbb{B}_{n+1}\right)\right\},$$

and so

$$(0_n,1) \in \text{cone}\left\{\sum_{i=1}^m \lambda_i\left((\overline{a}_i,\overline{b}_i) + r_i\mathbb{B}_{n+1}\right) : \lambda \in \mathcal{B}\right\}.$$

Then, by Proposition 2.1 part (a) , $r \notin C(\overline{A},\overline{b})$ (contradiction). Hence, $C(\overline{A},\overline{b})$ is bounded. Thus, the conclusion follows by the fact that $C(\overline{A},\overline{b})$ is radiant.

[Proof of (c)] Let us show the "[$\Longrightarrow$]" direction first. Assume that $\sigma_r^{\mathcal{B}}$ satisfies the Slater condition. As $r \in \mathbb{R}^m_{++}$, there exists $\xi > 0$ such that $r + \xi\mathbb{B}_m \subseteq \mathbb{R}^m_{++}$. Since $\mathcal{B}$ is a compact base (and so, $0_m \notin \mathcal{B}$), $\eta := \max_{\lambda\in\mathcal{B}} \max_{1\leq i\leq m} |\lambda_i|$ is a positive real number. Consider the mapping

$$\begin{array}{cccc} \Phi : \mathbb{R}^m \times & \overbrace{\mathbb{R}^{n+1} \times ... \times \mathbb{R}^{n+1}}^{(m)} & \longrightarrow & \mathbb{R}^{n+1} \\ \lambda & \left(z^1, ..., z^m\right) & & \sum_{i=1}^{m} \lambda_i z^i \end{array}$$

Since Φ is continuous (as it has quadratic components) and the set $D_r := \mathcal{B}\times \prod_{i=1}^{m} \left[\left(\overline{a}_i, \overline{b}_i\right) + r_i\mathbb{B}_{n+1}\right]$ is compact, $C_r := \Phi\left(D_r\right)$ is a compact subset of $\mathbb{R}^{n+1}$ too. By the Slater condition, there exists $x^0 \in \mathbb{R}^n$ such that $\left\langle \left(y, y_{n+1}\right), \left(x^0, 1\right)\right\rangle < 0$ for all $\left(y, y_{n+1}\right) \in C_r$. Let $\epsilon > 0$ be such that

$$\begin{aligned} -\epsilon &= \max\left\{\left\langle \left(y, y_{n+1}\right), \left(x^0, 1\right)\right\rangle : \left(y, y_{n+1}\right) \in C_r\right\} \\ &= \max\left\{\left\langle \Phi\left(d\right), \left(x^0, 1\right)\right\rangle : d \in D_r\right\}. \end{aligned}$$

Given $t \in \mathbb{R}^m_{++}$, we consider an element of D_t of the form

$$d^t = \left(\lambda^t, \left(\overline{a}_1, \overline{b}_1\right) + t_1 u^{1,t}, ..., \left(\overline{a}_m, \overline{b}_m\right) + t_m u^{m,t}\right),$$

with $\lambda^t \in \mathcal{B}$ and $u^{i,t} \in \mathbb{B}_{n+1}$, $i = 1, ..., m$, arbitrarily chosen. We define the vector $d^r := \left(\lambda^t, \left(\overline{a}_1, \overline{b}_1\right) + r_1 u^{1,t}, ..., \left(\overline{a}_m, \overline{b}_m\right) + r_m u^{m,t}\right) \in D_r$. Since

$$\left\|\Phi\left(d^t\right) - \Phi\left(d^r\right)\right\| = \left\|\sum_{i=1}^{m} \lambda_i \left(t_i - r_i\right) u^{i,t}\right\| \leq m\eta \left\|t - r\right\|,$$

$$\left|\left\langle \Phi\left(d^t\right) - \Phi\left(d^r\right), \left(x^0, 1\right)\right\rangle\right| \leq m\eta \left\|\left(x^0, 1\right)\right\| \left\|t - r\right\|,$$

and so $\left\langle \Phi\left(d^t\right), \left(x^0, 1\right)\right\rangle \leq m\eta \left\|\left(x^0, 1\right)\right\| \left\|t - r\right\| - \epsilon$. Hence, if

$$\left\|t - r\right\| \leq \min\left\{\frac{\epsilon}{2m\eta \left\|\left(x^0, 1\right)\right\|}, \xi\right\},$$

then, one has

$$\max\left\{\left\langle \left(y, y_{n+1}\right), \left(x^0, 1\right)\right\rangle : \left(y, y_{n+1}\right) \in C_t\right\} \leq -\frac{\epsilon}{2} < 0,$$

which shows that x^0 is a Slater point for $\sigma_t^{\mathcal{B}}$. This implies that $t \in C(\overline{A}, \overline{b})$. Thus, one has $r \in \operatorname{int} C(\overline{A}, \overline{b})$.

We now show the reverse direction "[$\Longleftarrow$]". Let $r \in \operatorname{int} C(\overline{A}, \overline{b})$ and for all $\lambda \in \mathcal{B}$, $\sum_{i=1}^{m} r_i\lambda_i > 0$. Then there exists $\epsilon > 0$ such that $r + \epsilon 1_m \in C(\overline{A}, \overline{b})$ and

$$\gamma := \min\left\{\sum_{i=1}^{m} r_i\lambda_i : \lambda \in \mathcal{B}\right\} > 0.$$

We consider perturbations of the coefficients of $\sigma_r^{\mathcal{B}}$ preserving the index set $\mathcal{B}\times \prod_{i=1}^{m} \left(r_i\mathbb{B}_{n+1}\right)$ and we measure such perturbations by means of the Chebyshev metric d_∞. The coefficients vector of $\sigma_r^{\mathcal{B}}$ can be expressed as

$$\sum_{i=1}^{m} \lambda_i \left[\left(\overline{a}_i, \overline{b}_i\right) + r_i u^i\right] = \sum_{i=1}^{m} \lambda_i \left(\overline{a}_i, \overline{b}_i\right) + \sum_{i=1}^{m} \lambda_i r_i u^i, \tag{17}$$

where $\lambda \in \mathcal{B}$ and $u^i \in \mathbb{B}_{n+1}$, $i = 1, ..., m$. Consider an additive perturbation δw, with $0 < \delta < \gamma\epsilon$ and $w \in \mathbb{B}_{n+1}$ of the coefficient vector in (17). Let $v := \frac{w}{\sum_{1\leq i\leq m} r_i\lambda_i}$. Since $\left\|u^i + \delta v\right\| \leq 1 + \delta \left\|v\right\| \leq 1 + \frac{\delta}{\gamma}$ for all i, one has

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i \left[\left(\overline{a}_i, \overline{b}_i\right) + r_i u^i\right] + \delta w &= \sum_{i=1}^{m} \lambda_i \left(\overline{a}_i, \overline{b}_i\right) + \sum_{i=1}^{m} \lambda_i r_i \left(u^i + \delta v\right)\\ &\in \sum_{i=1}^{m} \lambda_i \left(\overline{a}_i, \overline{b}_i\right) + \sum_{i=1}^{m} \lambda_i r_i \left(1 + \frac{\delta}{\gamma}\right) \mathbb{B}_{n+1}\\ &\subseteq \sum_{i=1}^{m} \lambda_i \left(\overline{a}_i, \overline{b}_i\right) + \sum_{i=1}^{m} \lambda_i r_i \left(1 + \epsilon\right) \mathbb{B}_{n+1}.\end{aligned}$$

So, the solution set of any perturbed system obtained from $\sigma_r^{\mathcal{B}}$ by summing up vectors of Euclidean norm less than $\gamma\epsilon$ to each the coefficient vector contains $F_r(\overline{A}, \overline{b}) \neq \emptyset$. In particular, summing up vectors of Chebyshev norm less than $\frac{\gamma\epsilon}{\sqrt{m}}$ to each coefficient vector of $\sigma_r^{\mathcal{B}}$ we get a feasible perturbed system. Hence, by [17, Theorem 6.1], $\sigma_r^{\mathcal{B}}$ has a strong Slater solution, which shows that $\sigma_r^{\mathcal{B}}$ satisfies the Slater condition.

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References

1. Anjos, M.: Conic linear optimization. In: T. Terlaky, M.F. Anjos, S. Ahmed (Eds.), *Advances and trends in optimization with engineering applications* (pp. 107-120). MOS-SIAM Ser. Optim. 24. Philadelphia: SIAM, (2017).
2. Boyd, S., Vandenberghe, L. *Convex optimization.* London: Cambridge U. P, (2004).
3. Ben-Tal, A., El Ghaoui, L., Nemirovski, A.: *Robust optimization.* Princeton: Princeton U.P, (2009).
4. Ben-Tal, A., Nemirovski, A.: Robust solutions of linear programming problems contaminated with uncertain data. *Math. Program.* 88A, 411-424, (2000).
5. Ben-Tal, A., Nemirovski, A.: *Lectures on modern convex optimization: analysis, algorithms, and engineering applications.* Philadelphia: SIAM, (2001).
6. Ben-Tal, A., Nemirovski, A.: Selected topics in robust convex optimization. *Math. Program.* 112A, 125-158 (2008).
7. Chuong, T.D., Jeyakumar, V.: An exact formula for radius of robust feasibility of uncertain linear programs. *J. Optim. Theory Appl.* 173, 203-226, (2017).
8. Goberna, M.A., Jeyakumar, V., Li, G., Vicente-Pérez.: J.: Robust solutions to multi-objective linear programs with uncertain data. *European J. Oper. Res.* 242, 730-743, (2015).
9. Liers, F., Schewe, L., Thürauf, J.: Radius of Robust Feasibility for Mixed-Integer Problems, (2020), to appear in INFORMS J. Comput. available at http://www.optimization-online.org/DB_FILE/2019/05/7219.pdf
10. Goberna, M.A., Jeyakumar, V., Li, G., Vicente-Pérez, J.: Robust solutions of uncertain multi-objective linear semi-infinite programming. *SIAM J. Optim.* 24, 1402-1419, (2014).
11. Chen, J., Li, J., Li, X., Lv, Y., Yao, J.-C.: Radius of robust feasibility of system of convex inequalities with uncertain data. *J. Optim. Theory Appl.* 184, 384-399, (2020).
12. Goberna, M.A., Jeyakumar, V., Li, G. Linh, N.: Radius of robust feasibility and optimality of uncertain convex programs. *Oper. Res. Letters* 44, 67-73, (2016).
13. Li, X.-B., Wang, Q.-L.: A note on the radius of robust feasibility for uncertain convex programs. *Filomat* 32, 6809-6818, (2018).

14. Carrizosa, E., Nickel, S.: Robust facility location. *Math. Meth. Oper. Res.* 58, 331-349, (2003).
15. Zhang, Q., Grossmann, I.E., Lima R.M.: On the relation between flexibility analysis and robust optimization for linear systems. *AIChE Journal* 62, 3109-3123, (2016).
16. Koch, T., Hiller, B., Pfetsch, M., Schewe, L., eds.: *Evaluating gas network capacities.* Philadelphia: SIAM, (2015).
17. Goberna, M.A., López.: *Linear semi-infinite optimization.* Chichester: Wiley, M.A. (1998).
18. Göpfert, A., Riahi, H., Tammer, Ch., Zălinescu, C.: *Variational methods in partially ordered spaces.* New York: Springer, (2003).
19. Beer, G.: *Topology on closed and closed convex sets.* Dordrecht-Boston: Kluwer, (1993).
20. Cánovas, M.J., López, M.A., Parra, J., Toledo, F.J.: Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems. *Math. Program.* 103A, 95-126, (2005).