Interface theory and Percolation

by

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Acknowledgments

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Declarations

This thesis is based on the articles Georgakopoulos and Panagiotis [2018, 2019a,b, 2020]; Haslegrave and Panagiotis [2020]. The first four articles are joint work with my supervisor Agelos Georgakopoulos, and the last article is joint work with John Haslegrave, a Research Fellow at the University of Warwick. I confirm that the thesis has not been submitted for a degree at another university.

Abstract

This thesis is mainly concerned with percolation on general infinite graphs, as well as the approximation of conformal maps by square tilings, which are defined using electrical networks.

The first chapter is concerned with the smoothness of the percolation density on various graphs. In particular, we prove that for Bernoulli percolation on $\mathbb{Z}^d$, $d \geq 2$, the percolation density is an analytic function of the parameter in the supercritical interval $(p_c(\mathbb{Z}^d), 1]$. This answers a question of Kesten [1981]. The analogous result is also proved for the Boolean model of continuum percolation in $\mathbb{R}^2$, answering a question of Last et al. [2017]. In order to prove these results, we introduce the notion of interfaces, which is studied extensively in the current thesis. For dimensions $d \geq 3$, we use renormalisation tecnhiques. Furthermore, we prove that the susceptibility is analytic in the subcritical interval for all transitive short- or long-range models, and that $p_c < 1/2$ for bond percolation on certain families of triangulations for which Benjamini & Schramm conjectured that $p_c \leq 1/2$ for site percolation. For the latter result, we use the well-known circle packing theorem of He and Schramm [1995], a discrete analogue of the Riemann mapping theorem.

In Chapter 2, we continue the study of interfaces, and in particular, we consider the exponential growth rate b_r of the number of interfaces of a given size as a function of their surface-to-volume ratio r. We prove that the values of the percolation parameter p for which the interface size distribution has an exponential tail are uniquely determined by b_r by comparison with a dimension-independent function $f(r) := \frac{(1+r)^{1+r}}{r^r}$. We also point out a formula for translating any upper bound on the percolation threshold of a lattice G into a lower bound on the exponential growth

rate of lattice animals $a(G)$ and vice-versa. We exploit this in both directions. We obtain the rigorous lower bound $p_c(\mathbb{Z}^3) > 0.2522$ for 3-dimensional site percolation. We also improve on the best known asymptotic lower and upper bounds on $a(\mathbb{Z}^d)$ as $d \to \infty$.

We also prove that the rate of the exponential decay of the cluster size distribution, defined as $c(p) := \lim_{n\to\infty} (\mathbb{P}_p(|C_o| = n))^{1/n}$, is a continuous function of p. The proof makes use of the Arzelà-Ascoli theorem but otherwise boils down to elementary calculations. The analogous statement is also proved for the interface size distribution. For this we first establish that the rate of exponential decay is well-defined.

In Chapter 3, we use interfaces to obtain upper bounds for the site percolation threshold of plane graphs with given minimum degree conditions. The results of this chapter are inspired by well-known conjectures of Benjamini and Schramm [1996b] for percolation on general graphs. We prove a conjecture by Benjamini and Schramm [1996b] stating that plane graphs of minimum degree at least 7 have site percolation threshold bounded away from 1/2. We also make progress on a conjecture of Angel et al. [2018] that the critical probability is at most 1/2 for plane triangulations of minimum degree 6. In the process, we prove tight new isoperimetric bounds for certain classes of hyperbolic graphs. This establishes the vertex isoperimetric constant for all triangular and square hyperbolic lattices, answering a question of [Lyons and Peres, 2016, Question 6.20].

Another topic of this thesis is the discrete approximation of conformal maps using another discrete analogue of the Riemann mapping theorem, namely the square tilings of Brooks et al. [1940]. This result is analogous to a well-known theorem of Rodin & Sullivan, previously conjectured by Thurston, which states that the circle packing of the intersection of a lattice with a simply connected planar domain Ω into the unit disc D converges to a Riemann map from Ω to D when the mesh size converges to 0. As a result, we obtain a new algorithm that allows us to numerically compute the Riemann map from any Jordan domain onto a square.

Preliminaries

We recall some standard definitions of graph theory and percolation theory used throughout this thesis in order to fix our notation. For more details, the reader can consult e.g. Grimmett [1999]; Lyons and Peres [2016]. For a higher-level overview of percolation theory, I recommend the recent survey by Duminil-Copin [2017].

Graph theory

Let $G = (V, E)$ be a graph. An *induced* subgraph H of G is a subgraph that contains all edges xy of G with $x, y \in V(H)$. Note that H is uniquely determined by its vertex set. The subgraph of G *spanned* by a vertex set $S \subseteq V(G)$ is the induced subgraph of G with vertex set S. The degree a of a vertex v is the number of edges incident to v. We say that G is *locally finite* if all vertex degrees are finite.

The vertex set of a graph G will be denoted by $V(G)$, and its edge set by $E(G)$. A graph G is *(vertex-)transitive* if for every $x, y \in V(G)$ there is an automorphism π of G mapping x to y, where an *automorphism* is a bijection π of $V(G)$ that preserves edges and non-edges. We say that G is *quasi-transitive* if there is a finite set $U \subset V(G)$ such that for every $x \in V(G)$, there exist $y \in U$ and an automorphism of G that maps x to y.

A *planar graph* G is a graph that can be embedded in the plane $\mathbb{R}^2$, i.e. it can be drawn in such a way that no edges cross each other. Such an embedding is called a *planar embedding* of the graph. A *plane graph* is a (planar) graph endowed with a fixed planar embedding.

A plane graph divides the plane into regions called *faces*, i.e. a face is the closure of a component of $\mathbb{R}^2 \setminus G$. Using the faces of a plane graph G we define its *dual graph* G^* as follows. The vertices of G^* are the faces of G, and we connect two vertices of G^* with an edge whenever the corresponding faces of G share an edge. Thus there is a bijection $e \mapsto e^*$ from $E(G)$ to $E(G^*)$.

Percolation

Let $G = (V, E)$ be a locally finite countably infinite graph, and let $\Omega := \{0, 1\}^E$ be the set of *percolation instances* on G. We say that an edge e is *vacant* or *closed* (respectively, *occupied* or *open*) in a percolation instance $\omega \in \Omega$ if $\omega(e) = 0$ (respectively $\omega(e) = 1$).

By *bond percolation* on G with parameter $p \in [0, 1]$ we mean the random subgraph of G obtained by keeping each edge with probability p and deleting it with probability $1 - p$, with these decisions being independent of each other.

More formally, we endow Ω with the σ-algebra $\mathcal{F}$ generated by the cylinder sets $C_e := \{\omega \in \Omega, \omega(e) = \epsilon\}_{e \in E, \epsilon \in \{0,1\}}$, and the probability measure defined as the product measure $\mathbb{P}_p := \Pi_{e \in E} \mu_e$, where $p \in [0, 1]$ is our *percolation parameter* and μ_e is the Bernoulli measure on $\{0, 1\}$ determined by $\mu_e(1) = p$.

The *percolation threshold* $p_c(G)$ is defined by

$$p_c(G) := \sup\{p \mid \mathbb{P}_p(|C_o| = \infty) = 0\},$$

where the *cluster* C_o of $o \in V$ is the component of o in the subgraph of G spanned by the occupied edges. It is easy to see that $p_c(G)$ does not depend on the choice of o.

To define *site percolation* we repeat the same definitions, except that we now let $\Omega := \{0, 1\}^V$, and let C_o be the component of o in the subgraph of G induced by the occupied vertices. The site percolation threshold is denoted by $\dot{p}_c$.

The graph G is a priori arbitrary. Some results will need assumptions on G such as vertex-transitivity or planarity, but these will be explicitly stated as needed.

Chapter 1

Analytic functions in Bernoulli Percolation

1.1 Introduction

In this chapter, we prove that several functions studied in percolation theory are analytic functions of the percolation parameter. We consider Bernoulli bond percolation on a variety of graphs, as well as general long-range models (defined in Section 1.2.2) preserved by a transitive group action. The *susceptibility* χ of a percolation model is the expected number of vertices in the cluster of a fixed vertex o. The *percolation density* θ is the probability that the cluster C_o of o is infinite. Let

$$p_{\mathbb{C}} := \inf\{p \leq 1 \mid \theta(p) \text{ is analytic in } (p, 1]\} \tag{1.1}$$

The main results of this chapter are

(i) For every quasi-transitive graph, and every quasi-transitive (1-parameter) long-range model, the susceptibility $\chi_o(p)$ is analytic in the subcritical interval $[0, p_c)$.

(ii) For every $d \geq 2$, the (Bernoulli, bond) percolation density $\theta(p)$ on $\mathbb{Z}^d$ is analytic in the supercritical interval $(p_c, 1]$ (in other words, $p_{\mathbb{C}} = p_c$). The corresponding result is proved for quasi-transitive lattices in $\mathbb{R}^2$ and continuum percolation in $\mathbb{R}^2$ as well.

(iii) For certain families of triangulations for which Benjamini and Schramm [1996b] and Benjamini [2015] conjectured that $p_c^{site} \leq 1/2$, we prove $p_c^{bond} \leq p_{\mathbb{C}} < 1/2$.

Perhaps the first occurrence of questions of smoothness in percolation theory dates back to the work of Sykes and Essam [1964]. Trying to compute the value of p_c for bond percolation on the square lattice $\mathbb{Z}^2$, Sykes & Essam introduced the fundamental idea of duality in planar percolation and obtained that the free energy (aka. mean number of clusters per vertex) $\kappa(p) := \mathbb{E}_p(|C_o|^{-1})$ satisfies the functional equation $\kappa(p) = \kappa(1-p) + \phi(p)$ for some polynomial $\phi(p)$. Under the assumption of smoothness of κ for every value of the parameter p other than p_c, at which it is conjectured that κ has a singularity, they obtained that $p_c = 1/2$ due to the symmetry of the functional equation around $1/2$. It is worth noting that extending this argument to bond percolation on other dual pairs (G, G^*) of planar quasi-transitive lattices (see Section 1.4 for the relevant definitions) is essentially straightforward, and under the same assumption one obtains the relation $p_c(G) + p_c(G^*) = 1$. In fact, the paper of Sykes & Essam contains the argument not only for the square lattice $\mathbb{Z}^2$ but also for the dual pair $(\mathbb{T}, \mathbb{H})$, where $\mathbb{T}$ denotes the triangular lattice and $\mathbb{H}$ the hexagonal lattice, and for site percolation on the self-dual case of $\mathbb{T}$. Their work generated considerable interest, and a lot of the early work in percolation was focused on the smoothness of functions like κ and χ that describe the macroscopic behaviour of its clusters. Kunz and Souillard [1978] proved that κ is analytic for small enough p. Grimmett [1981] proved that κ is C^∞ for $p \neq p_c$ in the case $d = 2$. A breakthrough was made by Kesten [1981], who proved that κ and χ are analytic on $[0, p_c)$ for all $d \geq 2$. Despite all the efforts, the argument of Sykes & Essam has never been made rigorous, and all proofs of the fact that $p_c(\mathbb{Z}^2) = 1/2$ use different methods, see e.g. Kesten [1980]; Bollobás and Riordan [2006]. The element still missing from making mathematically complete their argument is proving loss of regularity at p_c.

Except for the special case of κ on $\mathbb{Z}^2$ (and other planar lattices), smoothness results are harder to obtain in the supercritical interval $(p_c, 1]$, partly because the cluster size distribution $P_n := \mathbb{P}_p(|C_o| = n)$ has an exponential tail below p_c Menshikov [1986]; Aizenman and Barsky [1987] but not above p_c Aizenman et al. [1980]. Still, it is known that κ and θ are infinitely differentiable for $p \in (p_c, 1]$ (see Chayes et al. [1987] or [Grimmett, 1999, §8.7] and references therein). It is a well-known open question, dating back to Kesten [1981] at least, and appearing in several textbooks ([Kesten, 1982, Problem 6],Grimmett [1997, 1999]), whether θ is analytic for $p \in (p_c, 1]$ for percolation on the hypercubic lattice $\mathbb{Z}^d, d \geq 2$. In this chapter, we answer this question in the affirmative. We also answer the corresponding question, asked by Last et al. [2017], for the Boolean model in $\mathbb{R}^2$ (Theorem 1.7.1).

Part of the interest for this question comes form Griffiths [1969] discovery of models, constructed by applying the Ising model on 2-dimensional percolation clusters, in which the free energy is infinitely differentiable but not analytic. This phenomenon is since called a *Griffiths singularity*, see van Enter [2007] for an overview and further references.

The study of the analytical properties of the free energy is a common theme in several models of Statistical Mechanics. Perhaps the most famous such example is Onsager's exact calculation of the free energy of the square-lattice Ising model Onsager [1944]. A corollary of this calculation is the computation of the critical temperature, as well as the analyticity of the free energy for all temperatures other than the critical one. See also Kager et al. [2013] for an alternative proof of the latter result. The analytical properties of the free energy have also been studied for the q-Potts model, which generalizes the Ising model. For this model, the analyticity of the free energy has been proved for $d = 2$ and all supercritical temperatures when q is large enough van Enter et al. [1997].

Before our result, partial progress on the analyticity of the percolation density had been made by Braga et al. [2004, 2002], who showed that θ is analytic for p close enough to 1. Shortly after our paper Georgakopoulos and Panagiotis [2018] was released, Hermon and Hutchcroft [2019] proved that θ is analytic above p_c for every non-amenable transitive graph, by establishing that the cluster size distribution P_n has an exponential tail in the whole supercritical interval.

Kesten's method for the analyticity of χ (or κ) Kesten [1981] (see also [Grimmett, 1999, §6.4]) involves extending p and χ to the complex plane, and applying the standard complex analytic machinery of Weierstrass to the series $\chi(p) = \sum_{n\in\mathbb{N}} nP_n(p)$. This uses the fact that $P_n(p)$ can be expressed as a polynomial by considering all possible clusters of size n, and can hence be extended to $\mathbb{C}$. To show that this series converges to an analytic function $\chi(z)$, one needs upper bounds for $|P_n(z)|$ inside appropriate domains in order to apply the Weierstrass M-test. These bounds are obtained by combining the well-known fact due to Menshikov [1986]; Aizenman and Barsky [1987] that $P_n(z)$ decays exponentially in n for real z, with elementary complex-analytic calculations. Kesten's calculations involved the numbers of certain 'lattice animals', but we observe (Theorem 1.3.12) that this is not necessary and his proof can be simplified by making a direct comparison between the values of $P_n(z)$ for $z \notin [0, 1]$ and for $z \in [0, 1]$. An immediate benefit of this simplification is that the proof extends beyond $\mathbb{Z}^d$, to bond and site percolation on any quasi-transitive graph. The only ingredients needed are the appropriate exponential decay statement and elementary complex analysis. This summarises the proof of (i), which is given in detail in Section 1.3.4.

The technique we just sketched is used in our results (ii)–(iii) as well, but additional ingredients are needed. For (ii), we can write $\theta(p) = 1 - \sum_n P_n(p)$ by the definitions, but as P_n decays slower than exponentially for $p > p_c$ Kunz and Souillard [1978]; Grimmett [1999], the above machinery cannot be applied to this series. Therefore, instead of working with the size of C_o, we work with the 'area' of its boundary. To make this more precise, consider at first the 2-dimensional case and define the *outer interface* of the cluster C_o to be the pair $(\partial_{int}C_o, \partial_{ext}C_o)$, where $\partial_{int}C_o$ denotes the set of edges of C_o bounding its unbounded face, and $\partial_{ext}C_o$ denotes the set of vacant edges in the unbounded face incident with $\partial_{int}C_o$ (Figure 1.1). See Section 1.4.2 for a more formal definition. We say that such a pair of edge sets $I = (\partial_{int}C_o, \partial_{ext}C_o)$ *occurs* in some percolation instance if it is the outer interface of some cluster, in which case all edges in $\partial_{int}C_o$ are occupied and all edges in $\partial_{ext}C_o$ are vacant. For any plausible such I, the probability $P_I(p) := \mathbb{P}_p(I \text{ occurs})$ is just $p^{|\partial_{int}C_o|}(1-p)^{|\partial_{ext}C_o|}$ by the definitions, which is a polynomial we can extend to $\mathbb{C}$ hoping to apply our machinery. Moreover, these P_I exhibit the kind of exponential decay we need: $\partial_{ext}C_o$ gives rise to a connected subgraph of the dual lattice, and we can combine a well-known coupling between supercritical bond percolation on a lattice and subcritical bond percolation on its dual (see Theorem 1.4.2) with the aforementioned exponential decay of P_n.

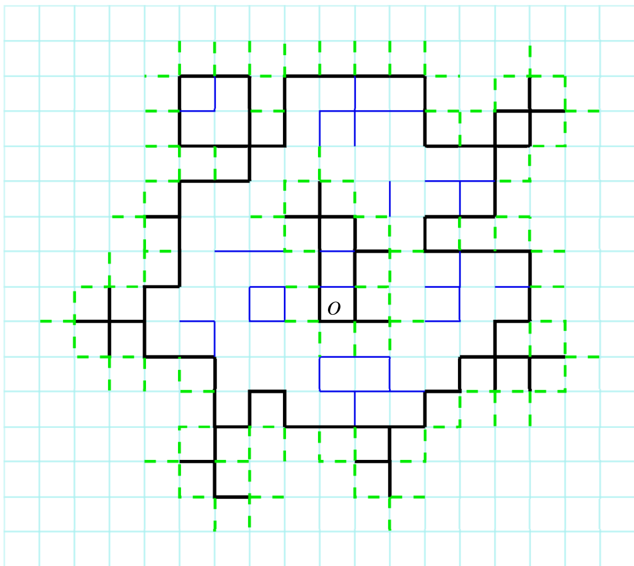

Figure 1.1: An example of two outer interfaces of percolation clusters, one nested inside the other. We depict $\partial_{int}C_o$ with bold lines and $\partial_{ext}C_o$ with dashed lines. The rest of the clusters is depicted in plain lines (blue if colour is shown).

Still, further challenges arise when trying to express θ in terms of the functions P_I because knowing that a certain outer interface I occurs does not imply that it is part of the cluster C_o: there could be other outer interfaces nested inside I, as exemplified in Figure 1.1. We overcome this difficulty using the Inclusion-Exclusion Principle, to express θ as

$$\theta(p) = 1 - \sum_{I \in \mathcal{MS}} (-1)^{c(I)+1} P_I, \tag{1.2}$$

where $\mathcal{MS}$ is the set of finite disjoint nonempty unions of outer interfaces, and $c(I)$ counts the number of outer interfaces in I. The problem now becomes whether the probability for such an $I \in \mathcal{MS}$ with n edges in total decays exponentially in n. All we know so far is that the probability to have an outer interface containing a fixed vertex x decays exponentially, which seems to be of little use given that there are many ways to partition n into smaller integers $n_1, \ldots n_k$, and construct an $I \in \mathcal{MS}$ out of k outer interfaces of lengths n_i, each rooted at one of many candidate vertices x_i. But there is a way to bring all these possibilities under control, and establish the desired exponential decay, by a certain combination of the following ingredients:
a) the Hardy–Ramanujan formula (Section 1.2.5), implying that the number of partitions of an integer n grows sub-exponentially;
b) some combinatorial arguments that restrict the possible vertices x_i at which the outer interfaces meet the horizontal axis, and
c) using the BK inequality (Theorem 1.2.2) to argue that for each choice of a partition of n, and vertices $x_1, \ldots x_k$, the probability of occurrence of an $I \in \mathcal{MS}$ complying with this data decays as fast as if we had a single outer interface of size n (which we already know to decay exponentially). This summarises the proof of (ii), which is given in detail in Section 1.4.

We remark that formula (1.2) can be thought of as a refinement of the well-known Peierls argument (see e.g. [Grimmett, 1999, p. 16]), where instead of an inequality we now have an equality. The price to pay is that the structures arising — of the form $(\partial_{int} C_o, \partial_{ext} C_o)$ instead of just $\partial_{ext} C_o$— are harder to enumerate, and the benefit is that the events we consider are mutually exclusive, hence the equality. We found this technique very convenient and it will be useful in the following chapters as well.

Our notion of interfaces can be generalised to higher dimensions in such a way that a unique interface is associated with any cluster. A slight modification of the above method still yields the analyticity of θ for the values of p close to 1, but not in the whole supercritical interval. The main obstacle is that for values of p in the interval $(p_c, 1 - p_c)$, the distribution of the size of the interface of C_o has only a stretched exponential tail, which follows from the work of Kesten and Zhang [1990].

As we will prove in Chapter 2, this behaviour holds for $p = 1 - p_c$ as well.

In the same paper, Kesten and Zhang introduced some variants of the standard boundary of C_o that are obtained by dividing the lattice $\mathbb{Z}^d$ into large boxes, and proved that these variants satisfy the desired exponential tail on the whole supercritical interval.[1] It is natural to try to apply our method to those variants, however, their occurrence does not prevent the origin from being connected to infinity. Instead, we expand these variants into larger objects that we call *separating components*. In Section 1.5 (Lemma 1.5.4) we prove that whenever a separating component S occurs, we can find inside S and its boundary $\partial_{\boxtimes} S$ an edge cut $\partial^b \mathcal{S}_o$ separating the origin from infinity. Conversely, some separating component occurs whenever C_o is finite (Lemma 1.5.2). Thus we can express θ in terms of the occurrence of separating components (see (1.20) in Section 1.5.3). In contrast to the behaviour of the boundary of C_o which has only a stretched exponential tail on the interval $(p_c, 1-p_c]$, $\partial^b \mathcal{S}_o$ has an exponential tail in the whole supercritical interval. We plug this exponential decay into a general tool (Corollary (1.3.15)), which rests on an application of the Weierstrass M-test to polynomials of the form $p^m(1-p)^n$, to obtain the analyticity of θ above p_c in Section 1.5.4. In Section 1.6 we use similar arguments to prove the analyticity of the k-point function τ and its truncation τ^f, as well as of χ^f and κ.

Typically, $\partial^b \mathcal{S}_o$ has size of smaller magnitude than the boundary of C_o, and it is obtained from the latter by 'smoothening' some of its parts with 'fractal' structure. As a corollary, we re-obtain, in Section 1.5.5, a result of Pete [2008] about the exponential decay of the probability that C_o is finite but sends a lot of closed edges to the infinite component.

Most of this chapter is concerned with analyticity results, but some of the methods developed can be applied to provide bounds on p_c as well. We display this in Section 1.8, where we prove that $p_c^{bond} < 1/2$ for certain families of triangulations for which Benjamini and Schramm [1996b], Benjamini [2015] and Angel et al. [2018] conjectured that $p_c^{site} \leq 1/2$ ((iii)).

1.2 Definitions and preliminaries

1.2.1 Cut sets and boundaries

Let $G = (V, E)$ be an infinite, locally finite and connected graph. For a finite connected subgraph H of G, we will define several notions of 'boundary'. We define

[1] The threshold $p_c(H^d)$ in Kesten's and Zhang's original formulation was proved later to coincide with $p_c(\mathbb{Z}^d)$ by Grimmett and Marstrand [1990].

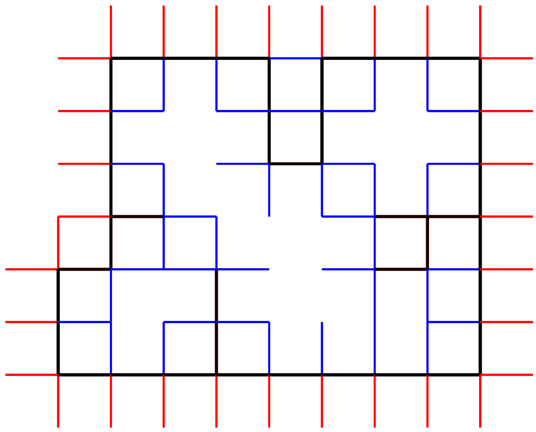

Figure 1.2: A connected sugraph H of $G = \mathbb{Z}^2$ shown in black. Among the edges in $G \setminus H$ that are incident to H (shown in blue or red), only the red ones belong to the corresponding minimal edge cut.

the *vertex boundary* $\partial^V H$ of H as the set of vertices in $V \setminus V(H)$ that have a neighbour in H. The *edge boundary* $\partial^E H$ is the set of edges in $E \setminus E(H)$ that are incident to H. The *internal boundary* ∂H of H is the set of vertices of H that are incident with an infinite component of $G \setminus H$ (recall that G is an infinite graph here).

Given a vertex x of G, we say that a set S of edges of G is a *minimal edge cut* separating x from infinity if x belongs to a finite component of $G \setminus S$, and S is minimal with respect to the inclusion relation. Analogously, we say that a set S of vertices of G is a *minimal vertex cut* separating x from infinity if x belongs to a finite component of $G \setminus S$, and S is minimal with respect to the inclusion relation. Note that any vertex x has several minimal edge (vertex) cuts. When working with a specific finite connected subgraph H of G, it will be useful to fix some minimal edge (vertex) cut. We define the minimal edge cut of H as the unique minimal edge cut, each edge of which has one end-vertex in H and one in an infinite component of $G \setminus H$. The minimal vertex cut of H is the unique minimal vertex cut, each vertex of which belongs to an infinite component of $G \setminus H$ and has a neighbour in H.

1.2.2 Long-range models

Long-range percolation is a generalisation of Bernoulli bond percolation where different edges become occupied with different probabilities, and each vertex can have infinitely many incident edges that can become occupied. In fact, the graph is often taken to be the complete graph on countably many vertices, and so its edges play a rather trivial role. Therefore, it is simpler to define our model with a set rather than a graph as follows.

Let $G = (V, E)$ be a countably infinite graph. We will typically write xy instead of $\{x, y\}$ to denote an element of E. Let $\mu : E \to \mathbb{R}_{\geq 0}$ be a function satisfying $\sum_{y \in V} \mu(xy) = 1$ for every $x \in V$ (in some occasions we allow more general μ, satisfying just $\sum_{y \in V} \mu(xy) < \infty$). The data V, μ define a random graph on V similarly to the previous definition, except that we now make each edge xy vacant with probability $e^{-\mu(xy)t}$, with our parameter t now ranging in $[0, \infty)$. The corresponding probability measure on $\Omega = \{0, 1\}^E$ is denoted by $\mathbb{P}_t$. (We like thinking of t as time, with each edge xy becoming occupied if vacant at a tick of a Poisson clock with rate $\mu(xy)$).

Analogously to p_c, one defines

$$t_c = t_c(V, \mu) := \sup\{t \mid \mathbb{P}_t(|C_o| = \infty) = 0\},$$

which again does not depend on the choice of $o \in V$.

We say that such a percolation model, defined by V and μ, is *transitive* if there is a group acting transitively on V that preserves μ. In other words, if for every $x, y \in V$ there is a bijection $\pi : V \to V$ mapping x to y and preserving edges and non-edges ($\pi(x)\pi(y)$ is an edge if and only if xy is an edge), such that $\mu(\pi(z)\pi(w)) = \mu(zw)$ for every edge zw.

Long-range percolation is a less standard topic that is not typically found in textbooks, and the term often refers to the special case where the group acting transitively is $\mathbb{Z}$, where it was used in order to come up with a model in which θ is discontinuous at t_c Aizenman and Newman [1986]. In the generality we work with it has been considered in e.g. Aizenman and Newman [1984]; Georgakopoulos and Haslegrave [2017].

1.2.3 Exponential tail of the subcritical cluster size distribution: the exponential decay property

An important fact that will be used throughout the chapter whenever we want to show the convergence of a series is the following exponential decay of the cluster size distribution $\mathbb{P}(|C_o| = n)$ (or equivalently, of $\mathbb{P}(|C_o| \geq n)$) in the subcritical regime.[2] For every vertex o, we define $\chi(p) = \chi_o(p) := \mathbb{E}_p(|C_o|)$, and we let $X(p) =$

[2] Some bibliographical remarks about Theorem 1.2.1: Kesten [1981] proved exponential decay when $\chi < \infty$ for lattices in $\mathbb{R}^d$, and Aizenman and Newman [1984] extended it to all models we are interested in. Menshikov [1986]; Aizenman and Barsky [1987] proved independently that $\chi < \infty$ below p_c on $\mathbb{Z}^d$. Antunović and Veselić [2008] extended this to all quasi-transitive models. Duminil-Copin and Tassion [2016] gave a shorter proof that $\chi < \infty$ below p_c (or β_c) for all independent,

$\sup_{o\in V} \chi_o(p)$.

Theorem 1.2.1 ([Aizenman and Newman, 1984, Proposition 5.1],Menshikov [1986]; Aizenman and Barsky [1987]; Antunović and Veselić [2008])**.** *For every bond, site, or long-range model, if $X(p) < \infty$, then*

$$\mathbb{P}_p(|C_o| \geq n) = O((e/n)^{1/2} e^{-n/(2X(p))^2}).$$

Moreover, for every quasi-transitive bond, site, or long-range model, if $p < p_c$, then $X(p) < \infty$.

We will say that a bond, site, or long-range model model satisfies the *exponential decay property* if for every $p < p_c$ and $o \in V$, there is a constant $c = c(p, o) > 0$ such that $\mathbb{P}_p(|C(o)| \geq n) \leq e^{-cn}$.

1.2.4 The BK inequality

We define a partial order on our space $\Omega = \{0, 1\}^{E(G)}$ of percolation instances as follows. For two instances ω and ω' we write $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for every $e \in E$.

A random variable X is called *increasing* if whenever $\omega \leq \omega'$, then $X(\omega) \leq X(\omega')$. An event A is called *increasing* if its indicator function is increasing. For instance, the event $\{|C_o| \geq m\}$ is increasing, where C_o, as usual, denotes the cluster of o.

For every $\omega \in \Omega$ and a subset $S \subset E$ we write

$$[\omega]_S = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for every } e \in S\}.$$

Let A and B be two events depending on a finite set of edges F. Then the disjoint occurrence of A and B is defined as

$$A \circ B = \{\omega \in \Omega : \text{ there is } S \subset F \text{ with } [\omega]_S \subset A \text{ and } [\omega]_{F\setminus S} \subset B\}.$$

Theorem 1.2.2. *(BK inequality)van den Berg and Kesten [1985]; Grimmett [1999] Let F be a finite set and $\omega = \{0, 1\}^F$. For all increasing events A and B on Ω we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

We remark that the $\circ$ operation is in general non-associative. In other words, for generic events $A_1, A_2, \ldots, A_m$, the notation $A_1 \circ A_2 \circ \ldots \circ A_m$ is ambiguous

transitive bond and site models.

without any reference to the order at which the operation is performed. However, if we assume that the events $A_1, A_2, \ldots, A_m$ are all increasing, then the $\circ$ operation is associative and in this case we have that

$$A_1 \circ A_2 \circ \ldots \circ A_m = \{\omega \in \Omega : \text{ there are pairwise disjoint sets } S_1, S_2, \ldots, S_m \subset F \\ \text{with } [\omega]_{S_i} \subset A_i \text{ for every } i = 1, 2, \ldots m\}.$$

Applying the BK inequality, we obtain $\mathbb{P}(A_1 \circ A_2 \circ \ldots \circ A_m) \leq \mathbb{P}(A_1)\mathbb{P}(A_2)\ldots\mathbb{P}(A_m)$.

1.2.5 Partitions of integers

A *partition* of a positive integer n is an unordered multiset $\{m_1, m_2, \ldots, m_k\}$ of positive integers such that $m_1 + m_2 + \ldots + m_k = n$. Let $p(n)$ denote the number of partitions of n. An asymptotic expression for $p(n)$ was given by Hardy & Ramanujan in their famous paper Hardy and Ramanujan [1918]. An elementary proof of this formula up to a multiplicative constant was given by Erdos [1942]. As customary we use $A \sim B$ to denote the relation $A/B \to 1$ as $n \to \infty$.

Theorem 1.2.3 (Hardy-Ramanujan formula). *The number $p(n)$ of partitions of n satisfies*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\Big(\pi\sqrt{\frac{2n}{3}}\Big).$$

The above asymptotic formula for $p(n)$ implies in particular that $p(n)$ grows sub-exponentially, and this is all we will need in our several applications of Theorem 1.2.3.

1.3 The basic technique

A common ingredient of our analyticity results is the following technique, the main idea of which is present in Kesten [1981] and was mentioned in the introduction. We express our function $f(p)$ as an infinite series $f(p) = \sum_{n\in\mathbb{N}} a_n f_n(p)$, where $f_n(p)$ is the probability of an event. For example, when $f = \chi$ is the expected size of the cluster C_o of o, then f_n is the probability that $|C_o| = n$, and $a_n = n$. To prove that $f(p)$ is analytic, our strategy is to extend the domain of definition of each f_n to complex values of p (we will usually write z instead of p when doing so). Our extended f_n will turn out to be complex-analytic, and so f is analytic if the series $\sum_{n\in\mathbb{N}} a_n f_n(p)$ converges uniformly by standard complex analysis (Weierstrass' Theorem 1.3.1). To show the latter, we employ the Weierstrass M-test (Theorem 1.3.2), using upper bounds on $|f_n(z)|$ inside appropriate discs (centred in the interval $[0, 1]$

where p takes its values). These upper bounds are obtained by Lemma 1.3.3 below for nearest-neighbour models, and by its counterpart Lemma 1.3.6 for long-range models.

Theorem 1.3.1. ***(Weierstrass Theorem)*** *Let f_n be a sequence of analytic functions defined on an open subset Ω of the plane, which converges uniformly on the compact subsets of Ω to a function f. Then f is analytic on Ω. Moreover, f_n' converges uniformly on the compact subsets of Ω to f'.*

Theorem 1.3.2. ***(Weierstrass M-test)*** *Let f_n be a sequence of complex-valued functions defined on a subset Ω of the plane and assume that there exist positive numbers M_n with $|f_n(z)| \leq M_n$ for every $z \in \Omega$, and $\sum_n M_n < \infty$. Then $\sum_n f_n$ converges uniformly on Ω.*

1.3.1 Nearest-neighbour models

The following lemma, and its generalisation Corollary 1.3.5 below, provides the upper bounds that we are going to plug into the M-test as explained above.

Let $\mathbb{P}_p$ denote the law of Bernoulli percolation with parameter p on an arbitrary graph G. Let $D(x, M)$ denote the disc with centre $x \in \mathbb{C}$ and radius $M > 0$ in $\mathbb{C}$. For a subgraph S of G, let ∂S be the set of edges of G that have at least one end-vertex in S but are not contained in $E(S)$.

In this lemma, x is to be thought of as a value of our parameter p near which we want to show the analyticity of some function, and we are free to choose the radius M of the disc we consider as small as we like.

Lemma 1.3.3. *For every finite subgraph S of G and every $o \in V(G)$, the function $P_S(p) := \mathbb{P}_p(C_o = S)$ admits an entire extension $P_S(z), z \in \mathbb{C}$, such that for every $1 > M > 0$, every $1 > x \geq 0$ with $x + M < 1$ and every $z \in D(x, M)$, we have*

$$|P_S(z)| \leq C^{|\partial S|} P_S(x + M),$$

where $C = C_{M,x} := \frac{1-x+M}{1-x-M}$.

Moreover, for every $1 \geq x > 0$, every $x > M > 0$ and every $z \in D(x, M)$, we have $|P_S(z)| \leq K^{|E(S)|} P_S(x - M)$, where $K = K_{M,x} := \frac{x+M}{x-M}$.

(The second sentence will be used to prove analyticity at $p = 1$; the reader who is only interested in analyticity for $p \in [0, 1)$ may ignore it and skip the last paragraph of the proof.)

Proof. By the definitions, we have

$$P_S(p) = (1-p)^{|\partial S|} p^{|E(S)|} \tag{1.3}$$

because the event $\{C_o = S\}$ is satisfied exactly when all edges in ∂S are absent and all edges in $E(S)$ present. This function, being a polynomial, admits an entire extension, which we will still denote by $P_S = P_S(z)$ with a slight abuse.

To prove the upper bound in our first statement —for $1 > x \geq 0$, and $z \in D(x, M)$— we will bound each of the two products appearing in (1.3) separately. Easily,

$$|z|^{|E(S)|} \leq (x+M)^{|E(S)|}$$

when $z \in D(x, M)$ because $|z| \leq x + |z - x| \leq x + M$.

Moreover, it is geometrically obvious that the distance $|1 - z|$ between 1 and z is maximised at $z = x - M$, which implies

$$|1-z|^{|\partial S|} \leq (1-x+M)^{|\partial S|}.$$

Plugging these two inequalities into (1.3) we obtain the desired inequality:

$$|P_S(z)| \leq (1-x+M)^{|\partial S|}(x+M)^{|E(S)|} =$$
$$\Big(\frac{1-x+M}{1-x-M}\Big)^{|\partial S|}(1-x-M)^{|\partial S|}(x+M)^{|E(S)|} = \Big(\frac{1-x+M}{1-x-M}\Big)^{|\partial S|} P_S(x+M),$$

where we also applied (1.3) with $p = x + M$.

For the second statement, let $x \in (0, 1]$, $0 < M < x$ $z \in D(x, M)$. Then $|z| \leq x + M$, and $|1 - z| \leq 1 - x + M$, and similarly to the above calculation we have

$$|P_S(z)| \leq (1-x+M)^{|\partial S|}(x+M)^{|E(S)|} =$$
$$(1-x+M)^{|\partial S|}\Big(\frac{x+M}{x-M}\Big)^{|E(S)|}(x-M)^{|E(S)|} = \Big(\frac{x+M}{x-M}\Big)^{|E(S)|} P_S(x-M).$$

□

Remark 1.3.4. *When G has maximum degree d, we have the crude bound $|\partial S| \leq d|S|$, with which Lemma* (1.3.3) *yields $|P_S(z)| \leq C_{M,x}^{d|S|} P_S(x+M)$.*

Note that in the proof of Lemma 1.3.3 we can replace $E(S)$ and ∂S with any two disjoint finite sets of edges $D, F \subset E(G)$, to obtain the following:

Corollary 1.3.5. *For every two disjoint finite sets of edges $D, F \subset E(G)$, the function $P(p) := \mathbb{P}_p(D \subseteq \omega$ and $F \cap \omega = \emptyset)$ (i.e. the probability that all edges in D are occupied and all edges in F are vacant) admits an entire extension $P(z), z \in \mathbb{C}$, such that*

$$|P(z)| \leq \Big(\frac{1-x+M}{1-x-M}\Big)^{|F|} P(x+M) \tag{1.4}$$

for every $M > 0$, $1 > x \geq 0$ with $x + M < 1$ and $z \in D(x, M)$. Moreover, for every $1 \geq x > 0$, every $x > M > 0$ and every $z \in D(x, M)$, we have

$$|P(z)| \leq \Big(\frac{x+M}{x-M}\Big)^{|D|} P(x-M). \tag{1.5}$$

□

1.3.2 Long-range models

We now prove the analogue of Lemma 1.3.3 for long-range models. Recall that in our long-range setup, we have a vertex set V and any two of its elements can form an edge. The parameters x, M now take their values in $[0, \infty)$, as this is the case for our percolation parameter t. Let ∂S be the set of pairs $\{x, y\} \subset V^2$ that are not contained in $E(S)$ but have at least one vertex in S.

Lemma 1.3.6. *For every finite graph S on a subset of V, and every $o \in V$, the function $P(t) := \mathbb{P}_t(C_o = S)$ admits an entire extension $P(z), z \in \mathbb{C}$, such that $|P(z)| \leq e^{2M|S|}P(x + M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x, M)$.*

The proof of this is similar to that of Lemma 1.3.3, but as our function $P(t)$ is not exactly a polynomial now we will need some reshuffling of terms and the following basic fact about complex numbers.

Proposition 1.3.7. *For every $\mu > 0$ and every $z \in \mathbb{C}$ we have*

$$|e^{\mu z} - 1| \leq e^{\mu|z|} - 1.$$

Proof. Expressing $e^{\mu z}$ via its Taylor expansion and using the triangle inequality yields

$$|e^{\mu z} - 1| = \left|\sum_{j=1}^{\infty} \frac{(\mu z)^j}{j!}\right| \leq \sum_{j=1}^{\infty} \frac{|z\mu|^j}{j!}. \tag{1.6}$$

Since $\mu > 0$, the last expression coincides with the Taylor expansion of $e^{\mu r} - 1$ evaluated at $r = |z|$, from which we obtain $|e^{\mu z} - 1| \le e^{\mu |z|} - 1$.

□

Proof of Lemma 1.3.6. Similarly to (1.3), we have

$$\mathbb{P}_t(C_o = S) = \prod_{e \in \partial S} e^{-t\mu(e)} \prod_{e \in E(S)} \left(1 - e^{-t\mu(e)}\right) \tag{1.7}$$

because the event $\{C_o = S\}$ is satisfied exactly when all edges in ∂S are absent and all edges in $E(S)$ present. Multiplying the second product by $\prod_{e \in E(S)} e^{t\mu(e)}$ and the first by its inverse, we obtain

$$\mathbb{P}_t(C_o = S) = \prod_{e \in \partial S \cup E(S)} e^{-t\mu(e)} \prod_{e \in E(S)} \left(e^{t\mu(e)} - 1\right) = e^{-t\mu(S)} \prod_{e \in E(S)} \left(e^{t\mu(e)} - 1\right), \tag{1.8}$$

where $\mu(S) := \sum_{e \text{ incident with } S} \mu(e) < \infty$ because the edges incident with S are exactly the elements of $\partial S \cup E(S)$. This function clearly admits an entire extension, which we will still denote by $P = P(z)$ with a slight abuse.

To prove the upper bound, we will bound each of the two products appearing in (1.8) separately. Easily,

$$|e^{-z\mu(S)}| \le e^{2M|S|} e^{-(x+M)\mu(S)}$$

when $z \in D(x, M)$ because $|z| \le x + |z - x| \le x + M$ and $\mu(S) \le |S|$. For the second product, we apply Proposition 1.3.7 to each factor to obtain

$$|e^{z\mu(e)} - 1| \le e^{|z|\mu(e)} - 1 \le e^{(x+M)\mu(e)} - 1 \tag{1.9}$$

for every for $z \in D(x, M)$.

Combining these two inequalities, and then applying (1.8) with $t = x + M$, we obtain the desired bound:

$$|P(z)| \le e^{2M|S|} e^{-(x+M)\mu(S)} \prod_{e \in E(S)} \left(e^{(x+M)\mu(e)} - 1\right) = e^{2M|S|} P(x + M).$$

□

Again, in this proof we can replace $E(S)$ and ∂S with any two disjoint finite sets of edges $D, F \subset E$, to obtain, in analogy with Corollary 1.3.5, the following statement:

Corollary 1.3.8. *For every two disjoint finite sets of edges $D, F \subset E$, the function $P(t) := \mathbb{P}_t(D \subseteq \omega$ and $F \cap \omega = \emptyset)$ (i.e. the probability that all edges in D are occupied and all edges in F are vacant) admits an entire extension $P(z), z \in \mathbb{C}$, such that $|P(z)| \leq e^{2M|V(D \cup F)|} P(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$, where $V(D \cup F)$ denotes the set of vertices that are incident with some edge in $D \cup F$.* □

1.3.3 Analyticity of the probability of a given cluster size

Next, we prove that $P_m(t) := \mathbb{P}_t(|C_o| = m)$ is analytic, in the full generality of our long-range models as above. For nearest-neighbour models, this is trivial because the corresponding probability can be expressed as a polynomial, but the long-range variant is more interesting. In addition to analyticity, the following result also provides the upper bound that we will plug into the Weirstrass M-test to deduce the analyticity of the susceptibility χ for subcritical long-range models (Theorem 1.3.12).

Theorem 1.3.9. *For every $m \in \mathbb{N}$ and every $o \in V$, the function $P_m(t) := \mathbb{P}_t(|C_o| = m)$ admits an entire extension $p_m(z), z \in \mathbb{C}$, such that*

$$|P_m(z)| \leq e^{2Mm} P_m(x+M)$$

for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$.

Proof. For $m \in \mathbb{N}$, let $\mathcal{G}_m(V)$ denote the set of finite graphs whose vertex set in a subset of V with m elements containing o (to be thought of as possible percolation clusters of o). For every such $S \in \mathcal{G}_m(V)$, Lemma 1.3.6 yields an entire extension P_S of $\mathbb{P}_t(C_o = S)$. We claim that the sum

$$\sum_{S \in \mathcal{G}_m(V)} P_S(z), \tag{1.10}$$

which for $t \in \mathbb{R}, t > 0$ coincides with $\mathbb{P}_t(|C_o| = m)$, converges uniformly on each closed disc $D(x,M), M > 0$, $x \geq 0$ to a function $p_m : \mathbb{C} \to \mathbb{C}$ which coincides with $\mathbb{P}_t(|C_o| = m)$ for $t \in \mathbb{R}, t > 0$. By Weierstrass' Theorem 1.3.1, this means that p_m admits an entire extension.

Indeed, this uniform convergence follows from the Weierstrass M-test: each summand P_S can be bounded by $|P_S(z)| \leq e^{2M|S|} P_S(x+M) = e^{2Mm} P_S(x+M)$ for every $M > 0$, $x \geq 0$ and $z \in D(x,M)$ by Lemma 1.3.6. Moreover, the sum of these bounds satisfies

$$\sum_{S \in \mathcal{G}_m(V)} e^{2Mm} P_S(x+M) = e^{2Mm} P_m(x+M) < \infty.$$

Thus the Weierstrass M-test can be applied to deduce that (1.10) converges uniformly on $D(x, M)$, and therefore on any compact subset of $\mathbb{C}$.

Finally, the above bounds also prove that $|P_m(z)| \leq e^{2Mm} P_m(x + M)$ as desired. □

Corollary 1.3.10. *For every $m \in \mathbb{N}$ and every $o \in V$, the function $f_m(t) := \mathbb{P}_t(|C_o| \geq m)$ admits an entire extension.*

Proof. It follows from the formula $\mathbb{P}_t(|C_o| \geq m) = 1 - \sum_{i=1}^{m-1} \mathbb{P}_t(|C_o| = i)$ and Theorem 1.3.9. □

1.3.4 Analyticity of χ in the subcritical regime

In this section, we prove that the *susceptibility* $\chi(t) := \mathbb{E}_t(|C_o|)$ of our models is an analytic function of the parameter in the subcritical interval. This applies to both nearest-neighbour and long-range models. For this, we need to assume that our model has the exponential decay property. We remark that the exponential decay property holds whenever our model is transitive Menshikov [1986]; Aizenman and Barsky [1987].

Theorem 1.3.11. *For every long-range model with the exponential decay property (in particular, for every quasi-transitive model), and every $o \in V$, $\chi_o(t)$ is analytic in the interval $[0, t_c)$.*

Theorem 1.3.12. *For every bounded-degree nearest-neighbour model with the exponential decay property (in particular, for every quasi-transitive graph), and every $o \in V$, $\chi_o(p)$ is analytic in the interval $[0, p_c)$.*

The proofs of these facts are very similar, and follow Kesten's proof of the corresponding statement for (nearest-neighbour) lattices in Z^d, except that we simplify it by avoiding any mention to lattice animals.

Proof of Theorem 1.3.11. Each summand in the definition $\chi_o(t) = \sum_{m=1}^{\infty} m\mathbb{P}_t(|C_o| = m)$ of χ_v admits an analytic extension to $\mathbb{C}$ by Theorem 1.3.9. By Weierstrass' Theorem 1.3.1, it suffices to prove that for every $x \in [0, t_c)$ there is an open disk D centred at x such that $\sum_{m=1}^{\infty} m\mathbb{P}_t(|C_o| = m)$ converges uniformly in D.

Pick an arbitrary $x \in [0, t_c)$ and $x < y < t_c$. Since we are assuming the exponential decay property, we have $\mathbb{P}_y(|C_o| \geq m) \leq e^{-cm}$ for some constant $c = c(y, o) > 0$. Since $\mathbb{P}_t(|C_o| \geq m)$ is an increasing function of t, we deduce that

$$P_m(t) = \mathbb{P}_t(|C_o| = m) \leq e^{-cm} \tag{1.11}$$

for every $t \leq y$. Pick $M > 0$ small enough that $x + M \leq y$ and $M < c/2$. Combined with Theorem 1.3.9, this implies that $|P_m(z)| \leq Ca^m$ for $z \in D(x, M)$, where C is a positive constant and $a < 1$. Since $\sum_{m=1}^{\infty} Cma^m < \infty$, we can use the Weierstrass M-test to conclude that the sum $\sum_{m=1}^{\infty} mp_m(z)$ converges uniformly on $D(x, M)$ and since each P_m is analytic the sum is also analytic. Moreover, this sum coincides with $\chi_o(t)$ for $t \in D(x, M) \cap [0, t_c)$, and so our statement follows. □

Proof of Theorem 1.3.12. This is similar to the above, but instead of Theorem 1.3.9, we use the corresponding statement for nearest-neighbour models. This is easier, as the sum (1.10) is finite. Applying Lemma 1.3.3 (using the bounded degree assumption, see also Remark 1.3.4) yields an upper bound of the form $|P_m(z)| \leq c^{dm} P_m(x + M)$ which we use instead of that of Theorem 1.3.9 in our application of the M-test. The rest of the proof is identical to that of Theorem 1.3.11. □

The above proofs show that there is an open disk centred at any subcritical value x of the parameter where p_m converges exponentially fast to 0. Easily, every higher moment $\mathbb{E}_t(|C_o|^k) = \sum_{m=1}^{\infty} m^k \mathbb{P}_t(|C_o| = m)$ (or for the same reason, the expectation of every sub-exponential function of $|C_o|$) admits an analytic extension on the same disk, and so we obtain

Corollary 1.3.13. *Every moment $\mathbb{E}_t(|C_o|^k)$ is an analytic function of the parameter t in the subcritical interval for all models as in Theorem 1.3.12 or Theorem 1.3.11.*

Let us summarize the ideas used to prove the analyticity of χ_o. Our proofs had little to do with χ_o itself. The main idea was to express χ_o as a sum of multiples of probabilities of events and use the exponential decay of those probabilities (Theorem 1.2.1) to counter the exponential growth of their complex extensions (as in Lemma 1.3.3) in small enough discs around every point p. The rest of the proof was standard complex analysis, namely the Weierstrass M-test and Theorem 1.3.1. As we are going to use the same proof structure several times, we reformulate it as the following corollary, which is a straightforward generalisation of the proof of Theorem 1.3.12. To formulate it, we need the following definition.

Definition 1.3.14. *We say that an event E —of a nearest-neighbour model on a graph G— has* complexity *n, if it is a disjoint union of a family of events $(F_i)_{i \in A}$ where each F_i is measurable with respect to a set of edges of G of cardinality n and A is a set of indices.*

Corollary 1.3.15. *Let $\mathbb{P}_p$ denote the law of a nearest-neighbour model, and let $f(p)$ be a function that can be expressed as $f(p) = \sum_{n \in \mathbb{N}} \sum_{i \in L_n} a_i \mathbb{P}_p(E_{n,i})$ in an interval*

$p \in (a, b) \subseteq [0, 1]$, where $a_n \in \mathbb{R}$, L_n is a finite index set, and each $E_{n,i}$ is an event measurable with respect to $\mathbb{P}_p$ (in particular, the above sum converges absolutely for every $p \in (a, b)$). Suppose that

(i) E_n has complexity of order $\Theta(n)$, and

(ii) there is a constant $0 < c < 1$ independent of p such that $\sum_{i \in L_n} |a_i| \mathbb{P}_p(E_{n,i}) = O(c^n)$ on (a, b).

Then there is a constant $\varepsilon > 0$ such that $f(p)$ is analytic in $(a - \varepsilon, b + \varepsilon)$.

(The analyticity on the larger interval $(a - \varepsilon, b + \varepsilon)$ is needed to handle the case $p = 1$. The proof shows that ((ii)) holds on the larger interval $(a - \varepsilon, b + \varepsilon)$ with c replaced by some other constant smaller than 1.)

Proof. We imitate the proof of Theorem 1.3.11, except that instead of the exponential decay property we use our assumption (ii), and instead of Lemma 1.3.3 we use its generalisation Corollary 1.3.5, which we apply to the sequence of events witnessing that $(E_{n,i})$ satisfies (i). (Note that the complexity of an event governs the exponential growth rate of the maximum modulus of the extension of its probability to a complex disc as a function of the radius of that disc.) For $p \in (a, b)$, we can use either (1.4) or (1.5). To obtain the analyticity at a neighbourhood of a we need to use (1.4), while to obtain the analyticity at a neighbourhood of b we need to use (1.5). □

Remark: A similar statement for long-range models can be formulated, and proved, along the same lines, except that we use the total μ-weight rather than the cardinality of an edge-set in Definition 1.3.14.

1.4 Analyticity above the threshold for planar lattices

A *planar quasi-transitive lattice* (in $\mathbb{R}^2$) is a connected, locally finite, plane graph L such that for some pair of linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$, translation by each v_i preserves L, and this action has finitely many orbits of vertices. This means that there exists a finite set U of vertices of L such that any vertex of L can be translated by a linear combination of v_1, v_2 to some vertex of U. In particular, L does not have accumulation points, i.e. there are finitely many vertices inside any every bounded region of the plane. It is not hard to see that we can draw L in the plane in such a way that the edges of L are piecewise linear curves. We can even assume that the edges of L are straight lines, which we can do for any planar graph Thomassen [1977].

In this section, we prove

Theorem 1.4.1. *For Bernoulli bond percolation on any planar quasi-transitive lattice we have* $p_{\mathbb{C}} = p_c$.

This result is new even for the standard square lattice $\mathbb{Z}^2$, i.e. the Cayley graph of $\mathbb{Z}^2$ with respect to the standard generating set $\{(0,1),(1,0)\}$. Slightly more effort is needed to prove it in the generality of planar quasi-transitive lattices. The reader that wants to see a simplest possible proof for the lattice $L = \mathbb{Z}^2$ is advised to:

- ignore Theorem 1.4.2, and just recall that $p_c(\mathbb{Z}^2) = 1/2$ and $\mathbb{Z}^{2*} = \mathbb{Z}^2$;
- skip the definition of X in Section 1.4.1, and instead take X to be the horizontal 'axis' of $\mathbb{Z}^2$, and X^+ the right 'half-axis' starting at the origin o; and
- notice that Proposition 1.4.4 holds trivially with $f = 1$.

We will use the following important fact about the relation between the percolation thresholds in the primal and dual lattice. The history of this result starts with the proof of the inequality $p_c(\mathbb{Z}^2) \geq 1/2$ by Harris [1960]. Soon after, Fisher [1961] extended this result to a wide class of dual pairs. This is the first paper where the formula $p_c(L) + p_c(L^*) \geq 1$ appears. Sykes and Essam [1964] established the relation $p_c(L)+p_c(L^*) = 1$ for some dual pairs with an unverified (and still open for proof) assumption in the argument. Kesten [1980] gave the first rigorous proof of the formula $p_c(\mathbb{Z}^2) = 1/2$. Kestens result was rigorously extended for a dual pair of graphs other than $\mathbb{Z}^2$ and for site percolation on $\mathbb{Z}^2$ and its matching pair by Russo [1981]. These results were further extended by Bollobas and Riordan [2008], and almost simultaneously the general case was proved by [Sheffield, 2005, Theorem 9.3.1] in a rather involved way. A shorter proof can be found in Duminil-Copin et al. [2019].

Theorem 1.4.2 (Sheffield [2005]; see also Duminil-Copin et al. [2019])**.** *For every planar quasi-transitive lattice* L, *we have* $p_c(L) + p_c(L^*) = 1$.

1.4.1 Preliminaries on planar quasi-transitive lattices

Consider a planar quasi-transitive lattice L. We will construct a 2-way infinite path X in any planar quasi-transitive lattice L, which can be thought of as a 'quasi-geodesic' of both L and L^*.

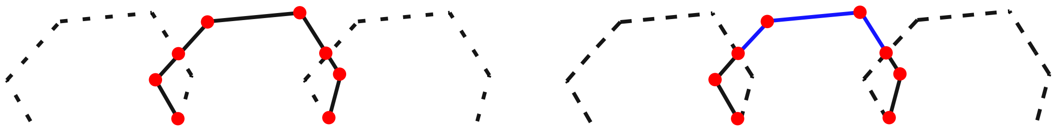

Figure 1.3: The path P (solid lines) and its translates $P + ktv_1$ (dashed lines). Translating the blue subpath (if colour is shown) by multiples of tv_1 we obtain X.

Since L is a plane graph, we naturally identify $V(L)$ with a set of points of $\mathbb{R}^2$. Let $o \in \mathbb{R}^2$ be a vertex of L and recall that $o + kv_1 \in V(L)$ for some non-zero vector $v_1 \in \mathbb{R}^2$ and every $k \in \mathbb{Z}$. Fix a path P from o to $o+v_1$. Note that the union $\bigcup_{k\in\mathbb{Z}}(P + kv_1)$ of its translates along multiples of v_1 contains a 2-way infinite path X. Indeed, consider the largest $t > 0$ such that $P + tv_1$ and P share a common vertex. Then only consecutive translates of P of the form $P + ktv_1$ overlap. Let W be the set of common vertices of P and $P - tv_1$, and notice that $W + tv_1$ lies in P. Now consider a vertex $u \in W$ such that for every other vertex $v \in W$, at most one of $v, v + tv_1$ belongs to the subpath Q of P connecting u to $u + tv_1$. It is easy to see that the union $\bigcup_{k\in\mathbb{Z}}(Q + ktv_1)$ is the desired 2-way infinite path X. See Figure 1.3. It follows from the construction that X is t-periodic, i.e. $X + tv_1 = X$. For convenience, we will assume that our reference vertex o belongs to X.

We will first show that X is a *quasi-geodesic*, i.e. there is a constant $c > 0$ such that for every $x_i, x_j \in X$ we have $d_L(x_i, x_j) \geq c|i - j|$, where d_L denotes the graph-distance in L.

Proposition 1.4.3. *Let L be a planar quasi-transitive lattice and X the infinite path defined above. Then X is a quasi-geodesic.*

Proof. First, notice that there are finitely many 'types' of edges of L up to translation by v_1, v_2, hence there is a constant $\ell > 0$ such that any edge of L has length at most ℓ. This implies that $d_L(x_i, x_j) \geq d(x_i, x_j)/\ell$, where $d(x_i, x_j)$ denotes the Euclidean distance between x_i and x_j, because any graph-geodesic connecting x_i to x_j has length at most $\ell d_L(x_i, x_j)$ when viewed as a curve in the plane.

Now to estimate $d(x_i, x_j)$, consider a large enough box B that contains entirely the path Q defined in the construction of X, the sides of which are parallel to v_1,v_2. Without loss of generality, we can assume that one of the sides of B coincides with v_1. Then $\bigcup_{k\in\mathbb{Z}}(B+kv_1)$ covers X, and furthermore, only consecutive translates of B overlap. Consider the adjacent translates $B + kv_1, B + (k + 1)v_1, \dots, B + nv_1$ of B connecting x_i to x_j, i.e. x_i lies in one of $B + kv_1, B + nk_1$ and x_j in the other. Projecting v_1 to the x and y axes we obtain two vectors with certain lengths.

Without loss of generality, we can assume that the projection to the x-axis has non-zero length, which we denote by s. We claim that $d(x_i, x_j) \geq s(n-k-1)$. To see this, write x'_i and x'_j for the x coordinate of x_i and x_j respectively. Then we have $d(x_i, x_j) \geq |x'_i - x'_j|$. Since the projection of $\bigcup_{j=k+1}^{n-1}(B + jv_1)$ to the x-axis has length $s(n-k-1)$, we deduce that $|x'_i - x'_j| \geq s(n-k-1)$, which proves the claim.

It remains to compare $n-k-1$ with $|i-j|$. To this end, notice that the boxes $B + kv_1, \ldots, B + nv_1$ contain all vertices of the subpath of X connecting x_i to x_j because B is so large that it contains Q. Write $v(B)$ for the number of vertices in B, which is finite by the definition of L. With this definition, the number of vertices in the union of the boxes $B + kv_1, \ldots, B + nv_1$ is at most $v(B)(n-k+1)$. In particular, $|i-j|$ is at most $v(B)(n-k+1)$. Combining all the above inequalities we obtain that $d_L(x_i, x_j) \geq c_1|i-j| - c_2$ for some constants $c_1, c_2 > 0$ which do not depend on i, j. Using the fact that $d_L(x_i, x_j) \geq 1$ whenever $x_i \neq x_j$, we see that there is a constant c which does not depend on i, j such that $d_L(x_i, x_j) \geq c|i-j|$. The proof is now complete. □

Let $X^+ = (x_0 = o, x_1, \ldots), X^- = (\ldots, x_{-1}, x_0 = o)$ denote the two 1-way infinite sub-paths of X starting at o.

Proposition 1.4.4. *Let L be a planar quasi-transitive lattice and X^+ the infinite path defined above. Then there is a constant $f = f(L) > 0$ such that every connected subgraph of L that surrounds o and has $N > 0$ edges, must contain one of the first fN vertices $x_0, x_1, \ldots, x_{fN-1}$ of X^+, and every connected subgraph of L^* that surrounds o and has at most N edges, must cross one of the first fN edges $x_0x_1, x_1x_2, \ldots, x_{fN-1}x_{fN}$ of X^+.*

Proof. Suppose that some connected graph $S \subset L$ surrounds o. Then S must separate o from infinity, and so it must contain a vertex x^+ in X^+, and a vertex x^- in X^- (x^+ and x^- may possibly coincide). If S has at most N edges, then the graph-distance between x^+ and x^- is at most N because S is a connected graph. Since X is a quasi-geodesic, the indices of x^+ and x^- differ by at most N/c, where c is the constant at the definition of a quasi-geodesic. We now see that x^+ is one of the first $1 + N/c$ vertices of X^+.

Suppose now that some connected graph $S^* \subset L^*$ with at most N edges surrounds o. Then some edges e^+ and e^- of X^+ and X^-, respectively, are crossed by S^*. Our aim is to define a connected graph H in the primal L that contains e^+ and e^- and has $O(N)$ edges. To this end, recall that each vertex of S^* corresponds to some face of L which is incident to a certain number of edges of L. Since S^* is connected, the set of all edges of L incident to (the dual face of) some vertex of S^*

defines a connected subgraph H of L. Notice that H surrounds o and contains both e^+ and e^-. We will show that the number of edges of H is of order N. Indeed, we first observe that every face of L is bounded: choose vectors v_1, v_2 as in the definition of a planar quasi-transitive lattice, fix a path P_1 from o to $o + v_1$ and a path P_2 from o to $o + v_2$, and note that the union of their translates $kP_i, k \in \mathbb{Z}$ contains a grid that separates $\mathbb{R}^2$ into bounded regions containing the faces of L. Therefore, since there are finitely many of orbits of vertices of L, there are also finitely many of orbits of faces, and so the number of edges in the boundary of any face is at most some $C \in \mathbb{N}$. Since S^* is connected and has at most N edges, it must have at most $N + 1$ vertices. Thus H has at most $C(N + 1)$ edges by its construction.

We can now argue as above to conclude that each of the two end-vertices of e^+ is some of the first $1 + C(N + 1)/c$ vertices of X^+. Hence e^+ is one of the first $1+C(N+1)/c$ edges of X^+. Choosing f large enough so that $1+C(N+1)/c \leq fN$, we obtain the desired result. □

1.4.2 Outer interfaces

A key element in the proof of analyticity of θ in 2-dimensions is the notion of *outer interface*. The definition we will give applies not only to planar quasi-transitive lattices but also to plane graphs without any accumulation points, i.e. every bounded region contains finitely many vertices. We remark that all planar graphs we will work with in this thesis, namely planar quasi-transitive lattices, triangulations of an open disk and regular tessellations of the hyperbolic plane, can be embedded in the plane without any accumulation points. For triangulations of an open disk, this follows e.g. from the Circle packing theorem (see Section 1.8 for more details). For regular tessellations of the hyperbolic plane, consider their standard embedding in the open unit disk and then map the open unit disk to the whole plane using a continuous bijection.

Thomassen [1977] proved that all plane graphs without any accumulation points can be embedded in the plane (without accumulation points) in such a way that all edges are straight lines.

Consider a connected plane graph G without any accumulation points and fix a vertex o. We will assume throughout that all edges of G are straight lines. Recall that a plane graph divides the plane into faces, i.e. a face of G is the closure of a component of $\mathbb{R}^2 \setminus E(G)$. To avoid any confusion, we emphasize that given a subgraph H of G, the faces of H are the closures of the components of $\mathbb{R}^2 \setminus E(H)$ (not the faces inherited from G). Notice that when H is finite, it has a unique unbounded face.

Definition 1.4.5. *An outer interface of G is a pair $(S, \partial S)$ of finite sets of edges of G with the following properties*

(i) The graph $H = (V(S), S)$, where $V(S)$ is the set of endpoints of S, is connected;

(ii) Each edge in S lies at the boundary of the (unique) unbounded face of H;

(iii) o lies either in H or inside a bounded face of H; and

(iv) ∂S is the set of edges in $E(G) \setminus S$ that have at least one endpoint in $V(S)$ and are contained in the (unique) unbounded face of H.

To simplify the notation, we will usually write S instead of $(S, \partial S)$. We will also refer to ∂S as the *boundary* of the outer interface. It is important to remember that ∂S may contain edges that have both their end-vertices in S; our proof will break down (at Lemma 1.4.8) if we exclude such edges from the definition of ∂S.

Given a finite connected subgraph Γ of G containing o, we can associate to Γ an outer interface S with $S \subset E(\Gamma)$ and $\partial S \subset (E(G) \setminus E(\Gamma))$. To this end, we let S consist of the edges lying at the boundary of the unbounded face of Γ. We define ∂S to be the set of edges in $E(G) \setminus S$ that have at least one endpoint in $V(\Gamma)$ and are contained in the (unique) unbounded face of H. We will call S the outer interface of Γ but we need to first verify that S is an outer interface.

It is clear that the pair $(S, \partial S)$ satisfies properties (ii) and (iii). To verify property (i), let D be a connected component of $H = (V(S), S)$. We have that every edge in $E(\Gamma) \setminus E(H)$ lies in a region bounded by a cycle of H. Hence the graph D' spanned by D and all edges of Γ lying in a region bounded by a cycle of D, forms a connected component of Γ, and so D' must coincide with Γ. Moreover, the only vertices of H that D' contains are those of D. Thus D coincides with H, i.e. S is a connected graph. Finally, it is now clear that ∂S satisfies (iv), because the unbounded faces of H and Γ coincide.

Given a realisation $\omega \in 2^{E(G)}$ of our Bernoulli percolation on G, we say that an outer interface S *occurs* in ω if S is the outer interface of some cluster of ω. This happens exactly when all edges of S are occupied and all edges in ∂S are vacant.

A *multi-interface* M is a finite set of pairwise vertex-disjoint outer interfaces.

Lemma 1.4.6. *For every outer interface S, the dual graph ∂S^* of its boundary ∂S spans a connected subgraph of G^* surrounding o.*

Proof. Let H be the graph $(V(S), S)$. We claim that there is a Jordan curve J disjoint from H such that H lies in the bounded side of J, and J is close enough to

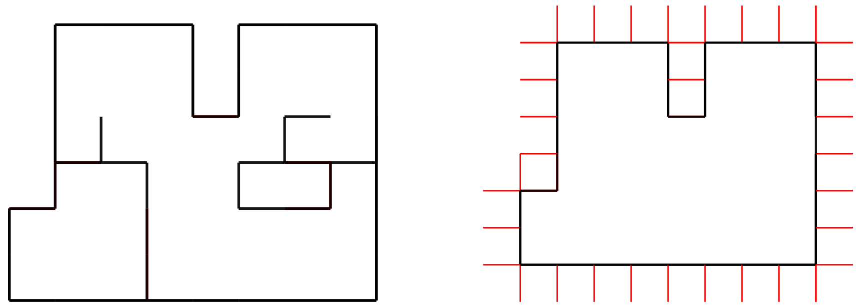

Figure 1.4: A connected graph and the corresponding outer interface.

H that it meets all edges in ∂S and no other edges of G. Indeed, let H' be the union of H (when viewed as a subset of the plane) with the bounded faces of $L\backslash H$. Then S coincides with the topological boundary of H'. Consider a collection $\{d_1, d_2, \ldots, d_k\}$ of open disks that cover the boundary S of H'. Since we have assumed that the edges of L are piecewise linear curves, we can choose the collection appropriately so that only consecutive disks d_i, d_{i+1} overlap (with the understanding that d_k,d_1 might overlap as well). This will ensure that the set A, which is defined as the union of H' with the disks $d_1, d_2, \ldots, d_k$, is a bounded simply-connected open set. Since the boundary of A is a finite union of circular arcs, it is a Jordan curve J. It is not hard to see that J satisfies the desired properties.

The cyclic sequence of faces and edges of G visited by J defines a closed walk in G^*. This easily implies that ∂S^* spans a connected subgraph of G^*. That this subgraph surrounds o is an immediate consequence of the definition of an outer interface. □

The following lemma, which is an easy consequence of the definitions, is the main reason why we define multi-interfaces to comprise vertex-disjoint outer interfaces.

Lemma 1.4.7. *If two occurring outer interfaces S_1, S_2 share a vertex, then they coincide. Moreover, if their boundaries $\partial S_1, \partial S_2$ share an edge, then they coincide.*

Proof. For the first assertion, let C_i, $i = 1, 2$ be the percolation cluster of S_i. That such clusters exist follows from the fact that each S_i spans a connected graph by Lemma 1.4.6. If S_1, S_2 share a vertex, then C_1 coincides with C_2. Now each of S_1, S_2 is the unique outer interface of $C_1 = C_2$. Hence S_1 and S_2 coincide.

For the second assertion, assume that $\partial S_1, \partial S_2$ share an edge. If S_1, S_2 share a common vertex, then they coincide, as proved above, so let us assume that they are vertex disjoint. Since o is a common vertex of the regions bounded by S_1, S_2,

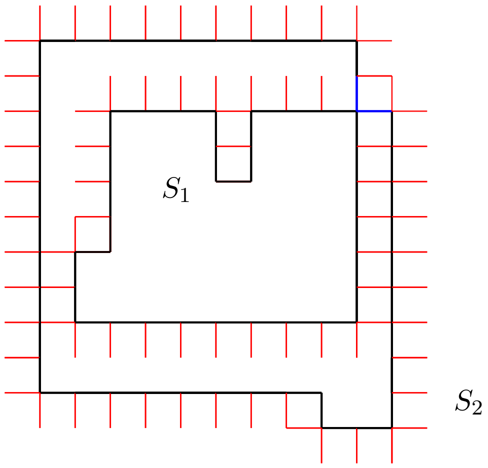

Figure 1.5: Two outer interfaces sharing a common vertex. The blue edges belong to both the boundary of the 'internal' outer interface S_1 and the 'external' outer interface S_2. If both S_1 and S_2 occur, then the blue edges need to be open and closed at the same time.

either S_1 lies in the interior of the region bounded by S_2 or vice versa. This implies that $\partial S_1, \partial S_2$ are disjoint. This contradiction shows that in any case, S_1 and S_2 coincide, as desired. □

1.4.3 Main result

We will now focus again on planar quasi-transitive lattices. The following lemma is one of the reasons why the proof of Theorem 1.4.1 only applies to lattices rather than arbitrary planar graphs. Let $|S|$, $|\partial S|$ be the number of edges in S and ∂S, respectively.

Lemma 1.4.8. *For every outer interface S we have $|\partial S| \geq |S|/k$ for some integer $k = k(L)$.*

For example, if L is the square lattice $\mathbb{Z}^2$, then $k = 2$. (And not $k = 1$ because it can happen that most edges in ∂S have both their end-vertices in S; for example, we can have a 'space filling' outer interface whose vertex set is an $n \times n$ box of $\mathbb{Z}^2$. The following proof will give a worse bound than $k = 2$ but we can afford to be generous.)

Proof. Recall that any face of L has at most C edges for some $C > 0$ (this was proved in Proposition 1.4.4), and also that S consists of the vertices and edges incident with the unbounded face F of some $H \subset L$. If we walk along the boundary

S of F, we will never traverse C or more edges of S without encountering an edge in ∂S, and we will encounter each edge in ∂S at most twice. Thus our assertion holds for $k = 2(C-1)$. □

Let $\mathcal{MS}$ denote the set of multi-interfaces of L. We say that $M \in \mathcal{MS}$ *occurs* if each of the outer interfaces it contains occurs. Let $|M| := \sum_{S_i \in M} |S_i|$ be the total number of edges in M. Let $\partial M := \bigcup_{S_i \in M} \partial S_i$, and let $\mathcal{MS}_n := \{M \in \mathcal{MS} \mid |\partial M| = n\}$ be the set of multi-interfaces with n boundary edges.

Lemma 1.4.9. *There is a constant $r \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ at most $r^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any percolation instance ω.*

Proof. Suppose $M \in \mathcal{MS}_n$ occurs in ω. Let us denote by D the subset of $\{x_0, x_1, \ldots\}$ comprising the first vertex of $\{x_0, x_1, \ldots\}$ that each component outer interface of M meets. Notice that M is uniquely determined by D because occurring outer interfaces are disjoint by Lemma 1.4.7. In other words, $M = \bigcup_{x_i \in D} S(x_i, \omega)$, where $S(x_i, \omega)$ denotes the occurring outer interface containing x_i.

Note that $|S(x_i, \omega)| > i/f$ for every $x_i \in D$ by Proposition 1.4.4. Since $kn \geq |M| = \sum_{x_i \in D} |S(x_i, \omega)|$ by Lemma 1.4.8 and the above remark, we deduce $fkn > \sum_{x_i \in D} i$. This means that the set $\{i \mid x_i \in D\}$ is a partition of a number smaller than fkn. Moreover, distinct occurring multi-interfaces in $\mathcal{MS}_n$ determine distinct subsets D of $\{x_0, x_1, \ldots\}$, and therefore distinct partitions. By the Hardy–Ramanujan formula, the number of such partitions is less than $r^{\sqrt{n}}$ for some $r > 0$. Thus less than $r^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in ω. □

If C_o is finite, then there is exactly one outer interface that occurs and is contained in C_o, namely the boundary of the unbounded face of C_o. We denote the probability of this event by P_S, that is, we set

$$P_S(p) := \mathbb{P}(S \text{ occurs and } S \subset C_o).$$

Thus we can write the probability $\theta_o(p)$ that C_o is finite by summing P_S over all $S \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of outer interfaces:

$$1 - \theta_o(p) = \textstyle\sum_{S \in \mathcal{S}} P_S(p) \tag{1.12}$$

for every $p \in (p_c, 1]$.

As usual, our strategy to prove the analyticity of θ, is to express θ as an infinite sum of functions that admit analytic extensions, namely, probabilities of events that depend on finitely many edges, and then apply Corollary 1.3.15. Formula

(1.12) is a first step in this direction, however, the functions P_S are not fit for our purpose: the event $\{S \text{ occurs and } S \subset C_o\}$ is not measurable with respect to the set of edges incident with S only. Therefore, we would prefer to express θ in terms of the simpler functions

$$Q_S := \mathbb{P}_p(S \text{ occurs}).$$

These functions have the advantage that comply with the premise of Corollary 1.3.5, and hence $|Q_S(p)|$ is bounded in $D(p, R)$ by $e^{C_{R,p}|S|}Q_S(p+R)$, where $C_{R,p}$ is independent of S. But when trying to write θ as a sum involving these Q_S, we have to be more careful: we have

$$1 - \theta_o(p) = \mathbb{P}_p(|C_o| < \infty) = \mathbb{P}_p(\text{at least one } S \in \mathcal{S} \text{ occurs})$$

by the definitions, but more than one $S \in \mathcal{S}$ might occur simultaneously. Therefore, we will apply the inclusion-exclusion principle to the set of events $\{S \text{ occurs}\}_{S \in \mathcal{S}}$.

For every multi-interface M, we define $Q_M := \prod_{S \in M} Q_S$, i.e. Q_M is the probability that M occurs.

Lemma 1.4.10. *For every $p \in (p_c, 1]$ we have*

$$1 - \theta_o(p) = \textstyle\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} Q_M(p), \tag{1.13}$$

where $c(M)$ denotes the number of outer interfaces in the multi-interface M.

Proof. To prove this, we need first of all to check that the sum in the right-hand side converges. This is implied by Lemma 1.4.11 below, which states that the sum $\sum_{M \in \mathcal{MS}_n} Q_M(p)$ decays exponentially in n, and therefore our sum converges absolutely. Then, we need to check that

$$\mathbb{1}_{\{\text{some S occurs}\}} = \sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} \mathbb{1}_{\{\text{M occurs}\}}.$$

To see this, notice that by the Borel-Cantelli lemma only finitely many outer interfaces occur in almost every ω. Furthermore, for every set M of outer interfaces, we have

$$\mathbb{P}(\text{every } S \in M \text{ occurs}) = 0$$

unless the elements of I are pairwise vertex-disjoint —that is, $M \in \mathcal{MS}$— by Lemma 1.4.7. Moreover, if for some $M \in \mathcal{MS}$, all outer interfaces $S \in M$ occur, then M occurs as well. Hence we have

$$\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} \mathbb{1}_{\{\text{M occurs}\}} = \sum_{k=1}^{N} \binom{N}{k} (-1)^{k+1},$$

where N is the number of occurring outer interfaces, because for every integer $1 \leq k \leq N$, all possible $\binom{N}{k}$ combinations of k occurring outer interfaces contribute $(-1)^{k+1}$ to $\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} \mathbb{1}_{\{\text{M occurs}\}}$. The binomial theorem implies that the sum in the right-hand side is equal to 1 if some outer interface occurs and 0 otherwise, i.e. coincides with $\mathbb{1}_{\{\text{some S occurs}\}}$. Finally, we have

$$\mathbb{E}_p\Big(\sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} \mathbb{1}_{\{\text{M occurs}\}} \Big) = \sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} Q_M(p)$$

by Fubini's theorem. □

The main part of our proof is to show that the probability for at least one multi-interface in $\mathcal{MS}_n$ to occur decays exponentially in n, which will imply the following lemma. The rest of the arguments used to prove Theorem 1.4.1 are identical to those of e.g. Theorem 1.3.12.

Lemma 1.4.11. *For every $p \in (p_c, 1]$ there are constants $c_1 = c_1(p), c_2 = c_2(p) > 0$ such that for every $n \in \mathbb{N}$,*

$$\textstyle\sum_{M \in \mathcal{MS}_n} Q_M(p) \leq c_1 e^{-c_2 n}. \tag{1.14}$$

Moreover, if $[a, b] \subset (p_c, 1]$, then the constants c_1 and c_2 can be chosen independently of p in such a way that (1.14) *holds for every $p \in [a, b]$.*

The proof of this is based on the fact that the size of the boundary of an outer interface S that contains a certain vertex x has an exponential tail. This is because ∂S is contained in a component of the dual L^* by Lemma 1.4.6, and as our percolation is subcritical on L^*, the exponential decay property holds. Still, the exponential tail of each $|\partial S|$ does not easily imply Lemma 1.4.11. First of all, the sum in the left-hand side of Lemma 1.4.11 is larger than the probability $\mathbb{P}(\mathcal{MS}_n \text{ occurs})$ that a multi-interface of $\mathcal{MS}_n$ occurs. Second, a multi-interface might consist of plenty of outer interfaces. Nevertheless, we will be able to overcome these difficulties. Using Lemma 1.4.9 we prove that the aforementioned sum does not grow too fast when compared with the probability that a multi-interface of $\mathcal{MS}_n$ occurs.

Proof of Lemma 1.4.11. We start by noticing that

$$\sum_{M\in\mathcal{MS}_n} Q_M(p) = \mathbb{E}_p(\sum_{M\in\mathcal{MS}_n} \mathbb{1}_{\{M \text{ occurs}\}}).$$

The number of multi-interfaces $M \in \mathcal{MS}_n$ that can occur simultaneously is bounded above by $r^{\sqrt{n}}$ for some $r > 0$ by Lemma 1.4.9. It follows that

$$\sum_{M\in\mathcal{MS}_n} \mathbb{1}_{\{M \text{ occurs}\}} \leq r^{\sqrt{n}} \mathbb{1}_{\{\mathcal{MS}_n \text{ occurs}\}}$$

which in turn implies that

$$\sum_{M\in\mathcal{MS}_n} Q_M(p) \leq r^{\sqrt{n}} \mathbb{P}_p(\mathcal{MS}_n \text{ occurs}).$$

Hence it suffices to show that $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$ decays exponentially in n. In order to do so, we will employ the exponential tail of the size of a certain (subcritical) cluster in the dual L^* given by the exponential decay property. For this, we will use the natural coupling of the percolation processes on L and L^*: given a percolation instance $\omega \in 2^{E(L)}$ on L, we obtain a percolation instance ω^* on L^* by changing the state of each edge, i.e. letting $\omega^*(e^*) = 1-\omega(e)$ for every $e \in E(L)$. Let $C(k)$ denote the event that there is a connected subgraph of ω^* which crosses one of the first fk edges of X^+, and has at least k edges, where f is the constant of Proposition 1.4.4. Note that $C(k)$ is an increasing event for ω^* that depends on finitely many edges. We claim that

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq \sum_{\{m_1,\ldots,m_k\}\in P_n} \mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)), \tag{1.15}$$

where $\circ$ means that the events occur edge–disjointly. Since the events $C(m_i)$ are increasing, the event $C(m_1) \circ \ldots \circ C(m_k)$ is well defined (see the discussion in Section 1.2.4). Here P_n is the set of partitions $\{m_1, \ldots, m_k\}$ of n. Once this claim is established, we will be able to employ the BK inequality (Theorem 1.2.2) to bound $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$.

To prove (1.15), we remark that each multi-interface $M \in \mathcal{MS}_n$ defines a partition $\{m_1, \ldots, m_k\}$ of n by letting m_i stand for the number of edges in the i-th component K_i of the subgraph of L^* spanned by ∂M^*. By Proposition 1.4.4 if M occurs, then K_i is a witness of $C(m_i)$, and these witnesses are pairwise edge-disjoint. Thus the occurrence of M implies the occurrence of the event $C(m_1) \circ \ldots \circ C(m_k)$ in ω^*. To conclude that (1.15) holds, we apply the union bound to the family of events of the latter form, ranging over all partitions $\{m_1, \ldots, m_k\} \in P_n$.

The BK inequality Grimmett [1999] states that

$$\mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)) \leq \mathbb{P}_{1-p}(C(m_1)) \cdot \ldots \cdot \mathbb{P}_{1-p}(C(m_k)).$$

Using the union bound and applying the exponential decay property, we obtain $\mathbb{P}_{1-p}(C(m_i)) \leq f m_i c^{m_i}$ for some constant $0 < c = c(p) < 1$. In addition, if $[a, b] \subset (p_c, 1]$, then the monotonicity of $\mathbb{P}_{1-p}(C(m_i))$ implies that the constant c can be chosen uniformly for $p \in [a, b]$. As $\mathbb{P}_{1-p}(C(n)) < 1$ for every n, we deduce that $\mathbb{P}_{1-p}(C(m_i)) \leq (c+\varepsilon)^{m_i}$ for some $\varepsilon > 0$ such that $c + \varepsilon < 1$; indeed, for any ε, this is satisfied for large enough m_i, and increasing ε we can make it true for the smaller values of m_i.

Combining all these inequalities starting with (1.15) we conclude that

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq |P_n|(c+\varepsilon)^n.$$

We have $|P_n| \leq h^{\sqrt{n}}$ for some constant h by the Hardy–Ramanujan formula (Theorem 1.2.3), and so

$$\mathbb{P}_p(\mathcal{MS}_n \text{ occurs}) \leq h^{\sqrt{n}}(c+\varepsilon)^n.$$

Thus $\mathbb{P}_p(\mathcal{MS}_n \text{ occurs})$ decays exponentially in n as claimed. □

Remark 1.4.12. *The BK-inequality is only used to prove the exponential decay of $\mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k))$. Here we briefly describe an alternative way to obtain this result. Let $t > 0$ be a fixed constant to be defined. If $m_i \geq tn$ for some $i = 1, 2, \ldots, k$, then because $\mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)) \leq \mathbb{P}_{1-p}(C(m_i))$ we obtain the desired exponential decay. If not, then we have that $m_i < tn$ for every $i = 1, 2, \ldots, k$. This implies that when the event $C(m_1) \circ \ldots \circ C(m_k)$ occurs, for every $i = 1, 2, \ldots, k$, there is a connected subgraph of ω^* which crosses one of the first ftn edges of X^+, and has at least m_i edges. Hence we can connect all these connected subgraphs by opening a path of length ftn in ω^* (closing the corresponding edges in ω) to create one connected subgraph of size at least n. In this case, the event $C(n)$ occurs. Computing the cost of opening this path, we obtain that $\mathbb{P}_{1-p}(C(m_1) \circ \ldots \circ C(m_k)) \leq K^{ftn}\mathbb{P}_{1-p}(C(n))$ for some constant $K > 0$ depending only on p. We can now choose $t > 0$ small enough so that $K^{ftn}\mathbb{P}_{1-p}(C(n))$ decays exponentially in n, which we can do because $\mathbb{P}_{1-p}(C(n))$ decays exponentially in n. In both cases, we obtain the desired exponential decay.*

We are now ready to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. As already explained, the inclusion–exclusion expression (1.13) holds by Lemma 1.4.11. The assertion follows if we can apply Corollary 1.3.15 for $L_n = \mathcal{MS}_n$, and $(E_{n,i})$ an enumeration of the events $\{M \text{ occurs}\}_{M \in \mathcal{MS}_n}$. So let us check that the assumptions of Corollary 1.3.15 are satisfied.

By definition, every $M \in \mathcal{MS}_n$ has n vacant edges. Moreover, $|M| \leq k(L)n$ by Lemma 1.4.8. Thus assumption (i) of Corollary 1.3.15 is satisfied. The fact that assumption (ii) is satisfied is exactly the statement of Lemma 1.4.11. □

1.5 Analyticity of θ in all dimensions

In this section, we will prove that for percolation on $\mathbb{Z}^d$, θ is analytic on the supercritical interval. Our proof works for $d = 2$ as well, providing an alternative statement.

We will write $\mathbb{L}^d$ for the hypercubic lattice $(\mathbb{Z}^d, E(\mathbb{Z}^d))$, the vertices of which have integer coordinates, and we connect two vertices when they have distance 1.

Theorem 1.5.1. *For Bernoulli bond percolation on $\mathbb{L}^d, d \geq 2$, the percolation density θ is analytic on $(p_c, 1]$.*

1.5.1 Setting up the renormalisation

We start by introducing some necessary definitions. Consider a positive integer N. For every vertex x of $\mathbb{Z}^d$, we let $B(x) = B(x, N)$ denote the box $\{y \in \mathbb{Z}^d : \|y - Nx\|_\infty \leq 3N/4\}$. With a slight abuse, we will use the same notation $B(x)$ to also denote the corresponding subset of $\mathbb{R}^d$, namely $\{y \in \mathbb{R}^d : \|y - Nx\|_\infty \leq 3N/4\}$.

The collection of all these boxes can be thought of as the vertex set of graph canonically isomorphic to $\mathbb{Z}^d$. We will denote this graph by $N\mathbb{L}^d$. Whenever we talk about percolation (clusters) from now on, we will be referring to percolation, with a fixed parameter $p > p_c$, on $\mathbb{L}^d$ and not on $N\mathbb{L}^d$; we will never percolate the latter.

For any percolation cluster C, we denote by $C(N)$ the set of boxes B such that the subgraph of C induced by its vertices lying in B has a component of diameter at least $N/5$. The boxes with this property will be called *C-substantial.* Notice that $C(N)$ is a connected subgraph of $N\mathbb{L}^d$. The internal boundary of $C(N)$ is denoted by $\partial C(N)$ following the terminology of Section 1.2.1. Notice that $\partial C(N)$ is not necessarily connected. For technical reasons, we would like it to be, and therefore we modify our lattice by adding the diagonals: we introduce a new graph $N\mathbb{L}^d_\boxtimes$, the vertices of which are the boxes $B(x), x \in \mathbb{Z}^d$, and we connect two boxes with an edge of $N\mathbb{L}^d_\boxtimes$ whenever they have non-empty intersection. When $N = 1$, the vertex set of $\mathbb{L}^d_\boxtimes$ is simply $\mathbb{Z}^d$. It is not too hard to show (see [Timár, 2007, Theorem 5.1]) that

$$\text{if } C \text{ is finite, then } \partial C(N) \text{ is a connected subgraph of } N\mathbb{L}^d_\boxtimes. \tag{1.16}$$

Given two diagonally opposite neighbours x, y of $\mathbb{L}^d$, we will write $B(x, y)$ for the intersection $B(x) \cap B(y)$. A percolation cluster C is a *crossing cluster* for some box $B(x)$ or $B(x, y)$, if C contains a vertex from each of the $(d-1)$-dimensional faces of that box. We say that a box $B(x)$ is *good* in a percolation instance ω if it has a crossing cluster C with the property that the intersection of C with each of the boxes $B(x, y)$ contains a crossing cluster (of $B(x, y)$), and every other cluster of $B(x)$ has diameter less than $N/5$. A box that is not good will be called *bad*. It is known [Grimmett, 1999, Theorem 7.61] that, for every $p > p_c$, the probability of having a crossing cluster and no other cluster of diameter greater than $N/5$ converges to 1 as $N \to \infty$. Combining this with a union bound we easily deduce that

for every $p > p_c$, the probability of any box being good converges to 1 as $N \to \infty$. (1.17)

We will say that a set of boxes is bad if all its boxes are bad.

Our definition of good boxes is slightly different than the standard one in that it asks for all boxes $B(x, y)$ to contain a crossing cluster. The reason for imposing this additional property is because now

every $N\mathbb{L}^d_\boxtimes$-component B of good boxes contains a unique percolation cluster C such that some box of B is C-substantial (and in fact all boxes of B are C-substantial). (1.18)

This follows easily once we notice that this holds for pairs of neighbouring boxes.

Observe that the boxes in $\partial C(N)$ are never good. Indeed, if some box $B \in \partial C(N)$ is good, then C connects all the $(d-1)$-dimensional faces of B, hence all $N\mathbb{L}^d$-neighbouring boxes of B contain a connected subgraph of C of diameter at least $N/5$, and so they lie in $C(N)$. This contradicts the fact that B belongs to $\partial C(N)$.

Having introduced the above definitions, our aim now is to find a suitable expression for $1 - \theta$ in terms of good and bad boxes surrounding o.

With the above definitions, we have that, conditioning on the event that C_o is finite and has diameter at least $N/5$, there is a non-empty $N\mathbb{L}^d_\boxtimes$-connected subgraph of bad boxes that separates o from infinity, namely $T := \partial C_o(N)$. However, the event $\{|C_o| < \infty\}$ is not necessarily measurable with respect to the instance inside T. In other words, we cannot express $1 - \theta$ in terms of just the instance inside T, and instead, we have to explore the instance inside the finite components surrounded by T. To this end, we will expand $\partial C_o(N)$ into a larger object.

1.5.2 Separating components

A *separating component* is an $N\mathbb{L}^d_\boxtimes$-connected set S of boxes, such that o lies either inside S or in a finite component of $N\mathbb{L}^d_\boxtimes \setminus S$. We will write $\partial_\boxtimes S$ for its vertex boundary —defined in Section 1.2.1— when viewed as a subgraph of $N\mathbb{L}^d_\boxtimes$. We say that S *occurs* in an instance ω if all the following hold:

(i) all boxes in S are bad;

(ii) all boxes in $\partial_\boxtimes S$ are good, and

(iii) there is an instance ω' which coincides with ω in $S \cup \partial_\boxtimes S$, such that $C_o(\omega')$ is finite, and S contains $\partial C_o(\omega')(N)$.

We will say that ω' is a *witness* for the occurrence of S if (i)–(iii) all hold.

One way to interpret (iii) is that there exists a minimal cut set F surrounding o with the property that all its edges inside $S \cup \partial_\boxtimes S$ are closed in ω. If there is an infinite path in ω starting from o, then it has to avoid the edges of F lying in $S \cup \partial_\boxtimes S$. As we will see, (ii) makes this impossible without violating that $C_o(\omega')$ is finite.

Note that (iii) implies that

$$\partial^V C_o(\omega') \text{ (and } C_o(\omega')\text{) does not share a vertex with the infinite component of } \mathbb{L}^d \setminus S. \tag{1.19}$$

1.5.3 Expressing θ in terms of the probability of the occurrence of a separating component

In this section, we show that C_o is finite exactly when some separating component occurs, unless $diam(C_o) < N/5$, which is a case that is easy to deal with. This will allow us to express $\theta(p)$ in terms of the probability of the occurrence of a separating component (see (1.20)). In the following section we will expand the latter as a sum (with inclusion-exclusion) over all possible separating components. The summands of this sum are well-behaved polynomials, that will allow us to apply Corollary 1.3.15 to deduce the analyticity of $\theta(p)$.

Lemma 1.5.2. *For every $p > p_c$ there is $N \in \mathbb{N}$ and an interval (a, b) containing p such that the following holds for every $q \in (a, b) \cap (p_c, 1]$. Conditioning on C_o*

being finite, and $diam(C_o) \geq N/5$, at least one separating component occurs almost surely.

Proof. Let S be the maximal connected subgraph of $N\mathbb{L}^d_\boxtimes$ that contains $\partial C_o(N)$ and consists of bad boxes only. This S exists whenever C_o is finite and $diam(C_o) \geq N/5$ because $\partial C_o(N)$ is connected by (1.16).

We claim that there is some N and an interval (a, b) containing p such that S is $\mathbb{P}_q$-almost surely finite for every $q \in (a, b) \cap (p_c, 1]$. For this, it suffices to show that for some large enough N, the probability $\mathbb{P}_q$(S has size at least n) converges to 0 as n tends to infinity for each such q. The latter follows by combining the union bound with Lemma 1.5.6 below, which states that

$$\sum_{T \text{ is a separating component of size } n} \mathbb{P}_q(\text{T is bad}) \leq e^{-tn}$$

for some constant $t = t(p) > 0$, for some N, and every q in an interval $(a, b) \cap (p_c, 1]$.

Note that conditions (i) and (ii) are automatically satisfied by the choice of S. The instance $\omega' := \omega$ satisfies condition (iii), since $C_o(\omega)$ is finite, and S contains $\partial C_o(\omega)(N)$ by definition. Thus S occurs in ω, as desired. □

Note that the proof of Lemma 1.5.2 finds a concrete occurring separating component whenever C_o is finite and $diam(C_o) \geq N/5$; we denote this separating component by $\mathcal{S}_o$ in this case.

The next two lemmas provide a converse to Lemma 1.5.2, namely that C_o is finite whenever some separating component occurs.

Whenever ω' is a witness for the occurrence of S, we let $R_o(\omega')$ denote the set of vertices of the infinite component of $\mathbb{L}^d \setminus C_o(\omega')$ lying in S.

Lemma 1.5.3. *Consider a separating component S, and assume that S occurs in ω. Let ω' be a witness of the occurrence of S. Then no vertex of $R_o(\omega')$ lies in $C_o(\omega)$.*

Proof. Assume that some vertex u of $R_o(\omega')$ lies in $C_o(\omega)$; we will obtain a contradiction.

Since $C_o(\omega)$ contains u, there must exist a path P in ω connecting o to u. This path cannot lie entirely in $S \cup \partial_\boxtimes S$ because ω and ω' coincide in that set of boxes and $u \notin C_o(\omega')$. Hence $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$ must have some finite component. Let E denote the minimal edge cut of $C_o(\omega')$. Clearly, P must meet E, since u lies in the infinite component of $\mathbb{L}^d \setminus C_o(\omega')$. Let e be an edge of E that P contains. Notice that no common edge of P and E lies in $S \cup \partial_\boxtimes S$ because the edges of E are closed in ω', the edges of P are open in ω, and the two instances coincide in $S \cup \partial_\boxtimes S$. Hence e must lie in one of the finite components $\mathcal{B}_{in}$ of $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$. Write $\mathcal{B}$ for the set of those boxes in $\partial_\boxtimes S$ that have a $N\mathbb{L}^d_\boxtimes$-neighbour in $\mathcal{B}_{in}$. (Thus $\mathcal{B}$ is the vertex boundary of $\mathcal{B}_{in}$.) See Figure 1.6.

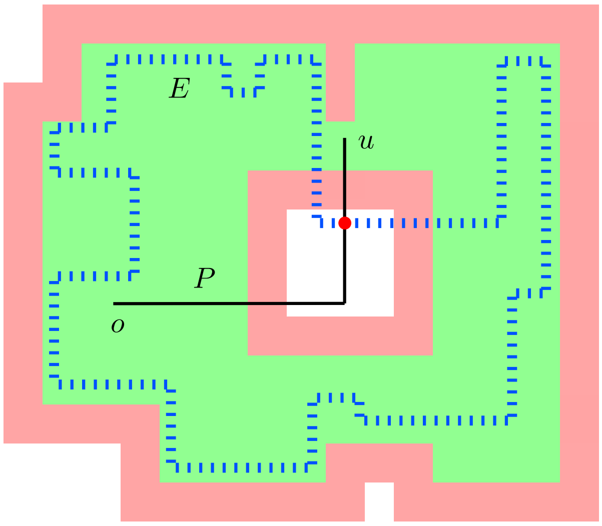

Figure 1.6: The situation in the proof of Lemma 1.5.3. The separating component S is depicted in green and its boundary $\partial_\boxtimes S$ in red (if colour is shown). When two boxes of S and $\partial_\boxtimes S$ overlap, their intersection is depicted also in green. The dashes depict the edges of the cut E, and e is highlighted with a (red) dot.

It is not hard to see that some box B of $\mathcal{B}$ is $C_o(\omega')$-substantial, which then implies that all boxes of $\mathcal{B}$ are $C_o(\omega')$-substantial because they are all good. Indeed, notice that one of the two end-vertices of e lies in $C_o(\omega')$ by the definition of the set E. As S contains a $C_o(\omega')$-substantial box, some box B of $\mathcal{B}$ must be $C_o(\omega')$-substantial, as claimed, because $\mathcal{B}$ is the vertex boundary of $\mathcal{B}_{in}$.

Our aim now is to show that we can connect u to the subgraph of $C_o(\omega')$ inside $\mathcal{B}$ with a path in ω' lying entirely in $S \cup \partial_\boxtimes S$. This will imply that u belongs to $C_o(\omega')$, contradicting that $u \in R_o(\omega')$.

For this, consider the subpath Q of P that starts at u and ends at the last vertex of the intersection of $\mathcal{B}_{in}$ and $\mathcal{B}$ (notice that although $\mathcal{B}_{in}$ and $\mathcal{B}$ are disjoint sets of boxes, the subgraphs of $\mathbb{L}^d$ inside them overlap). If Q is not contained in $S \cup \partial_\boxtimes S$, then we can modify it to ensure that it does lie entirely in $S \cup \partial_\boxtimes S$. Indeed, notice that each $N\mathbb{L}^d_\boxtimes$-component F of $\partial_\boxtimes S$ contains a unique ω-cluster C such that some box of F is C-substantial by (1.18) because all its boxes are good. Moreover, each time Q exits $S \cup \partial_\boxtimes S$, it has to first visit the unique such percolation cluster of some $N\mathbb{L}^d_\boxtimes$-component F of $\partial_\boxtimes S$, and then eventually revisit the same percolation cluster of F. We can thus replace the subpaths of Q that lie outside of $S \cup \partial_\boxtimes S$ by open paths lying entirely in $\partial_\boxtimes S$ that share the same end-vertices. Thus we may assume that Q is contained in $S \cup \partial_\boxtimes S$ as claimed.

Now notice that Q contains a subpath of diameter greater than $N/5$ lying entirely in some box B of $\mathcal{B}$. This box is $C_o(\omega')$-substantial, hence $C_o(\omega')$ and Q must meet. Then following the edges of Q, which are all open in ω', we arrive at u, and thus u belongs to $C_o(\omega')$, as desired. □

We now use this to prove

Lemma 1.5.4. *Whenever some separating component occurs in an instance ω, the cluster $C_o(\omega)$ is finite.*

Proof. We will prove the following slightly stronger statement: whenever a separating component S occurs in an instance ω, a minimal (finite) edge cut of closed edges occurs in ω which separates o from infinity and lies in $S \cup \partial_\boxtimes S$.

For this, consider a witness ω' of the occurrence of S, and let ω'' be the instance which coincides with ω (and ω') on every edge lying in $S \cup \partial_\boxtimes S$, and every other edge of ω'' is open. Note that S occurs in ω'' since it occurs in ω. Thus $C_o(\omega'')$ contains no vertex of $R_o(\omega')$ by Lemma 1.5.3. This implies that $C_o(\omega'')$ contains no vertex in the infinite component X of $N\mathbb{L}^d_\boxtimes \setminus S$ because any path P in $\mathbb{L}$ connecting o to X has to first visit $R_o(\omega')$.

We have just proved that $C_o(\omega'')$ can only contain vertices in S and the finite components of $N\mathbb{L}^d_\boxtimes \setminus S$. Since S is a finite set of boxes, $C_o(\omega'')$ is finite as well. Hence a minimal edge cut of closed edges separating o from infinity occurs in ω''. This minimal edge cut must lie entirely in $S \cup \partial_\boxtimes S$ because all edges not in $S \cup \partial_\boxtimes S$ are open. This is the desired minimal edge cut since it occurs in ω as well. We will denote it by $\partial^b \mathcal{S}_o$. □

Lemmas 1.5.2 and 1.5.4 combined allow us to express the event that C_o is finite in terms of the event that some separating component occurs. To do so, let us write D_N to denote the event $\{diam(C_o) < N/5\}$. Thus we have proved that

$$\begin{aligned} 1-\theta(p) =& \mathbb{P}_p(C_o \text{ is finite}) \\ =& \mathbb{P}_p(D_N) + \mathbb{P}_p(|C_o| < \infty, D_N^{\mathsf{c}}) \\ =& \mathbb{P}_p(D_N) + \mathbb{P}_p(\text{some separating component occurs}, D_N^{\mathsf{c}}). \end{aligned} \tag{1.20}$$

Here and below, the notation $X, Y, \ldots$ denotes the intersection of the events $X, Y, \ldots$.

1.5.4 Expanding θ as an infinite sum of polynomials

Notice that $\mathbb{P}_p(D_N)$ is a polynomial in p since the event D_N depends only on the state of finitely many edges.

Following our technique from Section 1.4.3, we will now use the inclusion-exclusion principle to expand the right-hand side of (1.20) as an infinite sum of polynomials, corresponding to all possible separating components that could occur.

Notice that any two occurring separating components are disjoint because they are connected, their boxes are bad, and they are surrounded by good boxes by definition.

Lemma 1.5.5. *For every $p > p_c$ there is some integer $N = N(p) > 0$ and an interval (a, b) containing p such that the expansion*

$$\mathbb{P}_q(\textit{some } S \textit{ occurs}, D_N^{\mathsf{c}}) = \sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{P}_q(S \textit{ occurs}, D_N^{\mathsf{c}}) \tag{1.21}$$

holds for every $q \in (a, b) \cap (p_c, 1]$, where MS^N denotes the set of all finite collections of pairwise disjoint separating components S, and $c(S)$ denotes the number of separating components of S.

Lemma 1.5.5 follows easily from the next lemma. We will use the notation MS_n^N to denote the set of those finite collections of pairwise disjoint separating components $\{S_1, S_2, \dots, S_k\}$ such that $|S_1| + |S_2| + \dots + |S_k| = n$. The superscript reminds us of the dependence of the boxes on N.

Lemma 1.5.6. *For every $p > p_c$, there are $N = N(p) > 0$, $t = t(p) > 0$ and an interval (a, b) containing p such that*

$$\sum_{S \in MS_n^N} \mathbb{P}_q(S \textit{ is bad}) \leq e^{-tn} \tag{1.22}$$

for every $n \geq 1$ and every $q \in (a, b) \cap (p_c, 1]$.

Proof. To prove the desired exponential decay we will use a standard renormalisation technique with a few modifications. We will first prove the exponential decay when $q = p$, and then we will use a continuity argument to obtain the desired assertion.

We will first show that there exists a constant $k > 0$ depending only on d such that for every $S \in MS_n^N$ we have

$$\mathbb{P}_p(\text{S is bad}) \leq c^{n/k},$$

where $c := \mathbb{P}_p(B(o)$ is bad). Indeed, it is not hard to see that there is a constant $k = k(d) > 0$ such that for every $S \in MS_n^N$ there is a subset Y of S of size at least n/k, all boxes of which are pairwise disjoint. As each box of Y is bad whenever S occurs, we have

$$\mathbb{P}_p(\text{S is bad}) \leq \mathbb{P}_p(\text{Y is bad}).$$

By independence $\mathbb{P}_p(\text{Y is bad}) = c^{n/k}$ and the assertion follows.

We will now find an exponential upper bound for the number of elements of $S \in MS_n^N$. Since $N\mathbb{L}_\boxtimes^d$ is isomorphic to $\mathbb{L}_\boxtimes^d$, there is a constant $\mu > 0$ depending only on d and not on N, such that the number of connected subgraphs of $N\mathbb{L}_\boxtimes^d$ with n vertices containing a given vertex is at most μ^n. However, an element of MS_n^N might contain multiple separating components, and there are in general several possibilities for the reference vertices that each of them contains. To remedy this, consider some axis $X = (\ldots, -x_1, x_0 = B(o), x_1, \ldots)$ of $N\mathbb{L}_\boxtimes^d$ that contains the box $B(o)$, and let X^+, X^- be its two infinite subpaths starting from $B(o)$. We will first show that any separating component of size n contains one of the first n elements of X^+. Indeed, consider an occurring separating component S of size n, and notice that S has to contain some vertex x^+ of X^+, and some vertex x^- of X^-. The graph distance between x^+ and x^- is at most n, as there is a path in S connecting them. This implies that x^+ is one of the first n elements of X^+, as desired.

Consider now a constant $M > 0$ such that $m\mu^m \leq M^m$ for every integer $m \geq 1$. Consider also a partition $\{m_1, m_2, \ldots, m_k\}$ of n. It follows that the number of collections $\{S_1, S_2, \ldots, S_k\}$ with $|S_i| = m_i$ is at most $m_1 m_2 \ldots m_k \mu^n \leq M^n$, since we have at most $m_i \mu^{m_i}$ choices for each S_i. Since there are at most $r^{\sqrt{n}}$ partitions of n by Theorem 1.2.3 (even an exponential bound would be good enough at this point), we can now deduce that the size of MS_n^N is at most $r^{\sqrt{n}} M^n$, implying that

$$\sum_{S \in MS_n^N} \mathbb{P}_p(\text{S is bad}) \leq r^{\sqrt{n}} M^n c^{n/k}.$$

Notice that in the right-hand side of the above inequality, only c depends on N. It is a standard result that c converges to 0 as N tends to infinity [Grimmett, 1999, Theorem 7.61]. Choosing N large enough so that $Mc^{1/k} < 1$, we obtain the desired exponential decay.

Now notice that $c(q) = \mathbb{P}_q(B(o)$ is bad) is a polynomial in q, hence a continuous function, since it depends only on the state of the edges inside $B(o)$. This implies that we can choose an interval (a, b) containing p such that $Mc(q)^{1/k} < 1$ for every $q \in (a, b) \cap (p_c, 1]$. This completes the proof. $\square$

We are now ready to prove Theorem 1.5.1.

Proof of Theorem 1.5.1. Consider some $p \in (p_c, 1]$. Let $N, t > 0$, and the interval (a, b) containing p, be as in Lemma 1.5.6. Then the expression

$$1 - \theta(q) = \mathbb{P}_q(D_N) + \sum_{n=1}^{\infty} \sum_{S \in MS_n^N} (-1)^{c(S)+1} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}})$$

holds for every $q \in (a, b) \cap (p_c, 1]$, and furthermore

$$\Big| \sum_{S \in MS_n^N} (-1)^{c(S)+1} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}}) \Big| \leq e^{-tn}$$

for every $q \in (a, b) \cap (p_c, 1]$. The probability $\mathbb{P}_q(D_N)$ is a polynomial in q, hence analytic, because it depends on finitely many edges. Moreover, the event $\{\text{S occurs}, D_N^{\mathsf{c}}\}$ depends only on the state of the edges lying in $S \cup \partial_{\boxtimes} S$ and the box $B(o, N)$. The number of edges of each box is $O(N^d)$, hence the event $\{\text{S occurs}, D_N^{\mathsf{c}}\}$ depends only on $O(N^d n)$ edges. The desired assertion follows now from Corollary 1.3.15. □

Remark 1.5.7. *It might seem surprising to the reader that the BK inequality is being used in the proof of the analyticity of θ in 2-dimensions but not in higher dimensions. It is worth pointing out that there is a proof of the higher dimensional result which uses the BK inequality but it is not simpler than the one we just presented. On the other hand, the BK inequality does simplify the argument in 2-dimensions (see Remark 1.4.12).*

1.5.5 Exponential tail of $\partial^b \mathcal{S}_o$

As we will now see, Lemma 1.5.6 easily implies that the size of $\partial^b \mathcal{S}_o$, as defined in the proof of Lemma 1.5.4, has an exponential tail:

Theorem 1.5.8. *For every $p > p_c$, there are constants $N = N(p) > 0$ and $t = t(p) > 0$ such that*

$$\mathbb{P}_p(|\partial^b \mathcal{S}_o| \geq n) \leq e^{-tn}$$

for every $n \geq N$.

Remark 1.5.9. *In the statement of Theorem 1.5.8, the dependence on N is implicit in the definition of $\partial^b \mathcal{S}_o$ in terms of boxes of $N\mathbb{L}^d_{\boxtimes}$.*

Proof. Assume that $|\partial^b \mathcal{S}_o| \geq n$, and consider the separating component S associated with C_o. Then the boxes of $S \cup \partial_{\boxtimes} S$ must contain at least n edges of $\mathbb{L}^d$. Hence the number of boxes of $S \cup \partial_{\boxtimes} S$ is at least cn/N^d for some constant $c > 0$. Moreover, we have $|\partial_{\boxtimes} S| \leq (3^d - 1)|S|$ because each box of $\partial_{\boxtimes} S$ has at least one neighbour in S, and each box in S has at most $3^d - 1$ neighbours. Therefore, S contains at least $cn/(3N)^d$ boxes. The desired assertion follows from Lemma 1.5.6. □

We recall that for every $p \in (p_c, 1-p_c)$, the probability $\mathbb{P}_p(|\partial C_o| \geq n)$ does not decay exponentially in n (Kesten and Zhang [1990]). This implies that for those values of p, $\partial^b \mathcal{S}_o$ has typically smaller order of magnitude than the standard minimal edge cut of C_o.

As a corollary, we re-obtain a result of Pete [2008] which states that when C_o is finite, the number of touching edges between C_o and the unique infinite cluster, which we denote C_∞, has an exponential tail. A *touching edge* is an edge in $\partial^E C_o \cap \partial^E C_\infty$. We denote the number of (closed) touching edges joining C_o to the infinite component C_∞ by $\phi(C_o, C_\infty)$.

Corollary 1.5.10. *For every $p > p_c$, there is some $c = c(p,d) > 0$ such that*

$$\mathbb{P}_p(|C_o| < \infty, \phi(C_o, C_\infty) \geq t) \leq e^{-ct}$$

for every $t \geq 1$.

Proof. The result follows from Theorem 1.5.8 by observing that C_∞ has to lie in the unbounded component of $\mathbb{L}^d \setminus \partial^b \mathcal{S}_o$, hence all relevant edges belong to $\partial^b \mathcal{S}_o$. □

1.6 Analyticity of τ

In the previous section we proved that θ is analytic above p_c. Some further challenges arise when one tries to prove that other functions describing the macroscopic behaviour of our model are analytic functions of p. The main obstacle is that events of the form $\{x \text{ is connected to } y\}$ are not fully determined, in general, by the instance inside $S \cup \partial_{\boxtimes} S$ of a separating component S (recall the relevant definitions introduced in Section 1.5.2). In this section, we show how one can remedy this issue, and we will prove that the k-point function $\tau = \mathbb{P}(x_1, x_2, \dots, x_k \text{ belong to the same cluster})$ and its truncated version $\tau^f = \mathbb{P}(x_1, x_2, \dots, x_k \text{ belong to the same finite cluster})$ are analytic functions above p_c. We will then deduce that the truncated susceptibility $\mathbb{E}(|C_o|; |C_o| < \infty)$ and the free energy $\mathbb{E}(|C_o|^{-1})$ are analytic functions as well. One can prove that the results of this section hold for planar quasi-transitive lattices as well. This can be done by using the notion of outer interfaces (see Georgakopoulos and Panagiotis [2018]).

Given a k-tuple $\boldsymbol{x} = \{x_1, \dots, x_k\}, k \geq 2$ of vertices of $\mathbb{Z}^d$, the function $\tau_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a cluster of Bernoulli percolation on $\mathbb{Z}^d$ with parameter p. Similarly, $\tau^f_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a *finite* cluster. We will write $MS^N(\boldsymbol{x})$ for the set of all finite collections of separating components surrounding some vertex of $\boldsymbol{x}$, and $MS^N_n(\boldsymbol{x})$ for the set of those elements of $MS^N(\boldsymbol{x})$ that have size n.

Arguing as in the proof of Lemma 1.5.6, we obtain the following:

Lemma 1.6.1. *For every $p > p_c$, there are $N = N(p) > 0$, $t = t(p) > 0$, and an interval (a, b) containing p, such that*

$$\sum_{S \in MS^N_n} \mathbb{P}_q(S \text{ occurs}) \leq e^{-tn} \tag{1.23}$$

for every $n \geq 1$ and every $q \in (a, b) \cap (p_c, 1]$.

We are now ready to prove that τ and τ^f are analytic.

Theorem 1.6.2. *For every $d \geq 2$ and every finite set $\boldsymbol{x}$ of vertices of $\mathbb{Z}^d$, the functions $\tau_{\boldsymbol{x}}(p)$ and $\tau^f_{\boldsymbol{x}}(p)$ admit analytic extensions to a domain of $\mathbb{C}$ that contains the interval $(p_c, 1]$.*

Moreover, for every $p \in (p_c, 1]$ and every finite set $\boldsymbol{x}$ such that $diam(\boldsymbol{x}) \geq N/5$, there is a closed disk $D(p, \delta), \delta > 0$ and positive constants $c_1 = c_1(p, \delta), c_2 = c_2(p, \delta)$ such that

$$|\tau^f_{\boldsymbol{x}}(z)| \leq c_1 e^{-c_2 diam(\boldsymbol{x})}$$

for every $z \in D(p, \delta)$ for such an analytic extension $\tau^f_{\boldsymbol{x}}(z)$ of $\tau^f_{\boldsymbol{x}}(p)$.

Proof. We start by showing that $\tau^f_{\boldsymbol{x}}(p)$ is analytic. Suppose $\boldsymbol{x} = \{x_1, \dots, x_k\}$, and let A be the event that $diam(C_{x_i}) \geq N/5$ for every $i \leq k$. We will write $\{\boldsymbol{x} \text{ is connected}\}$ to denote the event that all vertices of $\boldsymbol{x}$ belong to the same cluster. When $\{\boldsymbol{x} \text{ is connected}\}$ occurs, we will write $C_{\boldsymbol{x}}$ corresponding cluster. When both events $\{\boldsymbol{x} \text{ is connected}\}$ and A occur and $C_{\boldsymbol{x}}$ is finite, we will write $\mathcal{S}_{\boldsymbol{x}}$ for the separating component of the latter cluster, namely the $N\mathbb{L}^d_{\boxtimes}$-component of $\partial C_{\boldsymbol{x}}(N)$. The event {S occurs} is defined as in the previous section except that now C_o is replaced by $C_{\boldsymbol{x}}$, i.e. the event $\{\boldsymbol{x} \text{ is connected}\}$ occurs in a witness ω', and S contains $\partial C_{\boldsymbol{x}}(\omega')(N)$. With the above definitions, we have

$$\tau^f_{\boldsymbol{x}}(p) = \mathbb{P}_p(A^{\mathsf{c}}, \boldsymbol{x} \text{ is connected}) + \sum_{S} \mathbb{P}_p(A, \boldsymbol{x} \text{ is connected}, \mathcal{S}_{\boldsymbol{x}} = S),$$

where the sum ranges over all possible separating components separating all of $\boldsymbol{x}$ from infinity.

Our aim is to further decompose the events of the above expansion into simpler ones that we have effectively estimate, and then use the inclusion-exclusion principle. We will first introduce some notation. Given a separating component S as above, we first decompose $\boldsymbol{x}$ into two sets $\boldsymbol{x}_{out}$ and $\boldsymbol{x}_{in}$, where $\boldsymbol{x}_{out}$ denotes the set of those vertices of $\boldsymbol{x}$ lying in some finite component of $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$, and $\boldsymbol{x}_{in} := \boldsymbol{x} \setminus \boldsymbol{x}_{out}$ its complement. We write $\{\boldsymbol{x} \to S\}$ for the event that no separating component separating some $x_i \in \boldsymbol{x}$ from S occurs; to be more precise, the event $\{\boldsymbol{x} \to S\}$ means that for each $x_i \in \boldsymbol{x}_{out}$, no separating component that surrounds x_i and lies entirely in some of the finite components of $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$ occurs.

Consider now some vertex x in $\boldsymbol{x}_{out}$, and let F be the component of $\partial_\boxtimes S$ that separates x from S. We claim that when S and the events A, $\{\boldsymbol{x} \to S\}$ all occur, then x is connected to the unique large percolation cluster inside F. In particular, if another vertex of $\boldsymbol{x}$ lies in the same finite component of $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$ as x does, then both vertices are connected to the unique large cluster of F, hence they are connected to each other. To see that the claim holds, notice that C_x has to be finite because $S \cup \partial_\boxtimes S$ contains a minimal edge cut of closed edges that surrounds all vertices of $\boldsymbol{x}$, hence x. Now $\partial C_x(N)$ has to intersect S because it cannot lie entirely in $N\mathbb{L}^d_\boxtimes \setminus (S \cup \partial_\boxtimes S)$ by our assumption. This implies that x is connected to some vertex inside S, hence it must first visit the unique large cluster of F, as desired.

We now define $\mathcal{C}$ to be the event that

- all vertices of $\boldsymbol{x}_{in}$ are connected to each other with open paths lying in $S \cup \partial_\boxtimes S$,
- the unique large percolation clusters of the components F of $\partial_\boxtimes S$ that separate some $x_i \in \boldsymbol{x}_{out}$ from S are connected to each other with open paths lying $S \cup \partial_\boxtimes S$,
- all vertices of $\boldsymbol{x}_{in}$ are connected to all such percolation clusters with open paths lying in $S \cup \partial_\boxtimes S$.

(It is possible that either $\boldsymbol{x}_{in}$ or $\boldsymbol{x}_{out}$ is the empty set, in which case the third item and one of the first two are empty statements.) We claim that when S and the events A, $\{\boldsymbol{x} \to S\}$ and $\{\boldsymbol{x} \text{ is connected}\}$ all occur, then the event $\mathcal{C}$ occurs as well. Indeed, consider a vertex $x \in \boldsymbol{x}_{out}$, and let F be the component of $\partial_\boxtimes S$ that separates x from S, as above. Any open path connecting x to some vertex of $\boldsymbol{x}_{out}$ which does not lie in the same finite component of $N\mathbb{L}^d_\boxtimes$ that x does, has to first visit the unique large percolation cluster of F. Hence it suffices to prove that when two vertices x_i and x_j of $\boldsymbol{x}_{in}$ lie in the same cluster, there is always an open path connecting them lying entirely in $S \cup \partial_{\boxtimes} S$. To this end, assume that there is a path P in ω connecting x_i to x_j, which does not lie entirely in $S \cup \partial_{\boxtimes} S$. Arguing as in the proof of Lemma 1.5.3, we can modify P to obtain an open path P' connecting x_i to x_j which lies entirely in $S \cup \partial_{\boxtimes} S$. The desired claim follows now easily.

Combining the above claims, we conclude that the events $\{A, \boldsymbol{x}$ is connected, $\mathcal{S}_{\boldsymbol{x}} = S\}$ and $\{A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}\}$ coincide, and thus

$$\mathbb{P}_p(A, \boldsymbol{x} \text{ is connected}, \mathcal{S}_{\boldsymbol{x}} = S) = \mathbb{P}_p(A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}).$$

Using the inclusion-exclusion principle we obtain that

$$\begin{aligned} \mathbb{P}_p(A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}) = \mathbb{P}_p(A, \mathcal{C}, S \text{ occurs}) + \\ \sum_T (-1)^{c(T)} \mathbb{P}_p(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}), \end{aligned} \tag{1.24}$$

where the latter sum ranges over all finite collections T of separating components separating $\boldsymbol{x}$ from S. Collecting now all the terms we obtain that

$$\begin{aligned} \tau^f_{\boldsymbol{x}}(p) = \mathbb{P}_p(A^{\mathsf{c}}, \boldsymbol{x} \text{ is connected}) + \\ \sum_S \Big(\mathbb{P}_p(A, \mathcal{C}, S \text{ occurs}) + \sum_T (-1)^{c(T)} \mathbb{P}_p(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big). \end{aligned} \tag{1.25}$$

Notice that by combining S and T we obtain an element of $MS^N(\boldsymbol{x})$. Moreover, A depends on the state of a fixed finite set of edges, while $\mathcal{C}$ depends on the state of the edges inside $S \cup \partial_{\boxtimes} S$. Hence we can use Lemma 1.6.1, and then argue as in the proof of Theorem 1.5.1 to obtain that $\tau^f_{\boldsymbol{x}}$ is analytic above p_c.

We will now prove the analyticity of $\tau_{\boldsymbol{x}}$. Since $\tau^f_{\boldsymbol{x}}$ is analytic, it suffices to prove that $\tau_{\boldsymbol{x}} - \tau^f_{\boldsymbol{x}}$ is analytic. It is well-known that the infinite cluster is unique in our setup Burton and Keane [1989], and this implies that $\tau_{\boldsymbol{x}} - \tau^f_{\boldsymbol{x}} = \mathbb{P}(|C_{x_1}| = \infty, \ldots, |C_{x_k}| = \infty)$. The latter probability is complementary to $\mathbb{P}(\cup_{i=1}^k \{|C_{x_i}| < \infty\})$, which is in turn equal to

$$\mathbb{P}(\cup_{i=1}^k \{|C_{x_i}| < \infty\}) = \mathbb{P}(A^{\mathsf{c}}) + \mathbb{P}(\big(\cup_{i=1}^k \{|C_{x_i}| < \infty\}\big) \cap A).$$

Define the event {S occurs for some $x_i \in \boldsymbol{x}$} as in the previous section except that now we require the existence of a witness ω' such that S contains $\partial C_{x_i}(\omega')(N)$ for some $x_i \in \boldsymbol{x}$. We can expand the latter term as an infinite sum using the inclusion-exclusion principle to obtain

$$\mathbb{P}\big(\big(\cup_{i=1}^{k}\{|C_{x_i}|<\infty\}\big)\cap A\big)=\sum(-1)^{c(S)+1}\mathbb{P}(\text{S occurs for some } x_i\in\boldsymbol{x},A),$$

where now we require our separating components to surround some $x_i \in \boldsymbol{x}$. Arguing as in the proof of Theorem 1.5.1, we obtain that $\tau_{\boldsymbol{x}} - \tau_{\boldsymbol{x}}^{f}$ is analytic, as desired.

For the second claim of the theorem, notice that when $diam(\boldsymbol{x}) \geq N/5$, the probability $\mathbb{P}(A^{\mathsf{c}}, \boldsymbol{x}$ is connected) is equal to 0. Hence our expansion for $\tau_{\boldsymbol{x}}^{f}$ simplifies to

$$\tau_{\boldsymbol{x}}^{f}(p)=\sum_{S}\Big(\mathbb{P}(A,\mathcal{C},S\text{ occurs})+\sum_{T}(-1)^{c(T)}\mathbb{P}(A,T\text{ occurs},\mathcal{C},S\text{ occurs})\Big).$$

Our goal is to show that for every $p > p_c$, there are some constants $\delta, t > 0$ such that

$$\Big|\sum_{|S|=n}\Big(\mathbb{P}_z(A,\mathcal{C},S\text{ occurs})+\sum_{T}(-1)^{c(T)}\mathbb{P}_z(A,T\text{ occurs},\mathcal{C},S\text{ occurs})\Big)\Big|\leq e^{-tn} \tag{1.26}$$

for every $z \in D(p, \delta)$ for the analytic extensions of the above probabilities. Then the desired claim will follow easily from the observation that any plausible separating component S of $\boldsymbol{x}$ must have size $\Omega(diam(\boldsymbol{x}))$.

Notice that the event A depends only on finitely many edges. Moreover, the events $\mathcal{C}$ and $\{S$ occurs$\}$ depend on $O(|S|)$ edges, while the event $\{T$ occurs$\}$ depends on $O(|T|)$ edges. We can now use Lemma 1.3.3 to conclude that there is a constant $c = c(p, \delta, N) > 1$ (perhaps slightly larger than that of Lemma 1.3.3) such that

$$|\mathbb{P}_z(A,\mathcal{C},S\text{ occurs})|\leq c^{|S|}\mathbb{P}_{p'}(A,\mathcal{C},S\text{ occurs})$$

and

$$|\mathbb{P}_z(A,T\text{ occurs},\mathcal{C},S\text{ occurs})|\leq c^{|S|+|T|}\mathbb{P}_{p'}(A,T\text{ occurs},\mathcal{C},S\text{ occurs})$$

for every $z \in D(p, \delta)$, where $p' = p + \delta$ if $p < 1$, and $p' = 1 - \delta$ if $p = 1$. Moreover, we can always choose c in such a way that $c \to 1$ as $\delta \to 0$. Hence we have

$$\begin{aligned} \Big| \sum_{|S|=n} \Big(\mathbb{P}_z(A, \mathcal{C}, S \text{ occurs}) + \sum_T (-1)^{c(T)} \mathbb{P}_z(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big) \Big| \leq \\ c^n \sum_{|S|=n} \Big(\mathbb{P}_{p'}(A, \mathcal{C}, S \text{ occurs}) + \sum_T c^{|T|} \mathbb{P}_{p'}(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big) \end{aligned} \tag{1.27}$$

by the triangle inequality. It follows from Lemma 1.6.1 that the sum

$$\sum_{|S|=n} \Big(\mathbb{P}_{p'}(A, \mathcal{C}, S \text{ occurs}) + \sum_T \mathbb{P}_{p'}(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big)$$

decays exponentially in n, and by choosing δ small enough we can ensure that

$$c^n \sum_{|S|=n} \Big(\mathbb{P}_{p'}(A, \mathcal{C}, S \text{ occurs}) + \sum_T c^{|T|} \mathbb{P}_{p'}(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big)$$

decays exponentially in n as well, hence (1.26) holds. The proof is now complete. □

Using Theorem 1.6.2 we can now prove the following results.

Theorem 1.6.3. *For every $k \geq 1$ and every $d \geq 2$, the functions $\chi_k^f(p) := \mathbb{E}_p(|C_o|^k; |C_o| < \infty)$ are analytic in p on the interval $(p_c, 1]$.*

Proof. We observe that, by the definitions,

$$\chi_k^f(p) = \mathbb{E}_p\Big(\big(\sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{x \in C_o, |C_o| < \infty\}} \big)^k \Big) = \sum_{\boldsymbol{x}} \tau^f_{\boldsymbol{x}'},$$

where the latter sum ranges over all possible sequences $\boldsymbol{x}$ over $\mathbb{Z}^d$ of length k that contain o, and $\boldsymbol{x}'$ denotes the set of distinct elements in $\boldsymbol{x}$. The probabilities $\tau^f_{\boldsymbol{x}'}$ admit analytic extensions by Theorem 1.6.2, and so it suffices to prove that the sum $\sum_{\boldsymbol{x}} \tau^f_{\boldsymbol{x}'}$ converges uniformly on an open neighbourhood of $(p_c, 1]$. This follows from the fact that the polynomial growth of $\mathbb{Z}^d$ and the exponential decay of $|\tau^f_{\boldsymbol{x}}(z)|$ in the diameter of $\boldsymbol{x}$ proved in Theorem 1.6.2. □

Theorem 1.6.4. *For every $d \geq 2$, the free energy $\kappa = \mathbb{E}(|C_o|^{-1})$ is analytic in p on the interval $(p_c, 1]$.*

Proof. It is known Aizenman and Newman [1987] that κ is differentiable on $(p_c, 1)$ with derivative equal to

$$f(p) := \frac{1}{2(1-p)} \sum_{x \in N(o)} \big(1 - \tau_{\{o,x\}}(p)\big).$$

Since each $\tau_{\{o,x\}}$ is analytic on the interval $(p_c, 1]$, and $\tau_{\{o,x\}}(1) = 1$, f is analytic on $(p_c, 1]$ as well. So far we know that κ coincides with a primitive F of f only on $(p_c, 1)$, which implies that κ is analytic on that interval. In fact, κ coincides with F on the whole interval $(p_c, 1]$. Indeed, we simply need to verify that κ is continuous from the left at 1. To see this, notice that $\kappa(1) = 1 - \theta(1) = 0$ and $\kappa(p) \leq 1 - \theta(p)$. Since θ is continuous from the left at 1, which follows e.g. by Theorem 1.5.1, we have that κ is continuous from the left at 1 as well, hence coincides with F on the whole interval $(p_c, 1]$. It now follows that κ is analytic in p on the interval $(p_c, 1]$, as desired. □

1.7 Continuum Percolation

In this section, we will prove analyticity results for the Boolean model in $\mathbb{R}^2$ analogous to Theorem 1.4.1, answering a question of Last et al. [2017].

Let P_λ be a Poisson point process in $\mathbb{R}^d$ of intensity λ and let $\mathcal{N}(B)$ denote the number of points inside a bounded subset B of $\mathbb{R}^d$. The Boolean model is obtained by taking the union $\mathcal{Z}$ of disks of radius r, called *grains*, centred at the points of P_λ. The random radii are independent random variables and have the same distribution as another positive random variable ρ. They are also independent of P_λ. We denote (P_λ, ρ) the Boolean model with random radii sampled from ρ. If ρ is equal to a positive constant r we will write (P_λ, r).

The random set $\mathcal{Z}$ is called the *occupied region* and its complement $\mathcal{V}$ is called the *vacant region*. We will denote by $W(0)$ the connected component of $\mathcal{Z}$ containing 0 ($W(0) = \emptyset$ if 0 is not occupied) and $V(0)$ the connected component of $\mathcal{V}$ containing 0 ($V(0) = \emptyset$ if 0 is occupied).

It is well-known that for every non-negative random variable ρ, there is a critical value λ_c such that for every $\lambda > \lambda_c$ there is almost surely a (unique) occupied unbounded connected component Z_∞, but no unbounded connected components exist whenever $\lambda < \lambda_c$. It is possible that the critical value is equal to 0 or infinity. Under the assumptions that $\mathbb{E}(\rho^{2d-1}) < \infty$, where d is the dimension of our space, and $\mathbb{P}(\rho = 0) < 1$ we have $0 < \lambda_c < \infty$. An important tool in the study of Z_∞ is the *percolation density* $\theta_0 := \mathbb{P}_\lambda(0 \in Z_\infty)$ of Z_∞ (also called 'volume fraction' or 'percolation function'). For an introduction to the subject see Meester and Roy [1996]; Penrose [2003].

Under general assumptions on the grain distribution, θ_0 is continuous for every $\lambda \neq \lambda_c$ and $d \geq 2$, and $\theta_0(\lambda_c) = 0$ when $d = 2$ Meester and Roy [1996]. Similarly to the standard percolation model on $\mathbb{Z}^2$, it is expected that the latter holds for every $d \geq 3$ as well.

Much more is known about the behaviour of θ_0 on the interval (λ_c, ∞). Recently, it has been proved in Last et al. [2017] that θ_0 is infinitely differentiable on (λ_c, ∞) under general assumptions on the grain distribution. The authors asked whether θ_0 is analytic in that interval, and we answer this question in the affirmative when $d = 2$. For simplicity, we will assume that all discs have radius 1, although our proof easily extends to the case where the radii are bounded from above and below.

Theorem 1.7.1. *Consider the Boolean model $(P_\lambda, 1)$ in $\mathbb{R}^2$. Then θ_0 is analytic on (λ_c, ∞).*

The proof of Theorem 1.7.1 will follow the lines of that of Theorem 1.4.1. One of the main tools in the proof of the latter is the exponential decay of the probability $\mathbb{P}_p(\text{some } S \in \mathcal{MS}_n \text{ occurs})$, which follows from the exponential decay property, duality, and the BK inequality. In the case of the Boolean model, we will define another notion of outer interface and our goal once again is to show that the probability of having large multi-interfaces decays exponentially in their size. However, the Boolean model lacks a notion of duality which leads to certain complications. Nevertheless, it is still true that the probability $\mathbb{P}_\lambda(\mu(V(0)) \geq a)$, where $\mu(A)$ denotes the area of a set $A \subset \mathbb{R}^2$, decays exponentially in a for every fixed $\lambda > \lambda_c$, which we will combine with the more general Reimer inequality Gupta and Rao [1999], instead of the BK inequality, to show the desired exponential decay.

Before stating the Reimer inequality let us fix some notation. We denote a sample of the Boolean model (P_λ, ρ) by $\omega = \{(x_i, r_i) : i = 1, 2, \ldots\}$, where (x_i) is the sequence of points of the Poisson point process and (r_i) the associated sequence of radii. The *restriction* of ω to a set $K \subset \mathbb{R}^d$ is

$$\omega_K := \{(x_i, r_i) \in \omega : x_i \in K\}.$$

We also define

$$[\omega]_K := \{\omega' : \omega'_K = \omega_K\}.$$

We say that an event A *lives on* a set U if $\omega \in A$ and $\omega' \in [\omega]_U$ imply $\omega' \in A$. For A and B living on a bounded region U we define the event

$$\begin{aligned} A \square B = \{\omega :\ & \text{there are disjoint sets } K, L, \text{ each a finite union of} \\ & \text{rectangles with rational coordinates, with } [\omega]_K \subset A, [\omega]_L \subset B\}. \end{aligned} \tag{1.28}$$

When $A \square B$ occurs we say that A and B *occur disjointly*.

Theorem 1.7.2. *(Reimer inequality)Reimer [2000]; Gupta and Rao [1999] Let U be a bounded measurable set in $\mathbb{R}^d$. For any two events A and B living on U we have*

$$\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B).$$

We would like to apply the Reimer inequality to families of more than two events but the $\Box$ operation is not in general associative. For this reason, given events $A_1, A_2, \dots, A_m$ we define

$$A_1\Box A_2\Box\dots\Box A_m := \{\omega : \text{ there are disjoint sets } K_1, K_2, \dots K_m \text{ each a finite union of rectangles with rational coordinates, with } [\omega]_{K_i} \subset A_i \text{ for every } i = 1, 2, \dots, m\}.$$

Notice that the event $A_1\Box A_2\Box\dots\Box A_m$ is not in general equal to the event $((\dots(A_1\Box A_2)\Box A_3)\Box\dots\Box A_m)$ obtained by successively applying the $\Box$ operation, but the former event is always contained in the later. Hence the Reimer inequality implies that $\mathbb{P}(A_1\Box A_2\Box\dots\Box A_m) \leq \mathbb{P}(A_1)\mathbb{P}(A_2)\dots\mathbb{P}(A_m)$ whenever the events $A_1, A_2, \dots, A_m$ live on a bounded measurable set.

Before delving into the details of the proof of Theorem 1.7.1 let us give some more definitions. The area of an open set Ω is denoted by $\mu(\Omega)$, and the length of a curve γ by $\mathcal{L}(\gamma)$. We will write $\overline{D(x)}$ for the closed unit disk centred at x and $\overline{\mathbb{D}}$ for the closed unit disk centred at 0. The *Minkowski sum* of two sets $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ is defined as the set

$$\Omega_1 + \Omega_2 := \{a + b : a \in \Omega_1, b \in \Omega_2\}.$$

Given $Y = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^2$ we define

$$\Omega(Y) := \cup_{i=1}^n \overline{D(x_i)}.$$

In case $\Omega(Y)$ is not simply connected, consider the bounded connected components $C_1, C_2, \dots, C_k$ of its complement and define

$$\tilde{\Omega} = \tilde{\Omega}(Y) := (\cup_{i=1}^k C_k) \cup \Omega(Y).$$

Let us now focus on the function θ_0. If $0 \notin Z_\infty$, then there are two possibilities:

(i) either there is no point of P_λ in $\overline{\mathbb{D}}$

(ii) or there are points $x_1, x_2, \dots, x_n$ of P_λ in $W(0)$ such that $\Omega := \Omega(\{x_1, \dots, x_n\})$

is connected and there is no point of $P_\lambda \setminus \{x_1, \dots, x_n\}$ at distance $r \leq 1$ from $\partial\tilde{\Omega}$.

This observation leads to the following definition. Consider a subset $Y = \{x_1, x_2, \dots, x_n\}$ of $\mathbb{R}^2$ satisfying

(i) $\Omega := \Omega(Y)$ is connected;

(ii) $0 \in \tilde{\Omega}$; and

(iii) $\overline{D(x_i)} \cap \partial\tilde{\Omega}$ contains an arc of positive length for every $i = 1, \dots, n$.

Then we call $\partial\tilde{\Omega}$ an *outer interface* and we denote it by $J(Y)$. The set $S(Y) := J(Y) + \overline{\mathbb{D}}$ is called a *separating strip.* We say that a set Y as above *happens to separate in* P_λ if $Y \subset P_\lambda$ and no other point of $S(Y)$ belongs to P_λ. We say that $S(Y)$ *occurs* whenever Y happens to separate in P_λ.

There is a subtle point in the latter definition. It is possible that there are points x in $P_\lambda \setminus Y$ such that $\overline{D(x_i)} \cap J \neq \emptyset$ but does not contain an arc of positive length. However, this is an event of measure 0 and so we can disregard it.

To avoid such trivialities, we will always assume that no pair of points x_i, x_j of P_λ have distance 2, which implies that no pair of disks touch. We can do so as this event has measure 0.

The following lemma is an easy consequence of the definitions.

Lemma 1.7.3. *If Y_1 and Y_2 happen to separate in P_λ and $S(Y_1), S(Y_2)$ have non-empty intersection, then $Y_1 = Y_2$.* □

This leads us to define a *multi-interface* as a finite set of pairwise disjoint outer interfaces and a *separating multi-strip* as a finite set of pairwise disjoint separating strips. A separating multi-strip *occurs* if each of its separating strips occurs.

Using the above definitions we obtain

$$1 - \theta_0(\lambda) = \mathbb{P}_\lambda(0 \notin Z_\infty) = \mathbb{P}_\lambda(\text{some } S(Y) \text{ occurs})$$

for every $\lambda > \lambda_c$. The second equality follows from the fact that whenever $0 \notin Z_\infty$ and no Y happens to separate in P_λ, 0 belongs to an infinite vacant component, and this event has measure 0 for every $\lambda > \lambda_c$ Meester and Roy [1996].

Once again we intend to use the inclusion-exclusion principle to obtain the formula

$$\mathbb{P}_\lambda(\text{some } S(Y) \text{ occurs}) = \sum_{k=1}^{\infty} (-1)^{k+1} \mathbb{E}_\lambda(N(k))$$

for every $\lambda \in (\lambda_c, \infty)$, where $N(k)$ is the number of occurring separating multi-strips comprising k separating strips.

To prove the validity of the above formula we will show that the alternating sum converges absolutely. In order to do so, we first express the above expectations as an infinite sum according to the area of $S(Y_i)$, i.e.

$$\mathbb{E}_\lambda(N(k)) = \sum_{\{m_1,\dots,m_k\}} \mathbb{E}_\lambda(N(m_1,\dots,m_k)),$$

where the sum in the right-hand side ranges over all multi-sets of k positive integers, and $N(\{m_1,\dots,m_k\})$ is the number of occurring separating multi-strips $S = \{S_1,\dots,S_k\}$ with $\lfloor \mu(S_i) \rfloor = m_i$.

Let us define $\mathcal{MS}_n$ to be the set of separating multi-strips $S = \{S_1,\dots,S_k\}$ with $\lfloor \mu(S_1) \rfloor + \dots + \lfloor \mu(S_i) \rfloor = n$. We denote by N_n the number of occurring separating multi-strips of $\mathcal{MS}_n$. The analogue of Lemma 1.4.11 is

Lemma 1.7.4. *For every $\lambda \in (\lambda_c, \infty)$ there are constants $c_1 = c_1(\lambda)$ and $c_2 = c_2(\lambda)$ with $c_2 < 1$ such that for every $n \in \mathbb{N}$,*

$$\mathbb{E}_\lambda(N_n) \leq c_1 c_2^n. \tag{1.29}$$

Notice that whenever a separating strip S occurs, a subset of S is vacant. Thus we are lead to use the exponential decay in a of the probability $\mathbb{P}_\lambda(\mu(V(0)) \geq a)$ for every $\lambda > \lambda_c$ Meester and Roy [1996]. However, we cannot directly apply the aforementioned exponential decay as it is possible for the area of the vacant subset of S to be relatively small compared to the area of S.

In order to overcome this difficulty, we fix a $\lambda > \lambda_c$ and consider a small enough $1 > \varepsilon > 0$ such that $\lambda_c(B_{1-\varepsilon}) < \lambda$, where $\lambda_c(B_{1-\varepsilon})$ is the critical point of the Poisson Boolean model $(P_\lambda, 1-\varepsilon)$. We couple the two models by sampling a Poisson point process with intensity λ in $\mathbb{R}^2$ and placing two disks, one of radius 1 and another of radius $1-\varepsilon$, centred at each point of the process. We notice that whenever a separating strip $S = S(Y)$ occurs in $(P_\lambda, 1)$, the set $S(\varepsilon) := J(Y) + D(0, \varepsilon)$ is vacant in $(P_\lambda, 1-\varepsilon)$ in our coupling and our goal is to show that this happens with probability that decays exponentially in the area of S.

First, we need to show that $\mu(S(\varepsilon))$ and $\mu(S)$ are of the same order. We do so in the following purely geometric lemma.

Lemma 1.7.5. *Let $1 > \varepsilon > 0$. Then there is a constant $\gamma = \gamma(\varepsilon) > 0$, such that for* *every separating strip $S = S(Y)$ we have*

$$\mu\big(S(\varepsilon)\big) \geq \gamma\mu(S).$$

Proof. Let Y be the corresponding set of centres, and $J = J(Y)$ the corresponding outer interface of S. For each element y of Y, we choose a point in $\overline{D(y)} \cap J$. We denote the set of all those points by T. Clearly, $T + D(0,\varepsilon)$ lies in $S(\varepsilon)$, but it many happen that some disks of this collection overlap. For this reason, we consider a subset T' of T such that for any $x_1, x_2 \in T'$, the disks $D(x_1,\varepsilon)$ and $D(x_2,\varepsilon)$ are disjoint, and T' is maximal for this property. It follows from the definition of T' that

$$\mu\big(S(\varepsilon)\big) \geq \sum_{x\in T'} \mu\big(D(x,\varepsilon)\big).$$

Notice that for any $z \in T \setminus T'$, there is some $x \in T'$ such that the disks $D(z,\varepsilon)$ and $D(x,\varepsilon)$ overlap, by the maximality of T'. This implies that any $y \in Y$ is at distance at most $1 + 2\varepsilon$ from some element $x \in T'$ by the definition of T. We can now conclude that any $w \in S$ is at distance at most $2 + 2\varepsilon$ from some element $x \in T'$. Hence, the disks $D(x, 2 + 2\varepsilon)$, $x \in T'$ cover S, and the union bound implies that

$$\sum_{x\in T'} \mu\big(D(x, 2+2\varepsilon)\big) \geq \mu(S).$$

Combining the above inequalities we obtain that

$$\mu\big(S(\varepsilon)\big) \geq \sum_{x\in T'} \mu\big(D(x,\varepsilon)\big) = \gamma \sum_{x\in T'} \mu\big(D(x, 2+2\varepsilon)\big) \geq \gamma\mu(S),$$

where $\gamma = \gamma(\varepsilon) = \mu\big(D(0,\varepsilon)\big)/\mu\big(D(0, 2+2\varepsilon)\big)$. This proves the desired result. □

Notice that every $S(\varepsilon)$ has a non-empty intersection with the non-negative real line $[0,\infty)$ because S has this property. In fact, if x is the point of $J \cap [0,\infty)$ which has the greatest distance from 0, where J is the outer interface that defines S, then the interval $[x, x+\varepsilon)$ is contained in $S(\varepsilon) \cap [0,\infty)$. We conclude that $S(\varepsilon)$ contains one of the points $\{0, \varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$ for some $N \in \mathbb{N}$ depending on $S(\varepsilon)$. The next lemma provides a uniform upper bound for N that depends only on the area of S.

Lemma 1.7.6. *For every separating strip $S = S(Y)$ we have*

$$S \subset D(0, 3\mu(S)).$$

Proof. Let $J = J(Y)$ be the outer interface that defines S, and $\Omega = \Omega(Y)$ the closure of the Jordan domain bounded by J. We claim that the distance of any point (x, y) of J from 0 is bounded from above by $\mathcal{L}(J)$. Indeed, we can assume without loss of generality that both $x, y > 0$. Let x^- and y^- be the first points of $(-\infty, 0] \times \{0\}$ and $\{0\} \times (-\infty, 0]$, respectively, that J contains. Then there are two non-overlapping subarcs J_1, J_2 of J that connect (x, y) to x^-, y^-, respectively. Notice that $x \leq \mathcal{L}(J_1)$ and $y \leq \mathcal{L}(J_2)$. Since $\mathcal{L}(J_1) + \mathcal{L}(J_2) \leq \mathcal{L}(J)$, it follows that

$$\sqrt{x^2 + y^2} \leq \sqrt{\mathcal{L}(J_1)^2 + \mathcal{L}(J_2)^2} \leq \mathcal{L}(J_1) + \mathcal{L}(J_2) \leq \mathcal{L}(J),$$

as claimed.

Having proved the claim, we deduce that the distance of any point in S from 0 is bounded from above by $\mathcal{L}(J) + 1$. We now claim that

$$\mathcal{L}(J) \leq 2\mu(S) \tag{1.30}$$

Indeed, let us first partition J as follows. For every $x \in Y$, the intersection of J with the circle $C(x)$ of radius 1 centred at x may contain several connected components. Let (J_i) be an enumeration of all these connected components and (x_i) the corresponding sequence of centres, i.e. x_i is the centre of the arc J_i (some $x \in Y$ may appear more than once).

Every arc J_i has two endpoints A_i, B_i. Let $S(i)$ be the open sector of $D(x_i)$ enclosed by the radii $x_i A_i$, $x_i B_i$ and the arc J_i. Notice that $S(i)$ is a subset of S. Moreover, any two distinct $S(i), S(j)$ are disjoint. To see this, remove all disks from Ω except for $D(x_i)$ and $D(x_j)$. Since the arcs J_i and J_j lie in the boundary of Ω, they must lie in the boundary of $D(x_i) \cup D(x_j)$ as well. It is now clear that the sectors $S(i), S(j)$ are disjoint.

These observations imply that

$$\sum_i \mu(S(i)) \leq \mu(S).$$

An elementary computation yields $\mathcal{L}(J_i) = 2\mu(S_i)$, which implies that

$$\mathcal{L}(J) = \sum_i \mathcal{L}(J_i) = 2 \sum_i \mu(S(i)) \leq 2\mu(S)$$

establishing (1.30).

Clearly, $\mu(S) > 1$ because by definition S contains at least one disk of radius 1. Therefore, $\mathcal{L}(J) + 1 < 3\mu(S)$. This yields the desired assertion. $\square$

We deduce from Lemma 1.7.6 that N can be chosen to be $\lfloor 3\mu(S)/\varepsilon \rfloor$. We are now almost ready to prove the desired exponential decay. Before we do so we need to upper bound the number of occurring separating multi-strips of $\mathcal{MS}_n$.

Lemma 1.7.7. *There is a constant $R \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ at most $R^{\sqrt{n}}$ elements of $\mathcal{MS}_n$ can occur simultaneously in any ω.*

Proof. Notice that a separating strip $S = S(Y)$ contains an interval of the form $[x, x+1]$ for some $x \in [0, \infty)$. Combined with Lemma 1.7.6 this implies that S contains some element of the set $\{0, 1, \ldots, \lfloor 3\mu(S) \rfloor\}$. We can now proceed as in the proof of Lemma 1.4.9. □

We are now ready to prove Lemma 1.7.4.

Proof of Lemma 1.7.4. Since

$$N_n \leq R^{\sqrt{n}} \mathbb{1}_{\{\text{some } S \in \mathcal{MS}_n \text{ occurs}\}}$$

by Lemma 1.7.7, we conclude that

$$\mathbb{E}_\lambda(N_n) \leq R^{\sqrt{n}} \mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs}).$$

Hence it suffices to show that $\mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs})$ decays exponentially.

Recall our coupling between the Boolean models $(P_\lambda, 1)$ and $(P_\lambda, 1-\varepsilon)$, and the fact that whenever Y happens to separate in P_λ the set $S(\varepsilon)$ is a vacant connected subset of $(P_\lambda, 1-\varepsilon)$ in our coupling. For $m \in \mathbb{N}$, let V_m denote the event that there is a subset V of a vacant component with $\mu(V) \geq \gamma m$, where γ is the constant of Lemma 1.7.5, and some element of the set $\{0, \varepsilon, \ldots, \lfloor (3m+3)/\varepsilon \rfloor \varepsilon\}$ belongs to V, and V is contained in $D(0, 3m+3)$. We claim that

$$\mathbb{P}_\lambda(\text{some } S \in \mathcal{MS}_n \text{ occurs}) \leq \sum_{\{m_1, m_2, \ldots, m_k\} \in P_n} \mathbb{P}_{\lambda, 1-\varepsilon}(V_{m_1} \square \ldots \square V_{m_k}),$$

where as above $\square$ means that the events occur disjointly, P_n is the set of partitions of n, and the probability measure $\mathbb{P}_{\lambda,1-\varepsilon}$ refers to the Boolean model $(P_\lambda, 1-\varepsilon)$. Indeed, notice that if for some separating strip S we have $\lfloor \mu(S) \rfloor = m$, then $\mu(S) \leq m+1$, hence S lies in $D(0, 3m+3)$ by Lemma 1.7.6. The inequality follows now similarly to (1.15),

Reimer's inequality Gupta and Rao [1999] implies that

$$\mathbb{P}_{\lambda,1-\varepsilon}(V_{m_1} \square \ldots \square V_{m_k}) \leq \mathbb{P}_{\lambda,1-\varepsilon}(V_{m_1}) \cdot \ldots \cdot \mathbb{P}_{\lambda,1-\varepsilon}(V_{m_k}).$$

Combining the fact that $\mathbb{P}_{\lambda,1-\varepsilon}(\mu(V(0)) \geq a) \leq c^a$ Meester and Roy [1996] for every $\lambda > \lambda_c$ and some $c = c(\lambda) < 1$ with the union bound we obtain

$$\mathbb{P}_{\lambda,1-\varepsilon}(\mu(V_m)) \leq c_1(m){c_2}^m,$$

where $c_1(m) = \lfloor (3m+3)/\varepsilon \rfloor + 1$ and $c_2 = c^\gamma < 1$. We can now argue as in the proof of Lemma 1.4.11 to obtain the desired exponential decay. □

We proceed by establishing the analyticity and the necessary estimates of the functions involved in Lemma 1.7.4 that we will combine with their exponential decay to prove the analyticity of θ_0.

Lemma 1.7.8. *Let $\{m_1, m_2, \ldots, m_k\}$ be a composition of n. Then the function $f(\lambda) := \mathbb{E}_\lambda(N\{m_1, \ldots, m_k\})$ admits an entire extension satisfying*

$$|f(z)| \leq e^{4nM} f(\lambda + M) \tag{1.31}$$

for every $\lambda \geq 0$, $M > 0$ and $z \in D(\lambda, M)$.

Proof. To ease notation we will prove the assertion for $k = 2$ and $m_1 \neq m_2$. The general case can be handled similarly.

Given two disjoint sets $Y_1 = \{x_1, \ldots, x_{j_1}\}$ and $Y_2 = \{x_{j_1+1}, \ldots, x_{j_1+j_2}\}$, we let $L(x_1, \ldots, x_{j_1+j_2})$ denote the indicator function of the event that the sets Y_1 and Y_2 satisfy all three properties ((i))-((iii)) in the definition of a separating strip, and furthermore, $\lfloor \mu(S(Y_i) \rfloor = m_i$, $i = 1, 2$. The indicator function of the event $\{Y_i$ happens to separate in $P_\lambda\}$ is denoted by $\mathbb{1}_{Y_i}$. Let us also define the functions

$$g(x_1, \ldots, x_{j_1+j_2}) := \mu(S(x_1, \ldots, x_{j_1})) + \mu(S(x_{j_1+1}, \ldots, x_{j_2}))$$

and

$$h(x_1, \ldots, x_{j_1+j_2}) := L(x_1, \ldots, x_{j_1+j_2}) e^{-\lambda g(x_1, \ldots, x_{j_1+j_2})}.$$

First, we will find a suitable formula for f. We claim that

$$f(\lambda) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{(\lambda \mu(6nD))^{j_1+j_2}}{j_1! j_2!} f(\lambda, j_1, j_2), \tag{1.32}$$

where

$$f(\lambda, j_1, j_2) = \int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} h(x_1, \ldots, x_{j_1+j_2}). \tag{1.33}$$

Indeed, expressing f according to the size of Y_1 and Y_2 we obtain

$$f(\lambda) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \mathbb{E}_\lambda \Big(N\big(\{(m_1, j_1), (m_2, j_2)\}\big) \Big)$$

where $N\big(\{(m_1, j_1), (m_2, j_2)\}\big)$ denotes the number of sets Y_1, Y_2 that happen to separate with the property that $\lfloor \mu(S(Y_i)) \rfloor = m_i$, $|Y_i| = j_i$, $i = 1, 2$. This expression holds because we have assumed that $m_1 \neq m_2$, and so each $\{(m_1, j_1), (m_2, j_2)\}$ appears exactly once. Next notice that

$$\mu(S(Y_1)) + \mu(S(Y_2)) \leq (k_1 + 1) + (k_2 + 1) \leq 2k_1 + 2k_2 = 2n, \tag{1.34}$$

since $1 \leq k_1, k_2$, which combined with Lemma 1.7.6, implies that $N\big(\{(m_1, j_1), (m_2, j_2)\}\big)$ depends only on the points of the Poisson point process inside the disk $6nD$. Now regard $P_\lambda \cap 6nD$ as a finite Poisson point process whose total number of points has a Poisson distribution with parameter $\lambda\mu(6nD)$, each point being uniformly distributed over $6nD$. Notice that conditioned on the number of points $\mathcal{N}(6nD)$ inside $6nD$, the distribution of the sets Y_1, Y_2 depends only on their sizes.

Conditionally on the event $\{\mathcal{N}(6nD) = m\}$ and the sets $Y_1 = \{x_1, \ldots, x_{j_1}\}$ and $Y_2 = \{x_{j_1+1}, \ldots, x_{j_1+j_2}\}$ being contained in P_λ, the expectation of $\mathbb{1}_{Y_1}\mathbb{1}_{Y_2}$ is equal to

$$H_m(x_1, \ldots, x_{j_1+j_2}) := L(x_1, \ldots, x_{j_1+j_2}) \Big(\frac{\mu(6nD) - g(x_1, \ldots, x_{j_1+j_2})}{\mu(6nD)} \Big)^{m-j_1-j_2},$$

because every other point of the Poisson point process must lie outside of $S(Y_1)$, $S(Y_2)$. Hence expressing f according to the number of points of the Poisson process inside $6nD$ and the size of the sets Y_1, Y_2 we obtain

$$f(\lambda) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{m=j_1+j_2}^{\infty} e^{-\lambda\mu(6nD)} \frac{(\lambda\mu(6nD))^m}{m!} \binom{m}{j_1} \binom{m-j_1}{j_2} F(j_1, j_2, m),$$

where

$$F(j_1, j_2, m) = \int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} H_m(x_1, \ldots, x_{j_1+j_2}).$$

The factors $e^{-\lambda\mu(6nD)} \dfrac{(\lambda\mu(6nD))^m}{m!}$ and $\binom{m}{j_1}\binom{m-j_1}{j_2}$ correspond to the probability $\mathbb{P}_\lambda(\mathcal{N}(6nD) = m)$ and the number of ways to choose two disjoint subsets of size j_1 and j_2 from a set of size m (here the order of the sets matters because

$m_1 \neq m_2$), respectively. After changing the order of the second summation and integration, using the Taylor expansion

$$\sum_{m=j_1+j_2}^{\infty} \frac{(\lambda(\mu(6nD) - g(x_1, \ldots, x_{j_1+j_2})))^{m-j_1-j_2}}{(m - j_1 - j_2)!} = e^{\lambda(\mu(6nD)-g(x_1,\ldots,x_{j_1+j_2}))}$$

and cancelling some terms, we arrive at formula (1.33).

Using (1.32) we see that f extends to an entire function. Indeed, the assertion will follow from the standard tools once we have shown that every summand of (1.32) is an entire function and that the upper bound (1.31) holds for the summands of (1.32) in place of f.

First, we express $e^{-\lambda g(x_1,\ldots,x_{j_1+j_2})}$ via its Taylor expansion

$$e^{-\lambda g(x_1,\ldots,x_{j_1+j_2})} = \sum_{s=0}^{\infty} \frac{(-\lambda g(x_1, \ldots, x_{j_1+j_2}))^s}{s!}.$$

We will plug this into (1.33). We notice that the coefficient

$$\int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} L(x_1, \ldots, x_{j_1+j_2})(-g(x_1, \ldots, x_{j_1+j_2}))^s/s!$$

is bounded in absolute value by $(2n)^s/s!$, as $g(x_1, \ldots, x_{j_1+j_2}) = \mu(S(Y_1))+\mu(S(Y_2)) \leq 2n$ by (1.34) and $0 \leq L(x_1, \ldots, x_{j_1+j_2}) \leq 1$. Therefore the function defined by the Taylor expansion

$$\sum_{s=0}^{\infty} \lambda^s \int_{6nD} \frac{dx_1}{\mu(6nD)} \cdots \int_{6nD} \frac{dx_{j_1+j_2}}{\mu(6nD)} L(x_1, \ldots, x_{j_1+j_2})(-g(x_1, \ldots, x_{j_1+j_2}))^s/s!$$

is entire and by reversing the order of summation and integration we conclude that it coincides with $f(\lambda, j_1, j_2)$.

Now let $\lambda \geq 0$ and $M > 0$. Since $|z|^{j_1+j_2} \leq (\lambda + M)^{j_1+j_2}$ for every $z \in D(\lambda, M)$, inequality (1.31) will follow once we prove that

$$|f(z, j_1, j_2)| \leq e^{4nM} f(\lambda + M, j_1, j_2) \text{ for every } z \in D(\lambda, M). \tag{1.35}$$

Using once again (1.34) we obtain

$$|e^{-zg(x_1,\ldots,x_{j_1+j_2})}| \leq e^{-(\lambda-M)g(x_1,\ldots,x_{j_1+j_2})} =$$
$$e^{2Mg(x_1,\ldots,x_{j_1+j_2})}e^{-(\lambda+M)g(x_1,\ldots,x_{j_1+j_2})} \leq e^{4nM}e^{-(\lambda+M)g(x_1,\ldots,x_{j_1+j_2})}.$$

Hence (1.35) follows from the triangle inequality. This proves (1.31).

Combining (1.35) with (1.32) and the theorems of Weierstrass, we deduce that f is entire as well. □

We are finally ready to prove Theorem 1.7.1.

Proof of Theorem 1.7.1. Consider the functions

$$f(\lambda) = \sum_{k=1}^{\infty}(-1)^{k+1}\mathbb{E}_\lambda(N(k))$$

and

$$g_n(\lambda) := \sum_{\{m_1,m_2,\ldots,m_k\}\in P_n}(-1)^{k+1}\mathbb{E}_\lambda(N(\{m_1,\ldots,m_k\}),$$

where P_n is the set of partitions of n. Notice that

$$f = \sum_{n=1}^{\infty} g_n.$$

By Lemma 1.7.4 we have

$$\sum_{k=1}^{\infty}\mathbb{E}_\lambda(N(k)) < \infty$$

for any $\lambda > \lambda_c$. Hence f coincides with $1-\theta_0$ on the interval (λ_c,∞) by the inclusion-exclusion principle as remarked above. Combining Lemma 1.7.4 with Lemma 1.7.8 we conclude that for every $\lambda > \lambda_c$ there are constants $M = M(\lambda) > 0$, $c_1 = c_1(\lambda) > 0$ and $0 < c_2 = c_2(\lambda) < 1$ such that $|g_n(z)| \leq c_1 c_2{}^n$ for every $z \in D(\lambda, M)$. As usual, by the theorems of Weierstrass, we conclude that f, and thus θ_0, is analytic on the interval (λ_c,∞). □

1.8 Triangulations

1.8.1 Overview

In this section, we use the techniques we developed to provide upper bounds on p_c and $\dot{p}_c$ for certain families of triangulations. Although these bounds will apply to $p_{\mathbb{C}}$, we stress that the results of this section give the best known (or only) such bounds on $p_c, \dot{p}_c$ for these triangulations.

We will prove that $p_{\mathbb{C}} \leq 1/2$ for Bernoulli bond percolation on triangulations of an open disk that either satisfy a weak expansion property or are transient for simple symmetric random walk. Once again the analyticity of θ_o will follow by showing that the outer interfaces (interfaces) of o have an exponential tail for every $p > 1/2$.

The interest in the study of percolation on triangulations of an open disk was sparked by the seminal paper Benjamini and Schramm [1996b] of Benjamini & Schramm. They made a series of conjectures, the strongest one of which is that $\dot{p}_c(T) \leq 1/2$ on any bounded degree triangulation T of an open disk that satisfies a weak isoperimetric inequality of the form $|\partial^V A| \geq f(|A|)\log(|A|)$ for some function $f = \omega(1)$, where A is any finite set of vertices. More recently, Benjamini [2015] conjectured that $\dot{p}_c(T) \leq 1/2$ on any transient bounded degree triangulation T of an open disk.

Angel et al. [2018] proved that for any triangulation T of an open disk with minimum degree 6, the isoperimetric dimension of T is at least 2 and thus satisfies the assumption of the conjecture of Benjamini & Schramm. They also asked whether $p_c(T) \leq 2\sin(\pi/18)$ (and $\dot{p}_c \leq 1/2$), the critical value for bond percolation on the triangular lattice, for any such triangulation.

The main results of this section, which we now state, imply that in all aforementioned conjectures, the bound $p_c \leq 1/2$ is correct if one considers bond instead of site percolation.

Theorem 1.8.1. *Let T be a triangulation of an open disc such that every vertex has finite degree (not necessarily bounded) and*[3]

$$\text{for every finite set } A \text{ of vertices we have } |\partial^V A| \geq f(diam(A))\log(diam(A)) \text{ for some function } f = \omega(1), \tag{1.36}$$

then

$$p_c(T) \leq p_{\mathbb{C}}(T) \leq 1/2.$$

Theorem 1.8.2. *Let T be a transient triangulation of an open disc with degrees bounded above by d. Then*

$$p_c(T) \leq p_{\mathbb{C}}(T) \leq 1/2.$$

We will also prove the same bound for recurrent triangulations T with a uniform upper bound on the radii of the circles in any circle packing of T.

1.8.2 Proofs

We will first focus on proving Theorem 1.8.1, but many of the following arguments will also be valid for transient triangulations.

[3] The reader will lose nothing by replacing $diam(A)$ by $|A|$ in this statement, which only strengthens our assumptions.

Our proofs will follow the lines of that of Theorem 1.4.1. Recall the definitions of outer interface and multi-interface of Section 1.4. Again $\mathcal{MS}$ denotes the set of multi-interfaces of a chosen vertex o, while ∂M denotes the boundary of a multi-interface M and $\mathcal{MS}_n := \{M \in \mathcal{MS} \mid |\partial M| = n\}$.

Let T be a triangulation of an open disk and o a vertex in T. Once again we will utilise the inclusion-exclusion principle to express $1 - \theta_o$ as an infinite sum

$$1 - \theta_o(p) = \sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} Q_M(p) \tag{1.37}$$

for every p large enough, where $c(M)$ denotes the number of outer interfaces in the multi-interface M, and $Q_M(p) := \mathbb{P}_p(M \text{ occurs})$. The validity of the formula will follow as in the proof of Theorem 1.4.1 (recall (1.13)) once we establish an exponential tail for the corresponding probabilities, which is the purpose of the following lemma.

Lemma 1.8.3. *For every triangulation T of an open disk satisfying condition* (1.36) *of Theorem 1.8.1 and every $p \in (1/2, 1]$,*

$$\sum_{M \in \mathcal{MS}_n} Q_M(p) \leq c_1 c_2{}^n, \tag{1.38}$$

where $c_1 = c_1(p) > 0$ and $c_2 = c_2(p) > 0$ are some constants with $c_2 < 1$. Moreover, if $[a, b] \subset (1/2, 1]$, then the constants c_1 and c_2 can be chosen independent of p in such a way that (1.38) *holds for every $p \in [a, b]$.*

In order to prove the above lemma, we first pick an arbitrary infinite geodesic R starting from o. Our goal is to show that the outer interfaces M of o for which ∂M contains a fixed edge $e \in E(R)$ occur with exponentially decaying probability for every $p > 1/2$. Then we will upper bound the choices for $e \in R$.

In what follows we will be using the standard coupling between percolation on T and its dual T^* as in the proof of Lemma 1.4.11. Since T is a triangulation, the dual of any minimal edge cut of T is a cycle. The number of cycles in T^* of size n containing a fixed edge is at most 2^{n-1} because T^* is a cubic graph. Then the union bound shows that the probability that some minimal edge cut containing a fixed edge is vacant has an exponential tail for every $p > 1/2$. However, the boundary of an outer interface is not necessarily a minimal edge cut. Still, the dual of the boundary of any outer interface in T is a connected subgraph of T^*. The desired exponential tail will follow from Theorem 1.2.1 once we show that $\sup_{u \in V(T^*)} \chi_u(1-p) < \infty$ for every $p > 1/2$, where, as usual, $\chi_u(1-p)$ denotes the expected size of the percolation cluster of u in the dual graph T^*. The next lemma proves this statement.

Lemma 1.8.4. *Let T be a triangulation of an open disc. Then*

$$\chi^*(p) := \sup_{u \in V(T^*)} \chi_u(1-p) < \infty$$

for every $p \in (1/2, 1]$.

Proof. Let u be a vertex of T^*. Note that whenever some vertex v belongs to C_u there is a path from u to v with occupied edges. Hence we obtain $\mathbb{E}_{1-p}(|C_u|) \leq \mathbb{E}_{1-p}(P(u))$, where $P(u)$ is the number of occupied self-avoiding walks starting from u (including the self-avoiding walk with only one vertex). The number $\sigma_k(u)$ of k-step self-avoiding walks in T^* starting from u is at most $3 \cdot 2^{k-1}$. Consequently,

$$\mathbb{E}_{1-p}(P(u)) \leq \textstyle\sum_{k=0}^{\infty} 3 \cdot 2^{k-1}(1-p)^k < \infty \tag{1.39}$$

whenever $p > 1/2$. Since this bound does not depend on u the proof is complete. □

Using Theorem 1.2.1 we immediately obtain the desired exponential tail.

Corollary 1.8.5. *For every $p > 1/2$ there is a constant $0 < c = c(p) < 1$ such that for any triangulation T of an open disk and any vertex $u \in T^*$, we have $\mathbb{P}_{1-p}(|C_u| \geq n) \leq c^n$.*

Let R be a geodesic ray in T starting at any $o \in V(G)$. The following lemma converts condition (1.36) into a statement saying that every outer interface of T meets a relatively short initial subpath of R.

Define a function $g : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by letting $g(n)$ be the smallest integer l such that every outer interface of $\mathcal{S}_n$ contains at least one of the first l edges of R if such a l exists, and let $g(n) = \infty$ if no such l exists, with the convention that $g(n) = 1$ if no such edge-separator of size n exists. Clearly, we always have $g(n) < \infty$ for triangulations of an open disk satisfying condition (1.36).

Lemma 1.8.6. *Let T be a triangulation of an open disk satisfying condition* (1.36). *Then* $\limsup_{n\to\infty} g(n)^{1/n} = 1$.

Proof. Consider some outer interface M of $\mathcal{S}_n$. Let B be the minimal edge cut of M and $A = A_n$ be the component of o in $T \setminus B$. Our condition (1.36) says that

$$f(diam(A)) \log(diam(A)) \leq |\partial^V A| \leq |\partial^E A| = |B| \leq n.$$

Since $f = \omega(1)$, we obtain that $\log(diam(A)) = o(n)$. However, $diam(A)$ is greater than the initial segment of R that A contains because R is a geodesic ray. This implies that $\log(g(n)) = o(n)$, as desired. □

If a sequence satisfies the assertion of Lemma 1.8.6 we will say that it grows *sub-exponentially.*

The following lemma is the analogue of Lemma 1.4.9.

Lemma 1.8.7. *For every triangulation T of an open disk satisfying condition* (1.36) *of Theorem 1.8.1 the maximal number of elements of $\mathcal{MS}_n$ that can occur simultaneously in any percolation instance ω grows sub-exponentially.*

Proof. Let S be an element of $\mathcal{MS}_n$ comprising the outer interfaces $S_1, S_2, \dots, S_l$. Since any two distinct occurring outer interfaces are vertex disjoint by Lemma 1.4.7, the collection of all $m_i := |\partial S_i|$ define a partition of n. We call the multi-set $\{m_1, m_2, \dots, m_l\}$ the *boundary partition* of S. It is possible that more than one occurring multi-interfaces have the same boundary partition. In order to prove the desired assertion, we will show that for every partition $\{m_1, m_2, \dots, m_l\}$ of n, there are sub-exponentially many occurring multi-interfaces with $\{m_1, m_2, \dots, m_l\}$ as their boundary partition. Then the assertion follows by the Hardy–Ramanujan formula (Theorem 1.2.3).

Since occurring outer interfaces meet R and they are vertex-disjoint by Lemma 1.4.7, S is uniquely determined by the subset of R it meets. The number of occurring outer interfaces with boundary of size m_i is at most $g(m_i)$ by definition. Hence the number of occurring multi-interfaces with $\{m_1, m_2, \dots, m_l\}$ as their boundary partition is bounded above by $g(m_1)g(m_2)\dots g(m_l)$. It is not hard to see that this product grows sub-exponentially. Indeed, given a constant $c > 1$, for all but finitely many indices i, we have that $g(m_i) \leq c^{m_i}$ because g grows sub-exponentially. Hence $g(m_1)g(m_2)\dots g(m_l) \leq Kc^n$ for some large enough constant $K = K(c, g) > 0$ which does not depend on the partition. Letting now c converge slowly enough to 1 as $n \to \infty$ so that $K = O(n)$, we obtain that Kc^n grows sub-exponentially in n, which proves the desired assertion. □

We are now ready to prove Lemma 1.8.3.

Proof of Lemma 1.8.3. By Lemma 1.8.7 we have

$$\sum_{M \in \mathcal{MS}_n} Q_M(p) \leq h_n \mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs})$$

for some h_n growing sub-exponentially. Let r_m denote the m-th edge of R. For every dual edges r^*_m, we pick one of its two end-vertices and we denote it v_m (some of these end-vertices appear possibly more than once). Let $D(m)$ denote the event that one of the clusters of $v_1, \dots, v_{g(m)}$ contains at least m vertices. Arguing as in the proof of Lemma 1.4.11, we can deduce that

$$\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs}) \leq \sum_{\{m_1,\dots,m_k\}\in P'_n} \mathbb{P}_{1-p}(D(m_1)) \cdot \dots \cdot \mathbb{P}_{1-p}(D(m_k)),$$

where P_n is the set of partitions of n.

By Corollary 1.8.5 and the union bound we obtain

$$\mathbb{P}_p(\text{some } M \in \mathcal{MS}_n \text{ occurs}) \leq H_n c^n$$

for some H_n growing sub-exponentially, where c is the constant of Corollary 1.8.5. Hence $\sum_{M\in\mathcal{MS}_n} Q_M(p)$ decays exponentially in n for all $p > 1/2$. □

The following is an easy combinatorial exercise.

Lemma 1.8.8. *For every triangulation of an open disk T and every outer interface M we have $|E(M)| \leq 4|\partial M|$.*

Proof. Let H be a finite connected graph witnessing the fact that M is an outer interface. We claim that every edge $e \in M$ lies in a triangular face T_e of T such that at least one edge of $T_e - e$ lies in ∂M. Indeed, e lies in exactly two (triangular) faces of T, and we choose T_e to be one of them lying in the unbounded face of H; such a T_e exists because by definition the vertices and edges of M are incident with the unbounded face of H. As T_e lies in the unbounded face of H, one of the two other edges of T_e lies in ∂M.

Since any edge of ∂M lies in at most two of these triangular faces T_e, and each such face contains at most two edges of $E(M)$, the result follows. □

We have collected all the ingredients for the main result of this section.

Proof of Theorem 1.8.1. To obtain our precise result, note that, by definition, every $M \in \mathcal{MS}_n$ has n vacant edges. Moreover, $|E(M)| \leq 4n$ by Lemma 1.8.8. Hence we can now apply Corollary 1.3.15 for $(1/2, 1]$, $L_n = \mathcal{MS}_n$, and $(E_{n,i})$ an enumeration of the events $\{M \text{ occurs}\}_{M\in\mathcal{MS}_n}$, to deduce that $\theta_o(p)$ is analytic for $p > 1/2$. As usual, we then recall that $\theta_o(p)$ cannot be analytic at p_c, and so $p_c \leq p_{\mathbb{C}}$. □

Remark: The above proof uses some complex analysis (needed in Corollary 1.3.15) to prove $p_c \leq 1/2$. But the complex analysis can be avoided by using a refinement of the Peierls argument, as we will see in Chapter 3.

For the proof of Theorem 1.8.2 we just need to show that the size of the set of edges of a 1-way geodesic R that meets $\bigcup \mathcal{MS}_n$ grows sub-exponentially in

n. To this end, we will use the well-known theorem of He & Schramm stating that every graph as in our statement is the contacts graph of a circle packing whose carrier is the open unit disc $\mathbb{D}$ in $\mathbb{R}^2$; see He and Schramm [1995], where the relevant definitions can be found. We say that an edge e *meets* $\mathcal{MS}_n$, if there is $M \in \mathcal{MS}_n$ with $e \in \partial M$.

Lemma 1.8.9. *Let T be a triangulation of an open disk which is transient and has bounded vertex degrees. Let R be a geodesic ray in T starting at any $o \in V(G)$, and let R_n be the set of edges of R meeting $\mathcal{MS}_n$. Then $|R_n| = O(n^3)$.*

Proof. Let P be a circle packing for T whose carrier is the open unit disk $\mathbb{D}$, provided by He and Schramm [1995]. The main properties of P used in our proof are

(i) two vertices of T are joined with an edge if and only if the corresponding circles are tangent, and

(ii) there are no accumulation points of circles of P inside $\mathbb{D}$.

Assume that $|R_n| = \omega(n^3)$ contrary to our claim. Let R'_n be the set of vertices of R incident with an edge in R_n. Then $|R'_n| > |R_n| = \omega(n^3)$. For a vertex u of T, let x_u denote the corresponding circle of P.

For any $u \in R'_n$ Lemma 1.8.8 yields a connected subgraph G_u of T of at most $4n+1$ edges containing u and surrounding o; indeed, G_u can be obtained from any outer interface M witnessing the fact that $u \in R'_n$ by possibly adding the edge of u lying in ∂M in case u does not lie on M.

Let P_u denote the union of the disks of P corresponding to G_u. We claim that the area area(P_u) covered by P_u is at least r/n^2 for some constant $r = r(P)$. Indeed, P_u is the union of at most $4n+2$ disks ($|V(G_u)| \leq |E(G_u)| + 1$), and its diameter is greater than the diameter of x_o, and so at least one of its circles must have diameter of order at least $1/n$, hence area of order at least $1/n^2$.

For every n, pick a subset R''_n of R'_n such that any two vertices of R''_n lie at distance at least $8n+3$ along R, and therefore in T since R is a geodesic, and $|R''_n| = \omega(n^2)$. Such a choice is possible because $R'_n = \omega(n^3)$.

Note that for any two distinct elements $u, v \in R''_n$, the subgraphs G_u, G_v defined above are vertex disjoint: this is because we chose u, v to have distance at least $8n+3$ in T, and each of G_u, G_v has at most $4n+1$ edges and is connected. Moreover, recall that each P_u has area of order at least $1/n^2$. Combining these two facts we obtain $\sum_{u \in R''_n} \text{area}(P_u) = \omega(1)$, a contradiction since area$(\mathbb{D})$ is finite. □

Proof of Theorem 1.8.2. We repeat the arguments of the proof of Theorem 1.8.1, replacing Lemma 1.8.6 by Lemma 1.8.9. □

In the case of recurrent triangulations, the theorem of He & Schramm states that T is the contacts graph of a circle packing whose carrier is the plane $\mathbb{R}^2$ He and Schramm [1995]. Let P be such a circle packing. We will prove the analogue of Lemma 1.8.9 for recurrent triangulations of an open disk such that the radii of the circles of P are bounded from above. This, in turn, implies that $p_{\mathbb{C}} \leq 1/2$ for such triangulations by repeating the proof of Theorem 1.8.2.

Lemma 1.8.10. *Let T be a triangulation of an open disk which is recurrent and has bounded vertex degree. Assume that*

for some (and hence every) circle packing P of T, the radius of every disk in P is bounded from above by some constant M. (1.40)

Let R be a geodesic ray in T starting at any $o \in V(G)$, and let R_n be the set of edges of R contained in some outer interface of $\mathcal{MS}_n$. Then $|R_n| = O(n^5)$.

Proof. We will follow the proof of Lemma 1.8.9. Assume that $|R_n| = \omega(n^5)$ contrary to our claim. Recall the definitions of P_u, G_u and R'_n, and let R''_n be defined as in the proof of Lemma 1.8.9 with the additional property $\infty > |R''_n| = \omega(n^4)$. This is possible because $|R_n| = \omega(n^5)$. In the proof of Lemma 1.8.9, we utilised the finite area of $\mathbb{D}$ to derive a contradiction. However, the area of the plane is infinite. For this reason, we will construct a family of bounded domains (D_n) with the property that P_u is contained in D_n for most $u \in R''_n$.

Let u_n be the vertex of R''_n that attains the greatest graph distance from o. We claim that $G_n := G_{u_n}$ contains a cycle that surrounds o. Indeed, assuming that G_n does not contain any such cycle, we obtain that o lies in G_n. Consider now some $u \in R''_n$ other than u_n. Then G_u is vertex disjoint from G_n, as mentioned in the proof of Lemma 1.8.9. As G_u separates o from infinity, G_n must lie in a bounded face of G_u. This implies that $d(u, o) > d(u_n, o)$, which is a contradiction. Hence G_n contains a cycle C_n that surrounds o.

Let D_n be the domain bounded by C_n. Arguing as above, we can immediately see that each P_u for $u \in R''_n \backslash \{u_n\}$ lies in D_n. Moreover, C_n contains at most $4n$ edges by Lemma 1.8.8. Every edge of T has length at most $2M$ in P by our assumption, therefore, the length of C_n (as a curve in $\mathbb{R}^2$) is at most $8Mn$.

As in the proof of Lemma 1.8.9 if $u \in R''_n$, then some circle of P_u has area of order at least $1/n^2$. Hence we obtain $\sum_{u \in R''_n \backslash \{u_n\}} \text{area}(P_u) = \omega(n^2)$, since $|R''_n \setminus \{u_n\}| = \omega(n^4)$. Using the standard isoperimetric inequality of the plane, we derive

$$\sum_{u \in R''_n \backslash \{u_n\}} 4\pi \cdot \text{area}(P_u) \leq 4\pi \cdot \text{area}(D_n) \leq (8Mn)^2.$$

We have obtained a contradiction. □

Using an idea of Grimmett and Li [2015], we can slightly improve our results to obtain the strict inequality $p_c \leq p_{\mathbb{C}} < 1/2$ instead of $p_c \leq p_{\mathbb{C}} \leq 1/2$ in all above results. Indeed, it is not hard to see that for any bounded degree triangulation of an open disk T, $\sigma_k(o) \leq 3 \cdot 2^{d-1}(2^d - 2)^{\lfloor n/d \rfloor}$, where d is the maximum degree of T. This comes from the fact that for every vertex u and any edge e incident to u the number of d-step self-avoiding walks starting from u that do not traverse e is at most $2^d - 2$. Hence $p_c \leq p_{\mathbb{C}} < 1/2$ as claimed.

Chapter 2

Exponential growth rates

2.1 Introduction

2.1.1 Overview

The aim of this chapter is to study the properties of interfaces, in particular their exponential growth rates, and use them in order to obtain results for percolation on $\mathbb{Z}^d$ and other Euclidean lattices.

We point out a technique for translating any upper bound on the percolation threshold $\dot{p}_c(G)$ of a lattice G into a lower bound on the exponential growth rate $\dot{a}(G)$ of lattice animals and vice-versa. More precisely, we have

$$\dot{a}(G) \geq f(r(\dot{p}_c(G))), \tag{2.1}$$

where $f(r) := \frac{(1+r)^{1+r}}{r^r}$ and $r(p) := \frac{1-p}{p}$ are universal functions. *Percolation* for now refers to Bernoulli site percolation, and a *lattice animal* is an induced subgraph of G.

This is by no means the first time where such a formula has been observed (see e.g. Delyon [1980]; Hammond [2005]) but in all previous instances, the focus was on obtaining upper bounds for $\dot{a}(G)$ rather than lower bounds. Inequality (2.1) not only allows us to obtain lower bounds for $\dot{a}(G)$, but it also allows us to obtain lower bounds for $\dot{p}_c(G)$. Indeed, coupling (2.1) with a recent upper bound on $\dot{a}(G)$ Barequet and Shalah [2019] we obtain the lower bound $\dot{p}_c(\mathbb{Z}^3) > 0.2522$ for the site percolation threshold of the cubic lattice; see (2.25) in Section 2.8. This is higher than the predicted threshold for bond percolation, which is about 0.2488. The best rigorous bound previously known was about $\dot{p}_c(\mathbb{Z}^3) > 0.21225$, obtained as the inverse of the best known bound on the connective constant MacDonald et al.

[2000][1].

By combinatorial arguments, we obtain the upper bounds $\dot{a}(\mathbb{Z}^d) \leq 2de - 5e/2 + O(1/\log(d))$ that improve on those of Barequet and Shalah [2019], and plug them into (2.1) to deduce the asymptotic lower bounds $\dot{p}_c(\mathbb{Z}^d) \geq \dfrac{1}{2d} + \dfrac{2}{(2d)^2} - O(1/d^2 \log(d))$ (Section 2.8). Arguing conversely, and exploiting existing upper bounds on $\dot{p}_c(\mathbb{Z}^d)$ obtained using lace expansion, we also improve on the known lower bounds on $\dot{a}(\mathbb{Z}^d)$ from Barequet et al. [2010]; we obtain $\dot{a}(\mathbb{Z}^d) = 2de - O(1)$. See Section 2.7 for more.

There is a lot of room for improvement for bounds such as the above, and indeed we refine (2.1) into

$$b(G) \geq f(r(\dot{p}_c(G))), \tag{2.2}$$

where $b(G)$ denotes the exponential growth rate of the *interfaces* of G, a subfamily of the lattice animals that arises naturally in Peierls type arguments, on which we elaborate in Sections 2.3 and 2.4. To establish (2.2) we consider the exponential growth rate $b_r = b_r(G)$ of the number of interfaces of G with size n and volume-to-surface ratio approximating r, as a function of r. We consider the aforementioned results to be of independent interest; in fact, the main motivation for studying b_r is the duality relation Theorem 2.5.1 which was obtained before the above bounds were noticed. We summarize what we know about b_r in Figure 2.1.

One of the best known results of percolation theory is the exponential decay, as $n \to \infty$, of the probability $\mathbb{P}_p(|C_o| = n)$ of the cluster C_o of the origin having size n for p in the subcritical interval $[0, p_c)$ Menshikov [1986]; Aizenman and Barsky [1987]. In the supercritical case $p \in (p_c, 1]$ this exponential decay holds, for some but not all, lattices and values of p Aizenman et al. [1980]; Hermon and Hutchcroft [2019].

Letting $S_o \subseteq C_o$ denote the interface of C_o, we can analogously ask for which $p \in (0,1)$ we have exponential decay of the probability $\mathbb{P}_p(|S_o| = n)$. We prove that this is uniquely determined by the value $b_{r(p)}$, where $r(p) := \frac{1-p}{p}$ is a bijection between the parameter spaces of edge density p and volume-to-surface ratio r. More concretely, we identify a universal function $f(r) := \frac{(1+r)^{1+r}}{r^r}$ such that, firstly, $b_{r(p)}(G) \leq f(r(p))$ for every lattice G and every $p \in (0,1)$ (Proposition 2.4.4), and secondly, $\mathbb{P}_p(|S_o| = n)$ decays exponentially in n for exactly those values of p for which this inequality is strict (Figure 2.1):

[1] We thank John Wierman for this remark.

Theorem 2.1.1. *Let $G \in \mathcal{S}$. Then for every $p \in (0,1)$, the interface size distribution $\mathbb{P}_p(|S_o| = n)$ fails to decay exponentially in n if and only if*

$$b_{r(p)}(G) = f(r(p)).$$

The class of lattices $\mathcal{S}$ we work with includes the standard cubic lattice in $\mathbb{Z}_d$, $d \geq 2$, as well as all quasi-transitive planar lattices (see Section 2.3 for definitions). We expect our results to hold for all vertex-transitive 1-ended graphs but decided to restrict our attention to $\mathcal{S}$ to avoid technicalities that would add little to the understanding of the matter.

We emphasize that Theorem 2.1.1, and the function $f(r)$, is independent of the dimension.

Theorem 2.1.1 holds even if we replace the interface size distribution $\mathbb{P}_p(|S_o| = n)$ by the cluster size distribution $\mathbb{P}_p(|C_o| = n)$, and $b_{r(p)}(G)$ by its analogue $a_{r(p)}(G)$ counting lattice animals. This was proved by Hammond [2005] building on a result of Delyon [1980]. But it can also be seen as a special case of Theorem 2.1.1: our definition of interface entails some flexibility, as it is based on a choice of a basis $\mathcal{P}$ of the cycle space of G. Letting $\mathcal{P}$ contain all cycles identifies interfaces and lattice animals. However, b_r is a more interesting function when $\mathcal{P}$ is a sparser basis, in particular the set of squares of the cubic lattice in $\mathbb{R}^d$, and we will work with such bases in the rest of this chapter.

Incidentally, we also prove that the rate of the exponential decay of $\mathbb{P}_p(|C_o| = n)$, defined as $c(p) := \lim_n \left(\mathbb{P}_p(|C_o| = n)\right)^{1/n}$, is a continuous function of p (Theorem 2.9.1). Our proof makes use of the Arzelà-Ascoli theorem but otherwise boils down to elementary calculations not involving our notion of an interface.

2.1.2 Our interfaces and their growth rates

The term *interface* is commonly used in statistical mechanics to denote the common boundary of two components of a crystal or liquid that are in a different phase. The precise meaning of the term varies according to the model in question and the perspective of its study. In Chapter 1 we introduced a variant of the notion of interface for Bernoulli percolation and used it to prove the analyticity of the percolation density for supercritical planar percolation.

A famous and simple use of interfaces in percolation theory is Peierls' argument, which deduces an upper bound on $p_c(\mathbb{Z}^2)$ from an upper bound on the exponential growth rate of the number of cycles in the dual with size n separating the origin from infinity, see e.g. Grimmett [1999]. The bounds thus obtained are rather weak, mainly due to the use of a union bound over a large number of heavily dependent events. Our innovation of considering P as part of the definition of interface in Chapter 1 allowed us to reduce these dependencies, thus refining Peierls' argument into an exact formula for the percolation density $\theta(p) := \mathbb{P}_p(C_o \text{ is infinite})$:

$$1 - \theta(p) = \sum_{M \in \mathcal{MS}} (-1)^{c(M)+1} \mathbb{P}_p(\{M \text{ occurs}\}). \tag{2.3}$$

Equality (2.3) means that in principle we could answer any question about Bernoulli percolation (e.g. continuity of $\theta(p)$ at p_c, the numerical value of p_c etc.) if we could compute the numbers $c_{n,r}$ of (multi-)interfaces with $|M| = n$ edges and $|\partial M| = m$ boundary edges accurately enough. In practice, it is rather hopeless to compute these numbers but we will make some progress and obtain results about their order of magnitude, i.e. their exponential growth rate. It turns out that the *volume-to-surface ratio* n/m is bounded for every lattice G, and for a given ratio $r := n/m$, the number of interfaces with size n and ratio 'roughly' r grows exponentially in n, at a rate that we will denote by $b_r = b_r(G)$. *Size* here refers to the number of vertices or edges, depending on whether we are interested in site or bond percolation, respectively. We stress however that our notion of interface, and b_r, is defined without reference to any random experiment. Still, we have two variants, site- and bond-interfaces, and use the one or the other depending on whether we want to study site or bond percolation.

We now formally define b_r, the fundamental quantity of this chapter. For a possible 'size' $n \in \mathbb{N}$, 'volume-to-surface ratio' $r \in \mathbb{R}_+$, and 'tolerance' $\epsilon \in \mathbb{R}_+$, let $c_{n,r,\epsilon}$ denote the number of interfaces P with $|P| = n$ and $(r-\epsilon)n \leq |\partial P| \leq (r+\epsilon)n$. These numbers grow exponentially in n, and we define b_r to be their exponential growth rate as $\epsilon \to 0$:

$$b_r = b_r(G) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} c_{n,r,\epsilon}(G)^{1/n}.$$

Since $c_{n,r,\epsilon}$ decreases as $\epsilon \to 0$, it is unclear at first sight whether b_r can ever be greater than 1. But as we will see, there is some value of r such that b_r equals the exponential growth rate of *all* the interfaces of G (Proposition 2.4.3). By exploiting the fact that the number of partitions of an integer n grows sub-exponentially in n, we observed that the value of b_r is unaffected if instead of interfaces we count multi-interfaces (Lemma 2.4.2). Moreover, b_r is a continuous (Theorem 2.6.4), and log-concave (Theorem 2.6.3) function of r.

For 'triangulated' lattices as defined in (2.8), we show that

$$b_r = (b_{1/r})^r \tag{2.4}$$

in other words, the values of b_r for $r < 1$ determine those for $r > 1$ (Theorem 2.5.1). This is the technically most involved result of this chapter. It shows that considering interfaces rather than animals yields a more interesting function b_r, namely one where the values of b_r and $f(r)$ coincide for fewer values of r. One of the ideas involved in the proof of (2.4) is that one can reverse the roles of P and ∂P to define 'inner-interfaces', and a typical inner-interface can be turned 'inside out by changing relatively few edges to yield an interface, and vice-versa. Therefore, the exponential growth rates of interfaces and inner-interfaces coincide. Refining this statement by taking the corresponding ratios r into account yields (2.4). Amusingly, our universal function $f(r)$ also has property (2.4).

2.1.3 Using interfaces and percolation to count lattice animals

In Section 2.7 we combine (2.1) with known results on the asymptotic expansion of $\dot{p}_c(\mathbb{Z}^d)$ as $d \to \infty$ to deduce that $\dot{a}(\mathbb{Z}^d) = 2de - O(1)$ (Theorem 2.7.3). This improves on a result of Barequet et al. [2010] that $\dot{a}(\mathbb{Z}^d) = 2de - o(d)$, where a dot above p_c, a or b_r means that we are considering site-percolation, lattice site-animals, and site-interfaces respectively (most of our results have a bond and a site version). In the latter paper, it was conjectured that $\dot{a}(\mathbb{Z}^d) = 2de - 3e + O(1/d)$. Under the assumption that $\dot{p}_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{5}{2(2d)^2} + O(1/d^3)$ holds, as reported in Gaunt et al. [1976] based on numerical evidence, our method gives the conjectured lower bound $\dot{a}(\mathbb{Z}^d) \geq 2de - 3e + O(1/d)$. It is reasonable to expect that both $b(\mathbb{Z}^d) - \dot{b}_{r_d}(\mathbb{Z}^d) = O(1/d)$ and $\dot{a}(\mathbb{Z}^d) - b(\mathbb{Z}^d) = O(1/d)$ hold, which combined with the above assumption would imply the aforementioned conjecture $\dot{a}(\mathbb{Z}^d) = 2de - 3e + O(1/d)$. The case of bond lattice animals turns out to be a bit easier, and we obtain the analogous $a(\mathbb{Z}^d) = 2de - \frac{3e}{2} - O(1/d)$ using an asymptotic expansion for $p_c(\mathbb{Z}^d)$ for bond percolation obtained rigorously in Hara and Slade [1995]; Hofstad and Slade [2006] using lace expansion.

In simultaneous work, Barequet and Shalah [2019] prove $\dot{a}(\mathbb{Z}^d) \leq 2de - 2e + 1/(2d-2)$. In Section 2.8 we improve this asymptotically into $\dot{a}(\mathbb{Z}^d) \leq 2de - 5e/2 + O(1/\log(d))$, narrowing the gap towards the aforementioned conjecture of Barequet et al. [2010]. For this, we use direct combinatorial arguments that do not involve percolation. We then plug these bounds into (2.1) to obtain the bounds

$\dot{p}_c(\mathbb{Z}^d) \geq \frac{1}{2d} + \frac{2}{(2d)^2} - O(1/d^2 \log(d))$ (Theorem 2.8.4).

2.1.4 Comparison to Hammond's work

Several ideas and results of this chapter were previously obtained by Hammond [2005], with the difference that Hammond considered directly the exponential growth rate of the number of lattice animals, rather than interfaces, of surface-to-volume ratio r. Among other results, Hammond proved that these growth rates satisfy the statements analogous to our Propositions 2.4.4, 2.4.6 and 2.6.3. The two approaches have some similarities but certain additional combinatorial and geometric arguments are needed to prove our results. Our approach to defining b_r is simpler than that of Hammond, giving rise to simpler proofs. One additional difficulty that we were faced with is that, unlike lattice animals containing the origin, several interfaces can occur simultaneously in a percolation instance. Our results of Sections 2.5 (duality), 2.9 (continuity of decay exponents) and 2.7 (implications for counting lattice animals) have no analogues in Hammond [2005].

2.2 Definitions and preliminaries

2.2.1 Cycle space

The *edge space* of a graph G is the direct sum $\mathcal{E}(G) := \bigoplus_{e \in E(G)} \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{0, 1\}$ is the field of two elements, which we consider as a vector space over $\mathbb{Z}_2$. The *cycle space* $\mathcal{C}(G)$ of G is the subspace of $\mathcal{E}(G)$ spanned by the *circuits* of cycles, where a circuit is an element $C \in \mathcal{E}(G)$ whose non-zero coordinates $\{e \in E(G) \mid C_e = 1\}$ coincide with the edge-set of a cycle of G.

2.2.2 Convergence and continuity

Let (f_n) be a sequence of continuous functions on an interval $[a, b]$. The sequence is said to be equicontinuous if, for every $\epsilon > 0$ and x, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \epsilon$$

whenever $|x - y| < \delta$ for every n.

The Arzelà-Ascoli theorem Rudin [1964] gives necessary and sufficient conditions to decide whether a subsequence of functions converges uniformly.

Theorem 2.2.1 (Arzelà-Ascoli theorem). *Let (f_n) be a uniformly bounded and equicontinuous sequence of continuous functions on an interval $[a, b]$. Then there is a subsequence of (f_n) that converges uniformly on $[a, b]$.*

2.2.3 Quasi-transitive planar lattices

Recall that a planar quasi-transitive lattice is a locally finite, connected graph G embedded in $\mathbb{R}^2$ such that for some linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$, translation by each v_i preserves G, and the action defined by the translations has finitely many orbits of vertices. Although not part of the definition, we will always assume in this chapter that planar quasi-transitive lattices are 2-connected, so that the two definitions of interfaces coincide. This is only a minor assumption because the boundary of a face of G contains a cycle that surrounds every other boundary vertex of the same face. By deleting every vertex that does not lie in the surrounding cycle of some face of G, we obtain a 2-connected planar quasi-transitive lattice with the same p_c and many other common properties with the initial graph.

It is not hard to see that planar quasi-transitive lattices are quasi-isometric to $\mathbb{R}^2$, inheriting some of its geometric properties. More precisely any planar quasi-transitive lattice G

(1) has quadratic growth, i.e. there are constants $c_1 = c_1(G), c_2 = c_2(G) > 0$ such that
$$c_1 n^2 \leq |B(u, n)| \leq c_2 n^2$$
for every $u \in V(G)$ and every positive integer n, where $B(u, n)$ denotes the ball of radius n around u in either graph-theoretic distance or Euclidean distance,

(2) satisfies a 2-dimensional isoperimetric inequality, i.e. there is a constant $c = c(G) > 0$ such that for any finite subgraph $H \subset G$,
$$|\partial_V H| \geq c\sqrt{|H|}.$$

Here $\partial_V H$ denotes the minimal vertex cut of H, i.e. the minimal set of vertices, the deletion of which disconnects H from the infinite component of $G \setminus H$. Similarly, the minimal edge cut of H is the minimal set of edges, the deletion of which disconnects H from the infinite component of $G \setminus H$. It is denoted by $\partial_E H$. Any planar quasi-transitive lattice G is easily seen to satisfy the following properties as well:

(3) For some $o \in V(G)$, there is a 2-way infinite path $X = (\ldots, x_{-1}, x_0 = o, x_1, \ldots)$ containing o and a constant $f > 0$, such that $f|i - j| \leq d_G(x_i, x_j)$ for every

$i, j \in \mathbb{Z}$, where d_G denotes distance in G. Moreover, we can choose X to be periodic, i.e. to satisfy $X + tv_1 = X$ for some $t \in \mathbb{N}$. The existence of such a path was in fact proved in Chapter 1.

(4) The cycle space of G is generated by cycles of bounded length.

(5) G is 1-ended, i.e. for every finite subgraph H of G, the graph $G \setminus H$ has a unique infinite component.

2.3 Interfaces

In this chapter, we recall the notions of (bond-)interfaces and site-interfaces introduced in Georgakopoulos and Panagiotis [2018]. In most cases, we will work with the following families of graphs:

(a) planar quasi-transitive lattices,

(b) the standard cubic lattice $\mathbb{Z}^d$, $d > 1$,

(c) $\mathbb{T}^d$, the graph obtained by adding to $\mathbb{Z}^d$, $d > 1$ the 'monotone' diagonal edges, i.e. the edges of the form xy where $y_i - x_i = 1$ for exactly two coordinates $i \leq d$ and $y_i = x_i$ for all other coordinates ($\mathbb{T}^2$ is isomorphic to the triangular lattice).

Let us denote with $\mathcal{S}$ the set of all those graphs.

For each $G \in \mathcal{S}$, we will fix a basis $\mathcal{P} = \mathcal{P}(G)$ of the cycle space $\mathcal{C}(G)$ (defined in Section 2.2.1). If G is a planar quasi-transitive lattice , $\mathcal{P}$ consists of the cycles bounding the faces of G. For $G = \mathbb{Z}^d$ we can use the squares bounding the faces of its cubes as our basis $\mathcal{P}$, and for $G = \mathbb{T}^d$ we can use the triangles obtained from the squares once we add the 'monotone' diagonal edges. Our definition of the interfaces of G depends on the choice of $\mathcal{P}(G)$, and so in Georgakopoulos and Panagiotis [2018], we used the notation '$\mathcal{P}$-interface' to emphasize the dependence. Since we are fixing $\mathcal{P}(G)$ for each $G \in \mathcal{S}$, we will simplify our notation and just talk about interfaces.

In the 2-dimensional case, interfaces have already been defined in Chapter 1. With some thought, this notion can be generalised to higher dimensions in such a way that a unique interface is associated with any cluster. The reader may already have their own favourite definition of interface for $G = \mathbb{Z}^d$ or $G = \mathbb{T}^d$, and as long as that definition satisfies Theorem 2.3.3 below it will coincide with ours. For the remaining readers we offer the following abstract definition. For (site percolation on) $G = \mathbb{T}^d$ we offer a simpler alternative definition implicit in Proposition 2.3.5.

To define interfaces in full generality, we need to fix first some notation. From now on, all our graphs G will be

$$\text{1-ended and 2-connected.} \tag{2.7}$$

Every edge $e = vw \in E(G)$ has two *directions* $\overrightarrow{vw}, \overrightarrow{wv}$, which are the two directed sets comprising v, w. The head *head*$(\overrightarrow{vw})$ of $\overrightarrow{vw}$ is w. Given $F \subset E(G)$ and a subgraph D of G, let $\vec{F}^D := \{\overrightarrow{vz} \mid vz \in F, z \in V(D)\}$ be the set of directions of the elements of F towards D.

Let $\mathcal{P}$ denote a basis of $\mathcal{C}(G)$ (which we fixed at the beginning of this section). A $\mathcal{P}$*-path connecting* two directed edges $\overrightarrow{vw}, \overleftarrow{yx} \in \overleftrightarrow{E}(G)$ is a path P of G such that the extension $vwPyx$ is a subpath of an element of $\mathcal{P}$. Here, the notation $vwPyx$ denotes the path with edge set $E(P) \cup \{vw, yx\}$, with the understanding that the end-vertices of P are w, y. Note that P is not endowed with any notion of direction but the directions of the edges $\overrightarrow{vw}, \overleftarrow{yx}$ it connects do matter. We allow P to consist of a single vertex $w = y$.

We will say that P *connects* an undirected edge $e \in E(G)$ to $\overrightarrow{f} \in \overleftrightarrow{E}(G)$ (respectively, to a set $J \subset \overleftrightarrow{E}(G)$) if P is a $\mathcal{P}$-path connecting one of the two directions of e to $\overrightarrow{f}$ (resp. to some element of J).

Definition 2.3.1. *We say that a set $J \subset \overleftrightarrow{E}(G)$ is F-connected for some $F \subset E(G)$, if for every proper bipartition (J_1, J_2) of J, there is a $\mathcal{P}$-path in $G \setminus F$ connecting an element of J_1 to an element of J_2.*

Definition 2.3.2. *A (bond-)interface of G is a pair $(P, \partial P)$ of sets of edges of G with the following properties*

(i) ∂P separates o from infinity;

(ii) There is a unique finite component D of $G \setminus \partial P$ containing a vertex of each edge in ∂P;

(iii) $\partial \vec{P}^D$ is ∂P-connected; and

(iv) $P = \{e \in E(D) \mid$ there is a $\mathcal{P}$-path in $G \setminus \partial P$ connecting e to $\partial \vec{P}^D\}$.

We say that an interface $(P, \partial P)$ occurs in a bond percolation instance ω if the edges of P are occupied, and the edges of ∂P are vacant.

(Bond-)interfaces are specifically designed to study bond percolation on G. There is a natural analogue for site percolation. For an interface $(P, \partial P)$ of G , we let $V(P)$ denote the set of vertices incident to an edge in P, and we let $V(\partial P)$

denote the set of vertices incident to an edge in ∂P but with no edge in P. We say that an interface $(P, \partial P)$ is a *site-interface* if no edge in ∂P has both its end-vertices in $V(P)$. We say that a site-interface $(P, \partial P)$ occurs in a site percolation instance ω if the vertices of $V(P)$ are occupied, and the vertices of $V(\partial P)$ are vacant. We will still use P and ∂P to refer to $V(P)$ and $V(\partial P)$.

We say that $(P, \partial P)$ meets a cluster C of ω, if either $P \cap E(C) \neq \emptyset$, or $P = E(C) = \emptyset$ and $\partial P = \partial C$, where ∂C is the set edges in $E(G) \setminus E(C)$ with at least one end-vertex in C (in which case C consists of o only).

The following result applies to both bond- and the site-interfaces.

Theorem 2.3.3 ([Georgakopoulos and Panagiotis, 2018, Theorem 10.4.])**.** *For every finite (site) percolation cluster C of G such that C separates o from infinity, there is a unique (site-)interface $(P, \partial P)$ that meets C and occurs. Moreover, we have $P \subset E(C)$ and $\partial P \subset \partial C$.*

Conversely, every occurring (site-)interface meets a unique percolation cluster C, and ∂C separates o from infinity (in particular, C is finite).

Theorem 2.3.3 allows us to define the (site-)interface of a cluster C of a percolation instance ω as the unique occurring (site-)interface that meets C.

We remark that for every planar quasi-transitive lattice G, the notion of outer interface introduced in Chapter 1 coincides with the aforementioned notion of interface, once we choose as basis $\mathcal{P}$ of the cycle space $\mathcal{C}(G)$ the set of cycles at the boundary of each face of G. Indeed, it is clear that an outer interface $(S, \partial S)$ satisfies properties (i) and (ii). Properties (iii) and (iv) follow from the fact that ∂S^* is connected. Hence any outer interface is also a interface. On the other hand, any interface gives rise to a finite component D. We can associate to D a outer interface $(S, \partial S)$, as described in Section 1.4.5. As we explained, $(S, \partial S)$ is also a interface and by Theorem 2.3.3, it must coincide with $(P, \partial P)$, i.e. $(P, \partial P)$ is a outer interface, as desired.

Remark 2.3.4. *Let G be a graph the cycle space of which admits a basis consisting of cycles of length bounded by some constant $t > 0$. Then for every interface $(P, \partial P)$ of G, and any pair of edges in ∂P, there is a path contained in the $t/2$-neighbourhood of ∂P connecting the pair (see [Georgakopoulos and Panagiotis, 2018, p. 47]).*

We define a *multi-interface* to be a finite collection of pairwise disjoint interfaces, and a *site-multi-interface* to be a finite collection of pairwise disjoint site-interfaces.

In the case of an

$$\text{1-ended, 2-connected graph } G\text{, the cycle space of which is generated by its triangles,} \tag{2.8}$$

site-interfaces admit an equivalent definition that is more standard and easier to work with:

Proposition 2.3.5 ([Georgakopoulos and Panagiotis, 2019a, Proposition 3.7.]). *Let G be a graph satisfying* (2.8)*, and let D be a finite induced subgraph of G containing o. Let $\overline{D}$ be the union of D with the finite connected components of $G \setminus D$. Define P to be the set of vertices of $\overline{D}$ which have a neighbour not in $\overline{D}$, and let ∂P be the set of vertices of $G \setminus \overline{D}$ that have a neighbour in $\overline{D}$. Then $(P, \partial P)$ is the site-interface of D.*

Most of the time we will write P instead of $(P, \partial P)$ to simplify the notation.

2.4 Growth rates

In this section, we give the formal definition of b_r in its bond and site version, obtain some basic facts about it, and establish the connection to percolation.

Given a graph $G \in \mathcal{S}$, we let $I_{n,r,\epsilon} = I_{n,r,\epsilon}(G)$ denote the set of interfaces P with $|P| = n$ and $(r-\epsilon)n \leq |\partial P| \leq (r+\epsilon)n$. Here $|\cdot|$ counts the number of edges. Similarly, we let $MI_{n,r,\epsilon} = MI_{n,r,\epsilon}(G)$ denote the set of multi-interfaces P with $|P| = n$ and $(r-\epsilon)n \leq |\partial P| \leq (r+\epsilon)n$.

To avoid introducing a cumbersome notation, we will still write $I_{n,r,\epsilon}$ and $MI_{n,r,\epsilon}$ for the site-interfaces and site-multi-interfaces, respectively, of size n and boundary size between $(r-\epsilon)n$ and $(r+\epsilon)n$. Moreover, we will write $c^{\circ}_{n,r,\epsilon}$ and $c^{\odot}_{n,r,\epsilon}$ for the cardinality of $I_{n,r,\epsilon}$ and $MS_{n,r,\epsilon}$, respectively.

The definitions, results and proofs that follow apply to both (bond-)interfaces and site-interfaces unless otherwise stated.

Definition 2.4.1. *Define the (upper) exponential growth rate $b^{\circ}_r(G)$ of the (bond- or site-) interfaces of G with surface-to-volume ratio r by*

$$b^{\circ}_r = b^{\circ}_r(G) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} c^{\circ}_{n,r,\epsilon}(G)^{1/n}.$$

Similarly, we define the (upper) exponential growth rate $b^{\circ}_r(G)$ of the (site-)multi-

interfaces of G with surface-to-volume ratio r by

$$b_r^{\odot} = b_r^{\odot}(G) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} c_{n,r,\epsilon}^{\odot}(G)^{1/n}.$$

We remark that in Hammond's definition of the exponential growth rate of lattice animals with surface-to-volume ratio r, ϵ depends on n. The above definition simplifies the proofs of some of the following results.

We are going to study b_r° and $b_r^{\odot}$ as functions of r. As it turns out, these two functions coincide:

Lemma 2.4.2. *Let $G \in \mathcal{S}$. Then $b_r^{\circ}(G) = b_r^{\odot}(G)$.*

We postpone the proof until the next section where the necessary definitions and tools are introduced.

From now on, except for the proof of Lemma 2.4.2, we will drop the superscripts and we will simply write b_r and $c_{n,r,\epsilon}$. In our proofs, we will work with interfaces and site-interfaces instead of multi-interfaces and site-multi-interfaces.

Similarly to b_r, we define the (upper) exponential growth rate of all interfaces of G:

$$b = b(G) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} c_n(G)^{1/n}$$

where $c_n(G) := |\{ \text{ interfaces } P \text{ with } |P| = n\}|$. In the following proposition we prove that $b(G) = \max_r b_r(G)$.

Proposition 2.4.3. *Let $G \in \mathcal{S}$. Then there is some r such that $b(G) = b_r(G)$.*

Proof. Notice that there are no (site-)interfaces P with $|P| \geq 1$ and $|\partial P|/|P| > 2\Delta$, where Δ is the maximum degree of G. Recursively subdivide the interval $I_0 := [0, 2\Delta]$ into two subintervals of equal length. At each step j, one of the two subintervals I_j of I_{j-1} accounts for at least half of the (site-)interfaces P of size n with $|\partial P|/|P| \in I_{j-1}$ for infinitely many n. Hence there are at least $2^{-j} c_n$ (site-)interfaces of size n with $|\partial P|/|P| \in I_j$ for infinitely many n. By compactness, $[0, 2\Delta]$ contains an accumulation point r_0 of the $I_j, j \in \mathbb{N}$. Notice that for every $\epsilon > 0$ we have $\limsup_{n\to\infty} c_{n,r_0,\epsilon}^{1/n} = b$. Taking the limit as ϵ goes to 0, we obtain $b = b_{r_0}$, as desired. □

We will now obtain some bounds for b_r. Notice that both $\mathbb{Z}^d$, $\mathbb{T}^d$ contain a 2-way infinite geodesic, namely the x-axis.

Proposition 2.4.4. *Let $G \in \mathcal{S}$, and let $r > 0, 0 \leq p \leq 1$. Then we have $p(1-p)^r \leq 1/b_r(G)$.*

Proof. Let us first assume that G satisfies (3). Let N_n be the (random) number of occurring (site-)interfaces P with $|P| = n$ in a percolation instance ω. Consider a quasi-geodesic X containing o, and let X^+ be one of the two infinite subpaths starting from o. Arguing as in the proof of Proposition 1.4.4, we see that any occurring (site-)interface P has to contain one of the first fn vertices of X^+ for some constant $f > 0$. Since occurring (site-)interfaces are disjoint,

$$N_n \leq fn \tag{2.9}$$

for every n and any bond (site) percolation instance ω. Therefore, $\mathbb{E}_p(N_n) \leq fn$ for every $p \in [0,1]$. We now have $fn \geq \mathbb{E}_p(N_n) \geq c_{n,r,\epsilon}(p(1-p)^{r+\epsilon})^n$. Taking the n-th root, and then letting n go to infinity, and ϵ go to 0, we obtain $p(1-p)^r \leq 1/b_r$, as desired. □

Next, we observe that for any fixed r, equality in Proposition 2.4.4 can occur for at most one value of p, which value we can compute:

Proposition 2.4.5. *Let $G \in \mathcal{S}$. If $p(1-p)^r = 1/b_r(G)$ for some r, p, then $p = \frac{1}{1+r}$ (and so $r = \frac{1-p}{p}$ and $1/b_r(G) = p(1-p)^{\frac{1-p}{p}}$).*

Proof. Fix r and let $M := \max_{p\in[0,1]} p(1-p)^r$. If $p_0(1-p_0)^r = 1/b_r$ is satisfied for some $p_0 \in [0,1]$, then p_0 must attain M by Proposition 2.4.4, that is, we have $M = p_0(1-p_0)^r$. Thus $f'(p_0) = 0$. By elementary calculus, $f'(p) = (1-p)^r - rp(1-p)^{r-1}$, from which we obtain $r = \frac{1-p_0}{p_0}$ and $p_0 = \frac{1}{1+r}$. □

Combining Proposition 2.4.4 and Proposition 2.4.5 we obtain

$$b_r \leq f(r) := \frac{(1+r)^{1+r}}{r^r}. \tag{2.10}$$

Notice that the function $f(r)$ is reminiscent of the entropy function $p\log(p) + (1-p)\log(1-p)$ for Bernoulli random variables, where p and r are related via the formula $p = 1/(1+r)$. This is not a surprise, as the proofs of this section are essentially a matter of large deviations estimates.

Motivated by Proposition 2.4.5, we define the functions

$$p(r) := \frac{1}{1+r} \text{ and } r(p) := \frac{1-p}{p}.$$

These functions are $1-1$, strictly monotone decreasing, and the inverse of each other.

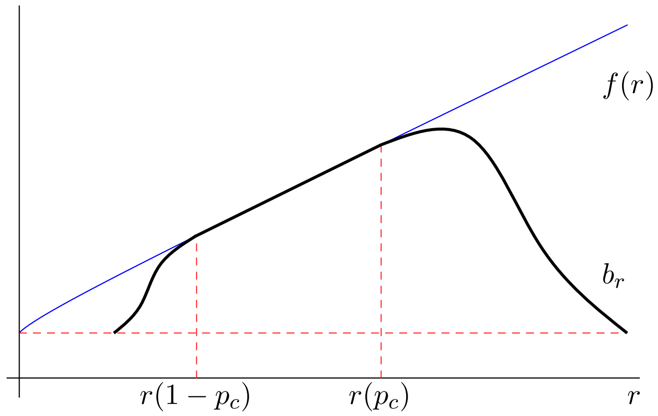

Figure 2.1: An approximate visualisation of $b_r(G)$ when G is a lattice in $\mathbb{R}^d, d \geq 3$. The graph of $b_r(G)$ (depicted in black, if colour is shown) lies below the graph of $f(r) := \frac{(1+r)^{1+r}}{r^r}$ (depicted in blue, if colour is shown). The fact that $f(r)$ plots (in Mathematica, in this instance) almost like a straight line can be seen by rewriting it as $(1+r)(1+1/r)^r$, which approximates the function $e(1+r)$. The fact that $b_r = f(r)$ for r in the interval $(r(1-p_c), r(p_c)]$, where $r(p) := \frac{1-p}{p}$, follows by combining a theorem of Kesten and Zhang [1990], saying that exponential decay of $\mathbb{E}_p(N_n)$ fails in that interval, with our Theorem 2.1.1. That $b_r < f(r)$ for $r > r(p_c)$ follows from the well-known exponential decay of $\mathbb{P}_p(|C_o| = n)$ for $p < p_c$ Menshikov [1986]; Aizenman and Barsky [1987].

Recall that N_n denotes the number of occurring multi-interfaces P with $|P| = n$. The next result says that equality is achieved in Proposition 2.4.5 (for some r) exactly for those p for which exponential decay in n of $\mathbb{E}_p(N_n)$ fails.

Proposition 2.4.6. *Let $G \in \mathcal{S}$ and $p \in (0,1)$. Then $\mathbb{E}_p(N_n)$ fails to decay exponentially in n if and only if $b_{r(p)}(G) = 1/p(1-p)^{r(p)}$ (that is, if and only if equality is achieved in Proposition 2.4.5).*

Proof. The backward implication is straightforward by the definitions.

For the forward implication, suppose to the contrary that

$$b_{r(p)} < 1/p(1-p)^{r(p)}.$$

The definition of b_r implies that there is $\epsilon > 0$ such that

$$c_{n,r(p),\epsilon} p^n (1-p)^{n(r(p)-\epsilon)} \leq (1-\epsilon)^n$$

for all but finitely n. Hence, if we denote by $N_{n,r(p),\epsilon}$ the (random) number of occurring (site-)multi-interfaces P with $|P| = n$ and $(r(p)-\epsilon)n \leq |\partial P| \leq (r(p)+\epsilon)n$, then for every large enough n,

$$\mathbb{E}_p(N_{n,r(p),\epsilon}) \leq c_{n,r(p),\epsilon} p^n (1-p)^{n(r(p)-\epsilon)} \leq (1-\epsilon)^n,$$

which implies the exponential decay in n of $\mathbb{E}_p(N_{n,r(p),\epsilon})$.

On the other hand, we claim that $\mathbb{E}_p(N_n - N_{n,r(p),\epsilon})$ always decays exponentially in n. Indeed, consider the function $g(q,r) = q(1-q)^r$. Notice that for every fixed r the function $g_r(q) := g(q,r)$ is maximised at $\frac{1}{1+r}$ and is strictly monotone on the intervals $[0, \frac{1}{1+r}]$ and $[\frac{1}{1+r}, 1]$. Recall that $p = \frac{1}{1+r(p)}$, and define

$$s = s(p,\epsilon) := \frac{1}{1 + r(p) + \epsilon}$$

and

$$S = S(p,\epsilon) := \frac{1}{1 + r(p) - \epsilon}.$$

It follows that there is a constant $0 < c = c(p,\epsilon) < 1$ such that $g(p, r(p) + \epsilon) \leq cg(s, r(p) + \epsilon)$ and $g(p, r(p) - \epsilon) \leq cg(S, r(p) - \epsilon)$ because $s < p < S$. Moreover, we have

$$\Big(\frac{1-p}{1-s}\Big)^r \leq \Big(\frac{1-p}{1-s}\Big)^{r(p)+\epsilon} \leq c$$

whenever $r \geq r(p) + \epsilon$, and

$$\Big(\frac{1-p}{1-S}\Big)^r \leq \Big(\frac{1-p}{1-S}\Big)^{r(p)-\epsilon} \leq c$$

whenever $r \leq r(p) - \epsilon$. This implies that $g(p,r) \leq cg(s,r)$ for every $r \geq r(p) + \epsilon$, and $g(p,r) \leq cg(S,r)$ for every $r \leq r(p) - \epsilon$. In other words, we have $\mathbb{P}_p(P \text{ occurs}) \leq c^n \mathbb{P}_s(P \text{ occurs})$ for every bond or site multi-interface$P$ with $|P| = n$, $|\partial P| > (r+\epsilon)n$ and $\mathbb{P}_p(P \text{ occurs}) \leq c^n \mathbb{P}_S(P \text{ occurs})$ for every bond or site multi-interface$P$ with $|P| = n$, $|\partial P| < (r - \epsilon)n$. This easily implies that

$$\mathbb{E}_p(N_n - N_{n,r(p),\epsilon}) \leq c^n(\mathbb{E}_s(N_n) + \mathbb{E}_S(N_n)).$$

Since both $\mathbb{E}_s(N_n), \mathbb{E}_S(N_n) \leq fn$, we conclude that $E_p(N_n - N_{n,r(p),\epsilon})$, and hence $E_p(N_n)$, decays exponentially in n, which contradicts our assumption. Therefore, $b_{r(p)} = 1/p(1-p)^{r(p)}$. □

Let S_o denote the (site-)interface of the cluster C_o of o if C_o is finite, and $S_o = \emptyset$ otherwise. We can now easily deduce that the statement of Proposition 2.4.6 holds for $P_p(|S_o| = n)$ in place of $\mathbb{E}_p(N_n)$, as stated in Theorem 2.1.1, which we repeat here for convenience:

Theorem 2.4.7. *Let $G \in \mathcal{S}$. Then for every $p \in (0,1)$, the cluster size distribution $\mathbb{P}_p(|S_o| = n)$ fails to decay exponentially in n if and only if*

$b_{r(p)}(G) = 1/p(1-p)^{r(p)} = f(r(p))$.[2]

Proof. Let X be a quasi-geodesic containing o such that $X+tv_1 = X$ for some $t \in \mathbb{N}$, and let X^+ be one of the two infinite subpaths of X starting at o. If G is $\mathbb{Z}^d$ or $\mathbb{T}^d$, we just let X be the horizontal axis. Write also Q for the subpath of X connecting o to $o+tv_1$. Notice that any (site-)interface P meets X^+ at some vertex. Write x^+ for the first of X^+ that it meets. Using a multiple ktv_1 of tv_1 for some integer k, we can translate x^+ to some vertex of Q. Then $P+ktv_1$ is a (site-)interface of o. To see this, notice that $o-ktv_1$ appears before x^+ in X^+ because o appears before $x^+ + ktv_1$ in X^+. Thus $o-ktv_1$ belongs to the finite component of $G \setminus \partial P$, and so o belongs to the finite component of $G \setminus \partial(P+ktv_1)$. On the event $A = A(P) := \{P + ktv_1 \text{ occurs}\} \cap \{\text{the subpath of } X^+ \text{ between } o \text{ and } x^+ + ktv_1 \text{ is open}\}$, we have $S_o = P + ktv_1$. Moreover,

$$p^M \mathbb{P}_p(P \text{ occurs}) \leq \mathbb{P}(A),$$

where M is the number of vertices of Q. Summing over all (site-)interfaces of size n with the property that the first vertex of X^+ they contain is x^+, we obtain

$$p^M \sum \mathbb{P}_p(P \text{ occurs}) \leq \mathbb{P}_p(|S_o| = n),$$

where the sum ranges over all such (site-)interfaces. Since there are at most fn choices for the first vertex of X^+, summing over all possible x^+ we obtain

$$p^M \mathbb{E}_p(N_n) \leq fn\mathbb{P}_p(|S_o| = n).$$

On the other hand, clearly

$$\mathbb{P}_p(|S_o| = n) \leq \mathbb{E}_p(N_n).$$

Therefore, $\mathbb{P}_p(|S_o| = n)$ decays exponentially if and only if $\mathbb{E}_p(X_n)$ does. The desired assertion follows now from Proposition 2.4.6. □

2.5 Duality

The main aim of this section is the proof of (2.4) (Theorem 2.5.1), and an analogous statement for planar bond percolation (Theorem 2.5.2). In this section, we study the properties of both interfaces and site-interfaces of graphs in $\mathcal{S}$.

If $G \in \mathcal{S}$ satisfies (2.8), we say that $(P, \partial P)$ is an *inner-interface* of G if

[2]That is, if and only if equality is achieved in Proposition 2.4.5.

$(\partial P, P)$ is a site-interface of G. We define b_r^* similarly to b_r, except that we now count inner-interfaces instead of site-interfaces. Note that both P and ∂P span connected graphs in this case. Since this operation inverts the surface-to-volume ratio, we have

$$b_r^* = b_{1/r}^r. \tag{2.11}$$

If G is a planar quasi-transitive lattice, we say that $(P, \partial P)$ is an inner-interface of G if $(\partial P^*, P^*)$ is an interface of G^*. Again define $b_r^*(G)$ similarly to $b_r(G)$, except that we now count inner-interfaces in the dual lattice G^*. Then (2.11) still holds in this case.

The main results of this section are:

Theorem 2.5.1. *Consider a graph $G \in \mathcal{S}$ satisfying* (2.8). *Then for the site-interfaces in G we have $b_r(G) = (b_{1/r}(G))^r$.*

Theorem 2.5.2. *Consider a planar quasi-transitive lattice G. Then for the interfaces in G and G^* we have $b_r(G) = (b_{1/r}(G^*))^r$.*

To prove Theorems 2.5.1 and 2.5.2 we need the following concepts. Given a graph $G \in \mathcal{S}$, let $v_1, v_2, \ldots, v_d \in \mathbb{R}^d$ be some linearly independent vectors that preserve G, and let $\mathcal{B}$ be the box determined by $v_1, v_2, \ldots, v_d$. For $\mathbb{Z}^d$ and $\mathbb{T}^d$ we can choose $v_1, v_2, \ldots v_d$ to be the standard basis of $\mathbb{R}^d$. Given a (site-)interface P of G, among all translates of $\mathcal{B}$ by an integer combination of $v_1, v_2, \ldots v_d$, consider those which intersect $P \cup \partial P$, and let $\mathcal{T}$ denote the set of all such translates. The *box* $B(P)$ of P is the smallest box with sides parallel to $v_1, v_2, \ldots, v_d$ containing $\mathcal{T}$. The *box size* $|B(P)|$ of P is the number of translates of $\mathcal{B}$ contained in $B(P)$ that intersect the topological boundary of $B(P)$. Define $\widetilde{b}_r$ like b_r, except that we restrict the (site-)interfaces we consider to a subfamily satisfying $|B(P)| = o(|P|)$, and take the supremum over all such subfamilies.

Our aim now is to prove that $\widetilde{b}_r = b_r$. In other words, (site-)interfaces with a 'fractal' shape have the same exponential growth rate as all (site-)interfaces. We will first consider the cases of $\mathbb{Z}^d$ and $\mathbb{T}^d$.

Proposition 2.5.3. *Let G be either $\mathbb{Z}^d$ or $\mathbb{T}^d$. Then $\widetilde{b}_r(G) = b_r(G)$.*

Proof. Let us first assume that $b_r > 1$. We will start by proving the assertion for (bond) interfaces. Let $n \in \mathbb{N}$, $\epsilon > 0$, $r > 0$, and let $P \in I_{n,r,\epsilon}$. Consider the associated box $B(P)$, and let a_i, b_i, $i = 1, 2, \ldots, d$ be integers such that $B(P) = \prod_{i=1}^d [a_i, b_i]$. Notice that $B(P)$ contains the graph $P \cup \partial P$ in its interior (no vertex

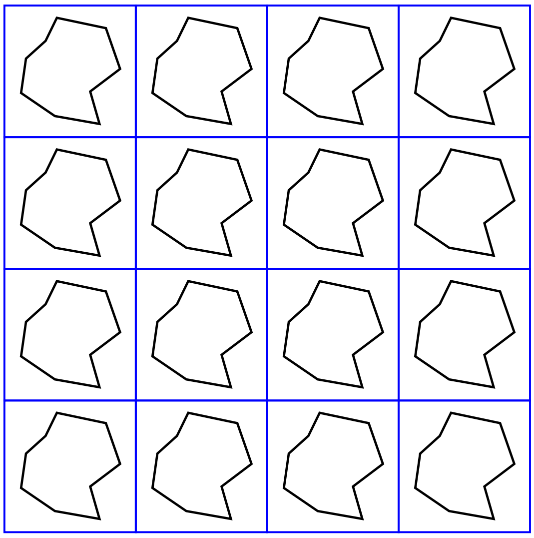

Figure 2.2: The grid B_n and the interfaces when $d = 2$ in the proof of Proposition 2.5.3.

of $P \cup \partial P$ lies in its topological boundary), and furthermore, each face of $B(P)$ is incident with ∂P. Order the vertices of G arbitrarily, and for each face f of $B(P)$, let v_f be the first vertex in our ordering that belongs to f and is incident with ∂P. We will call these vertices *extremal*. Now define the *shape* of an interface P to be the $4d$-tuple consisting of the numbers a_i, b_i, and the extremal vertices. Notice that the extremal vertices are in fact incident to a vertex of $\partial_V P$.

It is not hard to see that $-(n+1) \leq a_i \leq 0$ and $0 \leq b_i \leq n+1$ for every $i = 1, 2, \ldots, d$. This implies that there are at most $P_n := (n+2)^{2d}(2n+3)^{2d(d-1)}$ possible shapes for interfaces in $I_{n,r,\epsilon}$, since there are at most $n+2$ choices for each a_i, b_i, and each $(d-1)$-dimensional face has at most $(2n+3)^{d-1}$ vertices. On the other hand, there are exponentially many interfaces, hence we can choose n large enough to ensure that there is a non-empty set $K \subseteq I_{n,r,\epsilon}$ of cardinality at least

$$N := c_{n,r,\epsilon}/P_n$$

consisting of interfaces P with $|P| = n$ and $(r-\epsilon)n \leq |\partial P| \leq (r+\epsilon)n$ that have the same shape.

We now piece elements of K together to construct a large number of interfaces of arbitrarily high size that will contribute to b_r. We will construct a set K_n of cardinality about N^{n^d} of interfaces of size about n^{d+1}, of surface-to-volume ratio about r, and of small box-size.

Recall that all interfaces in K have the same shape, in particular, the same box B. Let B_n be a d-dimensional grid of n^d adjacent copies $B_{\mathbf{i}}$, $\mathbf{i} = (i_1, \ldots, i_d)$

of B (Figure 2.2). In each copy of B in B_n, we place an arbitrary element of K. We denote the copy of B placed in $B_{\mathbf{i}}$ with $K_{\mathbf{i}}$. Write S_k, $1 \leq k \leq n$ for the slab containing the boxes $B_{\mathbf{i}}$ with $i_1 = k$. Our aim is to connect the interfaces inside the boxes using mostly short paths. First, consider S_2 and notice that every box in S_2 shares a common face with a box in S_1. We can move S_2 using the vectors $v_2, \ldots, v_d$ in order to achieve that the 'rightmost' extremal vertices of S_1 coincide with the corresponding 'leftmost' extremal vertices of S_2 lying in a common face with them. This is possible because all interfaces in K have the same shape. Moving each slab S_k in turn, we can make the 'rightmost' and 'leftmost' extremal vertices of consecutive slabs coincide. We now connect all these extremal vertices with their corresponding interfaces by attaching paths of length 2 parallel to v_1. Finally, we connect the interfaces in the first slab as follows. If two boxes in the first slab share a common face, then we connect the two extremal vertices lying in the common face with a path of minimum length inside that face (hence of length $O(n)$). Also, we attach a path of length 2 connecting all those extremal vertices to the interface of their box (Figure 2.3).

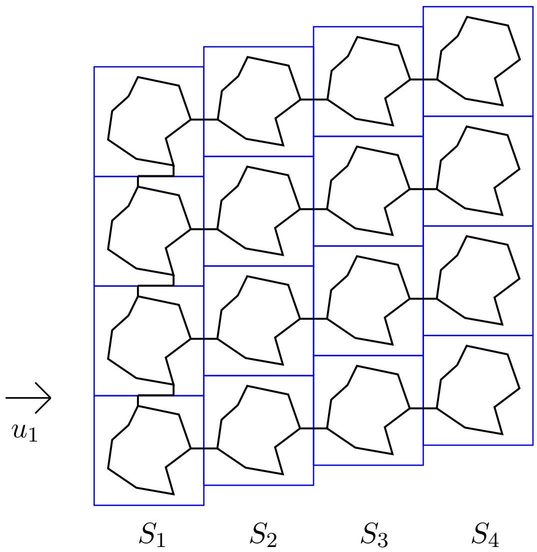

Figure 2.3: The interface Q in the proof of Proposition 2.5.3.

This construction defines a new graph Q. We claim that Q is an interface. Indeed, if $d = 2$ this follows easily from the topological definition of interfaces. For $d > 2$, since Q is a connected graph, there is an interface associated with it. We will verify that Q coincides with its interface, which is denoted Q'. Let $\partial_E Q$ denote the minimal edge cut of Q. Consider an interface $K_{\mathbf{i}}$. We will first verify that all edges of $\partial_E K_{\mathbf{i}} \setminus Q$ belong to $\partial_E Q$. Indeed, let $B'_{\mathbf{i}}$ be the smallest box containing $K_{\mathbf{i}}$ (but not necessarily its boundary). Any edge in $\partial_E K_{\mathbf{i}}$ that has not been attached to Q has either one end-vertex in the boundary of $B'_{\mathbf{i}}$ and one in the complement of $B'_{\mathbf{i}}$, or it can be connected with such an edge with a path lying in $G \setminus K_{\mathbf{i}}$. From there it can be connected to infinity without intersecting Q. Hence all edges of $(\partial_E K_{\mathbf{i}}) \setminus Q$ lie in $\partial_E Q$.

Next, we claim that all remaining edges of $(\partial K_{\mathbf{i}}) \setminus Q$ belong to the boundary of Q'. To see this, consider an edge e of $\partial K_{\mathbf{i}}$ that has been attached to Q. It is not hard to see that

any edge of $K_{\mathbf{i}} \cup \partial K_{\mathbf{i}}$ lying in the same basic cycle with e, lies in the same basic cycle with an edge of $\partial_E Q$. (2.12)

This easily implies that every edge in $\partial K_{\mathbf{i}} \setminus Q$ lies in the same ∂Q component of $\partial \vec{Q}^Q$ with some edge in $\partial_E Q$, which proves the claim.

It remains to show that every all edges of Q belongs to Q'. To see this, recall that any edge of $K_{\mathbf{i}}$ lies in the same basic cycle with an edge of $\partial K_{\mathbf{i}}$. Observation (2.12) implies that this remains true if we replace $\partial K_{\mathbf{i}}$ by $\partial Q'$, i.e. that any edge of $K_{\mathbf{i}}$ lies in the same basic cycle with an edge of $\partial Q'$. Hence all edges of $K_{\mathbf{i}}$ belong to Q'. It is easy to see that the remaining edges of Q are incident with $\partial_E Q$. Thus Q coincides with Q', hence Q is an interface, as desired.

It can be easily seen that Q has size roughly n^{d+1} and boundary size

$$(r - \epsilon')|Q| \leq |\partial Q| \leq (r + \epsilon')|Q|$$

for some $\epsilon' = \epsilon'(n)$ not necessarily equal to ϵ. Clearly, we can choose $\epsilon' = \epsilon + o(1)$, since the number of attached edges is $o(n^{d+1})$. The number of such Q we constructed is equal to $|K|^{n^d} \geq N^{n^d}$. This is because by deleting all attached paths we recover each $K_{\mathbf{i}}$, and we have $|K|$ choices for each $K_{\mathbf{i}}$.

Note that each slab S_k has been moved at distance at most $O(n^2) = o(|Q|)$ from its original position. Hence, $|B(Q)| = o(|Q|)$. The result follows by letting $n \to \infty$ and then $\epsilon \to 0$. In fact, we proved that the supremum in the definition of $\widetilde{b}_r$ is attained by some family.

Let us now consider the case of site-interfaces. Let K be a collection of at least N site-interfaces of $I_{n,r,\epsilon}$, all of which have the same shape. Arguing as above, we place the elements of K in a d-dimensional grid and we connect them in the same fashion to obtain a graph Q. For $\mathbb{Z}^d$, nothing changes since Q is an induced graph. This is not necessarily true for $\mathbb{T}^d$ because some end-vertices of the attached paths are possibly incident to multiple vertices of the same site-interface. This could potentially lead to an issue in the case that some boundary vertices cannot connect to infinity without avoiding the vertices of Q. See Figure 2.4. It is not hard to see that the latter is impossible in our case. Arguing as above, we obtain the desired result for site-interfaces as well.

It remains to consider the case where $b_r = 0$ or $b_r = 1$. If $b_r = 0$, there is nothing to prove. If $b_r = 1$, then we can argue as in the case $b_r > 1$, except that now we place the same interface at each box of the grid. □

The above arguments can be carried out for interfaces of any planar quasi-transitive lattice with only minor modifications that we will describe in Lemma 2.5.5. However, certain difficulties arise when studying site-interfaces on an arbitrary planar quasi-transitive lattice. Indeed, when we connect two site-interfaces P_1, P_2 with a path, it is possible that some of the vertices of ∂P_1 or ∂P_2 are now 'separated' from the remaining boundary vertices, see Figure 2.4. In fact, it is possible that most boundary vertices have this property. To remedy this, we will choose the path that connects P_1 and P_2 appropriately so that only a few of them, if any, are 'separated' from the remaining boundary vertices.

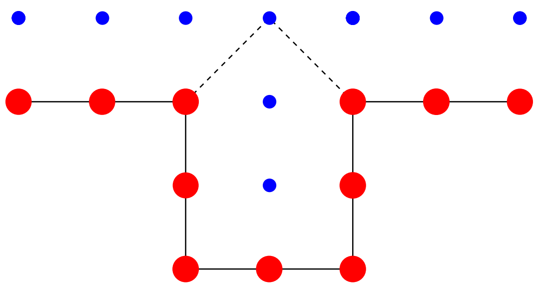

Figure 2.4: If the vertex incident to the two dashed lines is attached to the site-interface, the vertices of which are highlighted in red, then the new graph is not a site-interface anymore.

Lemma 2.5.4. *Let $G \in \mathcal{S}$. Let P be a site-interface of G. Then there are $|\partial_V P| - O(|P|^{1/4}) = \Omega(|P|^{1/2})$ vertices $u \in \partial_V P$ such that the site-interface of $P \cup \{u\}$ has size $|P| - O(|P|^{3/4})$ and boundary size $|\partial P| - O(|\partial P|^{3/4})$.*

Proof. Recall that the cycle space of G is generated by cycles of bounded length. We will write t for the maximal length of a cycle in our basis.

For every $v \in \partial_V P$, let P_v be the site-interface of the connected graph $P \cup \{v\}$ and let $Q_v := \partial P \setminus (\partial P_v \cup \{v\})$. Write L for the edges between P and ∂P, E_v for the edges between v and P, and L_v for the edges between P and Q_v. First, we claim that all Q_v are pairwise disjoint. Indeed, assuming that this is not true, we find a pair of distinct u, v such that $Q_u \cap Q_v \neq \emptyset$. Since the vertices of Q_z, $z \in \{u, v\}$ do not belong to ∂P_z, $\vec{E_z^P}$ separates $\vec{L_z^P}$ from the remaining edges of $\vec{L^P}$. Hence no vertex of Q_z lies in $\partial_V P$, as any path starting from a vertex of Q_z and going to infinity without intersecting P must intersect z. This implies that if X, Y are two overlapping components of $\vec{L}_u^P$, $\vec{L}_v^P$, respectively, then $X \cup Y$ is $L \setminus (E_u \cup E_v)$-connected, and thus X, Y coincide. Moreover, X is connected to $\vec{E}_u^P$ with a $\mathcal{P}$-path in $G \setminus L$, and Y is connected to $\vec{E}_v^P$ with a $\mathcal{P}$-path in $G \setminus L$. Therefore, u coincides with v, which is absurd. Hence, our claim is proved.

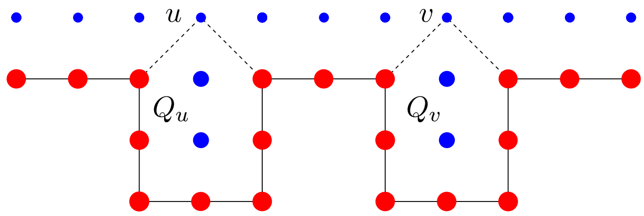

Figure 2.5: The situation in the proof of Lemma 2.5.4.

We can now conclude that $\sum_{v \in \partial_V P} |Q_v| \leq |\partial P| \leq \Delta |P|$, where Δ is the maximal degree of G. It follows that the number of $v \in \partial_V P$ such that $|Q_v| \geq |P|^{3/4}$ is at most $\Delta |P|^{1/4}$. By the isoperimetric inequality (2), which holds for all graphs in $\mathcal{S}$, there is $c > 0$ such that $|\partial_V P| \geq c|P|^{\frac{1}{2}}$, which implies the strict inequality $|\partial_V P| > \Delta |P|^{1/4}$ whenever $|P|$ is large enough. It is clear that P_u has size $|P| - O(|P|^{3/4})$ whenever $|Q_u| < |P|^{3/4}$ because $|P \setminus P_u| \leq \Delta^t |Q_u|$. The proof is now complete. □

The boundary vertices satisfying the property of Lemma 2.5.4 will be called *good*, and the remaining ones will be called *bad*.

In order to generalise Proposition 2.5.3 to all elements of $\mathcal{S}$, we will need the following definitions. Consider a quasi-transitive lattice G in $\mathbb{R}^2$. Given two linearly independent vectors $z, w \in \mathbb{R}^2$ we write $B(z, w)$ for the box determined by z and w. Given a side s of $B(z, w)$, we write $B^s(z, w)$ for the box that is congruent to $B(z, w)$, and satisfies $B^s(z, w) \cap B(z, w) = s$. It is not hard to see that there are vectors z_1, z_2, w_1, w_2 such that the following hold:

- For each $i = 1, 2$, z_i is parallel to v_i, and w_i is an integer multiple of z_i.
- For every side s of $B(z_1, z_2)$, there are vertices $u \in B(z_1, z_2)$ and $v \in B^s(z_1, z_2)$ that can be connected with a path lying in $B(z_1, z_2) \cup B^s(z_1, z_2)$.
- For every pair of vertices u, v in $B(z_1, z_2)$, there is a path in $B(w_1, w_2)$ connecting u to v.

We regard the tilings $\mathcal{T}_z$ and $\mathcal{T}_w$ of $\mathbb{R}^2$ by translates of $B(z_1, z_2)$ and $B(w_1, w_2)$, respectively, as graphs that are naturally isomorphic to $\mathbb{Z}^2$.

Lemma 2.5.5. *Consider a graph $G \in \mathcal{S}$. Then $\widetilde{b}_r(G) = b_r(G)$.*

Proof. We handled $\mathbb{Z}^d$ and $\mathbb{T}^d$ above, so it only remains to handle planar quasi-transitive lattices. We will focus on the case of site-interfaces with surface-to-volume ratio r such that $b_r > 1$, which is the hardest one.

Let $n \in \mathbb{N}$, $\epsilon > 0$, $r > 0$, and let $P \in I_{n,r,\epsilon}$. Recall that there is a $t > 0$ such that the cycles in our basis of $\mathcal{C}(G)$ have length at most t. Consider the set of boxes in $\mathcal{T}_w$ that either intersect the $2t$-neighbourhood of $P \cup \partial P$, or share a common face with such a box. Let $B^t(P)$ be the smallest box with sides parallel to w_1, w_2 containing all these boxes. Write s for a side of $B^t(P)$. Order the vertices of G arbitrarily. Among all vertices of $\partial_V P$ that are closest to s, there is one that is minimal. We call these vertices *extremal*. Each extremal vertex lies in some box of $\mathcal{T}_z$ that is called extremal as well (in case a vertex lies in more than one boxes of T_z, order the boxes arbitrarily and choose the minimal one). We define the shape of a site-interface P to be the tuple comprising the dimensions of the box $B^t(P)$, and the extremal vertices of $P \cup \partial P$. Using the polynomial growth of G, we immediately deduce that we have polynomially many choices $P(n)$ for the shape and auxiliary shape of any site-interface P. We define K as in the proof of Proposition 2.5.3.

By definition, all elements P of K have the same $B^t = B^t(P)$. It is not hard to see that at least one of the two dimensions of B^t is $\Omega(\sqrt{n})$. Indeed, B^t contains the vertices of P. For every vertex u of P there is a disk of small enough radius $r_u > 0$ contained in B^t, so that distinct disks are disjoint. The translation invariance of G implies that there are only finitely possibilities for r_u, hence $r := \inf_{u \in V} r_u > 0$. It follows that B^t has area $\Omega(n)$ because it contains n disjoint disks of radius at least r. This implies that at least one of the two dimensions of B^t is $\Omega(\sqrt{n})$. We can assume without loss of generality that the dimension parallel to v_1 has this property.

We start with a $n \times n$ grid of copies $B_{i,j}$ of B^t. We place inside every $B_{i,j}$ a site-interface $K_{i,j} \in K$. We write S_k for the k-th column of the grid. Similarly to the proof of Proposition 2.5.3, we move every column, except for the first one, in the direction parallel to v_2 in such a way that the 'rightmost' extremal boxes of S_k and the 'leftmost' extremal boxes of S_{k+1} can be connected in $\mathcal{T}_z$ by a straight path parallel to v_1.

For every pair $K_{i,j}$, $K_{i,j+1}$ of consecutive interfaces, there is an induced path in G of bounded length connecting their 'rightmost' and 'leftmost' extremal vertices. We can further assume that the path lies in $B_{i,j} \cup B_{i,j+1}$ by our choice of z_1, z_2, w_1, w_2 and the definition of B^t. Indeed, if $\Pi = B_1, B_2, \ldots, B_l$ is a straight path in $\mathcal{T}_z$ connecting the 'rightmost' and 'leftmost' extremal boxes of $K_{i,j}$, $K_{i,j+1}$, respectively, we first connect all consecutive boxes B_m, B_{m+1}, $m = 1, \ldots, l-1$

using paths Π_m in G lying in $B_m \cup B_{m+1}$. Then we connect the 'rightmost' and 'leftmost' end-vertices of consecutive paths Π_m, Π_{m+1}, respectively, using paths lying in boxes congruent to $B(w_1, w_2)$ containing those end-vertices. Finally, we connect the 'rightmost' and 'leftmost' extremal vertices of $K_{i,j}$, $K_{i,j+1}$ to Π_1 and Π_l using paths lying in boxes congruent to $B(w_1, w_2)$. In this way, we obtain a path that lies in $B_{i,j} \cup B_{i,j+1}$ because both $B_{i,j}$, $B_{i,j+1}$ contain a 'layer' of boxes of T_w surrounding $K_{i,j}$, $K_{i,j+1}$. The path is not necessarily disjoint from $K_{i,j}$, $K_{i,j+1}$ but it certainly contains a subpath that is disjoint from them and connects two boundary vertices of both site-interfaces. We can choose the subpath to contain exactly two boundary vertices, one from each of the two site-interfaces.

Let $\mathcal{W}$ be the path connecting $K_{i,j}$, $K_{i,j+1}$, and let $u_1 \in \partial_V K_{i,j}$, $u_2 \in \partial_V K_{i,j+1}$ be the end-vertices of $\mathcal{W}$. Adding u_1, u_2 to $K_{i,j}$, $K_{i,j+1}$ may result to much smaller site-interfaces. For this reason, we need to find two good boundary vertices. Consider the vertices x_1, x_2 at distance t from u_1, u_2, respectively, lying in $\mathcal{W}$. Write Q_1, Q_2 for the $(t-1)$-neighbourhood of $K_{i,j}$, $K_{i,j+1}$, respectively, and notice that both $\partial_V Q_1$, $\partial_V Q_2$ have distance t from $K_{i,j}$, $K_{i,j+1}$, respectively. Furthermore, we can connect any pair of vertices of $\partial_V Q_i$, $i = 1, 2$ with a path lying in the $t/2$ neighbourhood of $\partial_V Q_i$ by Remark 2.3.4, hence disjoint from $K_{i,j}$, $K_{i,j+1}$ and their boundaries. The isoperimetric inequality (2) gives $\partial_V Q_i = \Omega(\sqrt{n})$. Moreover, for every $k > 0$, the number of vertices of $\partial_V Q_i$ that can be connected to x_i with a path of length at most k lying in the $t/2$-neighbourhood of $\partial_V Q_i$ is $\Omega(k)$. On the other hand, Lemma 2.5.4 implies that $O(n^{1/4})$ boundary vertices of either $K_{i,j}, K_{i,j+1}$ are bad. Hence choosing $k = cn^{1/4}$ for some large enough constant $c > 0$, we can find two good vertices y_1, y_2 in $\partial_V K_{i,j}, \partial_V K_{i,j+1}$, that can be connected to x_1, x_2, respectively, in the following way: we first connect y_i to some vertex of $\partial_V Q_i$ with a path of length t, and then we connect the latter vertex with a path of length $O(n^{1/4})$ lying in the $t/2$ neighbourhood of $\partial_V Q_i$. Taking the union of these two paths with the subpath of $\mathcal{W}$ connecting x_1 to x_2, we obtain a path of length $O(n^{1/4})$ connecting y_1 to y_2 that lies in $B_{i,j} \cup B_{i,j+1}$. We attach this path to our collection of site-interfaces.

Consider now a site-interface $K_{i,j}$ with $2 \leq j \leq n-1$. Notice that exactly two paths emanate from $\partial K_{i,j}$, one of which has distance $O(n^{1/4})$ from the 'rightmost' extremal vertex of $K_{i,j}$, and the other has distance $O(n^{1/4})$ from the 'leftmost' extremal vertex of $K_{i,j}$. The two paths may possibly overlap, separating some vertices of $\partial K_{i,j}$ from infinity. However, the distance between the 'rightmost' and the 'leftmost' extremal vertex is $\Omega(\sqrt{n})$ because the dimension of $B_{i,j}$ that is parallel to v_1 is $\Omega(\sqrt{n})$. We can increase the value of n if necessary to ensure that the paths do not overlap.

Moreover, we connect, as we may, the boundaries of consecutive site-interfaces $K_{i,1}, K_{i+1,1}$ of the first column with induced paths of length $O(n)$ disjoint from any other site-interface, only the end-vertices of which intersect the boundary of $K_{i,1}, K_{i+1,1}$.

Taking the union of all site-interfaces $K_{i,j}$ and the attached paths we obtain a graph H. Let Q be the site-interface of the graph spanned by the vertices of H. We claim that Q has size $n^3(1-o(1))$, and boundary size between $(r-\epsilon')|Q|$ and $(r+\epsilon')|Q|$, for some $\epsilon' = \epsilon + o(1)$. Indeed, for every site-interface $K_{i,j}$ that does not lie in the first column, if $F_{i,j} \subset \partial K_{i,j}$ is the set of end-vertices of the attached paths that emanate from $\partial K_{i,j}$, then the site-interface of $K_{i,j} \cup F_{i,j}$ (which has size $n - O(n^{1/4})$) lies in the boundary of Q. Since we have $n^2 - n$ such $K_{i,j}$, the claim follows readily.

Each column S_k has been moved at distance $O(kn) = O(n^2) = o(|Q|)$ from its original position. Hence $|B(Q)| = o(|Q|)$. It remains to show that the number of constructed interfaces Q is roughly N^{n^2}. Notice that we have not necessarily used the same paths to connect our interfaces, and so given such a Q, we cannot immediately recover all possible sequences $(K_{i,j})$ giving rise to Q. Our goal is to restrict to a suitable subfamily of K^{n^2}.

We claim that there are only sub-exponentially many in n^3 possibilities for the attached paths. Recall that all elements of K have the same extremal vertices. The end-vertices of every attached path have distance $O(n^{1/4})$ from a pair of extremal vertices. Using the polynomial growth of G, we conclude that there are only polynomially many choices in n for each end-vertex. Moreover, the paths connecting interfaces of the first column have length $O(n)$, and the remaining paths have length $O(n^{1/4})$. There are at most $\Delta^{O(n)}$ choices for each path connecting site-interfaces of the first column, and at most $\Delta^{O(n^{1/4})}$ choices for each of the remaining paths, because any path starting from a fixed vertex can be constructed sequentially, and there are at most Δ choices at each step. In total, there are $\Delta^{O(n^{9/4})}$ possibilities for the attached paths. This proves our claim.

On the other hand, there are at least N^{n^2} sequences $(K_{i,j}) \in K^{n^2}$, hence for a subfamily of K^{n^2} of size at least $N^{n^2}/\Delta^{O(n^{9/4})}$, we have used exactly the same paths. Let us restrict to that subfamily. Since we have fixed the paths connecting the $K_{i,j}$, if we delete every vertex of the attached paths except for their end-vertices, then we can 'almost' reconstruct all site-interfaces producing Q. To be more precise, if $(K_{i,j})$ and $(K'_{i,j})$ are two sequences producing the same Q, then the site-interfaces of $K_{i,j} \cup F_{i,j}$ and $K'_{i,j} \cup F_{i,j}$ coincide. By Lemma 2.5.6 below, if we fix a sequence $(K_{i,j})$

producing Q, then for each i, j with $j > 1$, there are sub-exponentially many in n possible $K'_{i,j}$ as above. For each of the remaining i, j there are at most exponentially many in n possible $K'_{i,j}$ as above since there are at most exponentially many site-interfaces in total. Therefore, each Q can be constructed by sub-exponentially many in n^3 sequences $(K_{i,j})$. We can now deduce that we constructed roughly N^{n^2} site-interfaces Q, and taking limits we obtain $\widetilde{b}_r = b_r$, as desired. □

We now prove the lemma mentioned in the above proof.

Lemma 2.5.6. *Let G be a planar quasi-transitive lattice. Let P be a site-interface of size n in G, and $F \subset \partial_V P$. Assume that the site-interface of $P \cup F$ has size at least $n - O(n^{3/4})$. Then the number of site-interfaces P' of size n such that the site-interfaces of $P \cup F$ and $P' \cup F$ coincide, is $n^{O(n^{3/4})}$.*

Proof. Consider a site-interface P' of size n such that the site-interface X of $P \cup F$, and the site-interface of $P' \cup F$ coincide. Let k be the size of $P' \setminus X$. By our assumption $k = O(n^{3/4})$. Each connected component of $P' \setminus X$ is incident to some vertex of P, hence every vertex of $P' \setminus X$ has distance $O(n^{3/4})$ from P. By the polynomial growth of G, the number of vertices at distance $O(n^{3/4})$ from P is at most m for some $m = O(n^{3/2})$. There are

$$\binom{m}{k} \leq m^k = n^{O(n^{3/4})}$$

subsets of size k containing vertices having distance $O(n^{3/4})$ from P. Therefore, there are $n^{O(n^{3/4})}$ site-interfaces P' as above. □

We can now prove the main results of this section.

Proof of Theorem 2.5.1. We first prove that

$$b_r^* \geq b_r. \tag{2.13}$$

Combined with (2.11) this will easily yield the desired equality.

Assume first that G is a planar quasi-transitive lattice. Let $n \in \mathbb{N}$, $r > 0$, $\epsilon > 0$, and choose $P \in I_{n,r,\epsilon}$. By Proposition 2.5.5, we may assume that P satisfies $|B(P)| = o(|P|) = o(n)$. Recall that $B(P)$ contains $P \cup \partial P$ in its interior. It is not hard to see that there is a cycle C at a bounded distance from P that separates $B(P)$ from infinity, and has size $O(|B(P)|)$. Arguing as in the proof of Lemma 2.5.5, we find a good vertex $u \in \partial^V P$, and an induced path Π connecting u to C that has size $O(n^{1/4})$, and does not contain any other vertex of $P \cup \partial P$.

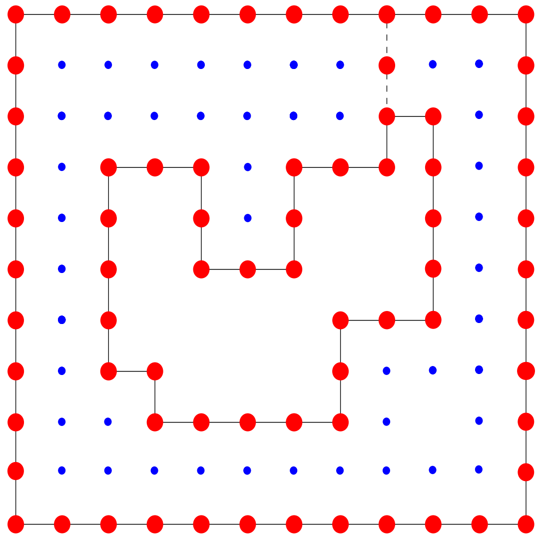

Figure 2.6: The vertices of Q are depicted with big dots, and the vertices of ∂Q are depicted with smaller dots. The edges spanned by P and C are depicted in solid lines, while the edges of Π are depicted in dashed lines.

Our aim now is to find a suitable inner-interface containing the site-interface of $P \cup \{u\}$, which we denote by X. Since the cycle space of G is generated by its triangles, the minimal vertex cut of P spans a cycle surrounding P and the remaining boundary vertices. Hence $\partial^V P \setminus \{u\}$ spans a connected graph. The graph $\Gamma := X \cup \Pi \cup C$ surrounds an open subset of the plane that contains $\partial^V P \setminus \{u\}$. Consider the connected component Y of $\partial^V P \setminus \{u\}$ in this open set. Write Q for the inner-interface of Y, i.e. the boundary of the site-interface of Y.

We claim that Q contains X and is contained in $X \cup \Pi \cup C$. To see that Q contains X, notice that all vertices of X are incident to Y because G is a triangulation, and lie in the external face of Y. Therefore, X is contained in Q. Moreover, if Q contains some vertex not in $X \cup \Pi \cup C$, then we can add this vertex to Y to obtain an even larger connected graph. This contradiction shows that there is no such vertex and proves our claim.

We now consider the case where $G = \mathbb{T}^d$. We can let C be the set of vertices in the boundary of $B(P)$, and Π be a path of length 2 connecting an extremal vertex of $B(P)$ to P. Let Y be the subgraph of G surrounded by $P \cup \Pi \cup C$. It is clear that Y is connected. Write Q for the inner-interface of Y. Every vertex of P is incident to Y and lies in the infinite component of $G \setminus Y$. Hence P lies in Q. Furthermore, Q contains only vertices of $P \cup \Pi \cup C$.

In both cases, Q has roughly n vertices and surface-to-volume ratio between $(r - \epsilon')|Q|$ and $(r + \epsilon')|Q|$ for some $\epsilon' = \epsilon + o(1)$. Moreover, each Q can be obtained from only sub-exponentially many P. This proves (2.13). Combining this with (2.11), we obtain the following:

$$b_r^* \geq b_r = (b_{1/r}^*)^r \geq b_{1/r}^r = b_r^*,$$

where both inequalities coincide with (2.13) and both equalities with (2.11). Thus we must have equality all along, and in particular $b_r = b_{1/r}^r$. □

We now prove Theorem 2.5.2.

Proof of Theorem 2.5.2. Choose $P \in I_{n,r,\epsilon}$ such that $|B(P)| = o(|P|) = o(n)$. Define C as in the proof of Theorem 2.5.1, and connect ∂P to C with a path Π of minimal length. Notice that $(\partial^E P)^*$ is a cycle, hence $(\partial P \setminus E(\Pi))^*$ is a connected graph.

Let X be the connected component of $(\partial P \setminus E(\Pi))^*$ in G^* that is surrounded by $P \cup \Pi \cup C$, and let Q be the interface of X. Arguing as in the proof of Theorem 2.5.1, we see that P^* lies in ∂Q, Q has size roughly n, and $(r - \epsilon')|Q| \leq |\partial Q| \leq (r + \epsilon')|Q|$ for some $\epsilon' = \epsilon + o(1)$.

Let $b_r^\bullet(G)$ be defined like $b_r(G)$ except that we now consider inner-interfaces. Thus we have

$$b_r^*(G) = b_r^\bullet(G^*) \tag{2.14}$$

by the definitions. The above construction now yields the inequality $b_r^\bullet(G) \geq b_r(G)$.

Combining this with (2.11), which we rewrite using (2.14), we obtain

$$b_r^\bullet(G) \geq b_r(G) = (b_{1/r}^\bullet(G^*))^r \geq b_{1/r}^\bullet(G^*)^r = b_r^\bullet(G),$$

as above, and again equality holds all along. In particular,
$b_r(G) = b_{1/r}^\bullet(G^*)^r = (b_{1/r}^*(G))^r$. □

The arguments in the proofs of Lemma 2.5.3 and Theorem 2.5.5 can be used to prove Lemma 2.4.2.

Proof of Lemma 2.4.2. The inequality $b_r^\circ \leq b_r^\odot$ is obvious.

For the reverse inequality, we will focus on the case of site-interfaces. We will construct an array of a certain number of boxes of possibly different sizes, then place the component site-interfaces of an arbitrary site-multi-interface inside the boxes, and connect them with short paths to obtain a new site-interface.

We claim that the number of choices for the shapes of the components of any site-multi-interface of size n grows sub-exponentially in n. Indeed, the number of choices for the shape of any site-interface grows polynomially in its size. Theorem 1.2.3 shows that there are most $s^{\sqrt{n}}$ choices for the component sizes of any site-multi-interface of size n, where $s > 0$ is a constant. Hence it suffices to show that a site-multi-interface of size n comprises $O(\sqrt{n})$ site-interfaces.

Let $X = (\ldots, x_{-1}, x_0 = o, x_1, \ldots)$ be a quasi-geodesic in G containing o and let $X^+ = (x_0, x_1, \ldots)$ be the one of the two 1-way infinite subpaths of X starting from o. Consider a site-multi-interface P of size n. As proved in Proposition 1.4.4, P contains at least one of the first fn vertices of X^+. We enumerate the component site-interfaces $P_1, P_2, \ldots, P_k$ of P according to the first vertex of X^+ that they contain. As the P_i's are disjoint, we have $l_i < l_{i+1}$, where l_i is the index of the first vertex of X^+ that P_i contains. Since $l_1 \geq 0$, we deduce that $l_i \geq i - 1$ for every i. Hence, we obtain

$$|P_i| \geq (i-1)/f$$

for every $i = 1, 2, \ldots, k$, which implies that

$$n = \sum_{i=1}^{k} |P_i| \geq \sum_{i=1}^{k} (i-1)/f = \frac{k(k-1)}{2f}.$$

The latter implies that $k = O(\sqrt{n})$, hence there are $(sn)^{O(\sqrt{n})}$ choices for the shapes of the components site-interfaces of any site-multi-interface of size n.

We can now restrict to a subfamily $K \subset MI_{n,r,\epsilon}$ of size at least

$$N := \frac{c_{n,r,\epsilon}}{(sn)^{O(\sqrt{n})}}$$

such that all site-multi-interfaces of K have the same component sizes $\{n_1, n_2, \ldots, n_k\}$, and corresponding component site-interfaces have the same shape. Let $B_1, \ldots, B_k$ be the boxes of the component site-interfaces. Instead of a grid, we construct an array by placing the above k boxes next to each other. Given an element of K, we place its component site-interfaces in their boxes. After moving the boxes, if necessary, we connect them with short paths, as described in the proof of Lemma 2.5.5. Arguing as in the proof of Lemma 2.5.5, we obtain $b_r^{\circ} \geq b_r^{\odot}$, as desired. □

Since $\mathbb{P}_{p_c}(|S_o| = n)$ does not decay exponentially in n, we conclude

Corollary 2.5.7. *Consider site percolation on a planar quasi-transitive lattice in $\mathbb{R}^d$ satisfying* (2.8). *Then $\mathbb{P}_{1-p_c}(|S_o| = n)$ does not decay exponentially in n.*

Proof. Notice that $r(1-p_c) = 1/r(p_c)$. The fact that $\mathbb{P}_{p_c}(|S_o| = n)$ does not decay exponentially in n implies that $b_{r(p_c)} = f(r(p_c))$. Theorem 2.5.1 shows that

$$b_{r(1-p_c)} = b_{r(p_c)}^{1/r(p_c)} = f(r(p_c))^{1/r(p_c)} = f(r(1-p_c)).$$

Using Theorem 2.1.1 we conclude that $\mathbb{P}_{1-p_c}(|S_o| = n)$ does not decay exponentially in n. □

2.6 Continuity

In this section, we study the analytical properties of b_r. To avoid repeating the arguments in the proof of Lemma 2.5.5 and considering cases according to whether we study interfaces or site-interfaces, we will prove the results for interfaces in $\mathbb{Z}^d$ and $\mathbb{T}^d$.

We first prove that the lim sup in the definition of b_r can be replaced by lim.

Proposition 2.6.1. *Let $G \in \mathcal{S}$. Then for every r such that $b_r > 1$ and for all but countably many $\epsilon > 0$ the limit $\lim_{n\to\infty} c_{n,r,\epsilon}(G)^{1/n}$ exists.*

Proof. We will first show that

$$\limsup_{n\to\infty} c_{n,r,\epsilon}^{1/n} = \liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n}$$

holds for any $\epsilon > 0$ at which the function $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n}$ is continuous. Since $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n}$ is an increasing function of ϵ, its points of discontinuity are countably many Rudin [1964].

Let ϵ be a point of continuity of $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n}$ and $n \in \mathbb{N}$. By combining elements of $I_{n,r,\epsilon}$ we will construct interfaces of arbitrarily large size and surface-to-volume ratio between $r - \epsilon'$ and $r + \epsilon'$ for some $\epsilon \leq \epsilon' = \epsilon + o(1)$. Let $0 \leq s \leq n+3$ be an integer. We repeat the idea of Proposition 2.5.3 but instead of a grid, we construct an array of m boxes for some $m > 0$. We place inside each box an element of $I_{n,r,\epsilon}$ and after moving the boxes, if necessary, we connect consecutive interfaces using paths of length 4, similarly to the proof of Proposition 2.5.3. We also attach a path of length $s + 4$, that is incident to the last interface and disjoint from any of the previous interfaces. In this way, we produce an element Q of $I_{k,r,\epsilon'}$, where $\epsilon' = \epsilon + o(1)$ and k is any integer of the form $k = m(n+4)+s$. There are roughly $c_{n,r,\epsilon}^m$ choices for Q. Since s ranges between 0 and $n + 3$, for every fixed n, all but finitely many k can be written in this form for some $m \geq 1$. Taking the k-th root and then the limit as $m \to \infty$ we conclude that $\liminf_{k\to\infty} c_{k,r,\epsilon'}^{1/k} \geq c_{n,r,\epsilon}^{1/(n+4)}$. Letting $n \to \infty$ we obtain $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n} \geq \limsup_{n\to\infty} c_{n,r,\epsilon}^{1/n}$. The above inequality follows from the fact that ϵ is a point of continuity of $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n}$. Hence $\liminf_{n\to\infty} c_{n,r,\epsilon}^{1/n} = \limsup_{n\to\infty} c_{n,r,\epsilon}^{1/n}$, as desired □

The following proposition follows directly from the definition of b_r:

Proposition 2.6.2. *Let $G \in \mathcal{S}$. Then $b_r(G)$ is an upper-semicontinuous function of r.*

Proof. Let $\epsilon > 0$ and $0 < \delta < \epsilon/2$. Then for every $r > 0$ and for every s with $|r - s| < \epsilon/2$, the interval $(s - \delta, s + \delta)$ is contained in $(r - \epsilon, r + \epsilon)$, and the site-interfaces P with $|\partial P|/|P| \in (s-\delta, s+\delta)$ are counted in the set of those site-interfaces with $|\partial P|/|P| \in (r - \epsilon, r + \epsilon)$ as well. Hence, $\limsup_{n\to\infty} c_{n,r,\epsilon}^{1/n} \geq \limsup_{n\to\infty} c_{n,s,\delta}^{1/n}$. Taking limits as $\delta \to 0$, $s \to r$ and finally $\epsilon \to 0$, we obtain $b_r \geq \limsup_{s\to r} b_s$. The latter shows that b_r is an upper-semicontinuous function of r. □

Next, we prove that b_r is a log-concave function of r:

Proposition 2.6.3. *Let $G \in \mathcal{S}$. Then for any $t \in [0, 1]$ and any r, s such that $b_r(G), b_s(G) > 1$, we have $b_{tr+(1-t)s}(G) \geq b_r(G)^t b_s(G)^{1-t}$.*

Proof. Pick an ϵ such that both $\lim_{n\to\infty} c_{n,r,\epsilon}^{1/n}$ and $\lim_{n\to\infty} c_{n,s,\epsilon}^{1/n}$ exist. Let (p_m/q_m) be a sequence of rational numbers converging to t such that $q_m \to \infty$. Consider subfamilies K, K' of $I_{p_m,r,\epsilon}$ and $I_{q_m-p_m,s,\epsilon}$, where the elements of both K and K' have the same shape (as defined in the proof of Proposition 2.5.3), and $|K| \geq c_{p_m,r,\epsilon}/P(p_m)$, $|K'| \geq c_{q_m-p_m,s,\epsilon}/P(q_m - p_m)$ for some polynomial $P(x)$. Note that the elements of K and K' share the same boxes B and B', respectively. Place two interfaces, one from K and another from K', in an array of two boxes parallel to B and B', and move the boxes, if necessary, in order to connect the interfaces with short paths. In this way, we obtain an interface Q of size roughly q_m and surface-to-volume ratio roughly $tr + (1 - t)s$. Notice that we have at least $c_{p_m,r,\epsilon}c_{q_m-p_m,s,\epsilon}/\big(P(p_m)P(q_m - p_m)\big)$ choices for Q. Taking the k-th root of the latter expression, where $k = |Q|$, and letting $m \to \infty$ gives

$$\lim_{n\to\infty} c_{n,r,\epsilon}^{t/n} \lim_{n\to\infty} c_{n,s,\epsilon}^{(1-t)/n}.$$

Letting $\epsilon \to 0$ along a sequence of points such that both $\lim_{n\to\infty} c_{n,r,\epsilon}^{1/n}$, $\lim_{n\to\infty} c_{n,s,\epsilon}^{1/n}$ exist, we obtain $b_{tr+(1-t)s} \geq b_r^t b_s^{1-t}$ as desired. □

We expect Proposition 2.6.3, and as a result Theorem 2.6.4 below, to hold in much grater generality than $G \in \mathcal{S}$, namely for all 1-ended Cayley graphs. In order to be able to put several interfaces close to each other to connect them with short paths as in the above proof, it could be handy to use [Bandyopadhyay et al., 2010, Lemma 6].

Let I be the closure of the set of r such that $b_r > 1$. Proposition 2.6.3, combined with Proposition 2.6.2, easily imply

Theorem 2.6.4. *Let $G \in \mathcal{S}$. Then $b_r(G)$ is a continuous function of r on I.*

Proof. By Proposition 2.6.3, I is an interval, and the only possible $r \in I$ such that $b_r = 1$, are its endpoints. For every r in I, we have $\limsup_{s\to r} b_s \leq b_r$ by Proposition 2.6.2. Using Proposition 2.6.3 for $t = 1/2$ we obtain $\liminf_{s\to r} b_{(r+s)/2} \geq \sqrt{b_r \liminf_{s\to r} b_s}$ for every r such that $b_r > 1$. This immediately implies that $\liminf_{s\to r} b_s \geq b_r$ and thus $\lim_{s\to r} b_s = b_r$.

On the other hand, if $b_r = 1$ for some of the endpoints of I, then Proposition 2.6.2 and the fact that $b_s > 1$ for s in the interior of I, give that

$$\lim_{\substack{s\to r\\ s\in I}} b_s = 1.$$

Therefore, b_r is a continuous function on I. □

Having proved that b_r is a continuous function, the next natural question is whether it is differentiable. It turns out that this holds everywhere except, perhaps, on a countable set.

Corollary 2.6.5. *Let $G \in \mathcal{S}$. Then $b_r(G)$ is differentiable for all but countably many r.*

Proof. By Proposition 2.6.3, $\log b_r$ is a concave function, hence differentiable everywhere except for a countable set Rockafellar [2015]. It follows immediately that this holds for b_r as well. □

2.7 Growth rates of lattice animals in $\mathbb{Z}^d$

In this section, we exploit the machinery developed above in order to obtain bounds on the exponential growth rates of lattice (site) animals in $\mathbb{Z}^d$.

A *lattice animal* in a graph G is a connected subgraph of G containing o. A *lattice tree* in G is a lattice animal that is also a tree. Let $a_n(G)$ be the number of all lattice animals of G with n edges, and let $t_n(G)$ be the number of all lattice trees of G with n edges. It is well-known that both $a(\mathbb{Z}^d) := \lim_{n\to\infty} a_n(\mathbb{Z}^d)^{1/n}$ and $t(\mathbb{Z}^d) := \lim_{n\to\infty} t_n(\mathbb{Z}^d)^{1/n}$ exist Klarner [1967]; Klein [1981].

A *lattice site-animal* in G is a set of vertices of G containing o that spans a connected graph. A *lattice site tree* in G is a lattice site-animal in G that spans a tree. Let $\dot{a}_n(G)$ be the number of all lattice site-animals of G with n vertices, and let $\dot{t}_n(G)$ be the number of all lattice trees of G with n vertices. We let $\dot{a}(G) := \lim_{n\to\infty} \dot{a}_n(G)^{1/n}$ and $\dot{t}(G) := \lim_{n\to\infty} \dot{t}_n(G)^{1/n}$ whenever the limits exist.

Our results allow us to translate any upper bound on $\dot{p}_c(G)$ into a lower bound on $\dot{a}(G)$, and conversely, any upper bound on $\dot{a}(G)$ into a lower bound on $\dot{p}_c(G)$. Indeed, we just remark that

$$\dot{a}(G) \geq \dot{b}(G) \geq \dot{b}_r(\dot{p}_c(G)) = f(r(\dot{p}_c(G))) \tag{2.15}$$

for every lattice G, where the two inequalities are obvious from the definitions (interfaces are a species of lattice animal), and the last equality is given by Theorem 2.4.7. To translate bounds on $\dot{p}_c(G)$ into bounds on $\dot{a}(G)$ and vice-versa, we just remark that $f(r)$ is monotone increasing in r and $r(p)$ is monotone decreasing in p. Inequality (2.15) and the above reasoning applies verbatim to $p_c(G)$ and $a(G)$.

In two dimensions we cannot hope to get close to the real value of $\dot{a}(G)$ with this technique, as we are only enumerating the subspecies of site-interfaces.[3] But as we will see in the next section, our lower bounds become asymptotically tight as the dimension d tends to infinity. In Section 2.8 we will argue conversely: we will prove upper bounds on $\dot{a}(\mathbb{Z}^d)$ and plug them into (2.15) to obtain lower bounds on $\dot{p}_c(\mathbb{Z}^d)$.

2.7.1 Lattice (site) animals in $\mathbb{Z}^d$

We start by computing the first terms of the $1/d$ asymptotic expansion of interfaces.

Theorem 2.7.1. *The exponential growth rate of the number of interfaces of $\mathbb{Z}^d$ satisfies $b(\mathbb{Z}^d) = 2de - \frac{3e}{2} - O(1/d)$.*

Proof. We claim that for any interface P of $\mathbb{Z}^d$ we have $|\partial P| \leq (2d-2)|P| + 2d$. Indeed, summing vertex degrees gives $\sum_{u\in V(P)} deg(u) \geq 2|P| + |\partial P|$, where $deg(u)$ is the degree of u in the graph $P \cup \partial P$, because the edges of P are counted twice, and the edges of ∂P are counted at least once. Since $deg(u) \leq 2d$ and $V(P) \leq |P| + 1$,

[3] Still, when G is the hexagonal (aka. honeycomb) lattice $\mathbb{H}$, the best known lower bound was $\dot{a}(\mathbb{H}) \geq 2.35$ Barequet et al. [2019a]; Rands and Welsh [1981], until this was recently improved to $\dot{a}(\mathbb{H}) \geq 2.8424$ Barequet et al. [2019b]. Plugging a numerical value for $\dot{p}_c(\mathbb{H})$, for which the most pessimistic (i.e. highest) estimate currently available is about 0.69704 Jacobsen [2014], we obtain $\dot{a}(\mathbb{H}) \geq 2.41073$. If those approximations were rigorous, this would have improved the bounds of Barequet et al. [2019a]; Rands and Welsh [1981].

we get

$$2|P| + |\partial P| \leq \sum_{u \in V(P)} deg(u) \leq 2dV(P) \leq 2d|P| + 2d.$$

By rearranging we obtain the desired inequality. It follows that $b_r = 0$ for every $r > 2d - 2$ which combined with Proposition 2.4.4 and the fact that $f(r)$ is an increasing function of r gives

$$b_r(\mathbb{Z}^d) \leq \frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}}$$

for $r \geq 0$. Using Proposition 2.4.3 we obtain that

$$b(\mathbb{Z}^d) \leq \frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}}.$$

Notice that for every $r > 0$,

$$\frac{(1+r)^{1+r}}{r^r} = (1+r)\Big(1 + \frac{1}{r}\Big)^r = (1+r)\exp\left(r \log\left(1 + \frac{1}{r}\right)\right).$$

Using the Taylor expansion $\log\left(1 + \frac{1}{r}\right) = \frac{1}{r} - \frac{1}{2r^2} + \frac{1}{3r^3} - O(1/r^4)$ we obtain

$$\frac{(1+r)^{1+r}}{r^r} = (1+r)\exp\left(1 - \frac{1}{2r} + \frac{1}{3r^2} - O(1/r^3)\right)$$

as $r \to \infty$. Now the Taylor expansion

$$\exp(1+x) = e\Big(1 + x + \frac{x^2}{2} + O(x^3)\Big) = e\Big(1 - \frac{1}{2r} + \frac{11}{24r^2} - O(1/r^3)\Big),$$

where $x = -\frac{1}{2r} + \frac{1}{3r^2} - O(1/r^3)$, gives

$$(1+r)\exp\left(1 - \frac{1}{2r} + \frac{1}{3r^2} - O(1/r^3)\right) = (1+r)e\Big(1 - \frac{1}{2r} + \frac{11}{24r^2} - O(1/r^3)\Big) = er + \frac{e}{2} - O(1/r).$$

Consequently,

$$\frac{(1+r)^{1+r}}{r^r} = er + \frac{e}{2} - O(1/r). \tag{2.16}$$

Plugging $r = 2d - 2$ in (2.16) we deduce that

$$\frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}} = 2de - 3e/2 - O(1/d). \tag{2.17}$$

Moreover, we have $b(\mathbb{Z}^d) \geq b_{r_d}(\mathbb{Z}^d)$ and $b_{r_d}(\mathbb{Z}^d) = f(r_d)$, where $r_d := r(p_c(\mathbb{Z}^d))$. It has been proved in Hara and Slade [1995]; Hofstad and Slade [2006] that

$$p_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{7}{2(2d)^3} + O(1/d^4), \tag{2.18}$$

hence

$$r_d = \frac{1 - p_c(\mathbb{Z}^d)}{p_c(\mathbb{Z}^d)} = \frac{16d^4}{8d^3 + 4d^2 + 7d + O(1)} - 1.$$

We can easily compute that

$$\frac{16d^4}{8d^3 + 4d^2 + 7d + O(1)} = 2d - \frac{8d^3 + 14d^2 + O(d)}{8d^3 + 4d^2 + 7d + O(1)} =$$
$$2d - \frac{8d^3 + 4d^2}{8d^3 + 4d^2 + 7d + O(1)} - O(1/d)$$

and

$$\frac{8d^3 + 4d^2}{8d^3 + 4d^2 + 7d + O(1)} = \frac{1}{1 + O(1/d^2)} = 1 - O(1/d^2).$$

Hence $r_d = 2d - 2 - O(1/d)$, which implies that

$$b_{r_d}(\mathbb{Z}^d) = \frac{(1 + r_d)^{1+r_d}}{r_d^{r_d}} = 2de - 3e/2 - O(1/d).$$

Therefore, $b(\mathbb{Z}^d) = 2de - \frac{3e}{2} - O(1/d)$ as desired. □

We remark that the asymptotic expansions of $\frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}}$ and b_{r_d} differ in their third terms, and so we are unable to compute the third term in the asymptotic expansion of $b(\mathbb{Z}^d)$. It follows from the proof of Theorem 2.7.1 above that $b(\mathbb{Z}^d) - b_{r_d}(\mathbb{Z}^d) = O(1/d)$, i.e. b_{r_d} is a good approximation of $b(\mathbb{Z}^d)$.

In the next theorem, using Theorem 2.7.1 and Kesten's argument Grimmett [1999], we obtain the first two terms in the asymptotic expansion of $a(\mathbb{Z}^d)$.

Theorem 2.7.2. $a(\mathbb{Z}^d) = 2de - \frac{3e}{2} - O(1/d)$.

Proof. Let C be a connected subgraph containing o, and let ∂C be the set of edges in $E(\mathbb{Z}^d) \setminus E(C)$ with at least one endpoint in C. Arguing as in the proof of

Theorem 2.7.1, we obtain that $|\partial C| \leq (2d-2)|E(C)| + 2d$. It follows that for every $p \in [0,1]$.

$$a_n(\mathbb{Z}^d)p^n(1-p)^{(2d-2)n+2d} \leq \mathbb{P}_p(|E(C_o)| = n) \leq 1.$$

Choosing $p = \frac{1}{2d-1}$ and dividing by $p^n(1-p)^{(2d-2)n+2d}$, we obtain that

$$a(\mathbb{Z}^d) \leq \frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}} = 2de - 3e/2 - O(1/d).$$

Since $b(\mathbb{Z}^d) \leq a(\mathbb{Z}^d)$, the desired assertion follows from Theorem 2.7.1. □

The behaviour of $a(\mathbb{Z}^d)$ and $t(\mathbb{Z}^d)$ has been extensively studied in the physics literature. The expansions

$$a(\mathbb{Z}^d) = \sigma e \exp\Bigg(-\frac{1}{2}\frac{1}{\sigma} - \Big(\frac{8}{3} - \frac{1}{2e}\Big)\frac{1}{\sigma^2} - \Big(\frac{85}{12} - \frac{1}{4e}\Big)\frac{1}{\sigma^3} - \Big(\frac{931}{20} - \frac{139}{48e} - \frac{1}{8e^2}\Big)\frac{1}{\sigma^4}$$
$$-\Big(\frac{2777}{10} + \frac{177}{32e} - \frac{29}{12e^2}\Big)\frac{1}{\sigma^5} + \cdots\Bigg)$$

and

$$t(\mathbb{Z}^d) = \sigma e \exp\left(-\frac{1}{2}\frac{1}{\sigma} - \frac{8}{3}\frac{1}{\sigma^2} - \frac{85}{12}\frac{1}{\sigma^3} - \frac{931}{20}\frac{1}{\sigma^4} - \frac{2777}{10}\frac{1}{\sigma^5} + \cdots\right), \tag{2.19}$$

where $\sigma = 2d-1$, were reported in Gaunt and Peard [2000], Harris [1982]; Peard and Gaunt [1995], respectively, but without any rigorous bounds on the error terms. Miranda and Slade [2011] proved that both $a(\mathbb{Z}^d)$ and $t(\mathbb{Z}^d)$ are asymptotic to $2de$. The first three terms of $a(\mathbb{Z}^d)$ and $t(\mathbb{Z}^d)$ have been computed rigorously by the same authors in Miranda and Slade [2013].

Since any lattice tree is an interface, we obtain that $t(\mathbb{Z}^d) \leq b(\mathbb{Z}^d) \leq a(\mathbb{Z}^d)$. Although the first two terms in the asymptotic expansions of each of them are the same, we believe that $b(\mathbb{Z}^d)$ lies strictly between $t(\mathbb{Z}^d)$ and $a(\mathbb{Z}^d)$.

Using (2.18) we can easily compute the first three terms of the $1/d$ expansion of $b_{r_d}(\mathbb{Z}^d)$, and check that they coincide with the corresponding terms of the $1/d$ expansion of $t(\mathbb{Z}^d)$. However, we expect that the fourth term of the asymptotic expansion of $b_{r_d}(\mathbb{Z}^d)$ is strictly smaller than the fourth term of the asymptotic expansion of $t(\mathbb{Z}^d)$, as suggested by (2.19) and the asymptotic expansion

$$p_c(\mathbb{Z}^d) = \frac{1}{\sigma} + \frac{5}{2\sigma^3} + \frac{15}{2\sigma^4} + \frac{57}{\sigma^5} + \cdots$$

that is reported in Gaunt and Ruskin [1978] without rigorous proof. This implies the strict inequalities $b_{r_d}(\mathbb{Z}^d) < t(\mathbb{Z}^d)$ and $b_{r_d}(\mathbb{Z}^d) < b(\mathbb{Z}^d)$ for every large enough value of d. We expect that these strict inequalities hold for every $d > 1$. For example, we know that $b_{r_2}(\mathbb{Z}^2) = 4$ because $p_c(\mathbb{Z}^2) = 1/2$ Kesten [1980]. On the other hand, for small enough numbers n, the value of $t_n(\mathbb{Z}^2)$ is known exactly, and a concatenation argument yields the lower bound $t(\mathbb{Z}^2) \geq 4.1507$ Gaunt et al. [1982]; Whittington and Soteros [1990].

We remark that for site percolation the expansion

$$\dot{p}_c(\mathbb{Z}^d) = \frac{1}{\sigma} + \frac{3}{2\sigma^2} + \frac{15}{4\sigma^3} + \frac{83}{4\sigma^4} + \cdots \tag{2.20}$$

was reported in Gaunt et al. [1976] without any rigorous bounds on the error terms.

For site-interfaces of $\mathbb{Z}^d$, we prove the following weaker asymptotic expansion.

Theorem 2.7.3. *The exponential growth rate $b(\mathbb{Z}^d)$ of the number of site-interfaces of $\mathbb{Z}^d$ satisfies $b(\mathbb{Z}^d) = 2de - O(1)$.*

Proof. Similarly to the proof of Theorem 2.7.1, we will show that for any site-interface P of $\mathbb{Z}^d$ we have $|\partial P| \leq (2d-2)|P| + 2$. Let k be the number of edges of the graph spanned by P, and let l be the number of edges with one end-vertex in P and one in ∂P. Notice that $k \geq |P| - 1$ and $l \geq |\partial P|$. Arguing as in the proof of Theorem 2.7.1, we obtain

$$2(|P| - 1) + |\partial P| \leq 2k + l \leq 2d|P|.$$

By rearranging we obtain the desired inequality. Arguing as in the proof of Theorem 2.7.1, we obtain

$$b(\mathbb{Z}^d) \leq \frac{(2d-1)^{(2d-1)}}{(2d-2)^{(2d-2)}} = 2de - O(1).$$

Moreover, we have that $b(\mathbb{Z}^d) \geq b_{\dot{r}_d}(\mathbb{Z}^d)$ and $b_{\dot{r}_d}(\mathbb{Z}^d) = f(\dot{r}_d)$, where $\dot{r}_d := r(\dot{p}_c(\mathbb{Z}^d))$. Hara and Slade [1995] proved that $\dot{p}_c(\mathbb{Z}^d) = \big(1 + O(1/d)\big)/2d$, hence

$$\dot{r}_d = \frac{1 - \dot{p}_c(\mathbb{Z}^d)}{\dot{p}_c(\mathbb{Z}^d)} = \frac{2d}{1 + O(1/d)} - 1.$$

Using (2.16) we obtain

$$b_{\dot{r}_d}(\mathbb{Z}^d) = \frac{(1 + \dot{r}_d)^{1+\dot{r}_d}}{\dot{r}_d^{\dot{r}_d}} = \frac{2de}{1 + O(1/d)} - e/2 - O(1/d).$$

Since $\frac{1}{1+O(1/d)} = 1 - O(1/d)$, we have

$$\frac{2de}{1+O(1/d)} - e/2 - O(1/d) = 2de\Big(1 - O(1/d)\Big) - e/2 - O(1/d) = 2de - O(1).$$

Therefore, $b_{\dot{r}_d}(\mathbb{Z}^d) = 2de - O(1)$, which implies that $b(\mathbb{Z}^d) = 2de - O(1)$ as desired.

□

Arguing as in the proof of Theorem 2.7.2, we can easily deduce that

Theorem 2.7.4. $\dot{a}(\mathbb{Z}^d) = 2de - O(1)$.

Barequet, Barequet and Rote proved the weaker result $\dot{a}(\mathbb{Z}^d) = 2de - o(d)$, and they conjectured that $\dot{a}(\mathbb{Z}^d) = 2de - 3e + O(1/d)$ in Barequet et al. [2010][4]. Under the assumption that $\dot{p}_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{5}{2(2d)^2} + O(1/d^3)$ holds, which is suggested by (2.20), our method gives the lower bound $\dot{a}(\mathbb{Z}^d) \geq 2de - 3e + O(1/d)$. Moreover, assuming that both $\dot{p}_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{5}{2(2d)^2} + O(1/d^3)$ and $\dot{a}(\mathbb{Z}^d) = 2de - 3e + O(1/d)$ hold, we obtain $b(\mathbb{Z}^d) - b_{\dot{r}_d}(\mathbb{Z}^d) = O(1/d)$.

2.8 Upper bounds for lattice site animals

In the previous section we used Kesten's argument in order to upper bound $\dot{a}(\mathbb{Z}^d)$. Another method that gives the same upper bounds for $\dot{a}(\mathbb{Z}^d)$ was introduced by Eden [1961]. Eden described a procedure that associates in a canonical way, a spanning tree and a binary sequence to every lattice site animal. This reduces the problem of counting lattice site animals to a problem of counting binary sequences with certain properties. Klarner and Rivest [1973] enhanced Eden's method in the case of $\mathbb{Z}^2$, proving that $\dot{a}(\mathbb{Z}^d) \leq 4.6496$. Recently, Barequet and Shalah [2019] extended this enhancement to higher dimensions, obtaining $\dot{a}(\mathbb{Z}^d) \leq 2de - 2e + 1/(2d-2)$.

In this section, we will utilise Eden's procedure to reduce the gap between the aforementioned inequality and the conjectured asymptotic expansion $\dot{a}(\mathbb{Z}^d) = 2de - 3 + O(1/d)$ mentioned in the previous section.

Theorem 2.8.1. *We have* $\dot{a}(\mathbb{Z}^d) \leq 2de - 5e/2 + O(1/\log(d))$.

Our result improves the bounds of Barequet and Shalah [2019] for every large enough d. In order to prove Theorem 2.8.1, we will show that a typical lattice site

[4]In fact, Barequet et al. [2010] offers the more detailed conjecture $\dot{a}(\mathbb{Z}^d) = 2de - 3e - \frac{31e}{48d} + O(1/d^2)$.

animal has surface-to-volume ratio that is bounded away from its maximal possible value, namely $2d-2$.

We will need the following definitions. Given a lattice site animal X of $\mathbb{Z}^d$, we write ∂X for the set of vertices of $\mathbb{Z}^d \setminus X$ that have a neighbour in X. We let $\dot{a}_{n,r,\epsilon}$ denote the number of lattice site animals X of $\mathbb{Z}^d$ containing o with $|X| = n$ and $(r-\epsilon)n \leq |\partial X| \leq (r+\epsilon)n$, and we define

$$\dot{a}_r = \dot{a}_r(\mathbb{Z}^d) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \dot{a}_{n,r,\epsilon}(\mathbb{Z}^d)^{1/n}.$$

As mentioned in the Introduction, using Kesten's argument, Hammond [2005] proved that

$$\dot{a}_r \leq f(r). \tag{2.21}$$

for every $r > 0$.

For the proof of Theorem 2.8.1, we will need the next lemma which bounds $\dot{a}_r(\mathbb{Z}^d)$ for r close to $2d-2$.

Lemma 2.8.2. *Consider some* $0 \leq x < 1$, *and let* $y = \min\{x, 1/2\}$. *Then*

$$\dot{a}_{2d-2-x}(\mathbb{Z}^d) \leq \frac{(2d-1)^{2d-1}}{y^y(1-y)^{1-y}x^x(2d-1-x)^{2d-1-x}}.$$

In particular, $\dot{a}_{2d-2}(\mathbb{Z}^d) = 1$.

Proof. Let us start by introducing some necessary definitions. The *lexicographical ordering* of $\mathbb{Z}^d$ is defined as follows. We say that a vertex $u = (u_1, u_2, \ldots, u_d)$ is smaller than a vertex $v = (v_1, v_2, \ldots, v_d)$ if there is some $i = 1, 2, \ldots, d$ such that $u_i \leq v_i$ and $u_j = v_j$ for every $j < i$. We also order the directed edges of the form $\overrightarrow{ou}$ in an arbitrary way. The latter ordering induces by translation a natural ordering of the set of directed edges with a common initial end-vertex v, where v is any vertex of $\mathbb{Z}^d$.

Consider some numbers $n \in \mathbb{N}$, and $\epsilon > 0$ with $x + \epsilon < 1$. We will start by describing Eden's procedure. Let X be a lattice site animal of size n in $\mathbb{Z}^d$ containing o, such that $(2d-2-x-\epsilon)n \leq |\partial X| \leq (2d-2-x+\epsilon)n$. We will assign to X a unique binary sequence $S = S(X) = (s_1, s_2, \ldots, s_{(2d-1)n-d+1})$ of length $(2d-1)n-d+1$. To this end, we will reveal the vertices of X one by one in a specific way. Let v_1 be the lexicographically smallest vertex of X, and notice that v_1 has at most d neighbours in X. For every $i = 1, \ldots, d$, we let s_i take the value 1 if the i-th directed edge of the form $\overrightarrow{u_1 v}$ in the above ordering lies in the set of directed edges $\overleftrightarrow{E}(X)$ of X, and 0 otherwise. The ordering of these directed edges induces an ordering on the neighbours of u_1 in P. We reveal the neighbours of u_1 in X one by one according to the latter ordering, and we let u_{j+1} be the j-th revealed vertex. Now we proceed to the lexicographically smaller neighbour of u_1 lying in X, denoted w. The valid directed edges starting from w are those not ending at u_1, and there are exactly $2d-1$ of them. The ordering of the whole set of directed edges starting from w induces an ordering of the set of valid directed edges starting from w. For every $i = d+1, \ldots, 3d-1$, we let s_i take the value 1 if the $(i-d)$-th valid directed edge of the form $\overrightarrow{wv}$ lies in $\overset{\leftrightarrow}{E}(X)$ and v has not been revealed so far (the latter is always true in this step but not necessarily in the following steps), and 0 otherwise. We reveal the corresponding neighbours of w in X one by one, and we label them $u_k, u_{k+1}\ldots$, where k is the smallest index not previously used. Now we proceed as before up to the point that all vertices of X have been revealed, and we set to 0 all the remaining entries of S that have not already been set to some value. Notice that S contains exactly $n-1$ 1's since P has size n.

The above construction defines naturally a spanning subtree T of X rooted at u_1, by attaching an edge $u_k u_l$, $k < l$ to T when u_l is one of the neighbours of u_k revealed when considering the valid directed edges starting from u_k. Given an edge uv of T with u being the ancestor of v, we say that uv is a *turn* of T if uv is perpendicular to the edge zu of T, where z is the (unique) ancestor of u. We denote by t the number of turns of T. We claim that

$$|\partial X| \leq (2d-2)n - t + 2. \tag{2.22}$$

Indeed, for every $k = 1, 2, \ldots, n$, let T_k be the subtree of T with $V(T_k) = \{u_1, u_2, \ldots, u_k\}$. Let also ∂T_k be the set of vertices in $\mathbb{Z}^d \setminus \{u_1, u_2, \ldots, u_k\}$ having a neighbour in $\{u_1, u_2, \ldots, u_k\}$. Write t_k for the number of turns of T_k. We will prove inductively that

$$|\partial T_k| \leq (2d-2)|T_k| - t_k + 2$$

for every $k = 1, 2, \ldots, n$. The claim will then follow once we observe that $|\partial X| = |\partial T_n|$, $|X| = |T_n| = n$ and $t = t_n$. For $k = 1$, the assertion clearly holds. Assume that it holds for some $1 \leq k < n$. Notice that we always have $|T_{k+1}| = |T_k| + 1$ and $|\partial T_{k+1}| \leq |\partial T_k| + 2d - 2$ because u_{k+1} lies in ∂T_k and at most $2d-1$ neighbours of u_{k+1} lie in ∂T_{k+1}. If $t_{k+1} = t_k$, then we get $|\partial T_{k+1}| \leq (2d-2)|T_{k+1}| - t_{k+1} + 2$ by our induction hypothesis, as claimed. Suppose that $t_{k+1} = t_k + 1$. Consider the ancestor u_l of u_{k+1} and the ancestor u_m of u_l. Since by adding u_{k+1} to T_k we create one more turn, u_{k+1}, u_l and u_m are three vertices of a common square. Let w be the fourth vertex. Notice that w lies in $T_k \cup \partial T_k$. Thus, at most $2d-2$ neighbours of u_{k+1} lie in $\partial T_{k+1} \setminus \partial T_k$. In this case, we have $|\partial T_{k+1}| \leq |\partial T_k| + 2d - 3$. Therefore, $|\partial T_{k+1}| \leq (2d-2)|T_{k+1}| - t_{k+1} + 2$, as desired. This completes the proof of (2.22).

We will now utilise (2.22) to prove the statement of the lemma. Our assumption $(2d-2-x-\epsilon)n \leq |\partial X|$ combined with (2.22) implies that $t \leq (x+\epsilon)n + 2$. Hence it suffices to find an upper bound for the number of lattice site animals Q of size n with $t \leq q := (x+\epsilon)n + 2$. We claim that the number c_n of such lattice site animals of size n satisfies

$$c_n \leq \sum_{i=1}^{d} \sum_{j=0}^{\min\{q,n-i-1\}} \binom{d}{i} \binom{(2d-1)(n-1)}{j} \binom{n-1}{n-i-j-1}. \tag{2.23}$$

Indeed, let i be number of neighbours of u_1 in Q, and let j be the number of 1's contributing to the number of turns. Let us apply the following steps in turn:

(i) Set i entries of $(s_1, \ldots, s_d)$ equal to 1,

(ii) Choose which entries of $S(Q)$ contribute to the number of turns,

(iii) Choose which bits, except for the first one, contain an additional 1.

After the first two steps, we have specified which entries of $S(Q)$ are set to 1, except for those that do not contribute to the number of turns. In order to specify which of the remaining entries are set to 1, assume that at some step we have revealed a vertex u of Q. Let v be the ancestor of v. Notice that there a unique edge adjacent to u which is not vertical to uv. Hence we can determine whether this edge belongs to Q or not by choosing whether the bit of the neighbours of u contains an additional 1 or not. Thus the three steps above uniquely determine Q. It is easy to see now that for every i and j, there are at most

$$\binom{d}{i} \binom{(2d-1)(n-1)}{j} \binom{n-1}{n-i-j-1}$$

possibilities for Q, and so (2.23) can be obtained by summing over all possible values of i and j.

We will now handle the sum in the right-hand side of (2.23). Since the binomial coefficient $\binom{m}{l}$ is an increasing function of l when $l \leq m/2$, we have

$$\binom{(2d-1)(n-1)}{j} \leq \binom{(2d-1)(n-1)}{q}.$$

Using Stirling's approximation $m! = \big(1+o(1)\big)\sqrt{2\pi m}(m/e)^m$ we obtain

$$\binom{(2d-1)(n-1)}{q} \approx \frac{(2d-1)^{(2d-1)n}}{(x+\epsilon)^{x+\epsilon}(2d-1-x-\epsilon)^{(2d-1-x-\epsilon)n}},$$

where $\approx$ denotes equality up to a multiplicative constant that is $O(c^n)$ for every $c > 1$. Clearly,

$$\binom{n-1}{n-i-j-1} \leq 2^n.$$

It follows that

$$\dot{a}_{n,2d-2-x,\epsilon} \lesssim 2^n \frac{(2d-1)^{(2d-1)n}}{(x+\epsilon)^{x+\epsilon}(2d-1-x-\epsilon)^{(2d-1-x-\epsilon)n}},$$

where $\lesssim$ denotes inequality up to a multiplicative constant that is $O(c^n)$ for every $c > 1$. Taking n-th roots and letting $n \to \infty$ and $\epsilon \to 0$ we obtain

$$\dot{a}_{2d-2-x} \leq 2\frac{(2d-1)^{2d-1}}{x^x(2d-1-x)^{2d-1-x}}.$$

The above bound can be improved when $x < 1/2$. Suppose that $x < 1/2$. We can choose $\epsilon > 0$ small enough, and increase the value of n, if necessary, to ensure that $q + d < n/2$. Since the binomial coefficient $\binom{m}{l}$ is a decreasing function of l when $l \geq m/2$, for every i and j, we have

$$\binom{n-1}{n-i-j-1} \leq \binom{n-1}{n-d-q},$$

because $n-i-j-1 \geq n-d-q-1 \geq n/2$. Using again Stirling's approximation, we deduce that

$$\binom{n-1}{n-d-q-1} \approx \big((x+\epsilon)^{x+\epsilon}(1-x-\epsilon)^{1-x-\epsilon}\big)^{-n}.$$

We can now conclude that

$$\dot{a}_{n,2d-2-x,\epsilon} \lesssim \frac{(2d-1)^{(2d-1)n}}{(x+\epsilon)^{(2x+2\epsilon)n}(1-x-\epsilon)^{(1-x-\epsilon)n}(2d-1-x)^{(2d-1-x)n}}.$$

Taking n-th roots and letting $n \to \infty$ and $\epsilon \to 0$ we obtain

$$\dot{a}_{2d-2-x} \leq \frac{(2d-1)^{2d-1}}{x^{2x}(1-x)^{1-x}(2d-1-x)^{2d-1-x}}.$$

□

Since site interfaces are also lattice site animals, we obtain

Corollary 2.8.3. *Consider some* $0 \leq x < 1$*, and let* $y = \min\{x, 1/2\}$*. Then*

$$b_{2d-2-x}(\mathbb{Z}^d) \leq \frac{(2d-1)^{2d-1}}{y^y(1-y)^{1-y}x^x(2d-1-x)^{2d-1-x}}.$$

In particular, $b_{2d-2}(\mathbb{Z}^d) = 1$*.*

The above bounds are in agreement with our plot of b_r.

We are now ready to prove Theorem 2.8.1.

Proof of Theorem 2.8.1. For every $0 \leq x \leq 1$, we let

$$g_d(x) = \frac{(2d-1)^{2d-1}}{y^y(1-y)^{1-y}x^x(2d-1-x)^{2d-1-x}},$$

where $y = \min\{x, 1/2\}$. It is not hard to see that there is a constant $C > 0$ such that $x^{-x} \leq C$ for every $x \in [0,1]$, and

$$\frac{1}{y^y(1-y)^{1-y}} \leq C$$

for every $y \in [0, 1/2]$. Moreover, for every $x \in [0,1]$ we have

$$\frac{(2d-1)^{2d-1}}{(2d-1-x)^{2d-1-x}} \leq \frac{(2d-1)^{2d-1}}{(2d-2)^{2d-1-x}}$$

by the monotonicity of $2d-1-x$ as a function of x, and

$$\frac{(2d-1)^{2d-1}}{(2d-2)^{2d-1-x}} = \frac{2d-1}{(2d-2)^{1-x}}\left(1 + \frac{1}{2d-2}\right)^{2d-2} \leq \frac{2d-1}{(2d-2)^{1-x}}e.$$

Thus,

$$g_d(x) \leq C^2 e \frac{2d-1}{(2d-2)^{1-x}}.$$

Since $\frac{2d-1}{(2d-2)^{1-x}}$ is an increasing function of x, it follows by Lemma 2.8.2 that for every

$$x \leq z := 1 - \frac{C^2}{\log\big(2d-2\big)}$$

we have

$$\dot{a}_{2d-2-x}(\mathbb{Z}^d) \leq g_d(x) \leq C^2 e \frac{2d-1}{(2d-2)^{1-x}} \leq C^2 e \frac{2d-1}{(2d-2)^{1-z}} = C^2 e^{1-C^2}(2d-1).$$

Using the standard inequality $e^{C^2} \geq 1 + C^2$ we obtain $e^{-C^2} \leq 1/(1 + C^2)$, hence

$$C^2 e^{1-C^2}(2d-1) \leq \frac{C^2 e}{1+C^2}(2d-1).$$

Plugging $r = 2d-2-z$ in (2.16) we obtain $f(2d-2-z) = 2de - 5e/2 + O(1/\log(d))$, and so

$$\dot{a}_{2d-2-x}(\mathbb{Z}^d) < f(2d-2-z) \tag{2.24}$$

for every d large enough. On the other hand, for every $r \leq 2d-2-z$ we have $\dot{a}_r(\mathbb{Z}^d) \leq f(2d-2-z)$, hence

$$\dot{a}(\mathbb{Z}^d) \leq f(2d-2-z) = 2de - 5e/2 + O(1/\log(d))$$

for every d large enough, which proves our claim. □

Combining this with (2.15) yields the following lower bound for $\dot{p}_c(\mathbb{Z}^d)$:

Theorem 2.8.4. $\dot{p}_c(\mathbb{Z}^d) \geq \frac{1}{2d} + \frac{2}{(2d)^2} - O(1/d^2 \log(d))$.

Proof. It follows from (2.24) that $b_r < f(2d-2-z) \leq f(r)$ for every $r \geq 2d-2-z$, where $z = 1 - \frac{C^2}{\log\big(2d-2\big)}$. Since $b_{\dot{r}_d}(\mathbb{Z}^d) = f(\dot{r}_d)$, we obtain

$$\dot{r}_d \leq 2d - 3 + \frac{C^2}{\log\big(2d-2\big)}.$$

Hence

$$\dot{p}_c(\mathbb{Z}^d) = \frac{1}{1+\dot{r}_d} \geq \frac{1}{2d-2+C^2/\log(2d-2)}.$$

It is not hard to see

$$\frac{1}{2d-2+C^2/\log(2d-2)} = \frac{1}{2d} + \frac{2-C^2/\log(2d-2)}{2d\big(2d-2+C^2/\log(2d-2)\big)} =$$
$$\frac{1}{2d} + \frac{2}{(2d)^2} - O(1/d^2\log(d)),$$

which proves the assertion. □

We remark that the well-known inequality $\dot{p}_c(\mathbb{Z}^d) \geq p_c(\mathbb{Z}^d)$ Grimmett [1999] and the asymptotic expansion $p_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{1}{(2d)^2} + O(1/d^3)$, mentioned in the previous section, give a weaker lower bound on $\dot{p}_c(\mathbb{Z}^d)$.

Recently, Barequet and Shalah [2019] proved that $\dot{a}(\mathbb{Z}^3) < 9.3835$. Plugging this into (2.15), we deduce

$$\dot{p}_c(\mathbb{Z}^3) > r^{-1} \circ f^{-1}(9.3835) > 0.2522. \tag{2.25}$$

As far as we know, the best rigorous bound previously known was about $\dot{p}_c(\mathbb{Z}^3) > 0.21225$, obtained as the inverse of the best known bound on the connective constant MacDonald et al. [2000].

Remark: In both Theorem 2.8.4 and (2.25) we made implicit use of Theorem 2.1.1 but it would have sufficed to use its variant for site lattice animals instead of interfaces. Thus adapting Delyon's result to site animals would have sufficed.

2.9 Continuity of the decay exponents

We prove that the rate of exponential decay $c(p) := \lim_{n\to\infty} \mathbb{P}_p(|C_o| = n)^{1/n}$ of the cluster size distribution —which is known to exist for every $p \in (0,1)$ Bandyopadhyay et al. [2010]; Grimmett [1999]— is a continuous function of p. This applies to bond and site percolation on our class of graphs $\mathcal{S}$.

We will also prove the analogous continuity result for the (upper) exponential growth rate of $\mathbb{E}_p(N_n)$, i.e. $\limsup_{n\to\infty} \mathbb{E}_p(N_n)^{1/n}$, where as before N_n denotes the number of occurring (site-)interfaces.

We will start by proving the continuity of $c(p)$.

Theorem 2.9.1. *Consider bond or site percolation on a graph in $\mathcal{S}$. Then $c(p)$ is a continuous function of $p \in (0,1)$.*

Proof. The proof is an easy application of the Arzelà-Ascoli theorem. Let I be a compact subinterval of $(0,1)$. Define $g_n(p) := \mathbb{P}_p(|C_o| = n)^{1/n}$, and notice that $g_n(p) \leq 1$. Moreover, g_n is a differentiable function with derivative equal to $g_n(p)\dfrac{\mathbb{P}'_p(|C_o| = n)}{n\mathbb{P}_p(|C_o| = n)}$, where $\mathbb{P}'_p(|C_o| = n)$ denotes the derivative of $\mathbb{P}_p(|C_o| = n)$. Expressing $\mathbb{P}'_p(|C_o| = n)$ via $\sum_P \left(\dfrac{n}{p} - \dfrac{|\partial P|}{1-p}\right) p^n (1-p)^{|\partial P|}$, where the sum ranges over all lattice (site) animals of size n, we conclude that there is a constant $c = c(I) > 0$ such that $|\mathbb{P}'_p(|C_o| = n)| \leq cn\mathbb{P}_p(|C_o| = n)$ for every $p \in I$. Therefore, g'_n is uniformly bounded on I. We immediately deduce that the sequence (g_n) is equicontinuous and bounded. The Arzelà-Ascoli theorem and the pointwise convergence of g_n to $c(p)$ give that every subsequence of g_n has a further subsequence converging uniformly on I to $c(p)$. Hence (g_n) converges uniformly on I to $c(p)$, and $c(p)$ is continuous on I. □

Define $B_p := \limsup_{n\to\infty} \mathbb{E}_p(N_n)^{1/n}$. Before proving the continuity of B_p, we will show that $\lim_{n\to\infty} \mathbb{E}_p(N_n)^{1/n}$ exists for every p.

Proposition 2.9.2. *Consider bond or site percolation on a graph in $\mathcal{S}$. Then for every $p \in (0,1)$, the limit $\lim_{n\to\infty} \mathbb{E}_p(N_n)^{1/n}$ exists.*

Proof. For simplicity, we will prove the assertion for interfaces in $\mathbb{Z}^d$ and $\mathbb{T}^d$. The remaining cases can be handled in a similar way.

Let m and n be positive integers. We will consider interfaces without any restriction on the surface-to-volume ratio. Arguing as in the proof of Proposition 2.6.1, we combine m interfaces $P_1, P_2, \dots, P_m$ of size n that have the same shape, and attach a horizontal path to P_m, to obtain an interface of size $k = m(n+4)+s$ for some s between 0 and $n+3$. Notice that the number of attached edges that were initially lying in some ∂P_i is equal to $2m-1$. The probability that the resulting interface occurs is equal to $p^k(1-p)^{M-(2m-1)+N}$, where $M = \sum_{i=1}^m |\partial P_i|$, and N is the number of remaining boundary edges of the interface. It is not hard to see that $N \leq Cm$ for some constant $C > 0$. Hence

$$p^k(1-p)^{M-(2m-1)+N} \geq p^{4m+s}(1-p)^{-(2m-1)+Cm} \prod_{i=1}^m p^n(1-p)^{|\partial P_i|}.$$

Summing over all possible sequences $(P_1, P_2, \dots, P_m)$ we obtain

$$\mathbb{E}_p(N_k) \geq p^{4m+s}(1-p)^{-(2m-1)+Cm}(\mathbb{E}_p(N_n))^m.$$

Taking the k-th root, and then letting $m \to \infty$ and $n \to \infty$, we obtain that $\liminf_{n\to\infty} \mathbb{E}_p(N_n)^{1/n} \geq \limsup_{n\to\infty} \mathbb{E}_p(N_n)^{1/n}$, which implies the desired assertion. □

The proof of Theorem 2.9.1 applies mutatis mutandis to B_p: instead of defining $g_n(p)$ as $\mathbb{P}_p(|C_o| = n)^{1/n}$, we define $g_n(p) := \mathbb{E}_p(N_n)^{1/n}$, and we use the fact that $\mathbb{E}_p(N_n) \leq fn$.

Corollary 2.9.3. *Consider bond or site percolation on a graph in $\mathcal{S}$. Then B_p is a continuous function of $p \in (0,1)$.*

Chapter 3

Site percolation on plane graphs

3.1 Introduction

In their highly influential paper Benjamini and Schramm [1996b], the authors made several conjectures that generated a lot of interest among mathematicians and led to many beautiful mathematical results Babson and Benjamini [1999]; Benjamini et al. [1999]; Benjamini and Schramm [2001]; Duminil-Copin et al. [2018]; Häggström and Y.Peres [1999]; Hutchcroft [2016], just to name a few. Despite the substantial amount of work, most of these conjectures are still open, while for a few of them, hardly anything is known. One of their conjectures states that the critical probability for site percolation, $\dot{p}_c(G)$, satisfies $\dot{p}_c(G) < 1/2$ on any planar graph G of minimal degree at least 7; they additionally conjecture that there are infinitely many infinite open clusters on the interval $(\dot{p}_c(G), 1 - \dot{p}_c(G))$. As Benjamini and Schramm observe in their paper, every planar graph of minimal degree at least 7 is non-amenable. The conjecture has been verified for the d-regular triangulations of the hyperbolic plane in Benjamini and Schramm [2001].

The connection between percolation thresholds and isoperimetric (or Cheeger) constants is well-known, and in Benjamini and Schramm [1996b] it is proved that the site percolation threshold for a graph G is bounded above by $(1 + \dot{h}(G))^{-1}$, where $\dot{h}(G)$ is the vertex isoperimetric constant. In their book Lyons and Peres [2016], Lyons and Peres give the edge isoperimetric constants for the regular hyperbolic tessellations $H_{d,d'}$, where $(d-2)(d'-2) > 4$ (which were established by Häggström et al. [2002]), and ask [Lyons and Peres, 2016, Question 6.20] for the corresponding vertex isoperimetric constants.

Angel, Benjamini and Horesh considered isoperimetric inequalities for plane triangulations of minimum degree 6 in Angel et al. [2018], and proved a discrete analogue of Weil's theorem, showing that any such triangulation satisfies the same isoperimetric inequality as the Euclidean triangular lattice $\mathbb{T}^2$. They conjectured that $\mathbb{T}^2$ is extremal in other ways which might be expected to have connections with isoperimetric properties. First, they conjecture that the connective constant $\mu(T)$ – that is, the exponential growth rate of the number of self-avoiding walks of length n on T – is minimised by $\mathbb{T}^2$ among triangulations of minimum degree at least 6. Secondly, they conjecture that percolation is hardest to achieve on $\mathbb{T}^2$ in the sense that both the critical probability for site percolation $\dot{p}_c(T)$ and the critical probability for bond percolation $p_c(T)$ are maximised by $\mathbb{T}^2$. Intuitively these conjectures are connected, in that if fewer long self-avoiding paths exist then long connections might be expected to be less robust, making percolation less likely to occur. See also Benjamini [2015] for several other conjectures regarding them.

In Chapter 1 we proved that the bond percolation threshold satisfies $p_c(T) \leq 1/2$ for any planar triangulation T with minimum degree at least 6, and a well-known result of Grimmett and Stacey [1998] shows that $\dot{p}_c(T) \leq 1 - (1/2)^{d-1}$ when the degrees in T are bounded by d. Unforunately, this bound converges to 1 as the maximal degree converges to infinity. We remark that Benjamini and Schramm [1996b] made an even stronger conjecture than that one of Angel, Benjamini and Horesh mentioned above, namely that $\dot{p}_c(T) \leq 1/2$ for any planar triangulation without logarithmic cut sets.

In Section 3.3 we consider the conjecture of Angel, Benjamini and Horesh for site percolation, and we prove the following theorem.

Theorem 3.1.1. *For any plane graph G without any accumulation points and with minimum degree at least* 6*,*

$$\dot{p}_c(G) \leq 2/3\,.$$

In section 3.4 we study planar graphs of minimal degree at least 7 and we prove the aforementioned conjecture of Benjamini and Schramm.

Theorem 3.1.2. *Let G be a plane graph without any accumulation points and with minimum degree at least $d \geq 7$. Then*

$$\dot{p}_c(G) \leq \frac{2+\alpha_d}{(d-3)(1+\alpha_d)}$$

where $\alpha_d = \frac{d-6+\sqrt{(d-2)(d-6)}}{2}$. In particular, for planar graphs with minimum degree at least 7

$$\dot{p}_c(G) \leq \frac{2+\alpha_7}{4(1+\alpha_7)} \approx 0.3455.$$

For plane graphs without faces of degree 3, a minimum vertex degree of 5 is sufficient to ensure non-amenability; we also give bounds on the site percolation threshold in this case.

Theorem 3.1.3. *Let G be a plane graph without any accumulation points and with minimum degree at least $d \geq 5$ and face degree at least 4. Then*

$$\dot{p}_c(G) \leq \frac{(2+\alpha_{d+2})(d-2)}{(1+\alpha_{d+2})(d^2-3d+1)}.$$

As far as we know, these results are new even for the d-regular triangulations and quadrangulations of the hyperbolic plane. In the process, we obtain best-possible bounds on the vertex Cheeger constants $\dot{h}(G)$ of such graphs.

Theorem 3.1.4. *Let G be a plane graph without any accumulation points and with minimum degree at least $d \geq 7$. Then*

$$\dot{h}(G) \geq \alpha_d.$$

If G is the d-regular triangulation of the hyperbolic plane, then we have equality.

Theorem 3.1.5. *Let G be a plane graph without any accumulation points and with minimum degree at least $d \geq 5$ and minimum face degree at least 4. Then*

$$\dot{h}(G) \geq \alpha_{d+2}.$$

If G is the d-regular quadrangulation of the hyperbolic plane, then we have equality.

The second halves of the last two theorems answer the aforementioned question of Lyons and Peres for all cases with $d' = 3$ and $d' = 4$. In particular, the vertex isoperimetric constant for both the 7-regular triangulation and the 5-regular quadrangulation, α_7, is the golden ratio.

3.2 Definitions and main technique

Let G be an infinite connected locally finite plane graph, and fix a root vertex o. Throughout this chapter, we assume that G is embedded in the plane without accumulation points, i.e. only finitely many vertices lie in any bounded region. We will also assume, as we may, that all egdes of G are straight lines. This is because any plane graph without accumulation points can be embedded in the plane (without accumulation points) in such a way that all edges are straight lines Thomassen [1977].

Let C_o be any finite connected induced subgraph containing o. Recall that outer interface of C_o consists of all vertices of C_o meeting the unbounded face of C_o. Deleting all vertices of C_o from G divides the remaining graph into components, at least one of which lies in the unbounded face of C_o. Write C_∞ for the union of all components lying in the outer face of C_o. The boundary of the outer interface is the set of vertices in C_∞ adjacent to C_o. Denote the outer interface by M and the outer boundary by B. By definition, M induces a connected subgraph of G and B forms a vertex cut separating o from infinity.

Note that, while (in general) neither M nor B uniquely determine C_o, each uniquely determines C_∞. In fact, it is the union of components of $G \setminus M$ lying in the outer face of M, and it is also the union of B and the set of vertices not connected to o in $G \setminus B$. Since M is also the outer interface of $G \setminus C_\infty$ and B coincides with the set of vertices in C_∞ adjacent to M, each of M and B uniquely determines the other. Let the set of feasible pairs (M, B) be $\mathfrak{O}$, and for each n let $\mathfrak{O}_n = \{(M, B) \in \mathfrak{O} : |B| = n\}$.

We say that a pair $(M, B) \in \mathfrak{O}$ *occurs* in a site percolation instance $\boldsymbol{\omega}$ if $\omega(m) = 1$ for each $m \in M$ and $\omega(b) = 0$ for each $b \in B$. Note that in a site percolation instance $\boldsymbol{\omega}$ with $\omega(o) = 1$, the occupied cluster of o is infinite if and only if no pair $(M, B) \in \mathfrak{O}$ occurs. Since each outer boundary forms a vertex cut separating o from infinity, the occurrence of a pair certainly precludes o being in an infinite cluster, whereas if o is in a finite cluster C_o then the outer interface and outer boundary of C_o form an occurring pair.

Our main technique is to upper bound the ratio $|M|/|B|$ for $(M, B) \in \mathfrak{O}$ and consequently to show that the probability of occurrence of any given pair is decreasing for p above a certain value. Provided G satisfies an isoperimetric inequality of moderate strength, we then deduce that with positive probability no pair occurs. This latter condition essentially requires non-positive curvature, and so the smallest minimum degree to guarantee this is 6 for the general case and 5 for the triangle-free case.

Theorem 3.2.1. *Suppose that there exist a real number α and a function $f(n)$ of sub-exponential growth with the following properties.*

(i) For each pair $(M, B) \in \mathfrak{O}$, we have $|M| \leq \alpha |B|$.

(ii) For each outer boundary B, the component of o in $G \setminus B$ contains at most $f(|B|)$ vertices.

Then $\dot{p}_c(G) \leq \frac{\alpha}{1+\alpha}$.

We will use a Peierls-type argument (see e.g. [Pete, 2008, Theorem 4.1]) to show that with positive probability o is in an infinite component for site percolation with any intensity $p > \frac{\alpha}{1+\alpha}$. It is sufficient to show that with positive probability no pair $(M, B) \in \mathfrak{O}$ occurs; in fact, we shall find it more convenient to work with a slightly weaker notion than occurrence. The key observations are that not too many pairs of any given size can occur for $p = \frac{\alpha}{1+\alpha}$, and that the expected number of occurring pairs at any given higher intensity is exponentially smaller.

In G, pick a geodesic R from o, going to infinity. For any $(M, B) \in \mathfrak{O}$, let R_B be the longest initial segment of R which does not intersect B (so the next vertex on R is the first intersection with B, which must exist since B is a vertex cut), and define $M' = M \setminus R_B$. We say that the pair (M, B) *almost occurs* in a percolation instance if the vertices of M' are occupied and the vertices of B are unoccupied.

We first need to show that (ii) gives a bound on the number of almost-occurring pairs.

Lemma 3.2.2. *At most $f(n)$ elements of $\mathfrak{O}_n$ almost occur in any instance $\boldsymbol{\omega}$.*

Proof. Fix an instance $\boldsymbol{\omega}$. Suppose $(M, B) \in \mathfrak{O}_n$ almost occurs in $\boldsymbol{\omega}$. Now define a new instance $\boldsymbol{\omega}'$ by setting each vertex in R_B to be occupied, leaving the states of other vertices unchanged. Note that (M, B) occurs in $\boldsymbol{\omega}'$; in fact, since $M \cup R_B$ induces a connected subgraph, (M, B) is the outer interface and boundary of the occupied cluster of o in $\boldsymbol{\omega}'$. Thus $\boldsymbol{\omega}'$ uniquely determines (M, B).

Since $o \in R_B$ and R_B lies entirely within the component of o in $G \setminus B$, by (ii) we have $1 \leq |R_B| \leq f(n)$. Thus, given $\boldsymbol{\omega}$, there are at most $f(n)$ possibilities for $\boldsymbol{\omega}'$ and hence at most this many pairs $(M, B) \in \mathfrak{O}_n$ almost occur. □

Proof of Theorem 3.2.1. We can always assume that f is a strictly increasing function as otherwise, we can work with $g(n) := n + \max_{k \leq n} f(k)$ which is a strictly increasing function and also satisfies the properties of f in the statement of the theorem.

Let $b_{n,m}$ be the number of pairs $(M, B) \in \mathfrak{O}_n$ for which $|M'| = m$. By (i), whenever $(M, B) \in \mathfrak{O}_n$ we have $|M| \leq \alpha n$, and so $b_{n,m} = 0$ for $m > \alpha n$.

Let X_n be the number of pairs $(M, B) \in \mathfrak{O}_n$ which almost occur, and write $q = \frac{\alpha}{1+\alpha}$. By Lemma 3.2.2, $X_n \leq f(n)$, and hence $\mathbb{E}_q(X_n) \leq f(n)$. Thus

$$\begin{aligned} f(n) &\geq \sum_{(M,B) \in \mathfrak{O}_n} \mathbb{P}_q((M, B) \text{ almost occurs}) \\ &= \sum_{m \leq \alpha n} b_{n,m} q^m (1-q)^n . \end{aligned}$$

Now for any $p > q$, we have

$$\begin{aligned}\mathbb{E}_p(X_n) &= \sum_{m \le \alpha n} b_{n,m} p^m (1-p)^n \\ &= \sum_{m \le \alpha n} b_{n,m} q^m (1-q)^n (p/q)^m \left(\frac{1-p}{1-q}\right)^n \\ &\le \sum_{m \le \alpha n} b_{n,m} q^m (1-q)^n . (p/q)^{\alpha n} \left(\frac{1-p}{1-q}\right)^n \\ &\le \left(\frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} p^\alpha (1-p)\right)^n f(n)\,.\end{aligned}$$

The arithmetic-geometric mean inequality implies $p^\alpha(1-p) < \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$. Consequently, since $f(n)$ is sub-exponential, $\sum_{n>0} \mathbb{E}_p(X_n)$ is finite, and in particular, there is some n_0 such that $\mathbb{E}_p\big(\sum_{n \ge n_0} X_n\big) < 1$.

Let $\boldsymbol{\omega}$ be a site percolation instance with intensity p. Notice that any pair (M, B) with $|R_B| \ge f(n_0)$ satisfies $|B| \ge n_0$ by the monotonicity of f. We can now deduce from the union bound that with positive probability no pair (M, B) with $|R_B| \ge f(n_0)$ almost occurs in ω; call this event E_1. Let E_2 be the event that the first $f(n_0)$ vertices along R are all occupied in $\boldsymbol{\omega}$; clearly $\mathbb{P}_p(E_2) > 0$. As E_1 and E_2 are determined by the states of disjoint sets of vertices, they are independent, and so their intersection also has positive probability. In the event $E_1 \cap E_2$, no pair (M, B) occurs in ω. Thus o is in an infinite cluster with positive probability, so $p \ge \dot{p}_c(G)$. □

Remark 3.2.3. *The reason for introducing the notion of almost occurrence of elements of $\mathfrak{O}$ is the following. In the proof of Theorem 3.2.1, it is crucial that the event $E_1 \cap E_2$ has positive probability, which holds because the events E_1 and E_2 depend on disjoint sets of vertices, hence they are independent. If instead of E_1 we were working with the event E_1' that no pair (M, B) with $|R_B| \ge f(n_0)$ occurs in ω, which also has positive probability, then it is a priori possible that the event $E_1' \cap E_2$ has probability zero. This is because E_1' does depend on the state of the first $f(n_0)$ vertices along R.*

Following Benjamini and Schramm [1996b], in this chapter, we use the notation ∂S, where S is a set of vertices, to denote the set of vertices which are not in S but are adjacent to some vertex in S, and we define the (site) Cheeger constant

$$\dot{h}(G) = \inf_{|S| < \infty} \frac{|\partial S|}{|S|}\,.$$

3.3 Graphs of minimum degree at least 6

In this section, we will consider the following problem of Angel et al. [2018].

Problem 3.3.1 (from [Angel et al., 2018, Problem 4.2])**.** *If T is a plane triangulation with all vertex degrees at least 6, is it necessarily true that $\dot{p}_c(T) \leq 1/2$?*

The authors ask also similar questions for bond percolation and self-avoiding walks.

While Angel et al. [2018] does not give a precise definition of "plane triangulation", it is clear from context that accumulation points are not permitted. Without this assumption, the answer to the question would be negative. Indeed, consider the graph of an infinite cylinder formed by stacking congruent antiprisms, which can be embedded in the plane with a single accumulation point. Notice that all vertices have degree 6, and furthermore this graph contains infinitely many disjoint cut sets of fixed size separating o from infinity, hence $\dot{p}_c = 1$ by the Borel-Cantelli lemma. We remark in passing that the same graph demonstrates that [Angel et al., 2018, Theorem 2.4] also assumes accumulation points are not permitted. With this assumption, we firmly believe that the answer to Problem 3.3.1 is affirmative.

The aim of this section is to give upper bounds for $\dot{p}_c$ on general plane graphs of minimum degree at least 6 that are not necessarily triangulations. As we will see, the general case can be easily reduced to the case of triangulations.

The following two results of Angel et al. [2018] about plane triangulations T with finitely many vertices will be used in our proofs. A *plane triangulation* is a plane graph without accumulation points in which every bounded face has degree 3. The *boundary* of T is the set of vertices incident with the unbounded face. The remaining vertices of T are called *internal*. The *total boundary length* of T counts all edges induced by the boundary vertices of T exactly once, except for those not incident with a triangular face of T which are counted twice. When the boundary vertices of T span a cycle, we will say that its boundary is *simple* and T is a *disc triangulation*.

Lemma 3.3.1. *Let T be a disc triangulation with a simple boundary of length n and at least one internal vertex. Let T' be the triangulation induced by the internal vertices of T and let m be the total boundary length of T'. Suppose all internal vertices of T have degree at least 6. Then $m \leq n - 6$.*

Lemma 3.3.2. *Any disc triangulation with k vertices and n boundary vertices, and with all internal vertices having degree at least 6, satisfies $k \leq \lfloor n^2/12 + n/2 + 1 \rfloor$.*

Consider now some plane graph G of minimum degree at least 6. For any pair $(M, B) \in \mathfrak{D}_n$, consider the subgraph of G induced by B and the finite components of $G \setminus B$. Write B° for the set of vertices in B which were adjacent to the infinite component of $G \setminus B$; note that $|B^\circ| \leq |B| = n$. By adding edges, if necessary, to the subgraph, we may obtain a plane triangulation $T = T(M, B)$ with finitely many vertices, say k, and boundary B°. We do this by first adding edges joining vertices of B° cyclically so that no other vertices meet the unbounded face, then adding internal edges to triangulate bounded faces of degree greater than 3. Each time that we triangulate a face between B and M, we do so by adding all diagonals meeting some vertex of B; this will ensure that in the final triangulation the subgraph induced by B is connected, and every vertex of M is adjacent to B. Let us fix such a triangulation T. Since M is connected, T must be a disc triangulation, and the choice of B° ensures that all internal vertices have degree at least 6. The next corollary follows now from Lemma 3.3.2, noting that the number of internal vertices is at most $k - n$.

Corollary 3.3.3. *For each pair $(M, B) \in \mathfrak{D}_n$, the component of o in $T \setminus B$ consists of at most $\lfloor n^2/12 - n/2 + 1 \rfloor$ vertices.*

We wish to apply Lemma 3.3.1 to bound $|M|$ in terms of $|B|$, for $(M, B) \in \mathfrak{D}$. However, the boundary of $T(M, B)$ does not necessarily coincide with B; to deal with this we show that the triangulation can be modified to give a disc triangulation with boundary which is not too much larger than $|B|$. First, we need a simple application of Euler's formula.

Lemma 3.3.4. *Consider a pair $(M, B) \in \mathfrak{D}$ and the corresponding triangulation $T = T(M, B)$. Let H be the subgraph of T induced by B, and f a face of H. Write ∂f for the boundary of f, and write $\partial^\circ f \subseteq \partial f$ for that part of the boundary which forms a cycle separating f from infinity. Follow ∂f clockwise, writing down a list of vertices visited (so the same vertex can appear in the list multiple times). The length of the list so obtained is at most $2|\partial f| - |\partial^\circ f|$, which is at most $2|\partial f| - 3$.*

Proof. We may assume that f is an internal face. Add a new vertex inside f, and join it to each vertex in the list in turn. This gives a plane multigraph H' with $|\partial f| + 1$ vertices in which each face incident with the new vertex has degree 3 and since H was simple, all other faces have degree at least 3. The unbounded face has degree $|\partial^\circ f|$, but there may be other faces inherited from H.

Suppose there are k internal faces. Then Euler's formula gives $e(H') - (k + 1) = |\partial f| - 1$. Since each edge is incident with two faces, the sum of face degrees coincides with $2e(H')$. Hence $3k + |\partial^\circ f| \leq 2e(H')$, so $k \leq 2|\partial f| - |\partial^\circ f|$. Since the length of the list is the number of faces incident with the new vertex, which is at most k, the result follows. □

We are now ready to bound $|M|$ in terms of $|B|$ and $|B^\circ|$.

Lemma 3.3.5. *For each pair $(M, B) \in \mathfrak{O}_n$ we have $|M| \leq 2n - |B^\circ|$.*

Proof. Fix such a pair, and let $T = T(M, B)$ be the plane triangulation fixed above. Recall that T is formed in such a way that B is internally connected, and every vertex of M is adjacent to B. Let C_o be the component containing o of $T \setminus B$. Removing the vertices of C_o would leave a face f with boundary vertices B.

We define an "unzipping" operation on B as follows. We imagine that each edge of B has positive width so that each such edge has two edge-sides, where each of them is incident with exactly one face. Moreover, each edge-side has two ends reaching the end-vertices of the corresponding edge. Proceed clockwise around the boundary of f along the edge-sides incident with f, recording the ends of edges-sides of f which are crossed in a cyclic ordering. Group these edge-ends by the vertex in B which they reach; since f is a face of $T \setminus C_o$, no edge of T inside f connects two vertices of B.

Note that since T is a triangulation, every time a vertex of B is encountered when proceeding around ∂f clockwise, at least one edge-end of T incident with that vertex is crossed. Thus the number of groups in the cyclic ordering of edge-ends is precisely the number of entries in the list constructed in Lemma 3.3.4; since $|B| = n$, this list has at most $2n - |B^\circ|$ entries.

We now "unzip" B by replacing vertices in B by the entries of the list, so that each vertex which appears more than once in the list is split into multiple vertices distinguished by list position. We also replace edges in ∂f by edges between consecutive entries in the list; this means that any edges of ∂f which were surrounded by f will also be split into two.

There is a one-to-one correspondence between groups of edge-ends of T inside f and entries in the list; we use this correspondence to replace every edge between C_o and B by an edge between C_o and a specific list entry. This will ensure that the graph obtained is still plane, and every vertex of T inside f retains its original degree. Finally, remove all vertices and edges which lie completely outside f. Figure 3.1 illustrates this unzipping operation.

This produces a disc triangulation with all internal vertices having degree 6. The simple boundary has at most $2n - |B^\circ|$ vertices, and M is precisely the boundary of the internal vertices. The required bound now follows from Lemma 3.3.1. □

We now have all the ingredients required for Theorem 3.1.1.

Proof of Theorem 3.1.1. We apply Theorem 3.2.1; by Lemma 3.3.5 we have (i) with $\alpha = 2$, and by Corollary 3.3.3 we have (ii) with $f(n) = \lfloor n^2/12 - n/2 + 1 \rfloor$. □

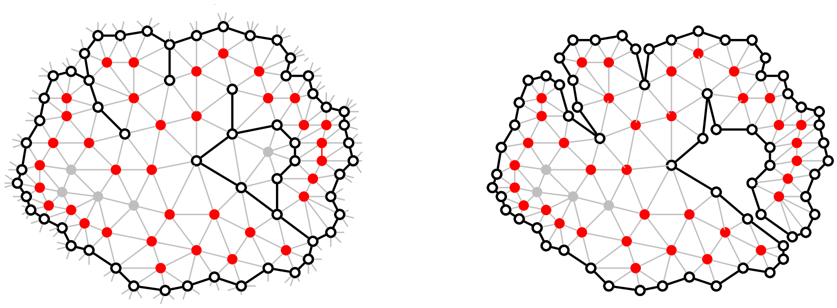

Figure 3.1: Unzipping an outer boundary B (bold vertices and edges); M is shown in red (if colour is shown).

If we further assume that G is non-amenable, we obtain better bounds.

Theorem 3.3.6. *Let G be a non-amenable plane graph without any accumulation points and with minimum degree at least* 6*, and set $\beta = \dot{h}(G) > 0$. Then $\dot{p}_c(G) \leq \frac{2+\beta}{3+3\beta}$.*

Remark. *This result is only interesting for $0 < \beta < 1$ since for $\beta \geq 1$ it is weaker than the bound of $\frac{1}{1+\beta}$, due to Benjamini and Schramm [1996b], which applies to general graphs. This is as we would expect since for $\beta \geq 1$ a stronger form of Lemma 3.3.1 may be deduced directly from the Cheeger constant.*

Proof. Consider some $(M, B) \in \mathfrak{D}_n$ and the corresponding triangulation T. Let $B^\circ \subseteq B$ be the minimum vertex cut separating o from infinity. Note that B° is the boundary of some set A consisting of those vertices of T separated from infinity by B°. In particular, $A \supseteq M \cup (B \setminus B^\circ)$. Consequently $|A| \geq |M| + n - |B^\circ|$, giving

$$|B^\circ| \geq \beta(|M| + n - |B^\circ|),$$

i.e.

$$|B^\circ| \geq \frac{\beta}{1+\beta}(|M| + n)\,. \tag{3.1}$$

Unzipping B as in the proof of Lemma 3.3.5 gives a disc triangulation with boundary of length at most $2n - |B^\circ|$ by Lemma 3.3.4, and so applying Lemma 3.3.1 and (3.1) gives $|M| \leq 2n - \frac{\beta}{1+\beta}(|M| + n)$, i.e. $|M| \leq \left(\frac{2+\beta}{1+2\beta}\right)n$. Applying Theorem 3.2.1 with $\alpha = \frac{2+\beta}{1+2\beta}$ gives the required result. $\square$

We remark that Angel et al. [2018] proved that for any $r > 0$, if a plane triangulation T has minimum degree at least 6 and every ball of radius r contains a vertex of degree greater than 6, then T is non-amenable.

3.4 Hyperbolic graphs

In this section, we consider more stringent degree conditions, motivated by regular hyperbolic graphs. Benjamini and Schramm [1996b] conjecture that a plane graph G of minimum degree at least 7 has $\dot{p}_c < 1/2$, and furthermore that there are infinitely many infinite open clusters for every $p \in (\dot{p}_c, 1 - \dot{p}_c)$. We show the first half of this conjecture. In the process we give tight bounds on the vertex Cheeger constant.

Let G be a plane graph with minimum degree at least $d \geq 7$, fixed throughout this section.

Lemma 3.4.1. *Fix a minimal vertex cut X separating o from infinity. Let Y be the vertices of the finite component of $G \setminus X$ which are adjacent to X, and Z be the other vertices of this component. Then $|X| \geq (d-5)|Y| + (d-6)|Z| + 5$.*

Proof. Consider the induced graph on $X \cup Y \cup Z$. We may add edges if necessary so that this is a disc triangulation with boundary X, and we may do this in such a way that the set of vertices in $G \setminus X$ which are adjacent to X is precisely Y. Notice that the vertices of X induce a cycle, and the graph induced by $Y \cup Z$ is a connected plane triangulation because X is a minimal vertex cut.

Now add a new vertex u in the unbounded face of this graph adjacent to all vertices in X to obtain a new graph H. We now have a triangulation of a sphere with $|X| + |Y| + |Z| + 1$ vertices, i.e. all faces, including the unbounded one, have degree 3. Since each edge is incident with two faces, we can easily deduce that $3F = 2E$, where F and E denote the number of faces and edges of H. Using Euler's formula we now obtain that $E = 3(|X| + |Y| + |Z|) - 3$.

Look at faces between X and Y in H, i.e. faces having two vertices from X and one from Y or vice versa. There are $|X|$ of the first type (one for each edge between vertices of X) because the vertices of X span a cycle, and at least $|Y| - 1$ of the second because there is at least one face for each edge between vertices of Y, and Y spans a connected graph (in fact we only need the -1 in the case $|Y| = 1$). When walking along X the faces between X and Y appear in a cyclic order because X spans a cycle, hence the number of edges between X and Y coincides with the number of faces between X and Y. It follows that there are at least $|X| + |Y| - 1$ edges between X and Y.

Thus we can calculate the sum of the degrees $d(v)$ of H to be

$$\sum_{v \in V(H)} d(v) \geq \sum_{v \in Y \cup Z} d(v) + |X| + |Y| - 1 + 4|X| \geq 5|X| + (d+1)|Y| + d|Z| - 1\,.$$

To see that the first inequality holds notice that the sum $\sum_{v \in X \cup \{u\}} d(v)$ counts the edges between X and Y only once and the remaining edges with at least one end-vertex in X exactly twice. Now the number of edges connecting vertices of X and the number of edges between X and u are both equal to $|X|$. In the second inequality, we used that $d(v) \geq d$. Since $\sum_{v \in V(H)} d(v)$ is equal to $2E = 6(|X| + |Y| + |Z|) - 6$ by the handshake lemma, the result follows. □

In particular, we have $(d-5)|Y| \leq |X| - 5$ and $(d-6)|Y \cup Z| \leq |X| - 5$. In fact, the second inequality can be improved.

Lemma 3.4.2. $\alpha_d |Y \cup Z| < |X| - 5$*, where*

$$\alpha_d = \frac{d - 6 + \sqrt{(d-2)(d-6)}}{2}\,.$$

In particular, $\alpha_7 \approx 1.618$ *is the golden ratio.*

Proof. We prove this by induction on $|Z|$. If $Z = \varnothing$ then the required inequality holds since $\alpha_d < d - 5$. Otherwise, Y contains $k \geq 1$ minimum vertex cuts which separate clusters of vertices in Z from infinity. Write $(Y_i)_{i=1}^k$ for the cuts (which may overlap) and $(Z_i)_{i=1}^k$ for the clusters; by Lemma 3.4.1 $|Y_i| \geq d$ for each i.

We may add edges, where necessary, between vertices of Y such that each of the cuts forms a cycle. The auxiliary graph H consisting only of these cycles has the property that all its belong to the unbounded face, by definition of Y. Notice that H has k faces bounded by the vertex cuts, one unbounded face, and possibly a few more faces that are formed when some vertex cuts overlap in a cyclic way. Let us denote by ℓ the number of the latter faces. Moreover, H has $|Y|$ vertices and its unbounded face has degree at least $|Y|$ because H is a connected graph and has at least one cycle. Since each edge of H is incident with two faces, the sum of face degrees coincides with $2e(H)$. It follows that $\sum_{i=1}^k |Y_i| \leq 2e(H) - |Y| - 3\ell$, and by Euler's formula, the right-hand side of the inequality is equal to $|Y| + 2k - \ell - 2$, implying that $\sum_{i=1}^k (|Y_i| - 2) \leq |Y| - 2$.

By the induction hypothesis, we have $\alpha_d |Z_i| \leq |Y_i| - 5$, and thus $\alpha_d |Z| = \alpha_d \sum_{i=1}^k |Z_i| \leq \sum_{i=1}^k (|Y_i| - 5) \leq |Y| - 2 - 3k \leq |Y| - 5$.

It follows that $|Y| > \frac{\alpha_d}{1+\alpha_d}(|Y \cup Z|)$, so applying Lemma 3.4.1 we get

$$\Big(d - 6 + \frac{\alpha_d}{1 + \alpha_d}\Big)|Y \cup Z| < |X| - 5\,,$$

and since $\alpha_d = d - 6 + \frac{\alpha_d}{1+\alpha_d}$ the result follows. □

It now follows that

Corollary 3.4.3. *Let W be any finite subset of $V(G)$. Then $|\partial W| \geq \alpha_d |W|$.*

Proof. We may assume every vertex in ∂W meets the infinite component of $G \setminus \partial W$, since otherwise we may find a larger set with smaller boundary. Now splitting ∂W into minimum vertex cuts surrounding clusters of W as above, and applying Lemma 3.4.2 to each cut, gives the required result. □

The following result, which shows that Lemma 3.4.2 is tight, may be of independent interest.

Theorem 3.4.4. *For each $d \geq 7$ the d-regular hyperbolic triangulation $H_{d,3}$ has vertex Cheeger constant $\dot{h}(H_{d,3}) = \alpha_d$.*

Proof. Corollary 3.4.3 immediately gives $\dot{h}(H_{d,3}) \geq \alpha_d$, and so it suffices to exhibit a sequence of sets S_n satisfying $|\partial S_n| = (\alpha_d + o(1))|S_n|$. In fact, the balls B_n have this property. Note that, for $n \geq 1$, ∂B_n forms a minimum vertex cut, the set of vertices on the external face of B_n is precisely $B_n \setminus B_{n-1}$, and the induced subgraph on this set is a cycle.

Thus, following the proof of Lemma 3.4.1, we obtain $|\partial B_n| = (d-5)|B_n \setminus B_{n-1}| + (d-6)|B_{n-1}| + 6$, or equivalently, noting that $B_n \cup \partial B_n = B_{n+1}$,

$$|B_{n+1}| - (d-4)|B_n| + |B_{n-1}| = 6\,. \tag{3.2}$$

Notice that the roots of the polynomial $x^2 - (d-4)x + 1 = 0$ are equal to $1 + \alpha_d$ and $(1+\alpha_d)^{-1}$. Standard techniques on recurrence relations imply that the solution of (3.2) is given by $|B_n| = a(1+\alpha_d)^n + b(1+\alpha_d)^{-n} + c$ for suitable constants a, b, c. Clearly in our case a has to be strictly positive, since the size of B_n converges to infinity. In particular, $|\partial B_n| = |B_{n+1}| - |B_n| = (\alpha_d + o(1))|B_n|$, as required. □

Lemma 3.4.2 therefore shows that among planar graphs with minimum degree $d \geq 7$, $H_{d,3}$ minimises the vertex Cheeger constant. This fact already implies an upper bound on the critical probability, using a result of Benjamini and Schramm [1996b] that $\dot{p}_c(G) \leq (1+\dot{h}(G))^{-1}$. However, combining these facts with our method of interfaces yields better bounds.

Theorem 3.4.5. *Let G be a plane graph without any accumulation points and with minimum degree $d \geq 7$. Then $\dot{p}_c(G) \leq \frac{2+\alpha_d}{(d-3)(1+\alpha_d)}$.*

Remark 3.4.6. *For $d = 7$, $(1+\alpha_d)^{-1} \approx 0.3820$, whereas $\frac{2+\alpha_d}{(d-3)(1+\alpha_d)} \approx 0.3455$.*

Proof. Let M, B be an outer interface and its boundary, let B° be the minimal vertex cut part of B, and fix a triangulation $T(M, B)$ as described in Section 3.3. We can unzip B as in Lemma 3.3.5 and then apply Lemma 3.4.1 to obtain

$$(d-5)|M| \leq 2|B| - |B^\circ|\,. \tag{3.3}$$

Also, applying Lemma 3.4.2 to the vertex cut B°, we have $\alpha_d(|M|+|B \setminus B^\circ|) \leq |B^\circ|$, i.e.

$$\alpha_d|M| \leq (1+\alpha_d)|B^\circ| - \alpha_d|B|\,. \tag{3.4}$$

Taking a linear combination of (3.3) and (3.4) we can cancel $|B^\circ|$ to obtain $(d-5+(d-4)\alpha_d)|M| \leq (2+\alpha_d)|B|$. Now Theorem 3.2.1 gives $\dot{p}_c(G) \leq \frac{2+\alpha_d}{(d-3)(1+\alpha_d)}$. □

3.5 Hyperbolic Quadrangulations

Let G be a plane graph without accumulation points, with no triangular faces, and with minimum degree $d \geq 5$, fixed throughout this section. While we will primarily be interested in the case where G is a quadrangulation, our results in this section apply more generally to any such graph, even though it is not necessarily possible to create a quadrangulation from such a graph by adding edges. Our first step is an analogue of Lemma 3.4.1.

Lemma 3.5.1. *Fix a minimal vertex cut X separating o from infinity. Let Y be the vertices of the finite component of $G \setminus X$ which are adjacent to X, and Z be the other vertices of this component. Then $|X| \geq (d-3)|Y| + (d-4)|Z| + 3$.*

Remark 3.5.2. *In fact, the proof gives $|X| \geq (d-3)|Y| + (d-4)|Z| + 4$ unless $|Y| = 1$.*

Proof. We may assume $|Y| > 1$ since otherwise the result is trivial. Take the induced subgraph on $X \cup Y \cup Z$ and add edges as necessary so that X is a cycle, giving a finite graph H. Note that H may have faces of degree 3. Furthermore, the unbounded face has degree $|X|$ and, by minimality of X, each other face meets X at one vertex, at two vertices with an edge between them, or not at all. Write A for the set of internal faces meeting X along an edge; note that $|A| = |X|$. Write A' for the set of faces meeting X at a single vertex. Each face in A has degree at least 3; let q be the number of faces in A of degree exactly 4, and p be the number of faces in A of degree at least 5.

Once again the sum of face degrees equals $2E$, hence we have $2E \geq |X| + 4(F - |X| - 1) + 3|A| + q + 2p = 4F - 4 + q + 2p$, where F and E are the number of faces and edges of H. By Euler's formula, $F = E - |X| - |Y| - |Z| + 2$. Thus $2E \leq 4(|X| + |Y| + |Z|) - q - 2p - 4$. Also, by the handshake lemma, $2E \geq d|Y| + d|Z| + \sum_{x\in X} d(x)$. The $|X|$ edges on the unbounded face are double-counted by this sum and there are as many edges between X and Y as faces because X induces a cycle. We can now deduce that $\sum_{x\in X} d(x) = 3|X| + |A'|$.

We now claim that $|A'| \geq |Y| - q - p$. Indeed, write $d'(y)$, $y \in Y$ for the number of edges between y and X. Since $|A| + |A'|$ coincides with the number of edges between X and Y, we have $|A| + |A'| = \sum_{y\in Y} d'(y)$. We can rewrite the latter sum as $|Y| + \sum_{y\in Y}(d'(y) - 1)$. Notice that $d'(y) > 1$ only when y is incident with a triangular face. Moreover, $d'(y) - 1$ is not less than the number of triangular faces incident with y because $|Y| > 1$ and so not all faces incident with y are triangular. Thus $\sum_{y\in Y}(d'(y) - 1) \geq |A| - p - q$, which implies that $|A'| \geq |Y| - p - q$.

Consequently we have $3|X| + (d+1)|Y| + d|Z| - q - p \leq 4(|X| + |Y| + |Z|) - q - 2p - 4$; since $p \geq 0$ the result follows. □

We next give an analogue of Lemma 3.4.2 for this setting; perhaps surprisingly, the same sequence of constants arises.

Lemma 3.5.3. *We have $\alpha_{d+2}|Y \cup Z| < |X| - 3$.*

Proof. We prove this by induction on $|Z|$. If $Z = \varnothing$ then the required inequality holds since $\alpha_{d+2} < d - 3$. Otherwise, as in the proof of Lemma 3.4.2, Y contains minimum vertex cuts $(Y_i)_{i=1}^k$ separating clusters $(Z_i)_{i=1}^k$, where $\sum_{i=1}^k |Z_i| = |Z|$ and $\sum_{i=1}^k (|Y_i| - 2) \leq |Y| - 2$.

By the induction hypothesis, we have $\alpha_{d+2}|Z_i| \leq |Y_i| - 3$, hence $\alpha_{d+2}|Z| = \sum_{i=1}^k \alpha_{d+2}|Z_i| \leq \sum_{i=1}^k (|Y_i| - 3) \leq |Y| - 2 - k \leq |Y| - 3$.

Consequently $|Y| > \frac{\alpha_{d+2}}{1+\alpha_{d+2}}(|Y \cup Z|)$, and applying Lemma 3.5.1 gives

$$\Big(d - 4 + \frac{\alpha_{d+2}}{1 + \alpha_{d+2}}\Big)|Y \cup Z| < |X| - 3\,,$$

whence the result follows since $\alpha_{d+2} = d - 4 + \frac{\alpha_d}{1+\alpha_d}$ □

Arguing as in the proof of Corollary 3.4.3, we obtain

Corollary 3.5.4. *Let W be any finite subset of $V(G)$. Then $|\partial W| \geq \alpha_{d+2}|W|$.*

Again, this result is best possible.

Theorem 3.5.5. *For each $d \geq 5$ the d-regular hyperbolic quadrangulation $H_{d,4}$ has vertex Cheeger constant $\dot{h}(H_{d,4}) = \alpha_{d+2}$.*

Proof. Again, it suffices to show that $|\partial B_n| = (\alpha_{d+2} + o(1))|B_n|$, where B_n is the ball of radius n.

Note that, for $n \geq 1$, ∂B_n forms a minimum vertex cut, the set of vertices in B_n adjacent to ∂B_n is precisely $B_n \setminus B_{n-1}$, and in the graph obtained by adding a cycle through ∂B_n to the induced subgraph on B_{n+1}, every vertex in B_n has degree d, every face meeting ∂B_n along an edge has degree 3, and every other face has degree 4. Thus, following the proof of Lemma 3.5.1, we have $q = p = 0$ and equality at every step, giving $|\partial B_n| = (d-3)|B_n \setminus B_{n-1}| + (d-4)|B_{n-1}| + 4$, or equivalently

$$|B_{n+1}| - (d-2)|B_n| + |B_{n-1}| = 4\,. \tag{3.5}$$

Again, it follows that $|B_n| = a(1+\alpha_{d+2})^n + b(1+\alpha_{d+2})^{-n} + c$ for suitable constants a, b, c with $a > 0$, and so $|\partial B_n| = |B_{n+1}| - |B_n| = (\alpha_{d+2} + o(1))|B_n|$, as required. □

Lemma 3.5.3 implies that $\dot{p}_c(G) \leq (1+\alpha_{d+2})^{-1}$ if G has all vertex degrees at least $d \geq 4$ and all face degrees at least 4. Our method yields again better bounds.

Theorem 3.5.6. *Let G be a plane graph without any accumulation points, with minimum degree $d \geq 5$ and no faces of degree 3. Then $\dot{p}_c(G) \leq \frac{(2+\alpha_{d+2})(d-2)}{(d^2-3d+1)(1+\alpha_{d+2})}$.*

Remark 3.5.7. *For $d = 5$, $(1+\alpha_{d+2})^{-1} \approx 0.3820$, whereas $\frac{(2+\alpha_{d+2})(d-2)}{(1+\alpha_{d+2})(d^2-3d+1)} \approx 0.3769$.*

Proof. Let M, B be an outer interface and its boundary, and let B° be the minimal vertex cut part of B. Delete all vertices of the infinite component of $G \setminus B$. Notice that any vertex of M belongs to a common face with a vertex of B. By adding edges if necessary, we can achieve that all faces between M and B have degree 4 or 5 and also preserve this property. Then every vertex of M has distance at most 2 from B.

We can argue as in the proof of Lemma 3.3.5 to unzip B and obtain a new graph H with at most $2|B| - |B^\circ|$ 'boundary' vertices in its unbounded face. To be more precise, we first add a set of edges, which we denote by S, to obtain the triangulation $T = T(M, B)$, and then we unzip B as in the proof of Lemma 3.3.5. This process gives rise to a correspondence between the edges of the graph obtained after unzipping B and the edges of T. The desired graph H is obtained after removing the pre-images of S. Write X for the 'boundary' vertices of H. Let Y be the set of vertices of $H \setminus X$ that are adjacent to B', and Z the remaining vertices of H. Lemma 3.5.1 implies that

$$2|B| - |B^\circ| \geq |X| \geq (d-3)|Y| + (d-4)|M \setminus Y| = |Y| + (d-4)|M|.$$

We now claim that $|Y| \geq (d-3)(|M|-|Y|)+3$. Indeed, if the graph induced by Z is connected we can apply Lemma 3.5.1 to $Y \cup Z$, noting that all vertices of $M \setminus Y$ are adjacent to some vertex of Y because M has distance at most 2 from B. If it is not connected we can split Y into minimal vertex cuts $(Y_i)_{i=1}^k$ and Z into its components $(Z_i)_{i=1}^k$, and define $M_i := (Y_i \cup Z_i) \cap M$. Then we have as above that $|Y_i| \geq (d-3)(|M_i|-|Y_i|)+3$. Arguing as in the proof of Lemma 3.4.2, we obtain that $\sum_{i=1}^k (|Y_i|-2) \leq |Y|-2$. The desired claim follows now from the fact that the sets $M_i \setminus Y_i$ partition $M \setminus Y$.

We can now deduce that $|Y| \geq \frac{d-3}{d-2}|M|$, which implies that

$$\Big(d-4+\frac{d-3}{d-2}\Big)|M| \leq 2|B|-|B^\circ|. \tag{3.6}$$

Also, applying Lemma 3.5.3 to the vertex cut B°, we have $\alpha_{d+2}(|M|+|B \setminus B^\circ|) \leq |B^\circ|$, i.e.

$$\alpha_{d+2}|M| \leq (1+\alpha_{d+2})|B^\circ| - \alpha_{d+2}|B|\,. \tag{3.7}$$

Taking a linear combination of (3.6) and (3.7) we obtain

$$\Big(\alpha_{d+2}+(1+\alpha_{d+2})\Big(d-4+\frac{d-3}{d-2}\Big)\Big)|M| \leq (2+\alpha_{d+2})|B|.$$

Using Theorem 3.2.1 we deduce that $\dot{p}_c(G) \leq \frac{(2+\alpha_{d+2})(d-2)}{(d^2-3d+1)(1+\alpha_{d+2})}$. □

Chapter 4

Square tilings

4.1 Introduction

In this chapter, we will discuss the convergence of certain discrete objects to their continuous counterparts. This is a topic with a long history (see e.g. Courant et al. [1928]; Lelong-Ferrand [1955]; Chelkak and Smirnov [2011] and the references therein) and deep connections with several areas of mathematics. In particular, this topic has recently emerged in percolation theory by the work of Smirnov [2001] on the conformal invariance of the scaling limit of critical percolation.

Thurston [1987] proposed the following method for approximating the Riemann map from a simply connected domain $\Omega \subset \mathbb{C}$ to the unit disc $\mathbb{D}$. Let $k \cdot \mathbb{T}$ denote the triangular lattice re-scaled by a factor of $k > 0$, and consider the plane graph $G_n := \Omega \cap 2^{-n} \cdot \mathbb{T}$. We can associate to G_n a circle packing (a collection of circles in the plane with disjoint interiors) so that the graph with vertex set the centres of the circles and edge set the straight lines connecting the centres of tangent circles, is isomorphic to G_n. In fact, there are more than one such circle packings. In order to fix one of them, we first add a new vertex u_n to G_n and we connect it to every vertex at the boundary of G_n. We require the circle corresponding to u_n to be the unit one. Moreover, we require the vertex closest to a given point p in Ω to correspond to the circle of the circle packing containing 0. There is a unique circle packing satisfying these properties. Consider now the map $f_n : \Omega \to \mathbb{D}$ defined by first sending the vertices of G_n to the centres of the circle packing, and then extending to the whole of Ω in a piecewise linear fashion. Thurston [1987] conjectured that f_n converges to a Riemann map from Ω to $\mathbb{D}$, and this was proved by Rodin and Sullivan [1987]. The aim of this chapter is to prove the analogous statement when circle packings are replaced by another discrete version of the Riemann mapping theorem, the square tilings of Brooks et al. [1940].

The theorem of Rodin & Sullivan has been extended in various directions. Convergence for lattices other than the triangular was proved by He and Rodin [1993], under the assumption of bounded degree. Stephenson [1996] proved that the convergence of f_n to the Riemann map is locally uniform. Doyle et al. [1994] improved the quality of convergence to convergence in C^2. He and Schramm [1996] gave an alternative proof of the convergence in C^2, and their proof works in further generality. In particular, it does not need the assumption of bounded degree of He and Rodin [1993]. Finally, for the triangular lattice, He and Schramm [1998] proved C^∞-convergence of circle packings to the Riemann map. Our result for square tilings also gives C^∞-convergence. We work with the square lattice $\mathbb{L}$ for convenience, but our proof applies to any lattice admitting a vertex-transitive action of the group $\mathbb{Z}^2$.

Theorem 4.1.1. *Consider a Jordan domain Ω in $\mathbb{C}$, and let $s_n : \Omega \to \mathbb{C}$ be defined by linear interpolation of the Brooks et al. square tiling map of $\Omega \cap 2^{-n} \cdot \mathbb{L}$. Then (s_n) converges in $C^\infty(\Omega)$ to a conformal map.*

Detailed definitions are given in the following sections, before the precise statement of Theorem 4.1.1 is given in Section 4.4.

Our result rests on a discrete version of the following remark. A classical result of Kakutani states that Brownian motion is conformally invariant [Mörters and Peres, 2010] (up to a time reparametrization that is irrelevant for our purposes). It is known that this conformal invariance is still true when the Brownian motion is reflected from $\partial\Omega$ back into Ω under the assumption that $\partial\Omega$ is in $C^{1,\alpha}$ [Pascu, 2002]. This allows one to describe a Riemann map from a Jordan domain Ω in $\mathbb{C}$ to a rectangle $H := [0, \ell] \times [0, 1]$ as follows. Consider four distinct points x_1, x_2, x_3 and x_4 in $\partial\Omega$ in clockwise ordering. These points subdivide $\partial\Omega$ into four subarcs T, R, B and L, appearing in that order along $\partial\Omega$. Let $f : \Omega \to H$ be the Riemann map mapping each x_i to a corner of H (where we tacitly use Caratheodory's theorem, see Section 4.2.2). Let ℓ denote the extremal length (see Section 4.2.2) in Ω between L and R. Then we have

Observation 4.1.2. *For every $z \in \Omega$ we have $f(z) = p_{LR} + ip_{TB}$, where p_{TB} is the probability that a Brownian motion started at z and reflected along $\partial\Omega$ will reach T before B, and similarly, p_{LR} equals the probability that a Brownian motion started at z and reflected along $\partial\Omega$ will reach L before R multiplied by ℓ.*

This follows immediately from the conformal invariance of reflected Brownian motion and the fact that the formula is correct when Ω is replaced by a rectangle

H. Observation 4.1.2 can be extended to all Jordan domains Ω by approximation by a sequence of Jordan subdomains with $C^{1,\alpha}$ boundaries. This follows from the weak convergence of the reflected Brownian motion on the subdomains to the reflected Brownian motion on Ω [Burdzy and Chen, 1998], and the converge of the corresponding conformal maps defined on the subdomains to f.

The construction of the square tilings of Brooks et al. can be thought of as a discrete variant of Observation 4.1.2 (and in fact our results can be used to obtain an alternative proof thereof), with reflected Brownian motion replaced by random walk on a mesh G_n: it assigns coordinates to vertices and edges of G_n similarly to the above function f. In contrast to circle packings where vertices correspond to circles, now edges correspond to squares. We use a compactness argument to obtain a convergent subsequence, and then verify that any limiting function f satisfies the Cauchy-Riemann equations (4.3) by noticing that s_n satisfies a discrete variant thereof. We then proceed to show that f is a bijection and determine its boundary behaviour. To this end, we first harness the combinatorial structure of our square tiling and utilise their probabilistic construction in terms of random walk on G_n. Certain complex analytic arguments are needed to determine the behaviour of f at each x_i. These properties uniquely define f and imply the convergence of the whole sequence (s_n).

The convergence of discrete functions like the ones we use to functions defined in the continuum is by no means a new idea. Courant et al. [1928] considered functions defined in discrete domains as the solutions of some discrete boundary value problems and proved convergence to their continuous counterparts. Since then several authors have considered similar approximation schemes, see e.g. Chelkak and Smirnov [2011]; Lelong-Ferrand [1955] and the references therein.

Apparently, part of the motivation for Thurston's question leading to the Rodin–Sullivan theorem came from approximating Riemann maps by computer, and he suggested an algorithm for doing so [Rodin and Sullivan, 1987, Appendix 2]. However, circle packing a given graph into a disc is a computationally challenging problem, and according to Collins and Stephenson [2003], *"In the numerical conformal mapping of plane regions, it is unlikely that circle packing can ever compete in speed or accuracy with classical numerical methods..."*. On the contrary, computing the square tiling boils down to solving a linear system of equations (Kirchhoff's laws) of size proportional to the number of vertices of the approximating graph G_n. These equations come from the probabilistic construction of the square tiling and they seem to be specific to squares. We are not yet sure to what extent our algorithm can compete with or complement existing numerical methods, but we did implement

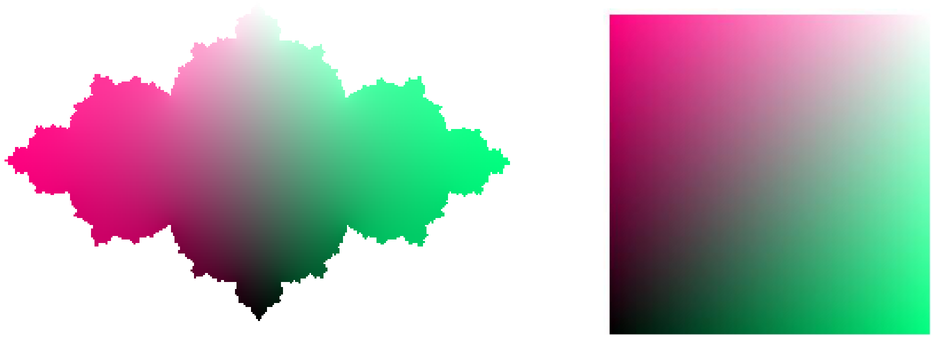

Figure 4.1: An approximation of a Riemann map between a Julia set and a square obtained by implementing our algorithm on Mathematica. Each point in one figure is the image, under the Riemann map, of the unique point in the other figure with the same colour.

it on a computer and Figure 4.1 shows an example of a resulting approximation of a Riemann map, while Figure 4.2 shows the corresponding square tiling.

According to Cannon et al. [1994], *"Riemann, in formulating his famous Riemann mapping theorem, surely relied on the physics of electrical networks and conducting metal plates for motivation."* Some biographical evidence about Riemann support this claim. He had a strong interest in the physics of electricity: *"To complete his Habilitation, Riemann had to give a lecture. He prepared three lectures, two on electricity and one on geometry.*[1] *"*. Both Riemann and Kirchhoff moved to Berlin in 1847[2], at a time when the latter was working on his laws of electricity (which we use in Section 4.3.2). Some of the ideas involved in the construction of square tilings and in our proof support the belief that the physics of electrical networks influenced Riemann in formulating his mapping theorem in his thesis in 1851. Indeed, the quantity $p_{TB} = p_{TB}(z)$ in Observation 4.1.2 coincides with the voltage $v(z)$ at z when a unit potential difference is imposed between T and B because both functions are harmonic and satisfy the same boundary conditions. The set of points z with $p_{LR}(z) = x \in (0, \ell)$ form a field line of the resulting electrical current.

4.2 Preliminaries

4.2.1 Notation

In this chapter, we will be working with the square lattice $\mathbb{L}$. Recall that its vertex set is the set of points of $\mathbb{R}^2$ with integer coordinates, and its edge-set comprises

[1] http://www-groups.dcs.st-and.ac.uk/history/Biographies/Riemann.html
[2] http://www-history.mcs.st-andrews.ac.uk/Biographies/Kirchhoff.html

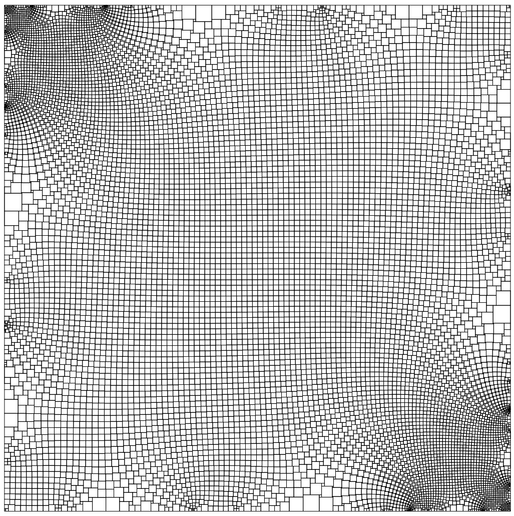

Figure 4.2: The square tiling of a mesh lying inside the Julia set of Figure 4.1. The dark regions consist of large amounts of squares.

the horizontal and vertical length 1 straight line segments connecting them. For an integer $n \geq 0$, we let $2^{-n} \cdot \mathbb{L}$ denote the plane graph obtained from $\mathbb{L}$ by multiplying the coordinates of each point by 2^{-n}. Thus each edge of $2^{-n} \cdot \mathbb{L}$ has length 2^{-n}. With a slight abuse, we denote both this graph and its vertex set by $2^{-n} \cdot \mathbb{L}$ for convenience. Notice that $\mathbb{L}$ is a *self-dual* graph, i.e. its dual graph $\mathbb{L}^*$ is isomorphic to $\mathbb{L}$.

4.2.2 Complex analytic definitions

Consider a *simply connected domain* $\Omega \subsetneq \mathbb{C}$, i.e. a connected open set such that its complement $\mathbb{C} \setminus \Omega$ is also connected. The Riemann mapping theorem states that there is a conformal map ϕ from Ω to the unit disk D, i.e. a holomorphic and injective function mapping Ω onto D. We will be working with bounded simply connected domains Ω whose boundary is a simple closed curve γ (a homeomorphic image of the unit circle). In this case, γ is called a *Jordan curve*, and Ω is called a *Jordan domain*. A homeomorphic image of the unit interval is called a *Jordan arc*.

Caratheodory studied the boundary behaviour of conformal maps, and established that ϕ witnesses the topological properties of the boundary of Ω. In particular, when Ω is a Jordan domain, Caratheodory's theorem (see e.g. Krantz [2006]; Pommerenke [1992]) states that ϕ extends to a homeomorphism between the closures $\overline{\Omega}$ and $\overline{D}$.

It follows from the Riemann mapping theorem that for every $M > 0$, there is a conformal map from Ω to the rectangle $(0, M) \times (0, 1)$. By Caratheodory's theorem, when Ω is a Jordan domain, ϕ extends to a homeomorphism between the closures $\overline{\Omega}$ and $[0, M] \times [0, 1]$. Consider now four distinct points $x_1, x_2, x_3, x_4 \in \partial\Omega$ in clockwise ordering, and let y_1, y_2, y_3, y_4 be the four corners of $[0, M] \times [0, 1]$ in clockwise ordering starting from the top left one. It is natural to ask whether there is a conformal map from Ω to $(0, M) \times (0, 1)$, with $\phi(x_i) = y_i$, $i = 1, 2, 3, 4$. As it turns out, three boundary points determine uniquely a conformal map [Pommerenke, 1992, Corollary 2.7], hence for each choice x_1, x_2, x_3, x_4 of boundary points, there is only one value of M (depending on these points) for which a conformal map with the desired property exists.

To determine the value of M, we recall the classical notion of *extremal length*. Let $\overline{x_i x_j}$ denote the arc of $\partial\Omega$ from x_i to x_j traversed in the clockwise direction. To define the extremal length between $\overline{x_1 x_2}$ and $\overline{x_3 x_4}$, given a Borel-measurable function $\rho : \Omega \to \mathbb{C}$ and a rectifiable curve γ in Ω connecting $\overline{x_1 x_2}$ to $\overline{x_3 x_4}$, we let

$$L_\rho(\gamma) := \int_\gamma \rho |dz|,$$

where $|dz|$ denotes the Euclidean element of length. We also define

$$A(\rho) := \iint_\Omega \rho^2 dx dy.$$

The extremal length between $\overline{x_1 x_2}$ and $\overline{x_3 x_4}$ is

$$\sup_\rho \inf_\gamma \frac{L_\rho(\gamma)^2}{A(\rho)},$$

where the infimum ranges over all rectifiable curves γ in Ω connecting $\overline{x_1 x_2}$ to $\overline{x_3 x_4}$, and the supremum ranges over all Borel-measurable functions $\rho : \Omega \to \mathbb{C}$ with $0 < A(\rho) < \infty$.

The extremal length between the sides $[0, M] \times \{1\}$ and $[0, M] \times \{0\}$ of the rectangle can be computed explicitly and is equal to $1/M$ [Ahlfors, 1973, p. 52-53]. Moreover, the extremal length is conformally invariant [Ahlfors, 1973, p. 52]. Therefore, M is the reciprocal of the extremal length between $\overline{x_1 x_2}$ and $\overline{x_3 x_4}$.

4.2.3 Simple random walk and electrical networks

A *walk* on G is a (possibly infinite) sequence $(v_n)_{n \in \mathbb{N}}$ of elements of V such that v_i is always connected to v_{i+1} by an edge. The *simple random walk* on G begins at some vertex and when at vertex x, traverses one of the edges $\overrightarrow{xy}$ incident to x

according to the probability distribution

$$p_{x\to y} := \frac{1}{d(x)},$$

where $d(x)$ denotes the *degree* of x.

There is a well-known correspondence between electrical networks and simple random walk. Given two vertices p and q of G, we connect a battery across the two vertices so that the voltage at p is equal to 0 and the voltage at q is equal to 1. Then certain currents will flow along the directed edges of G and establish certain voltages at the vertices of G. It is a standard fact that for every vertex u, the voltage at u is equal to the probability that the simple random walk from u visits q before p.

The physical notion of the electrical current can be defined in purely mathematical terms as follows. Let us first denote by $\overrightarrow{E}$ the set of ordered pairs (x, y) with $xy \in E$. We write $\overrightarrow{xy}$ to denote (x, y). We say that a function $f : \overrightarrow{E} \to \mathbb{R}$ is *antisymmetric*, and write $f : \overrightarrow{E} \hookrightarrow \mathbb{R}$, if $f(\overrightarrow{xy}) = -f(\overleftarrow{xy})$ for every $xy \in E$. Given two vertices p and q of G, we say that a function $f : \overrightarrow{E} \hookrightarrow \mathbb{R}$ is a *p-q flow* if it satisfies *Kirchhoff's node law*, which postulates that for every vertex x other than p and q,

$$\sum_{y\in N(x)} f(\overrightarrow{xy}) = 0,$$

where $N(x)$ denotes the set of neighbours of x, i.e. the vertices connected to x by an edge. The *p-q current* is the (unique) p-q flow $i : \overrightarrow{E} \hookrightarrow \mathbb{R}$ that satisfies *Kirchhoff's cycle law*. Kirchhoff's cycle law postulates that for every cycle $C = x_0e_{01}x_1e_{12}x_2 \ldots x_n$ in G, where the x_j are vertices, the e_{jk} are edges, and $x_n = x_0$, we have

$$\sum_{j=0}^{n-1} i(\overrightarrow{x_jx_{j+1}}) = 0.$$

The *intensity* I^* of i is the sum

$$\sum_{y\in N(x)} i(\overrightarrow{xy}).$$

The *effective resistance* R^{eff} between x and y admits several equivalent definitions, among which the most useful for us is

$$R^{\text{eff}} = 1/I^*. \tag{4.1}$$

Duffin [1962] proved that the effective resistance coincides with the notion of 'discrete extremal length'. We will utilise this fact later on.

4.2.4 Discrete partial derivatives and convergence in C^∞

Consider an integer $n \geq 0$. Any function g defined on a subset of $2^{-n} \cdot \mathbb{L}$ can be extended to the whole of $2^{-n} \cdot \mathbb{L}$ by setting $g(z) = 0$ on the remaining vertices z of $2^{-n} \cdot \mathbb{L}$. We will always assume that our functions are extended in this way to the whole of $2^{-n} \cdot \mathbb{L}$. For every vertex z of $2^{-n} \cdot \mathbb{L}$ we define the functions $\dfrac{\partial g}{\partial x}(z) := 2^n\big(g(u) - g(z)\big)$ and $\dfrac{\partial g}{\partial y}(z) := 2^n\big(g(v) - g(z)\big)$, where $u = z + 2^{-n}$ and $v = z + 2^{-n}i$. The functions $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ are called the *partial derivatives* of g with respect to x and y, respectively. For functions defined on the dual graph $\big(2^{-n} \cdot \mathbb{L}\big)^*$, the partial derivatives are defined analogously. As usually, by repeatedly applying the operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in any order and any number k of times, we define the partial derivatives of order k.

Consider now a domain $\Omega \subset \mathbb{C}$. We say that a sequence (f_n) of functions defined on $2^{-n} \cdot \mathbb{L}$ converges in $C^\infty(\Omega)$ to a smooth function $f : \Omega \to \mathbb{C}$ if for every closed disk $\Delta \subset \Omega$ and for every $k \geq 0$, the partial derivatives of f_n of order k converge uniformly in Δ to the corresponding partial derivatives of f of order k. For a sequence of functions defined on the dual graph $\big(2^{-n} \cdot \mathbb{L}\big)^*$, the definition is analogous.

4.3 Construction of the tiling

In this section, we will recall the construction of the square tiling of Brooks et al. [1940], and we will formally define the sequence (s_n) featuring in Theorem 4.1.1.

Consider a Jordan domain Ω in $\mathbb{C}$, and four distinct points x_1, x_2, x_3 and x_4 in $\partial\Omega$ in clockwise ordering. These points subdivide $\partial\Omega$ into four subarcs $T = \overline{x_1x_2}$, $R = \overline{x_2x_3}$, $B = \overline{x_3x_4}$ and $L = \overline{x_4x_1}$, where $\overline{x_ix_j}$ denotes the arc of $\partial\Omega$ from x_i to x_j traversed in the clockwise direction. We will refer to T, R, B and L as the *top, right, bottom and* left arc of $\partial\Omega$, respectively.

Without loss of generality we can assume that the origin lies in Ω. For every $n \geq 0$, we consider the subgraph of $2^{-n} \cdot \mathbb{L}$ determined by those vertices and edges lying entirely in Ω, and we define Ω_n to be the connected component of the origin in that graph. The *boundary* $\partial\Omega_n$ of Ω_n is the set of vertices z of Ω_n that are incident to an edge in $2^{-n} \cdot \mathbb{L}$ that intersects $\partial\Omega$. By rescaling Ω if necessary we can assume that for every $n \geq 0$, no pair of adjacent edges of $2^{-n} \cdot \mathbb{L}$ intersects opposite arcs of $\partial\Omega$ (T and B or R and L). In particular, no edge of $2^{-n} \cdot \mathbb{L}$ intersects opposite arcs of $\partial\Omega$. We can now define T_n, B_n to be the sets of vertices of $\partial\Omega_n$ that are incident to an edge intersecting T, B, respectively. We also define R_n, L_n to be the sets of vertices of $\partial\Omega_n \setminus (T_n \cup B_n)$ that are incident to an edge intersecting R, L, respectively. It will be useful for the construction of our tiling to identify the elements of T_n and B_n into single vertices, which we will denote by t_n and b_n, respectively. We will write $G_n = (V_n, E_n)$ for the graph obtained from Ω_n after these identifications. The *boundary* ∂G_n of G_n is the set of boundary vertices obtained after these identifications, i.e. $\partial G_n := \{t_n, b_n\} \cup R_n \cup L_n$.

4.3.1 The dual graph G_n^*

We consider G_n as a plane graph, in other words, V_n is now a set of points of $\mathbb{R}^2$ and E_n is a set of arcs in $\mathbb{R}^2$ each joining two points in V_n. The points in $\mathbb{R}^2$ occupied by the elements of V_n and E_n can be chosen in such a way that the vertices t_n and b_n are incident with the unbounded face of G_n, the position of every other vertex of V_n remains the same after the identifications, and the arcs connecting vertices of $V_n \setminus \{t_n, b_n\}$ are straight lines. It will be useful for the construction of the square tiling to associate to G_n a new graph G_n^* by slightly modifying the standard definition of the dual graph of G_n. First, let $t_n b_n$ be an arc in $\mathbb{R}^2$ connecting t_n with b_n, every interior point of which lies in the unbounded face of G_n. Consider the graph $G'_n = (V_n, E_n \cup \{t_n b_n\})$, and let $(V_n^*, E_n^* \cup \{l_n r_n\})$ be the dual graph of G'_n, where l_n is the face incident to L_n, and r_n is the face incident to R_n. Deleting the edge $l_n r_n$ we obtain the graph $G_n^* = (V_n^*, E_n^*)$.

Notice that there is a bijection $e \mapsto e^*$ from E_n to E_n^*. The orientability of the plane allows us to extend the bijection $e \mapsto e^*$ between E_n and E_n^* to a bijection between the directed edges of G_n and G_n^* in such a way that if $\overrightarrow{E}(u)$ is the set of edges incident to a vertex u of G_n directed towards u, then $\{\overrightarrow{e}^* \mid \overrightarrow{e} \in \overrightarrow{E}(u)\}$ is a cycle oriented in the counter-clockwise direction.

4.3.2 The tiling

For any vertex $u \in V(G_n)$, let $h_n(u)$ be the probability that simple random walk in G_n starting from u hits t_n before b_n. Thus $h_n(t_n) = 1$ and $h_n(b_n) = 0$. Notice that h_n is *harmonic* at every vertex in $V(G_n) \setminus \{t_n, b_n\}$, i.e.

$$h_n(u) = \frac{1}{d(u)} \sum_{v \in N(u)} h_n(v).$$

The values of h_n will be used as 'height' coordinates in the construction of the square tiling. Before defining the 'width' coordinates, let us consider the Ohm dual of h_n, namely the flow w_n given by the relation

$$w_n(\overrightarrow{xy}) = h_n(x) - h_n(y)$$

for every directed edge $\overrightarrow{xy}$ of G_n. Observe that w_n is antisymmetric, i.e. $w_n(\overrightarrow{xy}) = -w_n(\overrightarrow{yx})$, and satisfies Kirchhoff's laws.

Let us now define the functions w'_n and h'_n, the values of which will be used as 'width' coordinates. Given a directed edge $\overrightarrow{xy}$ in the dual graph G^*_n, we let

$$w'_n(\overrightarrow{xy}) = w_n(\overrightarrow{xy}^*),$$

where $\overrightarrow{xy}^*$ is the directed edge of G_n corresponding to $\overrightarrow{xy}$. It is a well-known consequence of the duality between Kirchhoff's laws in the primal and the dual graph that the function w'_n satisfies both Kirchhoff's cycle law and Kirchoff's node law. To define h'_n, set first $h'_n(l_n) = 0$. For every other vertex $z \in V(G^*_n)$, pick a path $P_z = z_0z_1 \ldots z_k$, where $z_0 = z$ and $z_k = l_n$, and let $h'_n(z) = \sum_{i<k} w'_n(\overrightarrow{z_iz_{i+1}})$. The value of $h'_n(z)$ does not depend on the choice of the path P_z because w'_n satisfies Kirchhoff's cycle law.

It is not hard to see that the pair h'_n, w'_n satisfies Ohm's law, i.e. $w'_n(\overrightarrow{xy}) = h'_n(x) - h'_n(y)$. Since w'_n satisfies Kirchhoff's node law, we deduce that h'_n is a harmonic function on the set $V(G^*_n) \setminus \{l_n, r_n\}$. Furthermore, it follows from the definition of h'_n that

$$h'_n(r_n) = I^*_n := \sum_{z \in N(t_n)} w_n(\overrightarrow{t_nz}),$$

since the directed edges $\overrightarrow{t_nz}^*$ form a directed path from r_n to l_n.

Having defined w'_n and h'_n, we can now specify the squares S_e of our square tiling indexed by the edges of G_n. Consider an edge $e = xy \in E(G_n)$ and assume that $h_n(x) \geq h_n(y)$. Then the square S_e has the form $I_e \times [h_n(x), h_n(y)]$. To define I_e, we consider the dual edge $e^* = x'y'$ of e, and we let I_e be the interval $[h'_n(x'), h'_n(y')]$, noting that $h'_n(y') \geq h'_n(x')$. For every $u \in V(G_n)$, we define

$$I_u = \cup_{e \in E(u)} I_e,$$

where $E(u)$ is the set of edges incident to u. It is easy to check that I_u is an interval. Brooks et al. [1940] proved that the interiors of the squares are disjoint and the union of the squares is the rectangle $[0, I^*_n] \times [0, 1]$. In other words, the

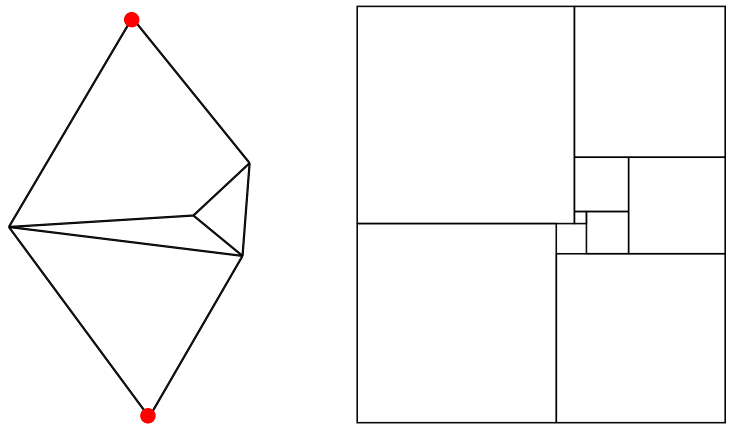

Figure 4.3: A graph and its square tiling with respect to the highlighted vertices.

collection $S = \{S_e, e \in E(G_n)\}$ is a tiling of the rectangle $[0, I_n^*] \times [0,1]$. Figure (4.3) gives an example of a square tiling. See also Benjamini and Schramm [1996a]; Georgakopoulos [2016] for a similar construction of a square tiling of a cylinder.

We remark that $h'_n(z)$ coincides with $I_n^* p_n(z)$ for every vertex z of G_n^*, where $p_n(z)$ denotes the probability that simple random walk from z visits r_n before l_n. This follows from observing that both functions are harmonic at every vertex $z \neq l_n, r_n$, and coincide at l_n and r_n because $p_n(l_n) = 0$ and $p_n(r_n) = 1$. This easily implies

Lemma 4.3.1. *The square tiling of G_n^* with respect to l_n, r_n coincides with that of G_n rotated by 90 degrees and re-scaled by $1/I_n^*$.*

4.3.3 Definition of the interpolation s_n

We will now define the functions $s_n : \Omega \to \mathbb{C}$. Given a vertex z of Ω_n that does not lie in $T_n \cup B_n$, we define the imaginary part of $s_n(z)$ to be equal to $h_n(z)$. For the real part of $s_n(z)$, let $f_1, f_2, \ldots f_k$ be the faces incident to z in G_n, where k denotes the number of such faces. The real part of $s_n(z)$ is defined to be the average horizontal coordinate $\sum_{i=1}^k h'_n(z_i)/k$. For those vertices z lying in $T_n \cup B_n$, we define $s_n(z)$ in terms of h_n and h'_n in a similar manner, except that we now replace z by t_n or b_n, as appropriate.

Notice that when I_z is not a single point, $s_n(z)$ belongs to the interior of $I_z \times \{h_n(z)\}$. To extend s_n to all of Ω, we first set s_n to be equal to 0 on the remaining vertices of $2^{-n} \cdot \mathbb{L}$, and then extend it to every point in Ω (in fact to every point in $\mathbb{C}$) by linear interpolation. Thus, if (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_2, y_1)

are the four corners of a square in $2^{-n} \cdot \mathbb{L}$ in counter-clockwise ordering with (x_1, y_1) being the bottom left one, and (x, y) is a point lying in that square, then

$$\begin{aligned} s_n(x,y) = 4^n\Big((x_2-x)(y_2-y)s_n(x_1,y_1) + (x-x_1)(y_2-y)s_n(x_2,y_1)+ \\ (x-x_1)(y-y_1)s_n(x_2,y_2) + (x_2-x)(y-y_1)s_n(x_1,y_2)\Big). \end{aligned} \tag{4.2}$$

We remark that every point $z \in \Omega$ is mapped under s_n in the rectangle $[0, I_n^*] \times [0, 1]$ since this holds for the lattice points $2^{-n} \cdot \mathbb{L}$ and the rectangle $[0, I_n^*] \times [0, 1]$ is a convex set.

4.4 Proof of main result

In this section, we prove Theorem 4.1.1. The proof is split into several smaller parts. Let us start by formulating it more precisely.

Theorem 4.4.1. *Consider a Jordan domain Ω in $\mathbb{C}$, and four distinct points x_1, x_2, x_3 and x_4 in $\partial\Omega$ in clockwise ordering. Let E be the extremal length between the arcs $\overline{x_1x_2}$ and $\overline{x_3x_4}$ of $\partial\Omega$, and y_1, y_2, y_3, y_4 be the four corners of the rectangle $[0, 1/E] \times [0, 1]$ in clockwise ordering with y_1 being the top left one. Then the sequence (s_n) converges in $C^\infty(\Omega)$ to the conformal map f mapping Ω onto the rectangle $(0, 1/E) \times (0, 1)$, with $f(x_i) = y_i$, $i = 1, 2, 3, 4$.*

4.4.1 Convergence to a holomorphic map

Since s_n is defined via h'_n and h_n, it will be useful to first establish the convergence of h'_n and h_n. The following lemma is our first step in that direction.

Lemma 4.4.2. *There is a strictly increasing sequence (k_n) of natural numbers such that both h'_{k_n} and h_{k_n} converge in $C^\infty(\Omega)$ to smooth functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$, respectively, and the intensities I_{k_n} converge to a non-negative real number I.*

For every $n \geq 0$, let D_n be the subgraph $\mathbb{L}$ spanned by the vertices (x, y) with both $|x| \leq 2^n$ and $|y| \leq 2^n$. In order to prove Lemma 4.4.2 above, we will utilise the next result about the partial derivatives of harmonic functions on D_n.

Theorem 4.4.3. *Brandt [1966]; Lawler [1991] There is a constant $C > 0$ such that for every harmonic function f on D_n we have*

$$\left|\frac{\partial f}{\partial x}(0)\right| \leq \frac{C\|f\|_\infty}{2^n} \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(0)\right| \leq \frac{C\|f\|_\infty}{2^n}.$$

Since by definition $\|h_n\|_\infty \leq 1$, Theorem 4.4.3 gives after a suitable rescaling and translation of D_n that for every closed disk $\Delta \subset \Omega$, the partial derivatives of h_n on Δ are bounded. It follows from the next lemma that h_n' is uniformly bounded too.

Lemma 4.4.4. *There is a constant $c = c(\Omega) > 0$ such that $I_n^* \leq c$ for every $n \geq 0$.*

Proof. Since I_n^* is by definition equal to the reciprocal of the effective resistance R_n^{eff} between t_n and b_n (4.1), it suffices to bound R_n^{eff} from below by a strictly positive real number.

Duffin [1962] proved that the effective resistance coincides with the *discrete extremal length*; applied to G_n, this statement becomes

$$R_n^{\text{eff}} = \max_W \min_P \frac{\left(\sum_{e \in E(P)} W_e\right)^2}{\sum_{e \in E(G_n)} W_e^2},$$

where the minimum ranges over all paths connecting t_n with b_n, and the maximum ranges over all assignments $W_e \in [0, \infty)$, $e \in E(G_n)$[3]. For every $n \geq 0$, we will assign to each edge of G_n some positive parameter $W_e = W_e(n)$, and we will show that

$$\frac{\left(\sum_{e \in E(P)} W_e\right)^2}{\sum_{e \in E(G_n)} W_e^2}$$

remains bounded from below for every path P connecting t_n with b_n. To this end, for every $e \in E(G_n)$, let $W_e = 2^{-n}$. Notice that every path P connecting t_n with b_n gives rise to a path in Ω_n connecting T_n to B_n which we will still denote by P. The sum $\sum_{e \in E(P)} W_e$ is now equal to the length of P. Hence this sum is bounded from below by the Hausdorff distance between T_n and B_n. This distance converges to the distance between the arcs T and B, which is strictly positive, showing that for every $n \geq 0$, $\left(\sum_{e \in E(P)} W_e\right)^2$ remains bounded from below by a strictly positive constant.

It remains to bound the denominator $\sum_{e \in E(G_n)} W_e^2$ from above. Associate to each edge $e \in E(G_n)$ the square of side length 2^{-n} that contains e (when viewed as an edge in Ω_n) and is dissected by e into two congruent rectangles. The area of each of these squares is equal to $W_e^2 = 4^{-n}$. It is not hard to see that the interiors of any pair of squares associated with distinct parallel edges of G_n (horizontal or

[3]The physical intuition behind this is the classical formula $R_n^{\text{eff}} = \frac{V^2}{E}$, where V is the potential difference of an electrical current and E its energy, combined with the fact that the electrical current is the energy minimiser among all functions W_e achieving a given potential difference between the source and the sink.

vertical) are disjoint. Moreover, all squares constructed in this way have distance at most 1 from a point in Ω, hence they lie in a bounded region Ω' independent of n. Therefore,

$$\sum_{e \in E(G_n)} W_e^2 \leq 2\text{area}(\Omega'),$$

where the factor 2 comes from considering the squares associated with horizontal edges and those associated with vertical edges. This completes the proof. □

Since $h'_n = I^*_n p_n$ and $\|p_n\|_\infty \leq 1$, Lemma 4.4.4 and Theorem 4.4.3 imply that for every closed disk $\Delta \subset \Omega$, the partial derivatives of h'_n on Δ are uniformly bounded. We are now ready to prove Lemma 4.4.2.

Proof of Lemma 4.4.2. The sequence (I^*_n) is bounded by Lemma 4.4.4, hence there is a subsequence $(I^*_{k_n})$ of (I^*_n) and a non-negative real number I such that I_{k_n} converges to I. Moreover, both h_{k_n} and p_{k_n} are positive harmonic functions bounded from above by 1. We will use Theorem 4.4.3 to prove that both (h_{k_n}) and (p_{k_n}) have further subsequences converging in $C^\infty(\Omega)$.

Extend the functions h_{k_n} and p_{k_n} in Ω by linear interpolation as in (4.2). Denote these extensions by H_{k_n} and P_{k_n}, respectively. Notice that for any point r lying in the interior of some horizontal edge zy of G_n, where $y = z + 2^{-n}$, we have

$$\frac{\partial H_{k_n}}{\partial x}(r) = \frac{\partial h_{k_n}}{\partial x}(z)$$

and

$$\frac{\partial P_{k_n}}{\partial x}(r) = \frac{\partial p_{k_n}}{\partial x}(z),$$

where in the left-hand side we have the standard partial derivative and in the right-hand side we have the discrete one. Similar equalities hold for the partial derivatives with respect to y. Moreover, for every point r in the interior of some square of $2^{-n} \cdot \mathbb{L}$, the partial derivatives of H_{k_n} and P_{k_n} at r are linear combinations of their partial derivatives at the boundary of the square. Theorem 4.4.3 now implies that the partial derivatives of H_{k_n} and $I^*_{k_n} P_{k_n}$ are locally bounded. Thus the sequences (H_{k_n}) and $(I^*_{k_n} P_{k_n})$ are locally bounded and equicontinuous. The Arzelà-Ascoli theorem now says that the sequences (H_{k_n}) and $(I^*_{k_n} P_{k_n})$, hence (h_{k_n}) and $(I^*_{k_n} p_{k_n})$, have further subsequences converging locally uniformly to some continuous functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$, respectively. For convenience, we will assume without loss of generality that the sequences converge along (k_n).

To deduce the convergence in $C^\infty(\Omega)$, we observe that if h is a harmonic function defined in a ball around a vertex of $2^{-n} \cdot \mathbb{L}$, then the function $g(z) =$

$h(z+2^{-n})$ is harmonic at every vertex of the ball except possibly for those at its boundary. It follows that the partial derivatives of h are also harmonic at every vertex of the ball except possibly for those at its boundary, as differences of harmonic functions. This implies that for every $k \geq 0$, all partial derivatives of h_{k_n} and p_{k_n} of order k, are harmonic functions on suitable subsets of $V(G_n)$. It is easy to prove inductively using Theorem 4.4.3 that all partial derivatives of order k of h_{k_n} and $I^*_{k_n} p_{k_n}$ are locally bounded. Arguing as above, we deduce that all partial derivatives of order k of h_{k_n} and $I^*_{k_n} p_{k_n}$ have a further subsequence that converges locally uniformly. It follows by Lemma 4.4.5 below that the limiting functions are the corresponding partial derivatives of order k of u and v, respectively. In other words, all subsequential limiting functions coincide with the corresponding partial derivatives of order k of u and v, respectively. This implies that all partial derivatives of order k of h_{k_n} and $I^*_{k_n} p_{k_n}$ converge locally uniformly to the corresponding partial derivatives of order k of u and v, respectively. This completes the proof. □

We now state the lemma mentioned in the proof of Lemma 4.4.2 above, which is an easy exercise.

Lemma 4.4.5. *Consider a sequence of piecewise continuously differentiable functions $f_n : [a,b] \to \mathbb{R}$. Assume that there are continuous functions $f, g : [a,b] \to \mathbb{R}$ such that (f_n) converges uniformly to f, and (f'_n) converges uniformly to g. Then f is differentiable with $f' = g$.*

We fix a sequence (k_n), smooth functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$, and a constant I as in Lemma 4.4.2, and we let $f : \Omega \to \mathbb{C}$ be the function defined as $f = u + iv$. In the next lemma, we show that (s_{k_n}) converges to f.

Lemma 4.4.6. *The sequence (s_{k_n}) converges in $C^\infty(\Omega)$ to f.*

Proof. Recall that for every vertex $z \in V(G_n)$, the real part of s_n is equal to $\sum_{i=1}^{k} h'_n(f_i)/k$. Every partial derivative of order j of $\sum_{i=1}^{k} h'_n(f_i)/k$ at z is a linear combination of the corresponding partial derivatives of order j of h'_n at f_i. Hence every partial derivative of order j of the real part of s_{k_n} converges locally uniformly to the corresponding partial derivative of order j of u by Lemma 4.4.2. The imaginary part of s_n is by definition equal to h_n, hence converges in $C^\infty(\Omega)$ to v. Thus we obtain the desired result. □

Our aim is to show that f is the conformal map of Theorem 4.4.1. We first show that f is holomorphic.

Lemma 4.4.7. *The function $f : \Omega \to \mathbb{C}$ is holomorphic.*

Proof. It follows from Lemma 4.4.2 that $f \in C^1(\Omega)$ (in fact $f \in C^\infty(\Omega)$). We will verify that f satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{4.3}$$

which implies that f is holomorphic.

Consider a point $z \in \Omega$ with *dyadic* coordinates $z = (k2^{-m}, l2^{-m})$, $k, l, m \in \mathbb{N}$. Notice that for every n large enough, z is occupied by a vertex of G_n of degree 4. Let $\overrightarrow{E}(z)$ be the four edges of G_n incident to z directed towards z. Let $C = C(n)$ be the dual directed cycle oriented in the counter-clockwise direction. Recall that for $\overrightarrow{rz} \in \overrightarrow{E}(z)$ we have

$$h'_n(r') - h'_n(z') = h_n(r) - h_n(z),$$

where r' and z' are the end-vertices of the dual directed edge $\overrightarrow{rz}^* \in E(C)$ of $\overrightarrow{rz}$. We can use this property to deduce that the pair h'_n, h_n satisfies the following 'discrete Cauchy-Riemann' equations

$$\frac{\partial h'_n}{\partial x}(z + (-1+i)2^{-n-1}) = \frac{\partial h_n}{\partial y}(z) \quad \text{and} \quad \frac{\partial h'_n}{\partial y}(z + (1-i)2^{-n-1}) = -\frac{\partial h_n}{\partial x}(z). \tag{4.4}$$

Taking limits along the sequence (k_n) of Lemma 4.4.2 we deduce that f satisfies the Cauchy-Riemann equations at z. The continuity of the partial derivatives of f combined with the density of the set of points with dyadic coordinates implies that f satisfies the Cauchy-Riemann equations at every point of Ω. □

4.4.2 Boundary behaviour

Consider a point $z \in \partial\Omega$. We say that $y \in \mathbb{C}$ is an f-limit point of z if there is a sequence (z_k) in Ω converging to z such that $f(z_k)$ converges to y. Write T' (resp. B', L', R') for the top (resp. bottom, left, right) side of the rectangle $[0, I] \times [0, 1]$.

Using the weak convergence of simple random walk to Brownian motion, we prove the following lemma about the f-limit points of $\partial\Omega \setminus \{x_1, x_2, x_3, x_4\}$. To be more precise, consider simple random walk on the graph $2^{-n} \cdot \mathbb{L}$, and let $S_n(t)$, $t = 0, 1, \ldots$ denote its position at time t. Extend $S_n(t)$ on the whole interval $[0, \infty)$ by linear interpolation, and define the process $W_n(t) = S_n(4^n t)$. It is an easy application of Donsker's invariance principle Mörters and Peres [2010] that W_n converges weakly in the locally uniform topology to the 2-dimensional Brownian motion W. Notice that since we are rescaling our lattice, we do not need to further scale S_n to obtain the convergence.

Lemma 4.4.8. *Consider a point $z \in \partial\Omega \setminus \{x_1, x_2, x_3, x_4\}$. If $z \in U$, where $U \in \{T, B, L, R\}$, then all f-limit points of z lie in U'.*

Proof. Consider a point $y \in \Omega$. We claim that

$$\begin{aligned} v(y) \geq \mathbb{P}_y(\tau_T = \tau_{\partial\Omega}), \quad v(y) \leq 1 - \mathbb{P}_y(\tau_B = \tau_{\partial\Omega}), \\ u(y) \geq I\mathbb{P}_y(\tau_R = \tau_{\partial\Omega}) \quad \text{and} \quad u(y) \leq I\big(1 - \mathbb{P}_y(\tau_L = \tau_{\partial\Omega})\big), \end{aligned} \tag{4.5}$$

where $\mathbb{P}_y$ denotes the probability measure of Brownian motion starting from y, and τ_S denotes the first hitting time of a set S. We will prove only the first inequality. The remaining ones follow similarly.

Assume first that y has dyadic coordinates, and let n be large enough that y is occupied by a vertex of G_n. Clearly, $h_n(y)$ is at least the probability for simple random in G_n to hit t_n before hitting $\partial G_n \setminus \{t_n\}$. Notice that simple random walk in G_n up to the fist hitting time of ∂G_n behaves like simple random in $2^{-n} \cdot \mathbb{L}$ up to the first hitting time of $\partial\Omega_n$. Hence $h_n(y)$ is at least the probability $\mathbb{P}_{n,y}(\tau_{T_n} = \tau_{\partial\Omega_n})$, where $\mathbb{P}_{n,y}$ denotes the probability measure of simple random walk in $2^{-n} \cdot \mathbb{L}$ starting from y.

We will now prove that

$$\mathbb{P}_{n,y}(\tau_{T_n} = \tau_{\partial G_n}) \text{ converges to } \mathbb{P}_y(\tau_T = \tau_{\partial\Omega}), \tag{4.6}$$

using the weak convergence in the locally uniform topology of W_n to Brownian motion. Indeed, there is a coupling of the simple random walk and Brownian motion in the same probability space, such that W_n converges almost surely to Brownian motion in the locally uniform topology by virtue of Skorokhod's representation theorem Billingsley [1999]. Notice that a priori it is possible for W_n to exit Ω_n at T_n for every n large enough, even though W exits Ω at $\partial\Omega \setminus T$. Our aim is to show that this is almost surely never the case.

To this end, let $U \in \{T, B, L, R\}$ denote the boundary arc first visited by Brownian motion, which is almost surely well-defined since W exits Ω at the set $\{x_1, x_2, x_3, x_4\}$ with probability 0. By the almost sure continuity of the Brownian paths, there is a number $\delta > 0$ such that the Hausdorff distance between the compact sets $\{W(t), t \in [0, \tau_{\partial\Omega} + \delta]\}$ and $(\partial\Omega \setminus U) \cup \{x_1, x_2, x_3, x_4\}$ is strictly positive. Hence for every n large enough, the distance between the sets $\{S_n(t), t \in [0, 4^n(\tau_{\partial\Omega} + \delta)]\}$ and $\partial\Omega_n \setminus U_n$ is strictly positive. For every point $p \in \partial\Omega$, Brownian motion from p

exits $\overline{\Omega}$ immediately, i.e.

$$\mathbb{P}_p(\inf\{t > 0 \mid W(t) \in \mathbb{C} \setminus \overline{\Omega}\} = 0) = 1$$

by Lemma 4.4.9 below. Therefore, there is some t between $\tau_{\partial\Omega}$ and $\tau_{\partial\Omega} + \delta$ such that $W(t)$ lies in the complement of $\overline{\Omega}$ by the strong Markov property. We can now deduce that for every n large enough, $S_n(t)$ lies in the complement of $\overline{\Omega}$ for some t between $4^n\tau_{\partial\Omega}$ and $4^n(\tau_{\partial\Omega} + \delta)$, hence hits $\partial\Omega_n$ before time $4^n(\tau_{\partial\Omega} + \delta)$. Consequently, $\tau_{U_n} = \tau_{\partial\Omega_n}$ for S_n when n is large enough because S_n does not hit $\partial\Omega_n \setminus U_n$ by time $4^n(\tau_{\partial\Omega} + \delta)$. This implies that the indicator of the event that W_n exits $\partial\Omega_n$ at U_n converges almost surely to the indicator of the event that W exits $\partial\Omega$ at U. Taking expectations we obtain (4.6).

Thus when y has dyadic coordinates, the desired inequality $v(y) \geq \mathbb{P}_y(\tau_T = \tau_{\partial\Omega})$ follows from the convergence of h_{k_n} to v. The continuity of both v and $\mathbb{P}_y(\tau_T = \tau_{\partial\Omega})$ gives the inequality for all y in Ω.

To obtain the assertion of the lemma, it remains to show the convergence of $\mathbb{P}_y(\tau_U = \tau_{\partial\Omega})$ to 1 as y tends to z. Consider the conformal map from Ω to $(0, M) \times (0, 1)$ of Theorem 4.4.1, where $M = 1/E$, and notice that it maps any arc $U \in \{T, B, L, R\}$ to U'. Since Brownian motion is conformally invariant Mörters and Peres [2010], it suffices to prove the assertion when Ω is the rectangle $(0, M) \times (0, 1)$, U is its top side, and z is some interior point of the top side. Write $y = y_1 + iy_2$, and let $W = W_1 + iW_2$ be our 2-dimensional Brownian motion, where W_1 and W_2 are independent 1-dimensional Brownian motions. Let also $\mathbb{P}_{W_i,y_i}$, $i = 1, 2$ denote the probability measure of W_i starting from y_i. Notice that if W_2 hits 1 before 0, and W_1 hits 0 or M after W_2 hits 1, then W exits the rectangle from the top. The probability $\mathbb{P}_{W_2,y_2}(\tau_1 < \tau_0)$ of the first event is equal to y_2 (see e.g. [Mörters and Peres, 2010, Theorem 2.45]), which converges to 1 as y tends to z. Moreover, for every $r > 0$ we have

$$\mathbb{P}_{W_2,y_2}(\tau_1 \leq r) = \mathbb{P}_{W_2,y_2}(|W_2(r)| \geq 1 - y_2) = 2\Phi\Big(\frac{1 - y_2}{\sqrt{r}}\Big)$$

$$\mathbb{P}_{W_1,y_1}(\tau_0 \leq r) = \mathbb{P}_{W_1,y_1}(|W_1(r)| \geq y_1) = 2\Phi\Big(\frac{y_1}{\sqrt{r}}\Big)$$

$$\mathbb{P}_{W_1,y_1}(\tau_M \leq r) = \mathbb{P}_{W_1,y_1}(|W_1(r)| \geq M - y_1) = 2\Phi\Big(\frac{M - y_1}{\sqrt{r}}\Big)$$

by the Reflection Principle (see [Mörters and Peres, 2010, Thoerem 2.18]), where Φ is the cumulative distribution of the standard Gaussian random variable. Choosing $r = 1 - y_2$ we obtain that $\mathbb{P}_{W_2,y_2}(|W_2(r)| \geq 1 - y_2)$ converges to 1 as y converges to

z. On the other hand, since both y_1 and $M - y_1$ remain bounded away from 0, the probabilities $\mathbb{P}_{W_1,y_1}(|W_1(r)| \geq y_1)$ and $\mathbb{P}_{W_1,y_1}(|W_1(r)| \geq M - y_1)$ converge to 0. By the union bound the probability $\mathbb{P}_{W_1,y_1}(\tau \leq r)$ converges to 0 as well, where τ is the minimum of τ_0 and τ_M. Thus with probability converging to 1 as y converges to z, W_2 hits 1 before 0, and W_1 hits 0 or M after W_2 hits 1. This proves the desired convergence. □

We now prove the lemma mentioned in the proof of Lemma 4.4.8 above.

Lemma 4.4.9. *For every $p \in \partial\Omega$ we have $\mathbb{P}_p(\inf\{t > 0 \mid W(t) \in \mathbb{C} \setminus \overline{\Omega}\} = 0) = 1$.*

Proof. Since $\partial\Omega$ is a Jordan curve, every boundary point $p \in \partial\Omega$ is regular for Brownian motion [Karatzas and Shreve, 1991, Problem 2.16], i.e.

$$\mathbb{P}_p(\inf\{t > 0 \mid W(t) \in \mathbb{C} \setminus \Omega\} = 0) = 1,$$

which is slightly weaker than the desired assertion. To remedy this, let γ be a Jordan curve passing through p with the property that every other point of γ lies in $\mathbb{C} \setminus \overline{\Omega}$. The existence of such a curve follows easily from the fact that p is accessible by an arc in $\mathbb{C} \setminus \overline{\Omega}$, since $\partial\Omega$ is a Jordan curve. Let Ω' be the bounded component of $\mathbb{C} \setminus \gamma$. We have that

$$\mathbb{P}_p(\inf\{t > 0 \mid W(t) \in \mathbb{C} \setminus \Omega'\} = 0) = 1$$

as above. Since every point of $\mathbb{C} \setminus \Omega'$ other than p lies in $\mathbb{C} \setminus \overline{\Omega}$, and Brownian motion almost surely never visits p after time 0, we obtain the desired result. □

4.4.3 Proof of injectivity

We now proceed with the proof of injectivity of f, for which we need the following lemma.

Lemma 4.4.10. *Consider a point $z \in \Omega$, and let $\Delta \subset \Omega$ be a closed disk centred at z. Let $W(n) = W(n, z, \Delta)$ be the maximum side length of a square S_e in the square tiling of G_n over those edges e with both end-vertices in Δ. Then $W(n)$ converges to 0.*

Proof. Notice that the side length of any square S_e in the square tiling of G_n is equal to either $2^{-n}\left|\frac{\partial h_n}{\partial x}(p)\right|$ or $2^{-n}\left|\frac{\partial h_n}{\partial y}(p)\right|$ for some vertex p of G_n, depending on whether e is a vertical or a horizontal edge. We know that there is a constant $C > 0$ such that $\left|\frac{\partial h_n}{\partial x}(p)\right| \leq C$ and $\left|\frac{\partial h_n}{\partial y}(p)\right| \leq C$ for every p in Δ by Theorem 4.4.3. The desired assertion follows. □

We now prove that f is injective.

Lemma 4.4.11. *The function $f : \Omega \to \mathbb{C}$ is injective.*

Proof. Suppose, to the contrary, there are points $z, y \in \Omega$ with $z \neq y$ and $f(z) = f(y) = h' + ih$ for some $h \in [0, I]$, $h' \in [0, 1]$.

$$\text{We have } h' + ih \in (0, I) \times (0, 1), \tag{4.7}$$

by (1.22) because all of $\mathbb{P}_y(\tau_T = \tau_{\partial\Omega})$, $\mathbb{P}_y(\tau_B = \tau_{\partial\Omega})$, $\mathbb{P}_y(\tau_L = \tau_{\partial\Omega})$ and $\mathbb{P}_y(\tau_R = \tau_{\partial\Omega})$ are strictly positive.

Our aim is to find a countable set $X \subset \Omega$ that accumulates to either z or y on which f is constant, as this contradicts the fact that f is a non-constant holomorphic function. The latter follows from Lemma 4.4.8.

For this, pick a sequence $(z_n)_{n\in\mathbb{N}}$ of vertices of G_n such that z_n is incident with a face or edge containing z. Pick a sequence $(y_n)_{n\in\mathbb{N}}$ of vertices of G_n^*, i.e. faces of G_n, such that y_n contains y in its closure. Let also $D_z, D_y \subset \Omega$ be two closed disks centred, at z, y, respectively. Then for every large enough n, z_n belongs to D_z, and y_n belongs to D_y. We will define a sequence of paths $P_n \subset \Omega$ in $G_n \cup G_n^*$ along which the values of f are closer and closer to $f(z) = f(y)$, and obtain the desired X as a set of accumulation points of P_n.

Lemma 4.4.2 implies that $\lim_n h_{k_n}(z_{k_n}) = h$. Let $a_n := Re(s_n(z_n))$ be the first coordinate of z_n in the interpolation s_n of the square tiling map as introduced in Section 4.3.3, and recall that a_n was defined as the average of the h'_n values of the faces incident with z_n. Therefore, by Lemma 4.4.2 combined with Lemma 4.4.10, we know that a_{k_n} converges to h'. Similarly, we have $\lim_n h'_{k_n}(y_{k_n}) = h$, and recalling that $b_n := H_n(y_n)$ is a convex combination of the h_{k_n} values of the vertices incident with y_n, we obtain $\lim_n b_{k_n} = h$ by Lemmas 4.4.2 and 4.4.10. We will assume without loss of generality that $Im(s_n(z_n)) \leq Im(s_n(y_n))$.

In order to construct P_n, we will consider two cases according to the behaviour of s_n on D_z, D_y. Consider at first the case that some I_w, $w \in \{z_n, y_n\}$ is trivial, i.e. a single point, and furthermore, that there is

$$\text{a path } P'_n \text{ in } G_n \text{ or } G_n^* \text{ connecting } w \text{ to the complement of } D_w\text{, such that } I_u \text{ is a single point for every vertex } u \text{ of } P'_n. \tag{4.8}$$

Then we let $P_n = P'_n$.

It remains to consider the case that no such path P'_n exists. We will obtain P_n by combining two paths $Q_n \subset G_n$ and $Q_n^* \subset G_n^*$. We start the construction of Q_n by the set of edges e whose square S_e in the square tiling of G_n has non-zero

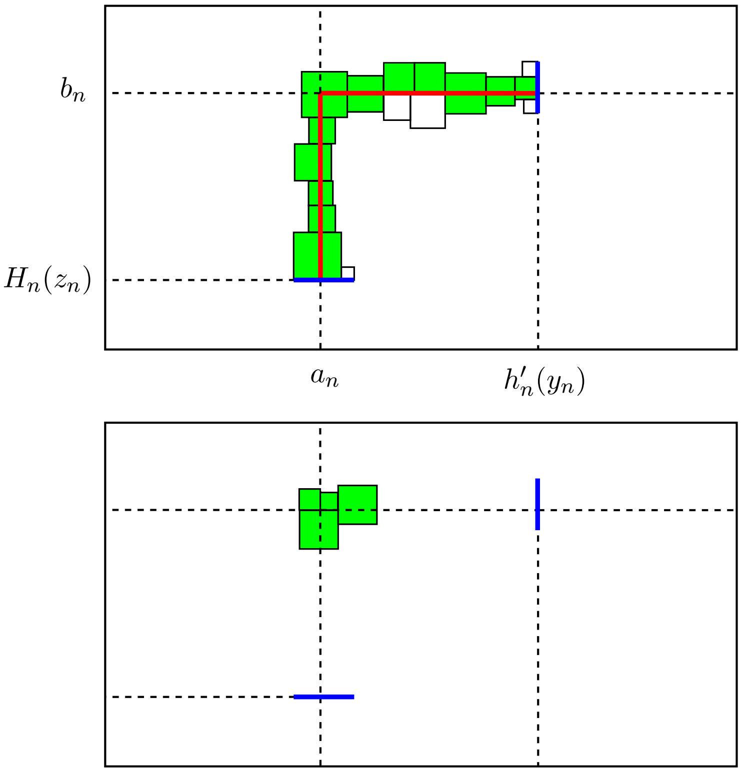

Figure 4.4: The situation in the proof of Lemma 4.4.11. The shaded squares depict the constructed path P_n. The top figure corresponds to the simpler case where (a_n, b_n) lies in the interior of a square of the tiling. The bottom figure shows how P_n is adapted locally in the other case.

width, and contains a point —possibly at its boundary— with coordinates (a_n, η) with $\eta \in [h_n(z_n), b_n]$. If for some value of η there are two such edges e, f, which can only be the case when both S_e, S_f are tangent on either side the vertical line L with first coordinate a_n, we only keep the edge that lies on the left of L. We let E_n denote the set of edges thus obtained. We remark that

E_n must contain an edge e such that S_e contains a point (ζ, η) with $\zeta < a_n$ and $\eta > b_n$, (4.9)

because some S_e contains (a_n, b_n) as well as such a point (see Figure 4.4).

Next, we claim that E_n spans a path of G_n. Indeed, we can linearly order the edges $e \in E_n$ by the second coordinates in S_e, and note that any two consecutive edges e_1, e_2 in this ordering share a vertex v, namely one with $h(v)$ coinciding with the vertical coordinate of $S_{e_1} \cap S_{e_2}$. We let Q'_n denote this path.

We take $Q_n = Q'_n$ if the latter happens to start at z_n, but this will fail if the interval I_{z_n} is a single point $I_{z_n} = \{a_n\}$. In this case, we let Z_n be the connected component of z_n in G_n consisting of vertices u such that I_u is trivial, and let ∂Z_n denote the set of those vertices of $G_n \setminus Z_n$ that have a neighbour in Z_n. It is not hard to see that $I_u = I_{z_n}$ for every $u \in V(Z_n)$. Moreover, for every vertex v of ∂Z_n we have that I_v is non-trivial and contains a_n, hence there is an edge e incident with v such that the square S_e is non-trivial, and contains $s_n(z_n)$ as well as a point with larger second coordinate than $s_n(z_n)$. Now our assumption that no path P'_n as defined in (4.8) exists, implies that Z_n lies in D_z. Thus for every large enough n, Z_n has at least two boundary vertices, and so there is a boundary vertex q_n and an edge e incident with q_n such that S_e is non-trivial, and contains $s_n(z_n)$ as well as a point (c, d) with $c < a_n$ and $d > Im(s_n(z_n))$. Pick now a path R_n in $Z_n \cup \{q_n\}$ connecting z_n to q_n. Then $Q_n := R_n \cup Q'_n$ is a path starting at z_n, and we have completed the first half of the definition of P_n.

The other half is now easy: we recall that the square tiling of G^*_n with respect to l_n, r_n coincides with that of G_n rotated by 90 degrees and re-scaled by $1/I^*_n$ by Lemma 4.3.1, and repeat the same construction with the role of z_n and y_n interchanged, to obtain the path $Q^*_n \subset G^*_n$ starting at y_n.

We claim that Q_n, Q^*_n intersect when viewed as subsets of Ω. Indeed, they both traverse the unique edge e such that S_e contains the point (a_n, b_n) as well as a point (ζ, η) with $\zeta > a_n, \eta > b_n$. Therefore, $Q_n \cup Q^*_n$ contains a z_n–y_n arc in Ω, which is our P_n. Write L_n for the curve in $(0, I) \times (0, 1)$ connecting $s_n(z_n)$ to $s_n(y_n)$ that lies in the union of the lines $x = a_n$ and $y = b_n$.

We can assume without loss of generality that in both cases Q_n contains the vertex z_n. Consider now a positive integer m, and let Δ_m denote the closed annulus centred at z of radii $1/2m$ and $1/(2m+1)$. Notice that for every m large enough, Δ_m lies entirely in D_z, and y_n lies in the unbounded component of $\mathbb{C} \setminus \Delta_m$. Moreover, for every n large enough, z_n lies in the bounded component of $\mathbb{C} \setminus \Delta_m$. Therefore, for every such m and n, there is a point of Q_n contained in Δ_m. Pick such a point and denote it by $x_n(m)$. We can choose $x_n(m)$ to be a vertex of either G_n or G^*_n.

For every fixed m, the sequence $(x_n(m))_{n \in \mathbb{N}}$ has an accumulation point in Δ_m, which we denote by $x(m)$. Let X be the set of all $x(m)$. Notice that all $x(m)$ are pairwise distinct because the annuli Δ_m are by definition disjoint, and furthermore, $x(m)$ converges to z as $m \to \infty$.

By construction, for every fixed m, the values of s_{k_n} at $x_{k_n}(m)$ are close to $f(z) = f(y)$: the points $s_{k_n}(z_{k_n})$ and $s_{k_n}(y_{k_n})$ converge to $f(z) = f(y)$, and the coordinates of every point of L_{k_n} are bounded from above and below by the coordinates of $s_{k_n}(z_{k_n})$ and $s_{k_n}(y_{k_n})$. Therefore, the points of L_{k_n} converge uniformly to $f(z) = f(y)$. Furthermore, the distance between $s_{k_n}(x_{k_n}(m))$ and L_{k_n} converges to 0 by Lemma 4.4.10, hence $s_{k_n}(x_{k_n}(m))$ converge to $f(z) = f(y)$. Since s_{k_n} converges uniformly in Δ_m to f, we have that $f(x(m))$ is an accumulation point of the sequence $\big(s_{k_n}(x_{k_n}(m))\big)_{n\in\mathbb{N}}$. Thus, $f(x(m)) = f(z) = f(y)$. This proves that f is constant on X, as desired. □

4.4.4 Behaviour at the designated boundary points

We will now determine the behaviour of f near x_1, x_2, x_3 and x_4. The proof of the next lemma is based purely on the boundary behaviour of f at $\partial\Omega \setminus \{x_1, x_2, x_3, x_4\}$, and the fact that f is a conformal map.

Lemma 4.4.12. *For each $i = 1, 2, 3, 4$, the only f-limit point of x_i is y_i.*

Proof. Let us assume without loss of generality that $i = 1$. Consider a Riemann map ϕ from the open unit disk D onto Ω, and recall that ϕ extends to a homeomorphism between their closures $\overline{D}$ and $\overline{\Omega}$ by Caratheodory's theorem. Let $X_1 = \phi^{-1}(x_1)$, and define $g = f \circ \phi$, which is a conformal map, as f is conformal by Lemma 4.4.7 and Lemma 4.4.11. For each $r > 0$, consider the curve $C(r) = D \cap \{z \in \mathbb{C} \mid |z - X_1| = r\}$, and let $l(r)$ be the length of the curve $g\big(C(r)\big)$. Write $t(r)$ for the set $\{t \in [0, 2\pi] \mid X_1 + re^{it} \in D\}$ on which the standard parametrization $X_1 + re^{it}$ of $C(r)$ is defined. Since g is a conformal map, the integral

$$\int_0^{1/2} \frac{l(r)^2}{r} dr$$

is finite. This follows from applying the Cauchy-Schwarz inequality to the formula

$$l(r) = \int_{t(r)} |g'(X_1 + re^{it})| r dt,$$

and then using the fact that

$$\iint_D |g'(z)|^2 dxdy = \text{area}(g(D)),$$

which is finite; see e.g. [Krantz, 2006, Lemma 5.1.3] for a detailed proof. On the other hand, the function $1/r$ is not integrable at 0, hence there is a sequence (r_k) of strictly positive real numbers converging to 0, such that the sequence $\big(l(r_k)\big)$ converges to 0 as well.

For every k large enough, $l(r_k)$ is in particular finite, implying that $g(C(r_k))$ extends to a continuous curve γ_k defined on the closure of $t(r_k)$, which clearly has the same length as $g(C(r_k))$. Furthermore, for every k large enough, one of the two endpoints of $C(r_k)$ lies in $\phi^{-1}(T)$, while the other lies in $\phi^{-1}(L)$. Hence one of the two endpoints of γ_k lies in T', while the other lies in L' by Lemma 4.4.8. Notice that the endpoints of γ_k may possibly coincide, in which case they coincide with y_1, but the curve is otherwise injective. Consequently, γ_k is either a Jordan arc or a Jordan curve. In both cases, γ_k divides $(0, I) \times (0, 1)$ into two components S_1 and S_2. We claim that one of them, say S_1, decreases to the empty set as $k \to \infty$. Indeed, if γ_k is a Jordan curve, then we let S_1 be the component bounded by γ_k, and otherwise, we let S_1 be the component whose boundary contains y_1. Since the length of γ_k converges to 0, the distance of its endpoints converges to 0 as well, which is possible only when the endpoints of γ_k converge to y_1. In both cases, the distance of each of the boundary points of S_1 from y_1 converges uniformly to 0. Our claim now follows easily.

Clearly, $C(r_k)$ divides D into two components C_1 and C_2 as well, with C_1 decreasing to the empty set and C_2 increasing to D as $k \to \infty$. Since g is injective, one of the sets $g(C_1)$, $g(C_2)$ lies in S_1, while the other lies in S_2. To decide which one lies in S_1, notice that $g(C_2)$ increases to $g(D)$. Therefore, $g(C_2)$ cannot lie in S_1 when k is large enough, hence $g(C_1)$ lies in S_1 for those k. This implies that all possible g-limit points of X_1, hence all possible f-limit points of x_1, belong to the closure of S_1. The closure of S_1 decreases to $\{y_1\}$ as we have seen, and the desired assertion follows. □

4.4.5 Completing the proof

We are now ready to prove Theorem 4.4.1.

Proof of Theorem 4.4.1. It follows from Lemma 4.4.7 and Lemma 4.4.11 that f is a conformal map. Moreover, f maps open sets to open sets by the Open mapping theorem for holomorphic functions. This shows that $f(\Omega)$ is an open set, and furthermore the only boundary points of $f(\Omega)$ are the f-limit points of $\partial\Omega$. We can now deduce from Lemma 4.4.8 and Lemma 4.4.12 that the boundary of $f(\Omega)$ lies at the boundary of the rectangle $[0, I] \times [0, 1]$. The set $f(\Omega)$ lies in $[0, I] \times [0, 1]$ because $s_n(\Omega)$ lies in $[0, I_n^*] \times [0, 1]$ for every $n \geq 0$, and s_{k_n} converges to f by Lemma 4.4.6. There is a unique set satisfying the aforementioned properties of $f(\Omega)$, namely $(0, I) \times (0, 1)$. Therefore, f maps Ω onto $(0, I) \times (0, 1)$.

Since $\partial\Omega$ is a Jordan curve, f extends to a homeomorphism between $\overline{\Omega}$ and $[0, I] \times [0, 1]$ by Caratheodory's theorem, and maps x_i, $i = 1, 2, 3, 4$ to y_i by Lemma 4.4.12. As mentioned in Section 4.2.2, I equals the reciprocal of the extremal length between T and B, and the above properties uniquely determine f.

Consequently, all subsequential limits of s_n coincide with f. Hence s_n converges in $C^\infty(\Omega)$ to f, as desired. □

Since the only limit point of (I_n^*) is I, we obtain that I_n^* converges to I. Moreover, I is the reciprocal of the extremal length between T and B by the discussion in Section 4.2.2, and $R_n^{eff} = 1/I_n^*$. As a corollary, we obtain

Corollary 4.4.13. *The effective resistance R_n^{eff} between T_n and B_n converges to the extremal length between T and B.*

Bibliography

L. H. Ahlfors. *Conformal invariants: topics in geometric function theory.* McGraw-Hill, 1973.

M. Aizenman and D. J. Barsky. Sharpness of the phase transition in percolation models. *Communications in Mathematical Physics*, 108:489–526, 1987.

M. Aizenman and C. M. Newman. Tree Graph Inequalities and Critical Behavior in Percolation Models. *Journal of Statistical Physics*, 36:107–143, 1984.

M. Aizenman and C. M. Newman. Discontinuity of the percolation density in one dimensional $1/|x-y|^2$ percolation models. *Communications in Mathematical Physics*, 107(4):611–647, 1986.

M. Aizenman and H. Kesten C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Communications in Mathematical Physics*, 111(4):505–531, 1987.

M. Aizenman, F. Delyon, and B. Souillard. Lower bounds on the cluster size distribution. *Journal of Statistical Physics*, 23(3):267–280, 1980.

O. Angel, I. Benjamini, and N. Horesh. An isoperimetric inequality for planar triangulations. *Discrete & Computational Geometry*, pages 1–8, 2018.

T. Antunović and I. Veselić. Sharpness of the Phase Transition and Exponential Decay of the Subcritical Cluster Size for Percolation on Quasi-Transitive Graphs. *Journal of Statistical Physics*, 130:983–1009, 2008.

E. Babson and I. Benjamini. Cut sets and normed cohomology with applications to percolation. *Proceedings of the American Mathematical Society*, 127(2):589–597, 1999.

A. Bandyopadhyay, J. Steif, and Á. Timár. On the cluster size distribution for percolation on some general graphs. *Revista Matemática Iberoamericana*, 26(2): 529–550, 2010.

G. Barequet and M. Shalah. Improved upper bounds on the growth constants of polyominoes and polycubes. *arXiv:1906.11447*, 2019.

G. Barequet, R. Barequet, and G. Rote. Formulae and growth rates of high-dimensional polycubes. *Combinatorica*, 30(3):257–275, 2010.

G. Barequet, G. Rote, and M. Shalah. An improved upper bound on the growth constant of polyiamonds. In *European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2019), Bratislava, August 2019, Editors: J. Nesetril and M. Skoviera (to appear)*, 2019a.

G. Barequet, M. Shalah, and Y. Zheng. An improved lower bound on the growth constant of polyiamonds. *Journal of Combinatorial Optimization*, 37(2):424–438, 2019b.

I. Benjamini. Percolation and Coarse Conformal Uniformization. *arXiv:1510.05196*, 2015.

I. Benjamini and O. Schramm. Random walks and harmonic functions on infinite planar graphs using square tilings. *The Annals of Probability*, 24(3):1219–1238, 1996a.

I. Benjamini and O. Schramm. Percolation beyond $\mathbb{Z}^d$, many questions and a few answers. *Electronic Communications in Probability*, 1:71–82, 1996b.

I. Benjamini and O. Schramm. Percolation in the hyperbolic plane. *Journal of the American Mathematical Society*, 14(2):487–507, 2001.

I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Critical percolation on any nonamenable group has no infinite clusters. *The Annals of Probability*, pages 1347–1356, 1999.

P. Billingsley. *Convergence of probability measures.* Wiley, 1999.

B. Bollobás and O. Riordan. A short proof of the Harris–Kesten theorem. *Bulletin of the London Mathematical Society*, 38(3):470–484, 2006.

B. Bollobas and O. Riordan. Percolation on dual lattices with k-fold symmetry. *Random Structures and Algorithms*, 32:463–472, 2008.

G. A. Braga, A. Proccaci, and R. Sanchis. Analyticity of the d-Dimensional Bond Percolation Probability Around $p = 1$. *Journal of Statistical Physics*, 107:1267–1282, 2002.

G. A. Braga, A. Procacci, R. Sanchis, and B. Scoppola. Percolation Connectivity in the Highly Supercritical Regime. *Markov Processes And Related Fields*, 10(4): 607–628, 2004.

A. Brandt. Estimates for difference quotients of solutions of Poisson type difference equations. *Mathematics of Computation*, 20(96):473–499, 1966.

R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. *Duke Mathematical Journal*, 7(1):312–340, 1940.

K. Burdzy and Z. Chen. Weak convergence of reflected brownian motions. *Electronic Communications in Probability*, 3(4):29–33, 1998.

R. M. Burton and M. Keane. Density and uniqueness in percolation. *Communications in Mathematical Physics*, 121(3):501–505, 1989.

J. W. Cannon, W. J. Floyd, and W. R. Parry. Squaring rectangles: The finite Riemann mapping theorem. Abikoff, William (ed.), The mathematical legacy of Wilhelm Magnus. Contemp. Math. 169, 1994.

J. T. Chayes, L. Chayes, and C. M. Newman. Bernoulli percolation above threshold: an invasion percolation analysis. *The Annals of Probability*, pages 1272–1287, 1987.

D. Chelkak and S. Smirnov. Discrete complex analysis on isoradial graphs. *Advances in Mathematics*, 228(3):1590–1630, 2011.

C. R. Collins and K. Stephenson. A circle packing algorithm. *Computational Geometry*, 25(3):233–256, 2003.

R. Courant, K. Friedrichs, and H. Lewy. Über die partiellen Differenzengleichungen der mathematischen Physik. *Mathematische annalen*, 100(1):32–74, 1928.

F. Delyon. *Taille, forme et nombre des amas dans les problemes de percolation, These de 3eme cycle*. Universite Pierre et Marie Curie, Paris, 1980.

P. Doyle, Z. X. He, and B. Rodin. Second derivatives of circle packings and conformal mappings. *Discrete & Computational Geometry*, 11(1):35–49, 1994.

R. J. Duffin. The extremal length of a network. *Journal of Mathematical Analysis and Applications*, 5(2):200–215, 1962.

H. Duminil-Copin. Sixty years of percolation. *arXiv:1712.04651*, 2017.

H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, 343(2):725–745, 2016.

H. Duminil-Copin, S. Goswami, A. Raoufi, F. Severo, and Ariel A. Yadin. Existence of phase transition for percolation using the Gaussian Free Field. *arXiv:1806.07733*, 2018.

H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. *Annals of Mathematics*, 189 (1):75–99, 2019.

M. Eden. A Two-dimensional Growth Process. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, volume 4, pages 223–239, 1961.

P. Erdos. On an elementary proof of some asymptotic formulas in the theory of partitions. *Annals of Mathematics*, pages 437–450, 1942.

M. E. Fisher. Critical probabilities for cluster size and percolation problems. *Journal of Mathematical Physics*, 2(4):620–627, 1961.

D. S. Gaunt and P. J. Peard. $1/d$-expansions for the free energy of weakly embedded site animal models of branched polymers. *Journal of Physics A: Mathematical and General*, 33(42):7515–7539, 2000.

D. S. Gaunt and H. Ruskin. Bond percolation processes in d dimensions. *Journal of Physics A: Mathematical and General*, 11(7):1369–1380, 1978.

D. S. Gaunt, M. F. Sykes, and H. Ruskin. Percolation processes in d-dimensions. *Journal of Physics A: Mathematical and General*, 9(11):1899–1911, 1976.

D. S. Gaunt, M. F. Sykes, G.M. Torrie, and S. G. Whittington. Universality in branched polymers on d-dimensional hypercubic lattices. *Journal of Physics A: Mathematical and General*, 15(10):3209–3217, 1982.

A. Georgakopoulos. The boundary of a square tiling of a graph coincides with the Poisson boundary. *Inventiones mathematicae*, 203(3):773–821, 2016.

A. Georgakopoulos and J. Haslegrave. Percolation on an infinitely generated group. *arXiv:1703.09011*, 2017.

A. Georgakopoulos and C. Panagiotis. Analyticity results in Bernoulli percolation. *arXiv:1811.07404*, 2018.

A. Georgakopoulos and C. Panagiotis. On the exponential growth rates of lattice animals and interfaces, and new bounds on p_c. *arXiv:1908.03426*, 2019a.

A. Georgakopoulos and C. Panagiotis. Convergence of square tilings to the Riemann map. *arXiv:1910.06886*, 2019b.

A. Georgakopoulos and C. Panagiotis. Analyticity of the percolation density θ in all dimensions. *arXiv:2001.09178*, 2020.

R. Griffiths. Nonanalytic behavior above the critical point in a random Ising ferromagnet. *Physical Review Letters*, 23(1):17, 1969.

G. Grimmett. On the differentiability of the number of clusters per vertex in the percolation model. *Journal of the London Mathematical Society*, 2(2):372–384, 1981.

G. Grimmett. Percolation and disordered systems. In *Lectures on probability theory and statistics*, pages 153–300. 1997.

G. Grimmett. *Percolation,* Second Edition. Grundlehren der mathematischen Wissenschaften. Springer, 1999.

G. Grimmett and Z. Li. Bounds on connective constants of regular graphs. *Combinatorica*, 35(3):279–294, 2015.

G. Grimmett and J. Marstrand. The supercritical phase of percolation is well behaved. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 430(1879):439–457, 1990.

G. Grimmett and A. Stacey. Critical probabilities for site and bond percolation models. *The Annals of Probability*, 26(4):1788–1812, 1998.

J. C. Gupta and B. V. Rao. van den Berg-Kesten inequality for the Poisson Boolean model for continuum percolation. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 337–346, 1999.

O. Häggström and Y.Peres. Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probability Theory and Related Fields*, 113(2):273–285, 1999.

O. Häggström, J. Jonasson, and Russell R. Lyons. Explicit isoperimetric constants and phase transitions in the random-cluster model. *The Annals of Probability*, 30 (1):443–473, 2002.

A. Hammond. Critical exponents in percolation via lattice animals. *Electronic Communications in Probability*, 10:45–59, 2005.

T. Hara and G. Slade. The self-avoiding-walk and percolation critical points in high dimensions. *Combinatorics, Probability and Computing*, 4(3):197–215, 1995.

G. H. Hardy and S. Ramanujan. Asymptotic Formulae in Combinatory Analysis. *Proceedings of the London Mathematical Society*, 17:75–115, 1918.

A. B. Harris. Renormalized $(1/\sigma)$ expansion for lattice trees and localization. *Physical Review B*, 26(1):337–366, 1982.

T. E. Harris. A lower bound for the critical probability in a certain percolation process. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 56, pages 13–20, 1960.

J. Haslegrave and C. Panagiotis. Site percolation and isoperimetric inequalities for plane graphs. *Random Structures and Algorithms (to appear)*, 2020.

Z. X. He and B. Rodin. Convergence of circle packings of finite valence to Riemann mappings. *Communications in Analysis and Geometry*, 1(1):31–41, 1993.

Z. X. He and O. Schramm. Hyperbolic and parabolic packings. *Discrete & Computational Geometry*, 14(2):123–149, 1995.

Z. X. He and O. Schramm. On the convergence of circle packings to the Riemann map. *Inventiones mathematicae*, 125(2):285–305, 1996.

Z. X. He and O. Schramm. The C^∞-convergence of hexagonal disk packings to the Riemann map. *Acta Mathematica*, 180(2):219–245, 1998.

J. Hermon and T. Hutchcroft. Supercritical percolation on nonamenable graphs: Isoperimetry, analyticity, and exponential decay of the cluster size distribution. *arXiv:1904.10448*, 2019.

R. Van Der Hofstad and G. Slade. Expansion in n^{-1} for Percolation Critical Values on the n-cube and $\mathbb{Z}^n$: the First Three Terms. *Combinatorics, Probability and Computing*, 15(5):695–713, 2006.

T. Hutchcroft. Critical percolation on any quasi-transitive graph of exponential growth has no infinite clusters. *Comptes Rendus Mathematique*, 354(9):944–947, 2016.

J. L. Jacobsen. High-precision percolation thresholds and Potts-model critical manifolds from graph polynomials. *Journal of Physics A: Mathematical and Theoretical*, 47(13):135001+78, 2014.

W. Kager, M. Lis, and R. Meester. The signed loop approach to the Ising model: foundations and critical point. *Journal of Statistical Physics*, 152(2):353–387, 2013.

I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 1991.

H. Kesten. The critical probability of bond percolation on the square lattice equals 1/2. *Communications in Mathematical Physics*, 74(1):41–59, 1980.

H. Kesten. Analyticity Properties and Power Law Estimates of Functions in Percolation Theory. *Journal of Statistical Physics*, 25(4):717–756, 1981.

H. Kesten. *Percolation theory for mathematicians.* Springer, 1982.

H. Kesten and Y. Zhang. The probability of a large finite cluster in supercritical Bernoulli percolation. *The Annals of Probability*, 18(2):537–555, 1990.

D. A. Klarner. Cell growth problems. *Canadian Journal of Mathematics*, 19:851–863, 1967.

D. A. Klarner and R. L. Rivest. A procedure for improving the upper bound for the number of n- ominoes. *Canadian Journal of Mathematics*, 25(3):585–602, 1973.

D.J. Klein. Rigorous results for branched polymer models with excluded volume. *The Journal of Chemical Physics*, 75(10):5186–5189, 1981.

S. G. Krantz. *Geometric function theory: explorations in complex analysis.* 2006.

H. Kunz and B. Souillard. Essential Singularity in Percolation Problems and Asymptotic Behavior of Cluster Size Distribution. *Journal of Statistical Physics*, 19(1): 77–106, 1978.

G. Last, M. D. Penrose, and S. Zuyev. On the capacity functional of the infinite cluster of a Boolean model. *The Annals of Applied Probability*, 27(3):1678–1701, 2017.

G. F. Lawler. Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proceedings of the London Mathematical Society*, 3(3):552–568, 1991.

J. Lelong-Ferrand. *Représentation conforme et transformations à intégrale de Dirichlet bornée*, volume 22. Gauthier-Villars, 1955.

R. Lyons and Y. Peres. *Probability on trees and networks*, volume 42. Cambridge University Press, 2016.

D. MacDonald, S. Joseph, D. L. Hunter, L. L. Moseley, N. Jan, and A. J. Guttmann. Self-avoiding walks on the simple cubic lattice. *Journal of Physics A: Mathematical and General*, 33(34):5973–5983, 2000.

R. Meester and R. Roy. *Continuum Percolation*, volume 119. Cambridge University Press, 1996.

M. V. Menshikov. Coincidence of critical-points in the percolation problems. *Doklady akademii nauk sssr*, 288(6):1308–1311, 1986.

Y. M. Miranda and G. Slade. The growth constants of lattice trees and lattice animals in high dimensions. *Electronic Communications in Probability*, 16:129–136, 2011.

Y. M. Miranda and G. Slade. Expansion in high dimension for the growth constants of lattice trees and lattice animals. *Combinatorics, Probability and Computing*, 22(4):527–565, 2013.

P. Mörters and Y. Peres. *Brownian motion.* Cambridge University Press, 2010.

L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Physical Review*, 65(3-4):117, 1944.

M. Pascu. Scaling coupling of reflecting Brownian motions and the hot spots problem. *Transactions of the American Mathematical Society*, 354(11):4681–4702, 2002.

P. J. Peard and D. S. Gaunt. $1/d$-expansions for the free energy of lattice animal models of a self-interacting branched polymer. *Journal of Physics A: Mathematical and General*, 28(21):6109–6124, 1995.

M. D. Penrose. *Random geometric graphs.* Number 5. Oxford university press, 2003.

G. Pete. A note on percolation on $\mathbb{Z}^d$: Isoperimetric profile via exponential cluster repulsion. *Electronic Communications in Probability*, 13:377–392, 2008.

C. Pommerenke. *Boundary Behaviour of Conformal Maps.* Springer-Verlag, 1992.

B. M. I. Rands and D. J. A. Welsh. Animals, trees and renewal sequences. *IMA Journal of Applied Mathematics*, 27(1):1–18, 1981.

D. Reimer. Proof of the van den berg–kesten conjecture. *Combinatorics, Probability and Computing*, 9(1):27–32, 2000.

R. T. Rockafellar. *Convex analysis.* Princeton University Press, 2015.

B. Rodin and D. Sullivan. The convergence of circle packings to the Riemann mapping. *Journal of Differential Geometry*, 26(2):349–360, 1987.

W. Rudin. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1964.

L. Russo. On the critical percolation probabilities. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 56(2):229–237, 1981.

S. Sheffield. *Random Surfaces.* Société mathématique de France, 2005.

S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences-Series I- Mathematics*, 333(3):239–244, 2001.

K. Stephenson. A probabilistic proof of Thurston's conjecture on circle packings. *Rendiconti del Seminario Matematico e Fisico di Milano*, 66(1):201–291, 1996.

M. F. Sykes and J. W Essam. Exact critical percolation probabilities for site and bond problems in two dimensions. *Journal of Mathematical Physics*, 5(8):1117–1127, 1964.

C. Thomassen. Straight line representations of infinite planar graphs. *Journal of the London Mathematical Society*, 2(3):411–423, 1977.

W. Thurston. The finite Riemann mapping theorem. In *Invited talk, An International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture*, 1987.

Á. Timár. Cutsets in Infinite Graphs. *Combinatorics, Probability and Computing*, 16:159–166, 2007.

J. van den Berg and H. Kesten. Inequalities with applications to percolation and reliability. *Journal of applied probability*, 22(3):556–569, 1985.

A. C. D. van Enter. Griffiths Singularities. In *Modern Encyclopedia of Mathematical Physics*, 2007.

A. C. D. van Enter, R. Fernández, R. H. Schonmann, and S. B. Shlosman. Complete analyticity of the 2D Potts model above the critical temperature. *Communications in mathematical physics*, 189(2):373–393, 1997.

S. G. Whittington and C. E. Soteros. Lattice animals: Rigorous results and wild guesses. In *Disorder in Physical Systems*, pages 323–335. Clarendon Press, 1990.