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Generalization Process by Second Grade Students

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Abstract: This study is part of a broader study on algebraic reasoning in elementary education. The research objective of the present survey, namely to describe generalization among second grade (7- to 8-year-old) students, was pursued through semi-structured interviews with six children in connection with a contextualized generalization task involving the function $y = x + 3$. Particular attention was paid to the structures recognized and the type of generalization expressed by these students as they reasoned. In all six, we observed three phases of inductive reasoning: (a) abductive, (b) inductive and (c) generalization. The students correctly recognized the structure at least once during the interview and expressed generalization in three ways.

Keywords: algebraic thinking; functional thinking; generalization; inductive reasoning

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1. Introduction

Inductive reasoning is a necessary process towards generalization because it both favors knowledge building by observing specific cases and enables the subject to verify a conjecture by working with such cases [1]. Inductive reasoning is introduced from the earliest years of schooling to help children acquire knowledge. In preschool, for instance, children are shown different objects of the same color until they are able to distinguish the ones of that color from those of other colors and ultimately understand what it means to "be that color". Inductive reasoning is a cognitive process that begins with working with specific cases, followed by formulating, and then verifying conjectures [2]. Generalization is pivotal to inductive reasoning because it is the pathway for generating knowledge, especially mathematical knowledge [1].

Generalization is also deemed a fundamental notion in algebraic contexts, in the lower grades especially. The general consensus in mathematics education is that generalization is pivotal to mathematical activity in general and algebraic thinking in particular (e.g., [3]) because it enables subjects to generate mathematical knowledge [1]. Introducing generalization in the lower grades enables students to distance themselves from the particulars inherent in arithmetic calculations and identify structure and the mathematical relationships involved [3].

Working from different approaches to algebraic thinking for very young children helps them to generalize by identifying regularities or patterns in a given mathematical situation [4]. From the various types of algebraic thinking dealing with basic algebraic notions in the lower grades, we focus here on functional thinking. Functional thinking revolves around the relationships between two (or more) covariant quantities. More specifically, it involves thinking processes evolving from specific relationships to their generalization [3] (p. 143). The general consensus in mathematics education is that generalization is a core component of mathematical knowledge and a key to algebraic thinking (e.g., [3]).

Functional thinking is a type of algebraic thinking that adopts the function as essential mathematical content [5]. Function is a key concept in secondary school curricula, where students move from the operational to the structural vision of the idea [6]. In elementary education, in contrast, functional thinking is deemed to include activities focusing on the generalization of the relationships among covarying quantities, the expression of such (functional) relationships using different manners of representation and the application of the expressions to analyze the behavior of a function [7]. Reference [8] explores verbal representations of the functional relationships recognized and the strategies used by (6- to 7-year-old) first graders when performing generalization tasks in a functional thinking context. They observed some pupils to be able to identify the covariant relationship at issue, finding functional relationships to be associated with students' operational strategies or counting. Earlier studies (e.g., [9]) reported elementary school students identifying general properties by building on specific situations involving a relationship between two sets of values. This general rule may be established in terms of inductive reasoning [10], in which generalization is the outcome of identifying regularity in inter-variable behavior.

Research on generalization among elementary school students has grown in recent decades (e.g., [10–13]). Whereas symbolizing general ideas helps students build a new platform from which to express and think about unknown situations [14], research has shown that their thinking process entails algebraic thinking, even where algebraic symbolism is lacking.

Particularly prominent in this regard are the studies conducted by [15] with elementary school children of different ages. These authors published a longitudinal study in which they characterized different levels of sophistication in children's reasoning around functional relationships in their learning process. Their results suggested that children can learn to think in general terms about relationships between functional data, challenging the standard curricular assumption that students in the lower grades can detect variation only in a single sequence of values. Different studies address generalization with first, second and third graders (e.g., [10,11]). However, for second grade, we have not found results about how these students express structures during a generalization process.

Our research objective is to describe how second graders reason as they evolve toward generalization (process) and how they express generalization (result) as part of inductive reasoning. Inductive reasoning is broken down into two phases, abduction and induction, to obtain greater insight into such evolution. Abductive reasoning is an initial stage in which subjects draw from knowledge and prior experience when trying to explain an event. Abductive explanation is a conjecture that must be tested before becoming belief [16]. In inductive reasoning (in this article, we use the terms "induction" and "inductive reasoning" indistinctly as synonyms for the same cognitive process), an abductive conjecture is tested with specific cases. Both abductive and inductive reasoning serves as support for subsequent generalization. In the present study of the reasoning process, we addressed the structures identified by the children when performing generalization tasks involving linear functions. Generalization tasks enable students to explore and express functional thinking. Such research is deemed necessary for in-depth functional thinking research [17], for it attempts to respond to questions still unresolved in the literature around how very young children generalize functional relationships between two quantities. More specifically, it explores the types of relationships expressed by children and the sophistication of their thinking about such relationships [15].

1.1. Abductive and Inductive Reasoning

Abduction was introduced by [18], who deemed it, like induction, to be a form of inference characterized by the generation of hypotheses. Abduction is the first and least certain stage of inference, for it entails building preliminary explanations [19]. For Peirce, abductive conclusions always have a lower epistemic status than inductive conclusions [20]. According to [21], to abduct is to generate a hypothesis or narrow a range of hypotheses based on a few specific cases, a process subsequently verified via inductive reasoning. From Peirce's perspective, induction consists not of creating but of verifying; i.e., inductive reasoning does not deliver new knowledge. Its purpose is to verify and, in some cases, amend theory. It primarily entails determining the verity of the hypothesis posed [22]. According to [23], the abduction phase is an initial period when testing is repeated and conjectures formulated. The expectation is that possible errors will be corrected in the inductive phase, which in turn evolves toward generalization based on the available evidence [21]. Induction is deemed a powerful knowledge-building resource that owes its potential essentially to the presence of generalization as one of its components. Generalization involves abstracting the systematic and common features of events [24].

Authors such as [25] contended in a study on patterns that abduction precedes generalization. During abductive reasoning, subjects discover regularity, enabling them to formulate initial conjectures when working with the first specific cases included in a task. In the abductive phase, a structure is accepted as valid based on the early specific cases, subject to verification with further cases, and ultimately inducing generalization. A conclusion drawn from an abductive inference carries less weight than one drawn from induction [26]. Induction entails confirming the conjecture premised. Regularity is accepted after such confirmation, enabling the subject to generalize [25]. The process is summarized in Figure 1.

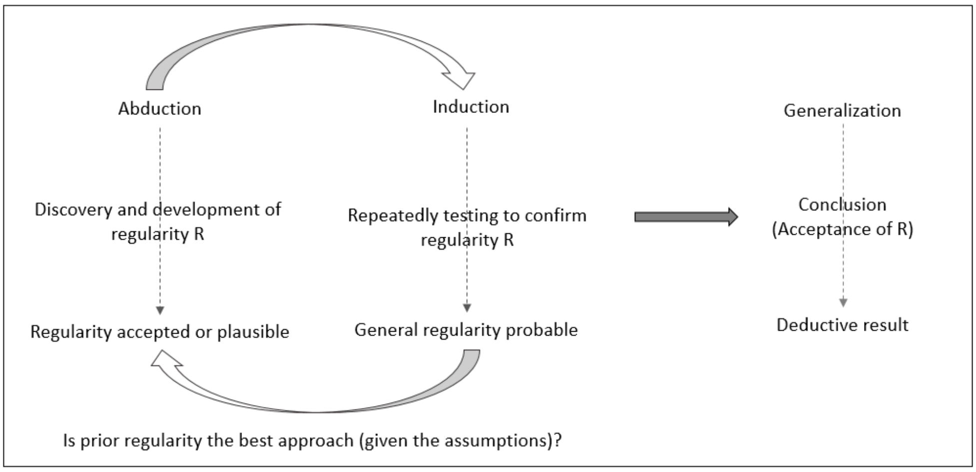

Figure 1. Abductive-inductive process [25]. Reproduced with permission of Rivera, F; Becker, J. R.

The author of [21] stressed that untested explanations may be put forward in the abduction phase, during which trial-and-error does not yield algebraic generalization because the specific cases involve numbers, which are specific values of the variables at issue [3]. In this article, Radford deemed abduction to form part of objectivation theory, which divides into stages the reasoning followed by students when working with patterns. The authors of [27] used abduction to explain the reasoning deployed by 9- and 10-year-olds when evolving toward algebraic thinking. Reference [8] associated the abductive phase with subsequent generalization based on representing structures. Structure is discussed in a later section of this article.

Reference [9] defined a seven-step inductive reasoning model (Figure 2) as part of the findings of research describing work with 359 secondary school students performing tasks that involved generalization.

1. *Working with specific cases* or examples, normally simple and readily identifiable, to initiate the process.
2. *Organizing specific cases* to arrange the data gathered in a way that helps to perceive patterns, in tables or rows and columns and ordered by some criterion.
3. *Seeking and predicting patterns* in the specific cases observed (which may or may not be organized) to formulate the next case on those grounds.
4. *Formulating conjectures* or premises assumed to be true subject to testing, which may lead to acceptance or rejection, the latter when an example is found for which the conjecture is not valid, for instance. In such cases, to use Popper's (1967) phrasing, the conjecture is refuted.
5. *Justifying conjectures*, meaning any reason put forward to defend the verity of an assertion, normally distinguishing between empirical and deductive conjectures. The former use examples as argument and additional specific cases for verification.
6. *Generalizing*, in which the conjecture is expressed in a way that refers to all the cases of a given type and implies extending reasoning beyond the specific cases explored.
7. *Proving* or formal validation establishing the unequivocal validity of the conjecture at issue.

Figure 2. Inductive reasoning model proposed by [9]. Reproduced with permission of Cañadas, M.C.; Castro.

The authors stressed that while all seven steps are useful for helping students progress toward generalization, the ultimate objective, all may not necessarily be present, appear in the order shown, or carry the same weight in inductive reasoning. Generalization is a key step, whereas organizing specific cases may prove helpful but is not routinely present. Some studies (e.g., [7]) used the model to design questionnaires exploring generalization in the lower grades; it has also been used at the university level. Reference [28] applied it to describe and characterize inductive reasoning among pre-service training elementary school teachers. The model was also used by [29] in a functional context, describing the steps followed by fourth graders working toward generalization.

1.2. Structures

As a rule, part of the inductive process in the evolution toward generalization entails observing regularities in the situation posed. In the aforementioned inductive model, that idea is captured in the third step—seeking and predicting patterns. The term pattern is normally associated with situations in which all the values in a given set are explicitly described. In mathematics education, the term structure is widely used with different meanings, although it always infers breaking an entity down into its inter-connected or inter-related component parts [30]. More specifically, structure expresses the relationships among numerical quantities, the properties of operations and inter-operational relationships [31].

Here we use the term to mean regularity in a functional algebraic thinking context. So defined, structure has to do with the terms comprising a functional algebraic expression, the signs inter-relating them, the order of the operations involved and the relationships among their elements. Structure is therefore associated with the manner in which the elements of an inter-variable regularity are organized and their inter-relationships [32]. Function is taken here to mean the numbers and numerical variables (expressed via different representation systems) as well as operations and inter-operational properties forming part of a regularity identified by the student [33]. The idea is analogous to the term "pattern" used in the [9] model, although the contexts in which the ideas are applied differ. Whereas patterns tend to deal with only one set of values with recurrence as the most obvious relationship, structures entail working with two or more sets of values.

Students may be able to detect structures on the grounds of how they represent them, both when working with specific cases and when generalizing [33]. Some researchers note that before being able to generalize, pupils must "see" the structure in a mathematical situation [8]. Identifying structure in a mathematical situation is therefore the key to induction-mediated generalization.

In the three specific cases illustrated in Figure 3, regularity may be identified as the recurrence of consecutive terms in a sequence. The position of each of the elements and the sequence in which they appear are the keys to this type of task, which differs from the generalization tasks explored here.

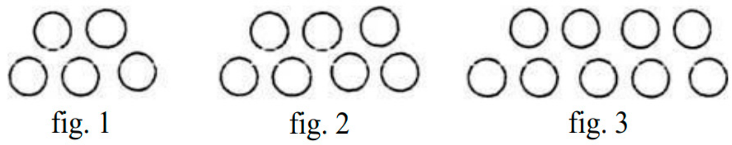

Figure 3. Pattern task [26]. Reproduced with permission of Vergel, R.

It also differs from the generalization task based on the relationship between the ages of two superheroes, given as $y = x + 4$ [34], shown in Figure 4.

> Two superheroes, Iron Man and Captain America, have birthdays on the same day.
> When Iron Man was 5 years old, Captain America was 9.
> When Iron Man was 7 years old, Captain America was 11....

Figure 4. Generalization involving superheroes' age.

Identifying structure calls for the existence of a relationship between two variables perceptible in specific cases. Such cases are characterized by the absence of sequence in the representation of the variables involved. In structure, both the domain and the range are explicit numerical sets.

Reference [8] showed that when students identify structures in mathematical tasks, they experience mathematics more deeply. Reference [33] concluded that when solving the tile problem involving the relationship $y = 2x + 6$, third graders (8- to 9-year-olds) invoked 17 different structures for a given regularity, five of which were correct. They also used structure to answer the items on the questionnaire. In a paper on structures involving the functions $y = 1 + 2x$, $y = x + 3$ and $y = 2x$, [34] observed consistency in students' replies, for they identified only one different structure for the same regularity with the functions $y = 2x + 1$ and $y = 2x$.

Evidence of the presence of structures does not in itself suffice, however, to determine whether students recognize the relationship in a specific situation or merely see it as an example of a general property applicable to different situations. As [8] noted, "because language is necessarily general, it is very difficult to tell from a learner's works whether they are dwelling totally in the specific and the particular, are vaguely aware of the particular as a special or specific case of something more general, or are aware of the particular as an instantiation of a general property" (p. 11).

1.3. Generalization

Generalization is a core element in inductive reasoning and is essential to mathematical reasoning, which entails seeing beyond the particularities of a mathematical situation to draw a conclusion [35].

In the area of research on children's algebraic thinking, generalization is a key process in the early years of schooling. Some authors [36] contend that children are naturally inclined to perceive and discuss regularity, a fundamental component of generalization, even when they lack the resources needed to represent general relationships. Generalization includes establishing general relationships between covarying quantities, expressing those relationships through different types of representation (verbal, symbolic, tabular or graphic, for instance) and reasoning with such representation to analyze the behavior of a function [5]. In particular, generalization is the core notion in inductive reasoning and forms part of the highest stratum of functional thinking. In order to generalize, subjects must identify the regularities in the behavior of a functional task. A conjecture established on the grounds of regularities identified and validated by detecting further regularities in a given situation may induce generalization of the behavior observed in the function at hand.

Generalization tasks entail building new specific cases on the grounds of one or several specific cases or the general term. They therefore necessitate identifying the structure or behavior pattern in specific cases. While letters are essential, inherent in any discussion of generalization in elementary education is the acknowledgement that students may express relationships not only in terms of algebraic symbols but also in natural language or by gesturing [11]. Verbal and pictorial representation may also be instrumental when working with children in the lower grades [15].

References [36,37] distinguish four types of algebraic generalization, three of which are algebraic and one arithmetic. In the fourth, generality is a similarity observed among certain cases that does not suffice to develop an expression valid for whatsoever term in the sequence. The author defines algebraic generalization of patterns as generalization involving (a) the awareness of a common property detected by working with a number of specific cases; (b) the application of that property to the following cases in the series; and (c) the ability to use that common property to deduce a direct expression with which to calculate the value of any term in the series. Factual, contextual and symbolic generalization are sub-types of algebraic generalization [11]. Factual generalization is based on actions performed with numbers, with actions being words, gestures and perception. It is expressed as specific action through work with numbers. In the factual type, generalization is implicit; students may point with their gaze, gesture with a pencil, say "here" or point with their fingers and say "plus three". Factual generalization is the first form of generalization, the form where perception, based on different mechanisms used by students to communicate, induces computation, which enables them to move from the particular to the abstract. It is generalization from which any specific case can be addressed, i.e., it is permanently associated with the specific. Contextual generalization is the abstraction of a specific action. It differs from factual generalization in that it does not involve operating with specific numbers. To put it another way, contextual generalization is the description of the general formulation in which gestures and words are replaced with "key" phrases, with students saying things like "always plus three". Symbolic generalization is the representation of sequences with algebraic alphanumerical symbols [37]. "Key" phrases are represented with algebraic alphanumerical symbols.

1.4. Research Objectives

The general research objective of this study is to describe the generalization process deployed by second-grade pupils. In particular, we lend special attention to the structures and the types of generalization evidenced by the students. The specific research objectives are the following.

1. To describe the generalization process deployed by the students.
2. To identify and describe the structures recognized by the students in their evolution toward generalization.
3. To characterize the types of generalization expressed by the students.

2. Materials and Methods

This exploratory, descriptive study employed qualitative data analysis. Working sessions consisting of individual interviews were conducted using a design research approach [38] with three aims: (a) to explore how students related the variables involved; (b) to identify structures on the grounds of the students' answers; and (c) to explore how the students generalized in their reasoning.

We worked with verbal productions of children. We were able to develop this research after obtaining permission from their parents and from the schools. The schools preserve the original documents, as the Spanish law dictates. This study was developed in the context of the project EDU2016-75771-P.

2.1. Students

Initially our subjects comprised a group of 24 students enrolled in the second grade (7- to 8-year-olds) in the academic year 2017/2018. The non-random sample was chosen on the grounds of the availability of a charter school in Granada (Spain), located in the northern district of the city. The participating institution implements a "learning communities" project to induce social and educational change geared to the profiles of the student body.

The curriculum of elementary education in Spain establishes the knowledge that students have to acquire at this stage. In order to complement this information, we interviewed the usual students' teacher about their mathematical knowledge. Concerning the topics related to the kind of tasks that we were posing, prior knowledge included the use of numbers from 0 to 399, numerical comparison, addition with carrying, and subtraction without borrowing. Their primary trait from the perspective of the study was that they had never worked with problems involving a linear function or generalization.

On the grounds of their replies to a questionnaire, the students were divided into three groups: beginning, intermediate, and advanced. All the students answered the written questionnaire individually. It included a generalization task in which the number of balls going into a machine was related to the number coming out by the functional relationship $f(x) = x + 3$. The questionnaire and interview questions were based on the inductive model specified by [9], i.e., the specific cases posed at the outset were designed to induce generalization. The first group included eight students who had not identified the structure. The nine students in the second group identified the structure involved in several (about 3 or 5 questions) of the questions, and the seven students in the third group proved able to generalize. The students' teacher chose two students from each group for the interviews (beginning: S1, S2; intermediate: S3, S4; advanced: S5, S6) based on their academic performance and classroom participation. This election was based on the teacher's assessment and knowledge of the students.

The students are identified here with the initial S and the numbers 1 to 6 to ensure anonymity. Individual interviews were conducted with all six students.

2.2. The Interview

The interviews were conducted on school premises and video-recorded. The aim was to explore students' replies in greater depth; for this purpose, we designed an interview protocol in which specific cases were distributed further to the [9] model to induce generalization.

As suggested by [39], the specific cases were presented in different ways. Items requiring the students to determine the next term in a series or one that could be found by counting were labelled as near cases, whereas those necessitating an understanding or the identification of the pattern or function were labelled as far cases. Far specific cases called for representing quantities that could not be drawn or were hard to find by counting with the students' existing academic skills. In addition to specific cases, indeterminate cases were posed in which the reply depended on recognizing a relationship. The values used were not consecutive, to discourage students from seeking a recurrent relationship that would mask the actual functional relationship. With the near specific cases, the idea was to obtain information on students' conjectures, first by observing the structure identified during the interview. After they worked with several specific cases, we asked "what did you do to find the answer or how did you to get it?". The approach is summarized in Figure 5.

Near specific cases
o If one ball goes into the machine, how many will come out?
o If three go in, how many will come out?
o If 12 go in, how many should come out?
o What did you do to find that answer?

Figure 5. Near specific cases used in the interview.

In the first stage of the interview, students were shown drawings representing non-consecutive specific cases (Figure 6), which were then discussed with them one by one.

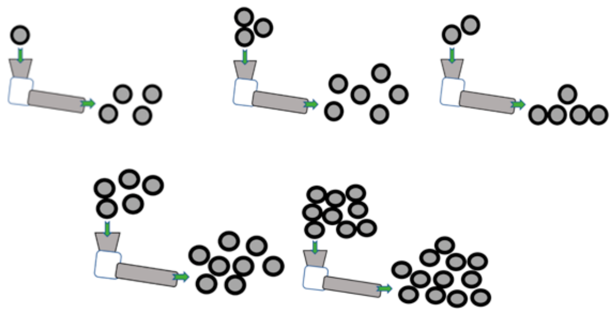

Figure 6. Near specific cases used in the interview.

The aim with the far specific cases was to determine whether students could confirm their earlier conjectures (when working with near specific cases) with justified arguments, while allowing them to vary the structure expressed about the relationship involved. Examples of far specific cases are listed in Figure 7.

Far specific cases
o If 50 balls go into the machine, how many will come out?
o If 350 go in, how many will come out?
o If one million go in, how many can come out?
o If three million go in, how many can come out?
o What did you do to find the answer?

Figure 7. Far specific cases used in the interview.

We used other specific cases, labelled "indeterminate". Here we used expressions such as "many" or "any number of" balls. As the aim was to observe in their answers whether students reaffirmed their conjecture and generalized, we ultimately asked them "how does the machine work?" (Figure 8).

Indeterminate cases and general case
o If a lot of balls go into the machine how many will come out?
o If any number goes in, how many do you think will come out?
o How does the machine work?

Figure 8. Indeterminate cases and general case used in the interview.

We also used other types of (near or far) specific cases as further stimulus. We drew from specific cases proposed by the students themselves and a specific case put forward by a classmate, aiming to induce conjecture validation and therefore justification, which would help us analyze students' functional thinking. Those questions were interspersed with the ones described above, as needed during the interview. Students' own specific cases were those formulated when they were asked to choose a number, any number, of balls to go into the machine. The specific case formulated by a classmate afforded another opportunity for pupils to justify their conjectures based on the situation proposed (Figure 9).

Other specific cases

- The student's own formulation
  - Choose a number (__). If you put that number of balls in, how many should come out? Explain how you got that answer.
- Classmate's formulation
  - One of your classmates says that "if XX balls go in, then YY should come out". Do you agree? How did you find that answer? (XX and YY symbolize the quantities used in the interview.)

Figure 9. Other specific cases used in the interview.

After the students worked with specific cases, we asked the following question to induce generalization: "Do you know how the machine works? How can you tell?".

The design of the tasks used in both the questionnaire and the interview (e.g., the functional relationship involved, the order of questions used) was inspired by studies on functional thinking previously cited. Contexts and vocabulary were chosen to be familiar to the participating students. The tasks were organised around the inductive reasoning model proposed by [9].

The structure of the questionnaire applied to select the sample and the interviews carried out was based on an inductive process, starting from particular cases and making progress to the general ones. The structure of the interview, and also that of the questionnaire, have been used in previous studies. In this way, the use of this structure to extract data from the work of elementary school students has been previously observed. Focusing on the validity of the instrument itself, we performed a pilot study using the same instrument to study the results with second graders. Thanks to this, we developed an approach to the analysis of the components. From this first approach, we modified some of the questions that seemed to be poorly understood by the students and made the instrument more valid and conducive to our particular investigation.

2.3. Data Analysis

The qualitative analysis run on the data contained in the transcriptions of the interviews was based on a combination of the inductive reasoning model of [9] and the definition of the abductive and inductive phases of such reasoning from [25]. The qualitative analysis of the data was carried out by coding the interviews after their transcription. The unit of analysis was sentences. From the keywords detected in the sentences about the relationship between variables through the particular cases and the generalization itself, we developed the categories that generated the explanations about the generalization process. We performed an expert individual triangulation made by the three authors of this article. In an independent way, we performed the analysis on part of the students' responses until we agreed on the categories for the analysis. As a result of the comparison between the authors, the categories extracted are more exhaustive and therefore more reliable.

In our analysis, generalization process was deemed to comprise three phases: (a) abduction; (b) induction; and (c) generalization. In each phase we considered the specific cases used, the conjectures posed, structures and their representation, and the acceptance of a structure previously defined that defined the inter-phase boundary.

The phases in the reasoning process were distinguished on the grounds of the specific cases the students worked with during the interviews. Inductive reasoning begins with abduction (Figure 10), a discovery phase needed to induce generalization, which requires students to formulate their initial conjectures and discover possible structures when working with a few near specific cases.

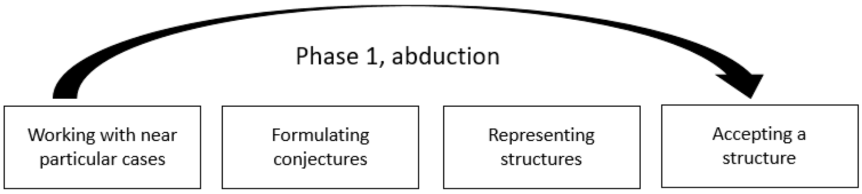

Figure 10. Abduction phase.

Representing structures is a key feature of this and the following phases, for it is the way that the students put forward their conjectures, affording us the opportunity to record them. We observed what students inferred when expressing their conjectures through structural representation. In the abductive phase, students might discover the structures with a few near specific cases. This phase concluded with a priori acceptance of a structure, in what [21] called hypothesis generation, subsequently confirmed in the induction phase.

In the induction phase (Figure 11), students worked with far specific cases, possibly reformulating conjectures, i.e., identifying another structure that need not necessarily have been recognizable in the abductive phase. The most prominent feature of this phase was that students had to confirm the viability of the structure described when working with a number of far specific cases.

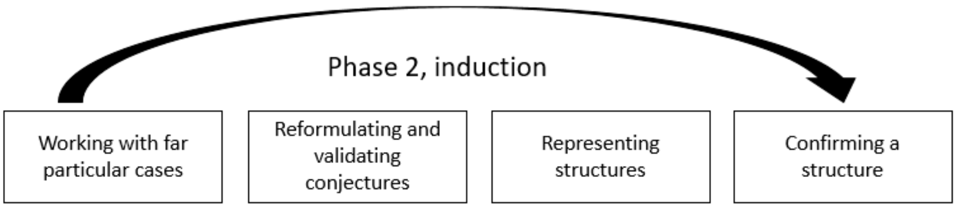

Figure 11. Induction phase.

In elementary education, a conjecture is deemed to be confirmed when a student recognizes the same structure on more than two occasions when working with far specific cases, thereby exhibiting awareness of the structure concerned.

This is when they may generalize (Figure 12); once confirmed, the structure can be applied to any other case. As [40] explains, when perceiving a trend in the inter-variable relationship with indeterminate cases or the general case, students may reaffirm the conjecture confirmed in the induction phase. Reaffirmation may then culminate in a generalized structure.

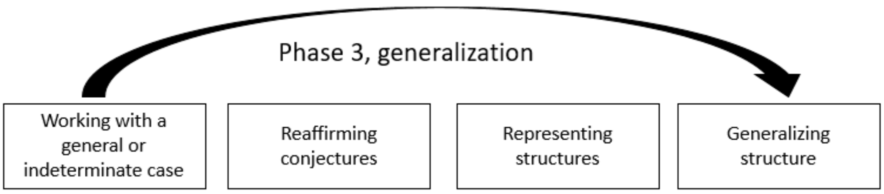

Figure 12. Generalization phase.

We analyzed students' reasoning while working with near and far specific cases and generalizing, and we determined the extent to which our analytical categories adapted to their inductive reasoning. We identified and described the structures expressed by students as they answered questions first involving specific cases and ultimately involving the general case. The types of generalization observed were also typed in accordance with the classification of [37].

3. Results and Discussion

The findings for the generalization process observed in the six students interviewed are discussed below in terms of the categories defined in the methodology.

3.1. Description of Generalization Process

Table 1 shows the presence or absence of each step in the three phases of reasoning and the order in which they appeared, student by student. The interviews were initially analyzed bearing in mind the order in which the steps were observed to appear. The data recorded included whether the students began by working with specific cases, whether or not they formulated a conjecture or represented a structure, whether or not they confirmed prior conjectures and whether or not they generalized, and if so, how. The table shows the order of the steps observed for each student in each phase.

Table 1. Steps in generalization process recognized in students.

		Student					
Phase	**Step**	**S1**	**S2**	**S3**	**S4**	**S5**	**S6**
Abductive	Working with near specific cases	1st	1st	1st	1st	1st	1st
	Representing structures (formulating conjectures) and accepting structure	2nd	2nd	2nd	2nd	2nd	2nd
Inductive							
	Working with far specific cases	3rd	3rd	3rd	3rd	3rd	3rd
	Representing structures (formulating conjectures)	4th	4th	4th	4th	4th	4th
	Confirming conjectures (formulating conjectures)	5th	5th	5th	5th	5th	5th
Generalization							
	Generalizing	6th	6th	6th	6th	6th	6th

The data in Table 1 show that all the students followed all the steps in the order proposed on the grounds of the logic underlying the categories. We deemed representing structure and formulating conjectures to form part of the same step, for structure is the expression of conjecture, and in the students' answers as we interpreted them, the two appeared simultaneously. An analysis revealed that all the students worked through the phases and steps of inductive reasoning leading to generalization in the order shown in Table 1. An example of the abductive phase is illustrated in the following excerpt from the interview with S4.

1. Interviewer (I): If three balls go into the machine, how many should come out?
2. S4: Six.
3. I: And if eight go in?
4. E4: 11.
5. I: Why 11? How do you know that?
6. S4: Because there could be three balls inside, and if you add eight, 11 come out.
7. I: How did you know there could be three balls inside?
8. S4: Because otherwise if you put eight in you can't get 11 out.
9. I: Three balls are going in now. What happens?
10. S4: Inside there are three more so six come out.
11. I: What is the machine always doing the same way?
12. S4: There are balls inside and more go in.
13. I: And how many balls are inside the machine?

14. S4: If three go in, well, there are three more inside, and they come out and have to be added.

Where a conjecture was expressed explicitly, we interpreted it to mean a structure was recognized. S4 formulated a conjecture in line 6 of the except: "because there could be three balls inside, and if you add eight, 11 come out". The student was uncertain about what happened in the machine (beginning the trial-and-error process). Based on the answer in line 6, we deduced that the structure represented by S4 was $y = x + 3$. In line 8, we observed that the student tried to justify the answer. In lines 10 and 14, we deduced the same structure, $y = x + 3$. Although a new conjecture appeared in line 12, in line 14 the student re-identified $y = x + 3$ (accepting the structure), a clear indication of the presence of a preliminary trial period (abductive phase) that preceded confirmation of the structure (induction phase). Both structure and generalization are discussed in greater detail in the following section.

The excerpt below from S5′s interview exemplifies abduction, induction and generalization.

1. Interviewer (I): If you put two balls in the machine, how many will come out?
2. S5: Five, no?
3. I: Correct. And if three go in?
4. S5: Six.
5. I: If nine balls go into the machine, how many will come out?
6. S5: Here 12, no? (counting the balls). Yes, 12.
7. I: Could you tell me how the machine works?
8. S5: Well if you put one ball in you get a triplet, four come out.
9. I: What do you mean by triplet?
10. S5: Well, three more.
11. I: OK. Now choose a larger number, any number you want, other than the ones we've seen.
12. S5: 52.
13. I: 52—how many would come out?
14. S5: 55.
15. I: And if 200 go in . . .
16. S5: 203 come out.
17. I: OK, look: now we're going to put a very large number of balls in (writing a very large number on a sheet of paper), how many would come out?
18. S5: If it's one million, one million three come out.
19. I: Fine. You're doing the same thing all the time, no?
20. S5: Yes, I just think plus three.

This student exhibited abduction in lines 2 and 6, answering hesitantly when faced with the first specific case and newly discovering the relationship between the variables. Line 8 provides evidence that S5 was conjecturing, recognizing the structure interpreted to be $y = x + 3$ (the same structure accepted earlier by the student in the abductive phase), as explicitly stated in line 10. As the interview then went on to far specific cases, we observed S5 to answer in lines 12 and 15 with the same structure recognized when working with near specific cases ($y = x + 3$). In other words, S5 confirmed the structure, applying it to far cases. When asked about indeterminate cases (a very large number of balls) in line 17, S5 reaffirmed the conjecture by answering with the structure initially recognized, $y = x + 3$ (line 18) and then went on to the generalization phase. Finally, in line 20, the student generalized the relationship involved: "I think plus three".

3.2. Structure Identification

We distinguished between students who identified structures while working with specific cases from those who did so when generalizing, defining structural identification in the general case as equivalent to generalizing. All students identified at least one structure in one scenario or the other, as summarized in the findings on structures set out in Table 2. Where students recognized different structures as they worked, they are listed in the table in chronological order.

Table 2. Structures recognized by students.

		Structure		
		Specific Cases		General Case
Group	Student	Near	Far	
Beginning	S1	$y = x + x$ $y = x + 3$	$y = x + 3$	$y = x + 3$
Intermediate	S2	$y = x + 1, x + 2$ (1,10)	$y = x + 2, x + 3$ (10 . . .)	$y = x + 3$
	S3	$y = x + 3$	$y = x + 3$	$y = x + 3$
	S4	$y = x + 3$ $y = x + x$	$y = x + 3$ $y = x + x$	$y = x + 3$
Advanced	S5	$y = x + 3$	$y = x + 3$	$y = x + 3$
	S6	$y = x + x$ $y = x + 3$ $y = x + x$	$y = x + 3$	$y > x$

Further to the data in Table 2, each student identified one to three structures for the specific cases during the interview. The four types of structures observed in all can be symbolized algebraically as $y = x + x$, $y = x + 1$, $y = x + 2$ and $y = x + 3$, although the students did not use such symbolism to represent the structures. The following excerpt illustrates the structures recognized by S2.

In light of the structures identified in the general case (Table 3), all six students were deemed to generalize the functional relationship. In their generalizations, we identified the structure $y = x + 3$ in five and $y > x$ in one. All generalizations were expressed verbally. We observed consistency in the results insofar as the same structure was identified in the specific and general cases at some point in the interview with five of the six students.

Table 3. Type of generalization.

Level	Student	Expression of Generalization	Type of Generalization
Beginning	S1	"The machine divides half in three balls and three more come out."	Contextual
Beginning	S2	"We put a few balls in and then if you want to put in seven or eight or however many you want, then more than that will come out. The machine takes the balls and adds three. The machine is always adding three, but I like to add one because it's more interesting."	Contextual
Intermediate	S3	"[With] one ball we get four, with two we get five, with four we get seven . . . You always have to add three. One million you get one million three."	Factual and contextual
Intermediate	S4	"You have to add the ones inside to the ones you put in. I think three inside because that's the number added most."	Contextual
Advanced	S5	"You put one ball in and you get three more. I think plus three."	Factual and contextual
Advanced	S6	"I know the machine returns the same [number of] balls, and what I know is that inside it adds lots more balls."	Incipient

3.3. Characterization of Types of Generalization

All the students in this group generalized. We perceived different types of generalization in their verbal descriptions. Three students (S2, S3 and S5) generalized by saying "you have to add three". The following extract from one of the interviews illustrates how S3′s thinking culminated in generalization.

I: How can you tell how many balls will come out of the machine if we don't care how many go in?
S3: Well, look: with one ball we get four, with two we get five, with three we get six and with four we get seven.
I: Fine. Then something's going on with the number of balls that comes out, right?
S3: It's like I said, you always have to add three.
I: And if one million balls go in?
S3: One million three come out.
I: And if I put three million balls in?
S3: Then three million three come out.
I: Then three more than you put in are always going to come out?
E3: Yes.

Three other students (S1, S4 and S6) realized that "more balls come out than go in". S1 and S4 later specified the number to be added, stating that they were thinking of a structure symbolized as $y = x + 3$. (We as authors, not the students themselves, are expressing the structure recognized by students symbolically, here as $y = x + 3$.) S4 said, for instance "more balls come out than go in. I always add three". S6, in contrast, exhibited incipient generalization, failing to quantify the number coming out: "I know the machine gives back the same balls and what I know is that inside it puts in lots more balls". Five students were found to identify the structure correctly ($y = x + 3$) in both the general and the specific cases.

All the students generalized verbally, using natural language to express mathematical ideas, in keeping with the nature of the data collection tool used, interviewing. Table 3 summarizes the characteristics of type of generalization observed under the classification of [37]. The table reproduces the transcripts of students' verbal expression of generalization as the grounds for identifying type.

In some cases, factual and contextual generalizations were observed to co-exist. No evidence of symbolic generalization was identified in the cases studied, however. Factual generalization never appeared alone, but always in conjunction with the contextual sort. Students S3 and S5 adopted the same approach. Factual generalization depending on the action performed with numbers was followed by the abstraction of specific actions (contextual generalization), ultimately expressed as "You always have to add three" or "I think plus three". Students initially identified the behavior pattern and described it in their answer to later specify the underlying general relationship. S2 and S4 exhibited contextual generalization, differing from the others in that they generalized the relationship directly with no reference to numbers. As noted in the section on data analysis, S6 identified a relationship between the number of balls going in and coming out of the machine but failed to recognize the functional relationship, $y = x + 3$ when generalizing. That differentiated S6′s answers from those of the other cases of contextual generalization. Attention is drawn to this case because it is not envisaged in any of the [37] categories used as our theoretical baseline. Unlike factual generalization, the contextual sort did not systematically appear together with any other type. More specifically, S1, S2 and S4 described the functional relationship without using numerical examples.

By way of summary of the findings on generalization as expressed by the students, we can say that five of the six generalized algebraically, for they used a common property to express a relationship with which the value of any term in the series could be calculated. S6, the exception, generalized incipiently, noting only that the dependent was larger than the independent variable, without quantifying the difference.

4. Conclusions

This article shows that second graders operationalize functional thinking by identifying structures, which enables them to generalize to most of the students in this study. Given the size of the sample, we emphasize that our intention is not to generalize the results.

Our empirical observations revealed that the generalization process deployed as they evolved toward generalization comprised three phases. The task designed for this study was based on the inductive reasoning model of [9], while students' answers were interpreted in keeping with the perspectives of [27] on the phases involved. The outcome is a description of how the thinking processes of the second graders in this study culminate in generalization.

Distinguishing abduction, induction and generalization as phases forming part of the generalization process constitutes a theoretical contribution to the field. Although some of these phases have been described previously, we emphasize the relationship between them and how they can be related in the generalization process.

Moreover, this contribution is useful from the teaching and learning viewpoint because it can be used for task design.

The model specified in [9] was useful for establishing the theoretical grounds for the study and its design. The data gathering tools used—Questionnaires and interviews—Were based on these guidelines. To analyze elementary school students' output, however, some of the broadly general steps in this model must be defined more precisely. To that end, we itemized the specific cases used in the interviews to describe students' reasoning in greater detail. That was one of the keys to our research, for the degree of thinking involved in each specific case varied. Our magnification of the model specified in [9], which was designed for secondary school, proved useful for determining the phases defined by [25]. Abduction, found to be a preliminary phase, appeared when addressing the first specific cases, as students formulated their initial conjectures and first detected structure. Unproved explanations, an element that according to [21] might form part of the inductive process, was observed in one of our students (S4). During induction, hypotheses were put forward that were not confirmed until the students solved other, far specific cases (inductive phase). They had then to identify an inter-variable relationship to continue the process, for at that age children are bereft of the skills to clearly visualize, count or draw very large quantities. We observed the possible confirmation of conjectures in the induction phase. A conjecture was deemed to be confirmed when a student, working with far specific cases, recognized the same structure on more than two occasions, denoting awareness of the structure concerned. Students were deemed to generalize when the confirmed conjecture was reaffirmed for the indeterminate or general cases.

The generalization process model illustrated in Figure 13 constitutes the contribution made by this study to an understanding of how stimulus (with specific cases) affects students' reasoning in a functional context.

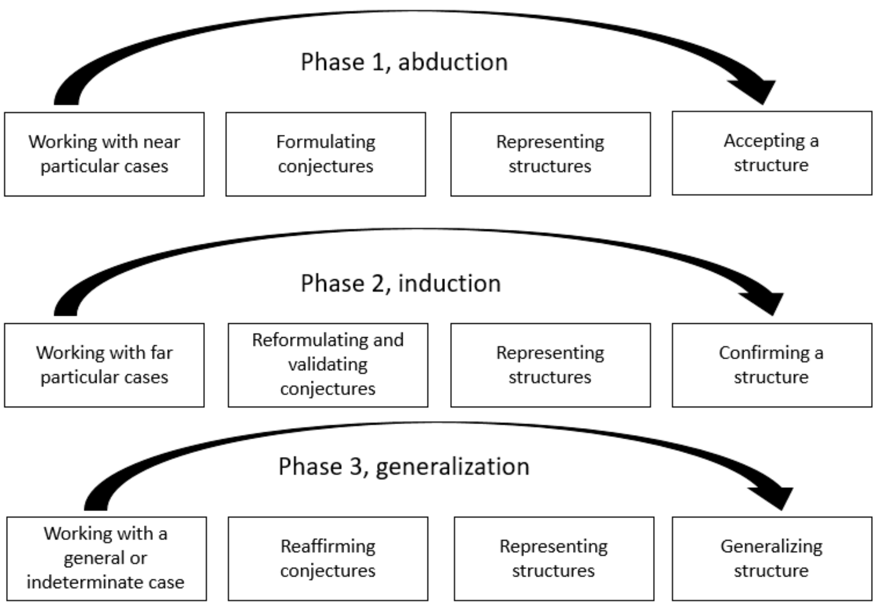

Figure 13. Adapted generalization process model.

The third step in the original model specified in [9], "seeking and predicting patterns", was re-labelled in this study as "representing structures", to highlight the role of the relationship between problem variables (number of balls entering and exiting the machine). A similar approach was adopted by [30], who used the model in a functional context to explore the evolution of reasoning toward generalization among fourth graders. Analogously, the term "regularity" as specified in [25] was replaced with "representing structures", to emphasize inter-variable covariation. This would make reasoning a feature suitably associated with functional thinking.

Taking generalization to be both a process and a product helped us understand how students of this study generalize by providing a means to follow their reasoning as described as well as observe how they ultimately expressed generalization. All the students generalized verbally, and the generalization observed was typed as factual and/or contextual, as defined by [38], with one exception: generalization as exhibited by student S6 was deemed incipient. S1, S2 and S4 generalized contextually only. S3 and S5 expressed a combination of factual and contextual generalization. Those findings were logical in light of the interview protocol, geared toward inductive reasoning, i.e., it was designed to stimulate the evolution from initial specifics to abstraction and generalization. Recording evidence of factual generalization necessitates careful analysis, for it has to do with children's perception when facing a task. For that reason, the more alert we are to the details communicated by students when first broaching the task, the more information can be drawn from our interpretation of their generalization.

In summary, five of the six students expressed generalization algebraically, identifying a common characteristic to formulate an expression that, while not symbolic, enabled them to calculate the value for any term in the series. In contrast, with merely incipient, non-algebraic generalization, S6 identified the structure in a way that obviated calculation of the value of whatsoever term in the series. That finding should be stressed, for with it we identified a form of generalization in which students fail to specify the functional relationship, merely noting (in this case) that the dependent was larger than the independent variable, without quantifying the difference. That led us to supplement the categories specified by [37] with a further form of generalization, differentiating a case in which the student recognized only that "more balls come out than go into the machine". According to the literature (e.g., [12,29]), some students do not generalize at this grade, even with this kind of interview or with interventions in the classrooms.

Another key to this study is the close relationship established between structure and the generalization process described. Representing structure is tantamount to explaining a conjecture. Structures help us interpret the inter-variable relationships identified by students. We found that conjectures were formulated at the same time as structures were represented in all the cases studied. Here, confirmation of the same structure in the inductive phase as accepted in the abductive phase guaranteed students' evolution to generalization in all cases. This provides insight into some students' ability to generalize. The potential found in inductive reasoning, however, was that it enabled us to observe when a structure was reformulated. We deemed reformulation to exist when different structures were identified, in keeping with the inductive phase as discussed in the section on structures in connection with students who identified different structures during the interview. Evidence was found of the use of four types of structure when the students worked with specific cases. When answering the interviewer's questions, the students used more than one structure, although the correct functional relationship, $y = x + 3$, was identified by all the interviewees and in the general case by five of the six. The present findings corroborate the results reported by [34] to the effect that students were inconsistent in their use of structure when solving the various specific cases involved in the problem. Drawing from a background paper by [8], we deem such inconsistency to denote non-completion of the algebraic process for want of stability in the identification of the structure during the process. We identified greater consistency here in terms of the structure detected in the specific cases and the general case, which was identical for most participants. Similar findings were reported by [35] in a study involving different functions.

In another vein, and although it was not one of the objectives pursued here, the possibility of a relationship between identification and the groups into which the students were divided on the grounds of academic performance might be envisaged. A cursory analysis revealed no noteworthy differences between the three groups in terms of structural identification, however. Except for S2, all the students recognized structure in the specific cases in a similar manner, i.e., $y = x + x$ or $y = x + 3$. We observed, in the advanced group, S6 change the structure initially identified to ultimately express the relationship as $y = x + x$. When generalizing, that same student limited recognition of the relationship to $y > x$, as noted above. In contrast, in that phase, the beginning and intermediate group students identified the structure to be $y = x + 3$. The conclusion drawn is that no relationship can be found between academic performance and students' ability to recognize structures in the inter-variable relationship studied here.

Regarding the reliability and validity of our research, [41] indicates that reliability consists of demonstrating that the analysis was systematic and exhaustive. In this sense, the analysis carried out is reliable given the degree of systematicity used, which allows us to affirm that the inferences we obtained are reliable. On the other hand, validity is related to the evidence that supports the reported results. In some way, the validity of the study is undermined by the antecedents, since among our results we obtained certain similarities with previous studies. Our research supports the results of previous studies. For example, the evidence on the achievement of generalization by students at an early age is something that has been observed before (e.g., [12,28,33]), as well as the identification of different structures when working with specific values (e.g., [5,32,33]). This corresponds to the validity of our research.

The categories defined here were useful for analyzing students' replies and may be applicable in future research. While based on earlier studies, the categories were complemented and adapted to the data collected. Part of the originality of this study rests in the categories proposed to analyze the generalization process followed, for this particular has not been previously addressed.

We consider of special interest to continue this research using application of the different phases with students at different grades of elementary school and comparing whether the pattern continues to correspond in the same way. This would also include the identification of structures involving different functions, so that the steps of each of the given phases can be corroborated from various perspectives.

5. Patents

We add this section to clarify patents since it is a study that is developed within a research process.

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