JOINT SOURCE CHANNEL CODING FOR NON-ERGODIC CHANNELS: THE DISTORTION SIGNAL-TO-NOISE RATIO (SNR) EXPONENT PERSPECTIVE

A Dissertation

by

KAPIL BHATTAD

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2008

Major Subject: Electrical Engineering

JOINT SOURCE CHANNEL CODING FOR NON-ERGODIC CHANNELS: THE DISTORTION SIGNAL-TO-NOISE RATIO (SNR) EXPONENT PERSPECTIVE

A Dissertation

by

KAPIL BHATTAD

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Krishna R. Narayanan
Committee Members,	Prabir Daripa
	Scott L. Miller
	Henry Pfister
	Alexander Sprintson
Head of Department,	Costas N. Georghiades

May 2008

Major Subject: Electrical Engineering

ABSTRACT

Joint Source Channel Coding for Non-Ergodic Channels:
The Distortion Signal-to-Noise Ratio (SNR) Exponent Perspective. (May 2008)

Kapil Bhattad, B.Tech., Indian Institute of Technology, Madras

Chair of Advisory Committee: Dr. Krishna Narayanan

We study the problem of communicating a discrete time analog source over a channel such that the resulting distortion is minimized. For ergodic channels, Shannon showed that separate source and channel coding is optimal. In this work we study this problem for non-ergodic channels.

Although not much can be said about the general problem of transmitting any analog sources over any non-ergodic channels with any distortion metric, for many practical problems like video broadcast and voice transmission, we can gain insights by studying the transmission of a Gaussian source over a wireless channel with mean square error as the distortion measure. Motivated by different applications, we consider three different non-ergodic channel models - (1) Additive white Gaussian noise (AWGN) channel whose signal-to-noise ratio (SNR) is unknown at the transmitter; (2) Rayleigh fading multiple-input multiple-output MIMO channel whose SNR is known at the transmitter; and (3) Rayleigh fading MIMO channel whose SNR is unknown at the transmitter.

The traditional approach to study these problems has been to fix certain SNRs of interest and study the corresponding achievable distortion regions. However, the problems formulated this way have not been solved even for simple setups like 2 SNRs for the AWGN channel. We are interested in performance over a wide range of SNR and hence we use the distortion SNR exponent metric to study this problem. Distortion SNR exponent is defined as the rate of decay of distortion with SNR in the high SNR limit.

We study several layered transmissions schemes where the source is first compressed in layers and then the layers are transmitted using channel codes that provide variable error protection. Results show that in several cases such layered transmission schemes are optimal in terms of the distortion SNR exponent. Specifically, if the bandwidth expansion (number of channel uses per source sample) is b, we show that the optimal distortion SNR exponent for the AWGN channel is b and it is achievable using a superposition based layered scheme. For the L-block Rayleigh fading $M \times N$ MIMO channel the optimal exponent is characterized for $b < (|N-M|+1)/\min(M,N)$ and $b > MNL^2$. This corresponds to the entire range of b when $\min(M,N) = 1$ and $L = 1$. The results also show that the exponents obtained using layered schemes which are a small subclass of joint source channel coding (JSCC) schemes are, surprisingly, as good as and better in some cases than achievable exponent of all other JSCC schemes reported so far.

To Jiji and Maa

ACKNOWLEDGMENTS

I am greatly indebted to my advisor, Dr. Krishna Narayanan. His positive influence has helped shape almost every aspect of my personal and professional life. He has been a very flexible advisor, giving me full freedom in selecting the problems to work on, and I really appreciate the confidence he has shown in me. His guidance was available at every step and none of this work would have been possible without his help. His insights and suggestions have helped immensely in solving research problems. His comments on my papers and presentations have greatly improved my writing and presentation skills. His friendly and personable nature has made my work life very enjoyable. I am grateful to him for taking me along during his sabbatical, during which I benefited from the exposure to different research groups. I would also like to thank him for supporting me financially during the duration of my Ph.D.

Thanks are due to Dr. Giuseppe Caire, Dr. Ralf Koetter, Dr. Vijay Kumar, Dr. Rudiger Urbanke, Niranjan Nayak, and Vishwambhar Rathi with whom I had worked closely during the course of my Ph.D.

I would like to thank Dr. Prabir Daripa, Dr. Scott Miller, Dr. Henry Pfister, and Dr. Alexander Sprintson for agreeing to be on my committee and for their interest in this work.

A warm thanks to Abhiram, Hari, Janath, Jing, Karthik, Makesh, Mothi, Nitin, Parimal, Raja, Salim, Vivek, and other graduate students in the WCL lab for several discussions on research, their feedback on papers, and for making graduate school fun. Thanks also to all my friends at TAMU, especially Abhiram, Mayank, and the friends at AID for being my family in College Station.

Finally, I would like to thank my parents, my brother Jitu, my wife Reena, and my close friend Shiva for their love and support.

TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

FIGURE Page

FIGURE Page

CHAPTER I

INTRODUCTION

In this dissertation we consider the problem of communicating a discrete time analog source S over a channel with as low distortion as possible. Here the channel refers to the communication medium. For the presentation here, it will be represented by a conditional probability $P_{Y|X}(y|x)$ which denotes the probability of receiving Y given that X was transmitted. The distortion measure is denoted by $d(S, \hat{S})$ where $\hat{S}$ is the reconstruction of the source at the receiver.

A channel is said to be ergodic if the conditional distribution of the received signal observed over time is same as $P_{Y|X}$. Consider two channels

$$\text{CH1}: \; Y_k = h_k X_k + N_k \tag{1.1}$$

and

$$\text{CH2}: \; Y_k = h X_k + N_k \tag{1.2}$$

where X_k and Y_k denote the transmitted and received signal at time index k, h_k's are independent and have same distribution as h, and N_k is additive white Gaussian noise (AWGN). For both CH1 and CH2 $P_{Y|X}$ is the same. However, CH1 is ergodic while CH2 is non-ergodic.

For ergodic channels, if we allow for infinite delay, the problem of communicating the analog source over the channel while minimizing distortion has been completely solved by Shannon. His famous source channel separation theorem states that the problem can be divided into two parts. The first part, source coding, is to compress the source stream S into a bit stream B at a suitably chosen rate R_s (bits per source

---

The journal model is *IEEE Transactions on Automatic Control.*

sample) such that $d(S;\hat{S})$ is minimized where $\hat{S}$ is the reconstruction of S from B. This minimum distortion is given by the distortion rate function

$$D_{DRF}(R_s) = \inf_{P(\hat{S},S):I(\hat{S};S)=R_s} d(S;\hat{S}) \tag{1.3}$$

where $I(U;V) = E\left[\log \frac{P(u,v)}{P(u)P(v)}\right]$ denotes the mutual information between U and V. The second part, channel coding, is to send the bit stream B reliably over the channel. This is possible if and only if the channel coding rate R_c (bits per channel use) is less than the capacity of the channel that is given by

$$C = \max_{P_X(x)} I(X;Y). \tag{1.4}$$

The performance limit is obtained when $R_c = C$. If we are allowed b channel uses per source sample then $R_s = bR_c$.

Although separation based schemes are optimal for ergodic channels, in several cases there are joint source channel coding schemes that are also optimal, have much lesser computational complexity, and have a lower latency. For example, if K samples of a Gaussian source have to be transmitted over K uses of an AWGN channel with the distortion metric being mean square error, a simple joint source channel coding scheme that involves just transmitting the source over the channel with appropriate power scaling is optimal [1, 2]. Some advantages and disadvantages of joint source channel coding are discussed in [3].

The advantages of joint source channel coding schemes become more apparent when we consider non-ergodic channels which is the case considered in this work. For non-ergodic channels, the instantaneous capacity C (mutual information corresponding to the current conditional distribution of the received signal) is a random variable. When we use a separation based scheme with a channel code of rate R_c for a non-ergodic channel, the transmitted message cannot be recovered when $C < R_c$. When $C > R_c$, although the transmitted message is reliably recovered, the amount of information transmitted is lesser than that supported by the channel. Due to these reasons it can be shown that separate source and channel coding, in general, is sub-optimal.

Finding the optimal joint source-channel coding scheme for non-ergodic channels is a challenging problem even for some very simple looking setups. In this dissertation, we consider three different problems that are all motivated from applications like video broadcast that involve transmitting an analog source over a multiple-input multiple-output (MIMO) channel. The problems differ in the way the channel and the associated side information at the transmitter is modelled.

A. Channel Models

P1 In the first problem, we consider a single-input single-output (SISO) additive white Gaussian noise (AWGN) channel whose signal-to-noise ratio (SNR), denoted by snr, is unknown at the transmitter. The received signal Y when the transmit signal is X is given by

$$Y = \sqrt{snr}\ X + N \tag{1.5}$$

where $N \sim \mathcal{N}(0, 1)$. This model closely resembles quite a few practical applications. To name a few: a wired cable network where users tap signals from different locations on the wires; stationary users in a wireless network with a strong line-of-sight component. Since SNR is not known at the transmitter, this channel is non-ergodic.

P2 In the second problem, we consider an application like video conference where an $M \times N$ MIMO channel is used to transmit an analog source to a particular user.

In this case, the wireless channel is often modelled as a Rayleigh fading channel given by

$$Y = \sqrt{snr}\ \mathbf{H}X + N \tag{1.6}$$

where: N is circularly symmetric complex Gaussian noise $N \sim \mathcal{CN}(0, I_{M\times M})$, and $\mathbf{H}$, the $N \times M$ channel gain matrix, is such that its elements $h_{i,j}$ are independent and identically distributed (i.i.d.) Gaussian random variables. We assume that the SNR is known at the transmitter but the channel realization $\mathbf{H}$ is unknown. This assumption is justified since we can use feedback to learn the SNR but the fading realization $\mathbf{H}$ changes with time. Due to the stringent delay constraints in video broadcast, the coding has to be done over a short block and so we model the channel as quasi static. That is, although $\mathbf{H}$ changes with time, we assume it is constant for the entire block and then changes randomly to a new value in the next block. We also consider extensions to block fading channel where the delay constraints allow us to code over multiple blocks of the quasi static fading channel. Since $\mathbf{H}$ is unknown at the transmitter this channel is non-ergodic.

P3 In the third problem, we again model the channel as a MIMO Rayleigh fading channel but here we assume that the transmitter does not know the SNR and the fading realization. This models applications like video broadcast over a wireless channel with many users with different SNR's. In such applications it is not practical to learn the SNR of every user. The system would hence be required to work well over a wide range of SNR.

In all the problems we assume that the fading realization and the SNR are known at the receiver.

B. Performance Metric

In this work we assume that the analog source can be modelled as a sequence of i.i.d Gaussian random variables. Also the distortion measure we use is mean square error

$$D_{\mathcal{SC}}(snr) = E[(S - \hat{S})^2] \tag{1.7}$$

where $\hat{S}$ is the estimate of S and the expectation is taken over the source sequence, noise variance, and the fading realization (for P2 and P3). The subscript $\mathcal{SC}$ denotes the joint source channel coding scheme used to transmit S. Note that with this definition, D is a function of snr. The assumptions on the source and the distortion measure makes the analysis tractable and the insights obtained apply to many other source-distortion measure pairs.

We are interested in studying how $D(snr)$ changes with snr. The traditional approach to study this problem has been to fix some values of snr that are of interest and study the achievable distortion region [4, 5]. For example, consider two SNR's say snr_1 and snr_2 and study the achievable distortion pair $D(snr_1)$ and $D(snr_2)$. Although this is probably the most rigorous way to study the problem, characterizing this region is a difficult problem and the exact region is not known even for the two SNR's case. From this approach it is also not clear how the performance improves if we increase the power since the "best" schemes obtained using this approach for a particular power level could be very different from the best scheme with an increased power level.

The performance metric we use is the distortion SNR exponent. The distortion SNR exponent of a scheme is given by

$$a_{\mathcal{SC}}(b) = \lim_{snr \to \infty} -\frac{\log D_{\mathcal{SC}}(snr)}{\log snr}. \tag{1.8}$$

This basically implies that in the high SNR regime,

$$D(snr) \approx snr^{-a_{\mathcal{SC}}(b)}. \tag{1.9}$$

We try to characterize the distortion SNR exponent of the channel which is the maximum achievable distortion SNR exponent and is given by

$$a(b) = \max_{\mathcal{SC}} a_{\mathcal{SC}}(b) \tag{1.10}$$

where the supremum is taken over all schemes. The definition of the distortion SNR exponent for the cases when the SNR is known at the transmitter is slightly different from the case where SNR is not known at the transmitter. We define them precisely in the following chapters.

Although this metric helps us identify schemes that are guaranteed to be good asymptotically (as $snr \to \infty$), as seen in many other problems where such asymptotic metrics are used (for example code design based on the diversity multiplexing tradeoff [6]), the identified schemes usually are good candidates for practical SNR's. Some examples, with performance for practical SNR's for the identified schemes, are shown for the AWGN channel in chapter II.

C. Overview of Results

Our approach to study these problems has been to compute the achievable exponent for different schemes and compare it with upper bounds that are obtained by assuming that the transmitter knows the SNR and the fading realization. Most of the schemes we study are layered source channel coding schemes, i.e., schemes in which the source is first compressed in layers of rate $R_{s,i}$ such that the distortion D_i when the first i layers are used for reconstruction is close to the minimum distortion corresponding to a source coding rate of $\sum_{j=1}^{i} R_{s,i}$. The source coded layers are then channel coded (a different channel code assigned for each layer) and sent over the channel with higher protection for lower layers. At the receiver, reconstruction is performed from all the source coding layers that have been perfectly recovered. Note that these are joint source channel coding schemes that are purely digital and that the separation based scheme is a special case of these schemes.

Some observations and our key results are summarized below:

- For the problems studied in this dissertation, we found that layered source channel coding schemes perform as well and better in some cases than all other currently known joint source channel coding schemes.

- We proved that the largest achievable distortion SNR exponent of the AWGN channel when the SNR is unknown at the transmitter is equal to the bandwidth expansion b. This is also the optimal rate of decay when the SNR is known at the transmitter and hence, for the AWGN channel, knowledge of SNR is not required in order to be optimal in terms of the distortion SNR exponent.

- For the $M \times N$ MIMO L-block Rayleigh fading channel where the SNR is known at the transmitter but the fading realization is unknown, we derived the optimal distortion SNR exponent corresponding to two schemes - the Broadcast Scheme (BS) and the Layered Scheme with Broadcast Layer at the end (LSBLEND), and an achievable exponent for a third scheme called the Box scheme. These exponents are currently the best known achievable exponent for this problem. The schemes are shown to be optimal for $b < (|N-M|+1)/\min(M,N)$ where the exponent is $\min(M,N)b$ and for $b > MNL^2$ where the exponent is MNL.

- For the $M \times N$ MIMO L-block Rayleigh fading channel where the SNR and the fading realization is not known at the transmitter we derive an achievable distortion SNR exponent for a superposition scheme. This exponent coincides with the exponent of the broadcast scheme which is the best superposition scheme that depends on SNR. The scheme is hence the optimal superposition scheme. Furthermore, for $b < (|N-M|+1)/\min(M,N)$ and for $b > MNL^2$ it is also an optimal scheme. Hence, for these values of b, knowledge of SNR is not necessary to be optimal in terms of the distortion SNR exponent.

- We defined the diversity versus decodable rate tradeoff that is useful for studying digital data transmissions and specified some achievable tradeoffs for the MIMO Rayleigh fading channel whose SNR is unknown at the transmitter. For the MIMO Rayleigh fading channel, with this tradeoff, we were able to identify a SNR and fading independent superposition scheme that has, in the high SNR regime, nearly the same throughput as the optimal scheme where the transmitter knows the fading and the SNR.

D. Organization of the Dissertation

Each chapter in this dissertation addresses problems that correspond to a particular assumption on the channel model. Since readers might be interested in only a particular channel model, the chapters are written so that they are self-contained. In chapter II we study the distortion SNR exponent corresponding to the AWGN channel. We also study extensions of this problem to the case when performance guarantees are required at a certain SNR in section H. In chapter III we study the distortion SNR exponent for the Rayleigh fading MIMO channel where SNR is known at the transmitter. The exponents for BS, LSBLEND, and Box scheme are derived for the quasi static fading case in section D, E, and F respectively, while extensions of these results to the block fading case are presented in section G. In chapter IV we derive some achievable distortion SNR exponent for the MIMO channels with schemes that do not depend on SNR. The diversity versus decodable rate tradeoff which is used to obtain the distortion SNR exponent is also defined and discussed here.

CHAPTER II

DISTORTION SNR EXPONENT FOR THE AWGN CHANNEL

In this chapter we consider joint source channel coding schemes for the AWGN channel with SNR unknown at the transmitter. We show that the achievable distortion SNR exponent for the AWGN channel is equal to the bandwidth expansion b. We also show that it is possible to be close to the optimal performance at some specified SNR snr_0 and still obtain a distortion SNR exponent of b. The results presented in this chapter have appeared in [7, 8].

The chapter is organized as follows. We introduce the distortion SNR exponent problem in section A. An upper bound on the exponent is derived in section B. Prior work related to this problem is discussed in section C. The main results are summarized in section D. The proposed scheme is described in section E and its performance is analyzed in section F. Some examples are given in section G. In section H we show how the scheme can be modified to meet certain distortion requirement at snr_0 and still obtain the optimal rate of decay at high SNR and finally we conclude in section I.

A. Introduction and Problem Statement

In this chapter, we consider the problem of conveying K samples of a Gaussian source over T uses of an AWGN channel with the minimum possible distortion. The distortion measure of interest is the mean square error.

We assume that the signal-to-noise ratio (SNR), snr, of the channel is unknown at the transmitter and known at the receiver. We are interested in schemes that have the best asymptotic performance, i.e., with $snr \to \infty$. Corresponding to a fixed bandwidth expansion $b = T/K$ and a particular scheme S, we use $D_S(snr)$ to denote

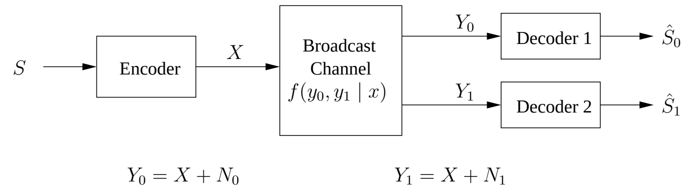

Fig. 1. Two User Broadcast Problem. $N_i \sim \mathcal{N}(0, 1/snr_i)$ and $D_i = E\left[\left(S - \hat{S}_i\right)^2\right]$ for $i = 0, 1$. $snr_1 > snr_0$.

the mean-square error when the SNR is snr. We define the distortion SNR exponent of this scheme as

$$a_S^{fixed}(b) = -\lim_{snr \to \infty} \frac{\log D_S(snr)}{\log snr}. \tag{2.1}$$

This implies that we can find a constant c such that $D_S(snr) \leq c \cdot snr^{-a_S^{fixed}(b)}$. Our aim is to characterize the distortion SNR exponent of the channel given by

$$a_*^{fixed}(b) = \sup a_S^{fixed}(b) \tag{2.2}$$

where the supremum is taken over all possible schemes S. We use the superscript fixed to differentiate between the distortion SNR exponent of schemes that could depend on the SNR which is discussed in chapter III with that for schemes that don't depend on SNR which is the case here.

In many applications, we require that the system meets a certain distortion requirement at the minimum operating SNR and offers performance improvements as the SNR increases. To model this requirement, we consider a generalization of the distortion SNR exponent problem, the two user broadcast problem shown in Fig. 1. In this setup, we have an AWGN broadcast channel with two users having an SNR of snr_0 and snr_1 respectively with $snr_1 > snr_0$. We assume that snr_0 is known at the transmitter while snr_1 is not known at the transmitter. We denote the achievable distortion pair by $D_0 = D(snr_0)$ and $D_1 = D(snr_1)$ for the weak user and the strong user respectively. For a specified distortion requirement at snr_0, $D_0 \leq d_0$, we are interested in characterizing the distortion SNR exponent corresponding to D_1.

B. Informed Transmitter Upper Bound

A simple upper bound on the exponent can be obtained by considering the case when the SNR is available at the transmitter. In this case, Shannon's separations theorem applies, and hence separate source and channel coding is optimal. The maximum channel coding rate supported by the channel is given by

$$R_c = \frac{1}{2}\log_2(1 + snr). \tag{2.3}$$

Since we have T uses of the channel, we can convey at most $\frac{T}{2}\log_2(1 + snr)$ bits. Therefore, the maximum source coding rate is

$$R_s = \frac{T}{2K}\log_2(1 + snr) = \frac{b}{2}\log_2(1 + snr). \tag{2.4}$$

The distortion rate function $D_{DRF}(R)$ for the Gaussian source is $D_{DRF}(R) = 2^{-2R}$. Hence, the minimum possible distortion is

$$D_{min}(snr) = 2^{-2R_s} = \frac{1}{(1 + snr)^b}. \tag{2.5}$$

If, for any scheme $\mathcal{S}$, $a_{\mathcal{S}}^{fixed}(b) > b$, then, at some high enough SNR, the distortion obtained using scheme S will be lesser than D_{min} which is impossible. Therefore,

$$a_*^{fixed}(b) \leq b. \tag{2.6}$$

We will refer to this bound as the informed transmitter upper bound since this bound is obtained by informing the transmitter about the SNR.

C. Prior Work

In [9], Ziv showed that for practical modulation schemes with peak power limitations and some continuity constraints the mean-square error cannot decay faster than $1/snr^2$, i.e., $a_*^{fixed}(b) \leq 2$.

Reznic, Feder, and Zamir [5] considered the problem of finding the achievable distortion pair (D_0, D_1) for the two user broadcast problem. For the bandwidth expansion case, i.e. with $b > 1$, for a scheme that is optimal at snr_0, i.e., $D_0 = D_{min}(snr_0)$, they showed that the distortion D_1 cannot decay at a rate faster than $1/snr_1$.

In [10], Santhi and Vardy considered the problem of transmitting a uniform source over the AWGN channel. They proposed a superposition scheme where each superposition layer contained refinement information about the source. They showed that the superposition scheme achieves a rate of decay of $1/(1+snr)^B$ for the mean-square error, where $B \geq 1$ is an integer. B is related to the bandwidth expansion by the equation $B = bR$ where R is the rate of a code used in their construction and can be chosen to be any value between 0 and 1 such that $B = bR$ is an integer. Therefore, in this case, the MSE can be made to decay as $1/(1+snr)^{\lfloor b \rfloor}$. The informed transmitter upper bound on the distortion SNR exponent for the uniform source is also b and therefore for integer values of b the scheme in [10] achieves the optimal exponent.

A special case of this problem, when the bandwidth expansion is 1, the optimal scheme is to transmit the source samples on the channel after scaling them to meet the power requirement [1]. This simple uncoded scheme achieves a distortion of $1/(1 + snr)$ without knowing the SNR, which is identical to that achievable with a separation based scheme when the SNR is known at the transmitter. A generalized study of when uncoded schemes are optimal is performed in [2] where the authors derive the necessary and sufficient conditions involving the source distribution, the distortion measure, the channel cost function, and the channel conditional distribution for uncoded transmissions to be optimal.

D. Main Results

We propose a layered joint source channel coding scheme where the layered channel coding scheme is a superposition scheme. By deriving a suitable rate and power allocation for this layered scheme we obtain the following results.

- For the superposition scheme we show that the exponent $a(b)$ can be made arbitrarily close to b which is also the upper bound. Therefore $a_*^{fixed}(b) = b$ for the AWGN channel.
  - Extends results of [10] to the Gaussian source.
  - Unlike in [10], the result derived here also applies for non-integer values of b.
- For the two user broadcast problem, using the superposition coding scheme, we can achieve $D(snr_0) \leq \alpha D_{min}(snr_0)$ for any constant $\alpha > 1$, and obtain a decay rate arbitrarily close to $1/snr_1^b$ for D_1. Therefore, $a_*^{fixed}(b) = b$.

E. Proposed Scheme : Layered Fixed Rate Scheme

A schematic of the proposed scheme is shown in Fig. 2. The scheme has two blocks. The first block is a layered source coder in which the source is compressed in layers of rates $R_{s,1}, R_{s,2}, \ldots, R_{s,N_s}$. Let $D_L(i)$ denote the distortion when the reconstruction is performed using the first i layers. For a successively refinable source it is possible to

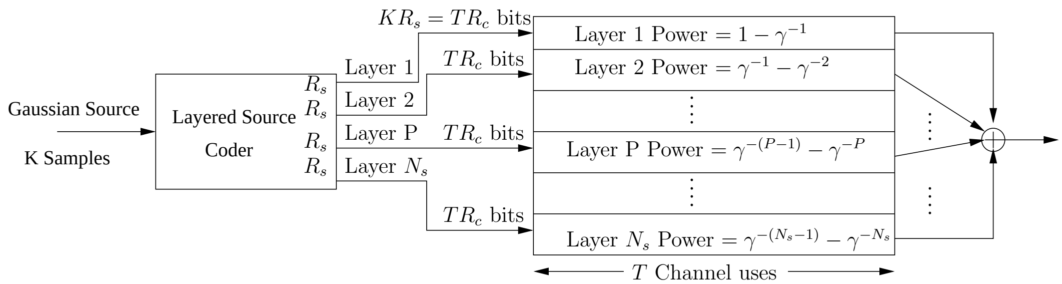

Fig. 2. Layered Fixed Rate Scheme.

compress in layers such that $D_L(i)$ is exactly equal to the $D_{DRF}(R_{s,1}+R_{s,2}+\ldots+R_{s,i})$. Here, $D_{DRF}(R)$ denotes the distortion rate function, that is, the minimum achievable distortion using any rate R source code. The Gaussian source with the mean square error distortion measure is successively refinable and in this case $D_L(i) = 2^{-2\sum_{j=1}^{i} R_{s,i}}$. Let m_i denote the $KR_{s,i}$ bits in the ith source coding layer.

The second block is a superposition based channel coding scheme. The transmitted signal X is chosen as

$$X = \sum_{i=1}^{N_s} \sqrt{P_i} X_i \tag{2.7}$$

where X_i is the codeword in a Gaussian codebook C_i of rate $R_{c,i}$ that corresponds to the message m_i. Note that the codebook used for different superposition layers are independent but have the same length T. The rate $R_{c,i}$ should satisfy $R_{c,i} = KR_{s,i}/T$ since the message m_i from the ith source coding layer is used to select the codeword transmitted in the ith superposition layer.

In our scheme, we choose $P_i = \gamma^{-i+1} - \gamma^{-i}$ with $\gamma > 1$ and $R_{c,i} = R_c = \frac{1}{2}\log\frac{1+a}{1+a/\gamma}$ for all i with $a > 0$. Hence we have $R_{s,i} = R_s = \frac{b}{2}\log\frac{1+a}{1+a/\gamma}$. We will also let $N_s = \infty$.

We assume that the decoding is performed using successive interference cancellation. That is, the first layer is decoded by treating all other layers as additional noise.

If the decoding is successful, its contribution is removed from the received signal and the second layer is decoded by treating layers 3 to N_s as noise and so on. The source is then reconstructed from the layers that were successfully recovered.

The motivation for choosing the rate and power allocation as specified is that such a choice ensures that i superposition layers are decoded when the SNR is $a\gamma^i$.

F. Performance Analysis

Lemma F.1 *For the superposition scheme with $P_i = \gamma^{-i+1} - \gamma^{-i}$, $\gamma > 1$, $R_{c,i} = R_c = \log \frac{1+a}{1+a/\gamma}$, and $N_s = \infty$, when the SNR is between $a\gamma^{i-1}$ and $a\gamma^i$, only the first i layers are decoded.*

We start by proving the following claim. The least SNR where the ith layer can be decoded is $snr_i = a\gamma^{i-1}$ if we assume that the previous layers (layers 1 to $i-1$) can be decoded at snr_i. Since the ith layer is decoded by removing the contribution of layers 1 to $i-1$ from the received signal and by treating the layers $i+1$ to N_s as additional noise, the ith layer is decoded at snr_i if

$$\begin{aligned} R_{c,i} &\leq \frac{1}{2}\log\left(1 + \frac{snr_i P_i}{1 + snr_i \sum_{j=i+1}^{N_s} P_j}\right) \\ &= \frac{1}{2}\log\frac{1 + snr_i \sum_{j=i}^{N_s} P_j}{1 + snr_i \sum_{j=i+1}^{N_s} P_j} \qquad (2.8) \\ &= \log\frac{1 + a\gamma^{i-1}\cdot a\gamma^{-i+1}}{1 + a\gamma^{i-1}\cdot a\gamma^{-i}} \qquad (2.9) \\ &= \log\frac{1+a}{1+a/\gamma}. \qquad (2.10) \end{aligned}$$

Since $R_{c,i} = R_c = \frac{1+a}{1+a/\gamma}$, (2.10) is satisfied with equality, which also implies that snr_i is the smallest SNR at which layer i can be decoded.

Now to complete the proof of the lemma, from the previous claim we see that layer 1 is decoded at $snr_1 = a$. Applying the claim for $i = 2$, we see that Layer 2 can be decoded at $snr_2 = a\gamma$ provided that layer 1 can be decoded at snr_2. Since $snr_2 > snr_1$, this is indeed the case. Proceeding in this manner we see that layer i can be decoded at $a\gamma^{i-1}$. This proves the lemma.

We have assumed in the above calculation that $N_s = \infty$. For the finite layers case, the decoder is again guaranteed to recover i layers when the snr is between $a\gamma^{i-1}$ and $a\gamma^i$ but it may be able to recover more than i layers.

Theorem F.1 *In the limit $\gamma \to 1$, the superposition scheme satisfies*

$$D(snr) \leq (snr/a)^{-b\frac{a}{a+1}} \tag{2.11}$$

The statement is trivially satisfied for $snr < a$ since the distortion is always less than the source variance.

For snr between $a\gamma^{i-1}$ and $a\gamma^i$, the decoder recovers i layers (Lemma F.1) which corresponds to a source coding rate of $iR_s = \frac{ib}{2} \log \frac{1+a}{1+a/\gamma}$. The corresponding distortion is 2^{-2iR_s}. Therefore for $a\gamma^{i-1} \leq snr < a\gamma^i$ we have

$$\begin{aligned} -\frac{\log D(snr)}{\log(snr/a)} &= \frac{2iR_s}{\log(snr/a)} && (2.12) \\ &\geq \frac{2iR_s}{i \log \gamma} \quad \text{(Since } snr/a < \gamma^i\text{)} && (2.13) \\ &= b\frac{\log \frac{1+a}{1+a/\gamma}}{\log \gamma} && (2.14) \end{aligned}$$

Applying the result for all $i \geq 1$ we have for all $snr > a$

$$-\frac{\log D(snr)}{\log(snr/a)} \geq b\frac{\log \frac{1+a}{1+a/\gamma}}{\log \gamma} \tag{2.15}$$

In the limit $\gamma \to 1$ we have

$$-\frac{\log D(snr)}{\log(snr/a)} \geq \lim_{\gamma\to 1} b\frac{\log\frac{1+a}{1+a/\gamma}}{\log\gamma} \tag{2.16}$$

$$= \lim_{\gamma\to 1} b\frac{-\frac{1}{1+a/\gamma}\cdot\frac{-a}{\gamma^2}}{1/\gamma} \quad \text{(L'Hospital's Rule)} \tag{2.17}$$

$$= b\frac{a}{a+1} \tag{2.18}$$

This proves the Theorem.

Theorem F.2 *The distortion SNR exponent of the AWGN channel is* $a_*^{fixed}(b) = b$.

For the superposition scheme $a_S^{fixed}(b) = b\frac{a}{a+1}$. By choosing a large a we can make $a_S^{fixed}(b)$ approach b. Recall that the informed transmitter upper bound on the distortion SNR exponent (2.6) obtained by assuming that the signal-to-noise ratio is known at the transmitter is also b. This proves the Theorem.

G. Examples

In Fig. 3 we plot $\log D$ against SNR for $\gamma = 3.12$. We see that the curve is like a staircase. This happens since $\log D$ decreases by $2R_s$ when the SNR increases by a factor of γ (SNR increases by $10\log_{10}\gamma$ dB). In Fig. 4 we plot distortion versus SNR for different values of a while choosing $\gamma = 1.01$. In this case, the step size is so small that we virtually get a straight line. We also see that as we increase a we get better slopes but the curves also shift away from $D_{min}(snr)$. This matches the behavior predicted by Theorem F.1.

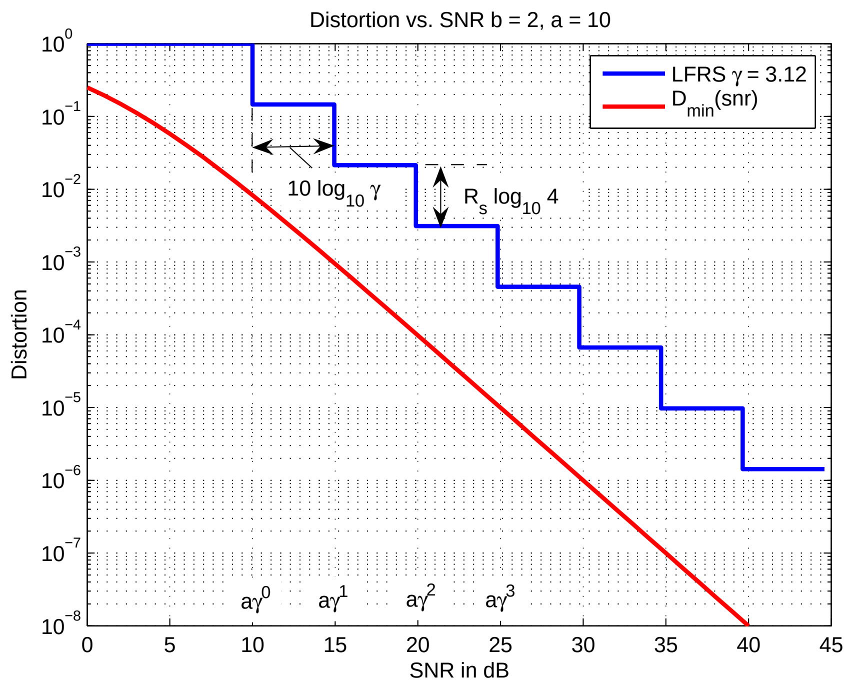

Fig. 3. $D(snr)$ versus snr for the Layered Fixed Rate Scheme.

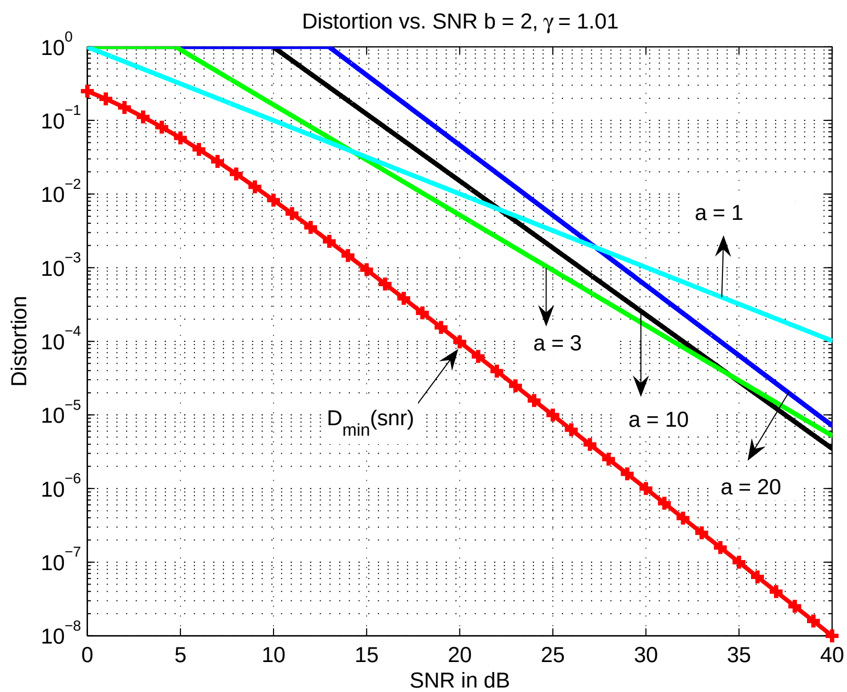

Fig. 4. $D(snr)$ versus snr for Proposed Layered Fixed Rate Scheme for Different Values of a.

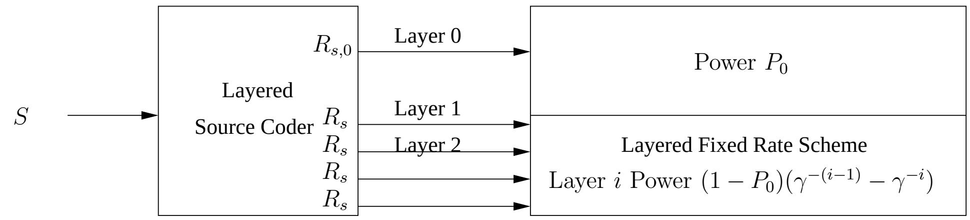

Fig. 5. Proposed Scheme for the Two User Broadcast Problem.

H. Broadcast Problem

In the following Theorem we consider the two user AWGN broadcast channel problem and characterize the achievable distortion SNR exponent for the stronger user given a desired distortion limit for the weaker user. That is, we characterize the maximum possible decay rate of D_1, $\max \lim_{snr_1\to\infty} -\frac{\log D_1}{\log snr_1}$, given that $D_0 = D(snr_0) \leq \alpha D_{min}(snr_0) = \alpha \cdot (1+snr_0)^{-b}$. Note that α has to be greater than 1 for $D(snr_0) \leq \alpha D_{min}(snr_0)$ to be possible.

Theorem H.1 *For the two user broadcast problem, D_1 can be made to decay as $1/snr_1^b$ while meeting the constraint $D_0 \leq \alpha D_{min}(snr_0)$ for any $\alpha > 1$.*

Consider the transmitted signal to be

$$X = \sqrt{P_0}x_0 + \sqrt{1-P_0}x_1 \tag{2.19}$$

where x_0 contains information for the weaker user and $x_1 = \sum \sqrt{P_i}X_i$ is the transmitted signal similar to that in the superposition scheme proposed earlier. The encoder uses a successively refinable source coder and maps the information to the codebook corresponding to x_0 followed by the layers in x_1. The decoder uses successive interference cancellation to decode the layers and decodes in the order x_0, X_1, X_2, and so on.

The maximum rate supported in Layer 0 when SNR is snr_0 is $R_{c,0} = \frac{1}{2}\log(1 + \frac{P_0 snr_0}{snr_0(1-P_0)+1})$. This corresponds to a source coding rate of $R_{s,0} = bR_{c,0}$. For the distortion $D_0 \leq \alpha D_{min}(snr_0)$, we have $2^{-2R_{s,0}} \leq \alpha D_{min}(snr_0)$. Therefore,

$$2^{-2bR_{c,0}} \leq \frac{\alpha}{(1+snr_0)^b} \tag{2.20}$$

$$\Rightarrow \left(1 + \frac{P_0 snr_0}{1 + snr_0(1-P_0)}\right)^{-b} \leq \frac{\alpha}{(1+snr_0)^b} \tag{2.21}$$

$$\Rightarrow 1 + snr_0(1-P_0) \leq \alpha^{1/b} \tag{2.22}$$

$$\Rightarrow 1 - P_0 \leq \frac{\alpha^{1/b}-1}{snr_0}. \tag{2.23}$$

We choose P_0 to be $\max(0, 1 - \frac{\alpha^{1/b}-1}{snr_0})$. For values of α that lead to $P_0 = 0$, i.e., $\alpha > (1+snr_0)^b$, the constraint on D_0 is trivially satisfied. For $\alpha < (1+snr_0)^b$, with the specified choice of P_0, $D_0 = \alpha D_{min}(snr_0)$.

The channel coding rate for the layers in x_1 are chosen identical to that used for the superposition scheme specified in section E, $R_{c,i} = \frac{1}{2}\log\frac{1+a}{1+\frac{a}{\gamma}}$. If the SNR of the stronger user, snr_1, is $a\gamma^{i-1}/(1-P_0)$, then the receiver can decode $x_0, X_1, \ldots, X_i$ provided that Layer 0 can be decoded at this SNR, equivalently $a\gamma^{i-1}/(1-P_0) > snr_0$. In this case, the decoder recovers a source coding rate $R_0 + iR_s$. We can then show that in the limit $\gamma \to 1$, for $snr > \max(a/(1-P_0), snr_0)$ we have

$$D(snr) \leq \alpha D_{min}(snr_0)((1-P_0)snr/a)^{-b\frac{a}{a+1}}. \tag{2.24}$$

Note that for $snr < snr_0$, $D(snr) = 1$, and for snr between snr_0 and $\max(a/(1-P_0), snr_0)$, $D(snr) \leq \alpha D_{min}(snr_0)$.

In the high SNR regime, we can make $D(snr)$ decay at a rate arbitrarily close to snr^{-b} by choosing a sufficiently large a. This therefore proves the Theorem.

This theorem implies that we can get arbitrarily close to the optimal distortion at any specified SNR, snr_0, while still getting a rate of decay arbitrarily close to

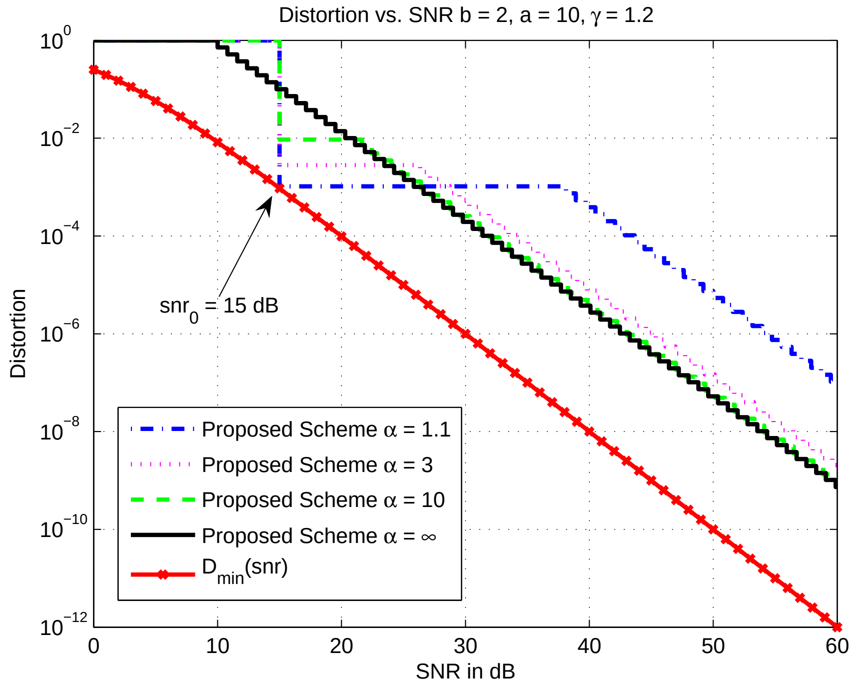

Fig. 6. $D(snr)$ vs snr for the Two User Broadcast Problem with SNR of Weaker User $snr_0 = 15$dB.

b. This contrasts sharply with the results in [5] where the authors showed that if $D_0 = D_{min}(snr_0)$ then D_1 cannot decay at a rate faster than $1/snr_1$.

The distortion obtained using the superposition scheme for the AWGN broadcast channel with $b = 2$ is shown in Fig. 6. We see that the superposition scheme has a rate of decay quite close to 2 which is the best possible decay rate. We also see that it is possible to be very close to the minimum possible distortion for the weaker user and still achieve a decay rate close to the optimal decay rate for the stronger user.

I. Concluding Remarks

We presented a simple superposition scheme that achieves the optimal distortion SNR exponent of b for the AWGN channel with a bandwidth expansion factor of b. This extends the results of [10] to the Gaussian source and to non-integer values of bandwidth expansion. The improvement on the earlier results by Ziv was obtained by removing the constraints on the encoder. We also showed that for the two user broadcast problem, it is possible to be arbitrarily close to the optimal distortion for the weaker user and still obtain a distortion SNR exponent of b for the stronger user. We believe that the results by Reznic *et al.* were pessimistic since they required the distortion of the weaker user to be exactly equal to its optimal value.

CHAPTER III

DISTORTION SNR EXPONENT FOR THE MIMO CHANNEL WITH SNR KNOWN AT THE TRANSMITTER

In this chapter we find achievable distortion SNR exponent for the block fading MIMO Rayleigh fading channel when the SNR is known at the transmitter. We study three different layered schemes namely the broadcast scheme (BS), the layered scheme with broadcast layer at the end (LSBLEND), and the box scheme. We show that the achievable exponent obtained using these schemes is better than the exponents reported in literature so far and that these schemes are optimal for certain range of the bandwidth expansion factor. The results reported in this chapter have appeared in [11, 12].

The chapter is organized as follows. We introduce the problem in Section A and discuss related prior work in section B. The main results presented in this chapter are summarized in section C. Achievable exponents for the three schemes BS, LSBLEND, and Box scheme are derived in section D, E, and F respectively for the MIMO quasi static fading case. Extensions of these results to the block fading case are discussed in section G. Some examples are shown in section H after which we conclude in section I.

A. Introduction and Problem Statement

In this chapter we consider the problem of transmitting K samples of a complex Gaussian source in $T = bK$ uses of an $M \times N$ MIMO channel with block fading where b, the bandwidth expansion factor, is the ratio of the channel bandwidth to the source bandwidth. The channel is given by

$$\mathbf{y}_t = \sqrt{\frac{\rho}{M}}\mathbf{H}_{\lceil \frac{Lt}{T} \rceil}\mathbf{x}_t + \mathbf{w}_t, \quad t = 1, \ldots, T \tag{3.1}$$

where: T is the duration (in channel uses) of the transmitted block; $\mathbf{H}_l \in \mathbb{C}^{N \times M}$, $l = 1, \ldots, L$, is the channel matrix for $\frac{(l-1)T}{L} < t \leq \frac{lT}{L}$ containing random i.i.d. elements $h^l_{i,j} \sim \mathcal{CN}(0, 1)$ (Rayleigh independent fading). The channel matrix for different blocks are independent; $\mathbf{x}_t$ is the transmitted signal at time t; the transmitted codeword, $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_T]$, is normalized such that $\mathrm{tr}(\mathbb{E}[\mathbf{X}^{\mathsf{H}}\mathbf{X}]) \leq MT$; $\mathbf{w}_t \sim \mathcal{CN}(0, I^{M \times M})$ is additive white Gaussian noise; ρ denotes the Signal-to-Noise Ratio (SNR), defined as the ratio of the average received signal energy per receiving antenna to the noise per-component variance. We also define $m = \min\{M, N\}$ and $n = \max\{M, N\}$.

In this chapter we assume that the SNR ρ is known at the transmitter and the receiver while the channel state information $\mathbf{H}$ is known only at the receiver. When the channel state information is available at both the transmitter and the receiver, Shannon's separation theorem applies and separate source and channel coding is optimal. However, when the channel state information is available only at the receiver, the separation theorem fails and the optimal scheme requires joint source and channel coding.

Consider a family of joint source-channel coding schemes $\mathcal{SC} = \{SC(\rho)\}$ corresponding to a bandwidth expansion factor of b, where $SC(\rho)$ denotes the scheme that operates at SNR ρ. Corresponding to the coding scheme $SC(\rho)$, let $D_{\mathcal{SC}}(\rho)$ denote the mean square error distortion averaged over the source, the noise, and the channel realization. The *distortion SNR exponent* of the family is defined as the limit

$$a_{\mathcal{SC}}(b) = -\lim_{\rho \to \infty} \frac{\log D_{\mathcal{SC}}(\rho)}{\log \rho}. \tag{3.2}$$

The distortion SNR exponent *of the channel*, denoted by $a^\star(b)$, is the supremum of $a_{\mathcal{SC}}(b)$ over all possible coding families. We are interested in characterizing $a^\star(b)$ for the block fading MIMO channel.

B. Prior Work

The diversity multiplexing tradeoff [6] is closely related to the problem considered here. In [6], Zheng and Tse consider the problem of transmitting digital information over a MIMO fading channel. For a family of coding schemes $\mathcal{C} = \{C(\rho)\}$ whose rate grows as $r \log \rho$, where r is referred to as the multiplexing rate, the diversity order of the family is defined as the limit

$$d_{\mathcal{C}}(r) = -\lim_{\rho \to \infty} \frac{\log P_e(\rho)}{\log \rho} \tag{3.3}$$

where $P_e(\rho)$ denotes the probability of decoding error corresponding to the coding scheme $C(\rho)$. The diversity order of the channel, $d^*(r)$, is the supremum of $d_{\mathcal{C}}(r)$ taken over all possible coding families. In [6], for the Rayleigh fading channel, the diversity order was determined to be

$$d^*(r) = (M-k)(N-k) - (M+N-1-2k)(r-k) \tag{3.4}$$

where $k = \lfloor r \rfloor$ for $0 < r < m$ and 0 for $r > m$.

The distortion SNR exponent problem has been considered previously by many researchers in [13–24]. Distortion SNR exponent was first defined by Laneman *et al.* in [22]. In [22–24] the authors compared the performance of two schemes for parallel fading channels (a) a separation based scheme and (b) a multiple description based scheme where the message sent on each channel corresponded to a description. If the multiplexing rate of the channel code is low the probability of outage is low.

However, the corresponding quantization error is large. When the multiplexing rate is increased quantization error decreases but outage probability increases. For these schemes, the optimal multiplexing rate is chosen such that it maximizes the distortion SNR exponent. Goldsmith and Holliday [20, 21] considered a separation based scheme for the MIMO channel and derived the optimal operating point (multiplexing rate of the channel code) that maximizes the distortion SNR exponent.

An upper bound on $a^{\star}(b)$ was derived by Caire and Narayanan [13–15] and by Gunduz and Erkip [16–19] by assuming that the transmitter is informed of the channel realization $\mathbf{H} = \{\mathbf{H}_1, \ldots, \mathbf{H}_L\}$. In this case, Shannon's separation theorem applies and the optimal distortion is given by $D(\mathbf{H}) = 2^{-bR(\mathbf{H})}$ where $R(\mathbf{H}) = \frac{1}{L}\sum_l \log\det(I + \frac{\rho}{M}\mathbf{H}_l\mathbf{H}_l^H)$. The distortion SNR exponent is then the exponent corresponding to $E_{\mathbf{H}}[D(\mathbf{H})]$. This has been computed in closed form for the Rayleigh fading channel in [13–19] and is given by

$$a^{*}(b) \leq \sum_{i=1}^{m} \min(b, (2i-1+n-m)L). \tag{3.5}$$

Note that this is an upper bound and is not known to be achievable. We refer to this bound as the informed transmitter upper bound.

The schemes by Laneman *et al.* [22–24] and Goldsmith and Holliday [20, 21] are far away from the informed transmitter upper bound. In [13–15], two hybrid digital analog (HDA) scheme were proposed for $b < 1/m$ and $b > 1/m$. For $b < 1/m$, in the HDA scheme, the transmitted signal was chosen to be a superposition of an analog layer with a digital layer. The analog layer is formed by a fraction mb of the source symbols. The remaining source symbols were quantized and transmitted in the digital layer. The scheme was shown to achieve the upper bound for $b < 1/m$. For $b > 1/m$, the HDA scheme involved transmitting in two "time" layers (i.e., two layers multiplexed in time). A digital layer of bandwidth $b - 1/m$ ($T - K/m$ channel uses) was used to transmit the quantized source and the quantization error was transmitted in an analog layer of bandwidth $1/m$. This scheme improved on the exponent obtained by the separation based scheme. However, the gap to the upper bound was still large.

In [16, 17, 19], Gunduz and Erkip proposed a hybrid layering scheme (HLS) that improved on the exponent obtained by the HDA scheme for $b > 1/m$ by allowing for multiple digital time layers instead of the single digital layer of the HDA scheme. They also proposed a broadcast scheme (BS) that involved transmitting a superposition of several digital layers. For the $L = 1$ case, the broadcast scheme was shown to achieve an exponent of MN for $b > MN$ which overlaps with the upper bound and is hence optimal. In this case, for the region $1/m < b < MN$, a characterization of the best achievable distortion SNR exponent is not available. The best known exponents prior to those reported in this chapter are obtained by the hybrid layering scheme and broadcast strategy of Gunduz and Erkip [17]. In [18], Gunduz and Erkip considered the broadcast scheme for parallel channels which corresponds to $M = N = 1$ and $L > 1$ in our model and they showed that the broadcast scheme achieves an exponent of L for $b > L^2$ and is hence optimal. Note that throughout this paper we refer to a superposition coding scheme as a broadcast scheme.

In other related work, Dunn and Laneman [25] consider the distortion to be of the form

$$D \approx C(b) \log(b\rho)^p \rho^{-a(b)} \tag{3.6}$$

and compare several schemes using this approximation.

C. Main Results

The main results presented in this chapter are summarized below.

1. We fully characterize the exponent achievable by any broadcast (superposition)

scheme. An achievable exponent and the corresponding rate and power allocation are specified in Theorem D.1. In Theorems D.2 and D.3, we show that no broadcast scheme can outperform the scheme in Theorem D.1.

2. We show that the broadcast scheme in [17] when used with a different power and rate allocation than that specified in [17] achieves the optimal exponent mb for $b < \frac{n-m+1}{m}$.

3. The broadcast scheme with the proposed power and rate allocation policy achieves the optimal exponent of MNL for $b > MNL^2$. Special cases of this result have been derived in [17] for the $L = 1$ case and in [18] for $M = N = 1$.

4. The proposed power and rate allocation policy for the broadcast scheme becomes identical to that specified in [17] for $MNL - (M + N - 1)L < b < MNL - (M + N - 1)(L - 1)$. For other $b < MNL^2$ the distortion SNR exponent obtained is larger than the broadcast scheme exponent of [17].

5. We propose a time layering scheme in which the last time layer is a broadcast layer, i.e, the last time layer is a superposition of several layers. The distortion SNR exponent obtained using this scheme is shown to be better than the exponent obtained using the HLS scheme of [17]. We refer to this scheme as LSBLEND as an abbreviation for Layered Scheme with a Broadcast Layer at the end.

6. We also propose a layering strategy, termed the Box scheme, which generalizes BS and LSBLEND proposed in this chapter and the strategies considered earlier in [13, 17] by allowing for superposition and time layers simultaneously. All previously known schemes are special cases of the Box scheme and hence the Box scheme performs at least as well as these schemes. However, the optimal

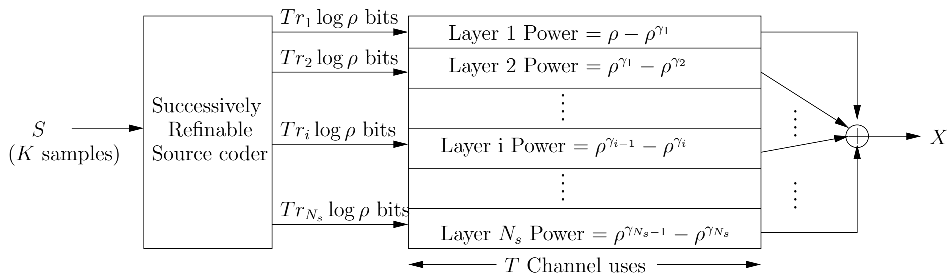

Fig. 7. Broadcast Scheme.

distortion SNR exponent for the Box scheme is difficult to obtain. We present a suboptimal algorithm to compute an achievable distortion SNR exponent. The scheme with the suboptimal algorithm is shown to outperform all previously known schemes, including BS and LSBLEND, for some range of b, whereas, for the considered examples, they are at least as good as previously known schemes for all b.

D. Digital Layering Using Superposition Only

1. Proposed Scheme

Consider the broadcast scheme shown in Fig. 7. The scheme has N_s superposition layers with the ith superposition layer being assigned a power level of $\rho^{\gamma_{i-1}} - \rho^{\gamma_i}$ where ρ is the signal-to-noise ratio and $\gamma_i \geq 0$ is a decreasing sequence with $\gamma_0 = 1$. The source is compressed into N_s layers such that it is successively refinable. The source coding rate in the ith refinement layer is $\frac{Tr_i \log\rho}{K} = br_i \log\rho$. Therefore, if a receiver estimates the source using the first J layers the resulting distortion would be $2^{-\sum_{i=1}^{J} br_i \log\rho} = \rho^{-b\sum_{i=1}^{J} r_i}$. The ith refinement layer is transmitted in the ith superposition layer. Since $Tr_i \log\rho$ bits have to be transmitted in T uses of the channel, the resulting channel coding rate corresponding to the transmission in the ith broadcast layer is $r_i \log \rho$. For mathematical convenience in deriving the expressions, we will assume that in the last layer (layer $N_s + 1$) the remaining power of $\rho^{\gamma_{N_s}}$ is used to transmit Gaussian noise. Therefore, $\gamma_{N_s+1} = 0$ and $r_{N_s+1} = 0$. The channel codes used in the broadcast layers are assumed to achieve the diversity multiplexing tradeoff [6] corresponding to that layer. Here achieving the diversity multiplexing tradeoff refers to achieving an error probability that decays as $\rho^{-d(r)}$ with a coding rate that grows as $r \log \rho$, where $d(r)$ is the optimal diversity multiplexing tradeoff function corresponding to that layer.

At the receiver, the decoder attempts to decode as many layers as it can using successive interference cancellation starting from the first layer. That is, it decodes layer 1 by treating the signal transmitted in layers 2 to N_s as noise. On successful decoding it removes the contribution of layer 1 from the received signal and repeats the process for layer 2 and so on. It then makes an estimate of the source using all the layers it is able to decode.

2. Diversity Multiplexing Tradeoff for the Layers in the Broadcast Scheme

To compute the distortion SNR exponent of the broadcast scheme, we first characterize the diversity multiplexing tradeoff of the broadcast scheme in the following lemma.

Lemma D.1 *If the multiplexing gain in the ith layer of the broadcast scheme is $r_i = k(\gamma_{i-1} - \gamma_i) + \delta$ where $k \in \{0, 1, \ldots, m-1\}$ and $0 \le \delta < \gamma_{i-1} - \gamma_i$, $\gamma_{i-1} > \gamma_i \ge 0$, then, the achievable diversity in the ith layer of the broadcast scheme, assuming that the message transmitted in the previous layers is available at the receiver, is given by*

$$d^\star(r_i, \gamma_{i-1}, \gamma_i) = (m-k)(n-k)\gamma_{i-1} - (m+n-1-2k)\delta. \tag{3.7}$$

That is, if

$$X = \frac{1}{\sqrt{\rho}}\left(\sum_{i=1}^{N_s}\sqrt{(\rho^{\gamma_{i-1}}-\rho^{\gamma_i})}X_i + \sqrt{\rho^{\gamma_{N_s}}}N_1\right), \tag{3.8}$$

where $X_i, N_1 \sim \mathcal{CN}(0, I^{M\times M})$*, is transmitted over a MIMO channel* $Y = \sqrt{\frac{\rho}{M}}\mathbf{H}X + N$*, then the probability of the outage event*

$$\mathcal{A}_i = \{H : I(X_i; Y|\mathbf{H} = H, X_1, \ldots X_{i-1}) < r_i \log\rho\} \tag{3.9}$$

is given by $P(\mathcal{A}_i) \doteq \rho^{-d^\star(r_i,\gamma_{i-1},\gamma_i)}$*. (Here* $A \doteq B$ *is used to denote that* A *and* B *are equal in exponential order, i.e.,* $\lim_{\rho\to\infty}\frac{\log A}{\log\rho} = \lim_{\rho\to\infty}\frac{\log B}{\log\rho}$*.) Note that the term* $\sqrt{\rho^{\gamma_{N_s}}}N_1$ *in* X *is the Gaussian noise transmitted in layer* $N_s + 1$ *and is introduced for mathematical convenience. It should not be confused with noise from the channel.*

We have

$$P(\mathcal{A}_i) = P\left(\log\frac{\det(I + \frac{1}{M}\rho^{\gamma_{i-1}}\mathbf{H}\mathbf{H}^H)}{\det(I + \frac{1}{M}\rho^{\gamma_i}\mathbf{H}\mathbf{H}^H)} < r_i\log\rho\right). \tag{3.10}$$

Let $\lambda_1, \ldots, \lambda_m$ denote the non-zero ordered eigenvalues of $\mathbf{H}\mathbf{H}^H$ with $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_m$. As in [6], let $\alpha_j = -\frac{\log\lambda_j}{\log\rho}$. Therefore, $\alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_m$. Then

$$P(\mathcal{A}_i) = P\left(\log\prod_{j=1}^{m}\frac{1 + \frac{1}{M}\rho^{\gamma_{i-1}-\alpha_j}}{1 + \frac{1}{M}\rho^{\gamma_i-\alpha_j}} < r_i\log\rho\right). \tag{3.11}$$

At high SNR, we obtain $P(\mathcal{A}_i) \doteq P(\mathcal{A}')$ where

$$\mathcal{A}' = \left\{\alpha : \sum_{j=1}^{m}(\gamma_{i-1} - \alpha_j)^+ - \sum_{j=1}^{m}(\gamma_i - \alpha_j)^+ < r_i\right\}. \tag{3.12}$$

Starting from Lemma 3 of [6] and following in the footsteps of [6] we obtain

$$P(\mathcal{A}') = \int_{\mathcal{A}'} p(\alpha)d\alpha \doteq \int_{\mathcal{A}'\cap\boldsymbol{\alpha}^+}\prod_{j=1}^{m}\rho^{(2j-1+n-m)\alpha_j}d\alpha \tag{3.13}$$

for the Rayleigh fading channel. Therefore the outage probability is given by $P(\mathcal{A}_i) \doteq \rho^{-d^\star(r_i,\gamma_{i-1},\gamma_i)}$ where

$$d^\star(r_i, \gamma_{i-1}, \gamma_i) = \inf_{\mathcal{A}' \cap \boldsymbol{\alpha}^+} \sum_{j=1}^{m} (2j - 1 + n - m)\alpha_j. \tag{3.14}$$

For $r_i = k(\gamma_{i-1} - \gamma_i) + \delta$ where $k \in [0, 1, \ldots, m-1]$ and $0 \leq \delta < \gamma_{i-1} - \gamma_i$, the infimum in (3.14) occurs when $\alpha = \alpha^*$ where

$$\alpha_j^* = \begin{cases} \gamma_{i-1}, & 1 \leq j < m - k; \\ \gamma_{i-1} - \delta, & j = m - k; \\ 0, & m - k < j \leq m. \end{cases} \tag{3.15}$$

Substituting α^* in (3.14) we obtain

$$\begin{aligned} & d^\star(r_i, \gamma_{i-1}, \gamma_i) \\ = & \left(\sum_{j=1}^{m-k-1} (2j - 1 + n - m) \right) \gamma_{i-1} + (2(m-k) - 1 + n - m)\,(\gamma_{i-1} - \delta) \\ = & \left(\sum_{j=1}^{m-k} (2j - 1 + n - m) \right) \gamma_{i-1} - (2(m-k) - 1 + n - m)\,\delta \\ = & (m-k) \left(2\frac{m-k+1}{2} - 1 + n - m \right) \gamma_{i-1} - (m + n - 1 - 2k)\delta. \end{aligned}$$

This then gives the desired result.

Note that the probability of the outage event $\mathcal{A}_i$ discussed in lemma D.1 is different from (a) the outage probability of layer i and (b) the outage probability of layer i given layers 1 to $i-1$ are decoded. Rather, it is the probability of outage of layer i with a genie aided decoder where the genie provides the signal that is transmitted in layers 1 to $i-1$.

We will refer to the rate of decay of $P(\mathcal{A}_i)$ with ρ, i.e., $\lim_{\rho\to\infty} \frac{\log P(\mathcal{A}_i)}{\log \rho} = d^*(r_i, \gamma_{i-1}, \gamma_i)$, as the diversity of layer i. In Fig. 8, as an example, the diversity

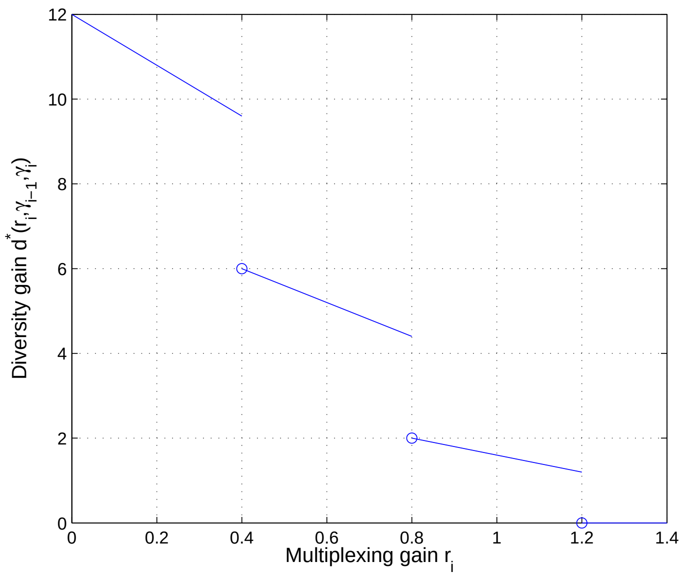

Fig. 8. Diversity Multiplexing Tradeoff Corresponding to a Broadcast Layer with $\gamma_{i-1} = 1$ and $\gamma_i = 0.6$ for a 3×4 MIMO System.

multiplexing tradeoff corresponding to a superposition layer in the broadcast scheme is plotted. Note that it is discontinuous.

Note that the diversity multiplexing tradeoff of Zheng and Tse [6] specified in (3.4) corresponds to the case when $\gamma_{i-1} = 1$ and $\gamma_i = 0$. From Lemma D.1 and (3.4) we can verify that $d^*(r_i, 1, 0) = d^*(r_i)$. To keep the notation brief, in such cases, we will use $d^*(r_i)$ in place of $d^*(r_i, 1, 0)$.

3. Achievability

The broadcast scheme considered in [17] used $r_i = \gamma_{i-1} - \gamma_i$ and optimized the power allocation, γ_i's, in order to maximize the distortion SNR exponent. With this rate and power allocation, the resulting scheme had a distortion SNR exponent equal to $\min(b, MN)$. We show that by using a different rate and power allocation than that specified in [17], we can obtain the optimal exponent of mb for any $b < \frac{n-m+1}{m}$. Notice that in this region the broadcast scheme with the rate and power allocation specified in [17] performs quite poorly. Our main result in this section is the following theorem.

Theorem D.1 *The broadcast scheme achieves a distortion SNR exponent of* $(k+1)b$, $k \in \{0, 1, \ldots, m-1\}$ *for* $\frac{(M-k-1)(N-k-1)}{k+1} < b < \frac{(M-k)(N-k)}{k+1}$ *with power and rate allocation*

$$\gamma_i = \left(\frac{b(k+1) - (M-k-1)(N-k-1)}{(M-k)(N-k) - (M-k-1)(N-k-1)} \right)^i \tag{3.16}$$

and

$$r_i = (k+1)(\gamma_{i-1} - \gamma_i) - \epsilon \tag{3.17}$$

for arbitrarily small $\epsilon > 0$.

The distortion is given by

$$D = \sum_{i=1}^{N_s} P(\text{Layer 1 to } i-1 \text{ decoded, layer } i \text{ decoding failed}) D_i + P(\text{All layers decoded}) D_{N_s+1} \tag{3.18}$$

where D_i is the distortion when only the first $i-1$ layers are used for reconstructing the source. If the layers $1, \ldots, i-1$ can be decoded, a source coding rate of $b \sum_{j=1}^{i-1} r_j \log \rho$ can be obtained. Therefore $D_i = \rho^{-b \sum_{j=1}^{i-1} r_j}$.

We have

$$\begin{aligned} & P(\text{Layer } 1, \ldots, i-1 \text{ decoded, layer } i \text{ decoding failed}) \\ &= P(\text{Layer } 1, \ldots, i-1 \text{ decoded, layer } i \text{ decoding failed} \mid X_1, \ldots, X_{i-1} \\ &\qquad \text{available to decode layer } i) \\ &\leq P(\text{Layer } i \text{ decoding failed} \mid X_1, \ldots, X_{i-1} \text{ available to decode layer } i) \\ &\doteq \rho^{-d^\star(r_i, \gamma_{i-1}, \gamma_i)}. \end{aligned} \tag{3.19}$$

If $d^\star(r_i, \gamma_{i-1}, \gamma_i) > 0$ for all i, then,

$$P(\text{All layers decoded})$$
$$= 1 - \sum_i P(\text{Layer } 1, \ldots, i-1 \text{ decoded, layer } i \text{ decoding failed}) \doteq \rho^0. \quad (3.20)$$

From (3.18), (3.19), and (3.20) we have

$$D \doteq \sum_{i=1}^{N_s} \rho^{-(b\sum_{j=1}^{i-1} r_j + d^*(r_i, \gamma_{i-1}, \gamma_i))} + \rho^{-b\sum_{i=1}^{N_s} r_i}. \quad (3.21)$$

Let

$$a(i) = b\sum_{j=1}^{i-1} r_j + d^*(r_i, \gamma_{i-1}, \gamma_i) \quad (3.22)$$

be the exponent corresponding to the case when the ith layer is in outage and $a(N_s + 1) = b\sum_{i=1}^{N_s} r_i$ the exponent when all layers are decoded. From (3.21), the distortion SNR exponent for the broadcast scheme is

$$a_{BS}(b) = \max_{r,\gamma} \min_i a(i). \quad (3.23)$$

In the following proof, we fix $r_i = (k+1)(\gamma_{i-1} - \gamma_i) - \epsilon$ and optimize the power allocation γ_i's for $i = 1$ to N_s in order to maximize the exponent. Note that $\gamma_0 = 1$.

In section 1 of the Appendix, using the Karush-Kuhn-Tucker (KKT) conditions, it is proved that for $b > \frac{(M-k-1)(N-k-1)}{k+1}$ the optimal exponent is obtained when all the exponents $a(i)$ are equal provided that the resulting solution satisfies $\gamma_i > \gamma_{i+1}$ for all i and $\gamma_{N_s} > 0$.

In order for $a(i) = a(i+1)$, from (3.22) we have

$$d^*(r_i, \gamma_{i-1}, \gamma_i) = br_i + d^*(r_{i+1}, \gamma_i, \gamma_{i+1}). \quad (3.24)$$

Since r_i is chosen to be $(k+1)(\gamma_{i-1}-\gamma_i)-\epsilon$, from (3.7) we have

$$d^*(r_i,\gamma_{i-1},\gamma_i) = (M-k)(N-k)\gamma_{i-1} - (M+N-1-2k)(\gamma_{i-1}-\gamma_i-\epsilon). \quad (3.25)$$

Substituting from (3.25) in (3.24) and using $r_i = (k+1)(\gamma_{i-1}-\gamma_i)$ we have

$$\begin{aligned}&(M-k)(N-k)\gamma_{i-1} - (M+N-1-2k)(\gamma_{i-1}-\gamma_i)\\ &\quad = b(k+1)(\gamma_{i-1}-\gamma_i) + (M-k)(N-k)\gamma_i - (M+N-1-2k)(\gamma_i-\gamma_{i+1}) + O(\epsilon).\end{aligned}$$

On simplifying we obtain

$$(\gamma_i-\gamma_{i+1}) = \alpha(\gamma_{i-1}-\gamma_i) + O(\epsilon) \quad (3.26)$$

where

$$\alpha = \frac{b(k+1)-(M-k-1)(N-k-1)}{M+N-1-2k}. \quad (3.27)$$

We can use (3.26) recursively to obtain

$$\gamma_i-\gamma_{i+1} = \alpha^i(\gamma_0-\gamma_1) + O(\epsilon) = \alpha^i(1-\gamma_1) + O(\epsilon). \quad (3.28)$$

Therefore,

$$1-\gamma_i = \sum_{j=1}^{i}(\gamma_{j-1}-\gamma_j) = \sum_{j=1}^{i}\alpha^{j-1}(1-\gamma_1) + O(\epsilon) = \frac{1-\alpha^i}{1-\alpha}(1-\gamma_1) + O(\epsilon). \quad (3.29)$$

Furthermore, if $b\sum_{j=1}^{N_s} r_j = a(1)$, we have

$$\begin{aligned} b\sum_{j=1}^{N_s}(k+1)(\gamma_{j-1}-\gamma_j) &= (M-k)(N-k) - (M+N-1-2k)(1-\gamma_1) + O(\epsilon)\\ \Rightarrow b(k+1)\frac{1-\alpha^{N_s}}{1-\alpha}(1-\gamma_1) &= (M-k)(N-k) - (M+N-1-2k)(1-\gamma_1) + O(\epsilon)\\ \Rightarrow (1-\gamma_1) &= \frac{(M-k)(N-k)(1-\alpha)}{b(k+1)(1-\alpha^{N_s}) + (M+N-1-2k)(1-\alpha)}.\end{aligned}$$

From (3.27) we have

$$(1-\gamma_1) = \frac{(M-k)(N-k)(1-\alpha)}{(M-k)(N-k)-b(k+1)\alpha^{N_s}}. \tag{3.30}$$

From (3.29)

$$(1-\gamma_i) = \frac{(M-k)(N-k)(1-\alpha^i)}{(M-k)(N-k)-b(k+1)\alpha^{N_s}} \tag{3.31}$$

$$\gamma_i = \frac{(M-k)(N-k)\alpha^i - b(k+1)\alpha^{N_s}}{(M-k)(N-k)-b(k+1)\alpha^{N_s}}. \tag{3.32}$$

Consider the case when $0 \leq \alpha \leq 1$, i.e., when $\frac{(M-k-1)(N-k-1)}{k+1} \leq b \leq \frac{(M-k)(N-k)}{k+1}$. Since $1 \geq \alpha^i$ and since $(M-k)(N-k)/(b(k+1)) > 1 > \alpha^{N_s}$, from (3.31) it follows that $\gamma_i \leq 1$. From (3.32), since $(M-k)(N-k)/(b(k+1)) > 1 > \alpha^{N_s-i}$, we have $\gamma_i \geq 0$ and we also observe that γ_i is a decreasing sequence in i. Therefore, this a valid power allocation.

The resulting exponent is $b(k+1)\frac{(M-k)(N-k)(1-\alpha^{N_s})}{(M-k)(N-k)-b(k+1)\alpha^{N_s}}$ and on taking the limit as $N_s \to \infty$ we obtain $b(k+1)$.

For the region $\frac{(M-k)(N-k)}{k+1} \leq b \leq \frac{(M-k)(N-k)}{k}$, Theorem D.1 does not specify any achievable exponent. But notice that the exponent corresponding to both $b = \frac{(M-k)(N-k)}{k+1}$ and $b = \frac{(M-k)(N-k)}{k}$ is $(M-k)(N-k)$. For this region, we can ignore the additional bandwidth $b - \frac{(M-k)(N-k)}{k+1}$ and use a power allocation corresponding to $b = \frac{(M-k)(N-k)}{k}$ to achieve an exponent of $(M-k)(N-k)$. The resulting achievable distortion SNR exponent curve is continuous and is flat in the region $\frac{(M-k)(N-k)}{k+1} \leq b \leq \frac{(M-k)(N-k)}{k}$ for $k = 1$ to $m-1$. and for $b > MN$.

Corollary D.1 *The optimal distortion SNR exponent for $b < (n-m+1)/m$ is mb.*

The result is obtained by comparing the upper bound in (3.5) with the achievable exponent specified in Theorem D.1 for the case when $k = (m-1)$.

4. Converse

For $b < (n-m+1)/m$ and $b > mn$, BS achieves the optimal exponent (it matches the informed transmitter upper bound) and hence the power and rate allocation specified in Theorem D.1 is optimal. For the region between these two values, the next two results prove that the exponent achieved in Theorem D.1 is the optimal exponent achievable by any superposition (broadcast) scheme. This is shown by finding an upper bound to the exponent of any superposition scheme, for any power allocation and number of layers, that matches the achievable exponent of Theorem D.1. This also calls for schemes that are not based on superposition alone in order to improve on the achievable exponent in this region (discussed in the next sections).

Theorem D.2 *For* $b \leq \frac{(M-k)(N-k)}{k}$*, the distortion SNR exponent of the broadcast scheme satisfies* $a_{BS}(b) \leq (M-k)(N-k)$*.*

Recall that the exponent of the broadcast scheme is given by $a_{BS}(b) = \min_i a(i)$ where $a(i)$ is as specified in (3.22).

Let us fix b and k such that $(M-k)(N-k)/k \geq b$. Let us assume that there exists a power and rate allocation such that the exponent $a_{BS}(b) > (M-k)(N-k)$. Then, since $a_{BS}(b) = \min_i a(i)$, for all i from 1 to $N_s + 1$ we must have $a(i) > (M-k)(N-k)$. As before, without loss of generality, let the rate used in the ith layer be $r_i = k_i(\gamma_{i-1} - \gamma_i) + \delta_i$, for some integer k_i and $0 \leq \delta_i < \gamma_{i-1} - \gamma_i$. We will now show that if $a_{BS}(b) > (M-k)(N-k)$ were to be true, then $k_i < k$ for all i.

The gist of the proof is as follows. If, to the contrary, $k_i \geq k$ for some i, then there must be a smallest value of i (say i^*) for which this is true. That is, there must be an $i^* \geq 1$, for which $k_{i^*} \geq k$ and $k_i \leq k-1$ for all $i = 1$ to $i^* - 1$. We will now show that $a(i^*)$ cannot be larger than $(M-k)(N-k)$.

We have

$$a(i^*) = b\sum_{i=1}^{i^*-1} r_i + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \tag{3.33}$$

Since, $r_i = k_i(\gamma_{i-1} - \gamma_i) + \delta_i$, clearly $r_i \leq (k_i + 1)(\gamma_{i-1} - \gamma_i)$. Therefore,

$$\begin{aligned}
a(i^*) &\leq b\sum_{i=1}^{i^*-1}(k_i+1)(\gamma_{i-1}-\gamma_i) + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \\
&\leq b\sum_{i=1}^{i^*-1}(k)(\gamma_{i-1}-\gamma_i) + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \quad (\because k_i \leq k-1, \text{ for } \; i < i^*) \\
&= bk(1-\gamma_{i^*-1}) + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \\
&\leq (M-k)(N-k)(1-\gamma_{i^*-1}) + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \left(\because b \leq \frac{(M-k)(N-k)}{k}\right) \\
&\leq (M-k)(N-k)(1-\gamma_{i^*-1}) + (M-k_{i^*})(N-k_{i^*})\gamma_{i^*-1} \quad (\because \delta_{i^*} \geq 0) \\
&\leq (M-k)(N-k)(1-\gamma_{i^*-1}) + (M-k)(N-k)\gamma_{i^*-1} \quad (\because k_{i^*} \geq k) \\
&= (M-k)(N-k).
\end{aligned}$$

For $a_{BS}(b) > (M-k)(N-k)$, we require $a(i) > (M-k)(N-k), \forall i$ and, hence, we must have that $k_i \leq k-1$, for all $i = 1, \ldots, N_s$. This implies that $r_i \leq k(\gamma_{i-1}-\gamma_i)$. But, in this case,

$$a(N_s+1) = b\sum_{i=1}^{N_s} r_i \leq bk(1-\gamma_{N_s}) \leq (M-k)(N-k).$$

Therefore, our assumption that $a_{BS}(b)$ can be greater than $(M-k)(N-k)$ for $b < (M-k)(N-k)/k$ is not valid. Hence proved. As pointed out in the discussion after Theorem D.1, the achievable exponent for $(M-k)(N-k)/(k+1) \leq b \leq (M-k)(N-k)/k$ is $(M-k)(N-k)$. This combined with the upper bound specified in Theorem D.2 proves that this is the best achievable exponent using any broadcast scheme for this range of b.

Theorem D.3 *For $b > (M-k-1)(N-k-1)/(k+1)$ the distortion SNR exponent of the broadcast scheme satisfies $a_{BS}(b) \leq b(k+1)$.*

Recall that the exponent of the broadcast scheme is given by $a_{BS}(b) = \min_i a(i)$ where $a(i)$ is as specified in (3.22). The idea of the proof is similar to that in the proof of the previous theorem. Again we fix b and k such that $b > (M-k-1)(N-k-1)/(k+1)$. Let us assume that there exists a power and rate allocation policy such that $a_{BS}(b) > b(k+1)$. Let the rate allocation be $r_i = k_i(\gamma_{i-1} - \gamma_i) + \delta_i$ as before.

Similar to the proof in the previous theorem, we start by showing that $k_i \leq k$ for all i. As before, let $i^* \geq 1$ be such that $k_i \leq k$ for $i = 1$ to $i^* - 1$ and $k_{i^*} \geq k+1$. We have

$$\begin{aligned}
a(i^*) &= b\sum_{i=1}^{i^*-1} r_i + d^*(r_{i^*}, \gamma_{i^*-1}, \gamma_{i^*}) \\
&\leq b(k+1)(1-\gamma_{i^*-1}) + (M-k_{i^*})(N-k_{i^*})\gamma_{i^*-1} \\
&\qquad\qquad (\because r_i \leq (k_i+1)(\gamma_{i-1}-\gamma_i) \leq (k+1)(\gamma_{i-1}-\gamma_i) \text{ for } i < i^*) \\
&\leq b(k+1)(1-\gamma_{i^*-1}) + (M-k-1)(N-k-1)\gamma_{i^*-1} \quad (\because k_{i^*} \geq k+1) \\
&= b(k+1) - \gamma_{i^*-1}(b(k+1) - (M-k-1)(N-k-1)) \\
&\leq b(k+1) \quad (\because b > (M-k-1)(N-k-1)/(k+1)).
\end{aligned}$$

This contradicts the assumption that $a_{BS}(b) > b(k+1)$. Therefore, the only other possibility is that $k_i \leq k$ for all i. In this case too, $a(N_s+1) = b\sum_{i=1}^{N_s} r_i \leq b(k+1)$ which implies that the assumption $a_{BS}(b) > b(k+1)$ is incorrect. Hence proved.

Note that for $(M-k-1)(N-k-1)/(k+1) \leq b \leq (M-k)(N-k)/(k+1)$ the achievable exponent in Theorem D.1 is also $b(k+1)$. Hence, Theorem D.1 along with Theorems D.2 and D.3 fully characterize the exponent achievable with any broadcast scheme.

5. Finite Number of Layers

In practice it is not possible to have infinitely many layers and it is important to study the performance of the broadcast scheme with a finite number of layers. The problem of finding the optimal distortion SNR exponent for a finite number of layers can be posed as the following optimization problem.

$$\begin{aligned}
\max \quad & a && (3.34)\\
\text{subject to:} \quad & \text{for all } i \in \{1, 2, \ldots, N_s\} \\
& \gamma_i \geq 0, \delta_i \geq 0, r_i \geq 0, k_i \in \{0, 1, \ldots, m-1\}, \\
& \gamma_{i-1} > \gamma_i, \gamma_0 = 1, \\
& \delta_i < \gamma_{i-1} - \gamma_i, \\
& r_i = k_i(\gamma_{i-1} - \gamma_i) + \delta_i, \\
& a \leq b \sum_{j=1}^{i-1} r_j + (m - k_i)(n - k_i)\gamma_{i-1} - (m + n - 1 - 2k_i)\delta_i, \\
& a \leq b \sum_{j=1}^{N_s} r_j.
\end{aligned}$$

For a fixed set of k_i's this reduces to a linear program. For small N_s, the optimum exponent can be found by using the linear program for all m^{N_s} choices of k_i's .

In Fig. 9 and Fig. 10, the distortion SNR exponent corresponding to the broadcast scheme proposed in Theorem D.1 is shown for a 3×4 and a 3×6 MIMO system. The optimal distortion SNR exponent corresponding to the broadcast scheme with 10 layers is also shown. We see that the exponent with finite layers is very close to the best achievable distortion exponent of the broadcast scheme for all b and the curves overlap for a large range of b. Also as proved in Theorem D.2 and D.3, the distortion exponent with finite layers does not improve on the achievable exponent specified in Theorem D.1.

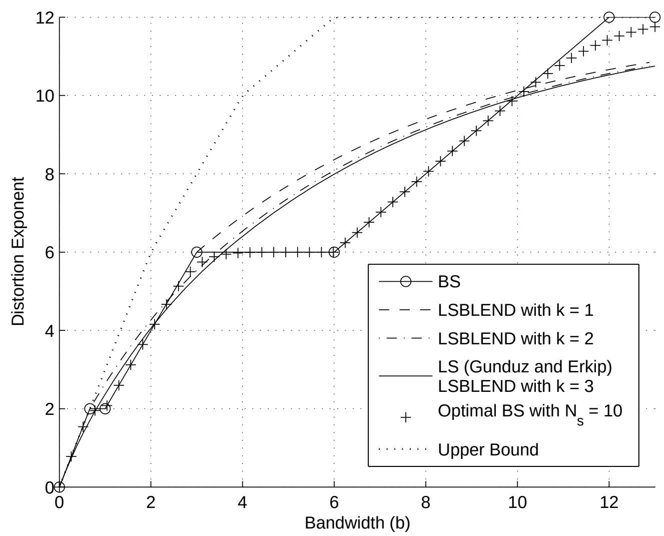

Fig. 9. Distortion SNR Exponent for $M = 3, N = 4$.

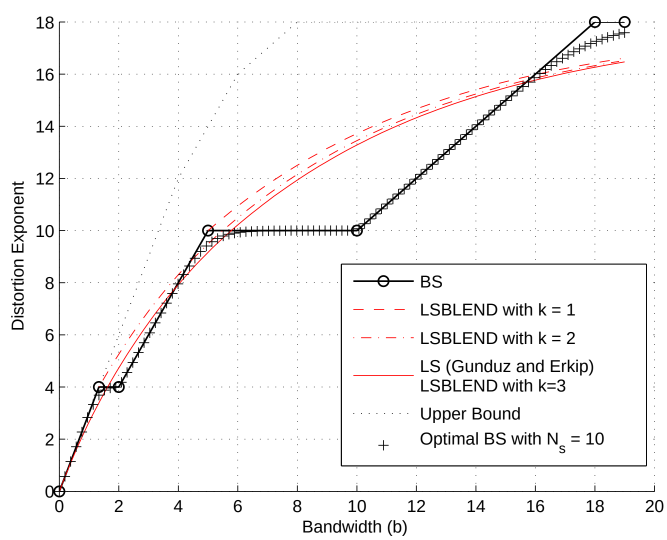

Fig. 10. Distortion SNR Exponent for $M = 3, N = 6$.

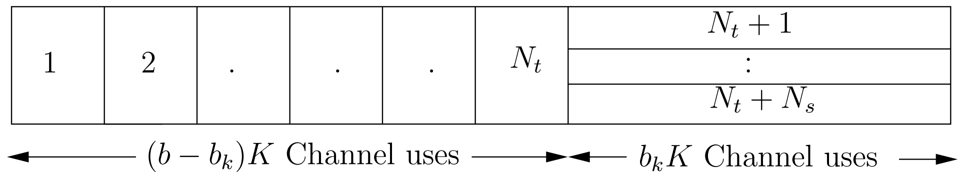

Fig. 11. Layered Schemes with Broadcast Layer at the End (LSBLEND).

E. Layering in Time with One Broadcast Layer at the End

Consider the scheme shown in Fig. 11. For $b > b_k = (m-k)(n-k)/(k+1)$, $k \in \{1, \ldots, m\}$, a bandwidth of $b - b_k$ is allocated for time layering and the remaining bandwidth of b_k is allocated to a broadcast scheme where the rate and power allocation for the broadcast layers are chosen as specified in Theorem D.1. The parameter k determines the bandwidth splitting between the broadcast layer and the time layers. The decoding proceeds by first decoding the time layers and then decoding the broadcast layers after all the time layers are decoded. This is similar to the HDA scheme of [13] and the HLS of [17] where the source is quantized and transmitted using time layering in a bandwidth of $b - 1/m$ and the quantization error is transmitted in an analog layer of bandwidth $1/m$. For the proposed scheme, the distortion SNR exponent obtained for a particular bandwidth splitting parameter k is given in the following theorem. The largest achievable distortion SNR exponent is then obtained by taking a supremum over all k.

Note that when $k = m$, the bandwidth allocated to the broadcast layer is 0, i.e., we have only time layering. This scheme, termed Layered Scheme (LS), was proposed and analyzed in [17]. The proof of the following theorem is similar to the derivation of the exponent for LS in [17].

Theorem E.1 *Let* $c_j = (m+n-1-2j)\log\frac{j+1}{j}$ *for* $j = 0, \ldots, m$. *Let* p *be such* *that* $\sum_{j=p+1}^{k-1} c_j \leq (b - b_k) < \sum_{j=p}^{k-1} c_j$. *Then, the best distortion SNR exponent* $a(b)$ *achievable using LSBLEND is given by*

$$a(b) = mn - p - p^2 - (m + n - 1 - 2p)(p + 1)e^{-\frac{b - b_k - \sum_{j=p+1}^{k-1} c_j}{m+n-1-2p}}. \tag{3.35}$$

Let N_t and N_s denote the number of time and superposition layers respectively. Let $a(i)$ for $i = 1, \ldots, N_t$ denote the distortion SNR exponent corresponding to the case when the time layers 1 to $i - 1$ are decoded and decoding of the ith time layer fails, $a(N_t + i)$ for $i = 1, \ldots, N_s$ denote the distortion SNR exponent corresponding to the case when all N_t time layers and the first $i - 1$ broadcast layers are decoded while decoding of the ith broadcast layer fails, and let $a(N_t + N_s + 1)$ denote the exponent corresponding to the case when all layers are decoded.

For the ith time layer, the probability of decoding failure is given by $P_e(i) \doteq \rho^{-d^*(r_i)}$ where r_i is the multiplexing rate of the ith time layer and $d^*(r_i)$ is the Zheng and Tse diversity multiplexing tradeoff function specified in (3.4). Note that power allocation to the time layer is $\rho^1 - \rho^0$ and $d^*(r_i) = d^*(r_i, 1, 0)$. The bandwidth allocated to a time layer is $(b - b_k)/N_t$. The distortion SNR exponent of the ith time layer is then given by

$$a(i) = \frac{b - b_k}{N_t} \sum_{j=0}^{i-1} r_j + d^*(r_i) \tag{3.36}$$

where $r_0 = 0$.

For the broadcast layer we use the rate and power allocation as specified in Theorem D.1. With that rate and power allocation it follows that the exponents

$a(N_t+1), a(N_t+2), \ldots a(N_t+N_s+1)$ are all equal and are given by

$$\begin{aligned} a(N_t+i) &= \frac{b-b_k}{N_t}\sum_{j=0}^{N_t} r_j + (k+1)b_k \\ &= \frac{b-b_k}{N_t}\sum_{j=0}^{N_t} r_j + (m-k)(n-k) \end{aligned} \tag{3.37}$$

for $i = 1$ to $N_s + 1$ in the limit $N_s \to \infty$. Note that we do not loose optimality here by fixing the rate and power allocation of the broadcast layer since $(m-k)(n-k)$ is the maximum possible contribution that the broadcast layer of bandwidth b_k can make to the exponent (see Theorem D.2 and Theorem D.3).

In the following proof, we optimize r_i's to maximize the distortion SNR exponent.

In section 2 of the Appendix we show that the exponent is maximized by choosing r_i's such that $a(1) = a(2) = \ldots = a(N_t) = a(N_t+1)$ provided that the resulting r_i's lie between 0 and m. By setting $a(N_t) = a(N_t+1)$ we obtain

$$d^*(r_{N_t}) = \frac{b-b_k}{N_t} r_{N_t} + (m-k)(n-k). \tag{3.38}$$

We will consider the limiting case when $N_t \to \infty$. From (3.38) we have, in the limiting case,

$$d^*(r_{N_t}) \to (m-k)(n-k). \tag{3.39}$$

Therefore,

$$(m-k_{N_t})(n-k_{N_t}) - (m+n-1-2k_{N_t})\delta_{N_t} \to (m-k)(n-k). \tag{3.40}$$

This happens when $k_{N_t} = k-1$ and $\delta_{N_t} \to 1$.

By setting $a(i-1) = a(i)$ we have

$$d^*(r_{i-1}) = \frac{b-b_k}{N_t} r_{i-1} + d^*(r_i). \tag{3.41}$$

$d^*(r)$ is a decreasing function and from (3.41) we have $d^*(r_{i-1}) \geq d^*(r_i)$. Therefore $r_{N_t} \geq r_{N_t-1} \geq ... \geq r_1$. Let r_i lie between t and $t+1$. We want to check if r_{i-1} also lies between t and $t+1$. To do so we assume that $k_{i-1} = t$ and solve for δ_{i-1}. If the resulting δ_{i-1} lies between 0 and 1, then the assumption $k_i = t$ is correct. From (3.41) we have

$$
\begin{aligned}
&(m-t)(n-t) - (m+n-1-2t)\delta_{i-1} = \\
&\qquad \frac{b-b_k}{N_t}(t+\delta_{i-1}) + (m-t)(n-t) - (m+n-1-2t)\delta_i. \\
&\Rightarrow \delta_{i-1}(m+n-1-2t+\frac{b-b_k}{N_t}) = \delta_i(m+n-1-2t) - \frac{b-b_k}{N_t}t. \\
&\Rightarrow \delta_{i-1} = \alpha\delta_i - (1-\alpha)t \qquad (3.42)
\end{aligned}
$$

where

$$
\alpha = \frac{m+n-1-2t}{m+n-1-2t+(b-b_k)/N_t} < 1. \qquad (3.43)
$$

On using recursion (3.42) N_l times we have

$$
\delta_{i-N_l} = \alpha^{N_l}\delta_i - \frac{1-\alpha^{N_l}}{1-\alpha}(1-\alpha)t = \alpha^{N_l}(t+\delta_i) - t. \qquad (3.44)
$$

The maximum number of times the recursion can be used such that the resulting δ is positive is given by

$$
\begin{aligned}
\alpha^{N_l}(t+\delta_i) &\geq t \\
\Rightarrow N_l \log\alpha &\geq \log\frac{t}{t+\delta_i} \\
\Rightarrow \frac{N_l}{N_t} &\leq \frac{1}{N_t\log\alpha}\log\frac{t}{t+\delta_i} \qquad \because \alpha < 1,\ \log\alpha < 0 \\
\Rightarrow \frac{N_l}{N_t} &\leq -\frac{1}{\log\left(\left(1+\frac{b-b_k}{(m+n-1-2t)N_t}\right)^{N_t}\right)}\log\frac{t}{t+\delta_i} \\
\Rightarrow \frac{N}{N_t} &\leq \frac{m+n-1-2t}{b-b_k}\log\frac{t+\delta_i}{t} \qquad (N_t \to \infty).
\end{aligned}
$$

For the proposed scheme, we start from $r_{N_t} = k$ ($k_{N_t} = k - 1$ and $\delta_{N_t} = 1$) and solve for r_{i-1} from r_i. Let $c_j = (m + n - 1 - 2j) \log \frac{j+1}{j}$. If p is such that $\sum_{j=p+1}^{k-1} c_j \leq (b - b_k) < \sum_{j=p}^{k-1} c_j$ then as i decreases from N_t, after a fraction $\sum_{j=p+1}^{k-1} c_j/(b - b_k)$ of the time layers, r_i decreases from k to $p + 1$. For the remaining fraction $(1 - \sum_{j=p+1}^{k-1} c_j/(b - b_k))$ of layers, as i decreases, r_i decreases but remains above p, i.e., k_i remains constant at p while δ_i decreases. From (3.44) we can calculate r_1 as

$$\begin{aligned} r_1 &= p + \lim_{N_t \to \infty} \alpha^{N_t(1 - \frac{1}{b - b_k} \sum_{j=p+1}^{k-1} c_j)} (p + 1) - p \\ &= (p + 1) e^{-(\frac{b - b_k}{m + n - 1 - 2p})(1 - \frac{1}{b - b_k} \sum_{j=p+1}^{k-1} c_j)}. \end{aligned} \tag{3.45}$$

The final exponent is given by

$$\begin{aligned} a(1) = d^*(r_1) &= (m - p)(n - p) - (m + n - 1 - 2p)(r_1 - p) \\ &= mn - p - p^2 - (m + n - 1 - 2p) r_1 \end{aligned}$$

which is the desired result.

Note that when $m = n$ and $k = m - 1$, the contribution to the distortion SNR exponent by the broadcast layer is $b_k(k+1) = 1$ and it uses a bandwidth of $b_k = 1/m$. In the HLS scheme, the analog layer uses a bandwidth of $b_0 = 1/m$ and it also has a contribution of $mb_0 = 1$ towards the exponent. Therefore, in this case, the distortion SNR exponent obtained with LSBLEND with $k = m - 1$ is identical to that with HLS. Therefore, the distortion SNR exponent obtained using LSBLEND becomes identical to that obtained using HLS when (a) $m = n$ and (b) the supremum occurs at $k = m-1$. It can be shown that LSBLEND is strictly better otherwise for $b > 1/m$.

1	2	3							N_t
N_t+1	N_t+2								
									N_tN_s

Fig. 12. Box Scheme.

F. Digital Layering in Time and Using Superposition

The source is encoded in such a way that it is successively refinable. The transmitted signal composes of N_t time layers where each time layer is a superposition of N_s layers. To the (i, j)th layer, i.e., the jth time layer and the ith superposition layer within it, we allocate a power level of $\rho^{\gamma_{i-1,j}} - \rho^{\gamma_{i,j}}$ and we use a rate of transmission of $r_{i,j} \log \rho$. This corresponds to a source coding rate of $(b/N_t) r_{i,j} \log \rho$. The order in which the source coded bits are mapped to the transmission layers is important. The source coded bits are successively mapped on to the transmitted layers from top left to bottom right going along each row. That is, in the order $(1, 1), \ldots, (1, N_t), (2, 1), \ldots, (2, N_t), \ldots, (N_s, N_t)$ (see Fig. 12). The decoding proceeds in the same order and when a layer cannot be decoded, the source is reconstructed using all the layers that have been successfully decoded up to that layer.

Let $\overline{r}_{(i-1)N_t+j} \log \rho = (b/N_t) \left(\sum_{p=1}^{i-1} \sum_{q=1}^{N_t} r_{p,q} + \sum_{q=1}^{j-1} r_{i,q} \right) \log \rho$ denote the cumulative source coding rate up to the (i, j)th layer. As in the broadcast scheme case, we can approximate the overall distortion up to an exponential order by

$$D \doteq \sum_i \sum_j \rho^{-d^\star(r_{i,j}, \gamma_{i-1,j}, \gamma_{i,j}) + \overline{r}_{(i-1)N_t+j}} + \rho^{-\overline{r}_{N_s N_t + 1}}. \tag{3.46}$$

Let r and γ denote the set of $r_{i,j}$'s and $\gamma_{i,j}$'s. For a given r, γ, the overall exponent of the scheme $a(b,r,\gamma)$ is then,

$$a(b,r,\gamma) = \min_{i,j}(d^\star(r_{i,j},\gamma_{i-1,j},\gamma_{i,j}) + \overline{r}_{(i-1)N_t+j}, \overline{r}_{N_sN_t+1}). \tag{3.47}$$

The best achievable exponent with this scheme $a(b)$ is then given by

$$a(b) = \max_{r,\gamma} a(b,r,\gamma). \tag{3.48}$$

If we allow for change in the bandwidth allocated to each layer, then both BS and LSBLEND become special cases of this scheme and therefore the exponent obtained from the maximization should be better than those reported earlier. We will now show that for the distortion SNR exponent, even with fixed bandwidth allocation to each layer, the Box scheme can be designed to perform at least as well as BS and LSBLEND.

Claim F.1 *The Box scheme with N_s superposition layer and N_t time layers has a distortion SNR exponent that is at least as good as that of the broadcast scheme with N_s layers.*

Let us denote the optimal rate and power allocation for the broadcast scheme by r_i, γ_i. The exponent corresponding to the ith broadcast layer is $a_{BS}(i) = b\sum_{j=1}^{i-1} r_j + d^*(r_i,\gamma_{i-1},\gamma_i)$. Now consider the Box scheme where the power allocation to the (i,j)th layer $\gamma_{i,j}$ is set to γ_i and the rate $r_{i,j} = r_i$. Then $a_{Box}(i,j) = b\sum_{j=1}^{i-1} r_j + \frac{b}{N_t}(j-1)r_i + d^*(r_i,\gamma_{i-1},\gamma_i)$. Clearly $\frac{b}{N_t}\sum_{i,j} r_{i,j} = b\sum r_i$ and $a_{Box}(i,j) \geq a(i)$. Therefore, $\min_{i,j}(a_{Box}(i,j), \frac{b}{N_t}\sum_{i,j} r_{i,j}) \geq \min_i(a_{BS}(i), b\sum r_i)$. In this case it is actually equal but if we optimize the power allocation of the box scheme it could possibly improve on the exponent.

Claim F.2 *In the limit as $N_t \to \infty$, the Box scheme has a distortion SNR exponent that is at least as good as that of LSBLEND.*

Consider the case when $(b - b_k)/N_t = b/N_{t,Box}$ where $N_t, N_{t,Box}$ are positive integers. Consider a power and rate allocation for the box scheme that is identical to the LSBLEND scheme for first N_t time layers. That is, the first N_t time layers have no superposition layers and the rate is identical to that of LSBLEND. For the remaining $N_{t,Box} - N_t$ layers we allocate power and rate with the procedure used in lemma F.1 and therefore its contribution to the exponent is identical to the contribution of the broadcast layer of LSBLEND. Therefore, this has an exponent that is identical to that of LSBLEND. Again, by optimizing the power and rate allocation of the box scheme we could possibly improve the exponent.

For the case when $\frac{b}{b-b_k}$ is irrational, the result still holds because the achievable exponent with LSBLEND and Box scheme is a continuous function of b.

The maximization in (3.48) is difficult to perform analytically and very quickly becomes difficult to perform even numerically. The procedure described in Algorithm 1 has been used to find a suboptimal set of r, γ. Remarkably, it turns out that for a range of b, this achieves performance very close to the informed transmitter upper bound $a_{IT}(b)$. Furthermore, for the considered examples, this scheme performs nearly as well as currently known schemes for all b while it is strictly better for some range of b.

For each (i, j) if we fix $k_{i,j} \in \{0, 1, \dots, m-1\}$ and let $r_{i,j} = k_{i,j}(\gamma_{i-1,j} - \gamma_{i,j}) + \delta_{i,j}$ where $0 \leq \delta_{i,j} < (\gamma_{i-1,j} - \gamma_{i,j})$, then, as before, the problem of finding the optimal exponent reduces to a linear program and hence by solving it for different $k_{i,j}$'s we would expect to find an exponent that is better than that obtained using Algorithm 1. However, in Step 4 of the algorithm, notice that we skip a layer if it is not possible to allocate a non zero rate. Therefore, this layer is never in outage. However, in the linear program, if we use lemma D.1 to compute $d^*(0, \gamma_{i-1,j}, \gamma_{i-1,j})$ we get 0 which means this layer is always in outage. Therefore, to obtain the optimal exponent, we

Algorithm 1 Algorithm to check if an exponent d is achievable using the proposed scheme.

Step 1: Initialization - Set $\gamma_{0,j} = 1 \ \forall j$ and $\overline{r}_1 = 0$.

Step 2: For $i = 1$ to N_s

Step 3: For $j = 1$ to N_t

Step 4: If $MN\gamma_{i-1,j} + \overline{r}_{(i-1)N_t+j} < d$, set $\gamma_{i,j} = \gamma_{i-1,j}$ and goto step 10.

Step 5: Find smallest $k_{i,j} \in \{0, 1, \ldots, m-1\}$ such that $0 \leq \delta_{i,j} < \gamma_{i-1,j}$ where $\delta_{i,j} = ((M - k_{i,j})(N - k_{i,j})\gamma_{i-1,j} + \overline{r}_{(i-1)N_t+j} - d)/(M + N - 1 - 2k_{i,j})$.

Step 6: If $i = N_s$ set $\gamma_{i,j} = 0$ else set $\gamma_{i,j} = \gamma_{i-1,j} - \delta_{i,j}$

Step 7: Set $r_{i,j} = k_{i,j}(\gamma_{i-1,j} - \gamma_{i,j}) + \delta_{i,j}$.

Step 8: Update $\overline{r}_{(i-1)N_t+j+1} = \overline{r}_{(i-1)N_t+j} + (b/N_t)r_{i,j}$

Step 9: If $\overline{r}_{(i-1)N_t+j+1} > d$, exponent d is achievable. return.

Step 10: End of j loop

Step 11: End of i loop

Step 12: Exponent d is not achievable using this scheme. return.

will need to allow for a layer to be skipped in addition to allowing for different values of k for that layer. The complexity thus grows as $(M+1)^{N_s N_t}$.

The achievable exponent with this scheme increases monotonically with N_S. Interestingly, the achievable exponent with this scheme may not increase monotonically with N_t.

We also considered the following variations, which provide some gain for finite number of layers. However, the gain diminishes as the number of layers increases.

Adding an Analog Layer : In this scheme, we start allocating rate and power levels to the layers as in Algorithm 1. Let us denote by $\mathcal{A}_{i,j} = \{(p,q) : p \leq i-1 \text{ AND } q \leq N_t\} \cup \{(i,q) : q < j\}$ the set of all layers for which a rate and power allocation has been found during the (i,j)th stage of the algorithm. Let $\mathbf{X}^a_{i,j}$ denote the analog quantization error in quantizing the source using $\overline{r}_{(i-1)N_t+j} \log \rho$ bits. We check if at least $\lceil \frac{N_t}{bm} \rceil$ layers are still available in $\mathcal{A}^c_{i,j}$ to transmit the analog quantization error such that the desired exponent can be achieved. If this is possible, we stop there and this becomes the overall transmission scheme. Otherwise, we allocate a power level $\gamma_{i,j}$ and rate $r_{i,j}$ corresponding to the (i,j)th layer as before and continue to the next layer. Note that this contains the HLS schemes of [13, 17] as a special case (when $N_s = 1$).

Ordering the layers based on available power : In this variation, we allocate rate and power as in Algorithm 1 but the order of selecting the layers is not sequential. At any stage of the algorithm, we select the time layer with the maximum available power (total power minus power already allocated to superposition layers in that time layer). A new superposition layer is added to this layer with power and rate allocation as specified in Algorithm 1. Note that this is the order in which the successive refinement information from the source coder is filled and therefore the decoder should decode the layers in this order.

G. Extensions to Multiple Block Fading Channels

In this section we extend the results derived for the MIMO channel to the L-block fading MIMO channel. We assume that K source samples are transmitted over L blocks of length T/L each. The fading coefficient for the different blocks are independent.

Lemma G.1 *If the multiplexing gain in the ith layer of the broadcast scheme is* $r_i = \frac{kL+a}{L}(\gamma_{i-1} - \gamma_i) + \delta$ *where* $k \in \{0, 1, \ldots, m-1\}$, $a \in \{0, 1, \ldots, L-1\}$ *and* $0 \leq \delta < \frac{\gamma_{i-1}-\gamma_i}{L}$, *then, the achievable diversity in the ith layer of the broadcast scheme, assuming that the message transmitted in the previous layers is available at the receiver, is given by*

$$d^\star(r_i, \gamma_{i-1}, \gamma_i) = L(m-k)(n-k)\gamma_{i-1} - (m+n-1-2k)(a\gamma_{i-1} + L\delta).$$

That is, if $X = \frac{1}{\sqrt{\rho}}(\sum_{i=1}^{N_s} \sqrt{\rho^{\gamma_{i-1}} - \rho^{\gamma_i}} X_i + \sqrt{\rho^{\gamma_{N_s}}} N_1)$, *where* $X_i, N_1 \sim \mathcal{N}(0, I^{M\times M})$, $Y_l = \sqrt{\frac{\rho}{M}} \mathbf{H_l} X + N$ *for* $l = 1, \ldots, L$, *and* $\mathcal{A}_i = \{H_1, \ldots, H_L : \frac{1}{L}\sum_{l=1}^{L} I(X_i; Y_l | \mathbf{H_1} = H_1, \ldots, \mathbf{H_L} = H_L, X_1, \ldots X_{i-1}) < r_i \log\rho\}$ *denotes the outage event set then* $P(\mathcal{A}_i) \doteq \rho^{-d^\star(r_i, \gamma_{i-1}, \gamma_i)}$.

We have

$$P(\mathcal{A}_i) = P\left(\frac{1}{L}\sum_{l=1}^{L} \log\frac{\det(I + \frac{1}{M}\rho^{\gamma_{i-1}}\mathbf{H_l}\mathbf{H_l}^H)}{\det(I + \frac{1}{M}\rho^{\gamma_i}\mathbf{H_l}\mathbf{H_l}^H)} < r_i \log\rho\right). \tag{3.49}$$

Let $\lambda_{1,l}, \ldots, \lambda_{m,l}$ denote the non-zero ordered eigenvalues of $\mathbf{H_l}\mathbf{H_l}^H$ with $\lambda_{1,l} \leq \lambda_{2,l} \leq \ldots \lambda_{m,l}$. As in [6], let $\alpha_{j,l} = -\frac{\log\lambda_{j,l}}{\log\rho}$. Therefore, $\alpha_{1,l} \geq \alpha_{2,l} \geq \ldots \alpha_{m,l}$. At high

SNR, $P(\mathcal{A}_i) \doteq P(\mathcal{A}')$ where

$$\mathcal{A}' = \left\{ \alpha : \frac{1}{L} \sum_{l=1}^{L} \sum_{j=1}^{m} \left((\gamma_{i-1} - \alpha_{j,l})^+ - (\gamma_i - \alpha_{j,l})^+ \right) < r_i \right\}. \tag{3.50}$$

Following in the footsteps of [6], we observe that the outage probability is given by $P(\mathcal{A}_i) \doteq \rho^{-d^\star(r_i, \gamma_{i-1}, \gamma_i)}$ where

$$d^\star(r_i, \gamma_{i-1}, \gamma_i) = \inf_{\mathcal{A}' \cap \boldsymbol{\alpha}^+} \sum_{j=1}^{m} \sum_{l=1}^{L} (2j - 1 + n - m)\alpha_{j,l}. \tag{3.51}$$

For $r_i = \frac{kL+a}{L}(\gamma_{i-1} - \gamma_i) + \delta$ where $k \in [0, 1, \ldots, m-1]$, $a \in [0, 1, \ldots, L-1]$, and $0 \leq \delta < \frac{\gamma_{i-1} - \gamma_i}{L}$, the infimum in (3.51) occurs when $\alpha = \alpha^*$ where

$$\alpha^*_{j,l} = \begin{cases} \gamma_{i-1}, & 1 \leq j < m-k; \\ \gamma_{i-1}, & j = m-k, 1 \leq l < L-a; \\ \gamma_{i-1} - L\delta, & j = m-k, l = L-a; \\ 0, & j = m-k, L-a < l \leq L; \\ 0, & m-k < j \leq m \end{cases} \tag{3.52}$$

Hence,

$$\begin{aligned} & d^\star(r_i, \gamma_{i-1}, \gamma_i) \\ &= \sum_{j=1}^{m-k-1} (2j - 1 + n - m)L\gamma_{i-1} + (2(m-k) - 1 + n - m)((L-a)\gamma_{i-1} - \delta) \\ &= \sum_{j=1}^{m-k} (2j - 1 + n - m)L\gamma_{i-1} - (2(m-k) - 1 + n - m)(a\gamma_{i-1} + L\delta) \\ &= L(m-k)(2\frac{m-k+1}{2} - 1 + n - m)\gamma_{i-1} - (m + n - 1 - 2k)(a\gamma_{i-1} + L\delta). \end{aligned}$$

This then gives the desired result.

Theorem G.1 *The broadcast scheme achieves a distortion SNR exponent of* $\frac{kL+a+1}{L}b$, $k \in \{0, 1, \ldots, m-1\}$, $a \in \{0, 1, \ldots, L-1\}$ *for* $\frac{L(M-k)(N-k)-(a+1)(M+N-1-2k)}{kL+a+1} < \frac{b}{L} <$

$\frac{L(M-k)(N-k)-a(M+N-1-2k)}{kL+a+1}$ *with power and rate allocation*

$$\gamma_i = \alpha^i \tag{3.53}$$

and

$$r_i = \frac{kL+a+1}{L}(\gamma_{i-1} - \gamma_i) - \epsilon \tag{3.54}$$

where

$$\alpha = 1 + a + \frac{b\frac{kL+a+1}{L} - L(M-k)(N-k)}{M+N-1-2k} \tag{3.55}$$

for arbitrarily small $\epsilon > 0$.

The power and rate allocation policy can be derived in a manner similar to that in Theorem D.1. Here we will just verify that the specified rate and power allocation policy indeed gives the specified exponent.

We first note that $\frac{L(M-k)(N-k)-(a+1)(M+N-1-2k)}{kL+a+1} < \frac{b}{L} < \frac{L(M-k)(N-k)-a(M+N-1-2k)}{kL+a+1}$ corresponds to $0 < \alpha < 1$ and therefore the specified rate and power allocation is a valid assignment.

As in Theorem D.1 the distortion SNR exponent is given by

$$a_{BS} = \min(b\sum_j r_j, a(1), \ldots, a(i), \ldots) \tag{3.56}$$

where

$$a(i) = b\sum_{j=1}^{i-1} r_j + d^*(r_i, \gamma_{i-1}, \gamma_i). \tag{3.57}$$

We have

$$\lim_{i\to\infty} b\sum_{j=1}^{i} r_j = b\frac{kL+a+1}{L}(1 - \lim_{i\to\infty}\gamma_i) = b\frac{kL+a+1}{L}. \tag{3.58}$$

Furthermore, from (3.57) and lemma G.1 we have

$$\begin{aligned}
a(i) &= b\frac{kL+a+1}{L}(1-\gamma_{i-1}) + L(m-k)(n-k)\gamma_{i-1} \\
&\qquad -(m+n-1-2k)((a+1)\gamma_{i-1}-\gamma_i) \\
&= b\frac{kL+a+1}{L} + (m+n-1-2k)\gamma_i \\
&\qquad +\gamma_{i-1}\left(L(m-k)(n-k) - (m+n-1-2k)(a+1) - b\frac{kL+a+1}{L}\right) \\
&= b\frac{kL+a+1}{L} + (m+n-1-2k)(\gamma_i - \alpha\gamma_{i-1}) = b\frac{kL+a+1}{L}.
\end{aligned}$$

Therefore, $a_{BS} = b\frac{kL+a+1}{L}$.

By comparing with the upper bound, we observe that the broadcast scheme achieves the optimal exponent of mb for $b < \frac{n-m+1}{m}$ and MNL for $b > MNL^2$. This has been shown earlier for the $M = N = 1$ case in [18] and for the $L = 1$ case in [17].

Theorem G.2 *Let* $c_{kL+a} = L(m+n-1-2k)\log\left(\frac{kL+a+1}{kL+a}\right)$ *for* $k \in \{0,1,\ldots,m-1\}$ *and* $a \in \{0,1,\ldots,L-1\}$*. Consider the time layering scheme with a broadcast layer of bandwidth* $b_{kL+a} = \frac{(m-k)(n-k)L^2 - aL(n+m-1-2k)}{kL+a+1}$ *at the end. Let* k_1 *and* a_1 *be such that* $\sum_{j=k_1L+a_1+1}^{kL+a-1} c_j < b - b_{kL+a} < \sum_{j=k_1L+a_1}^{kL+a-1} c_j$ *where* $k_1 \in \{0,1,\ldots,m-1\}$ *and* $a_1 \in \{0,1,\ldots,L-1\}$*. The distortion SNR exponent is then given by*

$$a_{kL+a}(b) = (m-k_1)(n-k_1)L - (r_1-k_1)L(n+m-1-2k_1) \tag{3.59}$$

where

$$r_1 = (k_1 + \frac{a_1+1}{L})e^{-\frac{b-b_{kL+a}-\sum_{i=k_1L+a_1+1}^{kL+a-1} c_i}{L(m+n-1-2k_1)}}. \tag{3.60}$$

The proof is similar to that of Theorem E.1 and is skipped here.

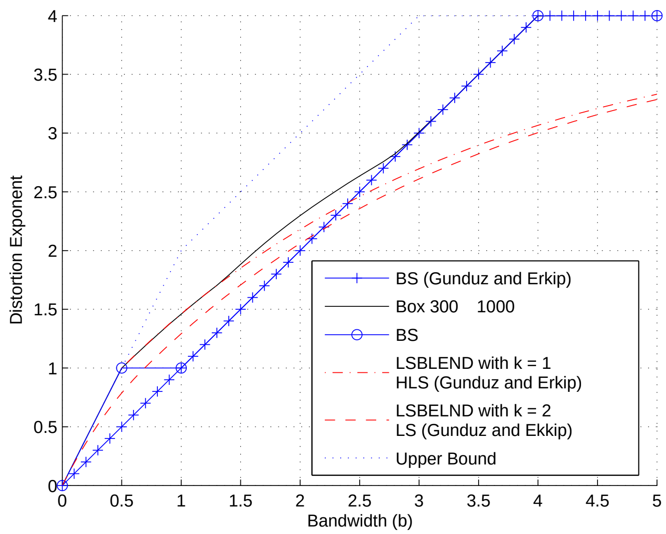

Fig. 13. Achievable Distortion SNR Exponent for $M = N = 2$.

H. Numerical Results

The achievable exponent using the proposed layering schemes along with that achievable by the HLS and broadcast schemes of [17] are shown in Fig. 13, Fig. 14, and Fig. 15. Note that the proposed schemes outperforms the schemes in [17] for all b, making this the best known achievable distortion SNR exponent.

The optimal exponent can be obtained for all $b < (n-m+1)/m$ using the purely digital BS. This is the first time a scheme has been shown to obtain the optimal exponent for $1/m < b < (n-m+1)/m$. Since BS is a special case of LSBLEND and the BOX scheme, the optimal exponent is achieved in this region by these schemes as well.

A plot of the distortion SNR exponent for $M = N = L = 2$ is shown in Fig. 16.

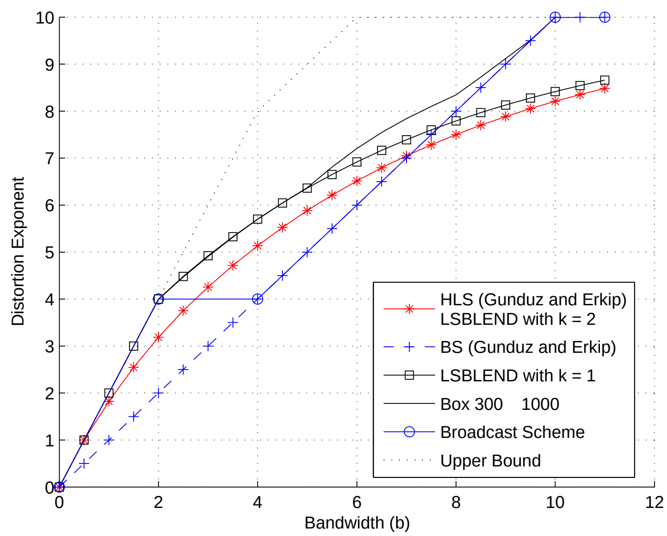

Fig. 14. Achievable Distortion SNR Exponent for $M = 2, N = 5$.

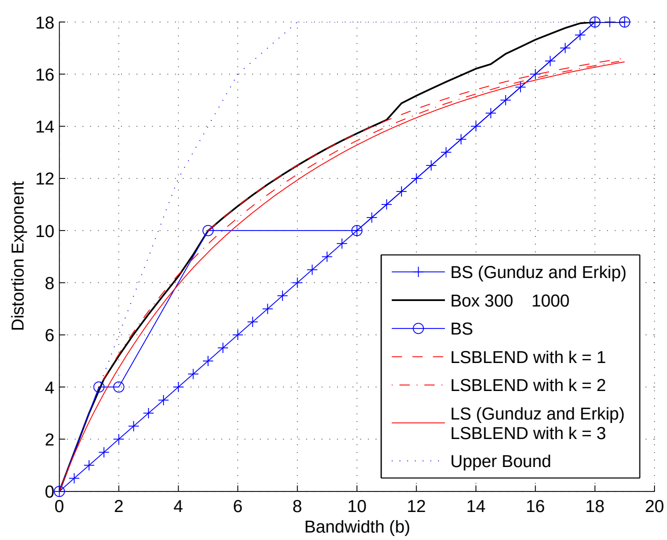

Fig. 15. Achievable Distortion SNR Exponent for $M = 3, N = 6$.

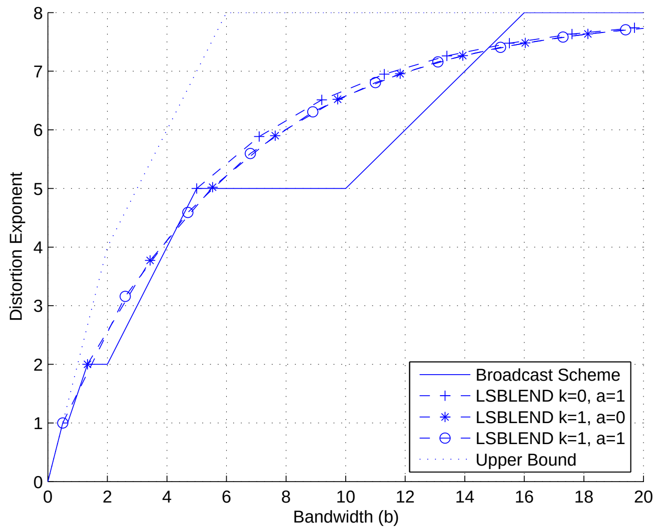

Fig. 16. Achievable Distortion SNR Exponent for $M = N = L = 2$.

I. Conclusion

We have proposed layering schemes for transmitting a discrete time analog source over a block fading MIMO channel. Achievable distortion SNR exponent using carefully selected rate and power allocation policies for these scheme have been studied. The achievable distortion SNR exponent obtained using these schemes is better than those reported in [13, 17] making this the best known distortion SNR exponent so far. Particularly, the optimal exponent is obtained for $b < (n-m+1)/m$ and $b > mnL^2$.

J. Appendix

1. Optimality of Equating Exponents for the BS

The exponent $a(b)$ is given by the following optimization problem

$$a(b) = -\min_{a,\gamma_1,\ldots,\gamma_{N_s}} -a$$

subject to:

$$C_i = a - b(k+1)(1-\gamma_{i-1}) - (m-k)(n-k)\gamma_{i-1} + (m+n-1-2k)(\gamma_{i-1}-\gamma_i);$$

$$C_i \leq 0 \quad \text{for } i = 1, 2, \ldots, N_s;$$

$$C_{N_s+1} = a - b(k+1)(1-\gamma_{N_s}) \leq 0;$$

$$\gamma_{i+1} \leq \gamma_i; \quad \gamma_{N_s} \geq 0; \quad \gamma_0 = 1.$$

We solve this optimization problem by ignoring the constraints $\gamma_{i+1} \leq \gamma_i$ and $\gamma_{N_s} \geq 0$. Any solution then is an upper bound on $a(b)$. Furthermore, if the solution satisfies the ignored constraints then the solution yields $a(b)$. Consider the function $F = -a + \sum_{i=1}^{N_s+1} \lambda_i C_i$. Setting $dF/d\gamma_i = 0$ we have for $i = 1$ to $N_s - 1$

$$\begin{aligned} \frac{dF}{d\gamma_i} &= -\lambda_i(m+n-1-2k) + \lambda_{i+1}(b(k+1) - (m-k)(n-k) + (m+n-1-2k)) \\ &= 0 \\ &\Rightarrow \lambda_i = \frac{b(k+1) - (m-k)(n-k) + (m+n-1-2k)}{m+n-1-2k}\lambda_{i+1} = \alpha\lambda_{i+1}. \end{aligned}$$

For $i = N_s$

$$\frac{dF}{d\gamma_{N_s}} = -\lambda_{N_s}(m+n-1-2k) + \lambda_{N_s+1} b(k+1) = 0. \tag{3.61}$$

Therefore, we have

$$\lambda_i = \alpha^{N_s-i}\frac{b(k+1)}{m+n-1-2k}\lambda_{N_s+1}. \tag{3.62}$$

Setting $dF/da = 0$ we have

$$-1 + \sum \lambda_i = 0. \tag{3.63}$$

We are interested in the region $b > (m-k-1)(n-k-1)/(k+1)$ and therefore $\alpha > 0$. So all λ's are strictly positive. Therefore, from the KKT conditions, it follows that the optimal solution satisfies $C_i = 0$ for $i = 1$ to $N_S + 1$.

2. Optimality of Equating Exponents for LSBLEND

The exponent $a(b)$ is given by the following optimization problem

$$\begin{aligned} a(b) \quad &= - \min_{a, r_1, \ldots, r_{N_t}} -a \\ &\text{subject to:} \\ &C_i = a - \frac{b - b_k}{N_t} \sum_{j=0}^{i-1} r_j - d^*(r_i) \leq 0 \qquad \text{for } i = 1, 2, \ldots, N_t; \\ &C_{N_t+1} = a - \frac{b - b_k}{N_t} \sum_{j=0}^{N_t} r_j + b_k(k+1) \leq 0; \\ &r_i \geq 0; \quad r_i < m; \quad r_0 = 0. \end{aligned}$$

As in Appendix 1 we ignore the constraints $0 < r_i < m$ and consider the function $F = -a + \sum_{i=1}^{N_t+1} \lambda_i C_i$. Setting $dF/dr_i = 0$ we have for $i = 1$ to N_t

$$\frac{dF}{dr_i} = -\frac{b - b_k}{N_t} \sum_{j=i+1}^{N_t+1} \lambda_j - \lambda_i \frac{d}{dr_i}(d^*(r_i)) = 0. \tag{3.64}$$

Note that $\frac{b-b_k}{N_t} > 0$ and $\frac{d}{dr_i}(d^*(r_i)) < 0$. Starting from $i = N_t$ and solving recursively for λ_i in terms of λ_{N_t+1} we observe that the λ_i's are of form $\alpha_i \lambda_{N_s+1}$ where $\alpha_i > 0$. By setting $dF/da = 0$ we have $\sum_{i=1}^{N_t+1} \lambda_i = 1$. Therefore, $\lambda_i > 0$ for all i and hence, from the KKT conditions, we conclude that the optimal solution satisfies $C_i = 0$ for $i = 1$ to $N_t + 1$.

CHAPTER IV

DISTORTION SNR EXPONENT FOR THE MIMO CHANNEL WITH SNR UNKNOWN AT THE TRANSMITTER

In this chapter we derive the distortion SNR exponent of layered schemes based on superposition for the MIMO Rayleigh fading channel whose SNR is not known at the transmitter. We first define the diversity versus decodable rate tradeoff. That is, we show that for SNR independent coding scheme of length T, the probability that a receiver with receive SNR ρ is unable to recover the first $Tr\log(1+\rho)$ information bits decays as $\rho^{-d(r)}$, where $d(r)$, referred to as the diversity, is a decreasing function of the decodable multiplexing rate r. We then show how the $d(r)$ versus r tradeoff can be used to compute the distortion SNR exponent.

We compute an achievable diversity for a superposition scheme. When the MIMO system has one degree of freedom, the obtained diversity is identical to the best achievable diversity corresponding to diversity multiplexing tradeoff of Zheng and Tse and the scheme is optimal. The distortion SNR exponents obtained for the superposition scheme are identical to the distortion SNR exponent of the Broadcast Scheme discussed in chapter III which is optimal for certain range of the bandwidth expansion factor. We can also conclude that for superposition based layering scheme, knowledge of SNR is not required to be optimal in terms on the distortion SNR exponent.

The results presented in this chapter have appeared in [26, 27].

The chapter is organized as follows. The problem statement and the channel model are discussed in section A. The diversity versus decodable rate tradeoff is described in section B. Related works are discussed in C. Section D summarizes the main contributions of this chapter. The proposed superposition scheme is described in section E and its corresponding diversity versus decodable rate tradeoff is derived in section F. The distortion SNR exponent is then computed is section G. An application of the diversity versus decodable rate tradeoff in analysis of digital data transmission is presented in section H and finally we conclude in section I

A. Introduction

In this chapter we are interested in studying schemes for applications like video broadcast over a wireless channel. Usually in such applications there are several users and each user has a different average receive SNR due to path loss and shadowing. The traditional approach for designing coding schemes for such a system is to use a strong source code to compress the video and then transmit the compressed bits using a good channel code. However, in this case, when the channel capacity is less that the rate of the channel code, the receiver gets no video. On the other end, even if the channel is very good, the user is stuck with the quantization error. We are interested in schemes that offer a graceful degradation of performance with SNR.

A good channel model to study this problem is the quasi static block Rayleigh fading multiple input multiple output (MIMO) wireless channel where the SNR and the fading realization are not known at the transmitter. The channel model we use is given by

$$y_t = \sqrt{\frac{\rho}{M}} \mathbf{H}_{\lceil \frac{Lt}{T} \rceil} x_t + w_t, \quad t = 1, \ldots, T \tag{4.1}$$

where: T is the duration (in channel uses) of the transmitted block; $\mathbf{H}_l \in \mathcal{C}^{N \times M}$, $l = 1, \ldots, L$, is the channel matrix for $\frac{(l-1)T}{L} < t \leq \frac{lT}{L}$ containing random independent and identically distributed (i.i.d.) elements $h^l_{i,j} \sim \mathcal{CN}(0,1)$ (Rayleigh independent fading). The channel matrix for different blocks are independent; x_t is the transmitted signal at time t; the transmitted codeword, $X = [x_1, \ldots, x_T]$, is normalized such that trace$(E[X^H X]) \leq MT$; $w_t \sim \mathcal{CN}(0, I^{M\times M})$ is additive white Gaussian noise; ρ denotes the signal-to-noise ratio (SNR), defined as the ratio of the average received signal energy per receiving antenna to the noise per-component variance. We also define $M^* = \min\{M, N\}$.

To make the problem analytically tractable we model the source, S, as an i.i.d complex Gaussian sequence. We assume that K source samples have to be transmitted over $T = bK$ uses of the channel where b is referred to as the bandwidth expansion factor. The aim is to minimize the end-to-end mean square error distortion. In this setup, since SNR is unknown at the transmitter, the scheme used to transmit the source has to be independent of SNR. For any such scheme, $\mathcal{SC}$, we use $D_{\mathcal{SC}}(\rho) = E[(S - \hat{S})^2]$ to denote the mean square error when the receive SNR is ρ. Here $\hat{S}$ denotes the estimate of S obtained from the received signal and the expectation is taken over the source statistics, the fading realization, and the noise. We are interested in schemes that are asymptotically optimal (as $\rho \to \infty$) and hence we use the distortion SNR exponent as the performance metric. The distortion SNR exponent for the scheme $\mathcal{SC}$ is given by

$$a_{\mathcal{SC}}^{fixed}(b) = -\lim_{\rho\to\infty} \frac{\log D_{\mathcal{SC}}(\rho)}{\log \rho}. \tag{4.2}$$

We use the superscript fixed to denote that the scheme here is fixed independent of SNR since we will also be referring to the distortion SNR exponent of schemes that could depend on SNR later in the text.

In this work, we consider coding schemes where the source is encoded using a successively refinable source coder and then the source coded bits are transmitted over the channel using digital codebooks. A successively refinable source is a source that can be compressed in layers such that the distortion achieved when the reconstruction is performed using the first P layers is identical to the best achievable distortion when

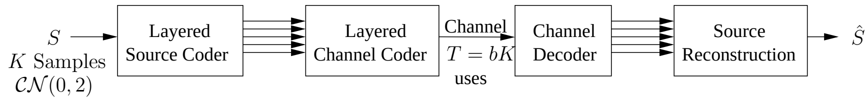

Fig. 17. Layered Source-Channel Coding Scheme.

a single source code of rate $r_1+r_2+\ldots+r_P$ is used, where r_i denotes the source coding rate corresponding to the ith layer. The Gaussian source is a successively refinable source and recent results suggest that all sources are nearly successively refinable [28]. In this case, the requirement that receivers with higher SNR receive a better quality video is equivalent to a requirement that a user with a higher SNR, on average, is able to recover more successive refinement layers than a user with lower SNR. Ideally we would like the users to recover the successive refinement layers at a rate close to their (instantaneous) capacity. Of course, this may not be possible. In this chapter, we present a framework for analyzing channel coding schemes that allow users with differing SNR's to recover information at a rate proportional to their capacity and show how it can be used to compute the distortion SNR exponent.

We will refer to channel coding schemes that allow us to decode the information bits in layers as layered channel coding schemes. Also, joint source-channel coding schemes that use layered source code followed by a layered channel code will be referred to as layered source-channel coding schemes.

B. Diversity versus Decodable Rate Tradeoff

We consider a system where bits $b_1, b_2, \ldots, b_K$ are transmitted using a fixed coding scheme $\mathcal{C}$ (independent of SNR) over T uses of an $M \times N$ MIMO channel. We say that the receiver decodes a rate of $R = k/T$ if the receiver is able to perfectly recover the bits $b_1, b_2, \ldots b_k$ from the received signal. We are interested in characterizing $P_{\mathcal{C}}(R, \rho)$, the probability that a receiver with receive SNR ρ is unable to decode a rate of R. To characterize the high SNR performance we pose a problem similar to that used in [6] to characterize the diversity multiplexing tradeoff. We consider $R = r\log(1+\rho)$ and define

$$d_{\mathcal{C}}(r) = \lim_{\rho\to\infty} \frac{P_{\mathcal{C}}(r\log(1+\rho), \rho)}{\log\rho}. \tag{4.3}$$

Therefore, $P_{\mathcal{C}}(r\log(1+\rho), \rho)$ decays as $\rho^{-d_{\mathcal{C}}(r)}$. Note that for the high SNR performance analysis we let $\rho \to \infty$ and consider a rate of $r\log(1+\rho)$ which essentially implies that the encoding scheme has to have an infinite rate to get a non zero $d_{\mathcal{C}}(r)$. In practice though, we could limit the scheme to have a large maximum rate, greater than $\min(M, N)\log\rho_{max}$, where ρ_{max} is the largest SNR of interest. $d_{\mathcal{C}}(r)$ will nevertheless be useful in predicting the high SNR behavior of the scheme.

Corresponding to each layered channel coding scheme $\mathcal{C}$ we have a $d_{\mathcal{C}}(r)$ versus r curve which we refer to as the diversity versus decodable multiplexing rate (DMR) curve. Comparing two schemes using the diversity versus DMR curve is not straightforward since it may be possible to have one scheme with a higher diversity than the second scheme for some values of DMR while the second scheme has better diversity for other values of DMR. Therefore, defining the best diversity versus DMR curve is not possible unless a scheme has better diversity than all other schemes for all values of DMR. However, it is possible to define the best diversity $d^*(r)$ for a particular DMR r, as

$$d^*(r) = \sup_{\mathcal{C}} \; d_{\mathcal{C}}(r) \tag{4.4}$$

where the supremum is taken over all coding schemes $\mathcal{C}$.

C. Related Work

In [6], Zheng and Tse considered a family of coding schemes $\{\mathcal{C}(\rho)\}$ where $\mathcal{C}(\rho)$ denotes the coding scheme corresponding to an SNR ρ. They showed that if the family of coding schemes has a multiplexing rate of r, i.e., the rate of the coding scheme grows as $r\log(1+\rho)$, then the average error probability decays at a rate of at most $\rho^{-d^*_{DMT}(r)}$ where

$$d^*_{DMT}(r) = L(M-k)(N-k) - L(M+N-1-2k)\delta \tag{4.5}$$

for $0 < r < \min(M,N)$ where $k = \lfloor r \rfloor$ is the largest integer less than r and $\delta = r-k$. One of the key features in the problem formulation of Zheng and Tse is that a different coding scheme is chosen for each SNR. We are interested in schemes that are independent of SNR but that allow us to decode at a rate of $r\log(1+\rho)$ when the receive SNR is ρ. It follows that $d^*_{DMT}(r)$ is an upper bound on $d^*(r)$ since diversity multiplexing tradeoff allows for the scheme to change depending on the SNR while the diversity versus decodable rate tradeoff doesn't.

In [29], Diggavi and Tse analyzed diversity embedded codes. The basic idea is to consider a family of coding schemes with a coding scheme for each SNR ρ such that the data in each code in embedded in layers with a rate of $r_i \log\rho$ in layer i. The receiver attempts to recover the layers in a sequential order. They proposed a superposition scheme that simultaneously achieves the best possible diversity for all layers when $\min(M,N) = 1$ with the diversity for layer i being $d_i = d^*_{DMT}(\sum_{j=i1}^{i} r_j)$. By considering image transmission over fading channels, the authors in [30] showed that the diversity embedded schemes can be used with layered source coding to get a lower distortion. Notice that in [29], the authors still consider a family of coding scheme whereas, in our problem, the scheme used has to be independent of SNR and hence the problem is very different from that considered in [29, 30].

Shamai and Steiner [31] considered the problem of maximizing average throughput of the superposition scheme for a particular SNR. Ng *et al.* [32, 33] and Tian *et al.* [34, 35] considered the problem of finding the power allocation for a layered superposition scheme that minimizes the average mean square error distortion. The optimal power allocation is derived for the SISO block fading channel. Again, the scheme depends on the SNR and it is not clear how a scheme designed for a particular SNR performs when the actual SNR is higher than the design SNR.

In [12, 14, 17, 21, 24], the problem of transmitting K samples of a Gaussian source using $T = bK$ uses of a MIMO channel when the SNR is known at the transmitter is studied. The aim is to find a family of coding scheme $\{\mathcal{SC}(\rho)\}$ that has the maximum distortion SNR exponent. The distortion SNR exponent is defined as the rate of decay of $D_{\mathcal{SC}}(\rho)$ with ρ and is given by

$$a_{\mathcal{SC}}(b) = \lim_{\rho \to \infty} \frac{-\log D(\rho)}{\log \rho} \tag{4.6}$$

where $D_{\mathcal{SC}}(\rho)$ denotes the mean square error corresponding to the coding scheme $\mathcal{SC}(\rho)$ at SNR of ρ. Clearly

$$a^*(b) = \sup_{\mathcal{SC}} a_{\mathcal{SC}}(b) \geq a_{\mathcal{SC}}^{fixed}(b). \tag{4.7}$$

Currently the best known distortion SNR exponent for this problem are those reported in Chapter III of this dissertation. Results also reported in [11, 12].

D. Main Results

In this chapter we propose a superposition based scheme and study the diversity versus DMR curve for this scheme. The main results that we have are summarized below.

- For the block fading MIMO channel, $d_{\mathcal{C}}(r)$ can be made arbitrarily close to $((M-c_1)(N-c_1)L - c_2(M+N-1-2c_1))\left(1-\frac{Lr}{c_1L+c_2+1}\right)$ for $r < \frac{c_1L+c_2+1}{L}$ for integers $c_1 < M^*$ and $c_2 < L$. The result specialized for the SIS0-SIMO-MISO, MIMO, and the SISO L-block fading case is specified below.
  - For the SISO / MISO / SIMO quasi static fading channel, i.e., when $\min(M,N) = 1$ and $L = 1$, $d^*(r)$ can be made arbitrarily close to $d^*_{DMT}(r) = \max(M,N)(1-r)$ for $r < 1$ and the proposed superposition scheme achieves it for all values of DMR r.
  - For the MIMO channel with $L = 1$, $d_{\mathcal{C}}(r)$ for the superposition scheme can be made arbitrarily close to $(M-k)(N-k)(1-\frac{r}{k+1})$ for $r < k+1$ and integer $k < M^*$.
  - For the block fading SISO channel, $d_{\mathcal{C}}(r)$ can be made arbitrarily close to $(L-k)(1-\frac{Lr}{k+1})$ for $r < (k+1)/L$ for integer $k < L$.
- A layered source coder followed by the proposed superposition scheme as the layered channel coding scheme can be designed to obtain an achievable distortion SNR exponent of $a^{fixed}_{\mathcal{SC}}(b)$ given by

$$a^{fixed}_{\mathcal{SC}}(b) = \min\left(b\frac{c_1L+c_2+1}{L}, (M-c_1)(N-c_1)L - c_2(M+N-1-2c_1)\right) \tag{4.8}$$

  for any integers c_1 and c_2 with $0 \leq c_1 < M^*$ and $0 \leq c_2 < L$.
  - This expression overlaps with the exponent achieved by the SNR dependent superposition scheme (termed broadcast scheme) in [12] which was also shown to be the best achievable exponent for any superposition scheme. Therefore, no superposition scheme can obtain a better exponent than that reported here.

– For $b < \frac{|N-M|+1}{M^*}$ and $b > MNL^2$, this achievable exponent coincides with an upper bound on the exponent obtained by assuming that the channel state information (channel realization $\mathbf{H}$ and the SNR ρ) is available at the transmitter and hence this is the optimal exponent for these values of bandwidth expansion.

– For the SISO-SIMO-MISO case with $L = 1$, the proposed scheme achieves the optimal exponent for all values of bandwidth expansion.

E. Proposed Scheme

The transmitted signal is chosen as

$$X^{M \times T} = \sum_{i=1}^{N_s} \sqrt{\gamma^{-(i-1)} - \gamma^{-i}} X_i^{M \times T} \tag{4.9}$$

where $\gamma > 1$ and X_i is a codeword chosen from a Gaussian codebook $\mathcal{C}_i$ of rate $R_c = \frac{1}{L} \log \frac{1+a}{1+a/\gamma}$. We will also study the case with $R_c = \frac{s}{L} \log \gamma$. $\mathcal{C}_i$ is such that $E[X_{i,j} X_{i,j}^H] = I^{M \times M}$ where $X_{i,j}$ is the jth column in X_i. Since we have T channel uses, TR_c information bits can be sent in each layer. The first TR_c information bits are used to select X_1, the next TR_c information bits are used to select X_2 and so on. In practice, if we are interested in schemes that perform well for some range of SNR bounded by ρ_{max}, it is sufficient to encode layers such that the total rate is $M^* \log \rho_{max}$, i.e., choose $N_s = \frac{M^* \log \rho_{max}}{R_c}$. However, in the analysis here, we will assume that $N_s = \infty$. A schematic of the encoding scheme is shown in Fig. 18.

The receiver decodes the layers using successive interference cancellation in the order X_1, X_2 and so on. That is, the receiver attempts to decode X_1 by treating the other X_i's as additional noise. If decoding is successful, it removes the contribution of X_1 from Y and attempts to decode X_2 treating X_i for $i > 2$ as noise. This process

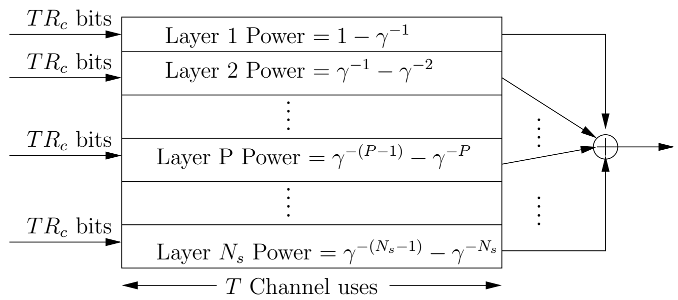

Fig. 18. Superposition Scheme with Each Layer Having Equal Rate of R_c.

is continued till decoding of a particular layer fails. If the receiver is able to decode n layers, it recovers nTR_c information bits which corresponds to a rate of nR_c.

In chapter II of this dissertation and in [7, 8], such a scheme was used along with a successively refinable source coder for transmitting K samples of a Gaussian source over $T = bK$ uses of an AWGN channel when the SNR is not known at the transmitter. It was shown that this scheme nearly achieves the optimal decay rate of SNR^{-b} in the high SNR regime for the minimum mean square error in estimating the source from the received signal.

F. Performance Analysis

To compute the diversity versus DMR curve for the proposed superposition scheme, we will need the following lemma's.

Lemma F.1 $\frac{1+c\gamma^{-(j-1)}}{1+c\gamma^{-j}}$ *is a decreasing function of* j *for* $c \geq 0$ *and* $\gamma > 1$.

Let $x = \gamma^{-(j-1)}$ and consider $f(x) = \frac{1+cx}{1+cx/\gamma}$.

$$\frac{df(x)}{dx} = \frac{c(1+cx/\gamma) - (1+cx)c/\gamma}{(1+cx/\gamma)^2} = \frac{c - c/\gamma}{(1+cx/\gamma)^2} \geq 0.$$

Therefore $f(x)$ is an increasing function of x and hence a decreasing function of j.

Lemma F.2 *Let* $Y = \sqrt{\frac{\rho}{M}}\mathbf{H}X + W$*, where* $\mathbf{H} \in \mathcal{C}^{N\times M}$*,* $X = \sum_{i=1}^{\infty} \sqrt{(\gamma^{-(i-1)} - \gamma^{-i})}X_i$ *with* $\gamma > 1$*,* $X_i, W \in \mathcal{C}^{M\times 1}$ *with* $X_{i,j}$ *and* W_j *being i.i.d. complex Gaussian with zero mean and unit variance. We have*

$$I(X_i; Y|X_1, \ldots, X_{i-1}) = \log \frac{det(I + \gamma^{-(i-1)}\frac{\rho}{M}\mathbf{H}\mathbf{H}^H)}{det(I + \gamma^{-i}\frac{\rho}{M}\mathbf{H}\mathbf{H}^H)}. \tag{4.10}$$

Let $Y_j = \sqrt{\frac{\rho}{M}}\mathbf{H}(\sum_{i=j}^{\infty} \sqrt{(\gamma^{-(i-1)} - \gamma^{-i})}X_i) + W$. Then, $Y_j \sim \mathcal{CN}(0, I + \gamma^{-(j-1)}\frac{\rho}{M}\mathbf{H}\mathbf{H}^H)$. We have

$$\begin{aligned} & I(X_i; Y|X_1, \ldots, X_{i-1}) \\ = \; & h(Y|X_1, \ldots, X_{i-1}) - h(Y|X_1, \ldots, X_i) \\ = \; & h(Y_i) - h(Y_{i+1}) \\ = \; & \log\det(I + \gamma^{-(i-1)}\frac{\rho}{M}\mathbf{H}\mathbf{H}^H) - \log\det(I + \gamma^{-i}\frac{\rho}{M}\mathbf{H}\mathbf{H}^H). \end{aligned} \tag{4.11}$$

This concludes the proof.

For a fixed l, let $\lambda_{m,l}$ for $m = 1, 2, \ldots, M^*$ denote the ordered eigenvalues of $\mathbf{H}_l\mathbf{H}_l^H$ with $\lambda_{m,l} \le \lambda_{m+1,l}$. Note that $\lambda_{m,l} \ge 0$.

Lemma F.3 *The probability that a receiver with a receive SNR of* ρ *is unable to decode a rate of* $r\log(1+\rho)$ *is given by*

$$P_{out}(r\log(1+\rho), \rho) = P\left(\prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1 + \lambda_{m,l}\frac{\rho\gamma^{-N_r+1}}{M}}{1 + \lambda_{m,l}\frac{\rho\gamma^{-N_r}}{M}}\right) < 2^{R_c L}\right) \tag{4.12}$$

where

$$N_r = \left\lceil \frac{r\log(1+\rho)}{R_c} \right\rceil. \tag{4.13}$$

The rate embedded in each layer is R_c. In order to recover a rate of $r\log(1+\rho)$ the receiver should be able to decode at least the first $N_r = \left\lceil \frac{r\log(1+\rho)}{R_c} \right\rceil$ layers.

We have,

$$
\begin{aligned}
&1 - P_{out}(r\log(1+\rho),\rho)\\
&= P(\text{Layers 1 to } N_r \text{ are decoded})\\
&= P\left(\cap_{j=1}^{N_r}\{\mathbf{H}_1,\ldots,\mathbf{H}_l : \text{Layer } j \text{ can be decoded given } X_1 \text{ to } X_{j-1}\}\right)\\
&= P\left(\cap_{j=1}^{N_r}\{\mathbf{H}_1,\ldots,\mathbf{H}_l : I(X_j;Y \mid X_1,\ldots,X_{j-1}) \geq R_c\}\right)\\
&\stackrel{(a)}{=} P\left(\frac{1}{L}\sum_{l=1}^{L}\log\frac{\det(I+\frac{\rho\gamma^{-(j-1)}}{M}\mathbf{H}_l\mathbf{H}_l^H)}{\det(I+\frac{\rho\gamma^{-j}}{M}\mathbf{H}_l\mathbf{H}_l^H)} \geq R_c \;\forall\, j\leq N_r\right)\\
&\stackrel{(b)}{=} P\left(\prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1+\lambda_{m,l}\frac{\rho\gamma^{-(j-1)}}{M}}{1+\lambda_{m,l}\frac{\rho\gamma^{-j}}{M}}\right) \geq 2^{R_cL} \;\forall\, j\leq N_r\right)\\
&\stackrel{(c)}{=} P\left(\prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1+\lambda_{m,l}\frac{\rho\gamma^{-N_r+1}}{M}}{1+\lambda_{m,l}\frac{\rho\gamma^{-N_r}}{M}}\right) \geq 2^{R_cL}\right)\\
&= 1 - P\left(\prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1+\lambda_{m,l}\frac{\rho\gamma^{-N_r+1}}{M}}{1+\lambda_{m,l}\frac{\rho\gamma^{-N_r}}{M}}\right) < 2^{R_cL}\right).
\end{aligned}
\tag{4.14}
$$

(a) is obtained by extending lemma F.2 to the block fading channel case. (b) follows since $\lambda_{m,l}$'s are eigenvalues of $\mathbf{H}_l\mathbf{H}_l^H$. Let

$$
A_j = \left\{\{\lambda_{m,l}\} : \prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1+\lambda_{m,l}\frac{\rho\gamma^{-(j-1)}}{M}}{1+\lambda_{m,l}\frac{\rho\gamma^{-j}}{M}}\right) \geq 2^{R_cL}\right\}. \tag{4.15}
$$

Since $\lambda_{m,l} \geq 0$, we have $\lambda_{m,l}\rho/M \geq 0$, and therefore from Lemma F.1 we have $\prod_{l=1}^{L}\prod_{m=1}^{M^*}\left(\frac{1+\lambda_{m,l}\frac{\rho\gamma^{-(j-1)}}{M}}{1+\lambda_{m,l}\frac{\rho\gamma^{-j}}{M}}\right)$ is a decreasing function of j. Therefore

$$
A_1 \supseteq A_2 \supseteq \ldots \supseteq A_j \supseteq A_{j+1} \ldots \supseteq A_{N_s}. \tag{4.16}
$$

We are interested in $P\left(\cap_{j=1}^{N_r} A_j\right) = P(A_{N_r})$. This proves the equality (c).

The above lemma implies that, for the proposed scheme, the probability of outage for decoding the first N_r layers is identical to the outage probability of a genie aided decoder attempting to decode the N_rth layer when the genie provides the signal transmitted in the first $N_r - 1$ layers. This makes the performance analysis considerably simpler.

Lemma F.4 *Let $\lambda_{m,l}$ for $m = 1, 2, \ldots, M^*$ be the eigenvalues of $\mathbf{H}_l\mathbf{H}_l^H$. Let $\lambda_{m,l} = \rho^{-\alpha_{m,l}}$. Let $\mathcal{A}$ be a set of α's satisfying $\alpha_{m,l} \geq \alpha_{m+1,l}$ $\forall m, l$. Then, for the Rayleigh fading channel,*

$$P(\alpha \in \mathcal{A}) \doteq \rho^{-d} \tag{4.17}$$

where

$$d = \inf_{\alpha \in \mathcal{A} \cap (\mathcal{R}^+)^{M^* L}} \sum_{l=1}^{L} \sum_{m=1}^{M^*} (2m - 1 + |N - M|)\alpha_{m,l}. \tag{4.18}$$

Here $A \doteq B$ denotes $\lim_{\rho\to\infty} \frac{\log A}{\log \rho} = \lim_{\rho\to\infty} \frac{\log B}{\log \rho}$.

This is specified in (12) in [6] for the $L = 1$ case. The result for $L > 1$ can be obtained by noting that the channel realizations corresponding to the different blocks are independent, hence the probabilities multiply, which corresponds to the sum of the exponents.

In the following subsections we evaluate $P_{out}(r\log(1+\rho), \rho)$ for different scenarios in the limit when $\gamma \to 1$. We first start with solving the SISO case where the outage probability expression can be obtained in terms of the non zero eigen value of $\mathbf{HH^H}$, λ_1. This gives us the insight necessary for solving the more general case. In the proof for this case we will carefully show how $P_{out}(r\log(1+\rho), \rho)$ behaves in the limit $\gamma \to 1$. Similar limits will be required in the other cases and detailed proofs with respect to behavior with $\gamma \to 1$ in those cases is avoided. We then solve the MIMO channel case with $L = 1$. Generalization to the block fading SISO channel and the MIMO block fading channel are relatively straightforward and are discussed after the MIMO channel case.

1. SISO / SIMO / MISO

Theorem F.1 *When $M^* = 1$, $L = 1$, and $R_c = \log \frac{1+a}{1+a/\gamma}$, in the limit $\gamma \to 1$, the diversity versus DMR curve achieved by the superposition scheme is given by*

$$d_{\mathcal{C}}(r) = d^*_{DMT}(r(1+1/a)) = \max(M,N)(1 - r(1+1/a)).$$

Since $M^* = 1$, we have only one non zero eigenvalue λ_1. Therefore (4.14) reduces to

$$\begin{aligned}
&P_{out}(r\log(1+\rho),\rho)\\
&= P\left(\frac{1+\lambda_1\frac{\rho\gamma^{-N_r+1}}{M}}{1+\lambda_1\frac{\rho\gamma^{-N_r}}{M}} < 2^{R_c}\right)\\
&= P\left(\lambda_1\frac{\rho\gamma^{-N_r+1}}{M}\left(1-\frac{2^{R_c}}{\gamma}\right) < 2^{R_c}-1\right)\\
&= P\left(\lambda_1\frac{\rho\gamma^{-N_r+1}}{M}\left(1-\frac{1+a}{(1+a/\gamma)\gamma}\right) < \frac{1+a}{1+a/\gamma}-1\right)\\
&= P\left(\lambda_1\frac{\rho\gamma^{-N_r+1}}{M}\left(\frac{\gamma-1}{(1+a/\gamma)\gamma}\right) < \frac{a(1-1/\gamma)}{1+a/\gamma}\right)\\
&= P\left(\lambda_1\frac{\rho\gamma^{-N_r+1}}{M} < a\right).
\end{aligned}$$

Since $N_r = \left\lceil \frac{r\log(1+\rho)}{R_c} \right\rceil$, we have $\frac{r\log(1+\rho)}{R_c} \le N_r < \frac{r\log(1+\rho)}{R_c} + 1$. Now since $\gamma > 1$ we have

$$P\left(\lambda_1\frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}+1}}{M} < a\right) \le P\left(\lambda_1\frac{\rho\gamma^{-N_r+1}}{M} < a\right) < P\left(\lambda_1\frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}}}{M} < a\right). \tag{4.19}$$

We are interested in $\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho)$. The probability density function of λ_1 is continuous, and hence from the continuity of the probability measure we have

$$\begin{aligned}\lim_{\gamma\to 1} P\left(\lambda_1 \frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}}}{M} < a\right) &= P\left(\lim_{\gamma\to 1}\lambda_1 \frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}}}{M} < a\right) \\ &= P\left(\lim_{\gamma\to 1}\lambda_1 \frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}+1}}{M} < a\right) \\ &= \lim_{\gamma\to 1} P\left(\lambda_1 \frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}+1}}{M} < a\right).\end{aligned}$$

Therefore in the limit as $\gamma \to 1$, the upper bound and lower bound in (4.19) coincide and hence we have

$$\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho) = P\left(\lim_{\gamma\to 1}\lambda_1 \frac{\rho\gamma^{-\frac{r\log(1+\rho)}{R_c}}}{M} < a\right). \tag{4.20}$$

Let

$$\begin{aligned} l &= \lim_{\gamma\to 1}\gamma^{-\frac{r\log(1+\rho)}{R_c}} \\ &= \lim_{\gamma\to 1}\gamma^{-\frac{r\log(1+\rho)}{\log(1+a)-\log(1+a/\gamma)}}.\end{aligned}$$

Then

$$\begin{aligned}\log l &= -\lim_{\gamma\to 1}\frac{r\log(1+\rho)}{\log(1+a)-\log(1+a/\gamma)}\log\gamma \\ &= -\lim_{\gamma\to 1}\frac{r\log(1+\rho)}{\frac{a/\gamma^2}{(1+a/\gamma)}}1/\gamma \quad \text{(L'Hospital's rule)} \\ &= -r(1+1/a)\log(1+\rho).\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho) &= P\left(\lambda_1 < aM\rho^{-1}(1+\rho)^{r(1+1/a)}\right) &\quad (4.21)\\ &= P\left(\|h\|^2 < aM\rho^{-1}(1+\rho)^{r(1+1/a)}\right). &\quad (4.22)\end{aligned}$$

In the high SNR regime,

$$\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho) \doteq P\left(\lambda_1 < aM\rho^{-1+r(1+1/a)}\right). \tag{4.23}$$

By setting $\lambda_1 = \rho^{-\alpha_1}$ we obtain

$$\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho) \doteq P(\alpha_1 > 1 - r(1+1/a)). \tag{4.24}$$

From Lemma F.4 we have $P_{out}(r\log(1+\rho),\rho) \doteq \rho^{-d}$ where

$$\begin{aligned} d &= \inf_{\alpha > 1-r(1+1/a)} (|N-M|+1)\alpha \\ &= (|N-M|+1)(1-r(1+1/a)) \\ &= \max(M,N)(1-r(1+1/a)). \quad (\because M^* = \min(M,N) = 1) \end{aligned}$$

In Fig. 19 and Fig. 20 we compare the outage probability of the proposed scheme for a 3×1 MIMO system specified in (4.22) with a lower bound obtained by choosing the best coding scheme for each SNR and each multiplexing rate. Note that, in Fig. 19, the slope of the two curves are nearly equal.

2. The General MIMO Channel

Theorem F.2 *For the $M\times N$ MIMO channel with $L=1$, the diversity versus DMR curve corresponding to the superposition coding scheme with $R_c = s\log\gamma$ is given by*

$$d_{\mathcal{C}}(r) = (M-k)(N-k)\left(1-\frac{r}{s}\right)$$

where $k = \lfloor s \rfloor$.

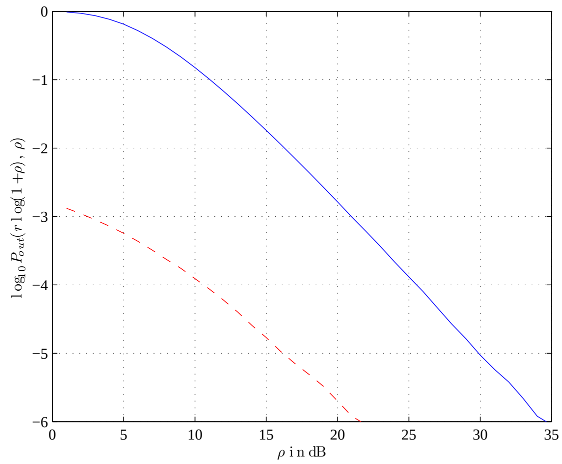

Fig. 19. $P_{out}(r\log(1+\rho),\rho)$ versus ρ for 3×1 MIMO System with $r=0.2$ and $a=5$.

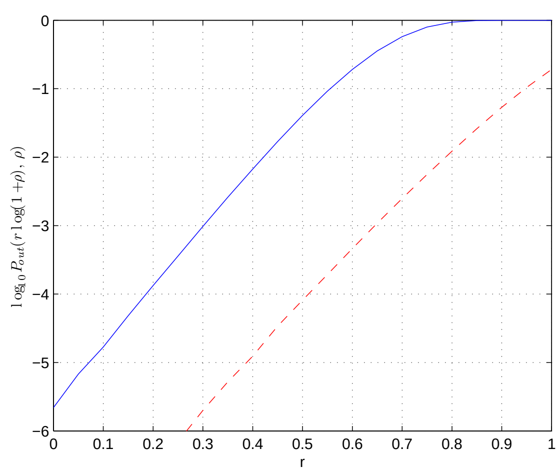

Fig. 20. $P_{out}(r\log(1+\rho),\rho)$ versus r for 3×1 MIMO System with $\rho=25$ db and $a=5$.

Starting from lemma F.3 we have,

$$
\begin{aligned}
&P_{out}(r\log(1+\rho),\rho)\\
&= P\left(\prod_{m=1}^{M^*}\left(\frac{1+\lambda_m\frac{\rho\gamma^{-N_r+1}}{M}}{1+\lambda_m\frac{\rho\gamma^{-N_r}}{M}}\right)<2^{R_c}\right)\\
&= P\left(\prod_{m=1}^{M^*}\left(1+\lambda_m\frac{\rho\gamma^{-N_r+1}}{M}\right)-2^{R_c}\cdot\prod_{m=1}^{M^*}\left(1+\lambda_m\frac{\rho\gamma^{-N_r}}{M}\right)+2^{R_c}-1<2^{R_c}-1\right)\\
&= P\Bigg(\sum_{m_1}\lambda_{m_1}\frac{\rho\gamma^{-N_r+1}}{M}\cdot\frac{1-\frac{2^{R_c}}{\gamma}}{2^{R_c}-1}+\sum_{m_1,m_2:m_1<m_2}\lambda_{m_1}\lambda_{m_2}\left(\frac{\rho\gamma^{-N_r+1}}{M}\right)^2\cdot\frac{1-\frac{2^{R_c}}{\gamma^2}}{2^{R_c}-1}\\
&\qquad +\ldots+\sum_{m_1,m_2,\ldots,m_m:m_1<m_2<\ldots<m_m}\lambda_{m_1}\ldots\lambda_{m_m}\left(\frac{\rho\gamma^{-N_r+1}}{M}\right)^m\cdot\frac{1-\frac{2^{R_c}}{\gamma^m}}{2^{R_c}-1}+\ldots\\
&\qquad +\lambda_1\ldots\lambda_{M^*}\left(\frac{\rho\gamma^{-N_r+1}}{M}\right)^{M^*}\cdot\frac{1-\frac{2^{R_c}}{\gamma^{M^*}}}{2^{R_c}-1}<1\Bigg)\\
&= P\left(\sum_{m=1}^{M^*}\sum_{m_1,m_2,\ldots,m_m:m_1<m_2<\ldots<m_m}\lambda_{m_1}\ldots\lambda_{m_m}\left(\frac{\rho\gamma^{-N_r+1}}{M}\right)^m\cdot\frac{1-\frac{2^{R_c}}{\gamma^m}}{2^{R_c}-1}<1\right)\\
&= P\left(\sum_{m=1}^{M^*}\Lambda_m\left(\frac{\rho\gamma^{-N_r+1}}{M}\right)^m\cdot\frac{1-\frac{2^{R_c}}{\gamma^m}}{2^{R_c}-1}<1\right)
\end{aligned}
\tag{4.25}
$$

where

$$
\Lambda_m=\sum_{m_1,\ldots,m_m:m_1<m_2<\ldots<m_m}\lambda_{m_1}\cdots\lambda_{m_m}. \tag{4.26}
$$

As in the proof of Theorem F.1, since $N_r=\lceil\frac{r\log(1+\rho)}{R_c}\rceil$, we can show that

$$
\lim_{\gamma\to 1}P_{out}(r\log(1+\rho),\rho)=P\left(\lim_{\gamma\to 1}\sum_m\Lambda_m\rho^m\ \gamma^{-\frac{mr\log(1+\rho)}{R_c}}\cdot\frac{1}{M^m}\cdot\frac{1-\cdot\frac{2^{R_c}}{\gamma^m}}{2^{R_c}-1}<1\right). \tag{4.27}
$$

Substituting for $R_c=s\log\gamma$, we note that

$$
\gamma^{-\frac{r\log(1+\rho)}{R_c}}=\gamma^{-\frac{r\log(1+\rho)}{s\log\gamma}}=(1+\rho)^{-r/s}. \tag{4.28}
$$

Let

$$\begin{aligned} c_m &= \frac{1}{M^m} \lim_{\gamma \to 1} \frac{1 - 2^{R_c}/\gamma^m}{2^{R_c} - 1} \\ &= \frac{1}{M^m} \lim_{\gamma \to 1} \frac{1 - \frac{\gamma^s}{\gamma^m}}{\gamma^s - 1} \\ &= \frac{1}{M^m} \lim_{\gamma \to 1} \frac{-(s-m)\gamma^{s-m-1}}{s\gamma^{s-1}} \quad \text{(L'Hospital's rule)} \\ &= \frac{1}{M^m} \frac{m-s}{s}. \end{aligned}$$

Then, we have,

$$\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) = P\left(\sum_{m=1}^{M^*} \Lambda_m c_m \rho^m (1+\rho)^{-m\frac{r}{s}} < 1 \right). \tag{4.29}$$

In the high SNR regime,

$$\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) \doteq P\left(\sum_{m=1}^{M^*} c_m \Lambda_m \rho^{m(1-r/s)} < 1 \right). \tag{4.30}$$

If $k < s < k+1$ for some integer $k < M^*$, then

$$\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) \doteq P\left(\sum_{m=k+1}^{M^*} a_m \Lambda_m \rho^{m(1-r/s)} < 1 + \sum_{m=1}^{k} a_m \Lambda_m \rho^{m(1-r/s)} \right) \tag{4.31}$$

where $a_m = |c_m| > 0$.

Before we simplify (4.31) we need to study how $a_m \Lambda_m \rho^{m(1-r/s)}$ behaves in the high SNR regime. As in [6], we let $\lambda_m = \rho^{-\alpha_m}$. Let $\beta_m = 1 - \frac{r}{s} - \alpha_m$. With this definition,

$$\begin{aligned} a_m \Lambda_m \rho^{m(1-r/s)} &= a_m \left(\sum_{m_1 < m_2 < \ldots < m_m} \rho^{-(\alpha_{m_1} + \alpha_{m_2} + \ldots + \alpha_{m_m})} \right) \rho^{m(1-r/s)} \\ &= a_m \sum_{m_1 < m_2 < \ldots < m_m} \rho^{(\beta_{m_1} + \beta_{m_2} + \ldots + \beta_{m_m})} \\ &\doteq \rho^{\max_{m_1 < m_2 < \ldots < m_m} (\beta_{m_1} + \beta_{m_2} + \ldots + \beta_{m_m})}. \end{aligned}$$

It then follows from (4.31) that for $0 \leq k < M^*$,

$$\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) \doteq P\left(\max_{m>k}(\max_{m_1 < m_2 \ldots < m_m} (\beta_{m_1} + \beta_{m_2} + \ldots + \beta_{m_m})) < \max(\max_{1 \leq m \leq k} (\max_{m_1 < m_2 \ldots < m_m} (\beta_{m_1} + \beta_{m_2} + \ldots + \beta_{m_m})), 0) \right). \quad (4.32)$$

Note that since $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{M^*}$, we have $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{M^*}$, and therefore $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{M^*}$. Therefore,

$$\max_{m_1 < m_2 \ldots < m_m} (\beta_{m_1} + \beta_{m_2} + \ldots + \beta_{m_m}) = \beta_{M^*-m+1} + \beta_{M^*-m+2} + \ldots + \beta_{M^*}. \quad (4.33)$$

This implies,

$$\begin{aligned} &\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) \\ \doteq\ & P\left(\max(\beta_1 + \beta_2 + \ldots + \beta_{M^*}, \beta_2 + \ldots + \beta_{M^*}, \ldots, \beta_{M^*-k} + \ldots + \beta_{M^*}) < \right. \\ & \left. \max(\beta_{M^*-k+1} + \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \ldots, \beta_{M^*}, 0)\right) \\ \overset{(a)}{=}\ & P(\beta_{M^*-k} < 0). \end{aligned} \quad (4.34)$$

To see the equality (a), note that if $\beta_{M^*-k} < 0$, then $\beta_j < 0$ for all $j \leq M^* - k$, and hence

$$\begin{aligned} & \max(\beta_1 + \beta_2 + \ldots + \beta_{M^*}, \beta_2 + \ldots + \beta_{M^*}, \ldots, \beta_{M^*-k} + \ldots + \beta_{M^*}) \\ =\ & \beta_{M^*-k} + \ldots + \beta_{M^*} \\ <\ & \beta_{M^*-k+1} + \beta_{M^*-k+2} + \ldots + \beta_{M^*} \quad (< 0 \text{ for } k = 0) \\ \leq\ & \max(\beta_{M^*-k+1} + \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \ldots, \beta_{M^*}, 0). \end{aligned}$$

Also if $\beta_{M^*-k} \geq 0$, then $\beta_j \geq 0$ for $j \geq M^* - k$, and hence

$$\begin{aligned}
&\max(\beta_{M^*-k+1} + \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \beta_{M^*-k+2} + \ldots + \beta_{M^*}, \ldots, \beta_{M^*}, 0) \\
&= \beta_{M^*-k+1} + \beta_{M^*-k+2} + \ldots + \beta_{M^*} \quad (= 0 \text{ for } k = 0) \\
&\leq \beta_{M^*-k} + \ldots + \beta_{M^*} \\
&\leq \max(\beta_1 + \beta_2 + \ldots + \beta_{M^*}, \beta_2 + \ldots + \beta_{M^*}, \ldots, \beta_{M^*-k} + \ldots + \beta_{M^*}).
\end{aligned}$$

Now we have, $\lim_{\gamma \to 1} P_{out}(r \log(1+\rho), \rho) \doteq P(\beta_{M^*-k} < 0) = P(1 - \frac{r}{s} - \alpha_{M^*-k} < 0) \doteq \rho^{-d}$ (from lemma F.4) where

$$\begin{aligned}
d &= \inf_{\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{M^*}, \alpha_{M^*-k} > 1 - \frac{r}{s}} \sum_{i=1}^{M^*} (2i - 1 + |N - M|)\alpha_i && (4.35) \\
&= \sum_{i=1}^{M^*-k} (2i - 1 + |N - M|) \left(1 - \frac{r}{s}\right) && (4.36) \\
&= (M^* - k)(M^* - k + |N - M|) \left(1 - \frac{r}{s}\right) && (4.37) \\
&= (M - k)(N - k) \left(1 - \frac{r}{s}\right) \qquad (\text{Recall } k = \lfloor s \rfloor). && (4.38)
\end{aligned}$$

This concludes the proof.

In Theorem F.2, note that, for a fixed k, the exponent specified is maximized when $s \to k+1$ with $s < k+1$. Therefore, the superposition scheme can achieve an exponent arbitrarily close to $(M-k)(N-k)\left(1 - \frac{r}{k+1}\right)$ for $r < k+1$ and for $k < M^*$. The diversity versus DMR curve that is achievable using the superposition scheme for MIMO system with $M = 3$, $N = 1$ and $L = 1$, and $M = N = 3$ and $L = 1$ for different values of s is shown in Fig. 21 and Fig. 22 respectively. The diversity multiplexing tradeoff, which is an upper bound on the diversity versus DMR curve, is also shown in the figure.

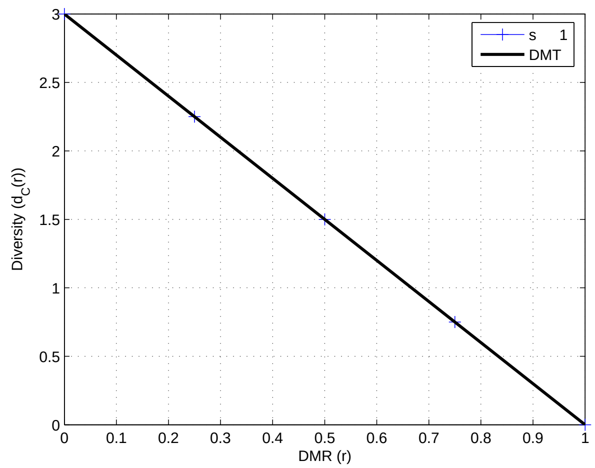

Fig. 21. Diversity versus DMR Curve for 3×1 MIMO System with $L = 1$.

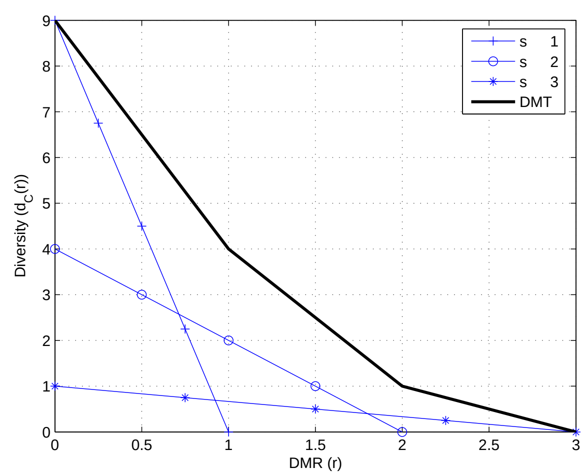

Fig. 22. Diversity versus DMR Curve for 3×3 MIMO System with $L = 1$.

3. Block Fading SISO Channel / Parallel SISO Channels

Theorem F.3 *When $M = N = 1$, the superposition scheme with $R_c = \frac{s}{L}\log\gamma$, can achieve an exponent of* $(L - \lfloor s \rfloor)(1 - \frac{r}{s})$.

Let $\lambda_l = |\mathbf{H}_l|^2$ be the eigenvalue corresponding to $\mathbf{H}_l$ and let $\lambda_l = \rho^{-\alpha_l}$. Let $\alpha_n^* = \max_n(\alpha_1, \alpha_2, \ldots, \alpha_L)$ denote the nth largest value in the set $\{\alpha_i, i = 1, \ldots, L\}$. By proceeding in a manner similar to the proof in Theorem F.2 for the MIMO channel case, for $k < s < k + 1$ for some integer k, we obtain

$$\lim_{\gamma \to 1} P_{out}(r \log(1 + \rho), \rho) \doteq P\left(1 - \frac{Lr}{s} - \alpha_{L-k}^* < 0\right) \doteq \rho^{-d} \tag{4.39}$$

where

$$d = \inf_{\alpha_{L-k}^* > 1 - \frac{Lr}{s}} \sum_{i=1}^{L} \alpha_i \tag{4.40}$$

$$= (L - k)\left(1 - \frac{Lr}{s}\right). \tag{4.41}$$

The infimum occurs when $L - k$ values of α are $1 - Lr/s$ and the remaining values are 0.

Note that the exponent in Theorem F.3 is maximized when s approaches $k + 1$ from below.

4. Block Fading MIMO Channel

Theorem F.4 *For the L-block fading $M \times N$ MIMO channel, the superposition scheme with $R_c = \frac{s}{L}\log\gamma$, achieves an exponent of*

$$d_{\mathcal{C}}(r) = ((M - c_1)(N - c_1)L - c_2(M + N - 1 - 2c_1))\left(1 - \frac{Lr}{s}\right) \tag{4.42}$$

where $c_1 = \lfloor \frac{s}{L} \rfloor$ and $c_2 = \lfloor s - c_1 L \rfloor$.

For the $M \times N$ block fading MIMO channel with L blocks, let $\alpha_n^* = \max_n(\{\alpha_{m,l}, l = 1,\ldots,L, m = 1,\ldots,M^*\})$ denote the nth largest value in the set $\{\alpha_{m,l} : \forall m, l\}$.

Let k be an integer such that $k < s < k+1$. Proceeding as in Theorem F.2 we have

$$\lim_{\gamma\to 1} P_{out}(r\log(1+\rho),\rho) \doteq P\left(1 - \frac{Lr}{s} - \alpha^*_{M^*L-k} < 0\right) \doteq \rho^{-d_{\mathcal{C}}(r)} \tag{4.43}$$

where

$$d_{\mathcal{C}}(r) = \inf_{\alpha^*_{M^*L-k}=\max_{LM^*-k}(\{\alpha_{m,l}\})>1-\frac{Lr}{s}} \sum_{l=1}^{L}\sum_{m=1}^{M^*}(2m-1+|N-M|)\alpha_{m,l}. \tag{4.44}$$

Let $k = c_1L + c_2$. Then $LM^* - k = (M^* - c_1 - 1)L + L - c_2$. One set of α's that corresponds to the infimum is specified below.

$$\alpha_{m,l} = \begin{cases} 1-\frac{Lr}{s}, & 1 \le m \le M^* - c_1 - 1; \\ 1-\frac{Lr}{s}, & m = M^* - c_1, 1 \le l \le L - c_2; \\ 0, & m = M^* - c_1, L - c_2 < l \le L; \\ 0, & M^* - c_1 < m \le M^*. \end{cases} \tag{4.45}$$

We have

$$\begin{aligned} d_{\mathcal{C}}(r) &= \sum_{m=1}^{M^*-c_1-1}(2m-1+|N-M|)L\left(1-\frac{Lr}{s}\right) + \\ &\qquad (L-c_2)(2(M^*-c_1)-1+|N-M|)\left(1-\frac{Lr}{s}\right) \\ &= ((M^*-c_1-1)(M^*-c_1-1+|N-M|)L+ \\ &\qquad (L-c_2)(2M^*-2c_1-1+|N-M|))\left(1-\frac{Lr}{s}\right) \\ &= ((M-c_1-1)(N-c_1-1)L+(L-c_2)(M+N-1-2c_1))\left(1-\frac{Lr}{s}\right) \\ &= ((M-c_1)(N-c_1)L-c_2(M+N-1-2c_1))\left(1-\frac{Lr}{s}\right). \end{aligned}$$

Note that the exponent specified in Theorem F.4 is maximized when $s \to c_1L+c_2+1$.

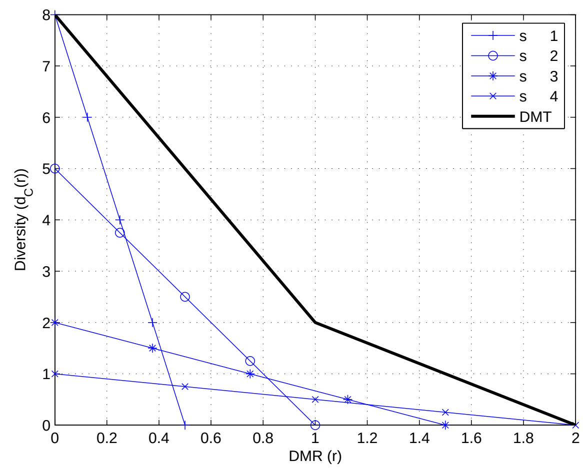

Fig. 23. Diversity versus DMR Curve for 2×2 MIMO System with $L = 2$.

The diversity versus DMR curve for $M = N = L = 2$ achievable using the superposition scheme is shown in Fig. 23.

G. Distortion SNR Exponent

In this section we show how the diversity versus decodable rate tradeoff developed in the previous sections can be used to obtain the distortion SNR exponent of layered transmission schemes. We also compute the achievable distortion SNR exponent of the superposition scheme.

Lemma G.1 *Consider a layered source channel coding scheme $\mathcal{SC}$ obtained from a layered channel coding scheme C whose DMR versus diversity tradeoff is $d_{\mathcal{C}}(r)$. The achievable distortion SNR exponent of scheme $\mathcal{SC}$ satisfies*

$$a_{\mathcal{SC}}^{fixed}(b) \geq \inf_{r \geq 0} d_{\mathcal{C}}(r) + br. \tag{4.46}$$

The bound becomes tight if $d'_{\mathcal{C}}(0) = \frac{d}{dr} d_{\mathcal{C}}(r) \mid_{r=0} < 0$.

Corresponding to the layered coding scheme $\mathcal{C}$, let $R_{\mathcal{C}}(\mathbf{H},\rho)$ denote the decodable rate where $\mathbf{H}$ denotes the current channel realization and ρ denotes the average receive SNR. The average distortion $D_{\mathcal{SC}}(\rho) \doteq \rho^{-a_{\mathcal{SC}}^{fixed}(b)}$ when the receive SNR is ρ is given by

$$\begin{aligned} D_{\mathcal{SC}}(\rho) &= \int_{\mathbf{H}} f_{\mathbf{H}}(\mathbf{H}) 2^{-bR_{\mathcal{C}}(\mathbf{H},\rho)}\, d\mathbf{H} \\ &= \int_{r=0}^{\infty} \left[\int_{\mathbf{H}:R_{\mathcal{C}}(\mathbf{H},\rho)=r\log(1+\rho)} f_{\mathbf{H}}(\mathbf{H})\, d\mathbf{H} \right] 2^{-br\log(1+\rho)}\, dr \qquad (4.47) \\ &\leq \int_{r=0}^{\infty} P(R_{\mathcal{C}}(\mathbf{H},\rho) \leq r\log(1+\rho))(1+\rho)^{-br}\, dr. \qquad (4.48) \end{aligned}$$

$P(R_{\mathcal{C}}(\mathbf{H},\rho) \leq r\log(1+\rho)) \doteq \rho^{-d_{\mathcal{C}}(r)}$. Therefore, from (4.48), we have

$$a_{\mathcal{SC}}^{fixed}(b) \geq \inf_{r\geq 0} d_{\mathcal{C}}(r) + br. \qquad (4.49)$$

Therefore an exponent of $\inf_{r\geq 0} d_{\mathcal{C}}(r) + br$ is achievable. We will now show that this bound is tight under the condition that $d'_{\mathcal{C}}(0) < 0$.

Let $f_r(r) = \int_{\mathbf{H}:R_{\mathcal{C}}(\mathbf{H},\rho)=r\log(1+\rho)} f_{\mathbf{H}}(\mathbf{H})\, d\mathbf{H}$ and let $\mathbf{r} = \{r : d'_{\mathcal{C}}(r) < 0\}$. Note that $d_{\mathcal{C}}(r)$ is always a decreasing function of r. The region $\mathbf{r}$ is obtained by just excluding the region where $d'_{\mathcal{C}}(r) = 0$. In region $\mathbf{r}$ we have,

$$\begin{aligned} f_r(r) &= \frac{d}{dr} P\left(R_{\mathcal{C}}(\mathbf{H},\rho) \leq r\log(1+\rho)\right) \\ &\doteq \frac{d}{dr}\rho^{-d_{\mathcal{C}}(r)} \\ &= \rho^{-d_{\mathcal{C}}(r)} \cdot \log\rho \cdot (-d'_{\mathcal{C}}(r)) \\ &\doteq \rho^{-d_{\mathcal{C}}(r)} \quad (\text{since } d'_{\mathcal{C}}(r) < 0 \text{ for } r \in \mathbf{r}\ ). \end{aligned}$$

From (4.47) we have

$$D_{\mathcal{SC}}(\rho) \geq \int_{\mathbf{r}} f_r(r)(1+\rho)^{-br}\, dr. \qquad (4.50)$$

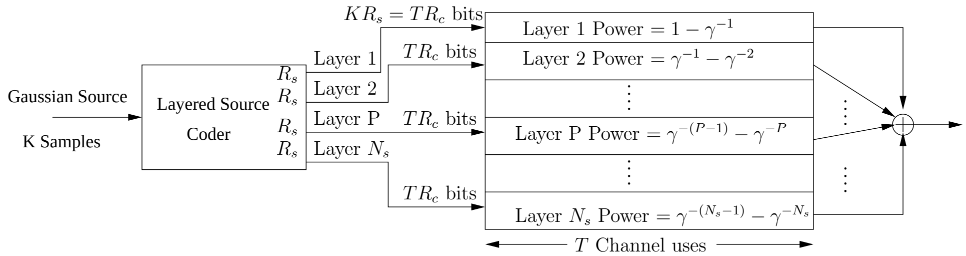

Fig. 24. Layered Fixed Rate Scheme.

It then follows that

$$a_{\mathcal{SC}}^{fixed}(b) \leq \inf_{r \in \mathbf{r}} d_{\mathcal{C}}(r) + br. \tag{4.51}$$

Consider a region $r_A < r < r_B$ not in $\mathbf{r}$ such that $r_A \in \mathbf{r}$. In this region $d_{\mathcal{C}}(r) = d_{\mathcal{C}}(r_A)$, since $d'_{\mathcal{C}}(r) = 0$. Hence $d_{\mathcal{C}}(r) + br > d_{\mathcal{C}}(r_A) + br_A$. Therefore,

$$\inf_{r \in \mathbf{r}} d_{\mathcal{C}}(r) + br = \inf_{r \in \mathbf{r} \cup (r_A, r_B)} d_{\mathcal{C}}(r) + br. \tag{4.52}$$

We can apply the above result for all regions in $r \in [0, \infty) - \mathbf{r}$ provided the point $r = 0$ lies in $\mathbf{r}$. It then follows that

$$\inf_{r \in \mathbf{r}} d_{\mathcal{C}}(r) + br = \inf_{r \geq 0} d_{\mathcal{C}}(r) + br. \tag{4.53}$$

Therefore the lower bound (4.49) and the upper bound (4.51) coincide when $d'_{\mathcal{C}}(0) < 0$. This concludes the proof.

We will now specialize the result for the proposed fixed rate superposition scheme.

Theorem G.1 *The distortion SNR exponent achievable by the fixed rate superposition scheme with* $R_c = \frac{s}{L} \log \gamma$ *where* $s = c_1 L + c_2 + 1$ *in the limit* $\gamma \to 1$ *is given* *by*

$$a_{\mathcal{SC}}^{fixed}(b) = \begin{cases} b\frac{c_1L+c_2+1}{L}, & \textit{for } 0 \leq b \leq L\frac{(M-c_1)(N-c_1)L-c_2(M+N-1-2c_1)}{c_1L+c_2+1} \\ (M-c_1)(N-c_1)L - c_2(M+N-1-2c_1), & \textit{otherwise} \end{cases}. \tag{4.54}$$

Consider the superposition scheme corresponding to $s = c_1L + c_2 + 1$. First note that for $r \geq \frac{c_1L+c_2+1}{L}$, $d_{\mathcal{C}}(r) = 0$, and therefore, for $r \geq \frac{c_1L+c_2+1}{L}$, $d_{\mathcal{C}}(r) + br \geq d_{\mathcal{C}}(r) + br \mid_{r=\frac{c_1L+c_2+1}{L}} = b\frac{c_1L+c_2+1}{L}$.

Therefore

$$a_{\mathcal{SC}}^{fixed}(b) = \inf_{r\in[0,\infty)} d_{\mathcal{C}}(r) + br = \inf_{r\in\left[0,\frac{c_1L+c_2+1}{L}\right]} d_{\mathcal{C}}(r) + br. \tag{4.55}$$

In the range of interest note that both $d_{\mathcal{C}}(r)$ and br are linear functions of r and therefore $d_{\mathcal{C}}(r) + br$ is a line. The infimum therefore occurs either at $r = 0$ or at $r = \frac{c_1L+c_2+1}{L}$. Therefore,

$$\begin{aligned} a_{\mathcal{SC}}^{fixed}(b) &= \min\left(d_{\mathcal{C}}(0), b\frac{c_1L+c_2+1}{L}\right) \\ &= \min\left((M-c_1)(N-c_1)L - c_2(M+N-1-2c_1), b\frac{c_1L+c_2+1}{L}\right). \end{aligned}$$

This proves the Theorem.

It is interesting to compare these results with some of the results derived for the distortion SNR exponent defined for family of schemes that depend on the SNR. An upper bound, referred to as the informed transmitter upper bound, was derived on $a^*(b)$ in [14, 17] by assuming that the channel state information (SNR and channel realization $\mathbf{H}$) is available at the transmitter. The bound is given by

$$a^*(b) \leq \sum_{i=1}^{\min(M,N)} \min\left(b, (2i - i + |N-M|)L\right). \tag{4.56}$$

From (4.7), it follows that this bound is also an upper bound for $a_{SC}^{fixed}(b)$. On comparing the upper bound with the achievable exponent, we see that the proposed scheme is optimal in terms of the exponent for $b < \frac{|N-M|+1}{M^*}$ and for $b > MNL^2$.

We also wish to point out that the maximum achievable distortion SNR exponent of the broadcast scheme discussed in Chapter III and in [12] is the same as the exponent of the superposition scheme proposed here. The broadcast scheme is a generalization of the superposition scheme considered here where the rate and power allocated to the superposition layers could be chosen based on the SNR. Hence, no superposition scheme, including those that depend on SNR, can perform better than the scheme proposed here in terms of the distortion SNR exponent.

The achievable distortion SNR exponent and the informed transmitter upper bound for 3×1, 2×2 MIMO systems with $L = 1$ and for 2×2 MIMO system with $L = 2$ are shown in Fig. 25, Fig. 26, and Fig. 27 respectively. In all these cases, we see that the upper bound and the achievable exponent coincide for $b < \frac{|N-M|+1}{M^*}$ and for $b > MNL^2$. For the 3×1 case with $L = 1$, the distortion SNR exponent is characterized for the entire range of bandwidth expansion.

H. Application in Digital Data Transmission

Before we conclude this chapter, we would like to mention that although our motivation for defining the diversity versus decodable rate tradeoff is from a joint source channel coding perspective, the tradeoff is useful even for studying digital data transmission. For example, consider a system where we have a long stream of data to be sent to a particular user over a MIMO fading channel and the receive SNR and the current channel realization is not known at the transmitter. We would like the coding scheme to be such that the user recovers a rate close to his capacity. In this

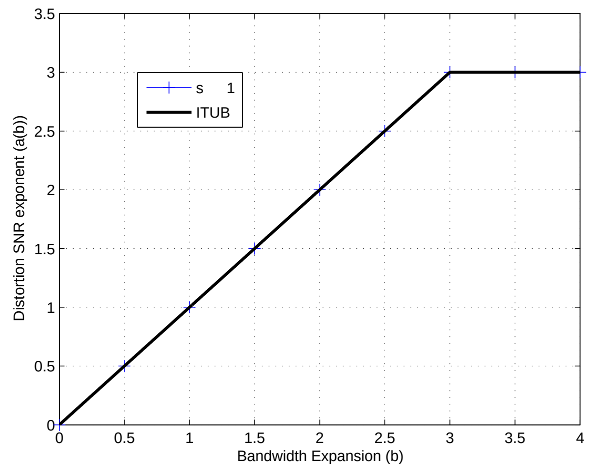

Fig. 25. Distortion SNR Exponent for 3×1 MIMO System with $L = 1$.

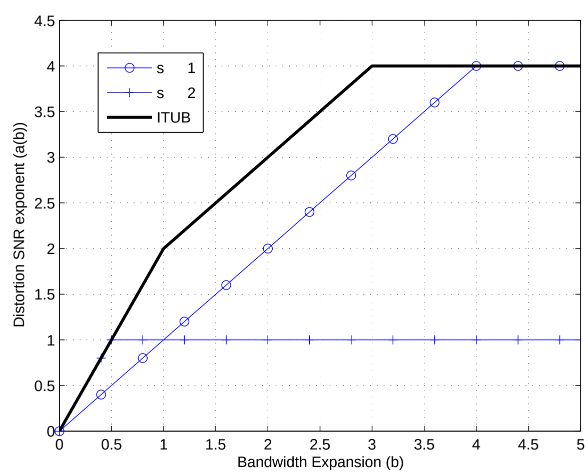

Fig. 26. Distortion SNR Exponent for 2×2 MIMO System with $L = 1$.

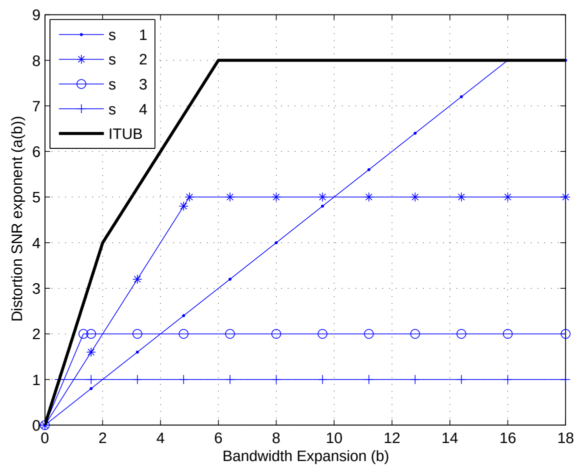

Fig. 27. Distortion SNR Exponent for 2×2 MIMO System with $L = 2$.

case, we could transmit the information using a layered channel code. Using some feedback, the transmitter could be informed about how much rate the receiver has recovered and then, in the next block, the transmitter attempts to transmit the remaining information. The diversity versus decodable rate tradeoff is a good measure to study the high SNR performance of such a system. To illustrate this point, we study the average throughput of a layered channel coding scheme and compare it with the throughput of a system in which the SNR and the fading realization are known at the transmitter.

For the MIMO channel, when the SNR and the fading realization is known at the transmitter, the throughput is the same as the average capacity which is given by

$$C_{av} = E\left[\sum_{m=1}^{M^*} \log(1 + \frac{\rho}{M}\lambda_m)\right] \approx M^* \log\rho. \tag{4.57}$$

The throughput above is shown for the case with no power control. With power control, we only get a power gain and hence the scaling with SNR is the same.

For a setup where the SNR and the fading realization is not known at the transmitter, we consider a layered channel coding scheme where the receiver feeds back the amount of data that is successfully recovered in the current block and then the transmitter attempts to transmit the remaining information. We consider a simple protocol where the receiver processes only the received symbols corresponding to the current block to decode the bits and ignores all previous transmissions. A simple bound on the throughput η is given below

$$\begin{aligned}
\eta &\geq P\left(R_{\mathcal{C}}(\mathbf{H},\rho) < r\log(1+\rho)\right)0 + \left(1 - P_{\mathcal{C}}\left(R(\mathbf{H},\rho) < r\log(1+\rho)\right)\right)\cdot r\log(1+\rho) \\
&\approx (1-\rho^{-d_{\mathcal{C}}(r)})r\log(1+\rho) \\
&\approx r\log(1+\rho) \quad \text{if } d_{\mathcal{C}}(r) > 0.
\end{aligned}$$

The above bound is valid for any layered coding scheme for any r where $d_{\mathcal{C}}(r) > 0$. We apply this bound for the superposition scheme with $R_c = s\log\gamma$ with $s \to M^*$. For the superposition scheme, $d_{\mathcal{C}}(r) \geq 0$ for any $r < s$. Thus we can get the throughput to scale as $M^*\log\rho$ without knowing the fading realization or the SNR.

I. Conclusion

In this chapter, we defined the diversity corresponding to a decodable multiplexing rate of r as the rate of decay of the probability that a receiver is unable to decode a rate of $r\log(1+\rho)$ where ρ is the SNR at the receiver and is unknown at the transmitter. We showed that the diversity versus DMR tradeoff is a suitable framework to analyze layered source channel codes. We proposed a superposition based scheme and showed that, for the $M \times N$ MIMO channel, it achieves a diversity arbitrarily close to $(M-k)(N-k)(1-\frac{r}{k+1})$ for integer $k < \min(M,N)$. This rate of decay is the best possible rate of decay when $\min(M,N) = 1$. We also computed the diversity versus

DMR curve for the block fading MIMO channel. We then showed how the diversity versus DMR tradeoff can be used to compute the distortion SNR exponent of the corresponding layered source channel coding scheme. We showed that we can find superposition schemes that do not depend on SNR that can achieve the same distortion SNR exponent as that of the broadcast scheme of [12]. The achievable distortion SNR exponent is also shown to be optimal for $b < \frac{|N-M|+1}{\min(M,N)}$ and for $b > MNL^2$.

CHAPTER V

CONCLUSION

In this dissertation we studied problems involving transmission of a discrete time Gaussian source over some frequently encountered non-ergodic channels. Specifically we considered the AWGN channel whose SNR is not known at the transmitter, Rayleigh fading MIMO channel whose fading realization is not known at the transmitter but SNR is known at the transmitter, and Rayleigh fading MIMO channel whose SNR and fading realization are not known at the transmitter. Since we were interested in the performance over a wide range of SNR we used distortion SNR exponent, defined as the rate of decay of the mean square error distortion with SNR, as the performance metric.

We showed that layered source channel coding schemes are very promising for all these problems. The achievable distortion SNR exponents obtained using layered schemes proposed in this dissertation were as good as and better in some cases than the achievable exponents of all other joint source channel coding schemes reported so far. The layered schemes were also shown to be optimal in certain cases. Specifically, for the AWGN channel, the optimal distortion exponent was shown to be equal to the bandwidth expansion b and was shown to be achievable using a layered scheme called the fixed rate superposition scheme. This scheme also achieved the optimal exponent for the L-block Rayleigh fading $M \times N$ MIMO channel with SNR unknown at the transmitter for $b < \frac{|N-M|+1}{\min(M,N)}$ and for $b > MNL^2$. Also for the case when the SNR is known at the transmitter, an SNR dependent superposition scheme called the broadcast scheme was shown to be optimal for these values of b. For the MIMO channel, for $\frac{|N-M|+1}{\min(M,N)} < b < MNL^2$, although the exponents reported in this dissertation are better than all the exponents reported so far, the problem is still open.

REFERENCES

[1] T. J. Goblick, "Theoretical limitations on the transmission of data from analog sources," *IEEE Transactions on Information Theory*, vol. 11, pp. 558–567, Oct. 1965.

[2] M. Gastpar, B. Rimoldi, and M. Vetterli, "To code, or not to code: lossy source-channel communication revisited," *IEEE Transactions on Information Theory*, vol. 49, no. 5, pp. 1147–1158, May 2003.

[3] J. L. Massey, *"Joint source and channel coding"*, J. K. Skwirzinski, editor, *Communication Systems and Random Process Theory*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1978.

[4] U. Mittal and N. Phamdo, "Hybrid digital-analog (HDA) joint source-channel codes for broadcasting and robust communications," *IEEE Transactions on Information Theory*, vol. 48, no. 5, pp. 1082–1102, May 2002.

[5] Z. Reznic, M. Feder, and R. Zamir, "Distortion bounds for broadcasting with bandwidth expansion," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3778–3788, Aug. 2006.

[6] L. Zheng and D. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Transactions on Information Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[7] K. Bhattad and K.R. Narayanan, "A note on the rate of decay of mean squared error with SNR for the AWGN channel," in *Allerton conference*, Monticello, IL, Sep. 2007.

[8] K. Bhattad and K. R. Narayanan, "A note on the rate of decay of mean-square error with SNR for the AWGN channel," *submitted to IEEE Transactions on Information Theory*, Nov. 2007.

[9] J. Ziv, "The behavior of analog communication systems," *IEEE Transactions on Information Theory*, vol. 16, no. 5, pp. 587–594, Sep. 1970.

[10] N. Santhi and A. Vardy, "Analog codes on graphs," *submitted to IEEE Transactions on Information Theory 2006, Arxiv preprint cs.IT/0608086*, Aug. 2006.

[11] K. Bhattad, K.R. Narayanan, and G. Caire, "On the distortion exponent of some layered transmission schemes," in *40th Asilomar Conf. on Sig., Syst. and Comp.*, Monterey, CA, Nov. 2006.

[12] K. Bhattad and K. R. Narayanan, "On the distortion SNR exponent of some layered transmission scheme," *submitted to IEEE Transactions on Information Theory*, March 2007.

[13] G. Caire and K.R. Narayanan, "On the SNR exponent of hybrid digital analog space time codes," in *Allerton Conf. Commun. Control and Computing*, Monticello, IL, Oct. 2005.

[14] G. Caire and K.R. Narayanan, "On the distortion SNR exponent of hybrid digital-analog space time coding," *IEEE Transactions on Information Theory*, vol. 53, no. 8, pp. 2867–2878, Aug. 2007.

[15] K.R. Narayanan and G. Caire, "Further results on the SNR exponent of hybrid digital analog space time codes," in *UCSD Workshop on Info. Theory and Its Applications*, San Diego, CA, Feb. 2006.

[16] D. Gunduz and E. Erkip, "Source and channel coding for quasi-static fading channels," in *39th Asilomar Conf. on Sig., Syst. and Comp.*, Monterey, CA, Nov. 2004.

[17] D. Gunduz and E. Erkip, "Joint source-channel codes for MIMO block-fading channels," *IEEE Transactions on Information Theory*, vol. 54, no. 1, pp. 116–134, Jan. 2008.

[18] D. Gunduz and E. Erkip, "Distortion exponent of parallel fading channels," in *ISIT*, Seattle, WA, July 2006.

[19] D. Gunduz and E. Erkip, "Distortion exponent of MIMO fading channels," in *ITW*, Punta del Este, Uruguay, March 2006.

[20] T. Holliday and A.J. Goldsmith, "Joint source and channel coding for MIMO systems," in *Allerton Conf. Commun. Control and Computing*, Monticello, IL, Oct. 2004, pp. 1302–1311.

[21] T. Holliday and A. Goldsmith, "Optimizing end-to-end distortion in MIMO systems," in *IEEE ISIT 2005*, Adelaide, Australia, Sep. 2005, pp. 1671–1675.

[22] J.N. Laneman, E. Martinian, G.W. Wornell, J.G. Apostolopoulos, and S.J. Wee, "Comparing application- and physical-layer approaches to diversity on wireless channels," in *Intl Conf Communications*, Seattle, WA, May 2003, pp. 2678–2882.

[23] J.N. Laneman, E. Martinian, and G.W. Wornell, "Source-channel diversity approaches for multimedia communication," in *ISIT*, Chicago, IL, July 2004.

[24] J.N. Laneman, E. Martinian, G.W. Wornell, and J.G. Apostolopoulos, "Source channel diversity for parallel channels," *IEEE Transactions on Information Theory*, vol. 51, no. 10, pp. 3518–3539, Oct. 2005.

[25] B. Dunn and J.N. Laneman, "Characterizing source-channel diversity approaches beyond the distortion exponent," in *Allerton Conf. Commun. Control and Computing*, Monticello, IL, Oct. 2005.

[26] K. Bhattad and K. R. Narayanan, "Tradeoff between diveristy and decodable-rate: Diversity multiplexing tradeoff for fixed encoding schemes," *submitted to IEEE Transactions on Information Theory*, Nov. 2007.

[27] K. Bhattad and K.R. Narayanan, "Tradeoff between diveristy and decodable-rate: Diversity multiplexing tradeoff for fixed encoding schemes," in *Information Theory Workshop*, Lake Tahoe, CA, Sep. 2007.

[28] L. Lastras and T. Berger, "All sources are nearly successively refinable," *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 918–926, March 2001.

[29] S. N. Diggavi and D. N. C. Tse, "Fundamental limits of diversity-embedded codes over fading channels," in *Proceedings of the International Symposium on Information Theory (ISIT)*, Adelaide, Australia, Sep. 2005, pp. 510–514.

[30] S. Dusad, S. N. Diggavi, and A. R. Calderbank, "Cross layer utility of diversity embedded codes," in *40th Annual conference on Information Sciences and Systems*, Princeton, NJ, March 2006, pp. 795–800.

[31] S. Shamai and A. Steiner, "A broadcast approach for a single-user slowly fading MIMO channel," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2617–2635, Oct. 2003.

[32] C. T. K. Ng, D. Gndz, A. J. Goldsmith, and E. Erkip, "Minimum expected distortion in Gaussian layered broadcast coding with successive refinement," in

IEEE International Symposium on Information Theory (ISIT), Nice, France, June 2007, pp. 2226–2230.

[33] C. T. K. Ng, D. Gunduz, A. J. Goldsmith, and E. Erkip, "Optimal power distribution and minimum expected distortion in Gaussian layered broadcast coding with successive refinement," *submitted to IEEE Transactions on Information Theory*, May 2007.

[34] C. Tian, A. Steiner, S. Shamai, and S. Diggavi, "Expected distortion for Gaussian source with a broadcast transmission strategy over a fading channel," in *IEEE Information Theory Workshop*, Lake Tahoe, CA, July 2007, pp. 1–5.

[35] C. Tian, A. Steiner, S. Shamai, and S. Diggavi, "Expected distortion for Gaussian source with a broadcast transmission strategy over a fading channel," *submitted to IEEE Transactions on Information Theory*, July 2007.

VITA

Kapil Bhattad obtained a B. Tech. degree in Electrical Engineering from Indian Institute of Technology, Madras, India in May 2002. He can be reached at 26/1 Padmanabhan Street, T. Nagar, Chennai - 600017, India or at kapilbhattad@gmail.com