The stable limit DAHA and the double Dyck path algebra

by

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The double Dyck path algebra (DDPA) is the key algebraic structure that governs the phenomena behind the shuffle and rational shuffle conjectures. Carlsson and Mellit[CM18] introduced the DDPA as part of their proof of the shuffle conjecture. Later Mellit[Mel16] used this algebra to prove the more general rational shuffle conjecture.

The structure emerged from their considerations and computational experiments (see, especially, [CM18, Sect. 4.1]) while attacking the conjecture. Nevertheless, the DDPA bears some resemblance to the structure of a type A double affine Hecke algebra (DAHA). While trying to address this resemblance, Carlsson and Mellit noted one aspect that differentiates the two structures and speculated on how they could be ultimately related [CM18, p. 693-694].

The goal of this project is to explain how the DDPA emerges naturally and canonically (as a stable limit) from the family of GL_n DAHA's. Our context is different from the one suggested by Carlsson and Mellit.

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Preface

This dissertation was inspired by the proof of the shuffle conjecture by Carlsson and Mellit. Throughout the writing of this dissertation I have received a great deal of support and assistance. I would like to thank my academic advisor Bogdan Ion, who constantly offers invaluable guidance for my research. I would like to thank my committee members, each of whom has provided patient advice throughout the research process. I would like to thank all people who have generously offered their help as I proceeded my research. Thank you all for your unwavering support.

1.0 Introduction

In 2018, Erik Carlsson and Anton Mellit published a proof [CM18] of the compositional shuffle conjecture of Haglund, Morse, and Zabrocki in [HMZ12], which is a refinement of the original shuffle conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov in [BG99]. The conjecture was originally stated in relation to the diagonal representation of the symmetric group S_n on the polynomial ring $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$ in n pairs of variables. Explicitly, it states that

$$\nabla e_n[X] = \sum_{\pi} \sum_{w \in \mathcal{WP}_\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi,w)} x_w,$$

where the distinguished ∇ operator is defined in [BGSLX16b] to act diagonally on the modified Macdonald basis $\{\tilde{H}_\mu\}$, and the right-hand side of the equation consists of various combinatorial quantities associated to Dyck paths which will be introduced in details in Chapter 4. Haglund, Morse, and Zabrocki later refined the conjecture as

$$\nabla C_\alpha[X;q] = \sum_{\mathrm{touch}(\pi)=\alpha} \sum_{w \in \mathcal{WP}_\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi,w)} x_w,$$

where C_α is a composition of creation operators for Hall-Littlewood polynomials with parameter $1/q$ acting on the constant polynomial 1.

To attack the conjecture, Carlsson and Mellit invented a new structure $\mathbb{A}_{q,t}$ called double Dyck path algebra. $\mathbb{A}_{q,t}$ together with its canonical representation is the key algebraic structure that governs the phenomena behind the shuffle conjecture. In this way, they successfully related steps in a Dyck path to operators in the corresponding double Dyck path algebra. They were able to recover a Dyck path from recursively applying operators on the constant polynomial 1. Furthermore, they discovered that the antilinear degree-preserving automorphism defined by

$$T_i \mapsto T_i^{-1}, \quad d_- \mapsto d_-, \quad d_+ \mapsto d_+^*, \quad y_i \mapsto z_i$$

maps intermediate polynomials in the recursion process into monomials in y_i. By using the automorphism, the conjecture can be proved by recursion.

Using the theory of the double Dyck path algebra, Mellit proved the rational compositional shuffle conjecture in [Mel16] and the compositional delta conjecture with D'Adderio in [DM20]. Considering the significant potential of this new algebra, we would like to understand the structure of the algebra $\mathbb{A}_{q,t}$. Carlsson and Mellit predicted there is a connection between $\mathbb{A}_{q,t}$ and double affine Hecke algebras $\mathcal{H}_k^+$. They left the problem of relating the two structures as an open problem.

In this dissertation, we will use the stabilization of the representation theory of the deformed double affine Hecke algebra $\tilde{\mathcal{H}}_k^+$ to explain the structure. The algebra $\tilde{\mathcal{H}}_k^+(k > 1)$ is defined to be the $\mathbb{Q}(t,q)$-algebra generated by $T_1, ..., T_{k-1}$, $X_1, ..., X_k, \bar{\omega}$ with the following:

1.
$$T_iT_j = T_jT_i, \text{ for } |i-j| > 1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad (T_i - 1)(T_i + t) = 0,$$
$$tT_i^{-1}X_iT_i^{-1} = X_{i+1}, \qquad X_iX_j = X_jX_i,$$
$$T_iX_j = X_jT_i, \text{ for } j \neq i, i+1,$$

2.
$$\bar{\omega}T_i = T_{i+1}\bar{\omega}, \text{ for } i = 1, ..., k-2,$$
$$\bar{\omega}X_i = X_{i+1}\bar{\omega}, \text{ for } i = 1, ..., k-1,$$

3. Denote

$$\gamma = \bar{\omega}^2 T_{k-1} - T_1 \bar{\omega}^2.$$

Then

$$\gamma T_{k-1} = -t\gamma, \quad T_1 \gamma = \gamma,$$

$$\gamma \bar{\omega}^{k-2} \gamma = \gamma \bar{\omega}^{k-1} \gamma = \gamma \bar{\omega}^k = 0.$$

We have discovered the algebra admits a representation on P_k^+, the polynomial ring $\mathbb{Q}(t,q)[x_1, ..., x_k]$. More specifically, we have proven the existence of the following action of $\tilde{\mathcal{H}}_k^+$ on P_k^+.

Theorem 1. *The algebra $\tilde{\mathcal{H}}_k^+$ has a representation on $\mathbb{Q}(t,q)[x_1, ..., x_k]$ defined as the following:*

$$X_i \mapsto x_i,$$
$$T_i \mapsto s_i + (1-t)x_i \frac{1-s_i}{x_i - x_{i+1}},$$
$$\tilde{Y}_i \mapsto t^{1-i+k} T_{i-1} ... T_1 \bar{\omega} T_{k-1}^{-1} ... T_i^{-1},$$

where

$$\bar{\omega}.f(x_1, ..., x_k) = p_1 \omega^{-1}.f(x_1, ..., x_k), \quad \omega f(x_1, ..., x_k) = f(q^{-1}x_k, x_1, ..., x_{k-1}),$$

for $f(x_1, ..., x_k) \in P_k^+$, and

$$p_1 : \mathbb{Q}(t,q)[x_1, ..., x_k] \to x_1 \mathbb{Q}(t,q)[x_1, ..., x_k],$$

$$x_1 f(x_1, ..., x_k) \mapsto x_1 f(x_1, ..., x_k), \;\; f(x_2, ..., x_k) \mapsto 0$$

is the projection map onto the subspace $x_1 \mathbb{Q}(t,q)[x_1, ..., x_k]$.

The main obstacle of the stabilization process is the fact that the deformed Cherednik operators $\tilde{Y}_i$ are still not fully compatible with the inverse system. Another difficult aspect of this structure is the non-commutativity of $\tilde{Y}_i$ operators. We overcome this second difficulty by formulating a concept of limit that takes into consideration not only the inverse system, but also the t-adic topology. As it turns out, the limit operators $\tilde{Y}_i$ act on the almost symmetric module $\mathcal{P}_{\mathrm{as}}^+$, a subspace of the inverse limit $\mathcal{P}_\infty^+$ of P_k^+. On $x_i\mathcal{P}_{\mathrm{as}}^+$ the action of the limit operator $\tilde{Y}_i$ coincides with the action of the Cherednik operator Y_i.

We relate the action of the limit operators on $\mathcal{P}_{\mathrm{as}}^+$ to the following algebra. Let $\mathcal{H}^+$ be the $\mathbb{Q}(t,q)$-algebra generated by the elements T_i,X_i, and Y_i, $i \geq 1$, satisfying the following relations:

$$T_iT_j = T_jT_i, \quad |i-j| > 1,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad (T_i - 1)(T_i + t) = 0, \quad i \geq 1,$$

$$tT_i^{-1}X_iT_i^{-1} = X_{i+1}, \quad i \geq 1$$
$$T_iX_j = X_jT_i, \quad j \neq i, i+1; \quad X_iX_j = X_jX_i, \quad i, j \geq 1,$$
$$t^{-1}T_iY_iT_i = Y_{i+1}, \quad i \geq 1$$
$$T_iY_j = Y_jT_i, \quad j \neq i, i+1; \quad Y_iY_j = Y_jY_i, \quad i, j \geq 1,$$
$$Y_1T_1X_1 = X_2Y_1T_1.$$

We call $\mathcal{H}^+$ the positive limit DAHA. In order to establish the limit action, we first proved the recovery of the commutativity of $\tilde{Y}_i$ operators in the limit.

Theorem 2. *Let $Y_i = \lim_k \tilde{Y}_i^{(k)}$. Then*

$$[Y_i, Y_j] = \lim_k [\tilde{Y}_i^{(k)}, \tilde{Y}_j^{(k)}]\Pi_k = 0.$$

The main technical result we proved is the following

Theorem 3. *The algebra $\mathcal{H}^+$ admits a representation on $\mathcal{P}^+_{\mathrm{as}}$ induced from the $\tilde{\mathcal{H}}^+_k$ action on P^+_k. More specifically,*

$$T_i = \lim_k T_i^{(k)}, \quad X_i = \lim_k X_i^{(k)}, \quad Y_i = \lim_k \tilde{Y}_i^{(k)}$$

defines an action on $\mathcal{P}^+_{\mathrm{as}}$.

We then prove the main theorem of this dissertation in Chapter 5, which states there exists an isomorphism defined as follows:

Theorem 4. *$\mathcal{P}(k)$ is an $\mathcal{H}^+_\infty$-subrepresentation of $\mathcal{P}^+_{\mathrm{as}}$ for any k. Define the algebraic isomorphism $\Phi = (\Phi_k)_{k\geq 0} : \mathcal{P}_* \to V_*$ as*

$$\Phi_k : \mathcal{P}(k) \to V_k; \quad x_i \mapsto y_i \ \text{ for } i \leq k; \quad \mathbf{X}_k \mapsto \frac{X}{t-1}.$$

Then under this definition we have

$$y_i \Phi_k = \Phi_k X_i, \qquad z_i \Phi_k = \Phi_k \tilde{Y}_i.$$

Furthermore, we will also be able to define explicitly all of the connecting maps as follows.

Theorem 5. *Now let $i_k : \mathcal{P}(k) \to \mathcal{P}(k+1)$ be the inclusion map. Define the connecting maps $\partial_k = \tilde{\omega}^{-1}_{k+1}.i_k$ and $\partial^*_k = \omega^{-1}_{k+1}.i_k$ as two operators from $\mathcal{P}(k)$ to $\mathcal{P}(k+1)$. Let $\partial^-_k : \mathcal{P}(k) \to \mathcal{P}(k-1)$ be defined as*

$$\partial^-_k f(x_1, ..., x_k) F[\mathbf{X}_k] = \tau_k f(x_1, ..., x_k) F[\mathbf{X}_k - x_k] Exp[(1-t) x_k^{-1} \mathbf{X}_k]\Big|_{\mathrm{const}(x_k)},$$

where τ_k is the alphabet shift $x_{i+1} \mapsto x_i$ for all $i \geq k$. Then $\mathbb{A}_{q,t}$ has a representation on $\mathcal{P}_*$ defined as the following:

$$T_i F = s_i F + (1-t) x_i \frac{F - s_i F}{x_i - x_{i+1}},$$

$$d_+F = \partial_k F, \qquad d_+^*F = \partial_k^* F, \qquad d_-F = \partial_k^- F,$$

where $F \in \mathcal{P}(k)$.

Then Φ is an isomorphism of $\mathbb{A}_{q,t}$ representations under this definition.

The theorem uses a simple isomorphism Φ to link the canonical DDPA representation with the modified polynomial DAHA representations. Therefore, the canonical DDPA representation can be fully characterized by studying the classical DAHA representations. As the theory of DAHA has been enthusiastically developed by mathematicians in various fields for years, one may expect the connection between the two kinds of algebras will considerably enrich the theory of the double Dyck path algebra and widen the range of its potential applications, as well as help resolve some still open conjectures generalized from or related to the shuffle conjecture.

2.0 Ring of symmetric functions

In this section we will introduce the preliminary facts on symmetric functions used in the thesis.

2.1 Combinatorics

2.1.1 Partitions

By definition, a partition λ of n is a sequence

$$\lambda = (\lambda_1, ..., \lambda_k)$$

such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k > 0$ and $\lambda_1 + ... + \lambda_k = n$. We will use the notation $\lambda \vdash n$ to denote that λ is a partition of n. We will call k the length of λ and denote by $l(\lambda) = k$. By convention, we will impose that $\lambda_i = 0$ for all $i > k$. Therefore, we have

$$\sum_{i \geq 1} \lambda_i = n.$$

For later use we define the quantities $n(\lambda)$ and z_λ below associated to the partition λ for later use.

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

$$z_\lambda = \prod_i (r_i! i^{r_i}),$$

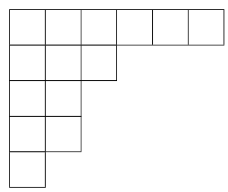
where r_i is the number of occurrences of i in λ.

2.1.2 Young diagrams and Young tableaux

We will use a family of combinatorial objects called Young diagrams to illustrate partitions. The Young diagram $D(\lambda)$ corresponding to the partition

$$\lambda = (\lambda_1, ..., \lambda_k)$$

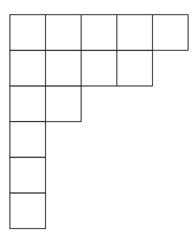
consists of k consecutive rows of squares such that the i-th row contains exactly λ_i number of squares. For instance, the following figure portraits the diagram corresponding to the partition $(6, 3, 2, 1, 1)$.



We say the box located in the j-th row from top to buttom and the i-th column from left to right has the coordinates (i, j). The diagram generator of λ is defined as

$$B_\lambda(q, t) = \sum_{(i,j)\in D(\lambda)} t^{i-1}q^{j-1}.$$

Given a partition λ, we define the conjugate λ' of λ to be the partition that corresponds to the diagram obtained by transposing $D(\lambda)$. For instance, the conjugate of the previous example $(6, 3, 2, 2, 1)$ is $(5, 4, 2, 1, 1, 1)$ as shown below.

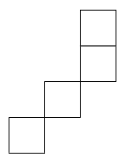


We will adopt two partial orders on the set of partitions of n. The dominance order is defined as

$$\lambda \trianglelefteq \mu \Longleftrightarrow \lambda_1 + ... + \lambda_k \leq \mu_1 + ... + \mu_k, \text{ for all } k \geq 1.$$

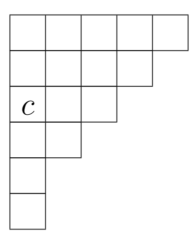
We say that μ dominates λ if $\lambda \trianglelefteq \mu$.

Another partial order is obtained from containment of Young diagrams. We will write $\lambda \subseteq \mu$ if $D(\lambda)$ is contained in $D(\mu)$, or equivalently, if $\lambda_i \leq \mu_i$ for all $i \geq 1$. If $\lambda \subseteq \mu$, we define the skew Young diagram μ/λ to be the collection of cells in $D(\mu) \setminus D(\lambda)$. The following figure shows the skew diagram μ/λ where $\lambda = (2, 2, 1)$ and $\mu = (3, 3, 2, 1)$.



We will associate four quantities to each cell in a Young diagram. The arm $\mathrm{arm}_\lambda(c)$, leg $\mathrm{leg}_\lambda(c)$, coarm $\mathrm{coarm}_\lambda(c)$, and coleg $\mathrm{coleg}_\lambda(c)$ of a cell c in the diagram $D(\lambda)$ will denote the number of cells to the east, south, west and north of c respectively. In the example below, we have

$$\mathrm{arm}_\lambda(c) = 2, \quad \mathrm{leg}_\lambda(c) = 3, \quad \mathrm{coarm}_\lambda(c) = 0, \quad \mathrm{coleg}_\lambda(c) = 2.$$



A Young tableau of a diagram $D(\lambda)$ is an assignment to each box a number in the set $\{1, 2, ..., n\}$ where $\lambda \vdash n$. A Young tableau is said to be standard if the entries are increasing along each row and each column. It is said to be semistandard if the entries are non-decreasing along each row and increasing along each column. The following are examples of a standard and a semistandard Young tableau of $D((5,2,1))$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 8 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 3 & 3 & 3 & 4 & 4 \\ \hline 4 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

We will then denote by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of $D(\lambda)$, and $\mathrm{SSYT}(\lambda)$ the set of semistandard Young tableaux of $D(\lambda)$.

2.2 Symmetric functions

2.2.1 Ring of symmetric functions

The polynomial $f(x_1, ..., x_n) \in \mathbb{Q}[x_1, ..., x_n]$ is said to be symmetric in $x_1, ..., x_n$ if we have

$$f(x_{\sigma_1}, ..., x_{\sigma_n}) = f(x_1, ..., x_n), \text{ for all } \sigma \in S_n.$$

All symmetric polynomials in $x_1, ..., x_n$ will form a ring under polynomial addition and multiplication. This is called the symmetric polynomial ring in X_n with the coefficient ring $\mathbb{Q}$, and denoted by $\mathrm{Sym}[X_n]$, where

$$X_n = x_1 + x_2 + ... + x_n$$

is called the alphabet of the symmetric polynomial ring.

Consider the projection map

$$p_{m,n}\mathbb{Q}[x_1, x_2, ..., x_m] \to \mathbb{Q}[x_1, x_2, ..., x_n] \quad (m > n)$$

by sending $x_{n+1}, ..., x_m$ to 0. The restriction map on $\mathrm{Sym}[X_m]$ will be also called $p_{m,n}$. Then the symmetric polynomial rings will form an inverse system with respect to the projection maps. We will define the ring of symmetric functions in the alphabet $X = x_1 + x_2 + ...$ to be the inverse limit, i.e.

$$\mathrm{Sym}[X] = \lim_{\leftarrow} \mathrm{Sym}[X_n]$$

At last we introduce the following notations for further use.

Notation. We define the following three alphabets

$$\bar{\mathbf{X}}_k = x_1 + ... + x_k, \quad \mathbf{X}_k = x_{k+1} + x_{k+2} + ..., \quad \bar{\mathbf{X}}_{[m,n]} = \bar{\mathbf{X}}_n - \bar{\mathbf{X}}_m.$$

2.2.2 Distinguished symmetric functions

There are distinguished families of symmetric polynomials and symmetric functions which will be useful. We will define them below.

Definition 1. (m_λ, e_μ, p_μ and h_μ) Let the alphabet $X_n = x_1 + x_2 + ... + x_n$. The monomial symmetric polynomial $m_\lambda[X_n]$ with respect to the partition λ for any $|\lambda| \le n$ is defined as

$$m_\lambda[X_n] = \sum x^\lambda = \sum_\alpha x_1^{\alpha_1} ... x_n^{\alpha_n},$$

where α runs over all distinct permutations of λ. The elementary symmetric polynomial $e_k[X_n]$, the power-sum symmetric polynomial $p_k[X_n]$ are then defined as

$$e_k[X_n] = m_{(1,1,...,1)}[X_n], \qquad p_k[X_n] = m_{(k)}[X_n],$$

where $(1, 1, ..., 1) \vdash k$. By convention, $e_k[X_n] = 0$ if $k > n$. Then for a partition $\mu \vdash m$, we define

$$e_\mu[X_n] = e_{\mu_1}[X_n] ... e_{\mu_m}[X_n], \qquad p_\mu[X_n] = p_{\mu_1}[X_n] ... p_{\mu_m}[X_n].$$

At last, the complete homogeneous symmetric polynomial $h_k[X_n]$ is defined to be the sum of all distinct monomials of degree k in X_n, and

$$h_\mu[X_n] = h_{\mu_1}[X_n]...h_{\mu_m}[X_n], \text{ for any } \mu.$$

Now let $X = x_1 + x_2 + ...$ Then we define

$$m_\lambda[X] = \sum_{Y \subset X,\ |Y|=|\lambda|} m_\lambda[Y],$$

and similarly for $e_\mu[X], p_\mu[X]$ and $h_\mu[X]$.

Example 1. We have

$$p_2[X_3] = x_1^2 + x_2^2 + x_3^2, \quad e_2[X_3] = x_1x_2 + x_1x_3 + x_2x_3, \quad h_2[X_3] = p_2[X_3] + e_2[X_3].$$

$$p_2[X] = x_1^2 + x_2^2 + ..., \quad e_2[X] = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + ...$$

We will then define another family of special symmetric functions called Schur polynomials/functions.

Definition 2. (Schur polynomials/functions) Let λ be a partition. Then the Schur polynomial $s_\lambda[X_n]$ is defined as

$$s_\lambda[X_n] = \sum_{T \in \mathrm{SSYT}(D(\lambda))} x^T = \sum_{T \in \mathrm{SSYT}(D(\lambda))} x_1^{t_1}...x_n^{t_n},$$

where t_i is the number of occurrences of the number i in the semistandard Young tableau T. For an infinite alphabet X, we define

$$s_\lambda[X] = \sum_{Y \subset X,\ |Y|=|\lambda|} s_\lambda[Y].$$

Example 2. We have

$$s_{(1,1,0)}[X_3] = x_1x_2 + x_2x_3 + x_1x_3 = e_2[X_3], \quad s_{(1,1)}[X] = e_2[X].$$

It is a classical result that these special kinds of symmetric functions will algebraically span the whole space. Furthermore, we have

Proposition 2.2.1. *Let Z be an alphabet, which is allowed to be either finite or infinite. Then each of the following sets*

$$\{p_\lambda[Z]\}_{|\lambda|\leq|Z|}, \quad \{e_\lambda[Z]\}_{|\lambda|\leq|Z|}, \quad \{h_\lambda[Z]\}_{|\lambda|\leq|Z|}, \quad \{m_\lambda[Z]\}_{|\lambda|\leq|Z|}, \quad \{s_\lambda[Z]\}_{|\lambda|\leq|Z|}$$

will form a basis for Sym[Z].

We will then define the involution ω on $\mathrm{Sym}[Z]$ on bases as the following

$$\omega e_k[Z] = h_k[Z], \quad \omega p_k[Z] = (-1)^{k-1} p_k[Z], \quad \omega s_\lambda[Z] = s_{\lambda'}[Z].$$

2.2.3 Hall inner product

We will then define the Hall inner product $\langle -, - \rangle$ to be the symmetric bilinear inner product on symmetric functions such that any of the following conditions is satisfied:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}, \quad \langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}, \quad \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}.$$

It is a fact that the three possible definitions are equivalent. Therefore with respect to the inner product, the Schur functions will form an orthonormal basis.

2.3 Plethysm

From now on in all of the remaining sections, we will use the symmetric function ring with coefficient ring $\mathbb{Q}(q,t)$, i.e. $\mathbb{Q}(q,t)\otimes \mathrm{Sym}[X]$. By an abuse of notation we will still use $\mathrm{Sym}[X]$ to represent $\mathbb{Q}(q,t)\otimes \mathrm{Sym}[X]$.

Let X be an alphabet and A be an expression of indeterminates. We define the plethystic evaluation $p_k[A]$ to be the expression obtained by substituting a^k for every indeterminate a that occurres in A. Note that we count the coefficients q,t as indeterminates as well. As we know $\{p_\lambda\}$ form a basis, we can algebraically extend the definition to $f[A]$ for any $f\in \mathrm{Sym}[X]$.

Example 3. We have

- $p_3[X+(t-1)z]=p_3[X]+(t^3-1)z^3$.
- $p_1p_2[\frac{X}{1-q}]=p_1[X]p_2[X](1-q)^{-1}(1-q^2)^{-1}$.
- $p_k[-X]=-p_k[X]$.
- $e_2[X+t]=\frac{1}{2}(p_1^2[X+t]-p_2[X+t])=e_2[X]+tp_1[X]$.
- $e_2[x]=0, \quad h_2[x]=x^2$.

It is worth noting that $p_k[-X]=(-1)^k\omega p_k[X]$ for all $k\geq 0$. Hence we have

$$f[-X]=(-1)^{deg(f)}\omega f[X]$$

for any homogeneous symmetric function f. This gives us the following useful facts:

$$h_k[-X]=(-1)^k e_k[X],\quad e_k[-X]=(-1)^k h_k[X],\quad s_\lambda[-X]=(-1)^n s_{\lambda'}[X],$$

where $\lambda\vdash n$.

Then we define the plethystic exponential function $\Omega[X]$ as

$$\Omega[X] = \sum_{n=0}^{\infty} h_n[X] = exp(\sum_{n=1}^{\infty} \frac{p_n[X]}{n})$$

Then by definition we have all exponential properties, i.e.

$$\Omega[X+Y] = \Omega[X]\Omega[Y], \quad \Omega[-X] = 1/\Omega[X].$$

Then we can rewrite the Cauchy identity in an exponential way as

Proposition 2.3.1. *Let $\{u_\lambda\}$ and $\{v_\mu\}$ be a pair of dual bases for Sym[X] with respect to the Hall inner product. Then we have*

$$\Omega[XY] = \sum_{\lambda} u_\lambda[X] v_\lambda[Y].$$

As a corollary, using the self-dual basis $\{s_\lambda\}$ we obtain

$$f[A] = \langle \Omega[AX], f[X] \rangle$$

for all symmetric $f[X] \in \mathrm{Sym}[X]$.

The following formulae will also be useful. We refer to [Hag08] for a detailed proof.

Proposition 2.3.2. *We have*

$$e_n[A+B] = \sum_{k=0}^{n} e_k[A]e_{n-k}[B], \quad e_n[A-B] = \sum_{k=0}^{n} e_k[A]e_{n-k}[-B],$$

$$h_n[A+B] = \sum_{k=0}^{n} h_k[A]h_{n-k}[B], \quad e_n[A-B] = \sum_{k=0}^{n} h_k[A]h_{n-k}[-B].$$

As a corollary which will be often used in later sections, we have

Corollary 2.3.3.

$$e_n[(t-1)x] = (-1)^{n-1}(t-1)x^n, \quad h_n[(t-1)x] = t^{n-1}(t-1)x^n, \ \textit{for any } n \geq 1.$$

Proof. Write $(t-1)x = tx - x$ and use the expansion formulae. $\square$

2.3.1 Macdonald polynomials

We will now define the q,t-Hall inner product. Let $f, g \in \mathrm{Sym}[X]$. Then we define

$$\langle f, g\rangle_{q,t} = \langle f[X], g[\frac{1-t}{1-q}X]\rangle.$$

According to [Mac15], the Macdonald polynomials $P_\lambda[X;t,q]$ can be uniquely defined by the following two conditions:

- $\{P_\lambda[X;t,q]\}$ form an orthogonal basis.
- $P_\lambda[X;t,q]$ is lower unitriangular with respect to m_λ. In other words we have

$$P_\lambda[X;t,q] = m_\lambda[X] + \sum_{\mu \lhd \lambda} c_{\mu\lambda}(t,q) m_\mu.$$

The integral form $\{J_\lambda[X;t,q]\}$ of the Macdonald polynomials is defined as

$$J_\lambda[X;t,q] = \prod_{c \in D(\lambda)} (1 - t^{\mathrm{arm}_\lambda(c)} q^{1+\mathrm{leg}_\lambda(c)}) P_\lambda[X;t,q],$$

and the transformed Macdonald polynomials $\{\tilde{H}_\lambda[X;t,q]\}$ are defined as

$$\tilde{H}_\lambda[X;t,q] = q^{n(\lambda)} J_\lambda[\frac{X}{1-q^{-1}}; t, q^{-1}].$$

Since the plethysm involved is invertible, the set $\{\tilde{H}_\lambda[X;t,q]\}$ will form a basis for $\mathrm{Sym}[X]$.

We have the following characterization of $\{\tilde{H}_\lambda[X;t,q]\}$.

Proposition 2.3.4. [Hai99] *The transformed Macdonald polynomials $\{\tilde{H}_\lambda[X;t,q]\}$ uniquely characterized by the following conditions:*

- $$\tilde{H}_\lambda[(1-t)X;t,q] \in \mathbb{Q}(t,q)\{s_\lambda : \lambda \geq \mu\},$$

- $$\tilde{H}_\lambda[(1-q)X;t,q] \in \mathbb{Q}(t,q)\{s_\lambda : \lambda \geq \mu'\},$$

- $$\langle \tilde{H}_\lambda, s_{(n)} \rangle = 1.$$

2.3.2 Remarkable operators on Sym$[X]$

We then define some remarkable operators on Sym$[X]$ and will discuss them in details in Chapter 4 and Chapter 5. These operators are studied in details in [BGSLX16b].

The nabla operator ∇ is an operator on Sym$[X]$ which acts diagonally on $\tilde{H}_\lambda[X;t,q]$ as

$$\nabla \tilde{H}_\lambda = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda.$$

The D_k operators are defined to be a family of operators on Sym$[X]$ satisfying

$$D_k F[X] = F[X + \frac{(1-t)(1-q)}{z}] \Omega[-zX]|_{z^k},$$

where $|_{z^k}$ means taking the coefficient of z_r. It has been proven in [BGSLX16b] that D_0 is also an diagonal operator with respect to the basis $\tilde{H}_\lambda[X;t,q]$. More specifically we have

$$D_0 \tilde{H}_\lambda = -D_\lambda(t,q) \tilde{H}_\lambda,$$

where

$$D_\lambda(t,q) = -1 + (1-q)(1-t) \sum_{c \in \lambda} q^{\mathrm{coleg}_\lambda(c)} t^{\mathrm{coarm}_\lambda(c)}.$$

Then we define delta operators Δ_f and Δ'_f as

$$\Delta_f \tilde{H}_\lambda = f[B_\lambda(q,t)] \tilde{H}_\lambda,$$

$$\Delta'_f \tilde{H}_\lambda = f[B_\lambda(q,t) - 1]\tilde{H}_\lambda.$$

The following results in [GHT99] are used for the proof of the shuffle conjecture. We will quote them for later use.

Proposition 2.3.5. *Let e_1 be the left multiplication operator by the elementary symmetric function $e_1[X]$. Then we have the following relations:*

$$\begin{aligned} D_k e_1 - e_1 D_k &= (1-q)(1-t)D_{k+1}, \text{ for all } k \geq 0, \\ D_0 e_1 - e_1 D_0 &= -(1-q)(1-t)\nabla e_1 \nabla^{-1}. \end{aligned} \tag{2.3.2.1}$$

Theorem 2.3.6. *Let P be a homogeneous symmetric polynomial of degree $k \geq 1$. Then we may write*

$$P = D_1 A + e_1 B,$$

where A and B are homogeneous symmetric polynomials of degree $k-1$. Therefore the operators D_1 and e_1 acting on 1 can generate the whole Sym[X].

At the end of this part, we define the C_r operators as

$$C_r F[X] = -t^{1-r} F[X + \frac{(t^{-1}-1)}{z}]Exp[zX]|_{z^r}.$$

Then we define $C_\alpha = C_{\alpha_1} ... C_{\alpha_l}$ for any composition α of length l.

As proven in [HMZ12], we have

Proposition 2.3.7.

$$\sum_{|\alpha|=n} C_\alpha[X;q] = (-1)^n e_n[X] \tag{2.3.2.2}$$

The C_α operators will be used to define the compositional shuffle conjecture.

2.3.3 Diagonal coinvariants

Let the symmetric group S_n act on $\mathbb{C}[x_1, y_1, x_2, y_2, ..., x_n, y_n]$ diagonally. More explicitly, the permutations will simultaneously permute the indices of x and y. R_n is defined to be the ring of coinvariants for the diagonal action. In other words,

$$R_n = \mathbb{C}[x_1, y_1, x_2, y_2, ..., x_n, y_n]/I,$$

where $I = ((x_1, y_1, ..., x_n, y_n) \bigcap \mathbb{C}[x_1, y_1, ..., x_n, y_n]^{S_n})$ is the ideal generated by all S_n-invariant polynomials without constant term. Note that the S_n-action preserves the double grading

$$R_n = \bigoplus_{r,s} (R_n)_{r,s}$$

with respect to degrees in x and y.

Now we define the Frobenius characteristic F to be the linear map from the space of S_n characters to symmetric functions by sending the irreducible character χ^λ to the Schur function $s_\lambda[X]$. The Frobenius series of R_n is defined as

$$\mathcal{F}_{R_n}[X; q, t] = \sum_{r,s} q^r t^s F(char(R_n)_{r,s}).$$

The following theorem by Haiman connects the Frobenius series of R_n and the nabla operator.

Theorem 2.3.8. [Hai01] *We have*

$$\mathcal{F}_{R_n}[X; q, t] = \nabla e_n[X].$$

One may expect a more explicit formula for $e_n[X]$. We will discuss it in details in Chapter 4.

2.3.4 Ring of almost symmetric functions

The positive k-symmetric space $\mathcal{P}(k)^+$ is defined as

$$\mathcal{P}(k)^+ = \{f(x_1, x_2, ...) \in \widehat{R^{(r)}} \mid f \text{ is symmetric in } x_{k+1}, x_{k+2}, ..., \text{ for some } r \geq 0\}.$$

We observe that $\mathcal{P}(k)^+ \subset \mathcal{P}(k+1)^+$. Then we may define the almost symmetric polynomial ring $\mathcal{P}_{\text{as}}^+$ to be the union of all $\mathcal{P}(k)^+$,i.e.

$$\mathcal{P}_{\text{as}}^+ = \bigcup_{k \geq 0} \mathcal{P}(k)^+.$$

Similarly we define the counterpart with respect to the alphabet $X^{-1} = x_1^{-1} + x_2^{-1} + ...$ We define $\mathcal{P}(k)^-$ and the ring of almost symmetric negative polynomials $\mathcal{P}_{\text{as}}^-$ analogously.

3.0 Double affine Hecke algebras

In this chapter we will define Double affine Hecke algebras and discuss the corresponding representation theory. Then we will introduce a stabilization procedure for the polynomial representations. The stabilization will be used to establish a connection between double affine Hecke algebras and the double Dyck path algebra.

3.1 Definitions

3.1.1 Affine Hecke algebras

Definition 3. The affine Hecke algebra AHA_k of type GL_k is a $\mathbb{Q}(t)$-algebra generated by

$$T_1, ..., T_{k-1}, X_1^{\pm 1}, ..., X_k^{\pm 1}$$

satisfying the following relations:

$$\begin{gathered} T_i T_j = T_j T_i, \text{ for } 1 \leq i < j \leq k-1 \text{ with } 1 < |i-j| < k-1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \text{ for } i = 1, ..., k-2, \\ (T_i - 1)(T_i + t) = 0, \text{ for } i = 1, ..., k-1, \end{gathered} \tag{3.1.1.1a}$$

$$\begin{gathered} t T_i^{-1} X_i T_i^{-1} = X_{i+1}, \text{ for } 1 \leq i \leq k-1 \\ T_i X_j = X_j T_i, \text{ for } 1 \leq i \leq k-1, 1 \leq j \leq k \text{ with } j \neq i, i+1, \\ X_i X_j = X_j X_i \text{ for } 1 \leq i, j \leq k. \end{gathered} \tag{3.1.1.1b}$$

The definition above is based on the Bernstein presentation of the affine Hecke algebra AHA_k. The following presentation of AHA_k will also be used.

Proposition 3.1.1. *The affine Hecke algebra AHA_k is generated by*

$$\tilde{T}_0, T_1, ..., T_{k-1}, \tilde{\omega}$$

with relations:

$$\begin{gathered} T_iT_j = T_jT_i,\ \textit{for } 1 \leq i < j \leq k-1 \textit{ with } 1 < |i-j| < k-1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ \textit{for } i = 1, ..., k-2, \\ (T_i - 1)(T_i + t) = 0,\ \textit{for } i = 1, ..., k-1, \\ T_i\tilde{T}_0 = \tilde{T}_0T_i,\ \textit{for } 2 \leq i \leq k-1, \\ \tilde{T}_0T_1\tilde{T}_0 = T_1\tilde{T}_0T_1, \\ (\tilde{T}_0 - 1)(\tilde{T}_0 + t) = 0, \end{gathered} \tag{3.1.1.2a}$$

$$\tilde{\omega}T_i\tilde{\omega}^{-1} = T_{i-1} \textit{ for } 1 \leq i \leq k-1, \quad \tilde{\omega}\tilde{T}_0\tilde{\omega}^{-1} = T_{k-1}. \tag{3.1.1.2b}$$

Remark. $\tilde{T}_0, \tilde{\omega}$ and $X_1, ..., X_k$ are related by

$$\tilde{\omega} = T_{k-1}...T_1X_1^{-1} = t^{k-1}X_k^{-1}T_{k-1}^{-1}...T_1^{-1},$$

$$\tilde{T}_0 = \tilde{\omega}^{-1}T_{k-1}\tilde{\omega} = \tilde{\omega}T_1\tilde{\omega}^{-1} = t^{k-1}X_1X_k^{-1}T_1^{-1}...T_{k-1}^{-1}...T_1^{-1}.$$

Notation. We denote by AHA_k^+ the subalgebra of AHA_k generated by T_i, $i \leq k-1$, and X_i, $i \leq k$, or equivalently, by T_i, $i \leq k-1$, and $\tilde{\omega}^{-1}$. Similarly, we denote by AHA_k^- the subalgebra of AHA_k generated by T_i, $i \leq k-1$, and X_i^{-1}, $i \leq k$, , or equivalently, by T_i, $i \leq k-1$, and $\tilde{\omega}$.

3.1.2 Double affine Hecke algebras

Now we will introduce the double affine Hecke algebras (DAHA).

Definition 4. The double affine Hecke algebra (DAHA) $\mathcal{H}_k$ of type GL_k is a $\mathbb{Q}(t,q)$-algebra generated by

$$T_1, ..., T_{k-1}, X_1^{\pm 1}, ..., X_k^{\pm 1}, Y_1^{\pm 1}, ..., Y_k^{\pm 1}$$

satisfying (3.1.1.1a), (3.1.1.1b) the following relations:

$$\begin{gathered} t^{-1}T_iY_iT_i = Y_{i+1}, \text{ for } 1 \leq i \leq k-1 \\ T_iY_j = Y_jT_i, \text{ for } 1 \leq i \leq k-1, 1 \leq j \leq k \text{ with } j \neq i, i+1, \\ Y_iY_j = Y_jY_i \text{ for } 1 \leq i, j \leq k. \end{gathered} \tag{3.1.2.1a}$$

$$\begin{gathered} Y_2X_1^{-1}Y_2^{-1}X_1 = t^{-1}T_1^2, \\ X_k...X_1Y_i = q^{-1}Y_iX_k...X_1. \end{gathered} \tag{3.1.2.1b}$$

Similarly, we will introduce a different presentation of $\mathcal{H}_k$.

Proposition 3.1.2. *The double affine Hecke algebra (DAHA) $\mathcal{H}_k$ is generated by*

$$T_0, T_1, ..., T_{k-1}, X_1^{\pm 1}, ..., X_k^{\pm 1}, \omega$$

satisfying (3.1.1.1a), (3.1.1.1b) and the following relations:

$$\begin{gathered} T_iT_0 = T_0T_i, \text{ for } 2 \leq i \leq k-1, \\ T_0T_1T_0 = T_1T_0T_1, \\ (T_0 - 1)(T_0 + t) = 0, \end{gathered} \tag{3.1.2.2a}$$

$$\begin{gathered} \omega T_i \omega^{-1} = T_{i-1} \text{ for } 2 \leq i \leq k-1, \\ \omega T_1 \omega^{-1} = T_0 \quad \omega T_0 \omega^{-1} = T_{k-1}, \end{gathered} \tag{3.1.2.2b}$$

$$\omega X_{i+1} \omega^{-1} = X_i \text{ for } 1 \leq i \leq k-1, \quad \omega X_1 \omega^{-1} = q^{-1} X_k. \tag{3.1.2.2c}$$

Remark. $Y_1, ..., Y_k$ and ω are related by

$$Y_i = t^{1-i} T_{i-1} ... T_1 \omega^{-1} T_{k-1}^{-1} ... T_i^{-1}.$$

Double affine Hecke algebras were first introduced by Cherednik[Che95]. We also refer to [SV11],[IS20] for alternative definitions.

Notation. For the sake of stabilization, we will also define the positive and negative subalgebras of $\mathcal{H}_k$. Let $\mathcal{H}_k^-$ be the subalgebra of $\mathcal{H}_k$ generated by

$$T_1, ..., T_{k-1}, X_1^{-1}, ..., X_k^{-1}, Y_1^{-1}, ..., Y_k^{-1},$$

and $\mathcal{H}_k^+$ the subalgebra of $\mathcal{H}_k$ generated by

$$T_1, ..., T_{k-1}, X_1, ..., X_k, Y_1, ..., Y_k.$$

We will then define the classical Laurent polynomial representation of the algebra $\mathcal{H}_k$.

Denote

$$\mathcal{P}_k^- = \mathbb{Q}(t,q)[x_1^{-1}, \dots, x_k^{-1}], \quad \mathcal{P}_k^+ = \mathbb{Q}(t,q)[x_1, \dots, x_k]$$

and

$$\mathcal{P}_k = \mathbb{Q}(t,q)[x_1^{\pm 1}, \dots, x_k^{\pm 1}].$$

The representation below is called the standard representation of $\mathcal{H}_k$ [Che95, Theorem 2.3]

Proposition 3.1.3 ([Che95], Theorem 2.3)**.** *The Laurent polynomial representation of the algebra $\mathcal{H}_k$ on $\mathcal{P}_k$ is defined as follows:*

$$
\begin{aligned}
T_i f(x_1, ..., x_k) &= s_i f(x_1, ..., x_k) + (1-t)x_i \frac{1-s_i}{x_i - x_{i+1}} f(x_1, ..., x_k),\\
\tilde{\omega} f(x_1, ..., x_k) &= T_{k-1} ... T_1 x_1^{-1} f(x_1, ..., x_k),\\
\omega f(x_1, ..., x_k) &= f(q^{-1} x_k, x_1, ..., x_{k-1}),
\end{aligned}
\tag{3.1.2.3}
$$

for any $f(x_1, ..., x_k) \in \mathcal{P}_k$. In particular, X_i will act on $\mathcal{P}_k$ as left multiplication by x_i.

Remark. It is easy to see $\mathcal{P}_k^-$ is an invariant subspace of $\mathcal{P}_k$ under the AHA_k^- and $\mathcal{H}_k^-$ actions, while $\mathcal{P}_k^+$ is an invariant subspace under the AHA_k^+ and $\mathcal{H}_k^+$ actions. Note that the actions mentioned are all faithful.

3.2 Stable limits of DAHA

3.2.1 Stable limit DAHA

Definition 5. The algebra $\mathcal{H}_\infty^+$ is defined to be the $\mathbb{Q}(t,q)$-algebra generated by $T_1, T_2, ..., X_1, X_2, ..., Y_1, Y_2, ...$ with the following relations:

$$
\begin{gathered}
T_i T_j = T_j T_i, \text{ for } |i-j| > 1,\\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad (T_i - 1)(T_i + t) = 0,
\end{gathered}
\tag{3.2.1.1a}
$$

$$
\begin{gathered}
t T_i^{-1} X_i T_i^{-1} = X_{i+1}, \qquad X_i X_j = X_j X_i,\\
T_i X_j = X_j T_i, \text{ for } j \neq i, i+1,
\end{gathered}
\tag{3.2.1.1b}
$$

$$\begin{gathered} t^{-1}T_iY_iT_i = Y_{i+1}, \qquad Y_iY_j = Y_jY_i, \\ T_iY_j = Y_jT_i, \text{ for } j \neq i, i+1, \end{gathered} \tag{3.2.1.1c}$$

$$Y_1T_1X_1 = X_2Y_1T_1. \tag{3.2.1.1d}$$

We will call the positive limit DAHA.

Note that, as opposed to the corresponding elements of $\mathcal{H}_k$, the elements X_i, Y_i are not invertible in $\mathcal{H}_\infty^+$. Similarly, we define the negative limit DAHA.

Definition 6. Let $\mathcal{H}_\infty^-$ be the $\mathbb{Q}(t,q)$-algebra generated by the elements T_i,X_i^{-1}, and Y_i^{-1}, $i \geq 1$, satisfying the following relations:

$$\begin{gathered} T_iT_j = T_jT_i, \text{ for } |i-j| > 1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad (T_i - 1)(T_i + t) = 0, \end{gathered} \tag{3.2.1.2a}$$

$$\begin{gathered} t^{-1}T_iX_i^{-1}T_i = X_{i+1}^{-1}, \quad i \geq 1 \\ T_iX_j^{-1} = X_j^{-1}T_i, \quad j \neq i, i+1, , \\ X_i^{-1}X_j^{-1} = X_j^{-1}X_i^{-1}, \quad i, j \geq 1, \end{gathered} \tag{3.2.1.2b}$$

$$\begin{gathered} tT_i^{-1}Y_i^{-1}T_i^{-1} = Y_{i+1}^{-1}, \quad i \geq 1 \\ T_iY_j^{-1} = Y_j^{-1}T_i, \quad j \neq i, i+1, \\ Y_i^{-1}Y_j^{-1} = Y_j^{-1}Y_i^{-1}, \quad i, j \geq 1, \end{gathered} \tag{3.2.1.2c}$$

$$X_1^{-1}T_1^{-1}Y_1^{-1} = T_1^{-1}Y_1^{-1}X_2^{-1}. \tag{3.2.1.2d}$$

We will call the negative limit DAHA.

Remark. There exists an anti-isomorphism of $\mathbb{Q}(t,q)$-algebras defined by mapping T_i, X_i, Y_i to T_i, Y_i^{-1}, X_i^{-1} respectively. Therefore, we may regard the two algebras as capturing the same structure and we will call both of them the stable limit DAHA.

3.2.2 Projective system for negative part

Let

$$\pi_k : \mathcal{P}_k^- \to \mathcal{P}_{k-1}^-$$

be the ring morphism that maps x_k^{-1} to 0 and acts as identity on all the other generators. The rings $\mathcal{P}_k^-$, $k \geq 1$ form an inverse system. We will use the notation $\mathcal{P}_\infty^-$ for the ring $\varprojlim \mathcal{P}_k^-$. We denote by $\Pi_k : \varprojlim \mathcal{P}_k^- \to \mathcal{P}_k^-$ the canonical morphism.

Let,

$$\iota_k : \mathcal{P}_{k-1}^- \to \mathcal{P}_k^-$$

the canonical inclusion ring morphism. The inductive limit $\varinjlim \mathcal{P}_k^-$ is canonically isomorphic with the ring $\mathbb{Q}(t,q)[x_1^{-1}, x_2^{-1}, \dots]$ of polynomials in infinitely many variables. As with the projective limit, the rings $\varinjlim_{k \geq n} \mathcal{P}_k^-$ and $\varinjlim \mathcal{P}_k^-$ are canonically isomorphic. We denote by $I_k : \mathcal{P}_k^- \to \varinjlim \mathcal{P}_k^-$ the canonical morphism.

The following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{P}_{k-1}^- & \longrightarrow & \mathcal{P}_{k-1}^- \\
{\scriptstyle \iota_k}\downarrow & & \uparrow{\scriptstyle \pi_k} \\
\mathcal{P}_k^- & \longrightarrow & \mathcal{P}_k^-
\end{array}$$

where each horizontal map represents the identity map. For a fixed $n \geq 1$, denote by $\iota_{n,k} : \mathcal{P}_n^- \to \mathcal{P}_k^-$, $n \leq k$, the canonical inclusion. The sequence of maps $\iota_{n,k} : \mathcal{P}_n^- \to \mathcal{P}_k^-$, $k \geq n$ is compatible with the structure maps π_k. Therefore, they induce a morphism

$$\mathcal{P}_n^- \to \varinjlim_{k \geq n} \mathcal{P}_k^- \cong \varinjlim \mathcal{P}_k^- .$$

Furthermore, these maps are compatible with the structure maps ι_n and therefore induce a morphism

$$J : \varinjlim \mathcal{P}_k^- \to \varprojlim \mathcal{P}_k^- .$$

By construction, $\Pi_k J I_k : \mathcal{P}_k^- \to \mathcal{P}_k^-$ is the identity function. We denote $J_k = JI_k : \mathcal{P}_k^- \to \mathcal{P}_\infty^-$.

For any $n \geq 1$, a sequence of operators $A_k : \mathcal{P}_k^- \to \mathcal{P}_k^-$, $k \geq n$, compatible with the inverse system induces a (limit) operator $\mathcal{A} : \mathcal{P}_\infty^- \to \mathcal{P}_\infty^-$. We have $A_k = \Pi_k \mathcal{A} J_k$.

An analogue structure emerges from the polynomial rings $\mathcal{P}_k^+$. We adopt the corresponding notation for all the relevant spaces and maps.

We will now recall some results from [Kno07] which allow for the stabilization of the negative DAHA.

The map π_k is partially compatible with the actions of AHA_k^- and AHA_{k-1}^-. More precisely we have

Proposition 3.2.1. [Kno07, Theorem 9.1] *Let $\pi_k : \mathcal{P}_k^- \to \mathcal{P}_{k-1}^-$. Then we have*

$$\begin{aligned}
\pi_k T_i &= T_i \pi_k, \quad 1 \leq i \leq k-2, \\
\pi_k T_{k-1} &= \pi_k s_{k-1}, \\
\pi_k \widetilde{\omega}_k &= 0, \\
\pi_k T_{k-1} \widetilde{\omega}_k &= \widetilde{\omega}_{k-1} \pi_k, \\
\pi_k X_i^{-1} &= X_i^{-1} \pi_k, \quad 1 \leq i \leq k-1, \\
\pi_k X_k^{-1} &= 0.
\end{aligned} \tag{3.2.2.1}$$

The compatibility with the actions of $\mathcal{H}_k^-$ and $\mathcal{H}_{k-1}^-$ can also be investigated, but the verification is more delicate. More precisely, we have the following result:

Proposition 3.2.2. [Kno07, Proposition 9.11] *For any $1 \leq i \leq k-1$, we have*

$$\pi_k Y_i = Y_i \pi_k. \tag{3.2.2.2}$$

Furthermore, the operator Y_i^{-1} stabilizes both $\mathcal{P}_k^-$ and $\mathcal{P}_{k-1}^-$ and

$$\pi_k Y_i^{-1} = Y_i^{-1} \pi_k. \tag{3.2.2.3}$$

3.2.3 Negative stable limit DAHA action

For example, for any $n \geq 1$, the sequence of operators $(A_k)_{k\geq 1}$ defined by

$$A_k := Y_n^{(k)}, \quad k \geq n,$$

induces the limit operator $Y_n : \varprojlim_{k\geq n} \mathcal{P}_k^- \to \varprojlim_{k\geq n} \mathcal{P}_k^-$. Since $\varprojlim_{k\geq n} \mathcal{P}_k$ and $\mathcal{P}_\infty^-$ are canonically isomorphic, we obtain an operator $Y_n : \mathcal{P}^- \to \mathcal{P}^-$.

Similarly, we obtain limit operators T_i, X_i^{-1}, and Y_i^{-1}, $i \geq 1$. The following result immediately follows.

Theorem 3.2.3. *The limit operators T_i, X_i^{-1}, and Y_i^{-1}, $i \geq 1$, define a $\mathcal{H}^-$ action on $\mathcal{P}_\infty^-$.*

Proof. All relations are satisfied because they are satisfied by the corresponding operators acting on each $\mathcal{P}_k^-$. □

It is important to remark that the operators Y_i^{-1} are invertible (with inverse Y_i).

3.2.4 $\mathcal{P}_{\mathrm{as}}^-$ subrepresentation

As it was pointed out in [Kno07, §10] the subrepresentation on the subspace $\mathcal{P}_{\mathrm{as}}^-$ is more canonical from a certain point of view. The reason for considering $\mathcal{P}_{\mathrm{as}}^-$ is the following. Each $\mathcal{P}_k^-$ is a parabolic module for the affine Hecke algebra AHA_k^- and has a standard basis (in the sense of Kazhdan-Lusztig theory) indexed by compositions with at most k parts. The sequences consisting of the standard basis elements indexed by the same composition (in all $\mathcal{P}_k^-$, $k \geq n$, for some n) give elements of $\mathcal{P}_\infty^-$ (see [Kno07, §9]) which are expected to play the role of a standard basis for the limit representation. However, these limits of standard basis elements do not span $\mathcal{P}_\infty^-$, but rather the smaller space $\mathcal{P}_{\text{as}}^-$.

From the previous section we know $\mathcal{P}(k)^-$ can be alternatively defined as

$$\mathcal{P}(k)^- = \{F \in \mathcal{P}_\infty^- \mid \forall i > k,\ T_i F = F\}.$$

We still define $\mathcal{P}_{\text{as}}^-$ as the inductive limit of $\mathcal{P}(k)^-$ with respect to the inclusion map. Then we have

Theorem 3.2.4. *The almost symmetric module $\mathcal{P}_{\text{as}}^-$ is a $\mathcal{H}^-$ submodule of $\mathcal{P}_\infty^-$.*

We call this representation the standard representation of $\mathcal{H}^-$. We expect this representation to be faithful.

As explained in [Kno07], a sequence on non-symmetric Macdonald polynomials indexed by the same composition gives rise to an element of $\mathcal{P}_{\text{as}}^-$, and such elements are common eigenfunctions for the action the operators Y_i^{-1}. These limit non-symmetric Macdonald polynomials do not span $\mathcal{P}_{\text{as}}^-$ and therefore the spectral theory of the operators Y_i^{-1} acting on $\mathcal{P}_{\text{as}}^-$ is not yet fully understood.

3.3 Modified DAHA

As we explained, the negative part of the DAHA admits a good stabilization. On the other hand, $Y_i^{\pm 1}$ fail to act consistently. Therefore we do not have a similar result for P_∞^+. The following example demonstrates the inconsistency of the action.

3.3.1 Motivation

Example 4. Consider the action of $Y_1^{(2)}$ and $Y_1^{(3)}$ on x_2. For $x_2 \in P_2^+$, we have

$$\begin{aligned} Y_1^{(2)} x_2 &= \omega^{-1} T_1^{-1} x_2 \\ &= x_2 + q(1-t^{-1})x_1, \end{aligned}$$

while for $x_2 \in P_3^+$, we have

$$\begin{aligned} Y_1^{(3)} x_2 &= \omega^{-1} T_2^{-1} T_1^{-1} x_2 \\ &= \omega^{-1} T_2^{-1}(x_1 + (1-t^{-1})x_2) \\ &= x_2 + qt^{-1}(1-t^{-1})x_1. \end{aligned}$$

Clearly we see $Y_1^{(2)}$ and $Y_1^{(3)}$ have different actions on x_2.

$Y_i^{\pm 1}$ will not act on $\mathcal{P}_{\text{as}}^+$ consistently. However, the rescaled compositions of operators $\{t^k Y_i X_i\} \subset \mathcal{H}_k^+$ have compatible actions on P_k^+. More specifically, we have

Proposition 3.3.1. $\pi_k t^k Y_i^{(k)} X_i^{(k)} = t^{k-1} Y_i^{(k-1)} X_i^{(k-1)} \pi_k. \qquad (i \leq k-1)$

Proof. First note that we have

$$\pi_k \omega_k^{-1} T_{k-1} = \omega_{k-1}^{-1} \pi_k,$$

which can be verified by direct computation. Hence we have

$$\begin{aligned}
\pi_k Y_i^{(k)} X_i &= t^{1-i} T_{i-1}...T_1 \pi_k (\omega_k^{-1} T_{k-1}^{-1}...T_i^{-1} X_i) \\
&= t^{1-k} T_{i-1}...T_1 \pi_k (\omega_k^{-1} X_k T_{k-1}...T_i) \\
&= t^{1-k} T_{i-1}...T_1 \omega_{k-1}^{-1} \pi_k (T_{k-1}^{-1} X_k T_{k-1}...T_i) \\
&= t^{-k} T_{i-1}...T_1 \omega_{k-1}^{-1} \pi_k (T_{k-1} X_k T_{k-1}...T_i) \\
&= t^{1-k} T_{i-1}...T_1 \omega_{k-1}^{-1} \pi_k (X_{k-1} T_{k-2}...T_i) \\
&= t^{-i} T_{i-1}...T_1 \omega_{k-1}^{-1} T_{k-2}^{-1}...T_i^{-1} X_i \pi_k \\
&= t^{-1} Y_i^{(k-1)} X_i \pi_k
\end{aligned}$$

Therefore we have $\pi_k t^k Y_i^{(k)} X_i = t^{k-1} Y_i^{(k-1)} X_i \pi_k$. □

This motivates us to consider a modification of Y_i operators to address the consistency problem. Therefore in order to correctly stabilize the action of the positive part, we modify the original algebra as explained in the following section

3.3.2 Deformed DAHA $\tilde{\mathcal{H}}_k^+$

Definition 7. The algebra $\tilde{\mathcal{H}}_k^+ (k > 1)$ is defined to be the $\mathbb{Q}(t,q)$-algebra generated by $T_1, ..., T_{k-1}$, $X_1, ..., X_k, \bar{\omega}$ with the following:

1. $T_1, ..., T_{k-1}, X_1, ..., X_k$ that satisfy the relations (3.1.1.1a), (3.1.1.1b).
2.
$$\begin{aligned}
\bar{\omega} T_i &= T_{i+1} \bar{\omega}, \text{ for } i = 1, ..., k-2, \\
\bar{\omega} X_i &= X_{i+1} \bar{\omega}, \text{ for } i = 1, ..., k-1,
\end{aligned} \tag{3.3.2.1}$$

3. Denote

$$\gamma = \bar{\omega}^2 T_{k-1} - T_1 \bar{\omega}^2.$$

Then

$$\begin{gathered} \gamma T_{k-1} = -t\gamma, \quad T_1\gamma = \gamma, \\ \gamma\bar{\omega}^{k-2}\gamma = \gamma\bar{\omega}^{k-1}\gamma = \gamma\bar{\omega}^k = 0. \end{gathered} \tag{3.3.2.2}$$

Remark. $\tilde{\mathcal{H}}_k^+$ can be seen as a deformation of $\mathcal{H}_k^+$ with the specification $\gamma = 0$. However, the generating relations of $\tilde{\mathcal{H}}_k^+$ do not yield commutativity for the analogues of the Cherednik operators.

Proposition 3.3.2. *Define* $\tilde{Y}_1, ..., \tilde{Y}_k$ *as*

$$\tilde{Y}_1 = t^k \bar{\omega} T_{k-1}^{-1} ... T_1^{-1}, \qquad \tilde{Y}_{i+1} = t^{-1} T_i \tilde{Y}_i T_i.$$

Then they do not form a commutative family of elements. In fact, we have

$$[\tilde{Y}_1, \tilde{Y}_2] = t^{2k-1}\gamma T_{k-1}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_2^{-1} = t^{2k-1}\gamma T_{k-2}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_1^{-1}.$$

Hence γ *can be seen as a measurement of the non-commutativity of* $\tilde{Y}_i$ *operators.*

Then we can define the following action of $\tilde{\mathcal{H}}_k^+$ on P_k^+.

Theorem 3.3.3. *The algebra* $\tilde{\mathcal{H}}_k^+$ *has a representation on* $\mathbb{Q}(t,q)[x_1, ..., x_k]$ *defined as the following:*

$$\begin{aligned} X_i &\mapsto x_i, \\ T_i &\mapsto s_i + (1-t)x_i \frac{1-s_i}{x_i - x_{i+1}}, \\ \tilde{Y}_i &\mapsto t^{1-i+k} T_{i-1}...T_1 \bar{\omega} T_{k-1}^{-1}...T_i^{-1}, \end{aligned} \tag{3.3.2.3}$$

where

$$\bar{\omega}.f(x_1, ..., x_k) = p_1 \omega^{-1}.f(x_1, ..., x_k) \qquad \text{for } f(x_1, ..., x_k) \in P_k^+,$$

and

$$p_1 : \mathbb{Q}(t,q)[x_1,...,x_k] \to x_1\mathbb{Q}(t,q)[x_1,...,x_k],$$

$$x_1 f(x_1,...,x_k) \mapsto x_1 f(x_1,...,x_k),\ \ f(x_2,...,x_k) \mapsto 0$$

is the projection map onto the subspace $x_1\mathbb{Q}(t,q)[x_1,...,x_k]$.

Proof. Only the consistency of the action of $\bar{\omega}$ needs to be verified.
First, from (3.1.2.2b) we have

$$\begin{aligned}
\bar{\omega}T_i.f &= p_1\omega^{-1}T_i.f \\
&= p_1(T_{i+1}\omega^{-1}.f) \\
&= T_{i+1}p_1\omega^{-1}.f \\
&= T_{i+1}\bar{\omega}.f,
\end{aligned}$$

for $f \in P_k^+$ and $1 \le i \le k-2$. Similarly from (3.1.2.2c) we have

$$\bar{\omega}X_i.f = X_{i+1}\bar{\omega}.f,$$

for $f \in P_k^+$ and $1 \le i \le k-1$.

It remains to verify the relations involving γ. By linearity it suffices to verify monomials. We present the explicit expression of the action of γ on monomials.

$$\gamma(x_1^{t_1}...x_{k-1}^{t_{k-1}}x_k^{t_k}) = \begin{cases} 0, & \text{if } t_{k-1} \neq 0, t_k \neq 0 \\ \\ (1-t)q^{t_{k-1}}(x_1^{t_{k-1}-1}x_2 + x_1^{t_{k-1}-2}x_2^2 + ... \\ +x_1x_2^{t_{k-1}-1})(x_3^{t_1}x_4^{t_2}...x_k^{t_{k-2}}), & \text{if } t_{k-1} \neq 0, t_k = 0 \\ \\ (t-1)q^{t_k}(x_1^{t_k-1}x_2 + x_1^{t_k-2}x_2^2 + ... \\ +x_1x_2^{t_k-1})(x_3^{t_1}x_4^{t_2}...x_k^{t_{k-2}}), & \text{if } t_{k-1} = 0, t_k \neq 0 \end{cases}$$

Note that the expression immediately implies $\gamma T_{k-1} = -t\gamma$ and $T_1\gamma = \gamma$. The rest may be also checked directly. □

Remark. The action of $\tilde{Y}_i X_i$ is the same as that of $t^k Y_i X_i$. In other words, the action of $t^{-k}\tilde{Y}_i$ on $x_i\mathbb{Q}(t,q)[x_1,...,x_k]$ is identical to that of Y_i on the same space. Therefore $\tilde{Y}_i$'s can be viewed as a modification of the Cherednik operators.

3.4 Stabilization of $\tilde{\mathcal{H}}_k^+$

3.4.1 Direct stabilization

The stability of this family of operators can be shown from the following theorem.

Theorem 3.4.1. *Let*

$$\iota_k : P_{k-1}^+ \to P_k^+$$

be the inclusion map. Then the following diagram is commutative:

$$\begin{array}{ccc} P_{k-1}^{+} & \xrightarrow{\tilde{Y}_i^{(k-1)}} & P_{k-1}^{+} \\ {\scriptstyle \iota_k}\big\downarrow & & \big\uparrow{\scriptstyle \pi_k} \\ P_k^{+} & \xrightarrow{\tilde{Y}_i^{(k)}} & P_k^{+} \end{array}$$

Equivalently, we have

$$\pi_k \tilde{Y}_i^{(k)} \iota_k.f = \tilde{Y}_i^{(k-1)}.f$$

for all $f \in P_{k-1}^{+}$.

Proof. Note that $p_1\omega_k^{-1} = \omega_k^{-1}p_k$. Hence we have

$$\begin{aligned}\pi_k \tilde{Y}_i^{(k)} \iota_k.f &= \pi_k t^{1-i+k} T_{i-1}...T_1 p_1 \omega_k^{-1} T_{k-1}^{-1}...T_i^{-1}.f \\ &= t^{k-i} T_{i-1}...T_1 p_1 \pi_k \omega_k^{-1} T_{k-1} T_{k-2}^{-1}...T_i^{-1}.f + \\ &\quad \pi_k t^{k-i}(t-1) T_{i-1}...T_1 p_1 \omega_k^{-1} T_{k-2}^{-1}...T_i^{-1}.f \\ &= t^{k-i} T_{i-1}...T_1 p_1 \omega_{k-1}^{-1} \pi_k T_{k-2}^{-1}...T_i^{-1}.f + \\ &\quad \pi_k t^{k-i}(t-1) T_{i-1}...T_1 \omega_k^{-1} p_k (T_{k-2}^{-1}...T_i^{-1}.f) \\ &= \tilde{Y}_i^{(k-1)} \pi_k.f + 0\end{aligned}$$

for all $f \in P_{k-1}^{+}$. □

The theorem implies the existence of limit operators $\tilde{Y}_i$.

Corollary 3.4.2. *For each* $i > 0$*, there exist an operator*

$$\tilde{Y}_i : \mathbb{Q}(t,q)[x_1, x_2, ...] \longrightarrow P_\infty^{+}$$

such that

$$\Pi_k \tilde{Y}_i I_k = \tilde{Y}_i^{(k)}, \qquad \forall k \geq i,$$

where $\Pi_k : P_\infty^{+} \to P_k^{+}$ *and* $I_k : P_k^{+} \to \mathbb{Q}(t,q)[x_1, x_2, ...]$ *are the projection map and the inclusion map respectively.*

Proof. Note that $\{P_k^+\}$ under the inclusion maps forms a direct system with the direct limit $\mathbb{Q}(t,q)[x_1, x_2, ...]$. $\square$

The domain of $\widetilde{Y}_i^{(\infty)}$ is smaller than the expected $\mathcal{P}_\infty^+$. The obstruction for the extension of $\widetilde{Y}_i^{(\infty)}$ to $\mathcal{P}_\infty^+$ can be precisely identified.

3.4.2 Extension of the stabilization

We define the $\mathbb{Q}(t,q)$-linear operators $W_i^{(k)} : \mathcal{P}_k^+ \to \mathcal{P}_k^+$, $i \geq 1$, as follows. Let $s \geq 0$, and let $i_1 < i_2 < \cdots < i_s \leq k$ and $t_1, \ldots, t_s$ be positive integers. We set

$$W_1^{(k)}(x_{i_1}^{t_1} \ldots x_{i_s}^{t_s}) = \begin{cases} 0, & \text{if } i_1 = 1 \text{ or } s = 0, \\ \\ t^{t_s}(1-t^{-1})q^{t_s}x_1^{t_s}x_{i_1}^{t_1}\ldots x_{i_{s-1}}^{t_{s-1}}, & \text{if } i_1 > 1. \end{cases} \tag{3.4.2.1}$$

Define

$$W_{i+1}^{(k)} = t^{-1}T_i W_i^{(k)} T_i, \quad 1 \leq i \leq k-1.$$

For $1 \leq i \leq k$, we also denote

$$\widetilde{Z}_i^{(k)} = \widetilde{Y}_i^{(k)} - W_i^{(k)}.$$

Lemma 3.4.3. *The following diagram is commutative:*

$$\begin{tikzcd} \mathcal{P}_k^+ \arrow[r, "\widetilde{Z}_i^{(k)}"] \arrow[d, "\pi_k"'] & \mathcal{P}_k^+ \arrow[d, "\pi_k"] \\ \mathcal{P}_{k-1}^+ \arrow[r, "\widetilde{Z}_i^{(k-1)}"] & \mathcal{P}_{k-1}^+ \end{tikzcd}$$

Proof. It is enough to prove the statement for $i = 1$. For any monomial $f \in \mathcal{P}_{k-1}^+$, we have

$$\pi_k W_1^{(k)} f = W_1^{(k-1)} \pi_k f$$

and the commutativity of the digram follows from Theorem 3.4.1.

Let $f \in x_k \mathcal{P}_k^+$ be a monomial and $g \in \mathcal{P}_k^+$. A direct check shows that

$$\pi_k \omega_k^{-1} \,\mathrm{p}_k\, T_{k-1}^{-1} \,\mathrm{p}_k\, g \neq 0$$

only if g is not divisible by x_{k-1}. On the other hand,

$$T_i^{-1} x_i \mathcal{P}_k^+ \subseteq x_{i+1} \mathcal{P}_k^+.$$

Now,

$$\begin{aligned}
\pi_k \widetilde{Y}_1^{(k)} f &= t^k \pi_k \bar{\omega}_k T_{k-1}^{-1} \dots T_1^{-1} f \\
&= t^k \pi_k \bar{\omega}_k T_{k-1}^{-1} \dots T_1^{-1} \,\mathrm{p}_k\, f \\
&= t^k \pi_k \omega_k^{-1} \,\mathrm{p}_k\, T_{k-1}^{-1} \,\mathrm{p}_k (T_{k-2}^{-1} \dots T_1^{-1} f).
\end{aligned}$$

Based on the previous remarks, $\pi_k \widetilde{Y}_1^{(k)} f \neq 0$ unless f is not divisible by x_1. Furthermore, the only monomial from $T_1^{-1} f$ that survives is $s_1 f$. Applying this repeatedly, we obtain

$$\pi_k \widetilde{Y}_1^{(k)} f = t^k \pi_k \omega_k^{-1} \,\mathrm{p}_k\, T_{k-1}^{-1} \,\mathrm{p}_k (s_{k-2} \dots s_1 f) = W_1^{(k)} f.$$

Therefore, $\pi_k \widetilde{Z}_i^{(k)} f = 0 = \widetilde{Z}_i^{(k-1)} \pi_k f$, as expected. $\square$

As a consequence, we obtain the following.

Proposition 3.4.4. *For any $i \geq 1$, the sequence of operators $(\widetilde{Z}_i^{(k)})_{k\geq 2}$ induces a map*

$$\widetilde{Z}_i^{(\infty)} : \mathcal{P}_\infty^+ \to \mathcal{P}_\infty^+,$$

such that $\Pi_k \widetilde{Z}_i^{(\infty)} = \widetilde{Z}_i^{(k)}$, for all $k \geq i$, $k \geq 2$.

This motivates the concept of limit we develop in what follows.

The order of vanishing at $t = 0$ of the rational function $R(t,q) = A(t,q)/B(t,q) \in \mathbb{Q}(t,q)$, with $A(t,q),\ B(t,q) \in \mathbb{Q}[t,q]$, denoted by

$$\operatorname{ord} R(t,q),$$

is the difference between the order of vanishing at $t = 0$ for $A(t,q)$ and $B(t,q)$.

We say that the sequence $(a_n)_{n\geq 1} \subset \mathbb{Q}(t,q)$ converges to 0 if the sequence $(\operatorname{ord} a_n)_{n\geq 1} \subset \mathbb{Z}$ converges to $+\infty$. The sequence $(a_n)_{n\geq 1} \subset \mathbb{Q}(t,q)$ converges to a if $(a_n - a)_{n\geq 1}$ converges to 0. We write,

$$\lim_{n\to\infty} a_n = a.$$

Definition 8. Let $(f_k)_{k\geq 1}$ be a sequence with $f_k \in \mathcal{P}_k^+$. We say that the sequence is convergent if there exists $N \geq 1$ and sequences $(h_k)_{k\geq 1}$, $(g_{i,k})_{k\geq 1}$, $i \leq N$, $h_k,\ g_{i,k} \in \mathcal{P}_k^+$, and $(a_{i,k})_{k\leq 1}$, $i \leq N$, $a_{i,k} \in \mathbb{Q}(t,q)$ such that

1. For any $k \geq 1$, we have $f_k = h_k + \sum_{i=1}^N a_{i,k} g_{i,k}$;
2. For any $i \leq N$, $k \geq 2$, $\pi_k(g_{i,k}) = g_{i,k-1}$ and $\pi_k(h_k) = g_{k-1}$. We denote by

   $$g_i = \lim_{k\to\infty} g_{i,k} \quad \text{and} \quad h = \lim_{k\to\infty} h_k$$

   the sequence $(g_{i,k})_{k\geq 1}$ and, respectively, $(h_k)_{k\geq 1}$ as an element of $\mathcal{P}_\infty^+$. We require that $g_i \in \mathcal{P}_{\text{as}}^+$.
3. For any $i \leq N$ the sequence $(a_{i,k})_{k\geq 1}$ is convergent. We denote $a_i = \lim_{k\to\infty}(a_{i,k})$.

If the sequence $(f_k)_{k\geq 1}$ is convergent we define its limit as

$$\lim_k (f_k) := h + \sum_{i=1}^{N} a_i g_i \in \mathcal{P}_\infty^+.$$

Example 5. The sequence

$$f_k = (1 + t + ... + t^k) e_i[\overline{\mathbf{X}}_k],$$

has the limit

$$\lim_k f_k = \frac{1}{1-t} e_i[X].$$

Similarly, the sequence

$$g_k = t^k e_i[\overline{\mathbf{X}}_k]$$

has limit 0.

We show that the limit of a sequence does not depend on the choice of the auxiliary sequences in Definition 8.

Proposition 3.4.5. *The concept of limit is well-defined.*

Proof. It suffices to show that the limit of the constant sequence 0 is zero, regardless of the auxiliary sequences in Definition 8. Consider sequences $(c_{i,k})_{k\geq 1}$ and $(q_{i,k})_{k\geq 1}$ such that

$$0 = \sum_{i=1}^{N} c_{i,k} q_{i,k} \in \mathcal{P}_k^+,$$

and

$$\lim_{k\to\infty} c_{i,k} = c_i \in \mathbb{Q}(t,q), \ \lim_{k\to\infty} q_{i,k} = q_i \in \mathcal{P}_{\text{as}}^+.$$

We need to show that $\sum_{i=1}^{N} c_i q_i = 0$.

Without loss of generality, we assume that $q_1, ..., q_N$ are $\mathbb{Q}$-linearly independent. Indeed, any linear relation between $q_1, ..., q_N$ must also hold for $q_{1,k}, ..., q_{N,k}$, for all k.

We can therefore substitute one of them, say $q_{1,k}$, with the same $\mathbb{Q}$-linear combination of $q_{2,k}, ..., q_{N,k}$ for all k. It is clear that the conclusion does not change after such a substitution.

Each $q_i \in \mathcal{P}_{\text{as}}^{+}$. We can find the $n \geq 1$, such that $q_i \in \mathcal{P}(n)$ for all $1 \leq i \leq N$. For the same reason as before, without loss of generality, we can assume that

$$q_i = f_i(x_1, ..., x_n) e_{\alpha_i}[\mathbf{X}_n]$$

and

$$\alpha_i = (\alpha_{i,1} \geq \cdots \geq \alpha_{i,s_i}), \quad 1 \leq i \leq N$$

are distinct partitions. Let

$$M = \max_{1 \leq i \leq N} \alpha_{i,1}.$$

For any $k > M + n$, we claim $q_{1,k}, ..., q_{N,k}$ are also $\mathbb{Q}$-linear independent. Indeed, if we have a linear relation

$$\sum_{i=1}^{N} a_i f_i(x_1, ..., x_n) e_{\alpha_i}[x_{n+1}, ..., x_k] = 0,$$

for some $a_i \in \mathbb{Q}$, then for any evaluation at $x_1 = b_1, ..., x_n = b_n$, we have

$$\sum_{i=1}^{N} a_i f_i(b_1, ..., b_n) e_{\alpha_i}[x_{n+1}, ..., x_k] = 0.$$

Note that because $k - n > M$, and the partitions α_i are distinct, the symmetric functions $e_{\alpha_i}[\mathbf{X}_n]$ are linearly independent. Therefore, for all $1 \leq i \leq N$,

$$a_i f_i(b_1, ..., b_n) = 0.$$

Since f_i are polynomials in finitely many variables we obtain, for all $1 \leq i \leq N$,

$$a_i f_i(x_1, ..., x_n) = 0,$$

which is a contradiction. Therefore, for k large enough, we have that $q_{1,k},...,q_{N,k}$ are $\mathbb{Q}$-linear independent.

We can now prove that for k large enough, all $c_{i,k}$ will necessarily be 0. Indeed, from the hypothesis we have

$$0 = \sum_{i=1}^{N} c_{i,k} q_{i,k}.$$

By multiplying both sides the common denominator of $c_{i,k}$ we may assume all $c_{i,k}$ are polynomials in two variables t, q. Again, for any evaluation $q = a, t = b$, we have

$$0 = \sum_{i=1}^{N} c_{i,k}(a,b) q_{i,k}.$$

But we already now $q_{1,k},...,q_{N,k}$ are $\mathbb{Q}$-linear independent for k large enough. Hence it forces $c_{i,k}(a,b) = 0$ for all i. Again, this implies that $c_{i,k}(t,q) = 0$, $1 \leq i \leq N$. $\square$

For later use, we record the following result. To set the notation, assume that $A_k : \mathcal{P}_k^+ \to \mathcal{P}_k^+$, $k \geq 1$, is a sequence of operators, with the following property: for any $f \in \mathcal{P}_{\text{as}}^+$, the sequence $(A_k \Pi_k f)_{k \geq 1}$ converges to an element of $\mathcal{P}_{\text{as}}^+$. Let $\mathcal{A}$ be the operator

$$\mathcal{A} : \mathcal{P}_{\text{as}}^+ \to \mathcal{P}_{\text{as}}^+, \quad f \mapsto \lim_k \mathcal{A}_k \Pi_k f.$$

We refer to $\mathcal{A}$ as the limit operator of the sequence $(A_k)_{k \geq 1}$.

Proposition 3.4.6. *Let $(f_k)_{k \geq 1}$, $f_k \in \mathcal{P}_k^+$ be a convergent sequence such that $f = \lim_k f_k \in \mathcal{P}_{\text{as}}^+$. Then, with the notation above, we have*

$$\mathcal{A} f = \lim_k A_k f_k.$$

Proof. By replacing f_k with $f_k - \Pi_k$, it is enough to prove the statement for the case $f = 0$. If is enough to assume that $f = 0$. In this case, there exist sequences $(c_{i,k})_{k\geq 1}$ and $(q_{i,k})_{k\geq 1}$ as in Definition 8 such that

$$f_k = \sum_{i=1}^{N} c_{i,k} q_{i,k} \in \mathcal{P}_k^+,$$

with

$$\lim_{k\to\infty} c_{i,k} = c_i \in \mathbb{Q}(t,q), \ \lim_{k\to\infty} q_{i,k} = q_i \in \mathcal{P}_{\text{as}}^+.$$

For each i such that $c_i \neq 0$, we may replace $c_{i,k}$ with $c_{i,k}/c_i$ and $q_{i,k}$ with $c_i q_{i,k}$. This allows us to assume that $c_i \in 0, 1$. Without loss of generality, we assume that $q_1, ..., q_N$ are $\mathbb{Q}$-linearly independent. Indeed, any linear relation between $q_1, ..., q_N$ must also hold for $q_{1,k}, ..., q_{N,k}$, for all k. We can therefore substitute one of them, say $q_{1,k}$, with the same $\mathbb{Q}$-linear combination of $q_{2,k}, ..., q_{N,k}$ for all k. It is clear that the conclusion does not change after such a substitution. It is now clear that for all i we have $c_i = 0$.

For any $k \geq 1$,

$$A_k f_k = \sum_{i=1}^{N} c_{i,k} A_k q_{i,k} = \sum_{i=1}^{N} c_{i,k} A_k \Pi_k q_i.$$

Since $\lim\limits_{k\to\infty} A_k \Pi_k q_i = \mathcal{A} q_i \in \mathcal{P}_{\text{as}}^+$, we have, by Definition 8,

$$\lim_k A_k f_k = \sum_{i=1}^{N} c_{i,k} \mathcal{A} q_i = 0.$$

This is precisely our claim. □

This result can be interpreted as a statement of continuity for the operator $\mathcal{A}$. Let $(B_k)_{k\geq 1}$ be another sequence of operators with the same property as $(A_k)_{k\geq 1}$ and denote by $\mathcal{B} : \mathcal{P}_{\text{as}}^+ \to \mathcal{P}_{\text{as}}^+$ the corresponding limit operator.

Corollary 3.4.7. *With the notation above, the operator $\mathcal{A}\mathcal{B}$ is the limit of the sequence $(A_k B_k)_{k\geq 1}$.*

Proof. Let $f \in \mathcal{P}^+_{\mathrm{as}}$. Since $\lim_k B_k \Pi_k f = \mathcal{B} f \in \mathcal{P}^+_{\mathrm{as}}$, we can apply Proposition 3.4.6 to obtain

$$\lim_k A_k B_k \Pi_k f = \mathcal{A}\mathcal{B} f \in \mathcal{P}^+_{\mathrm{as}},$$

which proves our claim. □

Let us examine the following situation.

Lemma 3.4.8. *Let $f \in \mathcal{P}^+_{\mathrm{as}}$ and $i \geq 1$. The sequence $W_i^{(k)} \Pi_k f$ converges. We define*

$$W_i^{(\infty)} : \mathcal{P}^+_{\mathrm{as}} \to \mathcal{P}^+_{\infty}, \quad W_i^{(\infty)} f = \lim_k W_i^{(k)} \Pi_k f \in \mathcal{P}^+_{\infty}.$$

Proof. It is enough to prove our claim for W_1 and $f = g(x_1, \ldots, x_n) m_\alpha[\mathbf{X}_n]$, for some $n \geq 1$, where g is a monomial and m_α is the monomial symmetric function in the indicated alphabet. If x_1 divides g, then $W_1 f = 0$. We assume that x_1 does not divide g.

We denote by $\ell(\alpha)$ the length of the partition α. Of course, for any $k \geq n + \ell(\alpha)$, we have

$$\Pi_k m_\alpha[\mathbf{X}_n] = m_\alpha[x_{n+1}, \ldots, x_k].$$

If $k \geq n + \ell(\alpha)$, denote by $a_1, \ldots, a_s$ the distinct parts of α. Let β_i, $1 \leq i \leq s$ the partition obtained by eliminating one part of size a_i from α. Then,

$$m_\alpha[x_{n+1}, \ldots, x_k] = \sum_{i=1}^{s} \sum_{j=n+\ell(\alpha)}^{k} m_{\beta_i}[x_{n+1}, \ldots, x_{j-1}] x_j^{a_i}.$$

Therefore, we have

$$\begin{aligned} W_1^{(k)}\Pi_k f &= g(x_1,\dots,x_n)\sum_{i=1}^{s}\sum_{j=n+\ell(\alpha)}^{k} W_1^{(k)} m_{\beta_i}[x_{n+1},\dots,x_{j-1}]x_j^{a_i} \\ &= g(x_1,\dots,x_n)\sum_{i=1}^{s}\sum_{j=n+\ell(\alpha)}^{k} (t^j - t^{j-1})q^{a_i}x_1^{a_i} m_{\beta_i}[x_{n+1},\dots,x_{j-1}]. \end{aligned}$$

For a monomial $m = x_{i_1}^{\eta_1}\cdots x_{i_M}^{\eta_M}$, $n < i_1 < \cdots < i_M$, $\eta_1,\dots,\eta_M \geq 1$, we denote $\ell(m) = i_M$ and we write $m \in [\beta]$ if $(\eta_1,\dots,\eta_M)$ is a permutation of the partition β and we write $m \in [\beta]_k$ if $m \in [\beta]$ and $\ell(m) \leq k$. With this notation, we have

$$\begin{aligned} W_1^{(k)}\Pi_k f &= g(x_1,\dots,x_n)\sum_{i=1}^{s}\sum_{m\in[\beta_i]_k} (t^k - t^{\ell(m)})q^{a_i}x_1^{a_i} m \\ &= g(x_1,\dots,x_n)\left(\sum_{i=1}^{s} t^k q^{a_i}x_1^{a_i} m_{\beta_i}[x_{n+1},\dots,x_k] - \sum_{i=1}^{s} q^{a_i}x_1^{a_i}\sum_{m\in[\beta_i]_k} t^{\ell(m)} m\right) \end{aligned}$$

According to Definition 8,

$$\lim_k W_1^{(k)}\Pi_k f = -\sum_{i=1}^{s} q^{a_i}x_1^{a_i} g(x_1,\dots,x_n)\sum_{m\in[\beta_i]} t^{\ell(m)} m \in \mathcal{P}_\infty^+,$$

which proves our claim. $\square$

We can now prove the following.

Proposition 3.4.9. *Let $f \in \mathcal{P}_{\text{as}}^+$ and $i \geq 1$. The sequence $Y_i^{(k)}\Pi_k f$ converges. We define*

$$Y_i : \mathcal{P}_{\text{as}}^+ \to \mathcal{P}_\infty^+, \quad Y_i f = \lim_k Y_i^{(k)}\Pi_k f \in \mathcal{P}_\infty^+.$$

Proof. Straightforward from Proposition 3.4.4 and Lemma 3.4.8. More precisely, $Y_i = \widetilde{Z}_i^{(\infty)} + W_i^{(\infty)}$. $\square$

It turns out that the image of Y_i is contained in $\mathcal{P}^+_{\mathrm{as}}$. We will use the following result.

Lemma 3.4.10. *Let $f \in \mathbb{Q}(t,q)[x_{k+1}, x_{k+2}, \dots]$ be a polynomial satisfying*

$$T_i f = f, \qquad \text{for } k+1 \le i \le k+m.$$

Let $g = T^{-1}_{k+m} \dots T^{-1}_{k+1} T^{-1}_k (x_k^s f)$, for some $s \ge 0$. Then,

$$T_i g = g \qquad \text{for } k \le i \le k+m-1.$$

Proof. The claim is a consequence of the braid relations for the elements T_i. $\square$

Lemma 3.4.11. *$\mathcal{P}(k)^+$ is stable under the action of Y_1. Therefore, we have*

$$Y_i : \mathcal{P}^+_{\mathrm{as}} \to \mathcal{P}^+_{\mathrm{as}}.$$

Proof. Let $f \in \mathcal{P}(k)^+$. For any $m > k+1$ we have

$$\widetilde{Y}_1^{(m)} \Pi_m f = t^m \bar{\omega}_m T^{-1}_{m-1} \dots T^{-1}_1 \Pi_m f.$$

Since $f \in \mathcal{P}(k)^+$, $\Pi_m f$ is fixed under the action of $T_{k+1}, \dots, T_{m-1}$. Now, we have

$$\bar{\omega}_m T^{-1}_{m-1} \dots T^{-1}_1 \Pi_m f = \bar{\omega}_m T^{-1}_{m-1} \dots T^{-1}_{k+1} T^{-1}_k (T^{-1}_{k-1} \dots T^{-1}_1 \Pi_m f),$$

and we may write

$$T^{-1}_{k-1} \dots T^{-1}_1 \Pi_m f = \sum_j c_j[x_1, \dots, x_{k-1}] x_k^{s_j} f_j$$

as a finite sum, where each f_j is a polynomial in $\mathbb{Q}(t,q)[x_{k+1}, x_{k+2}, \dots]$ satisfying

$$T_i f_j = f_j, \qquad \text{for } i = k+1, k+2, \dots, m-1.$$

Applying Lemma 3.4.10 for $T_{k-1}^{-1}\dots T_1^{-1}\Pi_m f$, we obtain that $T_{m-1}^{-1}\dots T_1^{-1}\Pi_m f$ is fixed under the action of the elements $T_k,\dots,T_{m-2}$. The relation (3.3.2.1) implies that $\bar{\omega}_m T_{m-1}^{-1}\dots T_1^{-1}\Pi_m f$ is fixed under the action of $T_{k+1},\dots,T_{m-1}$. Therefore, $\widetilde{Y}_1^{(m)}\Pi_m f$ is symmetric in $x_{k+1},\dots,x_m$ for all $m > k+1$. In conclusion, the limit

$$\lim_m Y_i^{(m)}\Pi_m f \in \mathcal{P}(k)^+,$$

proving our claim. □

We can now state the following.

Proposition 3.4.12. *The space $\mathcal{P}_{\mathrm{as}}^+$ carries an action of the limit operators T_i, X_i, Y_i, $i \geq 1$.*

As we will show in Theorem 3.4.16, these operators define an action of $\mathcal{H}^+$ on $\mathcal{P}_{\mathrm{as}}^+$.

Remark. It is important to note that on $x_i\mathcal{P}_{\mathrm{as}}^+$ the action of Y_i is the stable limit of the action of the sequence of Cherednik operators $Y_i^{(k)} \in \mathcal{H}_k$.

In fact, we can obtain a more precise description of the action of Y_1 on $\mathcal{P}(k)^+$. First, let us record the following technical result.

Lemma 3.4.13. *Let $x_1^n \in \mathcal{P}_{m+1}^+$. Then,*

$$t^m T_m^{-1}...T_1^{-1}x_1^n = \sum_{i=0}^{n-1} x_{m+1}^{n-i}h_i[(1-t)\overline{\mathbf{X}}_m].$$

Proof. We will prove by induction on m. By direct computation we obtain

$$\begin{aligned} T_1^{-1}x_1^n &= t^{-1}x_2^n + (t^{-1}-1)(x_2^{n-1}x_1 + x_2^{n-2}x_1^2 + ... + x_2x_1^{n-1}) \\ &= t^{-1}\sum_{i=0}^{n-1} x_2^{n-i}h_i[(1-t)x_1]. \end{aligned}$$

Assume our claim holds for $m-1$. Then, we have

$$\begin{aligned} t^m T_m^{-1} \dots T_1^{-1} x_1^n &= t T_m^{-1} \sum_{i=0}^{n-1} x_m^{n-i} h_i[(1-t)\overline{\mathbf{X}}_{m-1}] \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} x_{m+1}^{n-i-j} h_j[(1-t)x_m] h_i[(1-t)\overline{\mathbf{X}}_{m-1}] \\ &= \sum_{l=0}^{n-1} x_{m+1}^{n-l} \sum_{\substack{i+j=l \\ i,j\geq 0}} h_j[(1-t)x_m] h_i[(1-t)\overline{\mathbf{X}}_{m-1}] \\ &= \sum_{l=0}^{n-1} x_{m+1}^{n-l} h_l[(1-t)\overline{\mathbf{X}}_m], \end{aligned}$$

as expected. □

Recall that for all k, the multiplication map $\mathcal{P}_k^+ \otimes \mathrm{Sym}[\mathbf{X}_k] \cong \mathcal{P}(k)^+$ is an algebra isomorphism.

Proposition 3.4.14. *Let $n \geq 0$, $f(x_1, \dots, x_{k-1}) \in \mathcal{P}_{k-1}^+$, and $G[\mathbf{X}_{k-1}] \in \mathrm{Sym}[\mathbf{X}_{k-1}]$. We regard*

$$F = f(x_1, \dots, x_{k-1}) x_k^n G[\mathbf{X}_{k-1}]$$

as an element of $\mathcal{P}(k)^+$. Then,

$$Y_1 T_1 \cdots T_{k-1} F = \frac{t^k}{1-t} f(x_2, \dots, x_k) G[\mathbf{X}_k + q x_1](h_n[(1-t)(\mathbf{X}_k + q x_1)] - h_n[(1-t)\mathbf{X}_k]).$$

Proof. For any $m > k+1$, we have

$$\begin{aligned}
\widetilde{Y}_1^{(m)} T_1 \cdots T_{k-1} \Pi_m F &= t^m \bar{\omega}_m T_{m-1}^{-1} \dots T_k^{-1} f(x_1, \dots, x_{k-1}) x_k^n G[\overline{\mathbf{X}}_{[k,m]}] \\
&= t^k \bar{\omega}_m f(x_1, \dots, x_{k-1}) G[\overline{\mathbf{X}}_{[k,m]}] t^{m-k} T_{m-1}^{-1} \dots T_k^{-1} x_k^n \\
&= t^k \bar{\omega}_m f(x_1, \dots, x_{k-1}) G[\overline{\mathbf{X}}_{[k,m]}] \sum_{i=0}^{n-1} x_m^{n-i} h_i[(1-t)\overline{\mathbf{X}}_{[k,m-1]}] \\
&= t^k f(x_2, \dots, x_k) G[\overline{\mathbf{X}}_{[k+1,m]} + qx_1] \\
&\qquad \sum_{i=0}^{n-1} (qx_1)^{n-i} h_i[(1-t)\overline{\mathbf{X}}_{[k+1,m]}].
\end{aligned}$$

Therefore, $\lim_m Y_i^{(m)} T_1 \cdots T_{k-1} \Pi_m F$, equals

$$\begin{aligned}
t^k f(x_2, \dots, x_k) G[\mathbf{X}_k + qx_1] &\sum_{i=0}^{n-1} (qx_1)^{n-i} h_i[(1-t)\mathbf{X}_k] \\
&= \frac{t^k}{1-t} f(x_2, \dots, x_k) G[\mathbf{X}_k + qx_1] \\
&\qquad (h_n[(1-t)(\mathbf{X}_k + qx_1)] - h_n[(1-t)\mathbf{X}_k]),
\end{aligned}$$

proving our claim. □

We establish the following result in preparation for the proof of Theorem 3.4.16.

Lemma 3.4.15. *Let $f \in \mathcal{P}_{\mathrm{as}}^+$. Then,*

$$\lim_k [\widetilde{Y}_i^{(k)}, \widetilde{Y}_j^{(k)}] \Pi_k f = 0.$$

Proof. First note that we can apply the following two relations recursively

$$[\widetilde{Y}_i^{(k)}, \widetilde{Y}_j^{(k)}] = t^{-1}T_{j-1}[\widetilde{Y}_i^{(k)}, \widetilde{Y}_{j-1}^{(k)}]T_{j-1}, \text{ for } i > j,$$

$$[\widetilde{Y}_1^{(k)}, \widetilde{Y}_i^{(k)}] = t^{-1}T_{i-1}[\widetilde{Y}_1^{(k)}, \widetilde{Y}_{i-1}^{(k)}]T_{i-1}, \text{ for } i > 2.$$

Hence it suffices to prove the result for $[\widetilde{Y}_1^{(k)}, \widetilde{Y}_2^{(k)}]$.

Recall that we have

$$[\widetilde{Y}_1^{(k)}, \widetilde{Y}_2^{(k)}] = t^{2k-1}\gamma_k T_{k-1}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_2^{-1} = t^{2k-1}\gamma_k T_{k-2}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_1^{-1}.$$

Let $f(x_1, ..., x_m)F[X] \in \mathcal{P}(k)^+$, where $F[X]$ is fully symmetric and non-zero. Then for $k > m$ we have

$$\begin{aligned} &\gamma_k T_{k-1}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_2^{-1}\Pi_k f(x_1, ..., x_m)F[X] \\ &= \gamma_k T_{k-1}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_2^{-1} f(x_1, ..., x_m)F[\overline{\mathbf{X}}_k] \\ &= \gamma_k F[\overline{\mathbf{X}}_k](T_{k-1}^{-1}...T_1^{-1}T_{k-1}^{-1}...T_2^{-1} f(x_1, ..., x_m)) \\ &= 0. \end{aligned}$$

Therefore, in this case,

$$\lim_{k\to\infty} [\widetilde{Y}_i^{(k)}, \widetilde{Y}_j^{(k)}]\Pi_k f(x_1, ..., x_m)F[X] = 0.$$

It remains to compute the limit for $f(x_1, ..., x_m) \in \mathcal{P}_m^+$. Let $k > m+1$. Without loss of generality, we may assume that

$$T_{m-2}^{-1}...T_1^{-1}T_{m-1}^{-1}...T_1^{-1} f(x_1, ..., x_m)$$

is a monomial of the form $x_m^n x_{m-1}^{n'} g(x_1, ..., x_{m-2})$. If $n > 0$, we have

$$\begin{aligned}
[\widetilde{Y}_1^{(k)}, \widetilde{Y}_2^{(k)}] f(x_1, ..., x_m) &= \gamma_k T_{k-2}^{-1} ... T_1^{-1} T_{k-1}^{-1} ... T_1^{-1} f(x_1, ..., x_m) \\
&= \gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} T_{k-1}^{-1} ... T_m^{-1} \\
&\quad\ (T_{m-2}^{-1} ... T_1^{-1} T_{m-1}^{-1} ... T_1^{-1} f(x_1, ..., x_m)) \\
&= \gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} T_{k-1}^{-1} ... T_m^{-1} x_m^n x_{m-1}^{n'} g(x_1, ..., x_{m-2}) \\
&= g(x_1, ..., x_{m-2}) \gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} x_{m-1}^{n'} T_{k-1}^{-1} ... T_m^{-1} x_m^n.
\end{aligned}$$

If $n = n' = 0$ then $[\widetilde{Y}_1^{(k)}, \widetilde{Y}_2^{(k)}] f(x_1, ..., x_m) = 0$. If $n = 0$ and $n' > 0$, then by Lemma 3.4.13

$$\begin{aligned}
&\gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} x_{m-1}^{n'} \\
&= t^{m-k} \gamma_k \sum_{i=0}^{n'-1} x_{k-1}^{n'-i} h_i[(1-t)\overline{\mathbf{X}}_{[m-1,k-2]}] \\
&= t^{m-k} \sum_{i=0}^{n'-1} h_i[(1-t)\overline{\mathbf{X}}_{[m-1,k-2]}] \gamma_k x_{k-1}^{n'-i} \\
&= t^{m-k}(1-t) \sum_{i=0}^{n'-1} h_i[(1-t)\overline{\mathbf{X}}_{[m-1,k-2]}](x_1^{n'-i-1} x_2 + \cdots + x_1 x_2^{n'-i-1}).
\end{aligned}$$

Therefore, $\lim_k [\widetilde{Y}_1^{(k)}, \widetilde{Y}_2^{(k)}] f(x_1, ..., x_m) = 0$.

If $n' = 0$ and $n > 0$, then by Lemma 3.4.13

$$\gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} T_{k-1}^{-1} ... T_m^{-1} x_m^n = t^{m-k} \gamma_k T_{k-2}^{-1} ... T_{m-1}^{-1} \sum_{i=0}^{n-1} x_k^{n-i} h_i[(1-t)\overline{\mathbf{X}}_{[m,k-1]}].$$

By writing

$$h_i[(1-t)\overline{\mathbf{X}}_{[m,k-1]}] = \sum_{j=0}^{i} h_{i-j}[(1-t)\overline{\mathbf{X}}_{[m-1,k-1]}] h_j[(t-1)x_{m-1}]$$

and using again Lemma 3.4.13 for $T_{k-2}^{-1}...T_{m-1}^{-1}x_{m-1}^{j}$, we write

$$\gamma_k T_{k-2}^{-1}...T_{m-1}^{-1}T_{k-1}^{-1}...T_m^{-1}x_m^n$$

as

$$t^{m-k}\sum_{i=0}^{n-1}\gamma_k x_k^{n-i}h_i[(1-t)\overline{\mathbf{X}}_{[m-1,k-1]}]$$
$$= t^{m-k}(t-1)\sum_{i=0}^{n-1}h_i[(1-t)\overline{\mathbf{X}}_{[m-1,k-2]}](x_1^{n-i-1}x_2+\cdots+x_1x_2^{n-i-1}).$$

Therefore, $\lim_k[\widetilde{Y}_1^{(k)},\widetilde{Y}_2^{(k)}]f(x_1,...,x_m)=0$.

For the last case, $n,n'>0$, proceeding as in the previous case we obtain that

$$\gamma_k T_{k-2}^{-1}...T_{m-1}^{-1}x_{m-1}^{n'}T_{k-1}^{-1}...T_m^{-1}x_m^n$$

equals

$$t^{m-k}\gamma_k T_{k-2}^{-1}...T_{m-1}^{-1}x_{m-1}^{n'}\sum_{i=0}^{n-1}x_k^{n-i}\sum_{j=0}^{i}h_{i-j}[(1-t)\overline{\mathbf{X}}_{[m-1,k-1]}]h_j[(t-1)x_{m-1}].$$

Lemma 3.4.13 for $T_{k-2}^{-1}...T_{m-1}^{-1}x_{m-1}^{j}$ implies that $[\widetilde{Y}_1^{(k)},\widetilde{Y}_2^{(k)}]f(x_1,...,x_m)=0$. $\square$

We are now ready to prove our main result.

Theorem 3.4.16. *The operators T_i, X_i, and Y_i, $i\geq 1$, define a $\mathcal{H}^+$-module structure on $\mathcal{P}_{\mathrm{as}}^+$.*

Proof. The relations that hold in the algebra $\tilde{\mathcal{H}}_k$ also hold for the corresponding limit operators by the repeated application of Corollary 3.4.7. Recall that the relation 3.2.1.1d and the first two relations in 3.2.1.1c also hold inside the algebras $\tilde{\mathcal{H}}_k$. The only relations that do not transfer directly from those in $\tilde{\mathcal{H}}_k$ are the commutation relations between Y_i and Y_j. However, we do have

$$[\widetilde{Y}_i^{(k)}, \widetilde{Y}_j^{(k)}] = t^{-1}T_{j-1}[\widetilde{Y}_i^{(k)}, \widetilde{Y}_{j-1}^{(k)}]T_{j-1}, \quad i > j,$$
$$[\widetilde{Y}_1^{(k)}, \widetilde{Y}_i^{(k)}] = t^{-1}T_{i-1}[\widetilde{Y}_1^{(k)}, \widetilde{Y}_{i-1}^{(k)}]T_{i-1}, \quad i > 2.$$

These, by application of Corollary 3.4.7 imply the same relations for the limit operators

$$[Y_i, Y_j] = t^{-1}T_{j-1}[Y_i, Y_{j-1}]T_{j-1}, \quad i > j,$$
$$[Y_1, Y_i] = t^{-1}T_{i-1}[Y_1, Y_{i-1}]T_{i-1}, \quad i > 2.$$

Therefore, for any i, j, the commutativity of Y_i and Y_j follows from the commutativity of Y_1 and Y_2. Fix $f \in \mathcal{P}_{\mathrm{as}}^+$. Then, by Proposition 3.4.6 and Corollary 3.4.7,

$$[Y_1, Y_2]f = \lim_k [\widetilde{Y}_i^{(k)}, \widetilde{Y}_j^{(k)}]\Pi_k f,$$

which is 0 by Lemma 3.4.15. □

We call this representation the standard representation of $\mathcal{H}^+$. As with the standard representation of $\mathcal{H}^-$, we expect this representation to be faithful.

4.0 Double Dyck path algebra

4.1 Dyck paths

We will now introduce the double Dyck path algebra. We will first introduce the preliminaries about Dyck paths. Dyck paths are classical combinatorial objects which occur in many areas in combinatorics.

4.1.1 Definitions

Definition 9. (Dyck path) A Dyck path of order n is a path in the $n \times n$ lattice from $(0, 0)$ to (n, n) that consists of n unit-length north steps and n unit-length east steps, which stays weakly above the line $y = x$.

Note that a Dyck path may touch the main diagonal $y = x$ but shall not travel below the diagonal. The following figure shows a Dyck path of order 8.

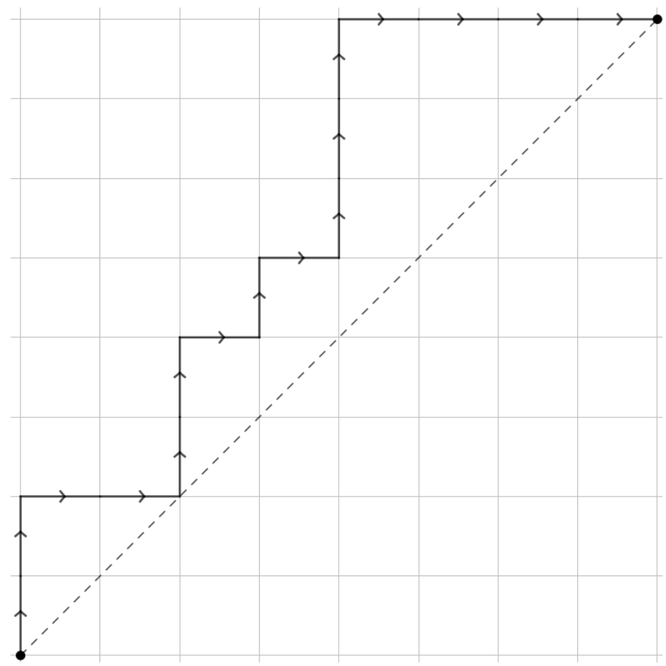


Figure 1: A Dyck path π of order 8

We will then define several combinatorial quantities associated to Dyck paths.

For a Dyck path π, we will denote by $|\pi|$ its order (length). In the previous case, we have $|\pi| = 8$. Then for each cell of the grid define its coordinates (i, j) to be the coordinates of the top right corner. For instance, in the previous example, the bottom left cell will have the coordinates $(1, 1)$, and the top right cell will have the coordinates $(8, 8)$.

Then the area of a Dyck path π is defined as:

$$\text{Area}(\pi) = \{(i, j) | i < j, (i, j) \text{ under } \pi\}, \quad \text{area}(\pi) = \#\text{Area}(\pi).$$

In other words, cells under the Dyck path π and strictly above the main diagonal $y = x$ will contribute to the area of π.

Now let a_j denote the number of cells (i, j) in the row j. We define

$$\begin{aligned}\mathrm{Dinv}(\pi) &= \{(j, j')|1 \leq j < j' \leq n,\ a_j = a_{j'}\} \cup \{(j, j')|1 \leq j' < j \leq n,\ a_{j'} = a_j + 1\}, \\ \mathrm{dinv}(\pi) &= \#\mathrm{Dinv}(\pi).\end{aligned}$$

More explicitly, the set $\mathrm{Dinv}(\pi)$ will consist of all pairs of row numbers satisfying either of the following two conditions: either the two corresponding rows have exactly same number of cells between the Dyck path and the main diagonal, or the higher corresponding row has exactly one cell less than the lower corresponding row. Furthermore, we use ordering to distinguish the two cases. In the previous case, we will arrange the two entries in ascending order. In the latter case, we will arrange them in descending order.

Let $(x_1, 1), (x_2, 2), ..., (x_n, n)$ be the cells immediately to the right of the North steps with respect to the Dyck path π. Then the set $\mathrm{Dinv}(\pi)$ can be equivalently interpreted as all pairs of numbers (j, j') satisfying either of the following two conditions: either $j < j'$, and the cells (x_j, j) and $(x_{j'}, j')$ are on the same subdiagonal, where a subdiagonal is defined to be a straight line determined by the equation $y = x + c$ for some constant c, or $j > j'$, and the cell (x_j, j) is on the subdiagonal one unit lower than that of the cell $(x_{j'}, j')$. This alternative interpretation of the definition of Dinv allows us to quickly count the number of elements in the set, namely $\mathrm{dinv}(\pi)$.

Let the touch compostion $\alpha = (\alpha_1, ..., \alpha_k)$ of π be defined as consisting of the gaps between the points where π touches the diagonal. We will call this composition $\mathrm{touch}(\pi)$.

As an exercise, in the previous example of a Dyck path, we have

- $\mathrm{Area}(\pi)=\{(1,2),(3,4),(4,5),(5,6),(5,7),(5,8),(6,7),(6,8),(7,8)\}$, $\mathrm{area}(\pi)=9$.

- $\mathrm{Dinv}(\pi)=\{(1,3),(2,4),(2,5),(2,6),(4,5),(4,6),(5,6)\} \cup \{(3,2)\}$, $\mathrm{dinv}(\pi)=8$.
- $\mathrm{touch}(\pi)=(2,6)$.

Definition 10. (Word parking function) Let π be a Dyck path of order n. A word parking function with respect to π is a function

$$w : \{1,2,...,n\} \rightarrow \mathbb{Z}_+,$$

such that whenever $x_j = x_{j+1}$ for some $j \in \{1,2,...,n-1\}$, we have $w(j) > w(j+1)$. We will also use w_j to denote the evaluation $w(j)$. The set of all word parking functions associated to the Dyck path π will be denoted by $\mathcal{WP}_\pi$.

Equivalently, we could regard a word parking function as an assignment of positive integers to the cells immediately to the right of the North steps, such that the cells in the same column must obey a strictly increasing order from top to bottom. The following figures illustrates an example of a word parking function with respect to the Dyck path π we discussed earlier.

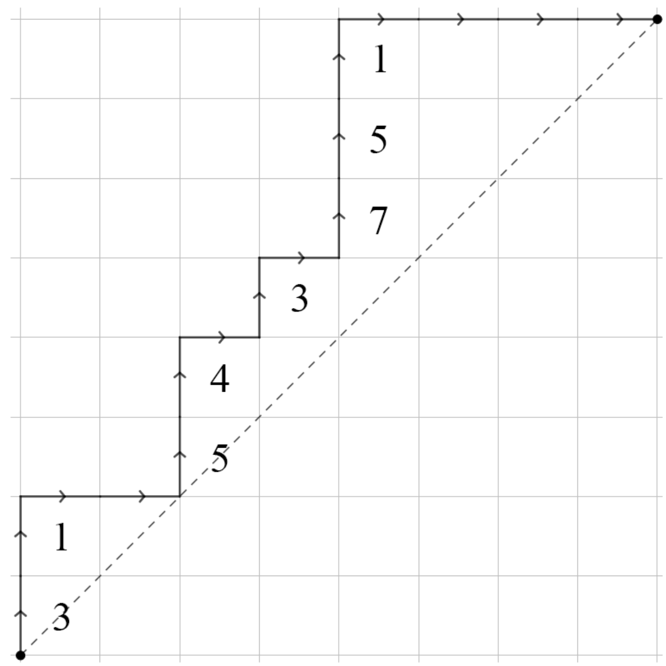


Figure 2: A word parking function w associated to π

We then define

$$\mathrm{Dinv}(\pi, w) = \{(j, j') \in \mathrm{Dinv}(\pi) | w_j > w'_j\}, \quad \mathrm{dinv}(\pi, w) = \#\mathrm{Dinv}(\pi, w)$$

More explictly, the set $\mathrm{Dinv}(\pi, w)$ is a subset of $\mathrm{Dinv}(\pi)$ satisfying the extra condition that the entry assigned the cell immediately to the right of the North step in the j-th row is strictly greater than that of the j'-th row. Therefore for the example above, we have

$$\mathrm{Dinv}(\pi, w) = \{(4, 5), (3, 2)\}, \quad \mathrm{dinv}(\pi, w) = 2.$$

Then we will introduced the original form of the shuffle conjecture.

Theorem 4.1.1.

$$\nabla e_n = \sum_{|\pi|=n} \sum_{w \in \mathcal{WP}_\pi} t^{area(\pi)} q^{dinv(\pi,\omega)} x_w.$$

For example, the previous example will contribute to the term $t^9 q^2 z_1^2 z_3^2 z_4 z_5^2 z_7$ on the right hand side of the equation.

We could have the following compositional version of the shuffle conjecture as a refinement of the original conjecture.

Theorem 4.1.2. [CM18]

$$\nabla C_\alpha(1) = \sum_{touch(\pi)=\alpha} \sum_{|\pi|=n} \sum_{w \in \mathcal{WP}_\pi} t^{area(\pi)} q^{dinv(\pi,\omega)} x_w.$$

Denote the right hand side of the equation by $D_\alpha(X; q, t)$. By Proposition 2.3.2.2, this is stronger than the original conjectrue. Carlsson and Mellit proved the composition version in [CM18] using the double Dyck path algebra structure. We will introduce their proof in the following sections.

4.2 Double Dyck path algebra

4.2.1 Definitions

Notation. We will first define two quivers $\dot{\mathbf{Q}}$ and $\ddot{\mathbf{Q}}$ and introduce some conventions.

$\dot{\mathbf{Q}}$ is defined to be the quiver with vertex set $\mathbb{Z}_{\geq 0}$, and for all $k \in \mathbb{Z}_{\geq 0}$ arrows d_+ from k to $k+1$, arrows d_- from $k+1$ to k, and for $k \geq 2$ loops $T_1, ..., T_{k-1}$ from k to k. Note that, to keep the notation as simple as possible, the same label is used to denote many arrows. To eliminate the possible confusion we adopt the following convention.

Convention. In all expressions involving paths in $\dot{\mathbf{Q}}$, unless specified otherwise, we assume that all the expressions involve non-zero paths (that is, the constituent arrows concatenate correctly to produce a non-zero path) that start at the node k (fixed, but arbitrary).

Then define $\ddot{\mathbf{Q}}$ as the quiver with vertex set $\mathbb{Z}_{\geq 0}$, and for all $k \in \mathbb{Z}_{\geq 0}$ arrows d_+ and d_+^* from k to $k+1$, arrows d_- from $k+1$ to k, and loops $T_1, ..., T_{k-1}$ from k to k. We will adopt the same labelling convention for paths in $\ddot{\mathbf{Q}}$.

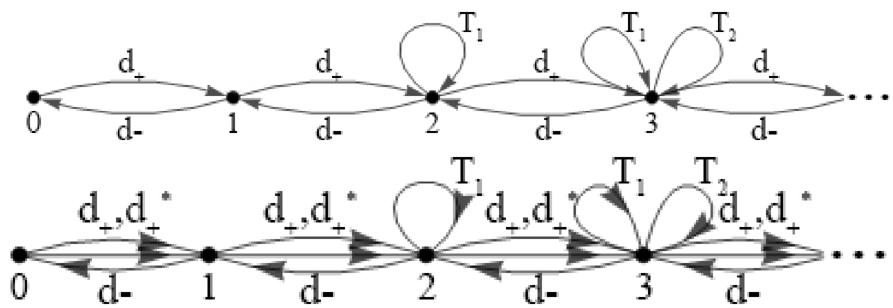


Figure 3: The quivers $\dot{\mathbf{Q}}$ and $\ddot{\mathbf{Q}}$

Definition 11. The Dyck path algebra $\mathbb{A}_t$ is defined to be the quiver path algebra over $\dot{\mathbf{Q}}$ subject to the following relations:

$$\begin{gathered} T_iT_j = T_jT_i, \text{ for } 1 \leq i < j \leq k-1 \text{ with } |i-j| > 1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \text{ for } i = 1, ..., k-2, \\ (T_i - 1)(T_i + t) = 0, \text{ for } i = 1, ..., k-1, \end{gathered} \tag{4.2.1.1a}$$

$$T_1 d_+^2 = d_+^2, \quad d_+ T_i = T_{i+1} d_+, \text{ for } i = 1, ..., k-2, \tag{4.2.1.1b}$$

$$d_-^2 T_{k-1} = d_-^2, \quad T_i d_- = d_- T_i, \quad 1 \leq i \leq k-2. \tag{4.2.1.1c}$$

$$d_-[d_+, d_-]T_{k-1} = t[d_+, d_-]d_-,$$
$$T_1[d_+, d_-]d_+ = td_+[d_+, d_-], \tag{4.2.1.1d}$$

where $[d_+, d_-] = d_+d_- - d_-d_+$.

To facilitate the comparison with the DAHA, the parameters t, q in [Mel16, Def. 3.1, 3.2] have been interchanged.

Definition 12. The double Dyck path algebra $\mathbb{A}_{t,q}$ is defined to be the quiver path algebra over $\ddot{\mathbf{Q}}$ subject to the following relations:

1. relations (4.2.1.1a), (4.2.1.1b), (4.2.1.1c) and (4.2.1.1d),
2. $$T_1(d_+^*)^2 = (d_+^*)^2, \quad d_+^*T_i = T_{i+1}d_+^*, \quad \text{for } i = 1, ..., k-2, \tag{4.2.1.2a}$$

$$td_-[d_+^*, d_-] = [d_+^*, d_-]d_-T_{k-1},$$
$$t[d_+^*, d_-]d_+^* = T_1d_+^*[d_+^*, d_-]. \tag{4.2.1.2b}$$

3. the cross relations:

$$d_+z_i = z_{i+1}d_+, \quad d_+^*y_i = y_{i+1}d_+^*, \quad \text{for } i = 1, ..., k-1,$$
$$z_1d_+ = -qt^{k+1}y_1d_+^*, \tag{4.2.1.3}$$

where loops y_i and z_i from k to k for $1 \leq i \leq k$ are defined as

$$y_1 = \frac{1}{t^{k-1}(t-1)}[d_+, d_-]T_{k-1}...T_1,$$
$$y_{i+1} = tT_i^{-1}y_iT_i^{-1}, \quad \text{for } 1 \leq i \leq k-1,$$
$$z_1 = \frac{t^k}{1-t}[d_+^*, d_-]T_{k-1}^{-1}...T_1^{-1},$$
$$z_{i+1} = t^{-1}T_iz_iT_i, \quad \text{for } 1 \leq i \leq k-1.$$

Remark. There exists an involution $\rho : \mathbb{A}_{t,q} \to \mathbb{A}_{t,q}$ defined as

$$T_i^{-1} \mapsto T_i,\ d_- \mapsto d_-,\ d_+^* \mapsto d_+,\ t^{-1} \mapsto t,\ q \mapsto q^{-1}.$$

Viewing $\mathbb{A}_t$ as a subalgebra of $\mathbb{A}_{t,q}$, we will denote by $\mathbb{A}_{t^{-1}}$ the image of this subalgebra under the involution ρ.

To shed light on the resemblance between $\mathbb{A}_t$ and affine Hecke algebras, we have the following proposition.

Proposition 4.2.1 ([Mel16],Proposition 3.1)**.** *The loops y_i's and z_i's satisfy the following relations:*

$$\begin{aligned} y_iT_j &= T_jy_i, \quad \textit{for } i \neq j, j+1 \\ y_{i+1} &= tT_i^{-1}y_iT_i^{-1}, \quad \textit{for } 1 \leq i \leq k-1, \\ y_iy_j &= y_jy_i, \quad \textit{for } 1 \leq i, j \leq k, \end{aligned} \tag{4.2.1.4}$$

$$\begin{aligned} y_id_- &= d_-y_i, \quad \textit{for } 1 \leq i \leq k-1 \\ d_+y_i &= T_1...T_iy_iT_i^{-1}...T_1^{-1}d_+, \quad \textit{for } 1 \leq i \leq k, \end{aligned} \tag{4.2.1.5}$$

$$\begin{aligned} z_iT_j &= T_jz_i, \quad \textit{for } i \neq j, j+1 \\ z_{i+1} &= t^{-1}T_iz_iT_i, \quad \textit{for } 1 \leq i \leq k-1, \\ z_iz_j &= z_jz_i, \quad \textit{for } 1 \leq i, j \leq k, \end{aligned} \tag{4.2.1.6}$$

$$\begin{aligned} z_id_- &= d_-z_i, \quad \textit{for } 1 \leq i \leq k-1 \\ d_+^*z_i &= T_1^{-1}...T_i^{-1}z_iT_i...T_1d_+^*, \quad \textit{for } 1 \leq i \leq k, \end{aligned} \tag{4.2.1.7}$$

Remark. Note that the relations (4.2.1.4) match the generating relations of the affine Hecke algebra AHA_k of type GL_k, which will be defined later. But since $y_1, ..., y_k$ are not defined to be invertible, $\mathbb{A}_t$ will only contain a copy of the polynomial part AHA_k^+ of AHA_k.

Similarly, we see that the loops $T_1^{-1}, ..., T_{k-1}^{-1}$ and $z_1, ..., z_k$ will generate a copy of AHA_k^+ with parameter t^{-1}.

The proof of the shuffle conjecture relies on two representations of $\mathbb{A}_t$ and $\mathbb{A}_{t,q}$ respectively defined as follows.

Proposition 4.2.2 ([Mel16], Proposition 3.2, 3.3)**.** *The following operations define an action of $\mathbb{A}_{t,q}$ on*

$$V_* = \bigoplus_k V_k = \bigoplus \mathbb{Q}(q,t)[y_1, ..., y_k] \otimes Sym[X].$$

$$\begin{aligned} T_i F &= s_i F + (1-t) y_i \frac{F - s_i F}{y_i - y_{i+1}}, \\ d_- F &= F[X - (t-1) y_k] \mathrm{Exp}[-y_k^{-1} X] \Big|_{\mathrm{const}(y_k)}, \\ d_+ F &= -T_1 ... T_k (y_{k+1} F[X + (t-1) y_{k+1}], \\ d_+^* F &= \gamma F[X + (t-1) y_{k+1}], \end{aligned} \tag{4.2.1.8}$$

where $F \in V_k$, $F|_{const(y_k)}$ means taking the constant term of F with respect to y_k, and all the operators correspond to arrows originating at node k. Note that the action defined above may restrict to an action of $\mathbb{A}_t$ on V_*.

Corollary 4.2.3. *Let the action of $\mathbb{A}_{t,q}$ on V_* be defined as above. Then y_i for $1 \leq i \leq k$ will act on V_k as the left multiplication by y_i.*

4.2.2 The shuffle theorem

We will then briefly explain the proof of the shuffle theorem in [CM18]. First we need to slightly modify the $\mathbb{A}_{t,q}$ representation on V_* as the following:

$$\begin{aligned} T_i F &= s_i F + (1-t) y_i \frac{F - s_i F}{y_i - y_{i+1}}, \\ d'_- F &= -y_k F[X - (t-1) y_k] \mathrm{Exp}[-y_k^{-1} X] \Big|_{\mathrm{const}(y_k)}, \\ d'_+ F &= T_1 ... T_k (F[X + (t-1) y_{k+1}], \\ d^*_+ F &= \gamma F[X + (t-1) y_{k+1}], \end{aligned} \tag{4.2.2.1}$$

Note that we use d'_- and d'_+ to distinguish the different actions. As explained in [Mel16], we have

Proposition 4.2.4. *Let $M : V_k \to V_k$ be the operator of multiplication by*

$$(-1)^k y_1 ... y_k$$

for each $k \geq 0$. Then we have

$$M d'_- = d_- M, \quad M d'_+ = d_+ M, \quad M y_i = y_i M.$$

Therefore, as M is injective, statements for the original representation could be used to deduce statements for the modified representation.

We will then quote the main recursion formula.

Proposition 4.2.5. *Let α be a composition of length l. Then we have*

$$D_\alpha(q,t) = (d'_-)^l (N_\alpha),$$

where $N_\alpha \in V_l$ is defined by the following recursion formulae:

$$N_\emptyset = 1, \quad N_{1\alpha} = d'_+ N_\alpha, \quad N_{a\alpha} = \frac{t^{a-1}}{q-1} [d'_-, d'_+] \sum_{\beta \vdash a-1} (d'_-)^{l(\beta)-1} N_{\alpha\beta}.$$

Now consider the antilinear automorphism $\mathcal{N}$ on V_* induced by the involution ρ. Then we have the following important result.

Theorem 4.2.6. *Let the antilinear automorphism $\mathcal{N} : V_* \to V_*$ induced by the involution ρ be defined as*

$$\mathcal{N}(1) = 1, \quad \mathcal{N}T_i = T_i^{-1}\mathcal{N}, \quad \mathcal{N}d'_- = d'_-\mathcal{N}, \quad \mathcal{N}d'_+ = d^*_+\mathcal{N}, \quad \mathcal{N}y_i = z'_i\mathcal{N}.$$

Then we have the following properties

1.

$$\mathcal{N}(y_\alpha) = t^{\sum(\alpha_i - 1)}N_\alpha,$$

2.

$$\mathcal{N}D_1 = -e_1\mathcal{N},$$

3. Let w_{inv} be the involution sending q, t, X to $q^{-1}, t^{-1}, -X$ respectively. Then

$$\mathcal{N}|_{V_0} = \nabla w_{inv}.$$

Therefore as by a simple computation

$$\begin{aligned} D_\alpha(q,t) &= (d'_-)^k(N_\alpha) = (d'_-)^k(\mathcal{N}(t^{|\alpha|-k}y_\alpha)) \\ &= \mathcal{N}(d'_-)^k(t^{|\alpha|-k}y_\alpha) \\ &= \mathcal{N}w_{\mathrm{inv}}C_\alpha(1) = \nabla C_\alpha(1). \end{aligned}$$

We see the conjecture is true.

5.0 Stable limit DAHA and DDPA

In this chapter we will describe the connection between the stable limit of the action of $\tilde{\mathcal{H}}_k^+$ on $\mathcal{P}_{\text{as}}^+$ and the action of $\mathbb{A}_{t,q}$ on V_*. Note that the elements in $\mathcal{P}(k)$ consist of elements of P which are symmetric in the alphabet $\mathbf{X}_k = x_{k+1} + x_{k+2} + \ldots$. Therefore each $\mathcal{P}(k)$ can be easily related to V_k. This leads to a relationship between $\mathcal{P}_{\text{as}}^+$ and V_*.

5.1 Induced $\mathbb{A}_{t,q}$ action

We will first have to construct an action of $\mathbb{A}_{t,q}$ on a complex of vector spaces extracted from $\mathcal{P}_{\text{as}}^+$.

5.1.1 Action on $\mathcal{P}_*$

We will use the standard representation of $\mathcal{H}^+$ to construct a quiver representation of $\mathbb{A}_{t,q}$. Let

$$\mathcal{P}_\bullet = (\mathcal{P}(k)^+)_{k\geq 0}.$$

For $k \geq 0$, recall that we denote by $\mathcal{H}^+(k)$ the subalgebra of $\mathcal{H}^+$ generated by T_i, X_i, and Y_i, $1 \leq i \leq k$. From Lemma 3.4.11 we know that each $\mathcal{P}(k)^+$ is stable under the action of $\mathcal{H}^+(k)$ through the standard representation of $\mathcal{H}^+$. For all k, the multiplication map $\mathcal{P}_k^+ \otimes \mathrm{Sym}[\mathbf{X}_k] \cong \mathcal{P}(k)^+$ is an algebra isomorphism.

The elements

$$\widetilde{\omega}_k^{-1}, \omega_k^{-1} \in \mathcal{H}_k^+$$

act on $\mathcal{P}_k^+$ via the standard representation of $\mathcal{H}_k^+$ (see Proposition 3.1.3). We extend their action to $\mathcal{P}(k)^+$ as $\mathrm{Sym}[\mathbf{X}_k]$-linear maps. Let

$$\iota(k) : \mathcal{P}(k)^+ \to \mathcal{P}(k+1)^+$$

be the canonical inclusion map. Denote

$$\partial_k = -\widetilde{\omega}_{k+1}^{-1}\iota(k) : \mathcal{P}(k)^+ \to \mathcal{P}(k+1)^+ \quad \text{and} \quad \partial_k^* = \omega_{k+1}^{-1}\iota(k) : \mathcal{P}(k)^+ \to \mathcal{P}(k+1)^+.$$

Recall that the Hall-Littlewood symmetric functions $P_\lambda(X,t)$ are a distinguished basis of the ring of symmetric functions $\mathrm{Sym}[X]$, indexed by partitions λ. There is a remarkable family of linear operators $\mathcal{B}_n$, $n \geq 0$, and $\mathcal{B}_\infty$ on $\mathrm{Sym}[X]$, defined as follows. $\mathcal{B}_\infty$ is the operator of left multiplication by the elementary symmetric function $e_1[X] = X$ and $\mathcal{B}_0$ is the operator defined by

$$\mathcal{B}_0 P_\mu(X, t^{-1}) = t^{\ell(\mu)} P_\mu(X, t^{-1}).$$

For $n \geq 0$, let $\mathcal{B}_{n+1} := [\mathcal{B}_\infty, \mathcal{B}_n]$.

The operator

$$\partial_k^- : \mathcal{P}(k)^+ \to \mathcal{P}(k-1)^+$$

is defined to be the $\mathcal{P}_{k-1}^+$-linear map which, on elements of the form $x_k^n F[\mathbf{X}_k]$ acts as

$$\partial_k^-(x_k^n F[\mathbf{X}_k]) = \mathcal{B}_n F[\mathbf{X}_{k-1}].$$

The operators $\mathcal{B}_n$, $n \geq 0$, are creation operators for the Hall-Littlewood symmetric functions. They are (modulo a change of variable) the vertex operators in [Jin91] (see also [Mac15, §III.5, Exp. 8]). They are particular cases of more general operators (as in [GHT99, BGSLX16b]) depending of both parameters q, t, which can be described more explicitly using plethystic substitution. In our case,

$$\mathcal{B}_n F[X] = (F[X - z^{-1}]\mathrm{Exp}[-(t-1)zX])\big|_{z^r},$$

where $\big|_{z^r}$ denotes the coefficient of z^r in the indicated expression. The expression for $\mathcal{B}_0$ implies the indicated formula for $\mathcal{B}_n$, $n \geq 1$ (see, e.g., [GHT99, Proposition 1.4]). $\mathcal{B}_0$ is the $q = 0$ specialization of the operator Δ' in [Hai99, (2.10)]. The fact that the Hall-Littlewood symmetric functions are eigenfunctions of this operator is proved in [Hai99, Corrolary 2.3]. This leads to the following compact expression for ∂_k^-. Let $f(x_1, \ldots, x_k) \in \mathcal{P}_k^+$ and $F[\mathbf{X}_k] \in \mathrm{Sym}[\mathbf{X}_k]$. Then,

$$\partial_k^- f(x_1, ..., x_k)F[\mathbf{X}_k] = \tau_k \mathfrak{c}_{x_k}(f(x_1, ..., x_k)F[\mathbf{X}_k - x_k]\mathrm{Exp}[-(t-1)x_k^{-1}\mathbf{X}_k]), \quad (5.1.1.1)$$

where τ_k denotes the alphabet shift $\mathbf{X}_k \mapsto \mathbf{X}_{k-1}$ (or $x_{i+1} \mapsto x_i$ for all $i \geq k$). This description makes it clear that, if $F[\mathbf{X}_{k-1}] \in \mathrm{Sym}[\mathbf{X}_{k-1}]$ then

$$\partial_k^- F[\mathbf{X}_{k-1}] = F[\mathbf{X}_{k-1}]. \quad (5.1.1.2)$$

By writing any $F[\mathbf{X}_k] \in \mathrm{Sym}[\mathbf{X}_k]$ as $F[\mathbf{X}_{k-1} - x_k]$ we see that the elements of $\mathcal{P}(k)^+$ can be written as finite sums of the form

$$\sum f_i(x_1, \ldots, x_{k-1})x_k^i G_i[\mathbf{X}_{k-1}]. \quad (5.1.1.3)$$

By (5.1.1.1), on such an expression, ∂_k^- acts as

$$\partial_k^- \sum_i f_i(x_1, \ldots, x_{k-1})x_k^i G_i[\mathbf{X}_{k-1}] = \sum_i f_i(x_1, \ldots, x_{k-1})G_i[\mathbf{X}_{k-1}]\partial_k^- x_k^i \quad (5.1.1.4)$$

To facilitate the comparison between the operators X_i, Y_i and $[\partial, \partial^-]$, $[\partial^*, \partial^-]$ we record the following formulas.

Lemma 5.1.1. *For any $k \geq 1$ we have*

$$\partial_{k-1}\partial_k^- - \partial_{k+1}^-\partial_k = (t-1)\widetilde{\omega}_k^{-1}. \tag{5.1.1.5}$$

Proof. The proof of this equality is identical to the one in the proof of [CM18, Lemma 5.4]; we include a brief explanation for the reader's convenience. Using the relations (4.2.1.1c), we have

$$\begin{aligned}\partial_{k-1}\partial_k^- - \partial_{k+1}^-\partial_k &= -T_1\cdots T_{k-1}X_k\partial_k^- + T_1T_2\cdots T_{k-1}\partial_{k+1}^-T_kX_{k+1}\\ &= T_1\cdots T_{k-1}X_k(-\partial_k^- + \partial_{k+1}^-T_k^{-1}).\end{aligned}$$

The operator $-\partial_k^- + \partial_{k+1}^- tT_k^{-1}$ acts on $\mathcal{P}(k)^+$ as scaling by $(t-1)$. By (5.1.1.4) this only needs to be for the action on x_k^n, $n \geq 0$. We have,

$$\begin{aligned}\partial_{k+1}^- tT_k^{-1}x_k^n - \partial_k^- x_k^n &= \partial_{k+1}^- x_{k+1}^n + (1-t)\sum_{i=1}^{n-1} x_k^i \partial_{k+1}^- x_{k+1}^{n-i} - \partial_k^- x_k^n\\ &= h_n[(1-t)\mathbf{X}_k] + (1-t)\sum_{i=1}^{n-1} x_k^i h_{n-i}[(1-t)\mathbf{X}_k] - h_n[(1-t)\mathbf{X}_{k-1}]\\ &= -(1-t)x_k^n.\end{aligned}$$

This proves our claim. $\square$

Lemma 5.1.2. *Let $k \geq 1$, $n \geq 0$, $f(x_1,\ldots,x_{k-1}) \in \mathcal{P}_{k-1}^+$, and $G[\mathbf{X}_{k-1}] \in \mathrm{Sym}[\mathbf{X}_{k-1}]$. We regard*

$$F = f(x_1,\ldots,x_{k-1})x_k^n G[\mathbf{X}_{k-1}]$$

as an element of $\mathcal{P}(k)^+$. Then,

$$(\partial_{k-1}^*\partial_k^- - \partial_{k+1}^-\partial_k^*)F = f(x_2,\ldots,x_k)G[\mathbf{X}_k + qx_1](h_n[(1-t)(\mathbf{X}_k + qx_1)] - h_n[(1-t)\mathbf{X}_k]). \tag{5.1.1.6}$$

Proof. Straightforward from (5.1.1.2) and the definition of ∂^*, ∂^-. $\square$

We then describe the $\mathbb{A}_{t,q}$-module structure on $\mathcal{P}_\bullet$.

Theorem 5.1.3. *The map that sends T_i, d_+, d_+^*, and d^- to T_i, ∂, ∂^*, and ∂^-, respectively, defines a $\mathbb{A}_{t,q}$-module structure on $\mathcal{P}_\bullet$. Under this action, the operators y_i, z_i act as X_i, Y_i.*

It is important to note that while the operators corresponding to arrows connecting different nodes are *local* (i.e. dependent on k), the operators T_i, X_i, Y_i that correspond to loops are *global* (i.e. independent of k, as they are restrictions of operators on $\mathcal{P}_{\mathrm{as}}^+$).

Proof. The fact that the operator

$$y_1 = \frac{1}{t^{k-1}(t-1)}[d_+, d_-]T_{k-1}\dots T_1,$$

acts on $\mathcal{P}(k)^+$ as X_1 follows from (5.1.1.5). Therefore, for all $i \geq 1$, the operator y_i acts as X_i.

Furthermore, from (5.1.1.3), (5.1.1.6), and Proposition 3.4.14 we obtain that

$$z_1 T_1 \cdots T_{k-1} = \frac{t^k}{1-t}[d_+^*, d_-]$$

acts on $\mathcal{P}(k)^+$ as $Y_1 T_1 \cdots T_{k-1}$. Therefore, z_1 acts on $\mathcal{P}(k)^+$ as Y_1 and, for all $i \geq 1$, the operator z_i acts as Y_i.

The verification of many of the relations in Definition 4.2.1.2 is virtually identical to the corresponding verification in [CM18]. We briefly indicate the main details.

The fact that (4.2.1.1a) holds is clear (also part of Theorem 3.4.16). For the first relation in (4.2.1.1c) see the proof of the corresponding relation in [CM18, Lemma 5.3]; the second set of relations in (4.2.1.1c) is clear from the definition of the maps involved. For (4.2.1.1b), recall the expressions for $\widetilde{\omega}_k^{-1}$. With this in mind, the first relation in (4.2.1.1b) follows from the equality

$$T_1 \cdots T_{k+1} T_1 \cdots T_k = T_2 \cdots T_{k+1} T_1 \cdots T_{k+1}, \tag{5.1.1.7}$$

which is a consequence of the braid relations. Indeed,

$$\begin{aligned} T_1 \partial_{k+1} \partial_k &= T_1 T_1 \cdots T_{k+1} T_1 \cdots T_k X_k X_{k+1} \\ &= T_1 T_2 \cdots T_{k+1} T_1 \cdots T_{k+1} X_k X_{k+1}, \end{aligned}$$

which on $\mathcal{P}(k)^+$ acts as $T_1 T_2 \cdots T_{k+1} T_1 \cdots T_k X_k X_{k+1} = \partial_{k+1} \partial_k$. The second set of relations in (4.2.1.1b) is again a straight consequence of the braid relations.

For the relations (4.2.1.1d), remark that (5.1.1.5), (4.2.1.1c), and the fact that ∂_k^- commutes with X_1 implies the first relation in (4.2.1.1d). The second relation in (4.2.1.1d) follows from (5.1.1.5) and (5.1.1.7).

The first relation on (4.2.1.2a) is a consequence of the fact that an element in the image of $\partial_{k+1}^* \partial_k^*$ is symmetric in x_1, x_2. The second relation in (4.2.1.2a) is essentially (3.1.2.2b).

For the first relation in (4.2.1.2b) can be seen (with the help of (4.2.1.1c)) to be equivalent to

$$Y_1 \partial_k^- = \partial_k^- Y_1. \tag{5.1.1.8}$$

The commutativity relation is not immediately clear from the definition of the operators involved. We proceed as in the proof of [CM18, Proposition 6.3]. More precisely,

$$
\begin{aligned}
&(t+1)(\partial^-[\partial^*,\partial^-]T_{k-1} - t[\partial^*,\partial^-]\partial^-)\\
&= (t+1)\partial^-\partial^*\partial^-(T_{k-1}+t) - t\partial^*(\partial^-)^2(T_{k-1}+t-T_{k-1}+1)\\
&\quad + (\partial^-)^2\partial^*(T_{k-1}+t-T_{k-1}+1)\\
&= ((t+1)\partial^-\partial^*\partial^- - t\partial^*(\partial^-)^2 + (\partial^-)^2\partial^*)(T_{k-1}+t)\\
&= (\partial^-[\partial^*,\partial^-] - t[\partial^*,\partial^-]\partial^-)(T_{k-1}+t).
\end{aligned}
$$

For the second equality we used the relations (4.2.1.1c) and (4.2.1.2a). The image of $T_{k-1}+t$ lies on the kernel of $T_{k-1}-1$. Therefore, it is enough to check that

$$\partial^-[\partial^*,\partial^-] = t[\partial^*,\partial^-]\partial^-$$

on $\mathcal{P}(k-1)^+ \subset \mathcal{P}(k)^+$. By (5.1.1.4), ∂^- acts as identity on $\mathcal{P}(k-1)^+ \subset \mathcal{P}(k)^+$. After examining the action of both sides of the relation on elements in $\mathcal{P}(k-1)^+ \subset \mathcal{P}(k)^+$ of the form (5.1.1.3) we see that it is enough to establish the equality for the action on the elements $x_{k-1}^n x_k^m + x_{k-1}^m x_k^n$, $n,m \geq 0$. The rest of the argument in [CM18, Proposition 6.3] applies to conclude the verification of the first relation in (4.2.1.2b).

For the second relation in (4.2.1.2b) we proceed as follows. By examining the action of both sides of the relation on elements of the form (5.1.1.3) we see that it is enough to establish the equality for the action on the elements x_k^n, $n \geq 0$. By direct computation,

$$[\partial^*,\partial^-]\partial^* x_k^n = h_n[(1-t)\mathbf{X}_{k+1} + (1-t)qx_1] - h_n[(1-t)\mathbf{X}_{k+1}]$$

and

$$T_1\partial^*[\partial^*,\partial^-]x_k^n = h_n[(1-t)\mathbf{X}_{k+1}+(1-t)qx_1+(1-t)qx_2] - T_1 h_n[(1-t)\mathbf{X}_{k+1}+(1-t)qx_1],$$

from which the claimed equality can be readily verified.

The second relation in (4.2.1.3) is precisely (3.1.2.2c). For the first relation in (4.2.1.3), it is enough to verify the case $i = 1$, that is

$$\widetilde{\omega}_{k+1}^{-1} Y_1 = Y_2 \widetilde{\omega}_{k+1}^{-1}.$$

Using the first expression for $\widetilde{\omega}_{k+1}^{-1}$, this reduces to (3.2.1.1d).

The last relation is proved as follows

$$\begin{aligned} qt^{k+1} X_1 \omega_{k+1}^{-1} &= t^{k+1} \omega_{k+1}^{-1} X_{k+1} \\ &= t^{k+1} \varpi_{k+1} X_{k+1} \\ &= Y_1 T_1 \cdots T_k X_{k+1} \\ &= Y_1 \widetilde{\omega}_{k+1}^{-1}. \end{aligned}$$

Therefore,

$$Y_1 \partial_k = -qt^{k+1} X_1 \partial_{k+1}^*,$$

as desired. □

5.1.2 Isomorphism

We conclude with the comparison of the representations of $\mathbb{A}_{t,q}$ in Proposition 4.2.1.8 and Theorem 5.1.3. Let us first define

$$\Phi_\bullet = (\Phi_k)_{k \geq 0} : \mathcal{P}_\bullet \to V_\bullet,$$

as follows

$$\Phi_k : \mathcal{P}(k)^+ \cong \mathcal{P}_k^+ \otimes \mathrm{Sym}[\mathbf{X}_k] \to V_k; \quad x_i \mapsto y_i, \quad 1 \leq i \leq k; \quad \mathbf{X}_k \mapsto \frac{X}{t-1}.$$

We note that each Φ_k is a $\mathbb{Q}(t,q)$-algebra isomorphism.

Theorem 5.1.4. $\Phi_\bullet$ *is an isomorphism of* $\mathbb{A}_{t,q}$*-representations.*

Proof. We clearly have $\Phi_k T_i = T_i \Phi_k$ for all $1 \leq i \leq k-1$. We will check that the action of d_+ satisfies the following equality

$$d_{+,k}\Phi_k = \Phi_{k+1}d_{+,k}.$$

Therefore we only need to prove the correspondence for d_+, d_+^*, d_-.

Let $F = x_1^{t_1}...x_k^{t_k} f[\mathbf{X}_k] \in \mathcal{P}(k)^+$. We have,

$$\begin{aligned}
d_{+,k}\Phi_k F &= d_{+,k} y_1^{t_1}...y_k^{t_k} f[\frac{X}{t-1}] \\
&= -T_1...T_k(y_1^{t_1}...y_k^{t_k} y_{k+1} f[\frac{X+(t-1)y_{k+1}}{t-1}]) \\
&= -T_1...T_k(y_1^{t_1}...y_k^{t_k} y_{k+1} f[\frac{X}{t-1}+y_{k+1}]) \\
&= -\Phi_{k+1}(T_1...T_k x_1^{t_1}...x_k^{t_k} x_{k+1} f[\mathbf{X}_{k+1}+x_{k+1}]) \\
&= -\Phi_{k+1}(T_1...T_k X_{k+1}(x_1^{t_1}...x_k^{t_k} f[\mathbf{X}_k])) \\
&= \Phi_{k+1} d_{+,k} F.
\end{aligned}$$

Similar computations show that $d_{+,k}^*\Phi_k = \Phi_{k+1}d_{+,k}^*$ and $d_{-,k}\Phi_k = \Phi_{k-1}d_{-,k}$. □

5.2 Operators on $\mathcal{P}(0)^+$

We will then describe the remarkable operators on $\mathcal{P}(0)^+$.

5.2.1 Remarkable operators $\tilde{\nabla}$ and $\tilde{D}_0$

Proposition 5.2.1. *Let*

$$\tilde{\nabla} = \Phi_0^{-1}\nabla\Phi_0 : \mathcal{P}(0) \to \mathcal{P}(0).$$

Then we have

$$\tilde{\nabla}P_\lambda[X;q,t^{-1}] = t^{n(\lambda)}q^{n(\lambda')}P_\lambda[X;q,t^{-1}].$$

Proof. Note that we have the identity

$$\begin{aligned} q^{n(\lambda)}J_\lambda[\frac{X}{1-q^{-1}};t,q^{-1}] &= \tilde{H}_\lambda[X;t,q] \\ &= \tilde{H}_{\lambda'}[X;q,t] \\ &= t^{n(\lambda')}J_{\lambda'}[\frac{X}{1-t^{-1}};q,t^{-1}]. \end{aligned}$$

Hence we have

$$\tilde{H}_\lambda[(t-1)X;t,q] = t^{n(\lambda')}J_{\lambda'}[tX;q,t^{-1}].$$

Therefore

$$\begin{aligned} \tilde{\nabla}J_{\lambda'}[tX;q,t^{-1}] &= \Phi_0^{-1}\nabla t^{-n(\lambda')}\tilde{H}_\lambda[X;t,q] \\ &= \Phi_0^{-1}t^{-n(\lambda')}t^{n(\lambda')}q^{n(\lambda)}\tilde{H}_\lambda[X;t,q] \\ &= t^{n(\lambda')}q^{n(\lambda)}J_{\lambda'}[tX;q,t^{-1}]. \end{aligned}$$

Now as a J-polynomial is a homogeneous symmetric polynomial coming from a scaling of the P-polynomial with the scalar only depending on the partition λ, the formula above leads to the following

$$\tilde{\nabla}P_\lambda[X;q,t^{-1}] = t^{n(\lambda)}q^{n(\lambda')}P_\lambda[X;q,t^{-1}].$$

□

With a similar argument, we have

Proposition 5.2.2. *Let*

$$\tilde{D}_0 = \Phi_0^{-1} D_0 \Phi_0 : \mathcal{P}(0) \to \mathcal{P}(0).$$

Then

$$\tilde{D}_0 P_\lambda[X; q, t^{-1}] = -D_\lambda(t,q) P_\lambda[X; q, t^{-1}],$$

where

$$D_\lambda(t,q) = -1 + (1-t)(1-q) \sum_{c \in \lambda} t^{coarm_\lambda(c)} q^{coleg_\lambda(c)}.$$

Meanwhile, we may also define the operator $\tilde{D}_k$ plethystically as follows.

Proposition 5.2.3. *We have*

$$\tilde{D}_k F[X] = \mathfrak{c}_k F[X - \frac{(1-q)}{z}] Exp[(1-t)zX].$$

Proof.

$$\begin{aligned} \tilde{D}_k F[X] &= \Phi_0^{-1} D_k F[\frac{X}{t-1}] \\ &= \Phi_0^{-1} \mathfrak{c}_k F[\frac{X + (t-1)(q-1)/z}{t-1}] Exp[-zX] \\ &= \mathfrak{c}_k F[X + \frac{(1-q)}{z}] Exp[-(t-1)zX]. \end{aligned}$$

□

As an analogy of 2.3.2.1, we could also have the following properties for $\tilde{D}_k$.

Proposition 5.2.4.

$$\begin{gathered} \tilde{D}_k e_1 - e_1 \tilde{D}_k = (1-q)(1-t)\tilde{D}_{k+1}, \ \textit{for all } k \geq 0, \\ \tilde{D}_0 e_1 - e_1 \tilde{D}_0 = -(1-q)(1-t)\tilde{\nabla} e_1 \tilde{\nabla}^{-1}. \end{gathered} \tag{5.2.1.1}$$

We would expect $\tilde{\nabla}$ can be similarly realized by an automorphism. Therefore the shuffle theorem can be analogously translated.

6.0 Bibliography

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