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PREDICTOR-CORRECTOR INTERIOR-POINT ALGORITHM BASED ON A NEW SEARCH DIRECTION WORKING IN A WIDE NEIGHBOURHOOD OF THE CENTRAL PATH

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Abstract. We introduce a new predictor-corrector interior-point algorithm for solving $P_*(\kappa)$-linear complementarity problems which works in a wide neighbourhood of the central path. We use the technique of algebraic equivalent transformation of the centering equations of the central path system. In this technique, we apply the function $\varphi(t) = \sqrt{t}$ in order to obtain the new search directions. We define the new wide neighbourhood $\mathcal{D}_\varphi$. In this way, we obtain the first interior-point algorithm, where not only the central path system is transformed, but the definition of the neighbourhood is also modified taking into consideration the algebraic equivalent transformation technique. This gives a new direction in the research of interior-point methods. We prove that the interior-point algorithm has $\mathcal{O}\left((1+\kappa)n \log\left(\frac{(\mathbf{x}^0)^T \mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity. Furtermore, we show the efficiency of the proposed predictor-corrector interior-point method by providing numerical results. Up to our best knowledge, this is the first predictor-corrector interior-point algorithm which works in the $\mathcal{D}_\varphi$ neighbourhood using $\varphi(t) = \sqrt{t}$.

JEL code: C61

Keywords. predictor-corrector interior-point algorithm; $P_*(\kappa)$-linear complementarity problems; wide neighbourhood; algebraic equivalent transformation technique.

1. Introduction

Starting from the field of *linear optimization* (LO), *interior-point algorithms* (IPAs) have spread around different fields of mathematical programming, returning to nonlinear (convex) programming, as well. For analysis of IPAs see the monographs of Roos et al. [51], Wright [61], Ye [62], Klerk [35], Kojima et al. [36], and Nesterov and Nemirovskii [42], respectively.

IPAs for (LO) have been extended to more general class of problems, such as *linear complementarity problems* (LCPs) [7, 28, 29, 31, 36, 38, 47], *semidefinite programming problems* (SDP) [18, 19, 35, 59], *smooth convex programming problems* (CPP) [42], and *symmetric cone optimization* (SCO) problems [32, 49, 53, 56, 58, 60].

LCPs have several applications in different fields, such as optimization theory, engineering, business and economics, etc [7, 20]. For example, the Arrow-Debreu competitive market equilibrium problem with linear and Leontief utility functions formulated as LCP

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[17, 63]. Testing copositivity of matrices also has connection with solvability of special LCPs [5]. In 2020, Darvay et al. [14] introduced a *predictor-corrector* (PC) IPA for $P_*(\kappa)$-LCPs and obtained very promising numerical results for testing copositivity of matrices using LCPs. Moreover, LCPs arise also in game theory, see [7, 54].

The monographs written by Cottle et al. [7] and Kojima et al. [36] summarize the most important results related to the theory and applications of LCPs. The solvability of the LCP is influenced by the properties of the problem's matrix. If the problem's matrix is skew-symmetric, see [51, 61, 62], or positive semidefinite, see [37], then LCPs can be solved in polynomial time by using IPAs. However, there is still an open question, whether the LCPs with other types of matrices can be solved in polynomial time [18]. In general, LCPs belong to the class of NP-complete problems, see [6]. The most important class of LCPs from the point of view of the complexity theory is the class of sufficient LCPs. This class was introduced by Cottle, Pang, and Venkateswaran [8]. The name *sufficient* comes from the observation that in case of LCPs this matrix property is sufficient in order to ensure that the solution set of the LCP is a convex, closed, bounded polyhedron [8]. The union of the sets $P_*(\kappa)$ for all nonnegative κ gives the P_* class. Väliaho [57] demonstrated that the class of P_*-matrices is equivalent to the class of sufficient matrices introduced by Cottle et al. [8]. It should be mentioned that LCPs can be extended to more general problems, such as general LCPs [29, 30] and $P_*(\kappa)$-LCPs over Cartesian product of symmetric cones [2, 3, 40, 52].

The predictor-corrector (PC) IPAs have ensured an efficient tool for solving LO and LCPs, respectively. They perform in a main iteration a predictor and several corrector steps. One of the first PC IPAs for LO was proposed by Sonnevend et al. [55]. Later on, Mizuno, Todd and Ye [41] intoroduced such PC IPA for LO in which only a single corrector step is performed in each iteration of the algorithm and whose iteration complexity is the best known in the LO literature. These types of methods are called Mizuno-Todd-Ye (MTY) PC IPAs. It should be mentioned that in order to use only one corrector step in each iteration, the centrality parameter and the update parameter should be properly synchronized. Illés and Nagy [27], Potra and Sheng [47, 48] and Gurtuna et al. [24] also introduced PC IPAs for $P_*(\kappa)$-LCPs.

We can classify the IPAs based on the length of the steps. In this way, there exist short- and long-step IPAs. The short-step algorithms generate the new iterates in a smaller neighbourhood, while the long-step ones work in a wider neighbourhood of the central path. Potra and Liu [39, 46] presented first order and higher order PC IPAs for solving $P_*(\kappa)$-LCPs using the $\mathcal{N}_\infty^-$ wide neighborhood of the central path. It should be mentioned that there was a gap between theoretical and practical behavior of these IPAs in the sense that in theory, short-step algorithms had better theoretical complexity, while the long-step algorithms turned out to be more efficient in practice. Peng et al. [43] were the first who reduced this gap by using self-regular barriers. After that, Potra [44] proposed a PC IPA for degenerate LCPs working in a wide neighbourhood of the central path having the same complexity as the best known short-step IPAs. Later on, Ai and Zhang [1] introduced a long-step IPA for monotone LCPs which has the same complexity as the currently best-known short-step interior-point methods. They decomposed the classical Newton direction as the sum of two other directions, corresponding to the negative and positive parts of the right-hand side. After that, Potra [45] generalized this algorithm to $P_*(\kappa)$-LCPs.

An important aspect in the analysis of the IPAs is the determination of the search directions. Peng et al. [43] used self-regular kernel functions and they reduced the theoretical complexity of long-step IPAs. Darvay [10] presented the technique of *algebraic* *equivalent transformation* (AET) of the centering equations of the central path system. The idea of this method is to apply a continuously differentiable, invertible, monotone increasing φ function on the nonlinear equation of the central path problem. The first PC IPAs using the AET method for determining search directions was given by Darvay [11, 12] for LO and linearly constrained convex optimization. Kheirfam [33] generalized these algorithms to $P_*(\kappa)$-horizontal LCPs. Note that the most widely used function for finding search directions using the AET technique is the identity map. Darvay [9, 10] used the square root function in the AET technique. Later on, Darvay et al. [15] proposed an IPA for LO based on the direction generated by using the function $\varphi(t) = t - \sqrt{t}$. In 2020, Darvay et al. [13, 14] introduced PC IPAs for LO and $P_*(\kappa)$-LCPs, that are based on this search directions. They also provided a new approach for introducing PC IPAs using the AET technique, which consists in the decomposition of the right hand side of the Newton-system into two terms: one depending and the other not depending on the parameter μ. Kheirfam and Haghighi [34] defined IPA for solving $P_*(\kappa)$-LCPs which uses the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique. Rigó [50] presented several IPAs that are based on the search directions obtained by using the function $\varphi(t) = t - \sqrt{t}$ in the AET technique. The broadest class of functions used in the AET technique was proposed by Haddou et al. [25]. However, the functions $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t - \sqrt{t}$ do not belong to the class of concave functions introduced by Haddou et al. An interesting research topic related to the AET technique would be to introduce a class of functions which contains the functions $\varphi(t) = \sqrt{t}$, $\varphi(t) = t - \sqrt{t}$ and $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ as well and for which polynomial-time IPAs can be introduced.

The purpose of this paper is to generalize the wide neighbourhoods $\mathcal{D}$ and $\mathcal{N}_\infty^-$ taking into consideration the transformed central path system using the AET approach. We also analyse the relationship between the new generalized neighbourhoods $\mathcal{D}_\varphi$ and $\mathcal{N}_{\infty,\varphi}^-$. We prove that in case of $\varphi(t) = t$ and $\varphi(t) = \sqrt{t}$ these neighbourhoods are the same. However, in case of $\varphi(t) = t - \sqrt{t}$, only the relation $\mathcal{D}_\varphi \subseteq \mathcal{N}_{\infty,\varphi}^-$ holds. Moreover, using the method given by Potra and Liu in [46] and the new approach proposed by Darvay et al. [14], we introduce a new first order PC IPA which works in the new wide neighbourhood $\mathcal{D}_\varphi$ using the function $\varphi(t) = \sqrt{t}$. This is the first PC IPA which works in the $\mathcal{D}_\varphi$ neighbourhood of the central path using $\varphi(t) = \sqrt{t}$ in the AET technique. We prove that the provided algorithm has $\mathcal{O}\left((1+\kappa)n\log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity, similarly to that of Potra and Liu [46]. Furthermore, by providing numerical results we also show the efficiency of the proposed PC IPA. We implemented the theoretical version of the IPA and followed the steps of the proposed PC IPA. We compared our PC IPA to the PC IPA using the function $\varphi(t) = \sqrt{t}$ in the AET technique and the neighbourhood $\mathcal{N}_{\infty,\varphi}^-(1-\beta)$ with the PC IPA of Potra and Liu proposed in [46], which corresponds to the $\varphi(t) = t$ case in our generalization of the wide neighbourhood.

The paper is organized in the following way. In the second section $P_*(\kappa)$-LCPs and the central path problem is presented. Section 3 contains the AET technique and the new generalized wide neighbourhoods used in this paper. In Section 4 we present the new PC IPA for solving $P_*(\kappa)$-LCPs. Section 5 is devoted to the analysis of the proposed PC IPA. In Section 6 we propose a new version of the PC IPA which does not depend on κ. In Section 7 we provide numerical results that show the efficiency of the introduced IPA. Finally, in Section 8 some concluding remarks are enumerated.

We use the following notations throughout the paper. Let $\mathbf{x}$ and $\mathbf{s}$ be two n-dimensional vectors. Then, $\mathbf{xs}$ denotes the componentwise product of the vectors $\mathbf{x}$ and $\mathbf{s}$. Furthermore, $\frac{\mathbf{x}}{\mathbf{s}} = \left[\frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots., \frac{x_n}{s_n}\right]^T$, where $s_i \neq 0$ for all $1 \leq i \leq n$. In case of an arbitrary function f and a vector $\mathbf{x}$ we use $f(\mathbf{x}) = [f(x_1), f(x_2), \ldots, f(x_n)]^T$. The vector $\mathbf{e} = [1, 1, \ldots, 1]^T$ denotes the n-dimensional all-one vector. The diagonal matrix obtained by the elements of the vector $\mathbf{x}$ is denoted by $diag(\mathbf{x})$. We denote by $\|\mathbf{x}\|$ the Euclidean norm and by $\|\mathbf{x}\|_\infty$ the infinity norm.

2. Linear Complementarity Problems (LCPs) and matrix classes

In this section we present some well known matrix classes and the linear complementarity problem (LCP).

A matrix $M \in \mathbb{R}^{n\times n}$ is a *P-matrix* (*P_0-matrix*), if all of its principal minors are positive (nonnegative), see [21, 22]. Furthermore, Cottle et al. [8] defined the class of *sufficient matrices*.

Definition 2.1. (Cottle et al. [8]) A matrix $M \in \mathbb{R}^{n\times n}$ is a *column sufficient matrix* if for all $\mathbf{x} \in \mathbb{R}^n$

$$X(M\mathbf{x}) \leq 0 \ \text{ implies } \ X(M\mathbf{x}) = 0,$$

where $X = diag(\mathbf{x})$. Analogously, matrix M is *row sufficient* if M^T is column sufficient. The matrix M is *sufficient* if it is both row and column sufficient.

Kojima et al. [36] defined the notion of $P_*(\kappa)$-matrices.

Definition 2.2. (Kojima et al. [36]) Let $\kappa \geq 0$ be a nonnegative real number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-matrix if

$$(1+4\kappa) \sum_{i\in I_+(\mathbf{x})} x_i(Mx)_i + \sum_{i\in I_-(\mathbf{x})} x_i(Mx)_i \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

where

$$I_+(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i > 0\} \ \text{ and } \ I_-(\mathbf{x}) = \{1 \leq i \leq n : x_i(Mx)_i < 0\}.$$

It should be mentioned that $P_*(0)$ is the set of positive semidefinite matrices. The handicap of the matrix M is defined in the following way:

$$\hat{\kappa}(M) := \min\{\kappa : \kappa \geq 0, M \text{ is } \mathcal{P}_*(\kappa)\text{-matrix}\}.$$

Definition 2.3. (Kojima et al. [36]) A matrix $M \in \mathbb{R}^{n\times n}$ is a P_*-matrix if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. Let $P_*(\kappa)$ denote the set of $P_*(\kappa)$-matrices. Analogously, we also use P_* to denote the set of all P_*-matrices, i.e.,

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa).$$

Kojima et al. [36] showed that a P_*-matrix is column sufficient and Guu and Cottle [23] proved that it is row sufficient, too. Therefore, each P_*-matrix is sufficient. Moreover, Väliaho [57] proved the other inclusion as well, showing that the class of P_*-matrices is the same as the class of sufficient matrices.
The linear complentarity problem (LCP) is the following:

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q} \quad \mathbf{x}, \mathbf{s} \geq \mathbf{0}, \quad \mathbf{xs} = \mathbf{0}, \tag{LCP}$$

where $M \in \mathbb{R}^{n\times n}$.

If M is $P_*(\kappa)$-matrix, then the corresponding LCP is called $P_*(\kappa)$-LCP. We define the feasibility set:

$$\mathcal{F} := \{(\mathbf{x},\mathbf{s}) \in \mathbb{R}^{2n}_{\oplus} : -M\mathbf{x}+\mathbf{s}=\mathbf{q}\},$$

the set of interior points:

$$\mathcal{F}^+ := \mathcal{F} \cap \mathbb{R}^{2n}_{+}$$

and the set of optimal solutions:

$$\mathcal{F}^* := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F} : \mathbf{x}\mathbf{s}=\mathbf{0}\}.$$

Throughout the paper we will assume that M is $P_*(\kappa)$-matrix. We also suppose that $\mathcal{F}^+ \neq \emptyset$. The *central path problem* is the following:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \mathbf{x}\mathbf{s}=\mu\mathbf{e}, \tag{CCP}$$

where $\mathbf{e}$ denotes the n-dimensional all-one vector and $\mu>0$. If M is a $P_*(\kappa)$-matrix, then the central path system has unique solution for every $\mu>0$, see [36].

3. Generalized wide neighbourhoods

In this section we define some new generalized neighbourhoods. Firstly, we present the AET technique of the centering equations of the central path system [10]. Let $\varphi : (\eta^2,\infty) \to \mathbb{R}$ be a continuously differentiable and invertible function, such that $\varphi'(t)>0$, for each $t \geq \eta^2$, where $\eta \in [0,1)$. Then, system (CCP_φ) can be transformed in the following way:

$$-M\mathbf{x}+\mathbf{s}=\mathbf{q} \qquad \mathbf{x},\mathbf{s}>\mathbf{0}, \qquad \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu}\right)=\varphi(\mathbf{e}). \tag{CCP_φ}$$

Let $(\mathbf{x},\mathbf{s}) \in \mathcal{F}$. Then, the *average duality gap* is defined as

$$\mu(\mathbf{x},\mathbf{s}) := \frac{\mathbf{x}^T\mathbf{s}}{n}. \tag{3.1}$$

Consider the following generalized *proximity measure*

$$\delta^-_{\infty,\varphi}(\mathbf{x},\mathbf{s}) := \left\| \left[\varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu(\mathbf{x},\mathbf{s})}\right) - \varphi(\mathbf{e})\right]^- \right\|_\infty .$$

Using the introduced proximity measure and the AET approach, we introduce the generalized *wide neighbourhood* of (CCP_φ):

$$\mathcal{N}^-_{\infty,\varphi}(\alpha) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \delta^-_{\infty,\varphi}(\mathbf{x},\mathbf{s}) \leq \alpha\}. \tag{3.2}$$

It should be mentioned that in case of $\varphi(t)=t$ we get the wide neighbourhood used by Potra and Liu [46]:

$$\mathcal{N}^-_{\infty}(\alpha) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \delta^-_{\infty}(\mathbf{x},\mathbf{s}) \leq \alpha\}. \tag{3.3}$$

We also introduce another, generalized *wide neighbourhood* of (CCP_φ):

$$\mathcal{D}_\varphi(\beta) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\mu(\mathbf{x},\mathbf{s})}\right) \geq \beta\varphi(\mathbf{e})\}. \tag{3.4}$$

Note, that in the special case when $\varphi(t)=t$, we get the wide neighbourhood used in [46]:

$$\mathcal{D}(\beta) := \{(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+ : \frac{\mathbf{x}\mathbf{s}}{\mu(\mathbf{x},\mathbf{s})} \geq \beta\mathbf{e}\}. \tag{3.5}$$

The following lemma represents a novelty of the paper. It plays important role in this theory, because it shows which functions used in the AET technique in the literature can be applied in this approach for introducing PC IPAs working in the generalized wide neighbourhood given in (3.2), but the analysis of the algorithm could be done in a simplier wide neighbourhood (3.4).

Lemma 3.1. *Let $(\boldsymbol{x},\boldsymbol{s})\in\mathcal{F}^+$ and $\alpha\in(0,1)$. Then, in case of $\varphi(t)=t$ and $\varphi(t)=\sqrt{t}$ we have $\mathcal{N}^-_{\infty,\varphi}(\alpha)=\mathcal{D}_\varphi(1-\alpha)$. In case of $\varphi(t)=t-\sqrt{t}$ we have $\mathcal{D}_\varphi(1-\alpha)\subseteq\mathcal{N}^-_{\infty,\varphi}(\alpha)$.*

Proof. Firstly we prove it in the case, when $\varphi(t)=t$:

$$\begin{aligned}(\mathbf{x},\mathbf{s})\in\mathcal{D}(1-\alpha) \quad &\iff \quad \mathbf{xs}\geq(1-\alpha)\mu(\mathbf{x},\mathbf{s})\mathbf{e}=\mu(\mathbf{x},\mathbf{s})\mathbf{e}-\alpha\mu(\mathbf{x},\mathbf{s})\mathbf{e}\\ &\iff \quad \frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}-\mathbf{e}\geq-\alpha\mathbf{e}\iff\left\|\left[\frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}-\mathbf{e}\right]^-\right\|_\infty\leq\alpha\\ &\iff \quad (\mathbf{x},\mathbf{s})\in\mathcal{N}^-_\infty(\alpha).\end{aligned}$$

Now we can consider the other cases. Then, we have

$$\begin{aligned}(\mathbf{x},\mathbf{s})\in\mathcal{N}^-_{\infty,\varphi}(\alpha) \quad &\iff \quad \left\|\left[\varphi\left(\frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}\right)-\varphi(\mathbf{e})\right]^-\right\|_\infty\leq\alpha\\ &\iff \quad \varphi\left(\frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}\right)-\varphi(\mathbf{e})\geq-\alpha\mathbf{e}\iff\varphi\left(\frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}\right)\geq\varphi(\mathbf{e})-\alpha\mathbf{e}\end{aligned}$$

and

$$(\mathbf{x},\mathbf{s})\in\mathcal{D}_\varphi(1-\alpha) \quad \iff \quad \varphi\left(\frac{\mathbf{xs}}{\mu(\mathbf{x},\mathbf{s})}\right)\geq(1-\alpha)\varphi(\mathbf{e})=\varphi(\mathbf{e})-\alpha\varphi(\mathbf{e}).$$

It is easy to see, that in case of $\varphi(t)=\sqrt{t}$ the $\varphi(\mathbf{e})=\mathbf{e}$ holds, so we obtain $\mathcal{N}^-_{\infty,\varphi}(\alpha)=\mathcal{D}_\varphi(1-\alpha)$. In case of $\varphi(t)=t-\sqrt{t}$ only $\mathcal{D}_\varphi(1-\alpha)\subseteq\mathcal{N}^-_{\infty,\varphi}(\alpha)$ holds. ☐

4. New predictor-corrector interior-point algorithm

In this paper we consider the $\varphi(t)=\sqrt{t}$ case in this generalized wide neighbourhood approach. Applying Newton's method to system (CCP_φ) with $\varphi(t)=\sqrt{t}$ we obtain the following transformed Newton system:

$$\begin{aligned}-M\Delta\mathbf{x}+\Delta\mathbf{s} &= \mathbf{0},\\ \mathbf{s}\Delta\mathbf{x}+\mathbf{x}\Delta\mathbf{s} &= 2\left(\sqrt{\mu\mathbf{xs}}-\mathbf{xs}\right).\end{aligned}\tag{4.1}$$

In the predictor step we use the approach given by Darvay et al [14]. In this way, we decompose the right hand side of (4.1) in two terms, one which depends on μ, the other which does not depend on μ. After that we set $\mu=0$, hence we obtain

$$\begin{aligned}-M\Delta^p\mathbf{x}+\Delta^p\mathbf{s}&=0,\\ \mathbf{s}\Delta^p\mathbf{x}+\mathbf{x}\Delta^p\mathbf{s}&=-2\mathbf{xs},\end{aligned}\tag{4.2}$$

where $(\Delta^p\mathbf{x},\Delta^p\mathbf{s})$ denote the predictor search directions. Now, we describe the main steps of the algorithm. Let $(\mathbf{x},\mathbf{s})\in\mathcal{N}^-_{\infty,\varphi}(1-\beta)=\mathcal{D}_\varphi(\beta)$, where $\beta\in(0,1)$. Then the predictor search direction $(\Delta^p\mathbf{x},\Delta^p\mathbf{s})$ can be calculated by using system (4.2). We want to compute the iterate in such a way, that $(\mathbf{x}^p(\theta),\mathbf{s}^p(\theta))\in\mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma)=\mathcal{D}_\varphi((1-\gamma)\beta)$ still stay true, where

$$\mathbf{x}^p(\theta)=\mathbf{x}+\theta\Delta^p\mathbf{x} \quad \text{and} \quad \mathbf{s}^p(\theta)=\mathbf{s}+\theta\Delta^p\mathbf{s}\tag{4.3}$$

and

$$\gamma=\frac{1-\beta}{(1+4\kappa)n+1}.\tag{4.4}$$

The step length in the predictor step is defined as

$$\theta_p = \sup\{\hat{\theta} > 0 : (\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma) = \mathcal{D}_\varphi((1-\gamma)\beta), \forall \theta \in [0,\hat{\theta}]\}, \tag{4.5}$$

After the predictor step we will have

$$(\mathbf{x}^p, \mathbf{s}^p) = (\mathbf{x}^p(\theta_p), \mathbf{s}^p(\theta_p)) = (\mathbf{x}+\theta_p\Delta^p\mathbf{x}, \mathbf{s}+\theta_p\Delta^p\mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma). \tag{4.6}$$

The output of the predictor step will be the input of the corrector step. Using system (4.1) we calculate the corrector direction $(\Delta^c\mathbf{x}, \Delta^c\mathbf{s})$ from the following system:

$$\begin{aligned} -M\Delta^c\mathbf{x} + \Delta^c\mathbf{s} &= 0, \\ \mathbf{s}^p\Delta^c\mathbf{x} + \mathbf{x}^p\Delta^c\mathbf{s} &= 2\left(\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p} - \mathbf{x}^p\mathbf{s}^p\right), \end{aligned} \tag{4.7}$$

where

$$\mu_p = \mu(\mathbf{x}^p, \mathbf{s}^p) = \frac{(\mathbf{x}^p)^T\mathbf{s}^p}{n}. \tag{4.8}$$

The corrector step length is defined in the following way:

$$\theta_c \coloneqq \arg\min\{\mu_c(\theta) : (\mathbf{x}^c(\theta), \mathbf{s}^c(\theta)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)\}, \tag{4.9}$$

where

$$\mu_c(\theta) = \mu(\mathbf{x}^c(\theta), \mathbf{s}^c(\theta)) = \frac{(\mathbf{x}^c(\theta))^T\mathbf{s}^c(\theta)}{n} \tag{4.10}$$

and

$$\mathbf{x}^c(\theta) = \mathbf{x}^p + \theta\Delta^c\mathbf{x}, \quad \mathbf{s}^c(\theta) = \mathbf{s}^p + \theta\Delta^c\mathbf{s}. \tag{4.11}$$

After the corrector step we get the following:

$$(\mathbf{x}^c, \mathbf{s}^c) = (\mathbf{x}^c(\theta_c), \mathbf{s}^c(\theta_c)) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta), \tag{4.12}$$

where $\mathbf{x}^c(\theta_c) = \mathbf{x}^p + \theta_c\Delta^c\mathbf{x}$ and $\mathbf{s}_c(\theta_c) = \mathbf{s}^p + \theta_c\Delta^c\mathbf{s}$.

As $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)$, we can set $(\mathbf{x}, \mathbf{s}) := (\mathbf{x}^c, \mathbf{s}^c)$ and start another predictor-corrector iteration. The obtained PC IPA is defined in Algorithm 1.

Algorithm 1: First-order predictor-corrector algorithm

Input:
Given $\kappa \geq \hat{\kappa}(M)$, $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)$, $\beta \in (0.9, 1)$
Calculate $\gamma = \frac{1-\beta}{(1+4\kappa)n+1}$
Let $\mu_0 = \mu(\mathbf{x}^0, \mathbf{s}^0)$ and $k = 0$
$\varepsilon > 0$ precision value.
Output: $(\mathbf{x}^k, \mathbf{s}^k) : \mathbf{x}^{kT}\mathbf{s}^k \leq \varepsilon$
begin
  while $n\mu \geq \varepsilon$ **do**
    (Predictor step);
    $\mathbf{x} := \mathbf{x}^k$, $\mathbf{s} := \mathbf{s}^k$;
    Step 1. Calculate affin direction from (4.2);
    Step 2. Calculate the predictor steplength using (4.5);
    Step 3. Calculate $(\mathbf{x}^p, \mathbf{s}^p)$ using (4.6);
    if $\mu(\boldsymbol{x}^p, \boldsymbol{s}^p) = 0$ **then**
      STOP; Optimal solution found;
    else
      if $(\boldsymbol{x}^p, \boldsymbol{s}^p) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta)$ **then**
        $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^p, \mathbf{s}^p)$, $\mu^{k+1} = \mu(\mathbf{x}^p, \mathbf{s}^p)$, $k = k+1$, RETURN;
      else
        (Corrector step);
        Step 4. Calculate centering direction from (4.7);
        Step 5. Calculate centering steplenght using (4.9);
        Step 6. Calculate $(\mathbf{x}^c, \mathbf{s}^c)$ using (4.12);
      end
      $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^c, \mathbf{s}^c)$, $\mu^{k+1} = \mu(\mathbf{x}^c, \mathbf{s}^c)$, $k = k+1$, RETURN;
    end
  end
end

We give a more detailed description on how the predictor and corrector step lengths could be determined. Using system (4.2) we determine the predictor search directions $(\Delta^p\mathbf{x}, \Delta^p\mathbf{s})$. After that we have to calculate the largest θ which will satisfy the following:

$$\sqrt{\frac{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)}{\mu_p(\theta)}} \geq (1-\gamma)\beta,$$

where $\mathbf{x}^p(\theta) = \mathbf{x} + \theta\Delta^p\mathbf{x}$, $\mathbf{s}^p(\theta) = \mathbf{s} + \theta\Delta^p\mathbf{s}$, $\mu := \mu(\mathbf{x}, \mathbf{s})$, and

$$\mu_p(\theta) := \mu(\mathbf{x}^p(\theta), \mathbf{s}^p(\theta)). \tag{4.13}$$

Using (3.1) and (4.2), after some calculations we have

$$\mathbf{x}^p(\theta)\mathbf{s}^p(\theta) = (1-2\theta)\mathbf{x}\mathbf{s} + \theta^2\Delta^p\mathbf{x}\Delta^p\mathbf{s}, \qquad \mu_p(\theta) = (1-2\theta)\mu + \frac{\theta^2\Delta^p\mathbf{x}^T\Delta^p\mathbf{s}}{n}. \tag{4.14}$$

We will use the following notations:

$$\mathbf{u} = \frac{\mathbf{x}\mathbf{s}}{\mu}, \qquad \mathbf{v} = \frac{\Delta^p\mathbf{x}\Delta^p\mathbf{s}}{\mu}. \tag{4.15}$$

Now from Lemma 5.1 and 5.2 using $\mathbf{a} = -2\mathbf{x}\mathbf{s}$ we can easily see that

$$-4\kappa n \leq \mathbf{e}^T\mathbf{v} \leq n \tag{4.16}$$

holds, hence the discriminant will always be nonnegative, which means the smallest root will be:

$$\theta_{p0} = \frac{2-\sqrt{4-4\frac{\mathbf{e}^T\mathbf{v}}{n}}}{2} = \frac{1}{1+\sqrt{1-\frac{\mathbf{e}^T\mathbf{v}}{n}}}. \tag{4.17}$$

Therefore,

$$\mu_p(\theta) > \mu_p(\theta_{p0}) = 0, \text{ for all } 0 \le \theta < \theta_{p0}. \tag{4.18}$$

Using (4.2) and (4.15), in case of $\varphi(t) = \sqrt{t}$ the relation

$$\varphi\left(\frac{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)}{\mu_p(\theta)}\right) \ge (1-\gamma)\beta$$

can be written as

$$(1-2\theta)(u_i - ((1-\gamma)\beta)^2) + \theta^2\left(v_i - ((1-\gamma)\beta)^2\frac{\mathbf{e}^T\mathbf{v}}{n}\right) \ge 0, \quad i = 1,\dots,n. \tag{4.19}$$

Since $(\mathbf{x},\mathbf{s}) \in \mathcal{N}^-_{\infty,\varphi}(1-\beta) = \mathcal{D}_\varphi(\beta)$, inequality (4.19) is satisfied for $\theta = 0$. The inequality (4.19) will be fulfilled if $\theta \in (0, \theta_{p_i}]$, where

$$\theta_{p_i} = \begin{cases} \infty, & \text{if } \Delta_i \le 0 \\ \frac{1}{2}, & \text{if } v_i - \frac{((1-\gamma)\beta)^2\mathbf{e}^T\mathbf{v}}{n} = 0 \\ \zeta, & \text{if } \Delta_i > 0 \text{ and } v_i - \frac{((1-\gamma)\beta)^2\mathbf{e}^T\mathbf{v}}{n} \ne 0, \text{ where} \end{cases} \tag{4.20}$$

$$\Delta_i = 4(u_i - ((1-\gamma)\beta)^2)^2 - 4(u_i - ((1-\gamma)\beta)^2)\left(v_i - \frac{((1-\gamma)\beta)^2\mathbf{e}^T\mathbf{v}}{n}\right)$$

and

$$\zeta = \frac{2(u_i - ((1-\gamma)\beta)^2) - \sqrt{\Delta_i}}{2\left(v_i - \frac{((1-\gamma)\beta)^2\mathbf{e}^T\mathbf{v}}{n}\right)} = \frac{2(u_i - ((1-\gamma)\beta)^2)}{2(u_i - ((1-\gamma)\beta)^2) + \sqrt{\Delta_i}}.$$

Taking

$$\theta_p = \min\{\theta_{p_i} : 1,\dots,n\} \tag{4.21}$$

will be a good ceiling for appropriate predictor steplengths. For all $0 \le \theta < \theta_p$ we will have

$$\sqrt{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)} \ge (1-\gamma)\beta\sqrt{\mu_p(\theta)} > (1-\gamma)\beta\sqrt{\mu_p(\theta_p)} \ge 0. \tag{4.22}$$

Using (4.2) and (4.14) we obtain that $-M\mathbf{x}^p(\theta) + \mathbf{s}^p(\theta) = \mathbf{q}$. Using standard continuity argument we obtain $\mathbf{x}^p(\theta) > \mathbf{0}$ and $\mathbf{s}^p(\theta) > \mathbf{0}$, for all $\theta \in (0, \theta_p)$, which means that $(\mathbf{x}^p, \mathbf{s}^p) \in \mathcal{F}^+$, where $(\mathbf{x}^p, \mathbf{s}^p)$ is defined in (4.6).

Using system (4.7), we determine the corrector search directions $(\Delta^c\mathbf{x}, \Delta^c\mathbf{s})$. We describe the way how we can calculate the corrector step length θ_c. Using (4.7), (4.10) and (4.11) we have

$$\mathbf{x}^c(\theta)\mathbf{s}^c(\theta) = (1-2\theta)\mathbf{x}^p\mathbf{s}^p + 2\theta\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p} + \theta^2\Delta^c\mathbf{x}\Delta^c\mathbf{s}, \tag{4.23}$$

$$\mu_c(\theta) = (1-2\theta)\mu_p + 2\theta\frac{\mu_p}{\sqrt{n}} + \frac{\theta^2\Delta^c\mathbf{x}^T\Delta^c\mathbf{s}}{n}. \tag{4.24}$$

Moreover, using (4.3) and (4.8) we consider the following notations:

$$\overline{\mathbf{u}} = \frac{\mathbf{x}^p\mathbf{s}^p}{\mu_p}, \qquad \overline{\mathbf{v}} = \frac{\Delta^c\mathbf{x}\Delta^c\mathbf{s}}{\mu_p}. \tag{4.25}$$

We want to reach

$$\sqrt{\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)}{\mu_c(\theta)}} \geq \beta. \tag{4.26}$$

Using (4.10) and the first equation of system (4.7), after some calculations we obtain that relation (4.26) is equivalent to the following system of quadratic inequalities in θ:

$$(1-2\theta)(\overline{u}_i-\beta^2)+2\theta\left(\sqrt{\overline{u}_i}-\frac{\beta^2}{\sqrt{n}}\right)+\theta^2\left(\overline{v}_i-\beta^2\frac{\mathbf{e}^T\overline{\mathbf{v}}}{n}\right)\geq 0,\ i=1,\dots,n. \tag{4.27}$$

We will use the following notations:

$$\overline{\alpha}_i=\overline{v}_i-\beta^2\frac{\mathbf{e}^T\overline{\mathbf{v}}}{n} \quad \text{and}$$

$$\overline{\Delta}_i=4\left(\overline{u}_i-\beta^2+\frac{\beta^2}{\sqrt{n}}-\sqrt{\overline{u}_i}\right)^2-4\left(\overline{v}_i-\beta^2\frac{\mathbf{e}^T\overline{\mathbf{v}}}{n}\right)(\overline{u}_i-\beta^2).$$

In the proof of Theorem 5.1 we will show that (4.27) has solution, hence the situation $\overline{\Delta}_i<0$ and $\overline{\alpha}_i<0$, $i=1,\dots,n$ cannot occur. If $\overline{\Delta}_i\geq 0$ and $\overline{\alpha}_i\neq 0$, the largest and the smallest root of the quadratic equation will be denoted as:

$$\overline{\theta}_i^- = \frac{\overline{u}_i-\beta^2+\dfrac{\beta^2}{\sqrt{n}}-\sqrt{\overline{u}_i}-\operatorname{sgn}(\overline{\alpha}_i)\sqrt{\overline{\Delta}_i}}{2\overline{\alpha}_i}$$

$$\overline{\theta}_i^+ = \frac{\overline{u}_i-\beta^2+\dfrac{\beta^2}{\sqrt{n}}-\sqrt{\overline{u}_i}+\operatorname{sgn}(\overline{\alpha}_i)\sqrt{\overline{\Delta}_i}}{2\overline{\alpha}_i}$$

After some calculations we obtain that the ith inequality of system (4.27) will be satisfied for all $\theta\in\mathcal{T}_i$, where

$$\mathcal{T}_i=\begin{cases}(-\infty,\infty), & \text{if } \overline{\Delta}_i<0, \overline{\alpha}_i>0\\ (-\infty,\overline{\theta}_i^-]\cup[\overline{\theta}_i^+,\infty), & \text{if } \overline{\Delta}_i\geq 0, \overline{\alpha}_i>0\\ [\overline{\theta}_i^-,\overline{\theta}_i^+], & \text{if } \overline{\Delta}_i\geq 0, \overline{\alpha}_i<0\\ \left(-\infty,\dfrac{\overline{u}_i-\beta^2}{2(\overline{u}_i-\beta^2-\sqrt{\overline{u}_i}+\frac{\beta^2}{\sqrt{n}})}\right], & \text{if } \overline{\alpha}_i=0, \overline{u}_i>\dfrac{\left(1+\sqrt{1+4\beta^2-\frac{4\beta^2}{\sqrt{n}}}\right)^2}{4}\\ \left[\dfrac{\overline{u}_i-\beta^2}{2(\overline{u}_i-\beta^2-\sqrt{\overline{u}_i}+\frac{\beta^2}{\sqrt{n}})},\infty\right), & \text{if } \overline{\alpha}_i=0, \overline{u}_i<\dfrac{\left(1+\sqrt{1+4\beta^2-\frac{4\beta^2}{\sqrt{n}}}\right)^2}{4}\\ (-\infty,\infty), & \text{if } \overline{\alpha}_i=0, \overline{u}_i=\dfrac{\left(1+\sqrt{1+4\beta^2-\frac{4\beta^2}{\sqrt{n}}}\right)^2}{4}\end{cases}$$

For all $\theta\in\mathcal{T}=\cap_{i=1}^n\mathcal{T}_i\cap\mathbb{R}^n_{\oplus}$, the inequality given in (4.26) will hold. We will show that $\mathcal{T}$ is nonempty.
In the following section we present the analysis of Algorithm 1.

5. Analysis of the algorithm

Firstly, we present some technical lemmas that will be used later in the analysis.

Lemma 5.1. *(Lemma 3.2 in [46]) Assume, that we have a $\mathcal{P}_*(\kappa)$-LCP and let $(\Delta\boldsymbol{x},\Delta\boldsymbol{s})$ be the solution of the following linear system:*

$$-M\Delta\boldsymbol{x}+\Delta\boldsymbol{s}=0,$$
$$\boldsymbol{s}\Delta\boldsymbol{x}+\boldsymbol{x}\Delta\boldsymbol{s}=\boldsymbol{a},$$

where $(\Delta\boldsymbol{x},\Delta\boldsymbol{s})\in\mathbb{R}^{2n}_+$ and $\boldsymbol{a}\in\mathbb{R}^n$ are given. If $\mathcal{I}_+=\{i:\Delta x_i\Delta s_i>0\}$,$\mathcal{I}_-=\{i:\Delta x_i\Delta s_i<0\}$ are defined in this way, then we have

$$\frac{1}{1+4\kappa}\|\Delta\boldsymbol{x}\Delta\boldsymbol{s}\|_\infty\le\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i\le\frac{1}{4}\left\|(\boldsymbol{x}\boldsymbol{s})^{-\frac{1}{2}}\boldsymbol{a}\right\|_2^2. \tag{5.1}$$

Lemma 5.2. *(Lemma 3.3 in [46]) Assume that we have a $\mathcal{P}_*(\kappa)$-LCP and let $(\Delta\boldsymbol{x},\Delta\boldsymbol{s})$ be the solution of the following linear system:*

$$-M\Delta\boldsymbol{x}+\Delta\boldsymbol{s}=0,$$
$$\boldsymbol{s}\Delta\boldsymbol{x}+\boldsymbol{x}\Delta\boldsymbol{s}=\boldsymbol{a},$$

where $(\Delta\boldsymbol{x},\Delta\boldsymbol{s})\in\mathbb{R}^{2n}_+$ and $\boldsymbol{a}\in\mathbb{R}^n$ are given. Then, the following inequality holds:

$$\Delta\boldsymbol{x}^T\Delta\boldsymbol{s}\ge-\kappa\left\|(\boldsymbol{x}\boldsymbol{s})^{-\frac{1}{2}}\boldsymbol{a}\right\|_2^2. \tag{5.2}$$

The following lemma will be used in the final theorem.

Lemma 5.3. *Let $\boldsymbol{u}=\frac{\boldsymbol{x}\boldsymbol{s}}{\mu}$, where $(\boldsymbol{x},\boldsymbol{s})\in\mathcal{N}^-_{\infty,\varphi}(1-\beta)=\mathcal{D}_\varphi(\beta)$ and let $\beta\in(0,1)$ and $\gamma=\frac{1-\beta}{(1+4\kappa)n+1}$. Then, we have*

$$u_i-((1-\gamma)\beta)^2\ge\beta^2\gamma.$$

Proof. Since before the predictor step $(\mathbf{x},\mathbf{s})\in\mathcal{D}_\varphi(\beta)$,

$$u_i-((1-\gamma)\beta)^2=u_i-\beta^2+2\beta^2\gamma-\beta^2\gamma^2\ge2\beta^2\gamma-\beta^2\gamma^2.$$

After that we have

$$2\beta^2\gamma-\beta^2\gamma^2\ge\beta^2\gamma, \tag{5.3}$$

hence we obtain $\gamma\ge\gamma^2$, which holds for all $\gamma<1$. Using the definition of γ in (4.4) and $0<\beta<1$ we obtain the final result.

□

Theorem 5.1. *Let $n\ge2$ and $\beta\in(0.9,1)$. Then, the PC IPA given in Algorithm 1 using the function $\varphi(t)=\sqrt{t}$ in the AET technique is well defined and*

$$\mu_{k+1}\le\left(1-\frac{3(1-\beta)\beta}{2((1+4\kappa)n+2)}\right)\mu_k,\quad k=0,1\dots$$

Proof. From Lemma 5.1 and Lemma 5.2 we have:

$$\|\mathbf{v}\|_\infty\le(1+4\kappa)n,\quad-4\kappa n\le\mathbf{e}^T\mathbf{v}\le\sum_{i\in\mathcal{I}_+}v_i\le n. \tag{5.4}$$

In the predictor step we have $(\mathbf{x},\mathbf{s})\in\mathcal{N}^-_{\infty,\varphi}(1-\beta)=\mathcal{D}_\varphi(\beta)$. Using (4.20) we get

$$\begin{aligned}\theta_{p_i}&\ge\frac{2(u_i-((1-\gamma)\beta)^2)}{2(u_i-((1-\gamma)\beta)^2)+\sqrt{4(u_i-((1-\gamma)\beta)^2)^2-4(u_i-((1-\gamma)\beta)^2)(v_i-((1-\gamma)\beta)^2\frac{\mathbf{e}^T\mathbf{v}}{n})}}\\&\ge\frac{2(u_i-((1-\gamma)\beta)^2)}{2(u_i-((1-\gamma)\beta)^2)+\sqrt{4(u_i-((1-\gamma)\beta)^2)^2+4(u_i-((1-\gamma)\beta)^2)(\|\mathbf{v}\|_\infty+((1-\gamma)\beta)^2)}}\end{aligned}$$

From Lemma 5.3 we have

$$u_i-((1-\gamma)\beta)^2\geq\beta^2\gamma.$$

Since the function $f(t)=\frac{2t}{2t+\sqrt{4t^2+4at}}$ is increasing in $(0,\infty)$ interval for each $a>0$, we have

$$\begin{aligned}\theta_{p_i}&\geq\frac{2\beta^2\gamma}{2\beta^2\gamma+\sqrt{4(\beta^2\gamma)^2+4(\beta^2\gamma)(\|\mathbf{v}\|_\infty+1)}}=\frac{1}{1+\sqrt{1+(\beta^2\gamma)^{-1}(\|\mathbf{v}\|_\infty+1)}}\\&\geq\frac{1}{1+\sqrt{1+(\beta^2\gamma)^{-1}((1+4\kappa)n+1)}}\\&=\frac{\beta\sqrt{1-\beta}}{\beta\sqrt{1-\beta}+\sqrt{\beta^2(1-\beta)+((1+4\kappa)n+1)^2}}.\end{aligned}$$

It can be seen, that $\beta\sqrt{1-\beta}\leq\frac{1}{2}$, hence

$$\begin{aligned}\beta\sqrt{1-\beta}\quad&+\quad\sqrt{\beta^2(1-\beta)+((1+4\kappa)n+1)^2}\\&\leq\quad\frac{1}{2}+\sqrt{((1+4\kappa)n+1)^2+\frac{1}{4}}<(1+4\kappa)n+2,\end{aligned}$$

that is why $\theta_{p_i}>\hat{\theta}:=\dfrac{\beta\sqrt{1-\beta}}{(1+4\kappa)n+2}$. Using the definition θ_p given in (4.21), from (4.16), (4.17), $\kappa>0$ and $n\geq2$ we have $\theta_{p_0}\geq\frac{1}{1+\sqrt{1+4\kappa}}>\hat{\theta}$. This means that in this case the step length defined in (4.21) satisfies $\theta_p>\hat{\theta}$. We have $\sqrt{\mathbf{x}^p(\theta)\mathbf{s}^p(\theta)}\geq(1-\gamma)\beta\sqrt{\mu_p(\theta)}>(1-\gamma)\beta\sqrt{\mu_p(\theta_p)}\geq0$. From (4.14) and (4.16) the following inequality holds:

$$\mu_p=\mu(\theta_p)<\mu(\hat{\theta})\leq\left((1-2\hat{\theta})+\hat{\theta}^2\right)\mu=(1-(2-\hat{\theta})\hat{\theta})\mu. \tag{5.5}$$

Assuming that $n\geq2$ and $\kappa>0$, we obtain

$$2-\hat{\theta}=2-\frac{\beta\sqrt{1-\beta}}{(1+4\kappa)n+2}\geq2-\frac{\beta\sqrt{1-\beta}}{4}\geq2-\frac{1}{8}=\frac{15}{8}$$

hence we have

$$\mu_p\leq\left(1-\frac{15\beta\sqrt{1-\beta}}{8((1+4\kappa)n+2)}\right)\mu \tag{5.6}$$

Now we are dealing with the corrector step. In this step $(\mathbf{x}^p,\mathbf{s}^p)\in\mathcal{N}^-_{\infty,\varphi}(1-\beta+\beta\gamma)$, so

$$\begin{aligned}\left\|2\left(\mathbf{e}-\sqrt{\frac{\mathbf{x}^p\mathbf{s}^p}{\mu_p}}\right)\right\|_2^2&=4\left(n-2\sum_{i=1}^n\sqrt{\frac{x_i^ps_i^p}{\mu_p}}+\sum_{i=1}^n\frac{x_i^ps_i^p}{\mu_p}\right)=8\left(n-\sum_{i=1}^n\sqrt{\frac{x_i^ps_i^p}{\mu_p}}\right)\\&\leq8(1-(1-\gamma)\beta)n=:\xi n.\end{aligned}$$

Using Lemma 5.1 with $\mathbf{a}=2(\sqrt{\mu_p\mathbf{x}^p\mathbf{s}^p}-\mathbf{x}^p\mathbf{s}^p)$ we get the following two inequalities:

$$\|\Delta^c\mathbf{x}\Delta^c\mathbf{s}\|_\infty\leq\frac{(1+4\kappa)\xi n}{4}\mu_p,\qquad\sum_{i\in\mathcal{I}_+}\Delta x_i^c\Delta s_i^c\leq\frac{\xi n}{4}\mu_p. \tag{5.7}$$

Using (4.23) we obtain

$$\begin{aligned}\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)}{\mu_p} &\geq (1-2\theta)((1-\gamma)\beta)^2+2\theta\,((1-\gamma)\beta)-\frac{(1+4\kappa)n}{4}\xi\theta^2\\ &= ((1-\gamma)\beta)^2+2\theta\left((1-\gamma)\beta-(1-\gamma)\beta)^2\right)-\frac{(1+4\kappa)n}{4}\xi\theta^2. \end{aligned} \tag{5.8}$$

Furthermore, from (4.24) and (5.7) we have

$$\mu_c(\theta)\leq\left(1-2\theta+\frac{2\theta}{\sqrt{n}}+0.25\xi\theta^2\right)\mu_p. \tag{5.9}$$

Using (5.8) and (5.9) we get

$$\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)-\beta^2\mu_c(\theta)}{\mu_p}\geq\frac{\mathbf{x}^c(\theta)\mathbf{s}^c(\theta)}{\mu_p}-\frac{\beta^2(1-2\theta+\frac{2\theta}{\sqrt{n}}+0.25\xi\theta^2)\mu_p}{\mu_p}\geq g(\theta), \tag{5.10}$$

where

$$\begin{aligned}g(\theta) &:= \beta^2-2\beta^2\gamma+\beta^2\gamma^2+2\theta(1-\gamma)\beta(1-(1-\gamma)\beta)-\beta^2+2\beta^2\theta\\ &- \frac{2\beta^2\theta}{\sqrt{n}}-0.25\xi((1+4\kappa)n+\beta^2)\theta^2. \end{aligned} \tag{5.11}$$

Using the definition of γ given in (4.4) we get the following

$$\xi=8\left(\frac{(1-\beta)(1+4\kappa)n+1-\beta^2}{(1+4\kappa)n+1}\right).$$

Then, $g(\theta)$ will be:

$$\begin{aligned}g(\theta) &= -2\frac{(1-\beta)\beta^2}{(1+4\kappa)n+1}+\frac{\beta^2(1-\beta)^2}{((1+4\kappa)n+1)^2}\\ &+ 2\theta\beta\left(\frac{(1+4\kappa)n+\beta}{(1+4\kappa)n+1}\frac{(1-\beta)(1+4\kappa)n+(1-\beta)(1+\beta)}{(1+4\kappa)n+1}+\beta-\frac{\beta}{\sqrt{n}}\right)\\ &- 2\theta^2\frac{(1-\beta)(1+4\kappa)n+(1-\beta)(1+\beta)}{(1+4\kappa)n+1}((1+4\kappa)n+\beta^2)\\ &\geq -\frac{(1-\beta)((1+4\kappa)n+\beta))}{2((1+4\kappa)n+1)^2}\cdot p\\ &\geq -\frac{(1-\beta)((1+4\kappa)n+\beta))}{2((1+4\kappa)n+1)^2}\cdot r\end{aligned}$$

where

$$\begin{aligned}p &= 4\beta^2\frac{(1+4\kappa)n+1}{(1+4\kappa)n+\beta}\\ &- 4\beta\theta\left(((1+4\kappa)n+1+\beta)+\frac{\beta}{1-\beta}\frac{((1+4\kappa)n+1)}{((1+4\kappa)n+\beta)}((1+4\kappa)n+1)\left(1-\frac{1}{\sqrt{n}}\right)\right)\\ &+ 4\theta^2\frac{((1+4\kappa)n+\beta^2)}{((1+4\kappa)n+\beta)}((1+4\kappa)n+1+\beta)((1+4\kappa)n+1). \end{aligned} \tag{5.12}$$

and

$$
\begin{aligned}
r &= 4\beta^2\frac{3}{2+\beta}-4\beta\theta\left(((1+4\kappa)n+1+\beta)+\frac{\beta((1+4\kappa)n+1)}{1-\beta}\left(1-\frac{1}{\sqrt{n}}\right)\right) \\
&+ 4\theta^2((1+4\kappa)n+1+\beta)((1+4\kappa)n+1). \qquad (5.13)
\end{aligned}
$$

We used that $p\le r$. Since $-4\beta^2\frac{3}{2+\beta}\ge -4\beta^2\frac{3}{2}=-6\beta^2$ and $n\ge 2$ we have

$$
g(\theta) \;\ge\; -\frac{(1-\beta)((1+4\kappa)n+\beta))}{2((1+4\kappa)n+1)^2}\cdot s, \qquad (5.14)
$$

where

$$
\begin{aligned}
s &= ((2(1+4\kappa)n+1)\theta-\beta)((2(1+4\kappa)n+1+\beta)\theta-\beta) \\
&- 2\beta^2\theta+5\beta^2-\frac{4\beta^2\theta(1+4\kappa)n+1)}{1-\beta}\left(1-\frac{1}{\sqrt{2}}\right). \qquad (5.15)
\end{aligned}
$$

If $\theta=\dfrac{\beta}{2((1+4\kappa)n+1)}$ and assuming $\beta\in(0.9,1)$ we obtain

$$
g\left(\frac{\beta}{2((1+4\kappa)n+1)}\right)\ge 0. \qquad (5.16)
$$

Hence, we have $\frac{\beta}{2((1+4\kappa)n+1)}\in\mathcal{T}$. From (5.9) and assuming that $n\ge 2$ we have

$$
\begin{aligned}
\mu_c &= \mu_c(\theta_c)\le \mu_c\left(\frac{\beta}{2((1+4\kappa)n+1)}\right) \\
&\le \left(1-\frac{\beta}{(1+4\kappa)n+1}+\frac{\beta}{\sqrt{n}((1+4\kappa)n+1)}+\frac{\beta^2(1-\beta)((1+4\kappa)n+1+\beta)}{2((1+4\kappa)n+1)^3}\right)\mu_p \\
&\le \left(1+\frac{\beta^2(1-\beta)((1+4\kappa)n+1+\beta)}{2((1+4\kappa)n+1)^3}\right)\mu_p.
\end{aligned}
$$

Since $\frac{(1+4\kappa)n+1+\beta}{2((1+4\kappa)n+1)}=\frac{1}{2}\left(1+\frac{\beta}{(1+4\kappa)n+1}\right)\le\frac{2}{3}$ we have

$$
\mu_c\le\left(1+\frac{2\beta^2(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu_p<\left(1+\frac{2\beta(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu_p. \qquad (5.17)
$$

Using (5.6) and (5.17) we obtain

$$
\begin{aligned}
\mu_c &\le \left(1-\frac{15\beta\sqrt{1-\beta}}{8(1+4\kappa)n+2}\right)\left(1+\frac{2\beta(1-\beta)}{3((1+4\kappa)n+1)^2}\right)\mu \\
&\le \left(1-\frac{15\beta(1-\beta)}{8(1+4\kappa)n+2}\right)\left(1+\frac{2\beta(1-\beta)}{3(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\frac{15\beta(1-\beta)}{8(1+4\kappa)n+2}+\frac{2\beta(1-\beta)}{3(1+4\kappa)n((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\left(\frac{15}{8}-\frac{2}{3(1+4\kappa)n}\right)\frac{\beta(1-\beta)}{((1+4\kappa)n+2)}\right)\mu \\
&\le \left(1-\frac{3(1-\beta)\beta}{2((1+4\kappa)n+2)}\right)\mu, \qquad (5.18)
\end{aligned}
$$

where the last inequality follows from the fact that $\frac{15}{8} - \frac{2}{3(1+4\kappa)n} \geq \frac{37}{24} > \frac{3}{2}$, where $n \geq 2$. Hence, we obtained the desired result. $\square$

The following corollary is a consequence of Theorem 5.1.

Corollary 5.1. *Let $n \geq 2$ and $\beta \in (0.9, 1)$. Then, Algorithm 1 produces a point $(\mathbf{x}^k, \mathbf{s}^k) \in \mathcal{N}_{\infty,\varphi}^{-}(1-\beta) = \mathcal{D}_\varphi(\beta)$ with $\mathbf{x}^k \mathbf{s}^k \leq \epsilon$ in at most $\mathcal{O}\left((1+\kappa) n \log\left(\frac{(\mathbf{x}^0)^T \mathbf{s}^0}{\epsilon}\right)\right)$ iterations.*

It should be mentioned that Algorithm 1 depends on a given parameter $\kappa \geq \hat{\kappa}(M)$ because of the parameter γ given in (4.4). It may be difficult and expensive to find on upper bound for the handicap $\hat{\kappa}(M)$ in case of many applications. That is why in the following section we present a PC IPA which does not depend on κ.

6. New version of predictor-corrector interior-point algorithm

We propose a new version of the PC IPA presented in Algorithm 1, which does not depend on κ. Firstly, we set $\kappa = 1$ and use Algorithm 1 for this value. If the algorithm fails to produce a point in $\mathcal{N}_{\infty,\varphi}^{-}(1-\beta) = \mathcal{D}_\varphi(\beta)$ with $\varphi(t) = \sqrt{t}$, then the current value of κ may be too small. Hence, we double the value of κ and restart Algorithm 1 from the last point produced in $\mathcal{D}_\varphi(\beta)$. In this way, we have to double the value of κ at most $\lceil \log_2 \hat{\kappa}(M) \rceil$ times. This new version of the algorithm is presented in Algorithm 2.

```
Algorithm 2: Predictor-corrector interior-point algorithm not depending on κ
Input:
(x^0, s^0) ∈ N^-_{∞,φ}(1−β), β ∈ (0.9, 1);
Set κ = 1
Let μ_0 = μ(x^0, s^0) and k = 0
ε > 0 precision value.
Output: (x^k, s^k) : x^{kT} s^k ≤ ε
begin
    while nμ ≥ ε do
        (Predictor step);
        x := x^k, s := s^k;
        Step 1. Calculate affin direction from (4.2);
        Step 2. Calculate the predictor steplength using (4.5);
        Step 3. Calculate (x^p, s^p);
        if μ(x^p, s^p) = 0 then
            STOP; Optimal solution found;
        else
            if (x^p, s^p) ∈ N^-_{∞,φ}(1−β) then
                (x^{k+1}, s^{k+1}) = (x^p, s^p), μ^{k+1} = μ(x^p, s^p), k = k+1, RETURN;
            else
                (Corrector step);
                Step 4. Calculate centering direction from (4.7);
                Step 5. Calculate centering steplenght using (4.9);
                Step 6. Calculate (x^c, s^c);
                if (x^c, s^c) ∈ N^-_{∞,φ}(1−β) then
                    (x^{k+1}, s^{k+1}) = (x^c, s^c), μ^{k+1} = μ(x^c, s^c), k = k+1, RETURN;
                else
                    κ = 2κ; (x^{k+1}, s^{k+1}) = (x^c, s^c), μ^{k+1} = μ(x^c, s^c), k = k+1,
                     RETURN;
                end
            end
        end
    end
end
```

Using Theorem 3.9 in [46] and Theorem 5.1, Corollary 5.1 we obtain the following.

Theorem 6.1. *Algorithm 2 produces a point* $(\mathbf{x}^k, \mathbf{s}^k) \in \mathcal{N}_{\infty,\varphi}^-(1-\beta) = \mathcal{D}_\varphi(\beta)$ *with* $\mathbf{x}^k\mathbf{s}^k \leq \epsilon$ *in at most* $\mathcal{O}\left((1+\hat\kappa(M))n \log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ *iterations.*

Proof. Consider $\bar\kappa$ as the largest value of κ used in Algorithm 2. Then, we have $\bar\kappa > 2\hat\kappa(M)$. Now we consider that at iteration k of Algorithm 2 we have $\kappa < \hat\kappa(M)$. If $(\mathbf{x}^c, \mathbf{s}^c) \in \mathcal{N}_{\infty,\varphi}^-(1-\beta)$, then $(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}) = (\mathbf{x}^c, \mathbf{s}^c)$. Using that Lemmas 5.1 and 5.2 hold for $\kappa = \hat\kappa(M)$ and the bound on the predictor step size depends on γ which is decreasing in κ, we obtain that

$$\mu_{k+1} \leq \left(1 - \frac{3(1-\beta)\beta}{2((1+4\hat\kappa(M))n+2)}\right)\mu_k \leq \left(1 - \frac{3(1-\beta)\beta}{2((1+8\hat\kappa(M))n+2)}\right)\mu_k.$$

Furthermore, if $\kappa \geq \hat\kappa(M)$, then the corrector step is never rejected. Hence, using Theorem 5.1 and the fact that $\kappa \leq \bar\kappa < 2\hat\kappa(M)$, we obtain

$$\mu_{k+1} \leq \left(1 - \frac{3(1-\beta)\beta}{2((1+4\kappa)n+2)}\right)\mu_k \leq \left(1 - \frac{3(1-\beta)\beta}{2((1+8\hat\kappa(M))n+2)}\right)\mu_k.$$

Using that there can be at most $\log_2(\bar\kappa)$ rejections we obtain the final result. □

7. Numerical results

We implemented a variant of the proposed PC IPA in the C++ programming language. We did all computations on a desktop computer with Intel quad-core 2.6 GHz processor and 16 GB RAM. Due to the fact that in many cases we do not have information about the value of κ, we used Algorithm 2 in our implementation. We set the values $\beta = 0.95$ and $\epsilon = 10^{-5}$. In spite of the fact that the complexity analysis of our PC IPA works for $\beta \in (0.9, 1)$, in the implementation we also consider the case when $\beta = 0.1$ and we obtain promising results.

It is important to mention that we implemented the theoretical version of the PC IPA and followed the main steps described in Section 4. Moreover, it should be mentioned that most of the numerical results related to $P_*(\kappa)$-LCPs are related to problems where the value of κ is zero, that lead to LO problems. Gurtuna et al. [24] and Asadi et al. [4] provided numerical results related to $P_*(\kappa)$-LCPs having positive handicap, by considering 2×2 or 3×3 matrices. They also analysed block diagonal matrices formed by the aformentioned ones. Darvay et al. [14] presented numerical results where they solved $P_*(\kappa)$-problems with matrices having positive κ generated by Illés and Morapitiye [26].

We tested the PC IPA on LCPs with sufficient matrices given by Illés and Morapitiye [26]. We generated the test problems in the following way: $\mathbf{q} := -M\mathbf{e} + \mathbf{e}$. We considered $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{s}^0 = \mathbf{e}$ as starting points for our PC IPA. In our computational study we compared our PC IPA using the function $\varphi(t) = \sqrt{t}$ in the AET technique and the neighbourhood $\mathcal{N}_{\infty,\varphi}^-(1-\beta) = \mathcal{D}_\varphi(\beta)$ with the PC IPA of Potra and Liu proposed in [46], which is based on the function $\varphi(t) = t$. Moreover, we also compared our IPA to the PC IPAs presented in [14] that use the neighbourhood

$$\mathcal{N}_2(\tau, \mu) := \left\{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \left\|\frac{\mathbf{a}_\varphi}{2\sqrt{\mu\mathbf{x}\mathbf{s}}}\right\| \leq \tau\right\}, \tag{7.1}$$

where $\mathbf{a}_\varphi = \mu\mathbf{e} - \mathbf{x}\mathbf{s}$ in case of $\varphi(t) = t$ and $\mathbf{a}_\varphi = 2(\sqrt{\mu\mathbf{x}\mathbf{s}} - \mathbf{x}\mathbf{s})$ in case of $\varphi(t) = \sqrt{t}$.

Table 1 contains the average of iteration numbers and CPU times (in seconds) with $\beta = 0.95$ for the given LCPs for each size n in case of the PC IPA proposed by Potra and Liu [46] based on the search direction using $\varphi(t) = t$ and in case of our PC IPA which uses the function $\varphi(t)=\sqrt{t}$ in the AET technique. Table 2 contains the average of iteration numbers and CPU times (in seconds) with $\beta=0.1$ in case of the two aformentioned PC IPAs. In these cases our PC IPA provided better results. In Table 3 we present the results given in [14] that are related to PC IPAs working in the neighbourhood given in (7.1) using the functions $\varphi(t)=t$ and $\varphi(t)=\sqrt{t}$, respectively. We can see that in case of small-sized problems the PC IPAs using the wide neighboruhoods $\mathcal{N}_{\infty,\varphi}^{-}(1-\beta)=\mathcal{D}_{\varphi}(\beta)$ provided usually better results, while in case of large-sized problems the PC IPAs presented in [14] were better.

n	$\varphi(t)=t$ with $\mathcal{D}_{\varphi}(0.95)$		$\varphi(t)=\sqrt{t}$ with $\mathcal{D}_{\varphi}(0.95)$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
10	10.6	0.2494	9	0.2584
20	13.6	0.5852	10.4	0.7474
50	7.4	1.7070	6.2	1.7478
100	9.8	8.8782	8.2	8.4858
200	11.2	31.4722	9	36.4954
500	14.2	258.7558	11.2	272.2758
700	16	534.0890	13	612.9010

Table 1. Numerical results with $\beta=0.95$ for $P_*(\kappa)$-LCPs from [26] having positive handicap.

n	$\varphi(t)=t$ with $\mathcal{D}_{\varphi}(0.1)$		$\varphi(t)=\sqrt{t}$ with $\mathcal{D}_{\varphi}(0.1)$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
10	4.8	0.1144	4.6	0.0978
20	5.4	0.2420	5.2	0.2276
50	4	0.9790	4	1.0752
100	4.4	3.7110	4.4	4.2046
200	5.2	14.8130	5.2	19.8456
500	5.6	93.2230	5.8	130.0336
700	6	195.2880	6	276.3360

Table 2. Numerical results with $\beta=0.1$ for $P_*(\kappa)$-LCPs from [26] having positive handicap.

In Table 4 we compare our PC IPA to the PC IPA proposed by Liu and Potra [46] and to the IPAs tha uses the neighbourhood appeared in (7.1) in case of five 10×10 sized problems from [26] with $\beta=0.95$ and $\beta=0.1$, respectively. We can see that in case of $\beta=0.1$ we obtained better results in case of both PC IPAs.

De Klerk and E.-Nagy [18] proved that the handicap of the matrix can be exponential in the size of the problem. They considered the following matrix which was proposed by Csizmadia:

$$M=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & \cdots & 0\\ -1 & -1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}, \tag{7.2}$$

n	$\varphi(t) = t$ with $\mathcal{N}_2(\tau, \mu)$		$\varphi(t) = \sqrt{t}$ with $\mathcal{N}_2(\tau, \mu)$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
10	23.9	0.024	25.7	0.026
20	6.1	0.037	8.0	0.050
50	5.1	0.317	5.1	0.317
100	5.4	2.212	5.4	2.216
200	5.8	17.564	6.0	18.165
500	6.2	279.968	6.4	288.993
700	7	898.662	7	899.404

TABLE 3. Numerical results given in [14] for $P_*(\kappa)$-LCPs from [26] having positive handicap.

	$\varphi(t) = t$ with $\mathcal{D}_\varphi(0.95)$	$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.95)$	$\varphi(t) = t$ with $\mathcal{D}_\varphi(0.1)$	$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.1)$	$\varphi(t) = t$ with $\mathcal{N}_2(\tau, \mu)$	$\varphi(t) = \sqrt{t}$ with $\mathcal{N}_2(\tau, \mu)$
MGS_10_1	18	16	7	6	9	9
MGS_10_2	13	10	5	6	7	14
MGS_10_3	8	7	4	6	72	86
MGS_10_4	17	13	6	4	7	9
MGS_10_5	16	14	7	6	50	11

TABLE 4. Numerical results for $P_*(\kappa)$-LCPs from [26] having positive handicap.

and they proved that $\hat{\kappa}(M) \geq 2^{2n-8} - 0.25$. However, we obtained promising results in this case as well. In this case we also compared our PC IPA with the IPA from [46]. The results are summarized in Tables 5 and 6.

n	$\varphi(t) = t$ with $\mathcal{D}_\varphi(0.95)$		$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.95)$	
	Nr. of Iter.	CPU (s)	Nr. of Iter.	CPU (s)
10	21	0.273	18	0.291
20	19	0.515	18	0.470
50	26	1.975	27	2.804
100	39	12.817	38	14.921
200	66	88.064	67	87.512
300	97	240.758	95	327.764
400	122	521.747	121	685.074

TABLE 5. Numerical results with $\beta = 0.95$ for $P_*(\kappa)$-LCPs with matrix given in (7.2)

M. E.-Nagy generated randomly $P_*(\kappa)$ matrices of different sizes from $n = 3, \ldots, 10$, see [16]. We tested our PC IPA in these cases as well and we compared our algorithm to the PC IPA of Potra and Liu [46]. The results are summarized in Table 7. It can be observed that in these cases the two PC IPAs using different search directions provide similar results.

Beside this, M. E.-Nagy also generated non-sufficient matrices of different sizes from $n = 3, \ldots, 10$, see [16]. We tested our PC IPA in these cases as well and we obtained promising results, which shows that this PC IPA can be used for solving more general LCPs as well. The results are presented in Table 8.

n	$\varphi(t) = t$ with $\mathcal{D}_\varphi(0.1)$		$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.1)$	
	Nr. of Iter.	CPU (s)	Nr. of Iter.	CPU (s)
10	8	0.154	7	0.081
20	10	0.297	9	0.248
50	16	1.892	15	1.601
100	25	11.02	24	10.356
200	47	72.755	43	69.919
300	66	174.835	63	227.014
400	87	406.000	82	531.613

TABLE 6. Numerical results with $\beta = 0.1$ for $P_*(\kappa)$-LCPs with matrix given in (7.2)

n	$\varphi(t) = t$ with $\mathcal{D}_\varphi(0.95)$		$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.95)$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
3	6.4	0.0634	4.8	0.0528
4	4.6	0.0576	4.2	0.0502
5	6.6	0.0942	5.6	0.0784
6	6.8	0.0948	5.6	0.0938
7	7.6	0.1162	6.0	0.1016
8	7.8	0.2132	6.2	0.1254
9	6.8	0.1274	5.8	0.1312
10	9.0	0.1952	7.0	0.1712

TABLE 7. Numerical results for $P_*(\kappa)$-LCPs from E.-Nagy having positive handicap.

n	$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.95)$		$\varphi(t) = \sqrt{t}$ with $\mathcal{D}_\varphi(0.1)$	
	Avg. Iter.	CPU	Avg. Iter.	CPU
3	6.0	0.0582	3.4	0.0318
4	6.0	0.0708	3.4	0.0428
5	6.2	0.0930	3.4	0.0502
6	5.6	0.0972	3	0.0520
7	6.8	0.1104	3.2	0.0574
8	5.8	0.1064	3.6	0.0694
9	6.4	0.1356	3.6	0.0752
10	9.4	0.2512	4.8	0.1118

TABLE 8. Numerical results with $\varphi(t) = \sqrt{t}$ for non-sufficient LCPs from E.-Nagy.

In the following section some concluding remarks are presented.

8. CONCLUSION

In this paper we proposed a new PC IPA for solving $P_*(\kappa)$-LCPs. The proposed IPA determines the new search directions by using the function $\varphi(t) = \sqrt{t}$ in the AET technique and works in a new wide neighbourhood. We proved that the PC IPA has $\mathcal{O}\left((1+\kappa)n\log\left(\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\epsilon}\right)\right)$ iteration complexity. We also provided numerical results where we compared our PC IPA to other ones that use different search directions or neighbourhoods. We also tested our PC IPA on LCPs, where the matrices are not sufficient and we obtained promising results. This leads to further research topic, because it shows that the PC IPA using this search direction can be used for solving general LCPs as well.

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