

## DOCTOR OF PHILOSOPHY

### Optimal control of functional differential systems with application to transmission lines

Davies, Iyai

*Award date:*  
2015

*Awarding institution:*  
Coventry University

[Link to publication](#)

#### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of this thesis for personal non-commercial research or study
- This thesis cannot be reproduced or quoted extensively from without first obtaining permission from the copyright holder(s)
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

#### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

# **OPTIMAL CONTROL OF FUNCTIONAL DIFFERENTIAL SYSTEMS WITH APPLICATION TO TRANSMISSION LINES**

By

**IYAI DAVIES**

**SEPTEMBER 2015**



# **OPTIMAL CONTROL OF FUNCTIONAL DIFFERENTIAL SYSTEMS WITH APPLICATION TO TRANSMISSION LINES**

By

**IYAI DAVIES**

**SEPTEMBER 2015**

*A thesis submitted in partial fulfilment of the University's requirements for  
the Degree of Doctor of Philosophy*

# Abstract

Robust control is an aspect of control theory which explicitly considers uncertainties and how they affect robust stability in the analysis and design of control decisions. A basic requirement for optimal robust guaranteed control in a real life scenario is the stabilization of systems in the presence of uncertainties or perturbations. In this thesis, the system uncertainties are embedded into a norm bounded uncertainty elements. The perturbation function is modelled as a class of nonlinear uncertainty influencing a neutral system with infinite delay. It is assumed to have delay in state and is input dependent; which implies the effect of control action can directly or indirectly influence the nonlinear perturbation function.

In recognition of the fact that stability and controllability are fundamental in obtaining the optimal robust guaranteed cost control design for neutral functional integro-differential systems with infinite delays (NFDSID), total asymptotic stability results were developed using Razumikhin technique, unique properties of eigenvalues, and the uniform stability properties of the functional difference operator for neutral systems. The new results, obtained using Razumikhin's technique, extend and complement basic stability results in neutral systems to NFDSID. Novel sufficient conditions were developed for the null controllability of nonlinear NFDSID when the controls are constrained. By exploring the knowledge gained through other controllability results; conditions are placed on the perturbation function. This guaranteed that, if the uncontrolled system is uniformly asymptotically stable, and the controlled system satisfies a full rank condition, then the control system is null controllable with constraint if it satisfies some algebraic conditions.

The investigation of optimal robust guaranteed cost control method has resulted in a novel delay dependent stability criterion for a nonlinear NFDSID with a given quadratic cost function. The new design is based on a model transformation technique, Lyapunov matrix equation and Lyapunov-Razumikhin stability approach. The Lyapunov-Razumikhin technique is adopted for this investigation because it is considered more scalable for optimal robust guaranteed cost control design for NFDSID. It is demonstrated that a memory less feedback control can be synthesized appropriately to ensure: (i) the closed-loop systems robust stability, and (ii) guarantee that the closed-loop cost function value remains within a specified bound. The problem of designing the optimal guaranteed cost controller is also realized in terms of inequalities. The Lyapunov-Krasovskii method is used to obtain stability conditions in comparison to the Razumikhin method. This method leads to linear matrix inequality (LMI) for the delay-independent case which is known to be conservative.

To illustrate the potential practical applicability of the theoretical results; a cascade connection of two fully filled chemical solution mixers, and an integrated lossless transmission line which has a capacitance, inductance, resistance and terminated by a nonlinear function are modelled. A neutral control system model for NFDSID is derived from each of these systems. Simulation studies on the transmission line system confirm the theoretical robust stability results. The new results and methods of analysis expounded in this thesis are explicit, computationally more effective than existing ones and will serve as a working document for the present and future generations in the comity of researchers and industries alike.

# Acknowledgements

Every natural object has got a beauty of control (Genesis1: 2-27)! It is pertinent to note that mathematics and control engineering was applied in the creation of the world. In fact, the discipline of mathematics and control engineering has existed much earlier than the time its name was coined. Indeed, at the time of creation mathematics and control measures were applied by God to ensure all variables are kept in robust optimal regime as desired and for His pleasure. Hence, to the king Eternal and only God; He that started application of mathematics and control engineering to creation be glorified for His continuous mercies, kindness and protection!

It is a pleasure to express my utmost gratitude to my Director of Studies, Dr. O. C. L. Haas and supervisor, Prof. K. J. Burnham; men of impeccable character and research icons, for giving me the opportunity to undertake this research degree project, and for their extremely valuable suggestions, guidance, focus, comments, and patience from the inception to the completion of this research degree program.

My estimable appreciations go to my wife, Mrs. Demeseme, I. Davies; whose love, encouragement and care throughout the duration filled me with inner warmth of exhilaration and a growing elation that facilitated the completion of this work. With equivalent warmth of indebtedness, I express my immeasurable gratitude to my children; Samuel, Shalom, and Sean for their understanding, encouragement and the fun times they provided throughout the duration. The support of these children has in no doubt reinvigorated the passion in me to complete this research program in record time.

My profound commendation is expressed to my mother, mother-in-law, father-in-law and all family members for their prayers, encouragement, advice and support throughout the duration of the program.

Finally, my appreciation goes to all my friends, colleagues and staff of the Control Theory and Applications Centre (CTAC) for their help and kindness. I especially would like to thank Joe Mahtani for his courses and support during the program.

To all these people and others may the Doyen of mathematics and control engineering, God Almighty, He who adaptively controlled all things by His words, protect and provide for them at the point of their needs. Amen!

# Table of contents

Abstract .....	i
Acknowledgements .....	iii
Table of contents.....	v
List of figures.....	ix
List of tables .....	x
Glossary of notation .....	xi
Nomenclature .....	xiii
Chapter 1 .....	1
Introduction and outline of approach .....	1
1.1. Introduction .....	1
1.2. Robust control problems and uncertainties .....	4
1.2.1. Stabilization.....	4
1.2.2. Disturbances .....	5
1.2.3. Unmodelled dynamics .....	6
1.3. Lyapunov's stability concepts .....	6
1.4. Control concepts for the systems .....	7
1.4.1. Feedback and robustness .....	7
1.5. Motivation for research .....	8
1.6. Research aim/objectives .....	11
1.7. Contributions of the thesis and peer reviewed works .....	11
1.8. Organisation and framework of the thesis.....	14
Chapter 2 .....	19
Literature review .....	19
2.1. Introduction .....	19
2.2. Neutral functional differential systems.....	19
2.3. Stability of neutral functional differential systems.....	20
2.3.1. Lyapunov functional method .....	21
2.3.2. Comparison method.....	23
2.3.3. Razumikhin technique .....	23



2.3.4.	Relationship between retarded and neutral functional differential systems.....	24
2.3.5.	Stability analysis of neutral functional differential systems .....	25
2.3.6.	Stability analysis of neutral systems with infinite delays .....	27
2.4.	Controllability methods in retarded functional differential systems.....	28
2.4.1.	Controllability methods in neutral functional differential systems .....	30
2.4.2.	Controllability methods in retarded and neutral integro-differential systems .....	32
2.4.3.	Controllability methods in neutral integro-differential systems with infinite delays ..	34
2.5.	Optimal control of neutral functional differential systems.....	36
2.5.1.	Advantages for optimal control of neutral functional differential systems .....	37
2.5.2.	Approaches in optimal control for neutral functional differential systems .....	37
2.5.3.	Cost function method for optimal control of neutral functional differential systems.	38
2.5.4.	Time-optimal control method for neutral functional differential systems .....	39
2.6.	Applications of neutral functional differential systems .....	40
2.7.	Concluding remarks.....	42
Chapter 3	.....	43
Potential application areas	.....	43
3.1.	Introduction .....	43
3.2.	Transmission lines modelling.....	43
3.2.1.	Example of transmission lines.....	44
3.2.2.	Lossy transmission lines.....	45
3.2.3.	Lossless transmission lines.....	48
3.3.	Population growth modelling .....	55
3.4.	Formulation of neutral control system model.....	57
3.5.	Other application areas of neutral functional differential system .....	59
3.6.	Concluding remarks.....	61
Chapter 4	.....	62
Stability methods	.....	62
4.1.	Introduction .....	62
4.2.	Model for a neutral system with infinite delay .....	62
4.3.	Razumikhin's approach for stability .....	71
4.3.1.	Application of the Razumikhin approach.....	75
4.4.	Lyapunov functional approach for stability.....	78
4.4.1.	Application of the Lyapunov functional approach .....	79
4.5.	Examples on stability methods .....	82

4.5.1.	Example using Razumikhin's approach.....	82
4.5.2.	Example using Lyapunov's functional approach .....	84
4.5.3.	Comparative example with other Lyapunov's results .....	85
4.6.	Concluding remarks.....	86
Chapter 5 .....		87
Control methods .....		87
5.1.	Introduction .....	87
5.2.	Control model for neutral system with infinite delay .....	89
5.3.	Necessary and sufficient conditions for controllability.....	93
5.3.1.	Controllability results .....	96
5.3.2.	Null controllability result .....	99
5.4.	Stabilisation of the system .....	104
5.5.	Examples on control methods .....	108
5.5.1.	Example on null controllability.....	108
5.5.2.	Example on stabilisation.....	110
5.6.	Simulation studies.....	111
5.7.	Concluding remarks.....	112
Chapter 6 .....		113
Optimal robust control for neutral systems .....		113
6.1.	Introduction .....	113
6.2.	Preliminaries and basic definitions for time optimal control problem .....	115
6.3.	Normal and completely controllable systems .....	118
6.3.1.	Time optimal control for neutral systems with an infinite delay .....	123
6.4.	Robust guaranteed cost control for the neutral system with infinite delay .....	126
6.4.1.	Designing a guaranteed cost controller.....	128
6.4.2.	Designing optimal robust controller that minimizes the guaranteed cost.....	134
6.5.	Examples on optimal control for neutral systems with infinite delays.....	134
6.5.1.	Example on time optimal control .....	135
6.5.2.	Example on robust guaranteed cost control problem.....	137
6.5.3.	Comparative results with other examples.....	138
6.5.4.	Effects of delay on simulation.....	140
6.6.	Concluding remarks.....	143
Chapter 7 .....		145
Application of results to lossless transmission line.....		145

7.1.	Introduction .....	145
7.2.	Operational conditions in transmission lines .....	146
7.2.1.	Voltage stability.....	147
7.3.	Application to lossless transmission lines .....	147
7.3.1.	Network of flip – flop circuit .....	148
7.4.	Simulation output studies for the wave patterns .....	156
7.4.1.	Changing resistance $R_0$ of the systems .....	156
7.4.2.	Changing capacitance $C$ of the systems.....	161
7.5.	Optimal robust control for transmission line systems .....	166
7.6.	Concluding remarks.....	172
Chapter 8	.....	174
Conclusions and further-work.....		174
8.1.	Conclusion .....	174
8.2.	Further-work.....	178
References	.....	182
Appendix I	.....	195
Appendix II	.....	198
Appendix III	.....	201
III1:	Simulink model for single transmission line without nonlinear function .....	201
III 2:	Simulink model for single transmission line with nonlinear function .....	201
Appendix IV	.....	202
IV1:	Simulink model for interconnected transmission lines without nonlinear functions .....	202
IV2:	Simulink model for interconnected transmission lines with nonlinear functions .....	202
Appendix V	.....	203

# List of figures

Figure 1.1: Illustration of disturbance on controlled system .....	6
Figure 1.2: Framework of Thesis structure/flow chart.....	18
Figure 3.1: Strip-line .....	44
Figure 3.2: Micro-strip .....	44
Figure 3.3: Two wire line .....	44
Figure 3.4: Coaxial cable .....	44
Figure 3.5: Optical fibre .....	45
Figure 3.6: A circuit representation of lossless transmission line .....	49
Figure 3.7: Schematic diagram of a cascade connection of two mixers .....	58
Figure 5.1: Simulation of control input and system states with perturbation function and linear control base system .....	111
Figure 6.1: Simulation example for $h = 0.1690$ .....	141
Figure 6.2: Simulation example for $h = 0.2306$ .....	142
Figure 6.3: Simulation example for $h = 7.4795$ .....	142
Figure 6.4: Control law for the system.....	143
Figure 7.1: Fundamental diagram of a flip-flop circuit.....	148
Figure 7.2: N mutually connected transmission line network.....	150
Figure 7.3: Current-voltage characteristics of the nonlinear function .....	151
Figure 7.4: Current and voltage waveforms for interconnected system without nonlinearity, $R_0 = 10$ .....	157
Figure 7.5: Current and voltage waveforms for interconnected system with nonlinearity $R_0 = 10$ .....	158
Figure 7.6: Current and voltage waveforms for system without nonlinearity, $R_0 = 10$ .....	158
Figure 7.7: Current and voltage waveforms for the nonlinear system with $R_0 = 10$ .....	159
Figure 7.8: Current and voltage waveforms for interconnected system without nonlinearity, $R_0 = 0.001$ .....	159
Figure 7.9: Current and voltage waveforms for interconnected system with nonlinearity, $R_0 = 0.001$ .....	160
Figure 7.10: Current and voltage waveforms for system without nonlinearity, $R_0 = 0.001$ .....	160
Figure 7.11: Current and voltage waveforms for the nonlinear system with $R_0 = 0.001$ ...	161
Figure 7.12: Current and voltage waveforms for interconnected system with nonlinearity, $C = 100$ .....	162
Figure 7.13: Current and voltage waveforms for interconnected system without nonlinearity, $C = 100$ .....	163
Figure 7.14: Current and voltage waveforms for system with nonlinearity, $C = 100$ .....	163
Figure 7.15: Current and voltage waveforms for system without nonlinearity, $C = 100$ ....	164

Figure 7.16: Current and voltage waveforms for interconnected system with nonlinearity, $C = 0.0001$ .....	164
Figure 7.17: Current and voltage waveforms for interconnected system without nonlinearity, $C = 0.0001$ .....	165
Figure 7.18: Current and voltage waveforms for system with nonlinearity, $C = 0.0001$ ....	165
Figure 7.19: Current and voltage waveforms for system without nonlinearity, $C = 0.0001$	166
Figure 7.20: Robust stability performance for single transmission line with $h = 0.0481$ ...	169
Figure 7.21: Robust stability performance for single transmission line with $h = 5.0295$ ...	169
Figure 7.22: Robust stability performance for interconnected transmission line, $h = 0.0481$ .....	171
Figure 7.23: Robust stability performance for interconnected transmission line, $h = 5.0295$ .....	172

## List of tables

Table 6.1: Maximum delay bound for $c$ when $f = G = B = 0$ .....	139
Table 6.2: Maximum delay bound ( $h^*$ ) comparison .....	140

# Glossary of notation

$E$	Euclidean space
$E^n$	$n$ – Dimensional Euclidean space
$W_2^{(0)}$	Lebesgue space of square-integrable functions
$W_2^{(1)}$	Sobolev space of all absolutely continuous functions
$\mathcal{B}$	Banach space
$\mathcal{L}$	Linear space
$\mathfrak{B}$	Functions of bounded variation
$\mathcal{M}$	Metric space of all nonempty compact subset of $E^n$
$\langle \cdot \rangle$	Inner product
$J$	Any interval in Euclidean space
$A$	For any square matrix
$A^T$	Transpose of the square matrix $A$
$\lambda_{\max}(A)$	Maximum eigenvalue of matrix $A$
$\lambda_{\min}(A)$	Minimum eigenvalue of matrix $A$
$\mu(A)$	Matrix measure for the matrix $A$
$I$	Identity matrix
$*$	Elements below the main diagonal of a symmetric block matrix
$\mathcal{C}$	Space of continuous function
$\mathcal{C}^m$	$m$ – dimensional unit cube in an Euclidean space
$\mathbb{C}$	Set of complex numbers
$\mathcal{U}$	Set of admissible controls
$\mathcal{S}$	Compact and convex subset of $\mathcal{C}$
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathcal{D}$	Differential operator for neutral systems
$D$	Difference differential operator for neutral systems
$\square$	End of proof
$\mathcal{G}$	Target set
$z_t$	Target point function
$\mathcal{A}$	Attainable set
$\mathcal{P}$	Reachable set
$R$	Constrained reachable set
$W$	Controllability matrix
$\mathfrak{U}$	Domain of null-controllability
$X$	Fundamental matrix
$J$	Quadratic cost function
$P, P_1$	Positive definite matrices
$\tilde{P}_i(\mathcal{D}), \hat{P}_i(\mathcal{S})$	Matrix operators
$\mathcal{S}$	Shift operator
$h$	Constant delay

$h_k$	Time delays
$\equiv$	Equivalence
$\ \cdot\ $	The Euclidean norm
$ \cdot $	Absolute value norm
C	Capacitor
R	Resistor
Z	Characteristics impedance of line
$b$	Propagative velocity of waves
$v$	Voltage
$\mathbf{v}$	$n$ dimensional row vector
$i$	Current
E	Alternating voltage source
L	Inductor
$\mathbf{x}$	State variable vector
$\mathbf{u}$	Control input vector
$\phi$	Initial function
$\sigma$	Initial time
$t_1, T$	Final times
$t^*$	Minimal time
$\partial$	Boundary of a set
$Z(\cdot)$	Matrix function
$\pi$	Support plane
$\eta$	A unit normal to a supporting plane
$K$	Linear or matrix operator
$V$	Lyapunov function
$V_m$	Volume of Mixer
$\Delta$	Parametric uncertainty
$Q_1^*, Q_2^*$	Flow intensities
$C_1(t), C_2(t)$	Total length of solutions in Mixers
$C_{in1}, C_{in2}$	Input concentrations of product
$L(t, \mathbf{x}, \mathbf{x}_t, \mathbf{u})$	Matrix function
${}^c D^\tau$	Fractional order derivative
NFDSID	Neutral functional integro-differential systems with infinite delays
LQR	linear quadratic regulators
SISO	single input single output
LMI	linear matrix inequality
HJB	Hamilton–Jacobi–Bellman
Adj	Transposed matrix of cofactors
$A_0, A_1, A_2$	State matrices
$B, B_1, B_2$	Input matrices
$\zeta$	Difference in birth and death rates
$g, f, F, G, H, L$	Continuous functions
$k, m, \sigma, c, d, \gamma$	Scalars
$\delta, \varepsilon, r, \alpha, \beta, \zeta, \tau$	
$\tau_0, \dots, \tau_6, \gamma_0, \dots, \gamma_2$	

# Nomenclature

Denote the Euclidean space by  $E = (-\infty, \infty)$  and  $E^n$  as a real  $n$  – dimensional Euclidean space with norm  $|\cdot|$  and let  $J$  be any interval in  $E$ .

Define the  $n$ -Euclidean norm  $\|\cdot\|$  on the length of a vector  $x = (x_1, x_2, \dots, x_n)$  by  $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$ . The absolute value norm  $|\cdot|$  is used to represent the norm in the various spaces as appropriate rather than using different symbols.

The notation  $W_2^{(0)}(J, E^n)$  will represent the Lebesgue space of square-integrable functions from  $J$  to  $E^n$ . The convention  $W_2^0(J, E^n) = L_2(J, E^n)$  is also adopted

Let  $\sigma \geq h \geq 0$  be any given real numbers ( $\sigma$  may be  $+\infty$ )

$W_2^{(1)}([-h, 0], E^n)$  is the Sobolev space of all absolutely continuous functions  $x: [-h, 0] \rightarrow E^n$  with the property that the function  $t \rightarrow \dot{x}(t) = (dx/dt)$  belongs to  $W_2^{(0)}([-h, 0], E^n)$

$C = C([-h, 0], E^n)$  represents the space of continuous function mapping the interval  $[-h, 0]$  into  $E^n$  with the norm  $\|\cdot\|$ , where  $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)|$ , for  $\phi \in C$ .  $S$  is any compact and convex subset of  $C$

$\mathcal{L}([-h, 0], E^n)$  is a linear space with norm  $\|\cdot\|$  defined by  $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)|$ .

$\mathcal{B}([- \sigma, 0], E^n)$  is the Banach space of functions which are continuous and bounded on  $[-h, 0]$  and such that  $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)| + \int_{-\sigma}^0 g(\theta) |\theta| d\theta < \infty$ , where  $g: [-\sigma, 0] \rightarrow (0, \infty)$  is Lebesgue integrable on  $[-\sigma, 0]$ , positive and non-decreasing.



The function  $g: [\alpha, \beta] \rightarrow E$  is said to be a bounded variation  $g \in \mathfrak{B}([\alpha, \beta])$  if  $\mathfrak{B}_\alpha^\beta(g) = \sup_{\mathcal{H} \in [\alpha, \beta]} \sum_{k=1}^n |g(x_k) - g(x_{k-1})|$ , where  $\mathcal{H} = \{\alpha = x_0 < \dots < x_n = \beta\}$ .  $\mathfrak{B}_\alpha^\beta(g)$  is the total variation of  $g$  on  $[\alpha, \beta]$ .

Let  $\tau \in E$ , so that  $x \in C([\tau - h, \tau + \sigma], E^n)$ , then given  $t \in [\tau, \tau + \sigma)$  define the symbol  $x_t, x_{t-} \in C$  or  $\mathcal{L}$  by  $x_t(s) = x(t + s)$  for  $-h \leq s \leq 0$  and  $x_{t-}(s) = x(t + s)$  for  $-h \leq s < 0$  respectively with  $x_{t-}(s) = x(t^-)$  for  $s = 0$ . The convention  $x_\sigma(s) = x(s)$  is adopted when  $t = 0$ . Also, the notations  $x, x_h$  and  $\dot{x}_h$  are used in some cases to denote  $x(t), x(t - h)$  and  $\dot{x}(t - h)$  respectively.

The controls  $u$  of special interest are square integrable functions with values in  $m$ -dimensional unit cube  $C^m$  of  $E^m$ , where  $C^m = \{u: u \in E^m, |u_j| \leq 1, j = 1, 2, \dots, m\}$ .

The set of admissible controls denoted by  $U$  are functions  $u: [\sigma, \infty) \rightarrow C^m$  which are square integrable on finite intervals with values in  $C^m$ .

The differential operator for neutral systems  $\mathcal{D}$  is defined by  $(\mathcal{D}x)(t) = \dot{x}(t) = dx(t)/dt$ , almost everywhere on bounded interval of  $J$ . Higher powers of the operator  $\mathcal{D}$  are defined inductively by  $\mathcal{D}^{k+1} = \mathcal{D}\mathcal{D}^k$ .

The difference differential operator for neutral system  $D$  is defined by  $D(t)x_t = x(t) - A_0x(t - h)$  and the restriction on  $D$  is given by  $D\phi = \phi(0) - A_0\phi(-h)$ .

$h_k$  represent time delays with  $h_k \in [\underline{h}_k, \bar{h}_k)$ , where  $0 < h_k < \underline{h}_k < \bar{h}_k \leq \infty, k = 1, \dots, N$ .

Whenever,  $N = 1$  the index is dropped and the delay is written as  $h \in [\underline{h}, \bar{h})$ . The notations  $N, n$  and  $m$  are always considered being positive numbers in  $E$ .

The matrix measure  $\mu(\cdot)$  for a matrix  $A$  is defined by  $\mu(A) = [A^T + A]/2$ , where  $A^T$  represents the transpose of  $A$ .

$d_1 = \frac{\partial}{\partial x_1}, d_2 = \frac{\partial}{\partial x_2}, \dots, d_n = \frac{\partial}{\partial x_n}$  denotes vector differentiation up to the  $n$  vector variables

The symbols  $u(s)$ ,  $v(s)$ , and  $w(s)$  represents continuous non-decreasing, and nonnegative functions. All other symbols for functions are appropriately defined to be linear, matrix or nonlinear functions

The target set function  $\mathcal{G}$  is assumed to be either a singleton or a smooth manifold in  $\mathcal{C}([-h, 0], E^n)$  represented by  $\mathcal{G} = \{(t, x) \in J \times E^n : t = \varphi(\varrho), \mathfrak{d}(\varrho), \varrho \in E^n\}$ , where  $\varphi$  and  $\mathfrak{d}$  are assumed to be continuously differentiable. That is, if  $\varphi: E^n \rightarrow E$  is the constant function  $\varrho \rightarrow \sigma$  and  $\mathfrak{d}: E^n \rightarrow E^n$  is the identity, then  $\mathcal{G} = \{t_1\} \times E^n$  (fixed final time, free final state). If  $\varphi: \varrho = (\varrho_1, \dots, \varrho_2) \rightarrow \varrho_1$  and  $\mathfrak{d}$  is a constant function  $\varrho \rightarrow x_{t_1}$  (free final time, fixed final state).

In circuit theory, dynamics of systems in lumped parameter are often considered to be function of time alone while the dynamics in distributed parameter are considered to be function of time and one or more variables. In the transmission lines modelled in this thesis, the voltage and currents are considered to depend on time  $t$  and on the length of line  $\xi$  (in meters). The state variables of the systems are represented by  $i(\xi, t)$  and  $v(\xi, t)$  where  $v$  is the voltage in Volts (V) across nodes and  $i$  is the current in Amperes (A) that flows through them, while  $i_0, v_0$  represents their initial current and voltage respectively. Other variables employed in the modelling are capacitances  $C, C_1, C_0$  in Farad (F), the inductance  $L, L_1$  in Henry (H), the resistance  $R$  in Ohms ( $\Omega$ ), and the impedance of the line  $Z$  in Ohms ( $\Omega$ ). Here  $v_{p\hbar}$  represents the potential difference between two nodes  $p$  and  $\hbar$  of a network.

# Chapter 1

## Introduction and outline of approach

### 1.1. Introduction

This thesis is devoted to the study of optimal robust control for a neutral functional differential system with infinite delays. Robust control explicitly considers uncertainties and how it affects the analysis and design of control decisions or rules governing a range of models. Using mathematical models in the analysis and design of such control decisions enables predictions to be made about the systems behaviour. It allows suitable analytical techniques and associated simulation tools to interpret systems behaviour predictions. In this Thesis the Lyapunov techniques will be explored in the analysis and design of robust control decisions while all simulations will be implemented using MATLAB<sup>®</sup> and Simulink<sup>®</sup> R2015b.

Robust control, which originated in the 1980s, from the applied mathematics and engineering branch of control theory is now one of the dominant approaches in control theory (Williams 2008). A practical requirement for robust control is to stabilize a system in the presence of uncertainties or perturbations which may take the form of noise or an external disturbance on the system. Disturbances may also be caused by internal parameters (known as parametric uncertainties) through variations in measurements of the physical parameters, ageing of the physical parameter or changes in the operational conditions of the physical parameters of the systems. Many methods seeking to design controllers for such imperfectly known systems, so that the system responses meet the desired properties and get stabilised, have evolved over the years in control theory. For example, the ideas of optimal control in the time-domain

which were introduced in the early 1960s and 1970s, largely through the works of Kalman on linear quadratic regulators (LQR) and filtering techniques (Williams 2008) has underwent a significant change. In effect, the LQR method uses a set of linear first order differential equations to represent the dynamics of the system to be controlled in state space model. The objective is to keep the state vector close to zero without excessive control effort. This is achieved through the minimization of a defined cost function. The overall solution is then provided by an optimal state feedback control whose state feedback matrix is obtained from an algebraic Riccati equation. Whilst the LQR approach was found to be robust in some model perturbations and the approach is still widely in use in some other forms, it is important to mention that the focus was on the optimality of the nominal system while the problems of plant uncertainty were largely ignored. The control approach started to change in the early 1970s and 1980s (Williams 2008) as theory and practice identified some of the shortcomings in the LQR approach. Doyle in 1978, as reported in Bhattacharyya et al. (1995), demonstrated by a counter example that all the robustness properties (gain and phase margins) and some other properties of the LQR design vanish in an output feedback implementation. See also (Dullerud and Paganini 2013) on the effect of feedback on stabilization. Doyle's observation spurred the research community to design feedback controllers that can have desirable robust properties.

This innovation made control scientists move away from the LQR approach to look for a more desirable and robust approach. About the same period, some significant results were reported on the analysis of multivariate systems in the frequency domain. Bhattacharyya et al. (1995) reported that some solutions to robust stabilization problems were also realized. In particular, the problem of determining a controller for a prescribed level of unstructured perturbations was given by Kimura in 1984 for single input single output (SISO) systems; the multivariable robust stabilization problem was solved by Vidyasagar, Kimura, and Glover in

1986 (Bhattacharyya et al. 1995). These results were a by-product of an important line of research initiated by Zames (1981) on the optimal disturbance rejection problem which can be summarised as the product of designing a feedback controller that minimizes worst case effects over a class of disturbances on the system outputs (Bhattacharyya et al. 1995). Zames fundamental paper of 1981 contained the solution of an  $H_\infty$  sensitivity minimization problem for a special case of a system with a single right half plane zero which also influenced the development of an  $H_\infty$  approach to control systems design as a more robust alternative to LQR. Unlike the LQR approach, where the quadratic cost could mean measuring the performances across frequencies with a 2-norm, the  $H_\infty$  approach looks at the peak of losses across frequencies using an  $\infty$ -norm. Again, uncertainties sets in  $H_\infty$  approach have no particular form but represent perturbations of the model which are bounded and are most often referred to as unstructured approach.

Bhattacharyya et al. (1995) observed that the problem of stability under large parameter perturbations was almost completely ignored by control theory researchers during the 1960s and 1970s. However, the situation changed dramatically with the advent of a remarkable theorem (Kharitonov 1979) from V.L. Kharitonov which was published in 1973 that led to a resurgence of interest in the study of robust stability under real parametric uncertainty (Bhattacharyya et al. 1995). Researchers then started to believe that the robust control problem for real parametric uncertainties could be approached without conservatism and over bounding, and with computational efficiency built right into the theory. The theory also revealed the effectiveness and transparency of methods which exploit the algebraic and geometric properties of the stability region in parametric space instead of the blind formulation of optimization problems. This has spurred many researchers in the field over the last few years to obtain interesting research results and has laid solid foundations for the future development of robust stability and control under various perturbations.

## 1.2. Robust control problems and uncertainties

This section outlines three basic scenarios associated with the discussions of robust control in Section 1.1. It serves to generate interest in the mathematical analysis and approach to be explored in dealing with stability and control of systems in this thesis.

### 1.2.1. Stabilization

Robustness in stability is a very important issue in design, analysis and evaluation of control systems and plays an important role in many other fields including economics, quantum mechanics, nuclear physics, numerical algorithms, mechanical and electrical engineering. Stability, literally speaking, means the ability of all signals in a system returning or decaying to zero when there are no excitations in the system. There are various stability properties identified in the literature whose implementation could be very impractical if the only way to determine them in a system were to do experiments or run simulations. Fortunately, these properties can be analysed using corresponding mathematical models. This thesis focuses on Lyapunov methods; see Section 1.3, on how to perform such analysis. The Lyapunov stable definition in simple terms implies that a solution starting close to an equilibrium point stays near that point forever. It is asymptotically stable if the equilibrium point is Lyapunov stable and every solution that starts near the equilibrium point eventually converges to it.

To illustrate the Lyapunov stability definitions consider the nonlinear system given by

$$\dot{x} = f(x, u), \tag{1.1}$$

where  $x$  is the state,  $u$  is the control input and suppose  $(0,0)$  is an equilibrium point of the system. The equilibrium point  $(0,0)$  is said to be (asymptotically) stable if zero is an (asymptotically) stable equilibrium point of  $\dot{x} = f(x, 0)$ . To stabilize the system, the first task is to investigate conditions under which it will be possible to stabilize such an equilibrium

point by using state feedback control. In this case, define a feedback control law  $u(t) = g(x(t))$ , where  $g$  is a continuously differentiable function so that the closed-loop  $\dot{x} = f(x, g(x(t)))$  is asymptotically stable. The problem now is that of finding a function  $g$  that maps the state  $x$  to the control action  $u$ . Assume first that such a  $g$  exists in order to examine some of its properties, and also assume that  $g(0) = 0$  in order to make zero an equilibrium point of the closed-loop. By applying the Jacobian linearization method (Dullerud and Paganini 2013), the linearization of the closed-loop system will be given by  $\dot{x} = (A + BF)x$ , where  $A = d_1 f(0,0)$ ,  $B = d_2 f(0,0)$ ,  $F = dg(0)$  and  $d_1, d_2$  denote vector differentiation by the first and second vector variables respectively. This shows that the closed-loop system is asymptotically stable if all the eigenvalues of  $A + BF$  lie on the left half of the complex plane. Conversely, if the matrix  $F$  exists such that  $A + BF$  has all the stability properties it requires, then the state feedback law  $g(x) = Fx$  is able to stabilize the closed-loop system.

### 1.2.2. Disturbances

Achieving robust stability for controlled systems has been a typical issue for control research, and has often led to the introduction of feedback in systems which are already stable to improve some aspects of the design parameters of the system in order to obtain a more acceptable behaviour (Dullerud and Paganini 2013). The effects of environmental influences are one of the important concerns for design of systems. One of the main objectives for introducing feedback in this research is to render a system less sensitive to such unknown environmental influences. For example, consider Figure 1.1, which shows a dynamical system with feedback using a control law.

The controller implements its actions based on the information it receives from the measurements. However, the output might show unexpected behaviour as a result of

disturbances acting on the systems if these were not considered during the control analysis and design.

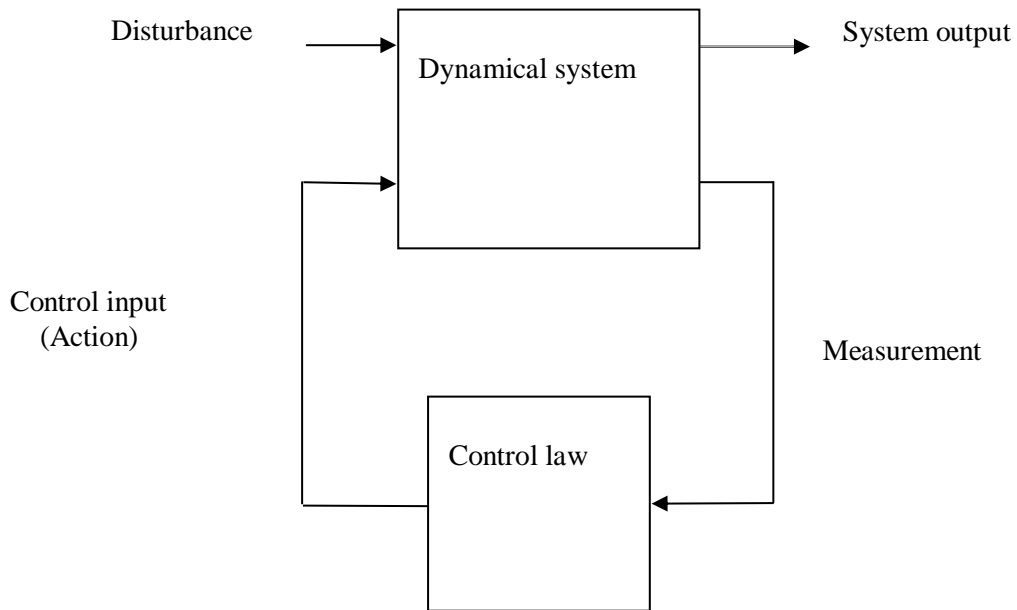


Figure 1.1: Illustration of disturbance on controlled system

### 1.2.3. Unmodelled dynamics

In most practical systems approximations are made to some physical parameters when modelling such systems with their nonlinearities. In some cases, where models are linearized or truncated and incorporated as norm-bounded operators, these approximations pose robust stability problems when applying feedback (Bhattacharyya et al. 1995). To develop effective and useful practical systems therefore model developers must ensure uncertainty or unpredictability in the system are adequately compensated for by feedback (Dullerud and Paganini 2013). The latter approach which ensures adequate compensation for uncertainties in systems is adopted in this thesis to model, analyse and design the systems.

### 1.3. Lyapunov's stability concepts

The concept of stability for non-autonomous nonlinear systems was first developed by Lyapunov in 1892. Lyapunov developed two methods; the direct and indirect methods for analysing the stability of systems. The Lyapunov's direct method or Lyapunov's second



method which is the focus of this research allows stability analysis to be extended to more general nonlinear non-autonomous systems, where the right hand side is allowed to depend explicitly on time. Unlike the root locus and frequency-response methods which are generally applicable to linear time invariant systems having single-input single-output structure, the Lyapunov's method can be applied to linear and nonlinear systems of any order and the systems stability can also be analysed without necessarily solving their state equations (see Burghes and Graham 1980).

The central idea of the Lyapunov's direct method is the concept of generating a function (Lyapunov function) that essentially represents the system energy. It is possible to define the total energy of the system in terms of the second law of thermodynamics or alternatively, as the principle of minimum total potential energy. The principle of minimum total potential energy asserts that "a structure or body shall deform or displace to a position that minimizes its total potential energy" (Liu 2011). The principle of total minimum potential energy implies that, in any stable region of a system, the total energy of the system decreases towards some local minimum along all its part of evolution. It also implies that any state of an object in a physical system can only be made stable to small disturbances or perturbations if it were a local minimum of the body's potential energy.

#### **1.4. Control concepts for the systems**

This section describes the fundamental properties of robustness that are inherent in the control design scheme adopted in this thesis.

##### **1.4.1. Feedback and robustness**

Feedback schemes are usually designed to deal with disturbances and provide robustness but can also be used to obtain an accurate account of every variation in a system despite disturbances and changes to internal parameters. Robustness therefore is considered as one of

the most valuable properties of feedback. Robustness in closed loop also depends on the structure of the feedback controller. For systems with error feedback only the error signals may be accessible for measurement through the output and all robustness issues can be completely dealt with using feedback. For other feedback system, it is possible to separate their reference and process output measurements, and then deal with their robustness and disturbance issues using feedback to finally obtain the desired response to their command signals through feedforward designs. Controlling unstable systems with delays is intrinsically very challenging. Such systems can be better controlled by integrating the fundamental systems dynamics and having knowledge of all the other elements necessary for the design.

In general feedback is known to:

- Give accurate control gain to systems
- Give account of all variations in a system
- Give a linearizing effect to systems

### **1.5. Motivation for research**

The existence of time delays in a dynamical system has been the source of poor system performance and even instability. Studies involving different time delays can be found in ship stabilization, control processes for pressure and heat transfer regulation. However, delays are sometimes deliberately introduced into feedback systems to improve system performances. See Kolmanovskii and Myshkis (1992) and references therein for details. There are well known developed fundamental theories for neutral delay differential systems, that is existence, uniqueness and continuous dependence of solution on parameters. However, unlike linear autonomous systems where Routh-Hurwitz criteria can be used as a standard sufficient condition for stability by checking the positivity of sequences of determinants in the principal sub-matrices, there is no standard sufficient condition of asymptotic

stability for neutral integro-differential systems because of the trivial nature of their solutions. In fact, some well-known results for linear autonomous ordinary and delay differential systems cannot be extended to neutral differential systems. For example; a linear neutral system can have unbounded solutions even when the roots of its characteristic equations are purely imaginary. It is also known, see Gopalsamy (1992) and references therein, that if all the roots of the equations of a linear neutral system have negative real parts only and if the roots are uniformly bounded away from the imaginary axis, then the asymptotic stability for the trivial solution of the corresponding linear autonomous equation can be asserted. However, verifying the uniform boundedness away from the imaginary axis of all the roots of the equation is a very difficult task. The Lyapunov function and functional method is an alternative resort to the investigation of stability for these neutral systems. However, the difficulty in the Lyapunov methods is the lack of generalized rules for constructing the functions; they are merely based on the researchers experience and techniques. The Lyapunov methods are classified into Krasovskii and Razumikhin approaches (Hale and Verduyn Lunel 1993). The Krasovskii's approach often leads to LMI results and can be applied to a wide range of problems. An important peculiarity of this method is in the consideration of a delay derivative upper bound constraint which is naturally excluded in Razumikhin's based approaches (Briat 2011). The Razumikhin's approach often leads to tedious manipulations and quasi-convex conditions but can yield structurally simpler and more scalable results, involving fewer variables, small matrix inequalities and simpler control design than the Krasovskii's approach (Briat 2011). Therefore, this Thesis will focus more on the Razumikhin's approach.

However, stability and robustness are key factors that guarantee the performance of a practical control application. Delays and disturbances in system performances may cause fatal and serious damage to control applications if the key factors are not well compensated

for in the development of the design. To develop a control scheme for neutral integro-differential systems with disturbances, it is therefore necessary to guarantee adequate level of stability and system performances. Control schemes are often designed for different purposes; for example, Kofman et al. (2008) presented a control design for perturbed multiple-input systems which guarantees any component-wise ultimate bound on the system state. He achieved his result by using eigenvalue/eigenvector assignment by state feedback and utilizing a component-wise bound computation procedure. This takes into account both the system and perturbation structures by performing component-wise analysis, thus avoiding the need for bounds on the norm of the perturbation. Soliman et al. (2011) designed a state feedback controller to investigate the guaranteed cost fault tolerant control with pole region constraints for power systems subject to actuator failures (either in power system stabilizers or flexible alternating current transmission systems). By using the linear matrix inequality technique, the feedback controllers ensured the closed-loop system achieves satisfactory oscillation damping and settling time with satisfactory cost performance. In recent years the advent of linear matrix inequality (LMI) optimization has significantly influenced the direction of research in robust control schemes. Lien (2006) considered a static output linear feedback control in stabilizing a class of uncertain neutral systems with time-varying delays via LMI and Lyapunov-Krasovskii approach, deriving a delay-dependent and delay-independent criteria for the stabilization of the system while Lien et al. (2015) proposed a delay-dependent criteria for the design of a guaranteed cost control and achieved the minimization of cost function for a class of Takagi-Sugeno fuzzy time-delay systems on the basis of Lyapunov-Krasovskii and the LMI optimization approach.

Inspired by the numerous applicable areas for neutral integro-differential systems with infinite delays, see Balachandran and Dauer (1996) and references therein, this research project aims to investigate the concept of optimal robust guaranteed cost control for such

systems and its perturbation in-line with the research aim and objectives given in Section 1.6 by proposing a novel control strategy that is robust and reliable.

### **1.6. Research aim/objectives**

The aim of this research is to investigate the optimal robust control of functional differential systems with infinite delays through the following objectives:

- Formulate a neutral control system and find its stability;
- Prove the system's controllability using rank and algebraic conditions if it is stable;
- Obtain the optimal control of the system with application to transmission lines;
- Demonstrate the applicability of the result through simulation studies.

### **1.7. Contributions of the thesis and peer reviewed works**

This section describes and then ranks the contributions of the thesis in terms of their significance:

- Stability results

A new mathematical model for a neutral functional differential delay control system is developed. A novel extension of basic stability results in functional differential equations to neutral functional integro-differential systems with infinite delays is achieved by the investigation of total asymptotic stability properties for neutral functional integro-differential systems with infinite delay using the basic Lyapunov-Razumikhin technique. Furthermore, a new delay-independent condition which is less conservative and sufficient to make the system uniformly asymptotically stable is developed using LMI and the Lyapunov-Krasovskii approach. The feasibility of the LMI which is sufficient to make the system uniformly asymptotically stable is solved by using the MATLAB's LMI Toolbox.

- Control Results

A novel null controllability result for neutral functional integro-differential systems with infinite delays is obtained by placing growth and continuity condition on the perturbation function. This condition guarantees that if the linear control base system has full rank with the condition that  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$  for every complex  $\lambda$ , where  $K(\lambda)$  is an  $n \times n$  polynomial matrix in  $\lambda$  constructed from the coefficient matrices of the control system and  $\xi(\exp(-\lambda h))$  is the transpose of  $[1, \exp(-\lambda h), \dots, \exp(-(n-1)\lambda h)]$ , and the functional difference operator for the system is uniformly stable, with the linear uncontrolled system uniformly asymptotically stable, then the perturbed neutral system with infinite delay is null controllable with constraint. Again, a new stabilization criterion and memory-less state feedback controllers are proposed using LMI and the Lyapunov-Krasovskii approach whose corresponding design procedures are used to stabilize the system.

- Robust optimal control results

Results on time optimal control are extended to neutral functional differential systems with infinite delays. A new delay-dependent result for optimal robust guaranteed cost control has been established for the system by defining a quadratic cost function and making the resulting closed-loop system uniformly asymptotically stable. The result was obtained using a model transformation technique, Razumikhin approach and Lyapunov matrix equation through a state feedback control design which guarantees adequate performance on the given performance index. Further, stabilization condition for the controller is also derived through solving an optimization problem whose constraints are given by a set of inequalities.

- Application

It has been shown that an interconnected network of lossless transmission lines which are each terminated by a nonlinear function in parallel with capacitance, resistance and an inductance is a natural model for a neutral functional differential system with infinite delays. Simulations studies are carried out in terms of stability and robust control on the model to show the applicability and effectiveness of the theoretical results and methods.

The following peer review contributions were also made:

**Davies and Haas (2015c)** ‘Null controllability of neutral system with infinite delays,’

*European Journal of Control*, <http://dx.doi.org/10.1016/j.ejcon.2015.09.001>

**Davies and Haas (2015e)** ‘Robust guaranteed cost control for a nonlinear neutral system with infinite delay,’ European Control Conference, July 14-17, Linz, Austria

**Davies and Haas (2015b)** ‘Delay-independent closed-loop stabilization of neutral systems with infinite delays,’ International Conference on Applied Mathematics and Computation (ICAMC), September, 17-18, Rome, Italy

**Davies and Haas (2015a)** ‘Delay-independent closed-loop stabilization of neutral systems with infinite delays,’ World Academy of Science, Engineering and Technology, International Science Index 105, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 9(9), 380-384.

**Davies and Haas (2015d)** ‘Stability of neutral systems with infinite delays,’ International Conference on Systems Engineering (ICSE) 2015, September, 8-10, Coventry, United Kingdom

**Davies and Haas** (2013) ‘Optimal control of functional differential systems,’ United Kingdom Automatic Control Council (PhD Presentations) Conference, October 31, 2013, Institution of Mechanical Engineers, Westminster, London

### **Contributions and their order of importance**

- The most important contribution is the robust guaranteed cost control for neutral system with infinite delays which was presented at European Control Conference in July, 2015, Linz, Austria
- The second most important contribution is Null controllability result for neutral system with infinite delays published in the European Journal of Control, 2015
- The third most important contribution is the demonstration of the applicability of the theoretical result with simulations which is being prepared for Journal submission.
- The fourth most important contribution is delay-independent closed-loop stabilization of neutral systems with infinite delays which was presented at International Conference on Applied Mathematics and Computation (ICAMC), 2015, Rome, Italy
- The fifth most important result is the extension of total stability results to neutral systems with infinite delays which was presented at International Conference on Systems Engineering (ICSE), 2015, Coventry, United Kingdom

### **1.8. Organisation and framework of the thesis**

This chapter has given a general overview of the investigations in this research project and subjects areas underpinning the research. The overview also gave clear indication of the importance of these subject areas in the development of some fundamental results in the thesis and in the proposed application of these results. The Lyapunov method, selected in this thesis to investigate the robust stability of the system has been introduced. The focus is on establishing robust guaranteed cost control for neutral integro-differential systems



with infinite delay. The approach will be first to ensure that the system is stable because of the trivial nature of their solution by obtaining total stability results and then controllability results for the systems which are the key issues that would guarantee optimal control of the system. The robust guaranteed cost control result of the system will then be obtained through feedback designs in order to compensate for all uncertainties on the system.

The structure for the rest of the thesis as shown in the flow chart (Figure 1.2) is organized in the following order:

Chapter 2 aims at providing literature exposition of some basic stability techniques and, control methods which are essential to this research project. Delay models as more realistic models than the principle of causality: that is future states of systems are determined solely by the present and are independent of their past states in the system. The classifications of delay equations and their importance in real life applications are discussed. The Razumikhin's method is identified as a more appropriate method than Lyapunov-Krasovskii for the stability analysis of neutral integro-differential equation with infinite delays. The advantages which stem from the difficulty posed in constructing a Lyapunov functional for the whole state space is identified from literature and is discussed. Furthermore, different optimal control approaches for neutral systems and their advantages are introduced; with time optimal control and cost function methods known from literature as most useful tools for analytic design and applications presented. The potential application areas, which include a cascade of chemical control solution for two mixers and lossless transmission line models, are identified.

Chapter 3 presents discussions on the relation between the potential application models identified in Chapter 2 and neutral functional differential equation. The derivation of a

general solution for an ideal lossless transmission line is given in terms of its voltage and current by representing the line equations of the transmission line as a system of first order partial differential equations. The solutions to this system of equations are then obtained by reducing the mixed boundary problem using D'Alembert's solution for wave equations. The chapter reviews how to mathematically analyse and predict controlled chemical solutions which support processes in microbiological growth as well as the evolution in their model development.

The chapter then develops a new mathematical model of a neutral functional differential control system by describing the state equations for a cascade connection of a two chemical solution control process to show one of the numerous applications of the type of system investigated in this thesis.

Chapter 4 studies and gives fundamental results for total asymptotic stability for neutral integro-differential systems with infinite delays modelled in Chapter 3. It uses the Lyapunov-Razumikhin technique discussed in Chapter 2. The study begins with a concise definition of terms, lemmas and theorems upon which the stability study hinges. It explores the basic Razumikhin stability theorems, the uniqueness property of the eigenvalues, and the Lyapunov matrix equation to establish these results. It continues by developing a delay-independent criterion for uniform asymptotic stability for the systems in terms of LMI using the Lyapunov-Krasovskii stability approach. The feasibility of the resulting LMI is solved by using the MATLAB's LMI Toolbox. The chapter contains numerical examples depicting the various approaches.

Chapter 5 examines the control methods for the neutral functional differential system with infinite delays based on the establishment of stability results in Chapter 4. It gives explicit algebraic conditions that can compute the controllability of such systems without the

knowledge of the controllability matrix. In particular necessary and sufficient conditions are developed for null controllability of the systems when the controls are functions which are square integrable on finite intervals with values in an  $m$ -dimensional unit cube. The chapter also examines the stabilization of the system by using the standard Lyapunov-Krasovskii approach to derive new sufficient conditions that stabilises the system in terms of LMI whose feasibility solutions are solved by using the MATLAB's LMI Toolbox. Some definitions, lemmas and theorems that are necessary for the investigation are given in their correct sequence. Numerical and simulated output examples are provided to illustrate the effectiveness of the results.

Chapter 6 investigates time optimal and robust guaranteed cost control problems for neutral functional differential control systems with infinite delays. Using key results of Chapters 4 and 5 on stability and controllability, easily computable criteria for the system to be normal and completely controllable are developed and the time optimal control for the neutral system with infinite delays formulated. Furthermore, methods for obtaining an optimal robust guaranteed cost control problem via state feedback control laws for the systems are presented using a transformation technique combined with the Lyapunov matrix equation and the Razumikhin approach. A guaranteed cost control gain for the system is also obtained through an optimization problem. The verification of the conditions developed in the chapter is simple; examples with simulated state outputs are also given to illustrate the robustness of the methods.

Chapter 7 makes use of an example of a lossless transmission line to demonstrate the applicability of the research results. Simulation studies confirms expected theoretical results prior to the conclusion and further studies in Chapter 8 which draw all the key findings in this research thesis together and makes recommendations for further work.

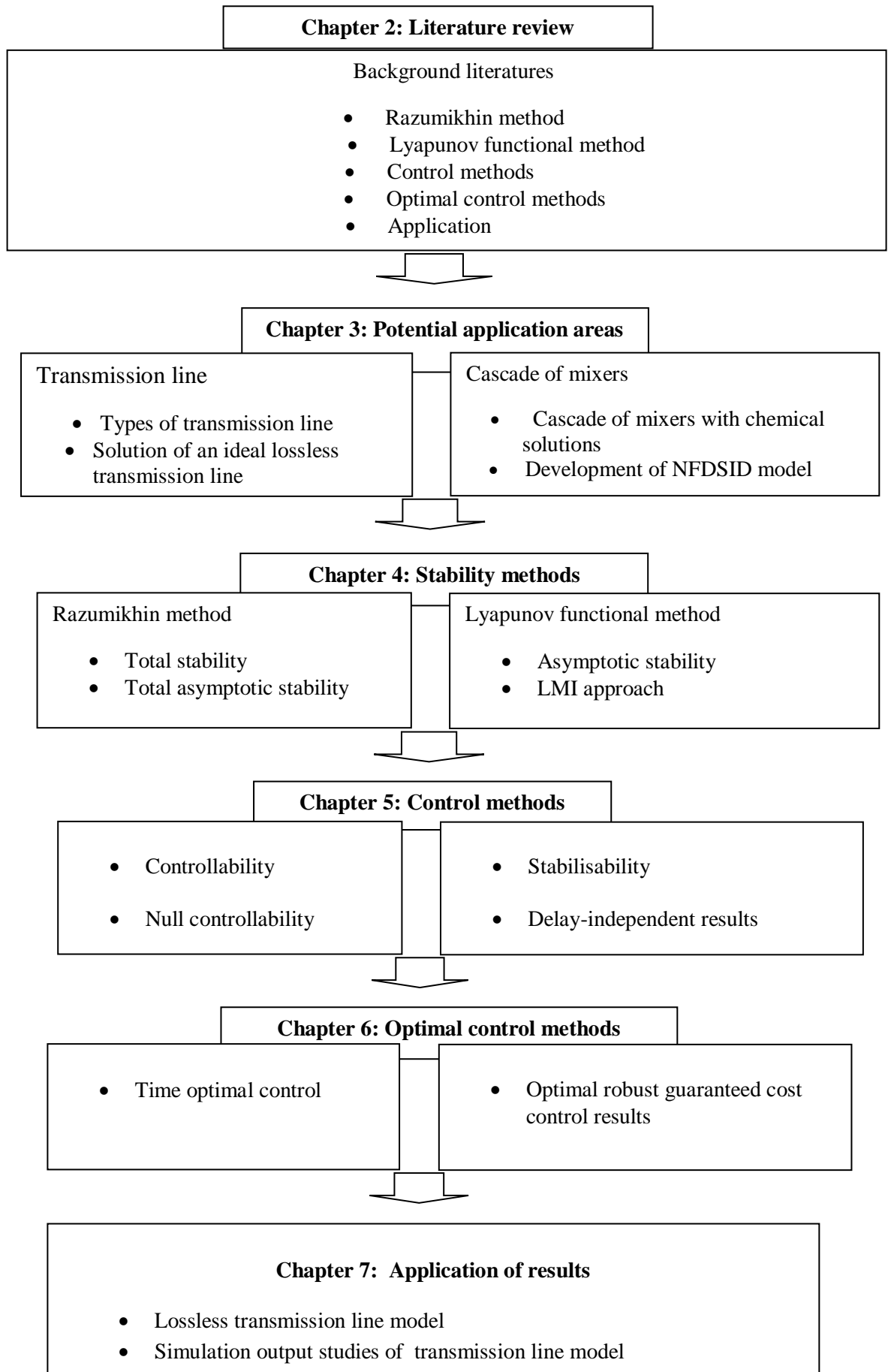


Figure 1.2: Framework of Thesis structure/flow chart

## **Chapter 2**

# **Literature review**

### **2.1. Introduction**

This chapter reviews relevant literature on stability, controllability and optimal control to justify the direction of the research. Relevant practical applications of neutral systems are reviewed leading to the selection of appropriate models to demonstrate the applicability of this work. First, stability methods are discussed leading to the selection and thorough review of Lyapunov stability methods in this research. Similarly, the types of controllable dynamical systems and appropriate controllability methods are discussed and the most appropriate to this research highlighted. The advantages associated with optimal control system designs are reviewed prior to describing the types of optimal control design. Finally, examples of application of neutral systems are given leading to the selection and detailed description of transmission line modelling and control.

### **2.2. Neutral functional differential systems**

As seen in Davies (2006) and the references therein differential equations, are an important tool that can harness interrelated systems' components, which otherwise might continue to remain independent of each other, into a single system. It provides the means to analyse the inter-relationships that exist between these different components of a physical system. Physical systems which express the present states of a situation are the most commonly encountered systems in the theory of differential equations. However, a more realistic system would encompass not only the present but also the past states or history of the system, otherwise referred to as the property of after-effect or time delays. To have a good grasp of

the state at present ( $t$ ), some knowledge of the past ( $t - r$ ),  $t \geq 0$ ,  $r > 0$  is important. Such systems were formulated by Volterra in 1928 with application to predator-prey models. This principle permeates various aspect of life and has lately influenced much research. It is now well known that the existence of time delays in a dynamical system has been the source of oscillation, instability and poor system performances.

In general, differential equations which involve the present as well as the past states of any physical system are called delay differential equations or functional differential equations. Research on the appearance of technical problems involving different delays can be found in the influence of hydro-shocks on the oscillations of turbines, feedback systems for hydroelectric power stations, ship stabilisation, control processes for pressure, heat transfer regulation, and time delays in feedback systems (see Kolmanovskii and Myshkis 1992 and references therein). However, they can also be introduced deliberately into feedback systems to improve system performance.

Delay differential equations can be classified into two broad types: retarded functional differential equations and neutral functional differential equations. This thesis will focus attention on the latter type, in which the evolution rate of the process described by such equations depends on the past as well as the present history. That is, one in which the derivatives of the past history or derivatives of functional of the past history are involved as well as the present states of the system.

Having introduced the concept of neutral functional differential systems, the following sections review stability methods, controllability results and optimal control of such system.

### **2.3. Stability of neutral functional differential systems**

Stability analyses for functional differential systems have attracted considerable research effort because of their applicability in various fields of research. The three most relevant

stability methods among stability theories for retarded functional differential systems (see Hale 1977, and Hale and Verduyn Lunel 1993) that will be reviewed in this Section are; the Lyapunov functional, comparison principle and Razumikhin method.

### 2.3.1. Lyapunov functional method

The Lyapunov functional method requires the construction of a Lyapunov function in terms of the rate of change of a functional along solution trajectories. The use of a functional is a natural generalization of the direct method of Lyapunov for ordinary differential equations. However, there are no general rules for constructing Lyapunov functions. The constructions are merely based on a researchers' experience and some particular techniques. Lyapunov functions were first developed in the 1950s by A. M. Lyapunov for the study of stability of systems described by ordinary differential equations in his famous studies '*Collected Works of Academician*' (Lyapunov 1967). Since then methods based on Lyapunov functions have been extended to study different kinds of stability of dynamical systems. For example, methods based on Lyapunov functions have been used to study:

(i) chaotic signals (Winful and Rahman 1990) in an array of coupled lasers given by the model

$$\left. \begin{aligned} \frac{dx_j}{dt} &= \frac{1}{2} \left[ \varphi(n_j) - \frac{1}{\sigma_p} \right] (1 - i\alpha)x_j + ik(x_{j+1} + x_{j-1}), \\ \frac{dn_j}{dt} &= \varphi_1 - \frac{n_j}{\sigma_s} - \varphi(n_j)|x_j|^2, \end{aligned} \right\} \quad (2.1)$$

where  $x_j$ ,  $n_j$  are the evolution of mode amplitude and the population in the  $j^{th}$  laser respectively,  $\varphi$  is the gain from lasing threshold of the uncoupled lasers,  $\sigma_p$ ,  $\sigma_s$  are the photon lifetime and lifetime of active populations respectively,  $\varphi_1$  is the pump rate,  $k$  is coupling strength between adjacent lasers and  $\alpha$  is a line width enhancement factor in semiconductor

lasers. See also (Pecora and Carroll 1990; 1991, and Carroll and Pecora 1991) for more studies on chaotic systems.

(ii) the stability of stochastic systems by Jeetendra and Vivin (2012) given by

$$\begin{aligned} dx(t) = & \left[ (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - h(t)) \right] dt \\ & + \left[ g(t, x(t), x(t - h(t))) \right] dw(t), \end{aligned} \quad (2.2)$$

where  $A, B$  are known real constant matrices with appropriate dimensions,  $w(t)$  is an  $m$  dimensional Brownian motion,  $h(t) \in [\underline{h}_1, \bar{h}_2]$ ,  $\dot{h}(t) \leq \mu < \infty$ ,  $\mu > 0$ ,  $g(\cdot) \in \mathbb{R}^{n \times n}$  is a nonlinear function,  $\Delta A(t)$  and  $\Delta B(t)$  are the parametric uncertainties with compatible dimensions. See also (Wei et al. 2007, Liu et al. 2009, and Deng et al. 2001) for more studies on the stability of stochastic systems.

(iii) impulsive systems of the form (2.3) where the system states changes abruptly at certain moments of time by Naghshtabrizi et al. (2008)

$$\begin{cases} \dot{x} = f_k(x(t), t), & t \neq \sigma_k, \quad \forall k \in \mathbb{N}, \\ x(\sigma_k) = g_k(x(\sigma_k^-), \sigma_k), & t = \sigma_k, \quad \forall k \in \mathbb{N}. \end{cases} \quad (2.3)$$

Here,  $f_k$  and  $g_k$  are locally Lipschitz functions from  $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $f_k(0, t)$ ,  $g_k(0, t)$  equals zero for all  $t \geq 0$ , and the impulse time sequence  $\{\sigma_k\}$  strictly increases in  $[\sigma_0, \infty)$  for some initial time  $\sigma_0$ . See also (Liu et al. 2012, Liu and Wang 2007, and Akhmet 2003) for similar studies on impulsive systems.

(iv) discrete time dynamical systems given by  $x_{k+1} = f(x_k)$  (Ahmadi and Parrilo 2008), where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  can be nonlinear, non-smooth, or uncertain. See also (Jiang and Wang 2002, Feng 2002, and Haddad and Bernstein 1994) for more studies on discrete systems.



### 2.3.2. Comparison method

The comparison principle requires finding an additional system with known stability properties, and then comparing that to the original time-delay system with an aim of establishing a comparison system free of delays from other previously reported stability criteria (see Liu and Marquez 2007). The comparison method has been used by a number of researchers; see for example (Knospe and Roozbehani 2003; 2006, Zhang et al. 2001; 2003). In particular, Knospe and Roozbehani (2006) demonstrated the use of the comparison principle to investigate stability of linear systems with multiple, time-invariant, independent and uncertain delays with each delay residing within a known interval outside zero given by the equation

$$\dot{x} = Ax(t) + \sum_{k=1}^N A_k x(t - h_k), \quad (2.4)$$

where  $A_k = H_k F_k$ ,  $H_k \in \mathbb{R}^{n \times q_k}$ ,  $F_k \in \mathbb{R}^{q_k \times n}$ , and  $h_k \in [\underline{h}_k, \bar{h}_k]$ . Establishing a delay free sufficient comparison system through the replacement of the elements with some parameters which satisfy certain conditions. It was also shown that robust stability of the comparison system guarantees stability of the original time-delay system without requiring any prior knowledge of the stability of the time-delay system for some fixed-delay.

### 2.3.3. Razumikhin technique

The Razumikhin technique is based on the application of Lyapunov functions. It essentially extends the stability theorem in Lyapunov's sense. It is considered to rehabilitate applications of Lyapunov functions on functional differential equation to a considerable extent in the sense that it uses functions which are much easier and natural to explore the possibility of using the rate of change of a function on the whole state space to determine sufficient conditions for stability. The Razumikhin technique has been found in some cases to be

simpler and more visual than an application of a general functional, and has been applied successfully by various authors in their stability problems for retarded functional differential systems (see Hale 1974; 1977, Hale and Verduyn Lunel 1993, and Myshkis 1995). In particular, the Razumikhin technique and Lyapunov functions have been employed to study (i) impulsive delay system (Liu and Ballinger 2001) of the form

$$\left. \begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \neq \sigma_k, \\ \Delta x(t) &= g(t, x_{t^-}), \quad t = \sigma_k, \end{aligned} \right\} \quad (2.5)$$

where  $f, g: J \times \mathcal{L}([-h, 0], E^n) \rightarrow E^n$  are given functionals with  $J \subset E_+$ ,  $\Delta x(t) = x(t) - x(t^-)$  and  $\sigma_k > \sigma_0 = 0$ ,  $k = 0, 1, 2, \dots$  satisfy  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ . See (Liu et al. 2006, Shen and Yan 1998, and Stamova and Stamov 2001) for more examples on impulsive systems, (ii) discrete delay system by Liu and Marquez (2007) of the form  $x(n+1) = f(n, x_n)$ ,  $n \geq n_0$ , where  $f \in C([-m, 0], E^n)$ ,  $m, n_0 \in \mathbb{N}$ . See also (Zhang and Chen 1998, Liu and Hill 2009, and Tsung-Lieh and Chien-Hua 1995) for more examples on discrete systems, (iii) stochastic delay system (Mao 1996) given by  $dx(t) = f(t, x_t)dt + g(t, x_t)dw(t)$ ,  $t \geq 0$ , where  $f: E_+ \times C([-h, 0], E^n) \rightarrow E^n$  and  $g: E_+ \times C([-h, 0], E^n) \rightarrow E^{n \times m}$  satisfy linear growth and local Lipschitz conditions. See (Liao and Mao (2000), Kolmanovskii and Myshkis (1992), and Kolmanovskii and Nosov (1986)) for similar studies on stochastic systems.

#### 2.3.4. Relationship between retarded and neutral functional differential systems

The analysis of different types of stability for linear systems of neutral functional differential equations is not as simple as that for retarded functional differential equations. Indeed, many surprising results occur even for autonomous systems (Hale and Verduyn Lunel 1993). It has been shown, in most cases, that stability of retarded systems would imply uniform stability for neutral systems. Again, one can have a result on asymptotic stability of a neutral system

and not have uniform asymptotic stability in the same system's analysis. This happens because uniform asymptotic stability of neutral systems is equivalent to exponential asymptotic stability. Hence, if the difference differential operator for a neutral system is stable, then the same relationship between the concepts of stability for linear autonomous equations for retarded functional differential equations can be obtained for neutral systems. That is, the difference differential operator plays a major role in neutral systems as in the case of the differential operator for retarded systems. It is important to know these relationships as a guide to deducing results that may appear from nonlinear neutral systems.

### **2.3.5. Stability analysis of neutral functional differential systems**

The stability analyses of retarded functional differential systems have been extended to neutral functional differential systems; see for example (Hale and Cruz 1969, Hu and Hu 1996, Hu et al. 2004, Li et al. 2007 and Yu 2013). This thesis will focus on the use of the Razumikhin technique because, for some neutral systems, it is difficult to construct the Lyapunov functional and the latter may require the analogue of the Razumikhin type technique. Classical results on neutral systems have been reported using the Razumikhin technique. For example, Haddock et al. (1994) generalized the results of Cruz and Hale (1970) on Lyapunov-Razumikhin asymptotic stability theorems to develop an invariance principle of Lyapunov-Razumikhin type for the equation  $d/dt(Dx_t) = f(x_t)$ , where,  $f: C([-h, 0], E) \rightarrow E^n$  is completely continuous and  $D: C([-h, 0], E^n) \rightarrow E^n$  is linear, continuous and atomic at zero in the sense used by Hale (1977: 50). This method has provided an effective tool for the study of asymptotic stability of neutral functional differential equations. Liu (2005), using some model transformation, the Lyapunov equation and decomposition technique, proposed delay-dependent criteria expressed in terms of Razumikhin-type theorem to derive a delay-dependent stability criteria for the neutral system

$$\dot{x}(t) - A_0 \dot{x}(t-h) = (A_1 + \Delta A)x(t) + (B + \Delta B)x(t-h), \quad (2.6)$$

where  $A_0, A_1, B$  are unknown constant matrices, and  $\Delta A, \Delta B$  are linear parametric uncertainties with given bounds. This method allows for model transformation and decomposition techniques that would guarantee an allowable bound on the time delay which could allow neutral systems to be tolerated if the time delays are less than the obtained constant delay bounds. For neutral stochastic functional differential equations, Mao et al. (1998) employed the Razumikhin technique to study the  $pth$  moment exponential stability for a neutral stochastic system of the form

$$d[x(t) - g_1(x_t)] = f(t, x_t)dt + g_2(t, x_t)dw(t), \quad (2.7)$$

where  $g_1: C([-h, 0], E^n) \rightarrow E^n$ ,  $g_2: E_+ \times C([-h, 0], E^n) \rightarrow E^{n \times m}$ , and  $f: E_+ \times C([-h, 0], E^n) \rightarrow E^n$  are all continuous functionals, deriving results for almost sure exponential stability from the  $pth$  moment exponential stability. By generalizing the Lyapunov-Razumikhin techniques, Lopes (1975) studied the existence of periodic solutions of a certain neutral functional differential system, where he gave sufficient conditions for uniform ultimate boundedness and proved the same for his neutral system given by

$$\frac{d}{dt}[x(t) - qx(t-h)] = g(t, x(t), x(t-h)). \quad (2.8)$$

Here,  $|q| < 1$  and  $g: E \times C([-h, 0], E^n) \rightarrow E^n$  is a continuous function. Some successful efforts have also been made by researchers through the use of the comparison method to investigate the stability of neutral systems. For example, Ionescu and Stefan (2009) have investigated the stability of a class of neutral systems given by

$$\frac{d}{dt}[x(t) - A_0 x(t-h)] = A_1 x(t) + A_2 x(t-h), \quad (2.9)$$

where  $A_0$ ,  $A_1$  and  $A_2$  are matrices of appropriate dimension using the comparison method to obtain two comparison systems whose robust stability gives a simple delay-dependent stability condition for the neutral system. These conditions guaranteed an a-priori upper bound for the degree of conservatism induced by the comparison method.

### **2.3.6. Stability analysis of neutral systems with infinite delays**

Unlike applications of Lyapunov–Razumikhin technique to neutral functional differential equations with finite delays, the transition from finite to infinite neutral functional differential equations has received little attention as it brings difficulties in the use of the technique and would require a new result (Haddock et al. 1994) which may involve:

- Comparison theorems using differential inequalities discussed in Section 2.3.2;
- Conditions for pre-compactness of positive orbits;
- Construction of various space phases and
- The natural relationship between the difference differential operator for the neutral system and differential operator for retarded system discussed in Section 2.3.4.

Having reviewed the research evolution it is now possible to use the natural relationship between neutral system and retarded systems to state the strategy adopted in investigating the total stability properties for NFDSID in this work. The main idea is to extend some basic stability results on Lyapunov-Razumikhin technique obtained by Murakami (1984) for the case of retarded functional differential systems to the case of NFDSID. This will be achieved by first applying a uniform stability property of the difference differential operator in the sense of Cruz and Hale (1970) to obtain new results for total stability. By decomposing a given neutral system with infinite delays into a sum of an equation with finite delays and its remainders, new perturbation result for the system with finite delays is first obtained using the Lyapunov-Razumikhin technique. The stability result of the original neutral system with

infinite delays is then analysed using the perturbation result of the system with finite delays. The comparison method will not be used in this thesis because the application in view will greatly depend on the stability of the actual system and not a comparative one.

#### **2.4. Controllability methods in retarded functional differential systems**

Controllability plays an important role in control of systems. It represents a major concept in modern control theory and its application. In this work, controllability is concerned with the possibility of steering the neutral control system with infinite delay from an arbitrary initial state to an arbitrary final state using set of admissible controls (see Klamka 2007). There are alternative definitions of controllability in the literature which depend on the kind of dynamical linear and nonlinear delay control systems (Klamka 2007). The investigation into controllability of linear and nonlinear delay control systems plays a central role, faces some fascinating challenges and approaches in real life application with some independent results obtained. Heemels and Camlibel (2007) extended classical results obtained from unconstrained and input-constrained linear systems by Kalman and co-workers in the 1960's (see Kalman et al. 1963). They established necessary and sufficient conditions for the controllability of a continuous-time linear system with input and state constraints of the form

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= A_1x(t) + B_1u(t), \end{aligned} \right\} \quad (2.10)$$

where all the matrices  $A, A_1, B, B_1$  are of appropriate dimensions by imposing the condition of right-invertibility on its transfer matrix. That is, fully characterizing controllability for the class of right-invertible linear systems having either state and input constraint or a combination of them in terms of algebraic conditions. Klamka (2007) studied controllability problems for finite-dimensional control systems described by linear stochastic ordinary differential state equations with a constant delay in the control

$$dx(t) = (Ax(t) + B_1u(t) + B_2u(t - h))dt + A_1dw(t), \quad (2.11)$$

by formulating and proving necessary and sufficient conditions for stochastic relative exact controllability in a prescribed time interval using techniques from deterministic controllability problems. Here  $A, A_1$  are  $n \times n$  dimensional constant matrices,  $B_1$  and  $B_2$  are  $n \times m$  dimensional constant matrices. It was shown that relative controllability of a deterministic linear associated dynamical system is equivalent to stochastic relative exact controllability and stochastic relative approximate controllability of the original linear stochastic dynamical system (2.11). Sikora (2003) has proved theorems concerning relative and approximate relative controllability with constrained controls for linear dynamical systems with multiple constant delays in the state of the form

$$\dot{x}(t) = \sum_{k=0}^M A_k x(t - h_k) + Bu(t), \quad t \geq 0, \quad (2.12)$$

where  $A_k, k = 0, 1, \dots, M$  are  $n \times n$  dimensional matrices with real elements,  $B$  is an  $n \times m$  dimensional matrix with real elements,  $0 = h_0 < h_k < h_m$ , by exploiting the notions of supporting function for attainable sets and the general permutation matrix. Dacka (1982) extended the methods used in studying controllability of nonlinear systems described by ordinary differential equations with implicit derivative to study systems of equations with delays in control given by the equation

$$\dot{x}(t) = A(t, x(t))x(t) + B(t, x(t))u(t) + B_1(t, x(t))u(w(t)) - f(t, x(t), \dot{x}(t)) \quad (2.13)$$

where  $A$  is an  $n \times n$  matrix,  $B, B_1$  are  $n \times m$  matrices,  $f$  is an  $n \times 1$  vector and the function  $w$  is an absolutely continuous and strictly increasing on  $[\sigma, t_1]$ ,  $t_1 \geq t$ , by using measure of non-compactness of a set and the Darbo's fixed-point theorem. Sinha and Yokomoto (1980)

have derived criterion for controllability of a nonlinear system with variable time delay of the form  $\dot{x}(t) = A(t)x(t) + f\left(t, x(t - h(t)), u(t)\right)$ , where  $A(t)$  is an  $n \times n$  matrix with continuous elements,  $-h(t) \leq t \leq 0$ , and  $f(\cdot)$  is a continuous function, by comparing a nonlinear system with and without delay, and in so doing examined the controllability of their systems. The comparison principle introduced in Section 2.3.2 can be applied to many physical systems. However, for controllability of nonlinear systems the attempt by Sinha and Yokomoto (1980) to determine sufficient conditions on a nonlinear function that would guarantee the domain of null controllability of the system to be the whole space was the first in the literature known to the author. A grasp of controllability of different retarded systems in this section is necessary to understand the controllability of neutral systems which is the focus of this research.

#### **2.4.1. Controllability methods in neutral functional differential systems**

The investigations on controllability of retarded functional differential systems have been extended to the controllability of linear and nonlinear neutral functional differential systems. See for example (Gahl 1978, Onwuatu 1984, Khartovskii 2012, Sakthivel et al. 2012, and Metel'skii and Minyuk 2007). Most corresponding controllability results from retarded systems were obtained by using the properties and the concepts of the difference differential operator for the neutral system involved. This concept is also explored in new technologies such as repetitive controls and will form part of the research technique in the development of null controllability results for the system to be investigated. Liu et al. (2007) have demonstrated the use of neutral functional differential systems in repetitive control by inserting artificial neutral delay into a control loop in order to boost the periodic signal control performance of the system. In the study of hyperbolic equations (Hale 1977: 7) has noted that sometimes the boundary control of linear hyperbolic equations can be more effectively studied by looking at the corresponding control problem for neutral functional



differential systems. A key difference between analysing nonlinear neutral control systems and nonlinear retarded control systems is that it is possible to reverse the time orientation for a large class of neutral control systems (Underwood and Chukwu 1988).

For linear autonomous systems of neutral type, Metel'skii and Minyuk (2008) have investigated the almost complete controllability of such systems of the form

$$\dot{x}(t) = Ax(t) + A_2x(t-h) + A_0\dot{x}(t-h) + Bu(t), \quad t > 0, \quad (2.14)$$

where  $A$ ,  $A_2$ ,  $A_0$  are  $n \times n$  constant matrices and  $B$  is a constant  $n \times m$  matrix, by showing that a spectral condition for the systems is necessary and sufficient for almost complete controllability of their systems. Khartovskii (2012) has obtained a criterion for the complete controllability of systems given by the equation

$$\dot{x}(t) - \sum_{k=1}^N A_{0k} \dot{x}(t-kh) = \sum_{k=0}^N A_k x(t-kh) + \sum_k B_k u(t-kh), \quad t \geq 0, \quad (2.15)$$

that involves solving boundary-value problems for ordinary linear differential equations with constant coefficients and calculating integrals of quasi-polynomial functions. Here  $A_{0k}$ ,  $A_k$  and  $B_k$  are constant matrices of an appropriate dimension. Recently Khartovskii and Pavlovskaya (2013) proposed a control method for such systems of the form (2.15) having commensurate delays in both state and control in cases where the complete controllability conditions are violated. That is, controlling such systems with an incomplete rank by using the existence of a full rank system for which the constructive identifiability problem is dual to the controllability problem for the incomplete rank system.

For nonlinear neutral systems, Gahl (1978) in his investigation of the controllability for nonlinear perturbation on a bounded interval for autonomous linear delay system of neutral type given by

$$\begin{aligned}\dot{x}(t) = & Ax(t) + A_2x(t-1) + A_0\dot{x}(t-1) + Bu(t) + B_1u(t-h) \\ & + f(t, x(t), \dot{x}(t), u(t)),\end{aligned}\tag{2.16}$$

have shown that if the linear delay neutral system is completely controllable then the perturbed system is completely controllable provided it satisfies certain growth and continuity conditions. Here,  $A$ ,  $A_2$ ,  $A_0$  are constant matrices with appropriate dimensions and  $f$  is a continuous function of  $t$  which satisfies certain growth and continuity conditions. Onwuatu (1984) established sufficient conditions for the null controllability in a function space of linear and nonlinear neutral systems using a similar approach to that Gahl (1978).

#### 2.4.2. Controllability methods in retarded and neutral integro-differential systems

The controllability of integro-differential systems has emerged in recent years with many researchers using the fixed point and other approaches to investigate such systems. Sivasundaram and Uvah (2008) gave sufficient conditions for the controllability of impulsive hybrid integro-differential systems in a finite interval given by the equation

$$\dot{x}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)ds + B(t)u(t)\tag{2.17}$$

where  $A(t)$ ,  $H(t) \in \mathcal{L}[E^+, E^{n^2}]$  and  $B(t) \in \mathcal{L}[E^+, E^{nm}]$ . By using the Schaefer fixed point theorem and controls whose initial and final values can be assigned in advance so that the set of points attainable by the trajectory of the control process make the whole state space.

Klamka (1999) has studied the relative controllability of semi-linear integro differential systems having infinite delays with both a linear and nonlinear part and with multiple lumped time varying delays in the control and state variables of the form

$$\begin{aligned}\dot{x}(t) = & L(t, x_t) + \int_{-\infty}^0 A(s)x(t+s)ds + \sum_{k=0}^N B_k(t)u(w_k(t)) \\ & + f(t, x(t), u(w_0(t)), u(w_1(t)), \dots, u(w_k(t)), \dots, u(w_N(t))),\end{aligned}\quad (2.18)$$

where the operator  $L$ , which is continuous in its first argument and linear in the other, is appropriately defined,  $A(s)$  is an  $n \times n$  matrix whose elements are square integrable on  $(-\infty, 0]$ ,  $B_k(t)$  are  $n \times m$  matrices which are continuous in  $t$ ,  $f$  is an  $n$ -dimensional vector function which is continuous in all its arguments and  $w_k(t)$  are twice continuously differentiable and strictly increasing functions on  $[\sigma, t_1]$ ,  $t_1 \geq t$ . His results were obtained using Schauder's fixed point theorem and information from the stability and relative controllability of the linear part.

For the study of neutral integro-differential systems, Balachandran et al. (1997) have developed sufficient condition for asymptotic null controllability in their investigation for the null controllability of nonlinear neutral Volterra integro-differential systems given by the equation

$$\begin{aligned}\frac{d}{dt} \left[ x(t) - \int_0^t H(t-s)x(s)ds - g(t) \right] \\ = Ax(t) + \int_0^t L(t-s)x(s)ds + B(t)u(t) + f(t, x(t), u(t))\end{aligned}\quad (2.19)$$

where  $H(t)$ ,  $L(t)$  are continuous  $n \times n$  matrix valued functions,  $B(t)$  is a continuous  $n \times m$  matrix valued function,  $A$  is a constant  $n \times n$  matrix,  $f: E_+ \times E^n \times E^m \rightarrow E^n$  and  $g: E_+ \rightarrow E^n$  are respectively continuous and absolutely continuous vector functions. The results were obtained by using the Leray-Schauder fixed point theorem. For other results on the controllability of neutral integro-differential system see (Balachandran and Balasubramaniam 1994, Balachandran and Anandhi 2003, and Korobov et al. 2001).

### 2.4.3. Controllability methods in neutral integro-differential systems with infinite delays

This research development is connected with the general theory of neutral functional differential equations. An understanding of the stability and control evolution of these systems is therefore necessary for the advancement of these results.

Balachandran and Dauer (1996) have studied the null controllability of nonlinear infinite delay systems with time varying multiple delays in control where they developed sufficient conditions for the null controllability of such systems described by the equation

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^0 A(s)x(t+s)ds + \sum_{k=0}^N B_k(t)u(h_k(t)) + f(t, x(t), u(t)), \quad (2.20)$$

where the operator  $L$  is appropriately defined and is continuous in its first argument, and linear in the other,  $A(s)$  is an  $n \times n$  matrix whose elements are square integrable on  $(-\infty, 0]$ ,  $B_k(t)$  are  $n \times m$  matrices which are continuous in  $t$ ,  $f$  satisfy some growth and continuity conditions and  $h_k(t)$  are twice continuously differentiable and strictly increasing functions on  $[\sigma, t_1]$ ,  $t_1 \geq t$ . The main idea used was to show that, if the uncontrolled system is uniformly asymptotically stable, and the linear system is controllable, then the nonlinear infinite delay system is null controllable. Davies (2006) has investigated the Euclidean null controllability of infinite neutral differential system of the form

$$\begin{aligned} \dot{x}(t) - A_0 \dot{x}(t-1) \\ = A_1 x(t) + A_2 x(t-1) + \int_{-\infty}^0 A(s)x(t+s)ds + Bu(t) \\ + B_1 u(t-h), \end{aligned} \quad (2.21)$$

establishing computable criteria for such systems by exploiting the stability of the free system and rank criterion for properness (that is being controllable). Here ,  $A_0$ ,  $A_1$ ,  $A_2$  are  $n \times n$

matrices,  $B, B_1$  are  $n \times m$  matrices,  $A(s)$  is an  $n \times n$  matrix whose elements are square integrable on  $(-\infty, 0]$ , and  $f$  is a continuous function which satisfies some growth condition. Onwuatu (1993) derived conditions for controllability of perturbed nonlinear systems with infinite delays in his study of null controllability for such neutral systems given by

$$\frac{d}{dt}D(t, x_t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^0 A(s)x(t+s)ds + f(t, x_t, u(t)), \quad (2.22)$$

where  $B$  is a  $n \times m$  matrix function,  $A(s)$  is an  $n \times n$  matrix whose elements are square integrable on  $(-\infty, 0]$ ,  $L, D$ , and  $f$  satisfy the smoothness conditions (Onwuatu 1993) imposed on them. His conditions were obtained by the study of the stability of the free linear base system and the controllability of the linear controllable base system, with an assumption that the perturbation function satisfies some smoothness and growth conditions. Dauer et al. (1998) extended on investigation of Onwuatu (1993) to establish sufficient conditions for null controllability of nonlinear neutral systems having both distributed and time-varying delays in control. By showing that, if the linear control system is proper and the free system without controls is uniformly asymptotically stable, then the systems are null controllable provided the perturbation function satisfies some growth conditions. Their results were established by using the Schauder fixed point theorem. Recently, Sun et al. (2013) developed sufficient conditions for controllability of a fractional neutral stochastic integro-differential system with infinite delays of the form

$${}^cD^\tau [x(t) + g(t, x_t)] = -Kx(t) + \int_0^t L(t, s, x_s)dw(s) + Bu(t) \quad (2.23)$$

where  ${}^cD^\tau$  is the fractional order derivative,  $\tau = (1/2, 1]$ ,  $K$  is an infinitesimal generator of an analytic semigroup of a bounded linear operator,  $g$  and  $L$  are appropriate mappings as specified in Sun et al. (2013), by using fractional calculus and Sadovskii's fixed point

principle. Bouzahir (2006) has proved a theorem on local existence and uniqueness of integral solutions for a class of partial neutral functional differential equations with infinite delays of the form

$$\frac{\partial}{\partial t} D x_t = K D x_t + B u(t) + f(t, x_t), \quad t \geq 0, \quad (2.24)$$

based on integrated semi-group theory and the Banach fixed point theorem. Here  $K$  is a linear operator in  $\mathcal{B}$ ,  $B$  is bounded linear operator from the space of admissible control functions into  $\mathcal{B}$ ,  $D$  is a bounded linear operator from the phase space of function into  $\mathcal{B}$  and  $f$  is a  $\mathcal{B}$ -valued nonlinear continuous mapping on  $E_+ \times \mathcal{B}$ .

From the foregoing, very little is known about the complete and null controllability of neutral integro-differential systems with infinite delays. This thesis aims to advance the results on null controllability through complete controllability for these systems by exploring the methods in the research evolution of Davies (2006), Dauer et al. (1998), Balachandran and Dauer (1996), Khartovskii and Pavlovskaya (2013), Khartovskii (2012), and Metel'skii and Minyuk (2007) as a key issue for settling the optimal control problems for such systems in this thesis.

## **2.5. Optimal control of neutral functional differential systems**

Optimal control theory is concerned with the determination of the best control signal that will cause the system of interest to satisfy some constraints, and at the same time minimise (or maximise) some performance criteria (see Kirk 1970: 3). Time optimal control of neutral functional differential equations is a subset of optimal control targeting systems such as transmission lines. See Section 2.6.1. Linear controllers of neutral systems are in most cases achieved by defining quadratic performance indices, see (Kent 1971). A justification for linear optimal control according to Anderson and Moore (1990) is that, many engineering

plants, prior to the addition of a controller to them, are linear. A linear controller is simple to implement physically, and will frequently suffice.

### **2.5.1. Advantages for optimal control of neutral functional differential systems**

The following are some advantages of finding a linear optimal control for neutral functional differential systems:

- Solutions for the linear forms of systems are easier to compute. Linear optimal control results may be applied to their nonlinear counterpart by replacing the nonlinear problem by a sequence of linear problems (Lewis 1986).
- Robustness properties, according to Anderson and Moore (1990), suggest that controller designs for nonlinear systems may sometimes be achieved by designing with assumption that the system is linear (even though it may not be a good approximation). By relying on this fact then, an optimally designed linear neutral system can tolerate nonlinearities without impairment of all its desirable properties. Linear optimal control of neutral systems can then provide a framework for a unified treatment of the classical control problems and extends the classes of neutral systems for which control designs may be achieved.

### **2.5.2. Approaches in optimal control for neutral functional differential systems**

There is a significant amount of research available on optimal control of neutral functional differential equation with different approaches; most are concerned with finding optimal control of these systems through the definition of a quadratic or other cost function (see Kent 1971, Banks and Kent 1972, Park and Kang 2001, and Chukwu 2001) and on the time optimal control method (see Chukwu 1988; 2001, Connor 1974, and Kent 1971).

### 2.5.3. Cost function method for optimal control of neutral functional differential systems

Banks and Kent (1972) have demonstrated the use of a quadratic cost function in finding an optimal control by investigating the optimal control of systems governed by functional differential equations of retarded and neutral type given by the equations

$$\left. \begin{aligned} \dot{x}(t) &= A_1(t)x(t) + A_2(t)x(t-h) + g(u(t), t), \\ \dot{x}(t) - A_0(t)\dot{x}(t-h) &= A_1(t)x(t) + A_2(t)x(t-h) + g(u(t), t), \end{aligned} \right\} \quad (2.25)$$

where  $A_0$ ,  $A_1$  and  $A_2$  are matrices of appropriate dimension and the function  $g: E^m \times E^1 \rightarrow E^n$  is continuous. Here, necessary conditions for optimality of problems concerned with a wide class of nonlinear neutral systems were obtained by using the approach introduced by Neustadt (1969). These conditions were applicable to problems with general restraints on the controls. The procedure applied was to split the end condition into two conflicting inequality constraints and use the methods of Neustadt (1969) to prove that the maximum principle is a necessary and, in the case of normality and convexity, also a sufficient optimal condition. Park and Kang (2001) derived conditions for the optimal control problem of a linear neutral differential equation with time varying delays of the form

$$\frac{d}{dt} \left[ x(t) - \sum_{k=1}^N A_{-1k} x(t-h_k) \right] = Kx(t) + \sum_{k=1}^N A_k x(t-h_k) + B(t)u(t), \quad (2.26)$$

by defining a quadratic cost function and dealing with properties of the fundamental solutions and its adjoint state equations. Here,  $A_{-1k}$  and  $A_k$  are well defined operators in  $W_2^{(0)}$ ,  $K$  is defined as a closed linear operator which the infinitesimal generator of a semigroup on  $W_2^{(0)}$ ,  $B \in L_\infty(J, W_2^{(0)})$ , and  $0 < h_1 < h_k < h_N$ . Chukwu (2001) studied an optimal control problem for a linear neutral control system of the form (2.26) in  $E^n$ , where he derived and proved conditions for the existence of an optimal control by defining an integral cost function



and using the properties of a fundamental matrix solution of the system. Here,  $A_{-1k}$  and  $A_k$  are well defined matrices in  $E^n$ ,  $B \in L_\infty(J, E^{nm})$ , and  $0 < h_1 < h_k < h_N$ . In economics, Boucek et al. (2012) have used two optimization methods to solve an optimal control problem for a linear neutral differential system arising in economics. The first one used was a variational method, while the second followed a dynamical programming approach through the reformulation of the latter as an ordinary differential equation in their appropriate state spaces. It was shown that the resulting Hamilton–Jacobi–Bellman (HJB) equation admitted a closed-form solution, and allowed for a finer characterization of the optimal dynamics when compared to the alternative vibrational method.

#### **2.5.4. Time-optimal control method for neutral functional differential systems**

The time-optimal control problem for neutral systems was implicitly touched upon by Kent (1971) as a consequence of a result on minimising a general cost function for nonlinear systems. In this investigation, he formulated a necessary condition for the time optimal control in the form of a maximum principle with no explicit general control law given. However, the first general rigorous solution of time optimal control for a linear system according to Chukwu (1988) was given by Bellman et al. (1956). Their approach, in terms of convex sets, has been the foundation for almost all subsequent investigation and included a proof of the existence of time optimal control which satisfies a maximum principle and a bang-bang principle.

This research has been extended to neutral systems, for example Connor (1974) has studied time optimal control of neutral systems of the form (2.14) in  $E^n$ , where  $A$ ,  $A_2$ ,  $A_0$  are continuous  $n \times n$  matrices and  $B$  is a continuous  $n \times m$  matrix by deriving a maximum principle for the time optimal problem for a linear neutral system having an integral constraint in the control. His results can be considered as an extension to those given by

Neustadt (1961) for ordinary differential systems. Chukwu (1988) has studied time optimal control problem for systems described by linear neutral systems given by

$$\frac{d}{dt}D(t, x_t) = L(t, x_t) + B(t)u(t), \quad t \geq 0 \quad (2.27)$$

where  $B$  is  $n \times m$  matrix function,  $L$  and  $D$  satisfy the smoothness conditions (Chukwu 1988) imposed on them. Here, he formulated a controllability condition for the systems, and developed criteria for the existence, form, uniqueness and general properties of the optimal control in function and Euclidean spaces.

Motivated by the works of Chukwu (1988; 2001), Neustadt (1961), Connor (1974), Banks and Kent (1972), this thesis will advance this investigation by considering the problem of reaching a continuously moving target in minimum time by a trajectory of the control system described by NFDSID.

## **2.6. Applications of neutral functional differential systems**

Neutral functional differential systems have applications in many fields. Before the 1960s, the stability of electric power systems was seldom threatened because of very conservative system designs which were based on fairly constant predictable load growth (Chukwu 2001). Environmental concerns, economic realities and other factors have now changed the situation, and led to the development of a new generation of equipment that is prone to cause network voltage collapse and acute instability (Chukwu 2001). Those responsible with power systems planning and operations in the real world have become increasingly concerned with the issue of stability for electric power systems. As the regions of stability for the equation of motion for such systems are now better understood, it is predicted that time-optimal control of voltage and current fluctuations of systems will receive greater importance and more research attention. The natural models for these voltage and current fluctuation of problems

arising from transmission lines takes the form of a neutral functional differential equation (see Slemrod 1971, Wu and Xia 1996; 1999, Angelov 2013, Chukwu 2001, and Salamon 1983).

It is also well known that the natural resources of this planet are not evenly distributed (Vemuri 1978) and behaviour of systems in the world is not always exemplary. For example, biological populations consume resources available to them unevenly and are subject to diseases, decay, and environmental pollution. The main aim for mathematical modelling of biological population is to better understand the functioning of their food chains, webs and the limits of robustness with respect to their dependence on internal and external conditions. However, most population models used to describe real concepts of contemporary ecological systems as observed in Vemuri (1978) are unrealistically simple, and may not effectively account for some intrinsic population cycle because of the randomness in natural phenomena. While the early mathematical model for population cycles was developed as a simple bilinear model (Morozov and Petrovskii 2009), later studies have shown that some important details of their system dynamics are not represented in the model and that even the smallest detail could have a crucial effect on a population cycle. Appropriate models that can effectively analyse, design, control and predict biological population dynamics has been a challenge for several decades now.

This thesis is aimed at developing such a model in the form of NFDSID through a cascade connection of two mixers with controlled chemical solutions in Chapter 3 and will form the basis of stability and controllability investigations in Chapters 4 and 5 respectively. In addition, this thesis aims to apply the theoretical result to distributed networks containing lossless transmission lines by first modelling them into NFDSID and then carry out simulation studies presented in Chapter 7.

## **2.7. Concluding remarks**

In this chapter relevant background literature to be used in developing the required theorems, algebraic methods and applications in this thesis has been provided. The classifications of delay equations and their importance in real life applications were discussed. Literatures about the neutral integro-differential equation with infinite delays, which has been identified as the subject for investigation in this thesis, were explicitly presented. The Razumikhin's method has been identified from literature as the most appropriate method for stability and stabilisation of the systems for this investigation and will be exploited in Chapter 3 and other chapters.

The control procedures have been reported and the approaches to controllability and null controllability of neutral systems have been presented. Furthermore, optimal control methods for neutral systems and their advantages have been introduced. Time optimal control and cost function methods known from literature as useful tools for analytic design and applications have been presented.

The potential application areas have been identified as transmission lines and cascade of controlled chemical solutions, they will be comprehensive analysed in Chapter 3, and chapter 7 respectively.

## **Chapter 3**

# **Potential application areas**

### **3.1. Introduction**

In this chapter relevant practical applications of neutral systems are reviewed leading to the selection of appropriate models to demonstrate the applicability of model proposed in this work and for a successful application of the theoretical background in Chapter 2. The procedure in this chapter is first to review some concepts on transmission lines. Literature on transmission line theories is then reviewed with a focus on the general solution for an ideal lossless transmission line representation derived in terms of voltage and current. Next, literature on how transmission of controlled chemical solution can support processes in the modelling of population control processes is reviewed, and a new mathematical model for a neutral control system is developed by using a cascade connection of two mixers with chemical solutions.

### **3.2. Transmission lines modelling**

A transmission line can be said to be a system of conductors whose cross-section made at any distance along the line remains constant, and are capable of providing a direct link to the energy transfer between the generator and the load. There are several types of transmission lines. Some common examples are given below.

### 3.2.1. Example of transmission lines

- Striplines and microstrips

These have short lengths not exceeding some centimetres. They are mostly used inside devices like amplifiers or filters. See Figure 3.1 and Figure 3.2 respectively.

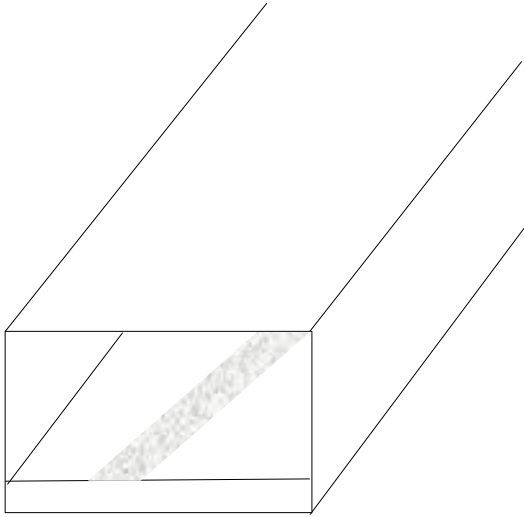


Figure 3.1: Strip-line

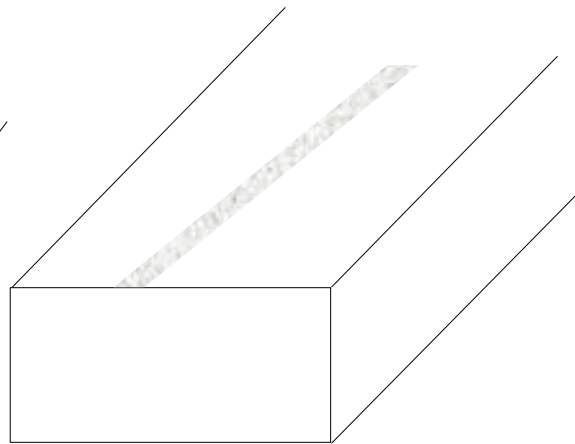


Figure 3.2: Micro-strip

- Twisted pairs and coaxial cables

These are commonly used for cabling of buildings, but coaxial cables are also used sometimes for inter-continental communications. See Figure 3.3 and Figure 3.4 respectively.

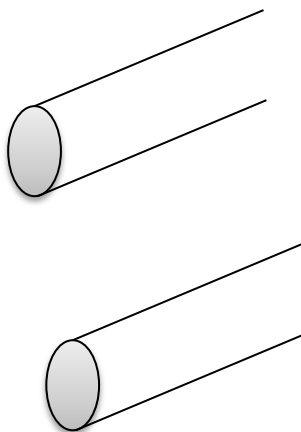


Figure 3.3: Two wire line

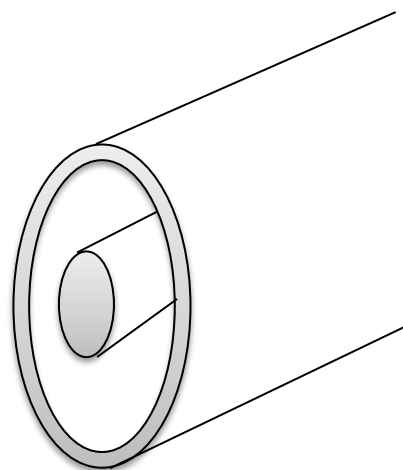


Figure 3.4: Coaxial cable

- Optical fibres

These are made from dielectric materials (see Figure 3.5) and are used to transmit microwave power over moderate distances.

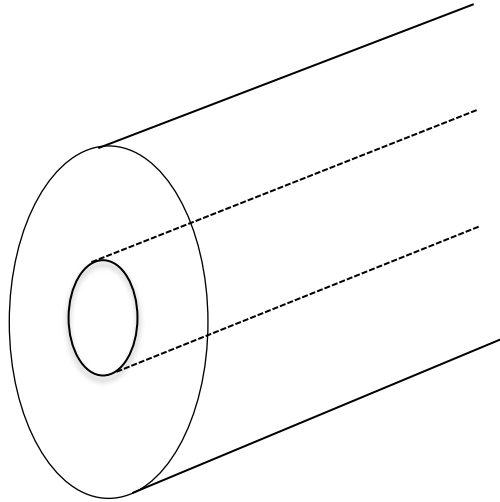


Figure 3.5: Optical fibre

This thesis will only consider structures consisting of two metal conductors, namely microstrips, strip-lines and coaxial cables. Parallel-wire line is used in most of the diagrams to represent the circuit connections for simplicity; however, the theory is the same for all types of transmission lines.

In order to derive the relationship between neutral differential systems and transmission lines in Chapter 7, this chapter will derive the differential behaviour of distributed circuits in terms of their voltage-current relationships to transmission lines.

### 3.2.2. Lossy transmission lines

Literature on transmission line theory very often deals with analysis for the ideal case without losses. Not addressing the transmission line theory for the general lossy case may be due to the fact that transmission systems require losses to be kept as low as possible to minimise its effect in the process of signal propagation. Losses refer to the amount of signal transmitted that does not reach the receiving end and they occur in different ways. They could be as a

result of impedance mismatch that leads to reflected energy, coupling of lossy transmission lines to adjacent traces, radiations, conduction and di-electric losses. Gago-Ribas and Carril-Campa (2012) have argued that, though transmission systems require very low losses in signal propagation, an analysis of the general lossy case would reveal that both the ideal lossless and low-lossy regime could be better explained and justified as a special case of the general lossy case. They added further that, a general analysis should allow for the parameterization of the effect of losses in behaviour of the parameters which would determine the final solution to a transmission line problem with specific boundary conditions. They stated that, parameterising the effect of losses in the system parameters can predict the ultimate behaviour of the problem and detect physical phenomenon associated with losses that may be of great practical interest. Gago-Ribas and Carril-Campa (2012) and Gago-Ribas et al. (2006) have identified complexity involved in analysing equations describing lossy transmission line models as a difficulty associated with studying lossy transmission line theories. They however gave a methodology that could be used to understand and predict the physical behaviour of the lossy transmission line problem by means of graphical representation which could help to avoid the complexity in analysing equations describing the model.

The difficulties involved in analysing the lossy transmission line model must become more complex when the interconnections are terminated with nonlinear loads, like diodes or transistors. The nonlinear terminators, together with coupled lossy transmission lines, could lead to a rather complicated simulation problem (see Dhaene and De Zutter 1992). The performance of high-speed transmission lines is usually determined and limited by the ability to transmit undisturbed and undistorted signals with the desired speed. Dhaene and De Zutter (1992) have given convolving transmission line impulse responses and use of lumped element equivalent circuits as the two basic ways of handling transmission lines in a transient-response simulation. They further formulated methodology for selecting the



minimal number of lumped elements needed to represent a lossy transmission line for a desired accuracy in a well-defined frequency range. The transient analysis of lossy transmission lines with nonlinear effects connected with semi-conductors becomes more interesting when the device is changing its state and/or when it is excited by a large-amplitude signal. Djordjevic et al. (1986) have investigated such nonlinear effects in multi-conductor transmission line systems by using time stepping and convolution with the transmission line impulse response method. The method was achieved by finding equivalent parameters of a suitable terminal (quasi-matched) multi-conductors transmission lines, which reduces the amount of computation required to obtain the final solution, and then computing the Green's functions.

For keeping desired accuracy in lossy transmission lines, Angelov and Hristov (2011) developed and obtained an analytical solution for a neutral system of the forms

$$\begin{aligned}
\frac{dv_0(t)}{dt} = & -\frac{di_0(t-T)}{dt} + \frac{R}{L}v_0(t) + \frac{R}{L}i_0(t-T) \\
& + \frac{(2e^{(R/L)/(t-T)}\sqrt{L}E_0(t-T) - (v_0(t) - i_0(t-T))Z_0)}{L_0} \\
& - \frac{2e^{(R/L)/(t-T)}\sqrt{L}R_0}{L_0} \left( \frac{e^{-(R/L)/(t-T)}}{2\sqrt{L}(v_0(t) + i_0(t))} \right) \\
& - \frac{2e^{(R/L)/(t-T)}\sqrt{L}C_0^{-1}}{L_0} \left( \int_T^t \frac{e^{-(R/L)/(t-T)}(v_0(t) + i_0(t-T))}{2\sqrt{L}} dt \right) \quad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{di_0(t)}{dt} = & -\frac{dv_0(t-T)}{dt} + \frac{R}{L}v_0(t-T) + \frac{R}{L}i_0(t) - \frac{2e^{(R/L)/(t-T)}\sqrt{L}E_1(t-T)}{L_1} \\
& + \frac{(v_0(t) - i_0(t-T))Z_0}{L_1} \\
& - \frac{2e^{(R/L)/(t-T)}\sqrt{L}}{L_1}R_1\frac{e^{-(R/L)/(t-T)}}{2\sqrt{L}}(v_0(t-T) - i_0(t)) \\
& - \frac{2e^{-(R/L)/(t-T)}\sqrt{L}C_0^{-1}}{L_1}\left(\int_T^t \frac{e^{-(R/L)/(t-T)}(v_0(t-T) + i_0(t))}{2\sqrt{L}}dt\right) \quad (3.2)
\end{aligned}$$

in terms of voltage and current respectively, where  $t \in [0, T]$ ,  $L_1$ ,  $C_1$  are characteristics of the nonlinear load on the line,  $R$ ,  $C$ ,  $L$  are specific parameters of the line,  $L_0$  is voltage on the inductor,  $C_0$  is the voltage of the condenser, and  $Z_0$  is the impedance of the line. Their result was obtained by developing conditions for the existence and uniqueness of periodic regimes. These conditions are then proved by finding an operator whose fixed points form a periodic solution of the neutral system. Angelov (2012) has formulated conditions for the existence-uniqueness of oscillatory regimes in lossy transmission lines terminated by in-series connected nonlinear RCL-load. This was achieved by transforming the mixed problem of hyperbolic systems to an initial value problem for a nonlinear neutral system similar to that in (3.1)-(3.2), and then introducing an operator representation of the oscillatory problem whose fixed points form an oscillatory solution of the initial value problem stated.

### 3.2.3. Lossless transmission lines

A transmission line is considered to be lossless if the conductors of the line have zero series resistance and the dielectric medium between the lines has infinite resistance. The equation of a lossless transmission line can be obtained from a circuit diagram (Figure 3.6 below) having conductance  $L$  per unit length and capacitance  $C$  per unit length, assuming that there are no losses in the line.

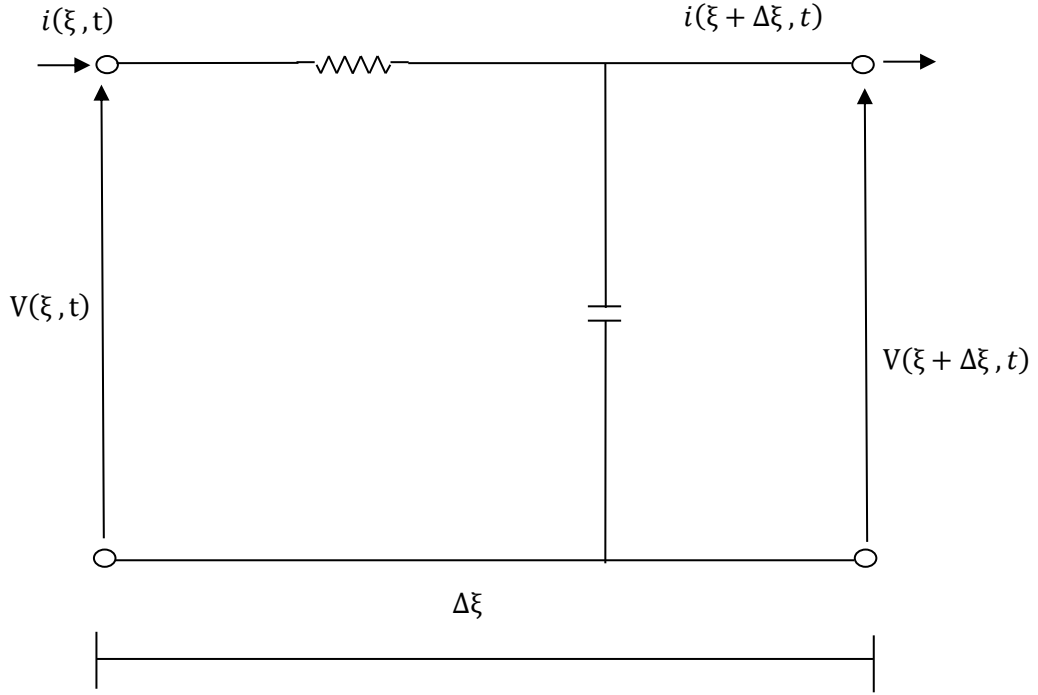


Figure 3.6: A circuit representation of lossless transmission line

The series inductance determines the variation of the voltage from input to output of the cell, and the current flowing through the shunt capacitance determine the variation of current from the input to output of the cell. The line equations can be represented by a system of first order partial differential equations (Telegrapher's equation) of the form (Orta 2012):

$$\frac{\partial v}{\partial \xi} + L \frac{\partial i}{\partial t} = 0 , \quad (3.3)$$

$$\frac{\partial i}{\partial \xi} + C \frac{\partial v}{\partial t} = 0 . \quad (3.4)$$

Differentiating equation (3.3) and (3.4) in terms of  $\xi$  and  $t$  respectively, gives

$$\left. \begin{aligned} \frac{\partial^2 v}{\partial \xi^2} + L \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial i}{\partial \xi} \right) &= 0 , \\ \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial i}{\partial \xi} \right) + C \frac{\partial^2 v}{\partial t^2} &= 0 . \end{aligned} \right\} \quad (3.5)$$

Making necessary substitution in terms of  $i(\xi, t)$ , a single order equation for the voltage  $v(\xi, t)$  alone is obtained as

$$\frac{\partial^2 v}{\partial \xi^2} - LC \frac{\partial^2 v}{\partial t^2} = 0 \quad (3.6)$$

The general solution for the voltage equation can be obtained from the wave equation (3.6). To solve (3.6), differentiate (3.3) and (3.4) in terms of  $t$  and  $\xi$  respectively and make the necessary substitution to obtain the initial conditions for the wave equation representing the ideal transmission line to get

$$v(\xi, 0) = v_0(\xi), \quad i(\xi, 0) = i_0(\xi). \quad (3.7)$$

Using a change of variable method, the solution of (3.6) using the initial conditions (3.7) can be obtained as follows. Define

$$\vartheta = \xi - v_{ph}t, \quad \rho = \xi + v_{ph}t,$$

so that,

$$\xi = \frac{1}{2}(\vartheta + \rho), \quad t = \frac{1}{2v_{ph}}(\rho - \vartheta).$$

Now writing the wave equation in terms of the new variables and making use of calculus (multivariable chain rule) gives,

$$\left. \begin{aligned} \frac{\partial v}{\partial \xi} &= \frac{\partial v}{\partial \vartheta} \frac{\partial \vartheta}{\partial \xi} + \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial \xi} = \frac{\partial v}{\partial \vartheta} + \frac{\partial v}{\partial \rho}, \\ \frac{\partial v}{\partial t} &= \frac{\partial v}{\partial \vartheta} \frac{\partial \vartheta}{\partial t} + \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial t} = -v_{ph} \left( \frac{\partial v}{\partial \vartheta} - \frac{\partial v}{\partial \rho} \right), \end{aligned} \right\} \quad (3.8)$$

and

$$\left. \begin{aligned} \frac{\partial^2 v}{\partial \xi^2} &= \frac{\partial}{\partial \vartheta} \left( \frac{\partial v}{\partial \vartheta} + \frac{\partial v}{\partial \rho} \right) + \frac{\partial}{\partial \rho} \left( \frac{\partial v}{\partial \vartheta} + \frac{\partial v}{\partial \rho} \right) \\ &= \frac{\partial^2 v}{\partial \vartheta^2} + 2 \frac{\partial^2 v}{\partial \vartheta \partial \rho} + \frac{\partial^2 v}{\partial \rho^2}, \\ \frac{\partial^2 v}{\partial t^2} &= v_{p\hbar} \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v}{\partial \rho} - \frac{\partial v}{\partial \vartheta} \right) v_{p\hbar} - \frac{\partial}{\partial \vartheta} \left( \frac{\partial v}{\partial \rho} + \frac{\partial v}{\partial \vartheta} \right) v_{p\hbar} \right] \\ &= v_{p\hbar} \left( \frac{\partial^2 v}{\partial \rho^2} - 2 \frac{\partial^2 v}{\partial \vartheta \partial \rho} + \frac{\partial^2 v}{\partial \vartheta^2} \right). \end{aligned} \right\} \quad (3.9)$$

Using (3.9), the wave equation in the new variable takes the form

$$\frac{\partial^2 v}{\partial \vartheta \partial \rho} = 0,$$

or

$$\frac{\partial}{\partial \rho} \left( \frac{\partial v}{\partial \vartheta} \right) = 0. \quad (3.10)$$

Let the solution of (3.10) be given by  $\partial v / \partial \vartheta = g(\vartheta)$ , where  $g$  is a constant with respect to  $\rho$ . Integrating (3.10) therefore gives

$$v(\vartheta, \rho) = \int g(\vartheta) d\vartheta + g_2(\rho), \quad (3.11)$$

where  $g_2$  is an arbitrary function of  $\rho$ . Denoting the integral of the arbitrary function  $g(\vartheta)$  as  $g_1(\vartheta)$ , the general solution of the wave equation can be written as

$$v(\vartheta, \rho) = g_1(\vartheta) d\vartheta + g_2(\rho), \quad (3.12)$$

Now, returning to the original variables and replacing  $g_1$  and  $g_2$  by  $v^+$  and  $v^-$  in order to derive the expression of the current, from (3.3) it is known from Orta (2012) that there exists a unique solution (D' Alembert solution), which is the general solution of the transmission line equation of the form

$$\left. \begin{aligned} v(\xi, t) &= v^+(\xi - v_{ph}t) + v^-(\xi + v_{ph}t), \\ i(\xi, t) &= Y_{\infty}v^+(\xi - v_{ph}t) - Y_{\infty}v^-(\xi + v_{ph}t), \end{aligned} \right\} \quad (3.13)$$

where  $Y_{\infty} = \sqrt{C/L}$  is the characteristic admittance of the line, and the symbols  $v^+$  is a constant with respect to  $\rho$ , that is, an arbitrary  $\vartheta$  and  $v^-$  represents an arbitrary function of  $\rho$ .

To derive the expression of the current, put (3.3) in the form

$$i(\xi, t) = -\frac{1}{L} \int \frac{\partial v(\xi, t)}{\partial \xi} dt. \quad (3.14)$$

Obtain an expression for  $\partial v / \partial \xi$  from (3.13) as

$$\frac{\partial v}{\partial \xi} = \dot{v}^+(\xi - v_{ph}t) + \dot{v}^-(\xi + v_{ph}t),$$

and substitute in (3.14) so that,

$$\begin{aligned} i(\xi, t) &= -\frac{1}{L} \left\{ \int \dot{v}^+(\xi - v_{ph}t) dt + \int \dot{v}^-(\xi + v_{ph}t) dt \right\} \\ &= -\frac{1}{L} \left\{ -\frac{1}{v_{ph}} \int \dot{v}^+(\vartheta) d\vartheta + \frac{1}{v_{ph}} \int \dot{v}^-(\rho) d\rho \right\} \\ &= Y_{\infty} \{ v^+(\xi - v_{ph}t) - v^-(\xi + v_{ph}t) \} \end{aligned}$$

The values of  $v^+(\vartheta)$  and  $v^-(\rho)$  can be obtained in such a way that the initial conditions are satisfied by writing equation (3.13) with  $t = 0$  to obtain

$$\left. \begin{aligned} v_0(\xi) &= v^+(\xi) + v^-(\xi), \\ i_0(\xi) &= Y_{\infty}v^+(\xi) - Y_{\infty}v^-(\xi). \end{aligned} \right\} \quad (3.15)$$

Rearranging (3.15) in terms of  $v^+(\xi)$  and  $v^-(\xi)$  gives

$$\left. \begin{aligned} v^+(\xi) &= v_0(\xi) + v^-(\xi), \\ v^-(\xi) &= v^+(\xi) - Z_{\infty}i_0(\xi). \end{aligned} \right\} \quad (3.16)$$

Solve the sum and difference of (3.16) to obtain

$$\begin{cases} v^+(\xi) = \frac{1}{2}(v_0(\xi) + Z_\infty i_0(\xi)), \\ v^-(\xi) = \frac{1}{2}(v_0(\xi) - Z_\infty i_0(\xi)). \end{cases} \quad (3.17)$$

By replacing the argument  $\xi$  in (3.17) with  $\xi - v_{ph}t$  in  $v^+$  and  $\xi + v_{ph}t$  in  $v^-$ , the overall solution for  $t > 0$  can be obtained by substituting the new values of  $v^+(\xi)$  and  $v^-(\xi)$  obtained in (3.17) into (3.13) and is given by

$$v(\xi, t) = \frac{1}{2}[v_0(\xi - v_{ph}t) + v_0(\xi + v_{ph}t)] = \frac{Z_\infty}{2}[i_0(\xi - v_{ph}t) + i_0(\xi + v_{ph}t)],$$

$$i(\xi, t) = \frac{Y_\infty}{2}[v_0(\xi - v_{ph}t) + v_0(\xi + v_{ph}t)] = \frac{1}{2}[i_0(\xi - v_{ph}t) + i_0(\xi + v_{ph}t)],$$

where  $Z_\infty = \sqrt{L/C}$  is called the characteristic impedance of the line.

An application of transmission lines regardless of the type means representation of some conductors into the circuit or sub-circuit, subject to some interconnections of distributed inductance and distributed capacitance, with or without resistance, resulting in the transfer of energy between a generator and a load. Significant research has been conducted on lossless transmission lines after Nagumo and Shimura (1961) derived a difference-differential equation from a transmission line with a tunnel diode on one end and presented self-oscillatory phenomena in transmission lines with a negative resistance element. For example, Shimura (1967) extended the work of Nagumo and Shimura (1961) to systems consisting of a lossless transmission line terminated with a tunnel diode and a lumped parallel capacitance on one end, where he obtained a nonlinear differential-difference equation which he analysed theoretically using available nonlinear techniques and obtained results for nonlinear phenomena on self-oscillation, synchronization and asynchronous quenching in his distributed systems. (Brayton 1966; 1967) introduced a more generalized difference-

differential equation of the neutral type where he obtained a set of uncoupled partial differential equations known as the wave equation. The solution to these wave equations was first obtained by D'Alembert in 1747. Using this equation, which describes the behaviour of voltage and current changes and the initial conditions at the terminals, he obtained D'Alembert's solution to the wave equation. He further analysed the existence of some periodic solutions of small amplitude that existed in transmission lines.

Subsequent to this derived difference-differential equation and its application to lossless transmission lines, this topic has been investigated by many authors (see Ferreira 1986, Hale 1977, Krawcewicz et al. 1993, Lopes 1975; 1976, Slemrod 1971, Wu and Xia 1996, Angelov 2013, Chukwu 2001, and Salamon 1983). In particular, Hale (1977) introduced a neutral functional differential equation with the stable operator  $D$  (the differential difference operator for neutral systems) into an application of lossless transmission lines in his discussion of simple oscillatory regimes present in such transmission lines systems. He assumed that, if interaction of the components of the coupled systems takes place immediately, then the connection between the systems can be described by a system of linear hyperbolic partial differential equations. These equations would have boundary conditions that lead to differential equations with delays in the highest order derivatives. Lopes (1976) studied problems like that of Brayton (1967) which is governed by the same physical problem using a new technique that involves the use of Lyapunov functionals and deduced, the problem of stability and uniform boundedness for a scalar ordinary differential inequality under some assumptions. Slemrod (1971) considered a network circuit with a lossless transmission line with specific inductance and capacitance in his study of the nonexistence of oscillations in a nonlinear distributed network. By reducing the distributed problem to a nonlinear functional differential equation of the neutral type, he analysed how asymptotic stability of the equilibrium state for his neutral equation guaranteed nonexistence of oscillation for the



distributed network. The approach of this application used in this thesis is to reduce an equation of nonlinear interconnected lossless transmission lines into the form of a NFDSID and finding its range of application through a simulation output study of the model.

### **3.3. Population growth modelling**

The transmission of controlled chemical solution can support processes in microbiological growth, production of useful products, and death. Because of these processes, their evolution can be more efficiently modelled in the form of neutral functional differential system as it depends on their past histories. Using NFDSID will help to account for various intricate factors in their evolution such as the distribution effect on juvenile birth rate which is related to the environmental suitability and sustainability of matured organisms, and the integral of the unknown function can be used to satisfy this relation.

It is shown in Gopalsamy (1992) that accumulation of metabolic products may seriously inconvenience a biological population, and one consequences of such accumulation can be a fall in the birth rate and an increase in the mortality rate. One of the simplest models describing species struggling for limited self-renewing food resources without consideration to migration, age heterogeneity, interaction with other species and other factors according to Kolmanovskii and Myshkis (1992) and references therein, is the logistic model

$$\dot{x}(t) = \zeta[1 - m^{-1}x(t - h)]x(t). \quad (3.18)$$

where,  $x(t)$  represents the population,  $h$  is the production time of food resources, the constant  $\zeta$  represents the difference birth and death rates and  $m$  represents the average population number. It has been observed in Kolmanovskii and Myshkis (1992), that a drop in birth rate caused by accumulation of metabolic products can be expressed by a power of the integral term in the logistic equation of (3.18).

Some control measures can be introduced into system models in order to account for some action of various factors on the system, such a model was given by the bilinear delay model (see Kolmanovskii and Myshkis (1992) and references therein)

$$\begin{cases} \dot{x}_1(t) = \zeta(t)x_1(t) - u(t)x_1(t) - \beta x_1(t-h), \\ \dot{x}_2(t) = \zeta(t)\alpha^{-1}x_1(t) - u(t)x_2(t) + mu(t), \end{cases} \quad (3.19)$$

Here, the first equation of (3.19) is a balance equation of biological substrate when bacteria are introduced into a vessel with an entrance to enable nourishment of substances and another for extraction of resulting product, while the second equation represents the production of resulting mass by the bacteria.  $x_1(t)$  represents the volume of microbiological substances,  $x_2(t)$  is the volume of the resulting product,  $u(t)$  is the volume of nourishing environment in the vessel,  $\zeta$  represents the rate of biological growth,  $x_1(t-h)$  is the account for loss of bacteria during a finite time  $h$ ,  $\beta$  and  $m$  are constant in the model, while  $\alpha$  is a rate of growth of the useful product. Sikora (2003) has presented a chemical solution control system where he developed a mathematical model for a dynamical system with delay in state given by

$$\dot{x}(t) = A_0x(t) - A_1x(t-h) + Bu(t), \quad (3.20)$$

where,  $x(t) \in E^n$ ,  $u(t) \in E^m$  are the state and control variables respectively,  $h$  is a constant delay,  $A_0$ ,  $A_1$  are  $n \times n$  matrices with real elements, while  $B$  is an  $n \times m$  matrix with real elements. Another method of modelling population control processes proposed in this thesis is by neutral functional differential equations. Baker et al. (2008) have illustrated with an example, roles that may be played by neutral system model which takes the form

$$\dot{x} = \zeta(\gamma; t, x(t), x(t-h), \dot{x}(t-h)) \quad (3.21)$$

where  $\gamma \in E^n$ ,  $h > 0$ , in their modelling of cell growth phenomena that display a time lag in reacting to events. See also Lu and Ge (2004), and Tang and Tang (2012) for other neutral functional differential equation population models, and Payam and Mansour (2014) for

modelling delay population control processes using neutral functional differential equation dynamics.

Having observed the use of mathematical analysis and design in the development of biological system models, the next section will be dedicated to the development of new NFDSID control system model that will enable the development, analysis and application of various stability and control techniques in-line with the research aim and objectives given in Section 1.6

### 3.4. Formulation of neutral control system model

One of the contributions of this thesis presented in this chapter is the development of a neutral differential delay system model. Following the methods of Sikora (2003), and Kolmanovskii and Myshkis (1992) a model for a neutral differential delay system is developed through a system of chemical solutions. Consider the cascade connection of two fully filled mixers according to the schematic diagram presented in Figure 3.7, where  $C_{in1}$  and  $C_{in2}$  are the input concentrations of the product,  $Q_1^*$  and  $Q_2^*$  denotes the constant flow intensities for the concentrations  $C_{in1}$  and  $C_{in2}$  respectively,  $V_{m1}$  and  $V_{m2}$  are the volumes of Mixer 1 and Mixer 2 respectively,  $C_1(t)$  and  $C_2(t)$  represent the total length of solutions in Mixer 1 and Mixer 2 respectively, and  $h$  is a constant delay arising from the tap or the reactor.

The chemical solution control process in Figure 3.7 below can be described by state equations as follows

$$\frac{V_m dC_1(t)}{dt} - \frac{V_m dC_1(t-h)}{2 dt} = Q_1^* C_{in1}(t) - Q_1^* C_1(t) - Q_1^* C_1(t-h),$$

$$\begin{aligned} \frac{V_m dC_2(t)}{dt} - \frac{V_m dC_2(t-h)}{2 \frac{dt}{dt}} \\ = Q_2^* C_{in2}(t) + Q_1^* C_1(t-h) - (Q_1^* + Q_2^*) C_2(t) - (Q_1^* + Q_2^*) C_2(t-h). \end{aligned}$$

After a transformation, the equations become

$$\frac{dC_1(t)}{dt} - \frac{1}{2} \frac{dC_1(t-h)}{dt} = \frac{Q_1^*}{V_m} C_{in1}(t) - \frac{Q_1^*}{V_m} C_1(t) - \frac{Q_1^*}{V_m} C_1(t-h) \quad (3.22)$$

$$\begin{aligned} \frac{dC_2(t)}{dt} - \frac{1}{2} \frac{dC_2(t-h)}{dt} \\ = \frac{Q_2^*}{V_m} C_{in2}(t) - \frac{(Q_1^* + Q_2^*)}{V_m} C_2(t) + \frac{Q_1^*}{V_m} C_1(t-h) \\ - \frac{(Q_1^* + Q_2^*)}{V_m} C_2(t-h). \end{aligned} \quad (3.23)$$

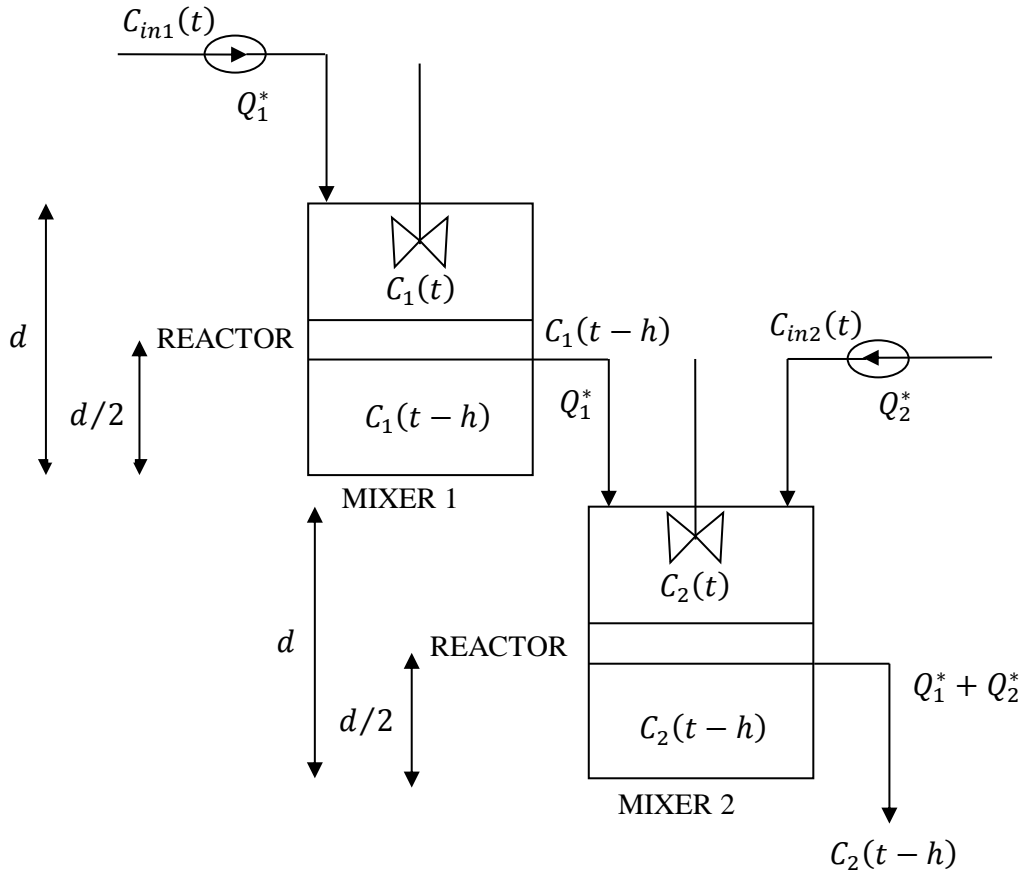


Figure 3.7: Schematic diagram of a cascade connection of two mixers

If it is assumed that  $C_1(t) = x_1(t)$ ,  $C_2(t) = x_2(t)$ ,  $C_{in1}(t) = u_1(t)$  and  $C_{in2}(t) = u_2(t)$ , where  $V_m = V_{m1} = V_{m2}$ , then a mathematical model for a neutral functional differential delay system is developed which is described by

$$\dot{x} - A_0 \dot{x}(t-h) = A_1 x(t) + A_2 x(t-h) + Bu(t), \quad (3.24)$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad A_2 = \begin{pmatrix} -Q_1^*/V_m & 0 \\ Q_1^*/V_m & -(Q_1^* + Q_2^*)/V_m \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -Q_1^*/V_m & 0 \\ 0 & -(Q_1^* + Q_2^*)/V_m \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} Q_1^*/V_m & 0 \\ 0 & Q_2^*/V_m \end{pmatrix}$$

If it is assumed that the input concentration of the product in the neutral model above is associated with the accumulation of metabolic products, then a more active reaction may take place in the upper half of Mixer 2. The terms in (3.24) can then be further transformed to include an integral term to get a NFDSID of the form (4.1) with  $u = 0$  or (5.2) with control measures.

### 3.5. Other application areas of neutral functional differential system

Neutral functional differential systems can also be found in such applicable areas as population ecology, heat exchangers and robots in contact with rigid environments (see Kolmanovskii and Myshkis 1992, Kolmanovskii and Nosov 1986, Kuang 1993, and Niculescu 2001). Examples of applications in engineering include air craft stabilisation, manual control, micro wave oscillators, laser models, neural networks, nuclear reactor and ship stabilisation (see Hale and Verduyn Lunel 1993, and Burnham and Ersanilli 2011). Examples from biology include predator-prey models, spread of measles in metropolitan areas models and model for the analysis of gonorrhoea (see Hale and Verduyn Lunel 1993). Applications in economics include dynamics of capital growth of global economy (see

Chukwu 2001), and applications in drilling include oil well drilling processes (see Saldivar Marquez et al. 2015).

These application areas have motivated extensive research in the area and are now extended to controllability of neutral functional control systems (see Onwuatu 1993, Chukwu 2001, Han 2002, Khartovskii and Pavlovskaya 2013, and Khartovskii 2012).

The study of integro-differential equations with infinite delays emerged in recent years as a branch of modern research owing to the difficulty that arises in presenting real-life situations in ecology, epidemics, population growth, and its connection with many fields of study such as continuum mechanics, system theory, viscoelasticity, and chemical oscillations (see Balachandran and Dauer 1996 and references therein).

The controllability of integro-differential systems has attracted the attention of many researchers in recent years because of its applications in many engineering and scientific disciplines. Some of the application areas include unsteady aerodynamics and aero-elastic phenomena, viscoelastic panels in supersonic gas flow, fluid dynamics, electrodynamics of complex media, population growth, polymer rheology, sandwich system identification, materials with fading memory, diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, lossless transmission lines, nuclear reactors and hereditary phenomena (see Sivasundaram and Uvah 2008). Balachandran et al. (1997) has noted that a nonlinear neutral Volterra integro-differential model is a good example for representing compartmental models such as the radiogram, where the two compartments correspond to the left and right ventricles of the heart and the pipes between these compartments represent the pulmonary and systematic circulation. Pipes coming out of and going back into the same compartment may represent shunts and the coronary circulation.

### **3.6. Concluding remarks**

This Chapter has reviewed applications of neutral system and focused on modelling two systems that will be exploited to validate and illustrate the theoretical results within this thesis. Following a review of transmission lines, it was found that voltage and current fluctuations arising from transmission lines can be conveniently modelled as neutral functional differential systems. The second application selected is addressing the determination of controlled chemical solution. A new neutral functional differential control system model with infinite delay was developed. The general form of the developed model in this chapter will form the fulcrum of the stability and control research analysis in Chapters 4 and 5 respectively. These stability and control analysis are also the fundamental issues for establishing optimal control in Chapter 6 and the application to transmission line in Chapter 7.

## Chapter 4

# Stability methods

### 4.1. Introduction

In this chapter, the Razumikhin approach, which is the main focus of application in this thesis, is used to investigate the stability of the developed neutral system with infinite delays model given in (4.1). By using the well-established Razumikhin's technique, new stability results are obtained which extend and complement basic stability results in functional differential equations to NFDSID.

The widely used Lyapunov-Krasovskii approach, which often leads to Linear Matrix Inequality (LMI) results, is also used to investigate the system modelled in Chapter 3. The approach is based on embedding the infinite delay into a norm-bounded uncertainty element and constructing a Lyapunov functional in order to obtain a novel less conservative stability condition in terms of LMI.

### 4.2. Model for a neutral system with infinite delay

This chapter will consider a neutral functional differential system with infinite delays of the form

$$\left. \begin{aligned} \frac{d}{dt}D(t)x_t &= A_1x(t) + A_2(t)x(t-h) + \int_{-\infty}^0 G(s, x_s)ds \\ x(t) &= \phi(t), \quad t \in (-\infty, 0] \end{aligned} \right\}, \quad (4.1)$$

and its difference differential operator



$$D(t)x_t = x(t) - A_0x(t-h).$$

The restriction on its difference differential operator is given by

$$D\phi = \phi(0) - A_0\phi(-h), \quad (4.2)$$

with the following assumptions:

- (i)  $A_0, A_1$  are  $n \times n$  constant matrices
- (ii)  $A_2(t)$  is an  $n \times n$  continuous matrix defined on  $[0, \infty)$ ,  
where  $\sup_{t \in [0, \infty)} \|A_2(t)\| \leq c$  is a constant for some  $c > 0$
- (iii)  $G: (-\infty, 0] \times E^n \rightarrow E^n$  is a continuous function which satisfies  $|G(t, x)| \leq M(t)\|x\|$   
for all  $(t, \phi) \in (-\infty, 0] \times E^n$ , where  $\int_{-\infty}^0 M(s)ds < \infty$ ,
- (iv) The constant time delay  $h$  is positive.

Throughout this chapter, it is assumed that  $G$  and  $D$  satisfy sufficient smoothness conditions to ensure that a solution of (4.1) exists through each  $(\sigma, \phi)$ , is unique, and depends continuously upon  $(\sigma, \phi)$  and can be extended to the right as long as the trajectory remains in a bounded set  $[\sigma, \infty) \times C$ . These conditions are given in (Hale and Cruz 1969).

Definitions which are required for this chapter will now be given

Consider the neutral system  $\frac{d}{dt}D(t)x_t = f(t, x_t)$ , and define the continuous function  $f: [\sigma, \infty) \times C \rightarrow E^n$  by using (4.1) as  $f(t, x_t) = A_1x(t) + A_2(t)x(t-h) + \int_{-\infty}^0 G(s, x_s)ds$ .

**Definition 4.1: (Total Stability)**

The solution  $x = 0$  of (4.1) is totally stable if for each  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon) > 0$  such that if  $|g(t, \phi)| < \delta(\varepsilon)$  for all  $(t, \phi) \in [\sigma, \infty) \times C$ , where  $g: [\sigma, \infty) \times C \rightarrow E^n$  is any

continuous matrix function, then the solution  $x(t, \sigma, \phi, f + g)$  of (4.1) satisfies  $\|x_t(\sigma, \phi, f + g)\| < \varepsilon$  for  $\sigma \in [\tau, \infty)$ ,  $t \geq \sigma$ ,  $\phi \in C$ .

**Definition 4.2: (Total Asymptotic Stability)**

The solution  $x = 0$  of (4.1) is totally asymptotically stable, if it is totally stable and there exist  $\delta_0 > 0$  and  $\gamma_0 > 0$  with the property that for any  $\varepsilon > 0$  there exist  $\gamma(\varepsilon) > 0$  and  $T(\varepsilon) > 0$  such that if  $|g(t, \phi)| < \gamma(\varepsilon)$  then  $\|x_t(\sigma, \phi, f + g)\| < \varepsilon$  for all  $t \geq \sigma$ ,  $\phi \in C$ .

**Remark 4.1**

Note also that, the zero solution of (4.1) is totally stable if and only if for any  $\varepsilon > 0$  there exists a  $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$  such that if  $(\sigma, \phi, p) \in [\tau, \infty) \times C \times C$  and  $\sup_{t \geq \sigma} |p(t)| < \bar{\delta}(\varepsilon)$ , then

$\|x_t(\sigma, \phi, f + p)\| < \varepsilon$  for all  $t \geq \sigma$ . Moreover, the zero solution of (4.1) is totally asymptotically stable if and only if it is totally stable and there exists a  $\bar{\delta}_0 > 0$  with the property that for any  $\varepsilon > 0$  there exists  $\bar{\gamma}(\varepsilon) > 0$  and  $\bar{T}(\varepsilon) > 0$  such that if

$(\sigma, \phi, p) \in [\tau, \infty) \times C \times C$  and  $\sup_{t \geq \sigma} |p(t)| < \bar{\gamma}(\varepsilon)$  then  $\|x(t, \sigma, \phi, f + p)\| < \varepsilon$  for all  $t \geq \sigma + \bar{T}(\varepsilon)$ . Note also that by employing the same arguments in Murakami (1984) and references therein  $\delta(\cdot)$ , with respect to  $\delta_0, \gamma_0, \gamma(\cdot), T(\cdot)$  satisfies the condition in Definition 4.1, if  $\delta(\cdot) = \bar{\delta}(\cdot)$ ,  $\delta_0 = \min(\bar{\delta}(1), \bar{\delta}_0)$ ,  $\gamma_0 = 1$ ,  $\gamma(\cdot) = \min(\bar{\gamma}(\cdot), \bar{\delta}(1))$  and  $T(\cdot) = \bar{T}(\cdot)$

Consider the system

$$\begin{cases} D(t)x_t = D\phi + K(t) - K(\sigma), & t \geq \sigma \\ x_\sigma = \phi, \end{cases} \quad (4.3)$$

where,  $K \in C([\tau, \infty), E^n)$ , the space of continuous functions taking  $[\tau, \infty)$  into  $E^n$ ,  $\sigma \in [\tau, \infty)$ ,  $\phi \in C$ , and  $D$  is the restriction on the difference operator defined in (4.2).

**Definition 4.3: (Uniform Stability)**

Suppose  $S$  is a subset of  $C([\tau, \infty), E^n)$ . The operator  $D(\cdot)$  is uniformly stable with respect to  $S$  if there are constants  $c, d$  such that for any  $\phi \in C, \sigma \in [\tau, \infty)$  and  $K \in S$ , the solution  $x(\sigma, \phi, K)$  of (4.3) satisfies

$$|x_t(\sigma, \phi, K)| \leq c|\phi| + d \max_{\sigma \leq s \leq t} |K(s) - K(\sigma)|, \quad t \geq \sigma, \quad (4.4)$$

**Definition 4.4: (Uniform Stability)**

If  $S = \{0\}$  and  $D(t, \phi) = D\phi$  is uniformly stable with respect to  $\{0\}$  then relation (4.4) implies that the solution of the homogeneous difference equation

$$\begin{cases} D(t)x_t = 0, & t \geq \sigma \\ x_\sigma = \phi, & D\phi = 0 \end{cases}$$

are uniformly stable (Cruz and Hale 1970).

**Definition 4.5 (Uniform Asymptotic Stability)**

The solution  $x = 0$  of (4.1) is uniformly asymptotically stable if and only if there exists constant  $c > 0, k > 0$  such that  $|x_t(\sigma, \phi)| \leq k \exp[-c(t - \sigma)]|\phi|$ , for all  $t \geq \sigma$ .

The next three lemmas are due to Cruz and Hale (1970). They are very important for the analysis and development of the properties for operator  $D(\cdot)$ , and for the overall stability result of this chapter in this section.

**Lemma 4.1**

Let  $A$  be an  $n \times n$  constant matrix. The operator  $D\phi = \phi(0) - A\phi(-h)$  is uniformly stable if all the roots of the equation  $\det[I - Ar^{-h}] = 0$  have moduli less than 1. This holds if

$$\|A\| < 1.$$

**Proof.** Assume in (4.3) that  $h = 1$ ,  $\sigma = 0$ , and let the matrix  $[I - A]$  be non-singular so that the  $\xi = [I - A]^{-1}D\phi$ , is well defined. If  $h(t) = K(t) - K(0)$ ,  $y_t = x_t - \xi$ ,  $\varphi(\theta) = \phi(0) - \xi$  in (4.3), then  $Dy_t = h(t)$ , where  $y_0 = \varphi$ ,  $D\varphi = 0$ . Setting  $z = (z^{(1)}, \dots, z^{(N)})$ ,  $z^{(k+1)}(t) = y(t - k)$ ,  $k = 0, 1, \dots, N - 1$ , then the system can be written as  $z(t) = Az(t - 1) + h^*(t)$ ,  $t \geq 0$ ,  $z_0 = \psi$ , where,  $N$  is an integer,  $|\psi| \leq L\varphi$  for some constant  $L$ ,  $h^* = (h, 0, \dots, 0, 0)$  and the eigenvalues of  $A$  have moduli less than 1. If  $k < t + \theta$  is the greatest integer then,  $z(t + \theta) = A^{k+1}\psi(t + \theta - k - 1) + h^*(t + \theta) + Ah^*(t + \theta - 1) + \dots + A^k h^*(t + \theta - k)$ , for  $-1 \leq \theta \leq 0$ . Therefore,

$$|z(t + \theta)| \leq |A^{k+1}| \cdot |\psi| + [1 + |A| + \dots + |A^k|] \sup_{0 \leq s \leq t} |h^*(s)|. \text{ More so, as } k \rightarrow \infty, |A^k|^{1/k} < 1 \text{ since the eigenvalues of } A \text{ have moduli less than 1. This implies the series converges and the lemma is proved. } \square$$

#### Lemma 4.2

The operator  $D\phi$  is uniformly stable if there are constants  $\alpha, \beta > 0$  such that for any  $\phi \in C$ ,  $\sigma \in [\tau, \infty)$ , the solution  $x(t, \sigma, \phi)$  of the homogeneous difference equation in Definition 4.4 satisfies  $\|x_t(\sigma, \phi)\| \leq \beta \|\phi\| e^{-\alpha(t-\sigma)}$ ,  $t \geq \sigma$ .

**Proof.** Suppose  $D(t)x_t = 0$ ,  $x_\sigma = \phi$ ,  $l(s)$  be scalar function which is continuous and non-decreasing for  $s \in [0, r]$ ,  $l(0) = 0$ ,  $-r \leq \theta \leq 0$  such that  $\left| \int_{-s}^0 d_\theta \mu(t, \theta) \varphi(\theta) \right| \leq l(s) \sup_{s \leq \theta \leq 0} |\varphi(\theta)|$ ,  $t \in [\tau, \infty)$ ,  $\varphi \in C$ , where  $\mu(t, \theta)$  is an  $n \times n$  matrix with bounded variation in  $\theta$ . Let  $\alpha > 0$  be any positive constant so that  $2l(r)(e^{\alpha h} - 1)e^{\alpha h} < 1$ . Assume in (4.4) that  $c = d$ , if  $y_t(\theta) = e^{\alpha(t+\theta-\sigma)}x_t(\theta)$ ,  $\varphi(\theta) = e^{\alpha\theta}\phi(\theta)$ ,  $-r \leq \theta \leq 0$ , then  $y_\sigma = \varphi$  and it is easy to see that  $D(t)y_t = D(\sigma)\varphi + h(t, y_t) - h(t, \varphi)$ ,  $h(t, \gamma) =$

$\int_{-r}^0 d_\theta \mu(t, \theta) (e^{-\alpha\theta} - 1) \gamma(\theta)$ , for every  $\gamma \in C$ . The choice of  $\alpha$  in the equation implies  $|h(t, \gamma)| \leq |\gamma| e^{-\alpha r} / c^2$ . Since  $|\varphi| \leq |\phi|$  and  $D(t)$  is a uniformly stable operator,  $|x_t| \leq c|\phi|$ , by the definition of  $y_t$  and relation (4.4),

$$\begin{aligned} |y_t| &\leq \left(c + \frac{1}{2}\right) |\phi| + \frac{e^{-\alpha r}}{2c} \sup_{\sigma \leq s \leq t} |y_s| \leq \left(c + \frac{1}{2}\right) |\phi| + \frac{e^{-\alpha r}}{2c} e^{\alpha(t-\sigma)} \sup_{\sigma \leq s \leq t} |x_s| \\ &\leq \left(c + \frac{1}{2}\right) |\phi| + \frac{e^{-\alpha r}}{2} e^{\alpha(t-\sigma)} |\phi|, t \geq \sigma. \end{aligned} \quad (4.5)$$

Since,  $|y_t| \geq e^{-\alpha r} e^{\alpha(t-\sigma)} |x_t|$ , this latter inequality would yield

$$|x_t| \leq \beta' e^{-\alpha(t-\sigma)} |\phi| + \frac{1}{2} |\phi|, \quad t \geq \sigma, \quad (4.6)$$

where,  $2\beta' = (2c + 1)e^{\alpha r}$ . Applying (4.6) in (4.5), gives the following

$$e^{\alpha r} |y_t| \leq \beta' |\phi| + \frac{1}{2c} \sup_{\sigma \leq s \leq t} \left( \beta' |\phi| + \frac{1}{2} e^{\alpha(s-\sigma)} |\phi| \right) \leq \beta' \left( 1 + \frac{1}{2c} \right) |\phi| + \frac{1}{2^2 c} e^{\alpha(t-\sigma)} |\phi|.$$

Therefore,  $|x_t| \leq \beta' \left( 1 + \frac{1}{2c} \right) e^{-\alpha(t-\sigma)} |\phi| + \frac{1}{2(2c)} |\phi|$ . Repeating the process gives

$$|x_t| \leq \beta' \left( 1 + \frac{1}{2c} + \frac{1}{(2c)^2} + \cdots + \frac{1}{(2c)^n} \right) e^{-\alpha(t-\sigma)} |\phi| + \frac{1}{2(2c)^n} |\phi|,$$

for all  $t \geq \sigma$  and every positive integer  $n$ . Since  $c \geq 1$ , it then implies that

$$|x_t| \leq \frac{2c\beta'}{2c-1} e^{-\alpha(t-\sigma)} |\phi|, \quad t \geq \sigma$$

which proves the lemma.  $\square$

**Lemma 4.3**

If  $D(t)$  is a uniformly stable operator with respect to  $C([\tau, \infty), E^n)$ , then there are positive constants  $\alpha, c, c_1$  and  $d$  such that for any  $g \in C([\tau, \infty), E^n)$ ,  $\sigma \in [\tau, \infty)$ , the solution  $x(\sigma, \phi, g)$  of the equation

$$\left. \begin{aligned} D(t)x_t &= g(t), \quad t \geq \sigma, \\ x_\sigma &= \phi, \end{aligned} \right\} \quad (4.7)$$

satisfies,

$$|x_t(\sigma, \phi, g)| \leq e^{-\alpha(t-\sigma)} \left( c_1 |\phi| + c \sup_{\sigma \leq v \leq t} |g(v)| \right) + d \sup_{\sigma \leq v \leq t} |g(v)|, \quad (4.8)$$

for all  $t \geq \sigma$ . The constants  $a, c, c_1$  and  $d$  can be chosen so that for any  $s \in [\sigma, \infty)$

$$|x_t(\sigma, \phi, g)| \leq e^{-\alpha(t-s)} \left( c_1 |\phi| + c \sup_{\sigma \leq v \leq t} |g(v)| \right) + d \sup_{s \leq v \leq t} |g(v)|, \quad (4.9)$$

for all  $t \geq \sigma + \tau$ .

**Proof.** By the method of proof in (Cruz and Hale 1970: 340), for any  $s \in [\sigma, \infty)$ , there exists a constant  $N$  which is independent of  $s, \sigma$  and an  $n \times n$  matrix  $\Phi$  depending on  $s$ ,  $\Phi = (\phi_1, \dots, \phi_n)$ ,  $\phi_i \in C$ ,  $|\phi_i| < N$  such that  $D(s)\Phi = I$ . If  $y: [s - \tau, \infty) \rightarrow E^n$  is defined by

$$y(t) = \begin{cases} \Phi(t)g(s), & s - \tau \leq t \leq s, \\ \Phi(s)g(t), & t \geq s, \end{cases} \quad (4.10)$$

then  $D(s)y_s = D(s)\Phi g(s) = g(s)$ . Therefore, for  $t \geq s$ ,

$$D(t)(x_t - y_t) = g^*(t), \quad (4.11)$$

where  $g^*(t)$  also depends upon  $s$  and satisfies  $g^*(s) = 0$ ,

$$|g^*(t)| \leq L \sup_{[\max(s, t-\tau)] \leq v \leq t} |g(v)| \quad (4.12)$$

for some constant  $L$  independent of  $s$  and  $\sigma$ . Also, from the definition of  $y$  in (4.10), it follows that

$$|y_t| \leq N \sup_{[\max(s, t-\tau)] \leq v \leq t} |g(v)|. \quad (4.13)$$

If  $z_t = x_t - y_t$ , then  $D(s)z_s = 0$ . The next objective is to estimate the function  $z_t$  satisfying the equation

$$D(t)z_t = g^*(t), \quad t \geq s, \quad z_s = \varphi, \quad D(s)\varphi = 0, \quad (4.14)$$

in terms of its value  $\varphi$  at  $s$  and  $|g^*(v)|$  for  $v \geq s$ . The solution  $z(s, \varphi, g^*)$  of (4.14) can be written as  $z_t(s, \varphi, g^*) = z_t(s, \varphi, 0) + z_t(s, 0, g^*)$ ,  $t \geq s$ . Since  $D(t)$  is uniformly stable with respect to  $C([\tau, \infty), E^n)$ , it follows then from (4.4) that  $|z_t(s, \varphi, g^*)| \leq c|\varphi| + d \sup_{s \leq v \leq t} |g^*(v)|$ ,  $t \geq s$ ,  $|z_t(s, 0, g^*)| \leq d \sup_{s \leq v \leq t} |g(v)|$ ,  $t \geq s$ . Also, from Lemma 4.2, there are  $\alpha, \beta > 0$  such that  $|z_t(s, \varphi, 0)| \leq \beta|\varphi|e^{-\alpha(t-s)}$ ,  $t \geq s$ . Consequently,  $|z_t(s, \varphi, g^*)| \leq \beta|\varphi|e^{-\alpha(t-s)} + d \sup_{s \leq v \leq t} |g^*(v)|$ ,  $t \geq s$ .

Now, letting  $z_t = x_t - y_t$ ,  $\varphi = x_s - y_s$  and using (4.11), (4.12) and (4.13), it is easily seen that there are positive constants  $c', d$  such that

$$|x_t(\sigma, \phi, g)| \leq e^{-\alpha(t-s)}[\beta|x_s(\sigma, \phi, g)| + c' \sup_{\sigma \leq v \leq t} |g(v)|] + d \sup_{[\max(s, t-\tau)] \leq v \leq t} |g(v)|,$$

$t \geq s$ . Since  $D(t)$  is uniformly stable with respect to  $C([\tau, \infty), E^n)$ ,  $|x_s(\sigma, \phi, h)|$  can be estimated uniformly in terms of  $|\phi|$  and  $\sup_{\sigma \leq v} |g(v)|$  to obtain constants  $c_1, c$  such that

$$|x_t(\sigma, \phi, g)| \leq e^{-\alpha(t-s)}[c_1|\phi| + c \sup_{\sigma \leq v \leq t} |g(v)|] + d \sup_{[\max(s, t-\tau)] \leq v \leq t} |g(v)|, \quad t \geq s.$$

For some  $s = \sigma$ , this gives (4.8) and for  $t \geq s + \tau$ , this gives (4.9) which completes the proof.  $\square$

The next lemma as proved is one of the contributions in this thesis and is developed following Theorem 1 of Sinha (1985) and Corollary 2 of Hale (1974) for functional differential equations with infinite delay; see also Corollary 3.8 of Davies (2006) and references therein for neutral functional differential systems with infinite delays. The lemma will play an important role in the control of the system (4.1).

**Lemma 4.4**

In system (4.1), assume that  $G$  is an  $n \times n$  matrix function whose elements are square integrable on  $(-\infty, 0]$ , and there is a  $c > 0$ , and a constant  $m$  such that

$$|G(\theta)| \leq m \exp(c\theta) \leq m \text{ for } \theta \in (-\infty, 0] \text{ and if}$$

$$\mathcal{H}(\lambda) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0, \det \Delta(\lambda) = 0\},$$

$$\Delta(\lambda) = \lambda(I - A_0 \exp(-\lambda h)) - A_1 - A_2 \exp(-\lambda h) - \int_{-\infty}^0 \exp(\lambda \theta) G(\theta) d\theta.$$

Then the solution of (4.1) is uniformly asymptotically stable if

$$|X(t, s)| \leq k \exp(-\alpha(t - \sigma)), \quad t \geq s \geq \sigma, \quad k > 0, \quad \alpha > 0$$

**Proof:** Let  $c > 0$  such that  $\exp(-c\theta)|G(\theta)|$  is Lebesgue integrable on  $(-\infty, 0]$  and if  $\exp(-c\theta)|G(\theta)| \leq \gamma(\theta) \leq m$ , then  $|G(\theta)| \leq m \exp(c\theta)$ ,  $\theta \in (-\infty, 0]$ . Let

$\bar{\gamma} = m \exp(c\theta)$ . Now defining the space  $\mathbb{C}$  using  $\bar{\gamma}$  rather than  $|\gamma(\theta)|$  such that if

$$\mathcal{H}(\lambda) = \{\lambda \in \mathbb{C}^n : \operatorname{Re} \lambda \geq 0, \det \Delta(\lambda) = 0\}, \text{ where}$$

$$\Delta(\lambda) = \lambda(I - A_0 \exp(-\lambda h)) - A_1 - A_2 \exp(-\lambda h) - \int_{-\infty}^0 \exp(\lambda \theta) G(\theta) d\theta,$$



then all conditions of Lemma 4.4 are satisfied and the solutions of (4.1) turns zero exponentially and uniformly following the conclusion of Sinha (1985). The hypothesis  $\mathcal{H}(\cdot)$  therefore implies that the spectrum of  $X$  is less than 1, and is contained in the disk of radius  $\exp(-\alpha t)$  and centre zero.  $\square$

### 4.3. Razumikhin's approach for stability

Here, we shall investigate the total stability of system (4.1) using a Razumikhin type argument.

For neutral functional differential systems it is natural to use Lyapunov functions of the form  $V(t, D\phi)$ , where  $V: [\tau, \infty) \times C \rightarrow E$  is continuous, and the derivative  $V$  along the solution of (4.1) can be defined by  $\dot{V}(t, D\phi) := \frac{\partial V(t, D\phi)}{\partial t} + \langle \text{grad } V(t, D\phi), f(t, \phi) \rangle$ ,

for  $t \in [\tau, \infty)$ ,  $\phi \in C$ . Obviously,  $\frac{d}{dt}V(t, D(t)x_t) = \dot{V}(t, D(t)x_t)$  for  $t \in [\tau, \infty)$ ,  $x_t \in C$ , where  $x(t)$  is a solution of (4.1) on  $[\sigma, \infty)$ .

The proofs of the next two theorems follow along the lines of the proofs of Theorem 1 and 2 of Murakami (1984) mutatis mutandis using properties of  $D\phi$  which are given in Lemma 4.2 and Lemma 4.3. The theorems are part of the contributions of the thesis in this chapter.

#### **Theorem 4.1**

Suppose there is a continuous function  $\alpha(\delta)$ ,  $\delta > 0$ , satisfying  $\nu(\beta\eta) \leq u(\alpha(\delta))$ , where  $\eta$  is a positive constant,  $\beta = \|D\|$ . Let  $D$  be uniformly stable,  $f: [\tau, \infty) \times C$  continuous and  $f$  maps  $[\tau, \infty) \times (\text{bounded sets of } C)$  into bounded set of  $E^n$ . Suppose there are constant  $k, r > 0$  and continuous non-decreasing, nonnegative functions  $u(s), \nu(s), w(s)$  with  $u(s), \nu(s), w(s) > 0$  for  $s > 0$  and  $u(0), \nu(0), w(0) = 0$ , and there is a continuous function

$V: [\tau, \infty) \times C \rightarrow E$  such that

$$(i) \quad u(|x|) \leq V(t, x) \leq v(|x|)$$

$$(ii) \quad |V(t, x) - V(t, y)| \leq k|x - y|,$$

$$t \in [\sigma, \infty), |x|, |y| \leq r$$

$$(iii) \quad \dot{V}(t, \phi) \leq -w(|D\phi|), \text{ for all } (t, \phi) \in [\tau, \infty) \times C \text{ satisfying}$$

$$V(t, D\phi) = \sup_{\theta \leq 0} V(t + \theta, \phi(\theta)).$$

Then the zero solution of system (4.1) is totally stable and  $\delta(\cdot)$  in Definition 4.1 can be chosen so that it depends on only the functions  $u$ ,  $v$ ,  $w$  and the constants  $k$ ,  $r$ .

**Proof:** Let  $\varepsilon$  be a positive number (which can be taken less than  $r$ ). Let  $\eta = \eta(\varepsilon)$  be a positive constant so that  $v(\beta\eta) < u(\alpha(\delta))$ . Set  $c := w(\beta\eta)$ , and define  $\delta = \delta(\varepsilon)$  by

$\delta(\varepsilon) := \min(\beta\eta, c/k)$ . Then by Remark 4.1, it suffices to show that if

$(\sigma, \phi, p) \in [\tau, \infty) \times C \times C$  and  $\sup_{t \leq \sigma} |p(t)| < \delta(\varepsilon)$ , then  $|x(t)| < \alpha(\delta) < \varepsilon$  for all  $t \geq \sigma$ , where  $x(t) = x_t(\sigma, \phi, f + p)$ . Suppose this is not true, then there exists a  $T > \sigma$  such that  $|x(t)| = \varepsilon$  and  $|x(t)| < \varepsilon$  for all  $t < T$ . Now, set  $V(t) = V(t, x(t))$ , and note that  $\sup_{\theta \leq 0} V(\sigma + \theta) < V(T)$ , since

$$\sup_{\theta \leq 0} V(\sigma + \theta) \leq \sup_{\theta \leq 0} v(|\phi(\theta)|) < v(\delta) < v(\beta\eta) < u(\alpha(\delta)) \leq V(T)$$

by (i). Hence, there is a  $T_0$  with  $\sigma < T_0 \leq T$  such that  $V(T_0) = \sup_{u \leq T} V(u) =: M$ . This implies  $\beta\eta \leq |x(T_0)| \leq \varepsilon$ , since  $|x(T_0)| \leq |x(T)| = \varepsilon$  and

$v(\beta\eta) < u(\varepsilon) \leq V(T) \leq V(T_0) \leq v(|x(T_0)|)$  by (i). Moreover,  $V(T_0) = M \geq V(u)$  for all

$u \leq T_0$ , which gives  $V(T_0, x_{T_0}) \leq -w(|D(T_0)x_{T_0}|) \leq -c$  by (iii), and consequently

$\dot{V}(T_0) \leq \dot{V}(T_0, x_{T_0}) + k|p(T_0)| < -c + k\delta(\varepsilon) < 0$  by (ii). Therefore, there is a  $T_1 < T_0$ ,

such that  $V(T_1) < V(T_0) = M$  which is a contradiction to the contraposition.  $\square$

### Theorem 4.2

Let  $V: [\tau, \infty) \times C \rightarrow E$  be the function satisfying conditions (i) and (ii) in Theorem 4.1, and if in addition there exists constant  $h > 0$ , a continuous non-decreasing, nonnegative function  $w(s) > 0$  for  $s > 0$ ,  $w(0) = 0$ , and a continuous function  $\rho(s) > s$  for  $s > 0$  such that condition (iii) in Theorem 4.1 is strengthened to

$$(iv) \quad \dot{V}(t, \phi) \leq -w(|D\phi|)$$

for all  $(\sigma, \phi) \in [\tau, \infty) \times C$  satisfying  $\rho(V(t, D\phi)) \geq \sup_{-h \leq \theta \leq 0} V(t + \theta, \phi(\theta))$ . Then the zero solution of (4.1) is totally asymptotically stable, and  $\delta_0$ ,  $\gamma_0$ ,  $\delta(\cdot)$ ,  $\gamma(\cdot)$ , and  $T(\cdot)$  in Definition 4.2 can be chosen so that  $\delta_0$ ,  $\gamma_0$ ,  $\delta(\cdot)$ , and  $\gamma(\cdot)$  depend only on the functions  $u$ ,  $v$ ,  $w$ ,  $\rho$  and the constants  $k$ ,  $r$  while  $T(\cdot)$  depends also on the constant  $h$ .

**Proof:** Since condition (iv) implies condition (iii) of Theorem 4.1, it follows from Definition 4.1 that, the zero solution of (4.1) is totally stable with  $\delta(\cdot)$  which depends only on the function  $u$ ,  $v$ ,  $w$  and the constants  $k$ ,  $r$ . Now, choose a positive constant  $\delta_0$  so that if  $(\sigma, \phi, p) \in [\tau, \infty) \times C \times C$  and  $\sup_{t \geq \sigma} |p(t)| < \delta_0$ , then  $|x_t(\sigma, \phi, f + p)| < r$  for all  $t \geq \sigma$ . Let  $\varepsilon$  be a given positive number (which can be chosen less than  $\delta_0$ ), and let  $\eta$  and  $c$  be the numbers as defined in the proof of Theorem 4.1. Take a number  $a > 0$  such that

$a = a(\varepsilon) := \inf\{\rho(s) - s: u(\varepsilon) \leq s \leq v(\beta\eta), 0 < \beta\eta \leq r\}$ . Let  $N = N(\varepsilon)$  be the first positive integer such that  $u(\varepsilon) - Na \geq v(\beta\eta)$  and set  $\gamma(\varepsilon) = \min(\delta_0, c/2k)$  and

$T(\varepsilon) = 2Nv(\beta\eta)/c + (N - 1)h$ . By Remark 4.1, it suffices to show that if

$(\sigma, \phi, p) \in [\tau, \infty) \times C \times C$  and  $\sup_{t \geq \sigma} |p(t)| < \gamma(\varepsilon)$ , then  $|x_t(\sigma, \phi, f + p)| \leq \varepsilon$  for all  $t \geq \sigma + T(\varepsilon)$ . Set  $x(t) = x(t, \sigma, \phi, f + p)$  and  $V(t) = V(t, x(t, \sigma, \phi, f + p))$ , and first show that

$$V(t_1) \leq u(\varepsilon) + (N - 1)a \text{ for } t_1 \in [\sigma, \sigma + 2v(\beta\eta)/c]. \quad (4.15)$$

Suppose that  $V(t) > u(\varepsilon) + (N - 1)a$  for all  $t \in [\sigma, \sigma + 2v(\beta\eta)/c]$ . It follows then that

$$\rho(V(t)) > V(t) + a > u(\varepsilon) + Na \geq v(\beta\eta) \geq \sup_{-h \leq \theta \leq 0} V(t + \theta) \text{ for some}$$

$t \in [\sigma, \sigma + 2v(\beta\eta)/c]$ , since  $|x(t)| \leq \beta\eta \leq r$ . It then follows from condition (iv) of

Theorem 4.2 that,  $\frac{d}{dt}V(t) \leq k|p(t)| - w(|D(t)x_t|) < k\gamma(\varepsilon) - c \leq -c/2$  for all

$t \in [\sigma, \sigma + 2v(\beta\eta)/c]$ . Hence,

$$V(\sigma + 2u(\beta\eta)/c) < V(\sigma) + (-c/2) \times 2v(\beta\eta)/c \leq V(\beta\eta) - V(\beta\eta) = 0.$$

This is a contradiction to the claim and therefore (4.15) holds. Next is to show that

$$V(t) \leq u(\varepsilon) + (N - 1)a, \text{ for } t \geq \sigma + 2v(\beta\eta)/c. \quad (4.16)$$

Suppose that  $V(t_2) > u(\varepsilon) + (N - 1)a$  for  $t_2 \in [\sigma + 2v(\beta\eta)/c, \infty)$ , then it follows from

(4.16) that there exists  $t_3 \in [t_1, t_2]$  which satisfies  $V(t_3) = u(\varepsilon) + (N - 1)a$  and  $\dot{V}(t_3) \geq 0$ .

Observe also that  $\rho(V(t_3)) \geq V(t_3 + \theta)$  for  $\theta \in [-h, 0]$  and  $|x(t_3)| \geq \beta\eta$ . Then, by the

condition (iv) of Theorem 4.2,  $\dot{V}(t_3) \leq k|p(t_3)| - w(|D(t_3)x_{t_3}|) \leq -c/2 < 0$ , which is a

contradiction to  $\dot{V}(t_3) \geq 0$  and therefore (4.16) holds. What is remains to show that

$$|x(t)| \leq \varepsilon, \text{ for all } t \geq \sigma + T(\varepsilon) \quad (4.17)$$

Now, if  $N = 1$ , then (4.16) implies  $V(t) \leq u(\varepsilon)$  for all  $t \geq \sigma + T(\varepsilon)$ , and hence the

inequality (4.17) holds. Suppose  $N \geq 2$ , repeating the same arguments as in the proof of

(4.15), it can be shown that  $V(t_4) \leq u(\varepsilon) + (N - 2)a$  for

$t_4 \in [\sigma + 2v(\beta\eta)/c + h, \sigma + 4v(\beta\eta)/c + h]$  by (4.16). Following the same type of

argument as in the proof of (4.16) therefore gives  $V(t) \leq u(\varepsilon) + (N - 2)a$  for all

$t \geq \sigma + 4v(\beta\eta)/c + h$ . Repeating the procedure over and again gives

$V(t) \leq u(\varepsilon) + (N - j)a$  for all  $t \geq \sigma + 2jv(\beta\eta)/c + (j - 1)h$  where  $j = 1, 2, \dots, N$ . It follows therefore that  $V(t) \leq u(\varepsilon)$  for all  $t \geq \sigma + T(\varepsilon)$ , and hence (4.17) holds.  $\square$

The results of Theorem 4.1 and Theorem 4.2 will now be used to investigate the total asymptotic stability for the system. It is clear from the conditions imposed on system (4.1) that, the function  $f: [\tau, \infty) \times C \rightarrow E^n$  can be defined by

$f(t, x_t) = A_1x(t) + A_2x(t - h) + \int_{-\infty}^0 G(s, x_s)ds$  is in  $E^n$ , as an application of Theorem 4.2, the asymptotic stability of the system (4.1) will now be investigated.

It is known from established result in matrix theory (Frommer and Hashemi 2013) that there is a symmetric positive definite matrix  $P$  such that the equation  $PA_1 + A_1^T P = -I$ , is called the Lyapunov matrix equation, where  $I$  is the identity matrix and  $A_1^T$  is the transpose of  $A_1$ . Let  $\lambda$  and  $\alpha$  be positive numbers such that  $\lambda^2$  and  $\alpha^2$  are the least and greatest eigen-values of  $P$  respectively. Then, it is clear that,

$$\lambda^2 |D(t)x_t|^2 \leq \langle PD(t)x_t, D(t)x_t \rangle \leq \alpha^2 |D(t)x_t|^2, \text{ for all } D(t)x_t \in E^n.$$

Making use of the assumptions on the system (4.1), and following the methods in Murakami (1984), a new theorem is developed as a contribution of the thesis in this chapter by using an idea from Theorem 8.2.6 in Burton (1983).

#### 4.3.1. Application of the Razumikhin approach

##### **Theorem 4.3**

Let all the assumptions on system (4.1) be satisfied, and suppose that

$$\|P\| \left( c + \int_{-\infty}^0 M(s) ds \right) < \lambda/2\alpha. \quad (4.18)$$

Then the zero solution of (4.1) is totally asymptotically stable.

**Proof.** Given relation (4.18), choose a constant  $\mu > 1$  so that

$$1 - \frac{2\mu\alpha\|P\| \left( c + \int_{-\infty}^0 M(s) ds \right)}{\lambda} = l > 0.$$

For any  $h \in (h_1, \infty)$ , consider the system

$$\frac{d}{dt}D(t)x_t = A_1x(t) + A_2(t)x(t-h) + \int_{-h}^0 G(s, x_s) ds. \quad (4.19)$$

Observe first that by Lemma 4.1,  $D\phi$  is uniformly stable. Further, Theorem 2.2 in Chukwu (1981) has demonstrated that, whenever it is required, the solution  $x \equiv 0$  of

$\frac{d}{dt}[x(t) - A_0x(t-h)] = f(t, x(t), x(t-h))$  is uniformly asymptotically stable, where  $f(t, 0, 0) = 0$ , and by Theorem 2 of Izé and Freiria (1981), this is totally stable.

Now, let  $V(D(t)x_t) = \langle PD(t)x_t, D(t)x_t \rangle$ . It is necessary to prove that  $V(D(t)x_t)$  satisfies all the conditions in Theorem 4.2 for system (4.19). It is obvious that conditions (i) and (ii) of Theorem 4.1 holds. Assume now that  $\mu^2 V(D\phi) \geq V(\phi(\theta))$ , so that  $\mu^2 \alpha^2 |D\phi|^2 \geq \lambda^2 |\phi(\theta)|^2$  and hence  $|\phi(\theta)| \leq \mu\alpha |D\phi|/\lambda$  for all  $\theta \in [-h, 0]$ . Then, the derivative  $V(t, D\phi)$  of  $V$  along the solution of equation (4.19) is given by

$$\begin{aligned}
\dot{V}(t, D\phi) &= \langle P \left[ A_1 \phi(0) + A_2 \phi(-h) + \int_{-h}^0 G(s, \phi(s)) ds \right], D\phi \rangle \\
&\quad + \langle PD\phi, A_1 \phi(0) + A_2 \phi(-h) + \int_{-h}^0 G(s, \phi(s)) ds \rangle \\
&= \langle (PA_1 + A_1^T P) \phi(0), D\phi \rangle + 2 \langle PD\phi, A_2 \phi(-h) + \int_{-h}^0 G(s, \phi(s)) ds \rangle \\
&\leq -|D\phi|^2 + 2|D\phi| \|P\| \left( c + \int_{-\infty}^0 M(s) ds \right) \cdot \sup_{-h \leq \theta \leq 0} |\phi(\theta)| \leq -l|D\phi|^2
\end{aligned}$$

Thus, the condition of Theorem 4.2 also holds as  $w(s) = ls^2$  and  $\rho(s) = \mu^2 s$ . Therefore, the zero solution of (4.19) is totally asymptotically stable with  $\delta_0, \gamma_0, \delta(\cdot), \gamma(\cdot)$  and  $T(h, \cdot)$ , where  $\delta_0, \gamma_0, \delta(\cdot)$  and  $\gamma(\cdot)$  are independent of  $h$ .

Now, let  $\varepsilon \in (0, \gamma_0)$  be given and select a constant  $h(\varepsilon) > h_1$ , such that

$\gamma_0 \cdot \int_{-\infty}^{-h(\varepsilon)} M(s) ds < \min(\delta(\varepsilon)/2, \gamma(\varepsilon)/2)$ . If  $Q \in E^n$  and  $|Q(t, \phi)| < \delta(\varepsilon)/2$  for all  $(t, \phi) \in [\sigma, \infty) \times C$ , then  $\left| \int_{-\infty}^{-h(\varepsilon)} G(s, \phi(s)) ds + Q(t, \phi) \right| < \varepsilon \cdot \int_{-\infty}^{-h(\varepsilon)} M(s) ds + \delta(\varepsilon)/2 \leq \delta(\varepsilon)$ , for all  $(t, \phi) \in [\sigma, \infty) \times C$ . Therefore, if  $(t, \phi, Q) \in [\sigma, \infty) \times C \times C$  and  $|Q(t, \phi)| < \delta(\varepsilon)/2$  for all  $(t, \phi) \in [\sigma, \infty) \times C$ , then from the total stability of the zero solution of (4.19) it follows that  $\|x_t(\sigma, \phi)\| < \varepsilon$  for all  $t \geq \sigma$ , where  $x(t, \sigma, \phi)$  denotes a solution of

$$\frac{d}{dt} D(t)x_t = L(t, x_t) + \int_{-h(\varepsilon)}^0 G(s, x_s) ds + \int_{-\infty}^{-h(\varepsilon)} G(s, x_s) ds + Q(t, x_t)$$

through  $(\sigma, \phi)$ . Hence, the zero solution of equation (4.1) is totally stable. Similarly, if  $(\sigma, \phi, Q) \in [\tau, \infty) \times C \times C$  and  $|Q(t, \phi)| < \gamma(\varepsilon)/2$  for all  $(t, \phi) \in [\sigma, \infty) \times C$ , then we

obtain  $\|x_t(\sigma, \phi)\| < \varepsilon$  for all  $t \geq \sigma + T(h(\varepsilon), \varepsilon)$ . Hence the solution of (4.1) is totally asymptotically stable.  $\square$

#### 4.4. Lyapunov functional approach for stability

Here, a delay-independent criterion for the asymptotic stability of the system (4.1) will be developed and proved in terms of LMI using the standard Lyapunov-Krasovskii approach. Some lemmas and definition that are required for the development of the criterion are also given

##### **Definition 4.6 (Linear Matrix inequality)**

LMI has the form

$$A(x) = A_0 + \sum_{k=1}^n x_k A_k > 0,$$

where,  $x \in E^n$ ,  $A_k = A_k^T \in E^{n \times n}$ ,  $k = 0, \dots, n$  are symmetric matrices and  $x^T A(x) x > 0$ , for  $x > 0$ . Also the set  $\{x: A(x) > 0\}$  is convex. Nonlinear (convex) inequalities can be converted to LMI using the basic ideas from Schur complements given in Lemma 4.6.

##### **Lemma 4.5**

For any matrices  $D$  and  $E$  with appropriate dimensions and any positive scalar  $\tau$ , then

$$D^T E + E^T D \leq \tau D^T D + \tau^{-1} E^T E$$

Proof: The proof is given in Khargonekar et al. (1990).  $\square$

##### **Lemma 4.6**

The linear matrix inequality (Boyd et al. 1994)

$$\begin{pmatrix} A(x) & A_1(x) \\ A_1^T(x) & A_2(x) \end{pmatrix} > 0$$



where  $A(x) = A^T(x)$ ,  $A_2(x) = A_2^T(x)$ , and  $A_1(x)$  depend affinely on  $x$ , is equivalent to  $A_2(x) > 0$ ,  $A(x) - A_1(x)A_2^{-1}(x)A_1^T(x) > 0$ . Here,  $A(x)$ ,  $A_2(x)$ , and  $A_1(x)$  are LMIs

**Proof:** The proof is given in Boyd et al. (1994).  $\square$

#### 4.4.1. Application of the Lyapunov functional approach

Using the Lyapunov-Krasovskii approach a new delay-independent criterion for the asymptotic stability of the system (4.1) will now be developed as a contribution of the thesis in this section

##### *Theorem 4.4*

Let system (4.1) be as defined with  $G$  satisfying the condition  $\|G(t, x_s)\| \leq M(t)\|x\|$  for all  $(t, \phi) \in (-\infty, 0] \times C$ , where  $\int_{-\infty}^0 M(s)ds = -l < \infty$ . System (4.1) is asymptotically stable for all  $h \geq 0$  if there exists positive symmetric matrices  $P$ ,  $P_1 > 0$ , and some positive scalars  $\tau_0, \tau_1, \tau_2 > 0$  which satisfy the following LMI

$$Z(X, P_1, \tau_0, \tau_1, \tau_2) = \begin{pmatrix} Z_{11} & Z_{12} & (A_2 + XA_1^T A_2) & (A_0 + XA_1^T A_0) \\ * & Z_{22} & 0 & 0 \\ * & * & A_2^T A_2 - R + \tau_1 A_2^T A_2 & A_2^T A_0 \\ * & * & * & A_0^T A_0 - I + \tau_2 A_0^T A_0 \end{pmatrix} < 0, \quad (4.20)$$

where,

$$Z_{11} = XA_1^T + A_1 X - 2lX,$$

$$Z_{12} = [XA_1^T \quad \tau_0 XA_1^T \quad XP_1 \quad LX \quad LX \quad LX \quad LX],$$

$$Z_{22} = \text{diag}\{-I, -\tau_0 I, -P_1, -I, -\tau_0 I, -\tau_1 I, -\tau_2 I\},$$

**Proof:** Let the Lyapunov function candidate be given by

$$V = V_1 + V_2 + V_3 \quad (4.21)$$

where,

$$V_1 = x^T(t)Px(t), \quad (4.22)$$

$$V_2 = \int_{-h}^0 \dot{x}^T(t+s)\dot{x}(t+s)ds, \quad (4.23)$$

$$V_3 = \int_{-h}^0 x^T(t+s)P_1x(t+s)ds \quad (4.24)$$

Taking the derivative of  $V$  along the solution of (4.1) gives

$$\dot{V}_1 = x^T(A_1^T P + PA_1)x + 2x^T PA_2 x_h + 2x^T PA_0 \dot{x}_h + 2x^T P \int_{-\infty}^0 G(t, x_s)ds. \quad (4.25)$$

$$\dot{V}_2 = \dot{x}^T \dot{x} - \dot{x}_h^T \dot{x}_h$$

$$\begin{aligned} &= x^T A_1^T A_1 x + x_h^T A_2^T A_2 x_h + \dot{x}_h^T A_0^T A_0 \dot{x}_h + \left( \int_{-\infty}^0 G(t, x_s)ds \right)^T \int_{-\infty}^0 G(t, x_s)ds \\ &+ 2x^T A_1^T A_2 x_h + 2x^T A_1^T A_0 \dot{x}_h + 2x_h^T A_2^T A_0 \dot{x}_h + 2x^T A_1^T \int_{-\infty}^0 G(t, x_s)ds \\ &+ 2x_h^T A_2^T \int_{-\infty}^0 G(t, x_s)ds + 2\dot{x}_h^T A_0^T \int_{-\infty}^0 G(t, x_s)ds - \dot{x}_h^T \dot{x}_h. \end{aligned} \quad (4.26)$$

$$\dot{V}_3 = x^T P_1 x - x_h^T P_1 x_h. \quad (4.27)$$

where  $x$ ,  $x_h$  and  $\dot{x}_h$  denote  $x(t)$ ,  $x(t-h)$  and  $\dot{x}(t-h)$  respectively. The term

$\left( \int_{-\infty}^0 G(t, x_s)ds \right)^T \int_{-\infty}^0 G(t, x_s)ds$  in (4.26) can be simplified using Jensen's Inequality (Gu

et al. 2003: 305) as follows,

$$\begin{aligned}
\left( \int_{-\infty}^0 G(t, x_s) ds \right)^T \int_{-\infty}^0 G(t, x_s) ds &= \left( \int_{-\infty}^0 \|G(t, x_s) ds\| \right)^T \int_{-\infty}^0 \|G(t, x_s) ds\| \\
&\leq \left( \int_{-\infty}^0 M(s) ds \|x\| \right)^T \int_{-\infty}^0 M(s) ds \|x\| \\
&\leq \left( \int_{-\infty}^0 |M(s) ds| \right) \int_{-\infty}^0 |M(s) ds| \|x\|. \|x\| \\
&\leq l \int_{-\infty}^0 M(s) ds \|x\|^2 \leq l^2 \|x\|^2 = l^2 x^T x.
\end{aligned} \tag{4.28}$$

Applying Lemma 4.5 with (4.28) to the following terms in equations (4.25) and (4.26) gives;

$$2x^T P \int_{-\infty}^0 G(t, x_s) ds \leq -2x^T P l x \tag{4.29}$$

$$\begin{aligned}
2x^T A_1^T \int_{-\infty}^0 G(t, x_s) ds &\leq \tau_0 x^T A_1^T A_1 x + \tau_0^{-1} \left( \int_{-\infty}^0 G(t, x_s) ds \right)^T \int_{-\infty}^0 G(t, x) ds \\
&\leq \tau_0 x^T A_1^T A_1 x + \tau_0^{-1} l^2 x^T x
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
2x_h^T A_2^T \int_{-\infty}^0 G(t, x_s) ds &\leq \tau_1 x_h^T A_2^T A_2 x_h + \tau_1^{-1} \left( \int_{-\infty}^0 G(t, x_s) ds \right)^T \int_{-\infty}^0 G(t, x) ds \\
&\leq \tau_1 x_h^T A_2^T A_2 x_h + \tau_1^{-1} l^2 x^T x
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
2\dot{x}_h^T A_0^T \int_{-\infty}^0 G(t, x_s) ds &\leq \tau_2 \dot{x}_h^T A_0^T A_0 \dot{x}_h + \tau_2^{-1} \left( \int_{-\infty}^0 G(t, x_s) ds \right)^T \int_{-\infty}^0 G(t, x_s) ds \\
&\leq \tau_2 \dot{x}_h^T A_0^T A_0 \dot{x}_h + \tau_2^{-1} l^2 x^T x
\end{aligned} \tag{4.32}$$

where  $\tau_0, \tau_1, \tau_2 > 0$  are scalars to be chosen.

The overall derivative of  $V$  along the solution of (4.1) can now be expressed as follows

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \leq \lambda^T(t) \mathfrak{M}(P, P_1, \tau_0, \tau_1, \tau_2) \lambda(t), \quad (4.33)$$

where

$$\mathfrak{M}(P, P_1, \tau_0, \tau_1, \tau_2) = \begin{pmatrix} \mathfrak{M}_{11} & (PA_2 + A_1^T A_2) & (PA_0 + A_1^T A_0) \\ * & M_{22} & A_2^T A_0 \\ * & * & A_0^T A_0 - I + \tau_2 A_0^T A_0 \end{pmatrix},$$

and  $\lambda(t) = [x^T, x_h^T, \dot{x}_h^T]^T$ , so that,

$$\mathfrak{M}_{11} = A_1^T P + PA_1 - 2Pl + A_1^T A_1 + \tau_0 A_1^T A_1 + P_1 + l^2 I + \tau_0^{-1} l^2 I + \tau_1^{-1} l^2 I + \tau_2^{-1} l^2 I$$

$$\mathfrak{M}_{22} = A_2^T A_2 - P_1 + \tau_1 A_2^T A_2.$$

Pre and post multiplying  $\mathfrak{M}(\cdot)$  by  $\Gamma^{-T}$  and  $\Gamma$ ; and now using the Schur complement gives

$\mathcal{Z}(X, P_1, \tau_0, \tau_1, \tau_2)$  where

$$\Gamma = \begin{pmatrix} X & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

It then follows that  $\dot{V}$  is negative definite since  $\mathfrak{M}(\cdot) < 0$  is equivalent to  $\mathcal{Z}(\cdot) < 0$ , which implies that the system (4.1) is asymptotically stable (see Hale and Verduyn Lunel 1993).  $\square$

#### 4.5. Examples on stability methods

In this section, numerical examples are given as contributions of the thesis to illustrate the applicability of the stability methods discussed in this chapter.

##### 4.5.1. Example using Razumikhin's approach

Consider the neutral system

$$\dot{x}(t) - A_0 \dot{x}(t-h) = A_1 x(t) + A_2(t)x(t-h) + \int_{-\infty}^0 G(s, x_s) ds, \quad (4.34)$$

where

$$G(t, x_t) = \left( \frac{-[\sin(x(t) + x(t-h))] \cdot x(t-h)}{1+t^2} \right),$$

$$A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 3/2 \\ 0 & -1/2 \end{pmatrix}.$$

The difference operator by definition is given by  $D(t)x_t = x(t) - A_0 x(t-h)$ , so that

$D\phi = \phi(0) - A_0 \phi(-h)$  for  $h > 0$  and the function

$$f(t, x(t), x(t-h)) = A_1 x(t) + A_2(t)x(t-h) + \int_{-\infty}^0 G(s, x_s) ds$$

is in  $E^n$ . Now, use Lemma 4.1 to check that, the operator  $D$  is uniformly stable as follows:

The condition  $\det[I - Ar^{-h}] = 0$  of Lemma 4.1 gives,

$$\begin{pmatrix} 1 & -0.5r^{-h} \\ -0.5r^{-h} & 1 \end{pmatrix} = 0,$$

which implies  $1 - 1/4 r^{-2h} = 0$ , and  $r = (1/2)^{\frac{1}{h}}$ . Hence, the operator  $D$  is uniformly stable

if  $h > 0$ . Let  $P = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ , be the symmetric positive definite matrix with  $\lambda^2 = 0.19$

and  $\alpha^2 = 1.31$  as the least and greatest eigen-values, and observe that

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

satisfies the Lyapunov matrix equation. Next check that the function  $G(t, x_t)$  satisfies all the

conditions and  $|G(t, x_s)| \leq M(t)\|x\|$ , where  $M(t) = -1/1+t^2$ , and

$$\int_{-\infty}^0 -dt/(1+t^2) = [-\tan^{-1} t]_{-\infty}^0 = -\pi/2 < \infty.$$

Moreover,  $\|P\| \int_{-\infty}^0 -dt/(1+t^2) < \lambda/2\alpha - c\|P\|$ , which satisfies the condition of Theorem 4.3. Also, let  $h = 1$  be arbitrarily chosen and let  $\mu = 2$ , it follows then that, the condition of Theorem 4.2 is also satisfied for

$$\dot{x}(t) - A_0\dot{x}(t-h) = A_1x(t) + A_2(t)x(t-h) + \int_{-1}^0 G(s, x_s)ds, \quad (4.35)$$

with  $h \in [0.5, \infty)$  and  $l = 0.86$ . Now, choose  $h(\varepsilon) = 2$ ,  $\gamma_0 = 4$ ,  $\delta(\varepsilon)/2 = 5/4$ , and  $\gamma(\varepsilon)/2 = 3/2$ , so that  $|Q(t, \phi)| < 5/4$  and  $\left| \int_{-\infty}^{-h(\varepsilon)} G(s, \phi(s))ds + Q(t, \phi) \right| \leq 5/2$ . Hence, from the total stability of equation (4.35), it follows that  $\|x_t(\sigma, \phi)\| < \varepsilon$ , where  $x(t, \sigma, \phi)$  represents a solution of

$$\frac{d}{dt}D(t)x_t = L(t, x_t) + \int_{-2}^0 G(s, x_s)ds + \int_{-\infty}^{-2} G(s, x_s)ds + Q(t, x_t).$$

Thus equation (4.34) is totally stable, and similarly totally asymptotically stable if  $|Q(t, \phi)| < 3/2$ .  $\square$

#### 4.5.2. Example using Lyapunov's functional approach

Consider the neutral system with infinite delay given by

$$\dot{x}(t) - A_0\dot{x}(t-h) = A_1x(t) + A_2x(t-h) + \int_{-\infty}^0 G(t, x_s)ds, \quad (4.36)$$

where,

$$A_0 = \begin{pmatrix} 0 & 0.4 \\ 0.4 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$G(t, x(t)) = \begin{pmatrix} 0 \\ -\exp(t-3) \times \sin x(t) \cdot x(t) \end{pmatrix}.$$

Note that, the function  $G(t, x_s)$  satisfies its conditions with,

$$M(t) = -\exp(t - 3) \times \sin x(t); \quad \int_{-\infty}^0 M(t)dt = -\exp(-3)/2 = l = -0.02489.$$

Now, to determine the stability bound of  $\alpha$  is to show that the delay-independent criterion in (4.20) of Theorem 4.4 satisfies the asymptotic stability for (4.36). By solving the LMI given in (4.20) of Theorem 4.4, the bound of  $\alpha$  for asymptotic stability is found to be  $|\alpha| \leq 0.8448$  and the solutions of the LMI for  $\alpha = 0.8448$  are given by

$$P = \begin{pmatrix} 1.0000 & 0 \\ 0 & 1.0000 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.8746 & 0 \\ 0 & 0.8746 \end{pmatrix}, \quad \tau_0 = 0.0249, \\ \tau_1 = 0.0249, \quad \tau_2 = 0.1500.$$

#### 4.5.3. Comparative example with other Lyapunov's results

In this section, the bound of  $\alpha$  for the asymptotic stability of (4.36) without the infinite delay term is compared with result obtained in Example 2 of Park and Won (2000) with other corresponding results as follows;

Li (1988):  $|\alpha| \leq 0.2$

Hu and Hu (1996):  $|\alpha| \leq 0.2$

Park and Won (2000):  $|\alpha| \leq 0.9165$

This Thesis (Theorem 4.4):  $|\alpha| \leq 0.9165$ .

It is observed that Theorem 4.4 gives a less conservative bound of  $\alpha$  than all the proposed methods contained in Park and Won (2000) and produces the same result as that of Park and Won (2000) for the case without the infinite delay term, that is when  $G = 0$ . The MATLAB code for this example is given in Appendix I.

#### **4.6. Concluding remarks**

In this chapter, total asymptotic stability results for neutral integro-differential systems having infinite delays are presented by using the Lyapunov-Razumikhin technique. The results were obtained by exploring the uniform stability properties of the functional difference operator for neutral systems, the basic Razumikhin stability theories, and the uniqueness property of the eigenvalues from the existence of symmetric positive definite matrix from Lyapunov matrix equation.

Furthermore, by using the Lyapunov-Krasovskii stability approach, a new delay independent condition which is sufficient to make the system uniformly asymptotically stable is developed. This new condition was then expressed in terms of LMI and solved by using the MATLAB's LMI Toolbox. The MATLAB code written for the problem in Example 4.5.2 is given in Appendix I. All the theoretical analyses in this chapter were illustrated with numerical example. These stability results play important role in the control methods and are utilized in Chapter 5.



## Chapter 5

# Control methods

### 5.1. Introduction

Having justified the stability of the system in-line with the research aim and objectives in Chapter 4, and the broadly reviewed literature in Chapter 2 which has led to the selection of appropriate models for the potential application of this work in Chapter 3; the aim of this chapter is to develop a neutral functional differential delay control system and investigate its controllability. Various controllability methods for the neutral control system are investigated. Relevant propositions, lemmas, theorems and definitions that would aid the development and computations of the results are stated and proved. Algebraic conditions are developed for the complete and null controllability results of the system. Some examples illustrating the design procedure and effectiveness of the theoretical results are given with some simulation output to illustrate the viability of the methods developed.

Since controllability is one of the most important structural properties of dynamical systems used to design model based controllers and estimators, this chapter aims at establishing the necessary results by focussing on the control of interest introduced in the Glossary of notation. It is common knowledge that controls can be assumed to be either (i) restrained or (ii) unrestrained; but is required only to be square integrable on finite intervals (Chukwu 1979). In the latter case, a non-singularity assumption for the controllability matrix of the system is a necessary and sufficient condition for null controllability. In the former case, which is the control of interest in this thesis, such conditions are no longer sufficient for null controllability and an additional condition of stability for the uncontrolled system is required

(Chukwu 1979). Dauer et al. (1998) has demonstrated this method by obtaining a null controllability result using the Schauder fixed point theorem. There result was based on the uniform asymptotic stability of the uncontrolled system and a properness assumption on the linear control system, the latter being equivalent to the non-singularity of the controllability matrix. However, evaluating controllability analytically for linear time varying systems, unlike time-invariant systems, is challenging even for very simple systems since it involves the evaluation of the controllability matrices. The controllability matrix may be calculated by computational methods provided that all the exact time-varying elements in the linear time varying systems are known. The control is assumed to be restrained in this chapter and the null controllability result obtained by Schauder's fixed point theorem method, the chapter aims to avoid the cumbersome computation of controllability matrices by introducing an equivalent rank condition which is simple to compute and generalizes to neutral systems. The method will extend the results from Underwood and Chukwu (1988), Jacobs and Langenhop (1976), Rivera Rodas and Langenhop (1978) to NFDSID. By using the Schauder fixed point theorem, growth and continuity conditions will be placed on the perturbation function which will guarantee that: if the linear control base system has full rank with the condition that  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$  for every complex  $\lambda$ , (where  $K(\lambda)$  is an  $n \times n$  polynomial matrix in  $\lambda$  constructed from the coefficient matrices of the control base system, and  $\xi(\exp(-\lambda h))$  is the transpose of  $[1, \exp(-\lambda h), \dots, \exp(-(n-1)\lambda h)]$ ), and the functional difference operator for the system uniformly stable, with the linear uncontrolled system uniformly asymptotically stable, then the perturbed neutral system with infinite delay is null controllable with constraint.

Further, by using the standard Lyapunov-Krasovskii approach, which often leads to Linear Matrix Inequality (LMI), a new delay-independent condition which is sufficient to make the neutral system with infinite delays uniformly asymptotically stable is developed. The novel

condition is obtained by embedding the infinite delay into a norm bounded uncertainty element, memory-less state feedback controllers are designed which stabilize the system using the feasible solution of the resulting LMI which is less conservative.

## 5.2. Control model for neutral system with infinite delay

This chapter considers neutral functional differential system with infinite delays of the form:

$$\left. \begin{aligned} \frac{d}{dt}D(t)x_t &= L(t, x, x_t, u) + \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta \\ x(t) &= \phi(t), \quad t \in (-\infty, 0] \end{aligned} \right\}, \quad (5.1)$$

and its perturbation

$$\frac{d}{dt}D(t)x_t = L(t, x, x_t, u) + \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta + f(t, x_t, u(t)) \quad (5.2)$$

through its linear base control system

$$\frac{d}{dt}D(t)x_t = L(t, x, x_t, u), \quad (5.3)$$

and its free system

$$\frac{d}{dt}D(t)x_t = L(t, x, x_t, 0) + \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta, \quad (5.4)$$

where the functional difference operator  $D: E \times C \rightarrow E^n$  for the system is defined Section 4.2

by  $D(t)x_t = x(t) - A_0(t)x(t-h)$ , and

$$L(t, x, x_t, u) = A_1(t)x(t) + A_2(t)x(t-h) + B(t)u(t),$$

with the following assumptions:

- (i)  $A_0(t)$ ,  $A_1(t)$  and  $A_2(t)$  are continuous  $n \times n$  matrices
- (ii)  $B(t)$ , is a continuous  $n \times m$  matrix
- (iii)  $A(\theta)$  is an  $n \times n$  matrix whose elements are square integrable on  $(-\infty, 0]$

(iv)  $f: [\sigma, \infty) \times W_2^{(1)} \times E^m \rightarrow E^n$  is a nonlinear continuous matrix function.

It is assumed that  $f$  satisfies sufficient smoothness conditions to ensure that: (i) a solution of (5.2) exists through each  $(\sigma, \phi)$ , (ii) it is unique, and (iii) it depends continuously upon  $(\sigma, \phi)$  and (iv) it can be extended to the right as long as the trajectory remains in a bounded set  $[\sigma, \infty) \times C$ . These conditions are given in Cruz and Hale (1970).

If  $K(t, \sigma): W_2^{(1)} \rightarrow W_2^{(1)}$  is defined by  $K(t, \sigma)\phi = x_t(\sigma, \phi)$ ,  $\phi \in W_2^{(1)}$ ,  $t \geq \sigma$ , where  $x(\sigma, \phi)$  is a solution of (5.3) with  $u = 0$ , then by the variation of constants formula given in (5.5), there exist an  $n \times n$  matrix function  $X(t, s)$  defined for  $0 \leq t \leq s$ ,  $t \in J = [\tau, \infty)$ , continuous in  $s$  from the right, and of bounded variation in  $s$ ;  $X(t, s) = 0$ ,  $t < s \leq t_1$ , such that  $X(t, s)$  satisfies

$$\frac{\partial X(t, s)}{\partial s} = L(t, X_t(\cdot, s), 0), \quad t \geq s.$$

Now, define the  $n \times n$  matrix function  $X_0$  as

$$X_0(s) = \begin{cases} 0, & -h \leq s < 0 \\ I, & s = 0. \end{cases}$$

Here  $X(t, t) = I$  is the identity matrix. Write  $K(t, s)X_0(s) = X(t + \sigma, s) = X_t(\cdot, s)$ , so that  $K(t, s)I = X(t, s)$ .

A solution  $x$  of (5.3) through  $(\sigma, \phi)$  satisfies the equation

$$x_t(\sigma, \phi, u) = K(t, \sigma)\phi + \int_{\sigma}^t T(t, \sigma)X_0B(s)u(s)ds,$$

or

$$x_t(\sigma, \phi, u) = x_t(\sigma, \phi, 0) + \int_{\sigma}^t X(t, s)B(s)u(s)ds. \quad (5.5)$$

The representation of the form (5.5) will be referred to as the variation of constants formula in this thesis. In a similar manner, any solution of system (5.2) will be given by

$$\begin{aligned} x_t(\sigma, \phi, u, f) = & x_t(\sigma, \phi, 0) + \int_{\sigma}^t X(t, s)B(s)u(s)ds + \int_{\sigma}^t X(t, s) \int_{-\infty}^0 A(\theta)x(t + \theta)d\theta ds \\ & + \int_{\sigma}^t X(t, s) f(s, x_s, u(s))ds, \end{aligned} \quad (5.6)$$

Define the matrix function  $Z(\cdot)$  by

$$Z(t, s) = X(t, s)B(s), \quad (5.7)$$

for  $t \geq s \geq \sigma$ , it follows then from (5.6) that

$$\begin{aligned} x_t(\sigma, \phi, u, f) = & x_t(\sigma, \phi, 0) + \int_{\sigma}^t Z(t, s)u(s)ds + \int_{\sigma}^t X(t, s) \int_{-\infty}^0 A(\theta)x(t + \theta)d\theta ds \\ & + \int_{\sigma}^t X(t, s) f(s, x_s, u(s))ds. \end{aligned} \quad (5.8)$$

Some definitions which underpin the subject of investigation in this chapter will now be given

**Definition 5.1: (Proper)**

The system (5.3) is proper on  $[\sigma, t_1]$  if  $\eta^T Z(t_1, s) = 0$  almost everywhere  $s \in [\sigma, t_1]$  implies  $\eta = 0$  for  $\eta \in E^n$ , where  $\eta^T$  is the transpose of  $\eta$ . If (5.3) is proper on each interval  $[\sigma, t_1]$ , then the system is said to be proper in  $E^n$ . Where  $\eta$  is as defined in the proof of Proposition

**Definition 5.2: (Controllable)**

System (5.3) is said to be controllable on  $[\sigma, t_1]$ , if for each function  $\phi \in W_2^{(1)}([-h, 0], E^n)$ , there is a control  $u \in L_2([\sigma, t_1], E)$  such that the  $x_{t_1}(\sigma, 0, u) = \phi$ .

**Definition 5.3: (Completely controllable)**

System (5.3) is said to be completely controllable on  $[\sigma, t_1]$ , if for each function  $\phi \in W_2^{(1)}$ ,  $x_1 \in E^n$  there is an admissible control  $u \in L_2([\sigma, t_1], E)$  such that the solution  $x(\sigma, \phi, u)$  of (5.3) satisfies  $x_\sigma(\sigma, \phi, u) = \phi$ ,  $x_{t_1}(\sigma, \phi, u) = x_1$ . It is completely controllable on  $[\sigma, t_1]$  with constraints, if the above holds with  $u \in U$ .

**Definition 5.4: (Null controllable)**

The system (5.2) is null-controllable on  $[\sigma, t_1]$  if for each  $\phi \in W_2^{(1)}([-h, 0], E^n)$ , there exists a  $u \in L_2([\sigma, t_1], E^m)$  such that the solution of (5.2) satisfies  $x_\sigma(\sigma, \phi, u, f) = \phi$ ,  $x_{t_1}(\sigma, \phi, u, f) = 0$ . The system (5.2) is null-controllable with constraints if the above holds with control  $u \in U$

**Definition 5.5: (Domain of null controllability)**

The domain  $\mathfrak{U}$  of null-controllability of (5.3) with constraints is the set of all initial functions  $\phi \in W_2^{(1)}$  for which the solution  $x(\sigma, \phi, u)$  of (5.3) with  $x_\sigma(\sigma, \phi, u) = \phi$ ,  $x_{t_1}(\sigma, \phi, u) = 0$  at some  $t_1$ ,  $u \in U$

**Definition 5.6: (Reachable set)**

The reachable set of (5.3) is a subset of  $E^m$  given by

$$\mathcal{P}(\sigma, t) = \left\{ \int_{\sigma}^t Z(t, s)u(s)ds : u \in L_2([\sigma, t], E^m) \right\}.$$

If the controls are in  $L_2([\sigma, t], C^m)$ , we define the constraint reachable set by

$$R(\sigma, t) = \left\{ \int_{\sigma}^t Z(t, s)u(s)ds : u \in L_2([\sigma, t], C^m) \right\},$$

where  $U = L_2^{loc}([\sigma, t], C^m)$ .

**Definition 5.7: (Controllability Matrix)**

The controllability matrix of (5.3) will be given by

$$W(\sigma, t) = \int_{\sigma}^t Z(t, s)Z^T(t, s)ds ,$$

where  $Z^T$  is the transpose of  $Z$ .

### 5.3. Necessary and sufficient conditions for controllability

This section develops and proves necessary and sufficient controllability and null controllability conditions for the system (5.2) by exploring the method described in Jacobs and Langenhop (1976), Rivera Rodas and Langenhop (1978). Some controllability results which are relevant to this investigation are also given.

**Lemma 5.1**

The system (5.3) is completely controllable if and only if  $W(\sigma, t_1)$  is non-singular

**Proof.** The proof is similar to that of Dauer and Gahl (1977) for retarded systems; it is done *mutatis mutandis* in applying to system (5.3). Assume  $W(\sigma, t_1)$  is non-singular. Let  $\phi$  be continuous on  $W_2^{(1)}$ , and let  $x_1 \in E^n$ . Let  $u$  be the admissible control function given by

$$u(t) = Z^T(t, s)W(\sigma, t)^{-1}[x_1 - x_t(t_1, 0)] ,$$

for  $t \in [\sigma, t_1]$ . Hence, from equation (5.5) it follows that

$$x_t(t_1, u) = x_t(t_1, 0) + \int_{\sigma}^{t_1} Z(t_1, s)Z(t_1, s)^TW(\sigma, t_1)^{-1} \times [x_1 - x_L(t_1, 0)]ds = x_1 .$$

Now, assume  $W(\sigma, t)$  is singular. Then, there exists a row vector  $v \neq 0$  such that  $vW(\sigma, t_1)v^T = 0$ . It follows that  $\int_{\sigma}^{t_1} vZ(t_1, s)(vZ(t_1, s))^T ds = 0$ . Therefore,  $vZ(t_1, s) = 0$  almost everywhere for all  $t \in [\sigma, t_1]$ . Therefore,  $v \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds = 0$ , for all admissible  $u$ . Since  $\left\{ \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds \mid u \text{ is admissible} \right\}$  is a vector space which is orthogonal to  $v$ , it cannot be equal to  $E^n$ . It follows from equation (5.5) that  $\{x_t(t_1, u) \mid u \text{ is admissible}\}$  cannot be equal to  $E^n$ . Therefore the system (5.3) is not completely controllable on  $[\sigma, t_1]$ , and the proof is complete.  $\square$

### **Proposition 5.1**

The system (5.3) is controllable on  $[\sigma, t_1]$  if and only if  $0 \in \text{int } R(\sigma, t)$

**Proof.** Since  $R(\sigma, t)$  is known to be a closed and convex subset of  $E^n$ , there exists a point  $y_1$ , on the boundary of  $R(\sigma, t)$ , which implies that there is a support plane  $\pi$  of  $R(\sigma, t)$  through  $y_1$ , that is,  $\eta^T(y - y_1) \leq 0$  for each  $y \in R(\sigma, t)$ , where  $\eta \neq 0$  is an outward normal to  $\pi$ . If  $u_1$  is the control corresponding to  $y_1$  then  $\eta^T \int Z(t, s)u(s)ds \leq \eta^T \int Z(t, s)u_1(s)ds$ , for each  $u \in C^m$ . Since the control function lie in an  $m$ -dimensional unit cube  $C^m$  in  $E^m$ , this last inequality holds for each  $u \in C^m$  if and only if

$$\eta^T \int Z(t_1, s)u(s)ds \leq \int |\eta^T Z(t_1, s)u_1(s)|ds = y_1 = \int |\eta^T Z(t_1, s)|ds,$$

and  $u_1(s) = \text{sgn } \eta^T Z(t_1, s)$ . Now  $0 \in R(\sigma, t)$  always, if  $0$  were not in the interior of  $R(\sigma, t)$ , then  $0$  would be on the boundary. Hence, from the preceding arguments, this implies that  $0 = \int |\eta^T Z(t_1, s)|ds$ , so that  $\eta^T Z(t_1, s) = 0$  almost everywhere  $s \in [\sigma, t_1]$ . This by the definition of properness implies that the system is not proper. Since  $\eta \neq 0$ , this completes the proof.  $\square$



**Proposition 5.2**

The system (5.3) is completely controllable with constraint on  $[\sigma, t_1]$  if and only if  $0 \in R(\sigma, t)$

**Proof.** Assume (5.3) is completely controllable, then by Lemma 5.1,  $W$  is non-singular. Note that  $W$  non-singular is equivalent to  $W$  being positive definite and this in turn is equivalent to  $\eta^T Z(t_1, s) = 0$  almost everywhere on  $[\sigma, t_1]$  which implies  $\eta = 0$ . This by definition implies that system (5.3) is proper. Hence by Proposition 5.1, this holds if and only if  $0 \in R(\sigma, t)$ .  $\square$

**Proposition 5.3**

The following are equivalent for system (5.3).

- (i)  $W(\sigma, t)$  is non-singular,
- (ii) System (5.3) is completely controllable on  $[\sigma, t_1]$ ,  $t_1 > \sigma$
- (iii) System (5.3) is proper on  $[\sigma, t_1]$ ,  $t_1 > \sigma$

**Proof.** The idea in this proof is to show that (i)  $\Rightarrow$  (ii), (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i). To show that

(i)  $\Rightarrow$  (ii): Define the operator  $K: L_2([\sigma, t_1], E^m) \rightarrow E^m$  by  $K(u) = \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds$ , where  $K$  is a continuous linear operator from a Hilbert space to another. Thus,  $R(K) \subseteq E^n$  is a linear subspace and its orthogonal complement satisfies the relation

$$\{R(K)\}^\perp = N(K^*) \quad (5.9)$$

where  $K^*$  is the adjoint of  $K$ ,  $K^*: E^n \rightarrow U \subseteq L_2$ . By the non-singularity of the controllability matrix  $W(\sigma, t_1)$ , the symmetric operator  $KK^* = W(\sigma, t_1)$  is positive definite and hence

$\{R(K)\}^\perp = \{0\}$ , and therefore  $N(K^*) = \{0\}$  by (5.9). For any  $c \in E^n$ ,  $u \in L_2$ ,

$$\langle c, Ku \rangle = \langle K^*c, u \rangle, \langle c, Ku \rangle = \langle c, \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds \rangle = \int_{\sigma}^{t_1} c^T [Z(t_1, s)]u(s)ds.$$

Thus,  $K^*$  is given by  $c \rightarrow c^T[Z(t_1, s)]$ ,  $s \in [\sigma, t_1]$ .  $N(K^*)$  is therefore the set of all  $c \in E^n$  such that  $c^T[Z(t_1, s)] = 0$ , almost everywhere in  $[\sigma, t_1]$ . Since  $N(K^*) = \{0\}$ , all such  $c$  are equal to zero, that is  $c = 0$ . This establishes the properness of system (5.3).

(ii)  $\Rightarrow$  (iii): The task now is to show that, system (5.3) is relatively controllable on each interval  $[\sigma, t_1]$ . Let  $c \in E^n$ , if system (5.3) is proper then  $c^T[Z(t_1, s)] = 0$  almost everywhere  $s \in [\sigma, t_1]$  for each  $t_1$  implies  $c = 0$ . Thus,  $\int_{\sigma}^{t_1} c^T[Z(t_1, s)]u(s)ds = 0$  for  $u \in L_2$ . It follows that the only vector orthogonal to the set  $R(\sigma, t_1) = \left\{ \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds : u \in L_2 \right\}$  is the zero vector. Hence,  $\{R(\sigma, t_1)\}^\perp = \{0\}$ . That is  $R(\sigma, t_1) = E^n$ . This establishes relative controllability on  $[\sigma, t_1]$  of system (5.3).

(iii)  $\Rightarrow$  (i): Next is to show that if system (5.3) is relatively controllable then  $W(\sigma, t_1)$  is non-singular. Assume for a contradiction that  $W(\sigma, t_1)$  is singular, then there exists an  $n$  vector  $v \neq 0$  such that  $vW(\sigma, t_1)v^T = 0$ . Then,  $\int_{\sigma}^{t_1} \|v[Z(t_1, s)]\|^2 ds = 0$ , this implies that  $\|v[Z(t_1, s)]\|^2 = 0$  almost everywhere  $s \in [\sigma, t_1]$ , hence  $v[Z(t_1, s)] = 0$ , almost everywhere  $s \in [\sigma, t_1]$ . This contradicts the assumption of properness of the system since  $v \neq 0$  and this completes the proof.  $\square$

### **Lemma 5.2**

The system (5.3) is completely controllable on  $[\sigma, t_1]$  if and only if it is controllable on  $[\sigma, t_1]$ .

**Proof.** The proof follows immediately from Proposition 5.1 and 5.2.  $\square$

#### **5.3.1. Controllability results**

Necessary and sufficient controllability conditions for systems (5.3) will now be developed in this section

Let  $x: [\alpha, \beta] \rightarrow E^q$ ,  $q$  a positive integer, be absolutely continuous and define the differential operator for neutral systems  $\mathcal{D}$  by  $(\mathcal{D}x)(t) = \dot{x}(t) = dx(t)/dt$ , almost everywhere on  $[\alpha, \beta]$ . Higher powers of the operator  $\mathcal{D}$  are defined inductively by  $\mathcal{D}^{k+1} = \mathcal{D}\mathcal{D}^k$ , with domain equal to all  $x: [\alpha, \beta] \rightarrow E^q$ , such that  $\mathcal{D}^k x$  is absolutely continuous on  $[\alpha, \beta]$ . Note that,  $\mathcal{D}^0$  refers to the identity  $(\mathcal{D}^0 x)(t) = x(t)$ ,  $t \in [\alpha, \beta]$ .

Define  $W_{2,0}^p(\tau, E^q)$ , where  $p$  is a nonnegative integer to be the collection of all  $x: (-\infty, \tau] \rightarrow E^q$  such that  $x(t) = 0$  for  $t \leq 0$  and the restriction of  $x$  on  $[0, \tau]$  is in  $W_2^p([0, \tau], E^q)$ , adopt the convention  $W_2^0([0, \tau], E^q) = L_2([0, \tau], E^q)$ . For  $f \in W_{2,0}^p(\tau, E^q)$  define the shift operator  $\mathcal{S}$  by

$$(\mathcal{S}f)(t) = f(t - h), \quad t \leq \tau. \quad (5.10)$$

Define  $\mathcal{S}^0$  to be the identity operator on  $W_{2,0}^p(\tau, E^q)$  and take  $\mathcal{S}^{k+1} = \mathcal{S}\mathcal{S}^k$ ,  $k = 0, 1, 2, \dots$  by inductively using (5.10), so that for any integer  $r \geq 1$ ,  $g^r: W_{2,0}^p(\tau, E^q) \rightarrow W_{2,0}^p(\tau, (E^q)^r)$  can be defined by  $g^r f = [\mathcal{S}^0 f, \mathcal{S}^1 f, \dots, \mathcal{S}^{r-1} f]^T$ .

Observe from the definition of the differential operator  $\mathcal{D}$ , and the shift operator  $\mathcal{S}$  that for  $p \geq 1$ , if the function space  $W_{2,0}^p(\tau, E^q)$  is taken as a common domain for the operators  $\mathcal{S}$  and  $\mathcal{D}$ , then  $\mathcal{S}$  and  $\mathcal{D}$  commute in this setting and each commutes with multiplication by a scalar (element in  $E$ ). The operators  $\mathcal{D}$ ,  $\mathcal{S}$  and multiplication by a scalar all commute with the coordinate projections; that is, if  $x \in W_{2,0}^p(\tau, E^q)$  and  $(x)_i = x_i$  denotes the  $i$ th component of  $x$ , then  $(\mathcal{S}x)_i = \mathcal{S}x_i$  and  $(\mathcal{D}x)_i = \mathcal{D}x_i$  for all  $i = 1, \dots, q$ . Note also that, for an operation with  $\alpha \in E$  on functions in  $W_{2,0}^p(\tau, E^q)$ ,  $\mathcal{D}\alpha = \alpha\mathcal{D}$  is different from a scalar with value  $\alpha$  for which the operation  $(\mathcal{D}\alpha)(t) = 0$ .

For any  $n \times n$  matrix  $A$  and  $n \times m$  matrix  $B$ , one can define  $n \times rm$  matrix by

$P_r[A, B] = [B, AB, \dots, A^{r-1}B]$ , for integers  $r \geq 1$ . Consider the neutral system

$$(d/dt)(x - A_0x(t-h)) = A_1x(t) + A_2x(t-h) + Bu(t), \quad (5.11)$$

with the assumption that  $A_i, i = 0, 1, 2$  are  $n \times n$  constant matrices, and  $B$  chosen to be  $n \times 1$  constant real matrix. Then, the solution  $x(\cdot, 0, u)$  of (5.11) is the restriction to  $[-h, 0]$  of the solution  $x \in W_{2,0}^{(1)}(\tau, E^n)$  of the equation

$$(ID - A_0SD - A_1 - A_2S)x = Bu.$$

Now define the matrix  $A(\mathcal{D}, \mathcal{S})$  by the equation

$$A(\mathcal{D}, \mathcal{S}) = ID - A_0SD - A_1 - A_2S,$$

and let

$$\check{P}(\mathcal{D}, \mathcal{S}) = \text{adj } A(\mathcal{D}, \mathcal{S}),$$

where “adj” denotes the transposed matrix of cofactors. Some basic relationship exists between these two operators which by Jacobs and Langenhop (1976) can be expressed as

$$\check{P}(\mathcal{D}, \mathcal{S}) = \sum_{i=0}^{n-1} \check{P}_i(\mathcal{D})\mathcal{S}^i = \sum_{i=0}^{n-1} \hat{P}_i(\mathcal{S})\mathcal{D}^i, \quad (5.12)$$

where the  $n \times n$  matrix polynomials  $\check{P}_i(\mathcal{D}), \hat{P}_i(\mathcal{S})$  are at most of degree  $n-1$  in their argument. Using the polynomial  $\check{P}_i(\mathcal{D})$  in (5.12) define a unique matrix operator by

$$K(\mathcal{D}) = [P_0(\mathcal{D})B, P_1(\mathcal{D})B, \dots, P_{n-1}(\mathcal{D})B].$$

Now, the operator  $K(\mathcal{D})$  can be written in the form of a polynomial to get

$$K(\mathcal{D}) = \sum_{i=0}^{n-1} K_i\mathcal{D}^{n-1-i}, \quad (5.13)$$

where,  $K_i, i = 0, 1, \dots, n-1$  are  $n \times n$  constant real matrices, and let  $\xi(\exp(-\lambda h))$  be the transpose of  $[1, \exp(-\lambda h), \dots, \exp(-(n-1)\lambda h)]$  for all complex numbers  $\lambda, h > 0$ .

**Theorem 5.1**

Let  $\tau > nh$ , then system (5.11) is controllable on  $[0, \tau]$  if and only if  $\text{rank } P_n[A_0, B] = n$  and  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$  for every complex  $\lambda$ .

**Proof.** This is Theorem 3.4 of Rivera Rodas and Langenhop (1978).  $\square$

**Corollary 5.1**

Let  $\tau > nh$ , and assume that system (5.3) satisfies the following

- (i)  $\text{rank } P_n[A_0, B] = n$ ;
- (ii)  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ , for every complex  $\lambda$ ,

Then, system (5.3) is completely controllable on  $[0, \tau]$ .

**Proof.** If condition (i) and (ii) holds, then by Theorem 5.1, the system (5.3) is controllable on  $[0, \tau]$ . This, by Lemma 5.2, implies that system (5.3) is completely controllable on  $[0, \tau]$ . Conversely, if system (5.3) is completely controllable on  $[0, \tau]$ , then it is controllable by Lemma 5.2, and by Theorem 5.1,  $\text{rank } P_n[A_0, B] = n$  and  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$  for every complex  $\lambda$ , and the proof is complete.  $\square$

**5.3.2. Null controllability result**

This section can now focus on the null controllability result for the neutral system with infinite delay as given by equation (5.2). The results of this section are part of the contributions of the thesis in this chapter.

**Theorem 5.2**

Consider system (5.1), and assume the following

- (i)  $A_0, A_1, A_2$  are  $n \times n$  constant matrices,  $B$  is  $n \times 1$  constant real matrix

- (ii) for  $\tau > nh$ ,  $\text{rank } P_n[A_0, B] = n$  ;
- (iii)  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ , for every complex  $\lambda$ ,
- (iv)  $\sup\{\text{Re}(\lambda), \det \Delta(\lambda) = 0\} < 0$ , with

$$\Delta(\lambda) = \lambda(I - A_0 \exp(-\lambda h)) - A_1 - A_2 \exp(-\lambda h) - \int_{-\infty}^0 \exp(\lambda \theta) A(\theta) d\theta$$

- (v) and  $D\phi = \phi(0) - A_0\phi(-h)$  is uniformly stable.

Then, system (5.1) is null controllable with constraints on  $(0, \sigma)$ ,  $\sigma > \tau$ .

**Proof.** Because of (i), (ii) and (iii), system (5.1) is controllable on  $[0, \tau]$  by Theorem 5.1.

Hence,  $0 \in \text{Int } R(0, \tau)$  by Proposition 5.1. By condition (iv), and (v) the system (5.1) with  $u = 0$  satisfies  $x_t(\cdot, \phi, 0) \rightarrow 0$  as  $t \rightarrow \infty$ . Hence, at some  $t_1 > 0$ ,  $x_{t_1}(0, \phi, 0) \in \text{Int } R(0, \tau)$  and hence  $0 \in \text{Int } \mathfrak{U}$ , the domain of null controllability of (5.1). Suppose for the contrary that  $0 \notin \text{Int } \mathfrak{U}$ . Since  $x = 0$  is a solution of (5.1) with  $u = 0$ , then  $0 \in \mathfrak{U}$ . This implies that, there exists a sequence  $\{\phi_i\} \subseteq W_2^{(1)}$  such that  $\phi_i \rightarrow 0$  as  $i \rightarrow \infty$  and  $\phi_i \notin \mathfrak{U}$ , for any  $i$ , therefore  $\phi_i \neq 0$ . It follows from the variation of constants formula (5.5) that:

$x_{t_i}(0, \phi_i, u) = x_t(0, \phi_i, 0) + \int_{\sigma}^{t_1} Z(t_1, s)u(s)ds$ . Let  $z_i = x_{t_1}(0, \phi_i, 0)$ . Then, since  $\phi_i \notin \mathfrak{U}$ ,  $x_{t_1}(0, \phi_i, u) \neq 0$ , for any  $i$ , and so  $z_i \notin R(0, t_1)$ , for any  $t_1 > 0$  and  $z_i \neq 0$ . However  $z_i \rightarrow 0$  as  $i \rightarrow \infty$ , and  $0 \notin \text{Int } R(0, t_1)$  which is a contradiction. Therefore,  $0 \in \text{Int } \mathfrak{U}$ , and hence there exists a ball  $S_2$  around 0 which is contained in  $\mathfrak{U}$ . Again, by (iv) there exists some  $t_2 < \infty$ ,  $x_{t_2}(\cdot, \phi, 0) \in S_2$ . Therefore, using  $t_2$  as initial point and  $x_{t_2}(\cdot, \phi, 0) \equiv \psi$  as initial function, there exists  $u \in U$  and  $t_3 > t_2$  such that, the solution  $x(t_2, x_{t_1}(\cdot, \phi, 0), u)$  of (5.1) satisfies  $x_{t_3}(\cdot, t_1, x_{t_1}, u) = 0$ , and the proof is complete.  $\square$

**Remark 5.1**

The conditions imposed on Theorem 5.2 constrain  $U$  in a box but these conditions can also allow the constraint set  $U$  to be an arbitrary compact set as shown in Theorem 5.3. This is possible because the null controllability of linear neutral system, in general, depends on the length of the time interval over which the system operates (Jacobs and Langenhop 1976). Therefore restrictions on interval could be made based on the requirements for the controllability of the linear controllable base system. Again, these conditions are made possible from the definition of  $U$  because by Theorem 5.1, if the conditions  $\text{rank } P_n[A_0, B] = n$ , and  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$  for every complex  $\lambda$  holds on  $[\sigma, t_1]$ ,  $t_1 \geq t$  then (5.3) is controllable. This means  $R(\sigma, t) = W_2^{(1)}$ . Now define a map  $K$  taking  $L_2([\sigma, t], E^m) \rightarrow W_2^{(1)}$  by  $K(u) = x_t(\cdot, \sigma, \phi, u)$ . Because  $K$  is a continuous linear transformation of  $L_2$  onto  $W_2^{(1)}$  it is open (Hale 1977). From the definition of  $U$ , there is an open ball  $S \subseteq U$  around zero, so that  $K(S) \subseteq K(U) = R(\sigma, t_1)$ . Therefore,  $0 \in K(S) \subseteq R(\sigma, t_1)$ . This implies that  $K(S)$  is open, since  $S$  is open and therefore  $0 \in \text{int } R(\sigma, t)$ . Moreover, if it is assumed that (5.3) is completely controllable on  $[\sigma, t_1]$  then by Lemma 5.1,  $W(\sigma, t_1)$  is non-singular which in turn is equivalent to  $\eta^T Z(t_1, s) = 0$  almost everywhere on  $[\sigma, t_1]$ , and  $\eta = 0$ , this implies properness of (5.3) by Definition 5.1, hence  $0 \in R(\sigma, t)$  by Proposition 5.2, and therefore completely controllable on  $(0, \sigma)$ ,  $\sigma > \tau$  with constraint.

**Theorem 5.3**

Assume for system (5.2) that

- (i) the constraint set  $U$  is an arbitrary compact set of  $E^m$
- (ii)  $D\phi = \phi(0) - A_0\phi(-h)$  is uniformly stable

- (iii) the system (5.4) is uniformly asymptotically stable; so that the solution of (5.4) satisfies  $\|x_t(\sigma, \phi, 0)\| \leq k \exp(-\alpha(t - \sigma))\|\phi\|$ ,  $\alpha > 0$ ,  $k > 0$ .
- (iv) the system (5.3) is completely controllable
- (v) The continuous function  $f$  satisfies  $|f(t, x(\cdot), u(\cdot))| \leq \exp(-bt) \pi(x(\cdot), u(\cdot))$ , for all  $(t, x(\cdot), u(\cdot)) \in [\sigma, \infty) \times W_2^{(0)} \times L_2$ , where  $\pi = \int_{\sigma}^{\infty} \pi(x(\cdot), u(\cdot)) ds \leq M < \infty$ , and  $b - \alpha \geq 0$ .

Then, the system (5.2) is null controllable.

**Proof.** By (iv),  $W^{-1}$  exists for each  $t_1 > \sigma$ . Assuming the pair of functions  $x, u$  forms a pair to the integral equations

$$u(t) = -Z(t_1, s)^T W^{-1}(\sigma, t_1) \left[ x_{t_1}(\sigma, \phi, 0) + \int_{\sigma}^{t_1} X(t_1, s) \int_{-\infty}^0 A(\theta) x(t + \theta) d\theta ds + \int_{\sigma}^{t_1} X(t_1, s) f(s, x(\cdot), u(\cdot)) ds \right], \quad (5.14)$$

for some suitably chosen  $t_1 \geq t \geq \sigma$ ,  $u(t) = v(t)$ ,  $t \in [\sigma - h, \sigma]$

$$x(t) = x_t(\sigma, \phi, 0) + \int_{\sigma}^t Z(t, s) u(s) ds + \int_{\sigma}^t X(t, s) \int_{-\infty}^0 A(\theta) x(t + \theta) d\theta ds + \int_{\sigma}^t X(t, s) f(s, x(\cdot), u(\cdot)) ds, \quad (5.15)$$

$$x(t) = \phi(t), \quad t \in [\sigma - h, \sigma].$$

Then  $u$  is square integrable on  $[\sigma - h, t_1]$  and  $x$  is a solution of (5.2) corresponding to  $u$  with initial state  $x_{\sigma}(t) = \phi$ . Also,  $x(t_1) = 0$ . It is necessary to show now that  $u: [\sigma, t_1] \rightarrow U$  is in the arbitrary compact constraint subset of  $E^m$ , that is  $|u(t)| \leq a_1$ , for some constant  $a_1 > 0$ .

By (ii) and (iii), and the continuity of  $B$  in compact intervals, it follows that for some

$$d_1 > 0, d_2 > 0, |Z(t_1, s)^T W^{-1}(\sigma, t_1)| \leq d_1,$$



$$\left| x_{t_1}(\sigma, \phi, 0) + \int_{\sigma}^{t_1} X(t_1, s) \int_{-\infty}^0 A(\theta) x(t + \theta) d\theta ds \right| \leq d_2 \exp(-\alpha(t_1 - \sigma)).$$

Hence,

$$|u(t)| \leq d_1 \left[ d_2 \exp(-\alpha(t_1 - \sigma)) + \int_{\sigma}^{t_1} k \exp(-\alpha(t - s)) \exp(-bs) \pi(x(\cdot), u(\cdot)) ds \right]$$

and therefore,

$$|u(t)| \leq d_1 d_2 \exp(-\alpha(t_1 - \sigma)) + d_1 k M \exp(-\alpha(t_1)), \quad (5.16)$$

Since  $b - \alpha \geq 0$  and  $s \geq \sigma \geq 0$ . Hence,  $t_1$  from (5.16) can be chosen sufficiently large such that  $|u(t)| \leq a_1$ ,  $t \in [\sigma, t_1]$ , showing that  $u$  is admissible control. It remains to prove the existence of a pair of the integral equations (5.14) and (5.15). Let  $\mathcal{B}$  represent the Banach space of all functions  $(x, u): [\sigma - h, t_1] \times [\sigma - h, t_1] \rightarrow E^n \times E^m$ , where

$x \in \mathcal{B}([\sigma - h, t_1], E^n)$ ;  $u \in L_2([\sigma - h, t_1], E^m)$  with the norm defined by

$$\|(x, u)\| = \|x\|_2 + \|u\|_2, \text{ where } \|x\|_2 = \left\{ \int_{\sigma-h}^{t_1} |x(s)|^2 ds \right\}^{1/2}; \|u\|_2 = \left\{ \int_{\sigma-h}^{t_1} |u(s)|^2 ds \right\}^{1/2}.$$

Define the operator  $K: \mathcal{B} \rightarrow \mathcal{B}$  by  $K(x, u) = (y, w)$ , where

$$\begin{aligned} w(t) = & -Z(t_1, s)^T W^{-1}(\sigma, t_1) \left[ x_{t_1}(\sigma, \phi, 0) + \int_{\sigma}^{t_1} X(t_1, s) \int_{-\infty}^0 A(\theta) x(t + \theta) d\theta ds \right. \\ & \left. + \int_{\sigma}^{t_1} X(t_1, s) f(s, x_s, u(s)) ds \right], \end{aligned} \quad (5.17)$$

for  $t \in J = [\sigma, t_1]$  and  $v(t) = w(t)$  for  $t \in [\sigma - h, \sigma]$ .

$$\begin{aligned} y(t) = & x_t(\sigma, \phi, 0) + \int_{\sigma}^t Z(t, s) u(s) ds + \int_{\sigma}^t X(t, s) \int_{-\infty}^0 A(\theta) x(t + \theta) d\theta ds \\ & + \int_{\sigma}^t X(t, s) f(s, x_s, u(s)) ds, \end{aligned} \quad (5.18)$$

for  $t \in J$  and  $y(t) = \phi(t)$  for  $t \in [\sigma - h, \sigma]$ .

It is clear from (5.16) that  $|v(t)| \leq a_1$  for  $t \in J$  and also  $v: [\sigma - h, \sigma] \rightarrow U$ , so that

$|v(t)| \leq k$ . Hence,  $\|v\|_2 \leq a_1(t_1 + h - \sigma)^{1/2} = b_0$ . Again,

$$|y(t)| \leq d_2 \exp(-\alpha(t_1 - \sigma)) + d_3 \int_{\sigma}^t |v(s)| ds + kM \exp(-\alpha(t_1)),$$

where  $d_3 = \sup|Z(t, s)|$ . Since  $\alpha > 0$ ,  $t \geq \sigma \geq 0$ , it follows that

$$|y(t)| \leq d_2 + d_3 a_1(t - \sigma) + kM = b_1, \quad t \in J \quad \text{and} \quad |y(t)| \leq \sup|\phi(t)| = \delta, \quad t \in [\sigma - \tau, \sigma].$$

Hence, if  $\lambda = \max[b_1, \delta]$ , then  $\|y\|_2 \leq \lambda(t_1 + h - \sigma)^{1/2} = b_2 < \infty$ . Let  $r = \max[b_1, b_2]$ .

Then letting  $Q(r) = \{(x, u) \in \mathcal{B}: \|x\|_2 \leq r, \|u\|_2 \leq r\}$ , it follows that  $K: Q(r) \rightarrow Q(r)$ .

Now, since  $Q(r)$  is closed, bounded and convex, by Riesz theorem (see Kantorovich and Akilov 1982), it is relatively compact under the transformation  $K$ . Hence, the Schauder's fixed point theorem implies that  $K$  has a fixed point, and therefore system (5.2) is null controllable.  $\square$

#### 5.4. Stabilisation of the system

This section investigates the stabilisation of the neutral functional differential system with infinite delays. A less conservative delay - independent stability criterion is developed in terms of LMI for the NFDSID. A state feedback controller is designed for the stabilisation of the system using the feasible solution of the resultant LMI which is solved using the LMI toolbox in MATLAB. The criterion developed in this section forms part of the contribution of the thesis in this chapter

The asymptotic stability result of the system (5.4) with  $A(\cdot)$  given as  $G(\cdot)$ , where  $G$  is as defined in Section 4.2 (iii) was established in Section 4.4 using the Lyapunov-Krasovskii approach. The interest now is to design a state feedback controller  $u(t)$  that will stabilize the control base system (5.1) with  $A(\cdot)$  in Section 5.2 (iii) now defined as  $G(\cdot)$  in Section 4.2 (iii). Let

$$u(t) = -B^T P x(t), \quad (5.19)$$

where  $P \in E^{n \times n}$  is a positive-definite matrix to be designated.

The closed-loop system design for system (5.1), using (5.19) is defined by

$$\frac{d}{dt}D(t)x_t = (A_1 - BB^T P)x(t) + A_2 x(t-h) + \int_{-\infty}^0 G(t, x_s) ds. \quad (5.20)$$

The task now is to ensure that system (5.20) is closed-loop asymptotically stable.

**Theorem 5.4**

Consider the system (5.1) and all its assumptions; if there exists positive symmetric matrices  $P$ ,  $P_1 > 0$ , some positive scalars  $\tau_4, \dots, \tau_6 > 0$  and a positive-definite symmetric matrix  $X \in E^{n \times n}$  which satisfy the following LMI

$$\begin{aligned} & \mathcal{Z}(X, P_1, \tau_4, \dots, \tau_6) \\ &= \begin{pmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} & (A_2 + XA_1^T A_2 - BB^T A_2) & (A_0 + XA_1^T A_0 - BB^T A_0) \\ * & \mathcal{Z}_{22} & 0 & 0 \\ * & * & A_2^T A_2 - P_1 + \tau_5 A_2^T A_2 & A_2^T A_0 \\ * & * & * & A_0^T A_0 - I + \tau_6 A_0^T A_0 \end{pmatrix} \\ &< 0, \end{aligned} \quad (5.21)$$

so that,

$$\mathcal{Z}_{11} = XA_1^T + A_1 X - 2BB^T - 2lX - 2XA_1^T BB^T - 2BB^T lX,$$

$$\mathcal{Z}_{12} = [XA_1^T \quad BB^T \quad \tau_4 XA_1^T \quad P_1 X \quad LX \quad LX \quad LX \quad LX],$$

$$\mathcal{Z}_{22} = \text{diag}\{-I, -I, -\tau_4 I, -P_1, -I, -\tau_4 I, -\tau_5 I, -\tau_6 I\},$$

where  $X = P^{-1}$ . Then, the system (5.1) is closed-loop asymptotically stable, and the input

$u(t) = -B^T P x(t)$  is a controller for the system (5.1).

**Proof.** Let the Lyapunov function be given by  $V = V_1 + V_2 + V_3$

where,

$$V_1 = x^T(t)Px(t),$$

$$V_2 = \int_{-h}^0 \dot{x}^T(t+s)\dot{x}(t+s)ds$$

and

$$V_3 = \int_{-h}^0 x^T(t+s)P_1x(t+s)ds$$

Taking the derivative of  $V$  along the solution of (5.1) gives

$$\begin{aligned} \dot{V}_1 = & x^T(A_1^T P + PA_1 - 2PBB^T P)x + 2x^T PA_2 x_h + 2x^T PA_0 \dot{x}_h \\ & + 2x^T P \int_{-\infty}^0 G(t, x_s)ds . \end{aligned} \quad (5.22)$$

$$\begin{aligned} \dot{V}_2 = & \dot{x}^T \dot{x} - \dot{x}_h^T \dot{x}_h \\ = & x^T A_1^T A_1 x + x^T PBB^T BB^T Px + x_h^T A_2^T A_2 x_h + \dot{x}_h^T A_0^T A_0 \dot{x}_h \\ & + \left( \int_{-\infty}^0 G(t, x_s)ds \right)^T \int_{-\infty}^0 G(t, x_s)ds - 2x^T A_1^T BB^T Px + 2x^T A_1^T A_2 x_h \\ & + 2x^T A_1^T A_0 \dot{x}_h - 2x^T PBB^T A_2 x_h - 2x^T PBB^T A_0 \dot{x}_h + 2x_h^T A_2^T A_0 \dot{x}_h \\ & + 2x^T A_1^T \int_{-\infty}^0 G(t, x_s)ds - 2x^T PBB^T \int_{-\infty}^0 G(t, x_s)ds + 2x_h^T A_2^T \int_{-\infty}^0 G(t, x_s)ds \\ & + 2\dot{x}_h^T A_0^T \int_{-\infty}^0 G(t, x_s)ds - \dot{x}_h^T \dot{x}_h . \end{aligned} \quad (5.23)$$

$$\dot{V}_3 = x^T P_1 x - x_h^T P_1 x_h. \quad (5.24)$$

Applying Lemma 4.5 with (4.28) to the term  $2x^T PBB^T \int_{-\infty}^0 G(t, x_s) ds$  in equation (5.23) gives;

$$-2x^T PBB^T \int_{-\infty}^0 G(t, x_s) ds \leq -2x^T PBB^T l x, \quad (5.25)$$

Using (5.25) and inequalities (4.28) – (4.32) in Section 4.4.1 of Chapter 4, the overall derivative of  $V$  along the solution of (5.1) can be expressed as  $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \leq \lambda^T(t) \mathfrak{M}(P, P_1, \tau_4, \dots, \tau_6) \lambda(t)$ ,

where

$$\mathfrak{M}(P, P_1, \tau_4, \tau_5, \tau_6) = \begin{pmatrix} \mathfrak{M}_{11} & (PA_2 + A_1^T A_2 - PBB^T A_2) & (PA_0 + A_1^T A_0 - PBB^T A_0) \\ * & A_2^T A_2 - P_1 + \tau_5 A_2^T A_2 & A_2^T A_0 \\ * & * & A_0^T A_0 - I + \tau_6 A_0^T A_0 \end{pmatrix},$$

and  $\lambda(t) = [x^T, x_h^T, \dot{x}_h^T]^T$ , so that,

$$\begin{aligned} \mathfrak{M}_{11} = & A_1^T P + PA_1 - 2PBB^T P - 2Pl - 2A_1^T BB^T P - 2PBB^T l + A_1^T A_1 + PBB^T BB^T P \\ & + \tau_4 A_1^T A_1 + R + l^2 I + \tau_4^{-1} l^2 I + \tau_5^{-1} l^2 I + \tau_6^{-1} l^2 I \end{aligned}$$

Pre and most multiplying  $\mathfrak{M}(\cdot)$  by  $\Gamma^{-T}$  and  $\Gamma$ ; and now using the Schur complement gives

$$\mathcal{Z}(X, P_1, \tau_4, \tau_5, \tau_6)$$

where

$$\Gamma = \begin{pmatrix} X & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

It follows then that  $\dot{V}$  is negative definite since  $\mathfrak{M}(\cdot) < 0$  is equivalent to  $\mathcal{Z}(\cdot) < 0$ , which implies that the system (5.1) is closed-loop asymptotically stable (see Hale and Verduyn Lunel 1993).  $\square$

### **Remark 5.2**

The problems in Theorem 4.4 and 5.4 are feasibility problems. The solution can be found by solving it in the form of a generalized eigenvalue problem see Boyd et al. (1994) for details. In this chapter, the solutions were found by utilizing the MATLAB's LMI Control Toolbox (Gahinet et al. 1995) which implements the interior point algorithm.

## **5.5. Examples on control methods**

In this section, the numerical examples used in Chapter 4 are given to illustrate the applicability of the control methods discussed in this chapter. These examples are part of the contributions made to the thesis in this chapter.

### **5.5.1. Example on null controllability**

Consider the neutral control system

$$\begin{aligned} (d/dt)(x - A_0x(t-h)) &= A_1x(t) + A_2(t)x(t-h) + C_0 \int_{-\infty}^0 \exp(v\theta)x(t+\theta)d\theta \\ &+ Bu(t) \end{aligned} \quad (5.26)$$

and its perturbation

$$\begin{aligned} (d/dt)(x(t) - A_0x(t-h)) &= A_1x(t) + A_2x(t-h) + C_0 \int_{-\infty}^0 \exp(v\theta)x(t+\theta)d\theta \\ &+ Bu(t) + f(t, x(t), x(t-h), u(t)). \end{aligned} \quad (5.27)$$

Its linear control base system is given by

$$(d/dt)(x(t) - A_0 x(t-h)) = A_1 x(t) + A_2 x(t-h) + Bu(t), \quad (5.28)$$

and its free system

$$(d/dt)(x(t) - A_0 x(t-h)) = A_1 x(t) + A_2 x(t-h) + C_0 \int_{-\infty}^0 \exp(v\theta) x(t+\theta) d\theta, \quad (5.29)$$

where,

$$A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$f(t, x(t), x(t-h), u(t)) = \begin{pmatrix} 0 \\ \exp(-\alpha t) \sin(x(t) + x(t-h)) \cdot \cos u(t) \end{pmatrix}, \quad \alpha > 0,$$

The uniform stability of the system  $D\phi = \phi(0) - A_0\phi(-h)$  for  $h > 0$  has been computed in Example 4.5.1 of Chapter 4. The total asymptotic stability of (5.29) is similarly computed in Example 4.5.1. Next, the characteristic root of (5.29) is given by

$$(4 - \exp(-2\lambda h))\lambda^2 + (12 - 2\exp(-\lambda h) - \exp(-2\lambda h))\lambda + 4 + (\lambda + 1) \int_{-\infty}^0 \exp(\lambda + v)\theta d\theta = 0, \quad (5.30)$$

and all the roots of (5.29) have negative real part. Hence by Lemma 4.4 of Chapter 4, system (5.29) is uniformly asymptotically stable.

Finally, check that system (5.28) is controllable as follows:

$$\text{rank}[B, A_0 B] = \text{rank} \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} = 2,$$

$$P_0(\lambda) = \text{adj}(I\lambda - A_1)$$

$$= \text{adj} \begin{pmatrix} \lambda + 1 & -1 \\ -1 & \lambda + 2 \end{pmatrix} = \begin{pmatrix} \lambda + 2 & 1 \\ 1 & \lambda + 1 \end{pmatrix},$$

$$P_1(\lambda) = \text{adj}(-A_0\lambda - A_2)$$

$$= \text{adj} \left[ \begin{pmatrix} 0 & -\lambda/2 \\ -\lambda/2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix} \right] = \begin{pmatrix} 1/2 & (\lambda+1)/2 \\ \lambda/2 & 0 \end{pmatrix},$$

$$K(\lambda) = [P_0(\lambda)B, P_1(\lambda)B] = \begin{pmatrix} 1 & (\lambda+1)/2 \\ 1+\lambda & 0 \end{pmatrix},$$

$$K(\lambda)\xi(\exp(-\lambda h)) = \begin{pmatrix} 1 & (\lambda+1)/2 \\ 1+\lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \exp(-\lambda h) \end{pmatrix}.$$

Observe that for all complex  $\lambda$ ,  $h > 0$ ,

$$K(\lambda)\xi(\exp(-\lambda h)) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore system (5.28) is controllable on  $(0, \tau)$ ,  $\tau > 2h$

Moreover,

$$|f(t, x(t), x(t-h), u(t))| = |\exp(-\alpha t) \sin(x(t) + x(t-h)) \cdot \cos u(t)| \leq \exp(-\alpha t) \cdot 1$$

Hence, all the conditions of Theorem 5.3 are satisfied and system (5.27) is null controllable.

### 5.5.2. Example on stabilisation

Using the modelled system (4.1) in Section 4.2 of Chapter 4 with an assumption that the systems matrices are equivalent to the following

$$A_0 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with,

$$G(t, x_s) = \begin{pmatrix} 0 \\ -\exp(t-3) \times \sin x(t) \cdot x(t) \end{pmatrix}.$$

Solving the LMI (5.21) gives  $\tau_4 = 0.1849$ ,  $\tau_5 = 54.6079$ ,  $\tau_6 = 1.6422$ ,

$$X = \begin{pmatrix} 0.1449 & 0 \\ 0 & 0.1449 \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} 387.8617 & 0 \\ 0 & 387.8617 \end{pmatrix}.$$

Therefore the stabilizing feedback controller  $u(t)$  for the system (5.26) is



$$u(t) = -B^T P x(t) = -B^T X^{-1} x(t) = -\begin{pmatrix} 6.9027 & 0 \\ 0 & 6.9027 \end{pmatrix} x(t).$$

## 5.6. Simulation studies

The stability and controllability of the open-loop system (5.27) can be illustrated using Simulink® and MATLAB® based simulation studies. The simulation model parameters are given as defined in (5.27) with the default parameter setting and a square wave input where  $\alpha$  and  $\nu$  are chosen to be 2 and 1 respectively with  $h = 0.25s$ . Figure 5.1 depicts the stability and controllability of the states when the simulation is performed with the linear control base system i.e. (5.28), and when the simulation is carried out with the perturbation function (see (5.27)). The amplitude of the internal state  $x_1$  from the system response is observed to be slightly higher with the perturbation function whilst it exhibits a faster response when the simulation is done without the perturbation function. The settling times for the systems without the perturbation function are also observed to be faster; this is as expected and depends on the assumptions placed on the perturbation function. The simulation showed that, the system (5.29) is stable and the overall control system (5.27) is controllable.

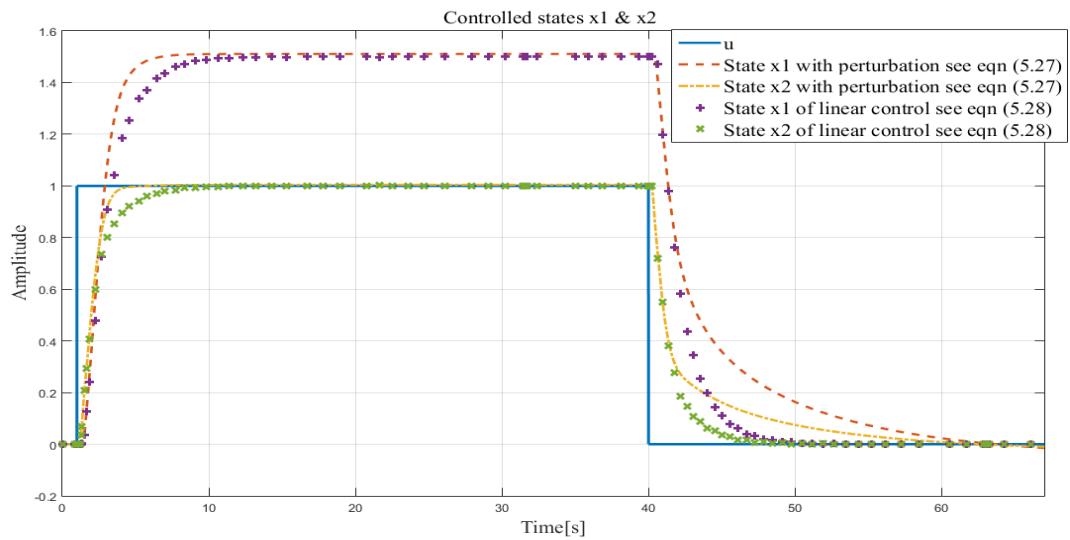


Figure 5.1: Simulation of control input and system states with perturbation function and linear control base system

## 5.7. Concluding remarks

Necessary and sufficient conditions have been developed and proved for the complete controllability and null controllability of neutral functional differential control system with infinite delays, when the controls are functions that are square integrable on finite intervals with values in an  $m$ -dimensional unit cube. It has been proved that when the system has full rank, a sufficient condition for the complete controllability of the system is that  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ , for every complex  $\lambda$ , where  $K(\lambda)$  is an  $n \times n$  polynomial matrix in  $\lambda$  constructed from the coefficient matrices of the system and  $\xi(\exp(-\lambda h))$  is the transpose of  $[1, \exp(-\lambda h), \dots, \exp(-(n-1)\lambda h)]$ . Furthermore, it has been shown that if the above controllability conditions hold and the uncontrolled system is uniformly asymptotically stable, then the neutral functional differential control system is null controllable with constraints. Null controllability has an important relationship with stability in the development of modern control systems and will play an essential role in establishing the optimal control of the system in Chapter 6.

Furthermore, new sufficient conditions are derived for the stabilisation of the neutral systems with infinite delays. The new stabilization conditions were obtained by using the Lyapunov stability approach which are then expressed in terms of LMI and solved by using the MATLAB's LMI Toolbox. The stabilization of the system was obtained by designing a state feedback control law which is presented in terms of LMI and solved by using the MATLAB's LMI Toolbox. The MATLAB code written for the problem in Example 5.5.2 is given in Appendix II

Numerical and simulated output examples were provided to demonstrate the effectiveness of the new results.

## Chapter 6

# Optimal robust control for neutral systems

### 6.1. Introduction

Having settled the stability and controllability problems which are key issues for the NFDSID in Chapters 3 and 4 respectively, this chapter investigates the time optimal control problem of the system. The robust guaranteed cost control problem for the neutral system having infinite delay with a given quadratic cost function is also presented. A delay-dependent stability criterion is proposed based on a model transformation technique. A state feedback control law is then designed using the Razumikhin stability approach and the Lyapunov matrix equation to ensure not only the closed-loop systems robust stability but guarantee that the closed-loop cost function value remains within a specified bound. The problem of designing the optimal guaranteed cost controller is also given in terms of inequalities.

The approach in this chapter will be first to focus on settling the time optimal control for the neutral control systems with infinite delays. This is because finding an optimal control for neutral systems is quite challenging even though interesting results are expected in this chapter. Because of the challenges most, studies conducted on optimal control for neutral systems are based on relevance and are required to achieve specific objectives. For example, the optimal control study by Mordukhovich and Wang (2004) was based on dynamical systems that linearly depend on delayed velocity variables which are governed by neutral functional – differential inclusion models.

However, investigation into time optimal control problems for linear neutral control systems deserves particular attention because of their significant theoretical results and number of important applications. The main idea in time optimal control theory is to steer the system within the shortest time interval from some point in the given allowable initial states to a suitable point on the target set of allowable final states. Chukwu (1988) has demonstrated this by formulating a controllability condition for such systems, and developing criteria for existence, form, uniqueness and general properties of an optimal control in function and Euclidean spaces. Exploring the methods in Chukwu (1988), the time optimal control problem for the neutral control system with infinite delays will be investigated in this chapter.

Next, the optimal robust guaranteed cost control for the NFDSID is investigated. This is to enable the design of the control systems to be not just stable but guarantee an adequate level of performance in the presence of uncertainties in the system. This will be done by defining a quadratic cost function that will provide a bound on the performance index so that the performance degradation will lie within this range. Some interesting efforts have been made by Lien (2006), Park (2003), Xu et al. (2003), and Fernando et al. (2013) to address this robust performance problem. In particular, Lien (2006) obtained stabilisation results with guaranteed cost using linear matrix inequality (LMI) and Krasovskii approach for a class of uncertain neutral systems with time-varying delay. Using a similar approach Park (2003) designed a feedback control system that was robustly stable for a closed-loop cost function value within a specified upper bound for an uncertain neutral type equation with a nonlinear parameter uncertainty. The motivation for the optimal robust guaranteed cost control of NFDSID investigation in this chapter is connected with its wide range of applicable areas as observed in Balachandran and Dauer (1996) and references therein. This chapter will therefore use a model transformation technique to derive, in terms of the Razumikhin approach and the Lyapunov matrix equation, a novel simple delay-dependent stability

condition which is sufficient to make the closed-loop system uniformly asymptotically stable and guarantee adequate level of performance. Furthermore, a stabilisation criterion for the guaranteed cost controller is derived by conversion into a constrained optimization problem with constraints given by a set of inequalities.

## 6.2. Preliminaries and basic definitions for time optimal control problem

Consider the neutral functional differential control system with infinite delays given by

$$\left. \begin{aligned} \frac{d}{dt}D(t)x_t &= L(t, x, x_t, u) + \int_{-\infty}^0 G(t, x_s)dt \\ x(t) &= \phi(t), \quad t \in (-\infty, 0] \end{aligned} \right\}, \quad (6.1)$$

Its linear base control system

$$\frac{d}{dt}D(t)x_t = L(t, x, x_t, u), \quad (6.2)$$

and its perturbation

$$\left. \begin{aligned} \frac{d}{dt}D(t)x_t &= L(t, x_t, u) + \int_{-\infty}^0 G(t, x_s)dt + f(t, x(t), x(t-h)) \\ x(t) &= \phi(t), \quad t \in (-\infty, 0] \end{aligned} \right\} \quad (6.3)$$

Here, the set of admissible controls considered are measureable  $u: [\sigma, t_1] \rightarrow C^m$ ,  $t_1 > \sigma$ , where  $C^m$  is the unit cube in  $E^m$ , such controls will simply be denoted by  $u \in C^m$ ,  $x(t) \in E^n$  is the state variable, with  $D(t)x_t = x(t) - A_0x(t-h)$ , and  $L(t, x_t, u) = A_1x(t) + A_2(t)x(t-h) + Bu(t)$ , so that the following assumptions holds:

$H_0$ :  $A_0, A_1, A_2$  are  $n \times n$  constant matrices

$H_1$ :  $B$  is an  $n \times m$  constant matrix

$H_2$ :  $G: (-\infty, 0] \times C \rightarrow E^n$  is a continuous matrix function which satisfies  $\|G(t, x_s)\| \leq M(t)\|x\|$  for all  $(t, \phi) \in (-\infty, 0] \times C$ , where  $\int_{-\infty}^0 M(t)ds = \gamma_0 < \infty$ .

$H_3$ :  $f: E \times C \times C \rightarrow E^n$ , is a continuous matrix function which satisfies the condition

$$\|f(t, x(t), x(t-h))\| \leq \gamma_1\|x\| + \gamma_2\|x(t-h)\|.$$

where,  $\gamma_0$  defined in  $H_2$ , and  $\gamma_1, \gamma_2$  as defined in  $H_3$  are positive constants less than  $\delta$ , where  $\delta$  is as defined in (6.10).

$H_4$ :  $h$  is a constant delay with  $\dot{h} = 0$

It is assumed that the continuous matrix functions  $G$  and  $f$  satisfy some smoothness conditions to ensure that a solution of (6.3) exists through each  $(\sigma, \phi)$ ,  $t \geq \sigma \geq 0$ , is unique, depends continuously upon  $(\sigma, \phi)$  and can be extended to the right as long as its trajectory remains in a bounded set  $[\sigma, \infty) \times C$ . These conditions are given in Cruz and Hale (1970).

The properties of the reachable sets given in Definition 5.2 of Section 5.2 will now be summarized as a proposition because of their importance in the next section as follows;

**Proposition 6.1**

- (i)  $0 \in R(\sigma, t)$  for each  $t \geq \sigma$ ,
- (ii)  $X(t, s)R(\sigma, s) \subseteq R(\sigma, t)$ , for  $\sigma \leq s \leq t$ ,
- (iii)  $R(\sigma, t)$  is compact in  $E^n$ , and convex
- (iv)  $t \rightarrow R(\sigma, t)$  is continuous in the Hausdorff metric (Chukwu 1988).
- (v)  $0 \in \mathcal{P}(\sigma, t)$  for each  $t \geq \sigma$ ,
- (vi)  $X(t, s)\mathcal{P}(\sigma, s) \subseteq \mathcal{P}(\sigma, t)$ , for  $\sigma \leq s \leq t$ ,
- (vii)  $\mathcal{P}(\sigma, t)$  is compact in  $C$
- (viii)  $t \rightarrow \mathcal{P}(\sigma, t)$  is continuous in the Hausdorff metric.

Also

- (ix)  $\mathcal{A}(\sigma, t)$  is compact and convex, where  $\mathcal{A}(\sigma, t)$  is the attainable set given in Definition 6.1

**Proof.** See Chukwu (1988).

**Definition 6.1: (Attainable set)**

Let  $S$  be a compact and convex subset of  $C$ . The attainable set  $\mathcal{A}(\sigma, t)$  of (6.2) at time  $t$  is defined by

$$\mathcal{A}(\sigma, t) = \{x(t, \phi, u): \phi \in S, u \in C^m\} \subset E^n$$

**Definition 6.2: (Main time optimal control problem)**

Let  $\mathcal{M}$  represent the metric space of all nonempty compact subsets of  $E^n$  with the metric  $\rho$  defined as follows: The distance of a point  $x$  from  $\mathcal{A}(\sigma, t_1)$  is given by

$$d_{\mathcal{A}}(x) = \inf\{|x - \alpha|: \alpha \in \mathcal{A}(\sigma, t)\},$$

$$N_{\mathcal{A}}(\varepsilon) = \{x \in E^n: d_{\mathcal{A}}(x) \leq \varepsilon\},$$

$$\rho(\varepsilon_1, \varepsilon_2) = \inf\{\varepsilon: \mathcal{A}(\sigma, t_1) \subseteq N_{\mathcal{A}}(\varepsilon) \text{ and } \mathcal{A}(\sigma, t_2) \subseteq N_{\mathcal{A}}(\varepsilon)\}.$$

The target set in system (6.2) is the continuous set function  $\mathcal{G}: [\tau, \infty) \rightarrow \mathcal{M}$ . The problem of reaching  $\mathcal{G}$  in minimum time will be called the main time optimal control problem, where  $\mathcal{G}$  is as defined in the Nomenclature.

**Definition 6.3: (Extremal control)**

The control  $u \in C^m$  is an extremal control on  $[\sigma, t_1]$  if for some  $\phi \in C$  and each  $t \in [\sigma, t_1]$  the solution  $x(\sigma, \phi, u)$  of (6.2) through  $\sigma$ ,  $\phi$  belongs to the boundary  $\partial\mathcal{A}(\sigma, t)$  of  $\mathcal{A}(\sigma, t)$ .

**Definition 6.4: (Convex)**

Let  $S \subseteq \mathcal{C}$  be a set.  $S$  is strictly convex if for every  $x_1, x_2 \in S, x_1 \neq x_2$ , the open line segment  $\{x_1 + (1 - \lambda)x_2: 0 < \lambda < 1\}$  is in the interior of  $S$ .

**Definition 6.5: (Controllable to target)**

Let  $z_t \in C([-h, 0], E^n)$  be a target point function which is time varying. The system (6.2) is controllable to the target if for each  $\phi \in \mathcal{C}$  there exists a  $t_1 \geq \sigma$  and admissible control  $u \in L_2([\sigma, t_1], C^m)$  such that the solution of equation (6.1) satisfies  $x_\sigma(\sigma, \phi, u) = \phi$ ,  $x_{t_1}(\sigma, \phi, u) = z_{t_1}$ .

**6.3. Normal and completely controllable systems**

This section derives necessary and sufficient conditions for an autonomous system, which is a special case of system (6.2), to be normal and completely controllable. It then gives conditions for the existence of time optimal control. The result of this section follows the pattern of Chukwu (2001). Some theorems and propositions from Chukwu (2001) that are necessary for the development of our results are also given. The main result of this Section is then formulated and given in the form of a theorem.

**Lemma 6.1**

The set  $\mathcal{A}(\sigma, t)$  is convex and compact. Also,  $R(\sigma, t)$  is convex and compact and satisfies the monotonicity relation

$$X(t, s)R(\sigma, s) \subseteq R(\sigma, t), \quad \sigma \leq s \leq t, \quad (6.4)$$

also  $0 \in R(\sigma, t)$  for each  $t \geq \sigma$ .



**Proof.** The convexity of  $\mathcal{A}(\sigma, t)$  follows trivially from that of  $S$  and  $C^m$ . That of  $R(\sigma, t)$  follows from the convexity of  $C^m$ . The compactness of  $S$  and continuity of  $x(t, \sigma, \cdot, 0)$  implies that  $x(t, \sigma, S, 0)$  is bounded. Also, since  $Z(t, s)$  is integrable and  $u \in C^m$ ,  $\mathcal{A}(\sigma, t)$  is bounded in  $E^n$  and therefore  $R(\sigma, t)$  is also bounded. It follows from the weak compactness argument and the compactness of  $S$  that  $\mathcal{A}(\sigma, t)$  is closed in  $E^n$ ; The same weak compactness argument implies that  $R(\sigma, t)$  is closed. It is now necessary to prove relation (6.4) to complete the proof. If  $r \in R(\sigma, s)$ ; then for some  $u \in C^m$ ,

$r = \int_{\sigma}^s Z(s, \tau)u(\tau)d\tau$ . Define the control

$$u^*(\tau) = \begin{cases} u(\tau), & \sigma \leq \tau \leq s \\ 0, & s < \tau \leq t. \end{cases}$$

Then  $u^*(\tau) \in C^m$ . Now, consider the point

$$\begin{aligned} p = X(t, s)r &= \int_{\sigma}^s X(t, s)Z(s, \tau)u(\tau)d\tau \\ &= \int_{\sigma}^s X(t, s)Z(s, \tau)u(\tau)d\tau + \int_s^t X(t, s)Z(s, \tau)u(0)d\tau \\ &= \int_{\sigma}^s X(t, s)Z(s, \tau)u(\tau)d\tau + \int_s^t X(t, s)Z(s, \tau)0d\tau \\ &= \int_{\sigma}^t Z(s, \tau)u^*(\tau)d\tau \in R(\sigma, t) \end{aligned}$$

Hence, the relation (6.4) holds and the proof is complete.  $\square$

**Lemma 6.2**

Let  $C^{0m} = \{u \in C^m: |u_k| = 1, k = 1, 2, \dots, m, \}$  be the bang-bang control on  $[\sigma, t]$ . If

$$\mathcal{A}^0(\sigma, t) = \{x(t, \phi, u^0): \phi \in S, u^0 \in C^{0m}\}, \text{ and } R^0(\sigma, t) = \left\{ \int_{\sigma}^t Z(t, s) u^0(s) ds: u^0 \in C^{0m} \right\},$$

then  $R^0(\sigma, t) = R(\sigma, t)$  and  $\mathcal{A}^0(\sigma, t) = \mathcal{A}(\sigma, t)$  for each  $t \geq \sigma$ .

**Proof.** Observe that the matrix  $Z(t, s) = X(t, s)B(s) \in L_2([\sigma, t], E^{n \times m})$ , because  $X(t, s) \in C([\sigma, t], E^{n \times n})$  and  $B(\cdot) \in L_2([\sigma, t], E^{n \times m})$ . It follows from Chukwu (2001) that

$$R^0(\sigma, t) = R(\sigma, t) \text{ for each } t \geq \sigma \text{ and therefore } \mathcal{A}^0(\sigma, t) = \mathcal{A}(\sigma, t). \quad \square$$

**Lemma 6.3**

The attainable set  $\mathcal{A}(\sigma, t): [\sigma, \infty) \rightarrow r^n$  is continuous

**Proof.** The idea is to show that there exists a constant  $m > 0$  such that  $\|x_t(\sigma, \phi, u)\| \leq m$  for  $t \in [\sigma, t_1]$ ,  $\phi \in S$ ,  $u \in C^m$ , the rest of the proof follows the method of Chukwu (2001) by showing that  $\mathcal{A}(\sigma, t)$  is an equicontinuous subset of  $C([\sigma - h, t_1], E^n)$ . As a consequence  $t \rightarrow x(t, \sigma, \phi, u)$  is uniformly continuous in  $\phi$  and  $u$ , the continuity of  $t \rightarrow \mathcal{A}(\sigma, t)$  follows similarly from this argument.  $\square$

**Proposition 6.2**

Assume for the main time optimal control problem that the pair  $\phi \in S$ ,  $u \in C^m$  exists such that  $x(t, \phi, u) \in \mathcal{G}$  for some time  $t$ , then there is an optimal pair  $\phi \in S$ ,  $u \in C^m$ .

**Proof.** Assume that  $\mathcal{A}(t) \cap \mathcal{G}(t) \neq \emptyset$  for some  $t \geq \sigma$ ,  $\phi \in S$ . Now define the a minimal time function,  $t^*(S) = \inf\{t \geq \sigma: \mathcal{A}(t) \cap \mathcal{G}(t) \neq \emptyset\}$ , where  $\mathcal{A}(t) = \mathcal{A}(t, S)$ , by using the compactness and continuity of  $\mathcal{A}(t)$  and  $\mathcal{G}(t)$ , it follows from Chukwu (2001) that  $\mathcal{A}(t^*) \cap \mathcal{G}(t^*) \neq \emptyset$ .  $\square$

**Proposition 6.3**

Assume that system (6.2) is controllable to the target. Then there exists an optimal control that stabilises (6.2).

**Proof.** From the variation of constants formula (5.5) in Section 5.2 of Chapter 5, system (6.2) controllability to the target is equivalent to  $x_{t_1}(\sigma, \phi, u) = z_{t_1}$ , for some  $t_1$ , that is

$w_{t_1} = z_{t_1} - K(t, \sigma)\phi - \int_{\sigma}^t K(t, \sigma)X_0B(s)u(s)ds$ . This is equivalent to  $w_{t_1} \in R(\sigma, t_1)$ . Let  $t^* = \inf\{t: w_t \in R(\sigma, t)\}$ . Now  $\sigma \leq t^* \leq t_1$ . There is a non-increasing sequence of times  $t_n$  converging to  $t^*$ , and a sequence of controls  $u^n \in L_2([\sigma, t_1], C^m)$  with  $w_{t_n} = y(t_n, u^n) = \int_{\sigma}^{t_n} X_{t_n}(\cdot, s)B(s)u^n(s)ds \in R(\sigma, t_n)$ . Also,

$$\|w_{t^*} - y(t^*, u^n)\| \leq \|w_{t^*} - w_{t_n}\| + \|w_{t_n} - y(t^*, u^n)\| \leq \|w_{t^*} - w_{t_n}\| + I$$

where  $I$  as defined in Section 5.2 of Chapter 5 will be given by

$$\begin{aligned} I &\leq \left\| \int_{\sigma}^{t_n} X_{t_n}(\cdot, s)B(s)u^n(s)ds - \int_{\sigma}^{t^*} X_{t_n}(\cdot, s)B(s)u^n(s)ds \right\| \\ &\quad + \int_{\sigma}^{t^*} \|X_{t_n}(\cdot, s)B(s)u^n(s) - X_{t^*}(\cdot, s)B(s)u^n(s)\|ds \\ &\leq \int_{t^*}^{t_n} \|X_{t_n}(\cdot, s)B(s)u^n(s)\|ds + \int_{\sigma}^{t^*} \|[X_{t_n}(\cdot, s) - X_{t^*}(\cdot, s)]B(s)u^n(s)\|ds. \end{aligned}$$

Because  $X_{t_n}(\cdot, s)B(s)u^n(s)$  is integrable and  $[t_n, t^*] < \infty$ , the first term on the right hand side of the inequality tends to zero as  $t_n \rightarrow t^*$ . Now  $\|X_{t_n}(\cdot, s)\| \leq \beta < \infty$  for all  $t_n, s$  for some  $\beta$  from Chukwu (2001) and references therein, also  $X_{t_n}(\cdot, s) \rightarrow X_{t^*}(\cdot, s)$  in the uniform topology of  $C$ . By the bounded convergence theorem, the second summand on the left hand side tends to zero as  $n \rightarrow \infty$ . Again, from continuity of solution in time and the continuity of the target,  $\|w_{t^*} - w_{t_n}\| \rightarrow 0$  as  $t_n \rightarrow t^*$  therefore,  $w_{t^*} = \lim_{n \rightarrow \infty} y(t^*, u^n)$ . Because  $R(\sigma, t^*)$  is closed and  $y(t^*, u^n) \in R(\sigma, t^*)$ ,  $w(t^*) = y(t^*, u^n)$  for some  $u^* \in L_2([\sigma, t_1], C^m)$  and by definition  $t^*, u^*$  is optimal.  $\square$

Observe from Proposition 6.3 that at the time of hitting a target  $w_t \in C$  in system (6.2)

$$w_t - K(t, \sigma)\phi \equiv z(t) = \int_{\sigma}^t K(t, \sigma)X_0B(s)u(s)ds.$$

That is reaching  $w_t$  in time  $t$  corresponds to  $w_t - K(t, \sigma)\phi \equiv z(t) \in R(\sigma, t)$

**Proposition 6.4**

If  $u^*$  is the optimal control that is used to hit  $w_t$  in minimum time  $t^*$  then  $z(t^*) \in \partial R(\sigma, t^*)$ , that is  $z(t^*)$  is on the boundary ( $\partial$ ) of the constrained reachable set.

**Proof.** See Chukwu (1988).

**Theorem 6.1**

Assume that the solution of (6.1) is pointwise complete, and the conditions on Proposition 6.1 hold. Let  $u^*$  be optimal on  $[\sigma, t^*]$ . Then  $u^*$  is an extremal control on  $[\sigma, t^*]$  and there is a nonzero  $n$  dimensional row vector  $v^*$  depending on  $t^*$  and  $\phi^* \in S$  such that

$$\{u^*(t)\}_j = \text{sgn}\{v^*X(t^*, t)B(t)\}_j, \quad \sigma \leq t \leq t^*, \quad (6.5)$$

for each  $1 \leq j \leq m$  for which  $\{v^*X(t^*, t)B(t)\}_j \neq 0$ .

**Proof.** Suppose  $u^*$  is extremal on  $[\sigma, t^*]$ , note then that  $x(t^*, \sigma, \phi^*, u^*) \in \partial \mathcal{A}(\sigma, t^*)$ ,  $\sigma \leq t \leq t^*$ . Since,  $\mathcal{A}(\sigma, t^*)$  is convex and closed, there is a supporting hyperplane  $\pi$  through  $x^* = x(t^*, \sigma, \phi^*, u^*)$  such that  $\mathcal{A}(\sigma, t^*)$  lies on one side of  $\pi$ . Let  $\eta$  be a unit normal to  $\pi$  which is directed away from  $\mathcal{A}(\sigma, t^*)$ . Clearly, for each  $u \in C^m$ ,  $x(t^*) = x(t^*, \sigma, \phi^*, u^*) \in \mathcal{A}(\sigma, t^*)$  such that

$$\langle \eta, x^* \rangle = \sup\{\langle \eta, x \rangle | x \in \mathcal{A}(\sigma, t^*)\} \quad (6.6)$$

It follows from the variation of constants formula (5.5) in Section 5.2 that, after the cancelling out of the  $x_0$  terms it is equivalent to the following

$$\langle \eta, \int_{\sigma}^{t^*} Z(t^*, s)u(s)ds \rangle \leq \langle \eta, \int_{\sigma}^{t^*} Z(t^*, s)u^*(s)ds \rangle = \int_{\sigma}^{t^*} \eta^T Z(t^*, s)u^*(s)ds,$$

for all  $u \in C^m$ . Define  $\gamma(s) = \eta^T Z(t^*, s)$  so that,

$\int_{\sigma}^{t^*} \gamma(s)u^*(s)ds \leq \sum_{i=1}^m \int_{\sigma}^{t^*} \gamma_i(s) \text{sgn } \gamma_i(s)ds$ . Hence, it is clear that on any interval of positive length where  $\gamma(s) \neq 0$ , it must be that  $u_i^*(s) = \text{sgn } \gamma_i(s)$  for  $i = 1, \dots, m$ ,  $0 \leq t \leq t^*$  and the theorem is proved.  $\square$

### **Corollary 6.1**

If the system (6.2) is normal on  $[\sigma, t^*]$  then  $u^*(t)$ , the optimal control is uniquely determined by (6.5) and is bang-bang.

**Proof.** See Chukwu (2001).  $\square$

#### **6.3.1. Time optimal control for neutral systems with an infinite delay**

The time optimal control for the neutral system with an infinite delay, which is the main result of this section, will now be formulated.

Consider an autonomous system of the form (6.2) defined by

$$\left. \begin{aligned} \frac{d}{dt}(x(t) - A_0 x(t-h)) &= A_1 x(t) + A_2 x(t-h) + Bu(t), \quad t \geq 0 \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \right\} \quad (6.7)$$

where  $A_0$ ,  $A_1$  and  $A_2$  are  $n \times n$  matrices and  $B$  is an  $n \times m$ ,  $u \in C^m$  and  $\phi \in C([-h, 0], E^n)$ .

For each admissible control  $u \in L_1([-h, 0], C^m)$  on the above equation, there exists a unique solution to (6.7) on  $[-h, \infty)$  through  $\phi$  (Chukwu 2001). Furthermore, by Hale (1977) if  $A_0 \neq 0$ , this solution exists on  $E$  and is unique. The fundamental matrix of

$$\frac{d}{dt}(x(t) - A_0x(t-h)) = A_1x(t) + A_2x(t-h), \quad (6.8)$$

is a solution of the equation (6.8) with initial data

$$X_0(t) = \begin{cases} 0, & t < 0 \\ I, & t = 0, \end{cases} \quad (I \text{ is the identity})$$

for which  $X(t) - A_0X(t-h)$  is continuous and satisfies (6.8) for  $t \geq 0$  except at  $kh$ ,  $k = 0, 1, 2, \dots$ . Indeed  $X(t)$  has a continuous first derivative on each interval  $(kh, (k+1)h)$ ,  $k = 0, 1, 2, \dots$ , the right and left hand limits of  $X(t)$  exists at each  $kh$ ,  $k = 0, 1, 2, \dots$ , so that  $X(t)$  is of bounded variation on each compact interval and satisfies

$$X(t) - A_0X(t-h) = A_1X(t) + A_2X(t-h),$$

$$t \neq kh, \quad k = 0, 1, 2, \dots.$$

Again, if  $X(t)$  is the fundamental matrix solution of (6.8), then the solution  $x(\phi, u)$  of (6.7) is given by

$$x(t, \phi, u) = x(t, \phi, 0) + \int_0^t X(t-s)Bu(s)ds,$$

where,

$$\begin{aligned} x(t, \phi, 0) = & X(t)(\phi(0) - A_0\phi(-h)) + A_2 \int_{-h}^0 X(t-s-h)\phi(s)ds \\ & - A_0 \int_{-h}^0 dX(t-s-h)\phi(s). \end{aligned}$$

A computational criterion for complete controllability of (6.1) will now be developed in the next theorem as part of the contributions of the thesis in this session and will be used in the

proof of Theorem 6.2 which the main result of this session. To develop this criterion, introduce an algebraic notation by following the method used in Chukwu (2001) and references therein for neutral systems as;

$$Q_k(s) = A_1 Q_{k-1}(s) + A_2 Q_{k-1}(s-h) + A_0 Q_k(s-h),$$

$$k = 0, 1, 2, \dots; \quad s = 0, h, 2h, \dots$$

$$Q_0(0) = I, \quad Q_0(s) \equiv 0, \quad \text{if } s < 0.$$

### ***Theorem 6.2***

A necessary and sufficient condition for the system (6.2) to be normal on the interval  $[0, t_1]$  is that for each  $r = 1, 2, \dots, m$ , the matrix

$$Q_k(t_1) = \{Q_k(s)b_r, k = 0, 1, \dots, n-1, s \in [0, t_1]\},$$

where  $b_r$  is the  $r^{th}$  component of  $B$ .

Observe that Theorem 6.2 is the algebraic condition for complete controllability proof given in Chukwu (2001) for properness on  $[0, t_1]$ , and by Proposition 5.3 of Section 5.3 in Chapter 5 this is equivalent to the system being completely controllable on  $[0, t_1]$ . It is also completely controllable for  $t_1 = \tau$  by Corollary 5.1 of Section 5.3.1 in Chapter 5. The main result will now be formulated using the conditions for normal and complete controllable systems.

### ***Theorem 6.3***

Consider (6.1), and assume the following

- (i)  $A_0, A_1, A_2$  are  $n \times n$  constant matrices,  $B$  is  $n \times 1$  constant real matrix
- (ii) for  $\tau > nh$ ,  $\text{rank } P_n[A_0, B] = n$ ;

(iii)  $K(\lambda)\xi(\exp(-\lambda h)) \neq 0$ , for every complex  $\lambda$ ,

(iv)  $\sup\{\operatorname{Re}(\lambda), \det \Delta(\lambda) = 0\} < 0$ , with

$$\Delta(\lambda) = \lambda(I - A_0 \exp(-\lambda h)) - A_1 - A_2 \exp(-\lambda h) + \int_{-\infty}^0 \exp(\lambda \theta) A(\theta) d\theta$$

(v) and  $D\phi = \phi(0) - A_0\phi(-h)$  is uniformly stable.

Then there is a time optimal control which drives any  $\phi \in C([-h, 0], E^n)$  in minimum time  $t^*$  and is given by  $u_i^*(t) = \operatorname{sgn}(c^T X(t^* - t)B)_i$ ,  $i = 1, 2, \dots, m$ ,  $0 \leq t \leq t^*$ .

**Proof.** By (i) – (iii) system (6.1) is completely controllable by Corollary 5.1 of Section 5.3.1 and therefore normal. By condition (iv) and (v), system (6.1) with  $u = 0$  satisfies Lemma 4.2 and 4.4 of Section 4.2 in Chapter 4, that is  $x_t(\cdot, \phi, 0) \rightarrow 0$  as  $t \rightarrow \infty$ , and therefore null controllable by Theorem 5.2 of Section 5.3.2 in Chapter 5. It follows from the null controllability of (6.1) that an optimal control exists by Proposition 6.2 which is extremal by Theorem 6.1 and is determined uniquely by (6.5) because of Corollary 6.1.  $\square$

#### 6.4. Robust guaranteed cost control for the neutral system with infinite delay

This section forms part of the contributions of the thesis in this chapter and is concerned with finding guaranteed optimal control for the neutral system with infinite delay through the definition of a quadratic cost function. The main results will be given in terms of theorems and proofs which are based on the Razumikhin approach and the Lyapunov matrix equation.

Consider the neutral functional differential control system with infinite delays and its perturbation given in (6.3). Now, let the initial time be zero and  $x(t)$  be the solution of (6.3) through  $(0, \phi)$ . Since  $x(t)$  is extendable for  $t \geq 0$  one can use the model transformation



technique (see Gu et al. 2003) to write,  $x(t) - x(t - h) = \int_{-h}^0 \dot{x}(t + \theta) d\theta$  for  $t \geq h$ . So that,

(6.3) using this expression can be written in the form

$$\begin{aligned} \dot{x}(t) - A_0 \dot{x}(t - h) \\ = (A_1 + A_2)x(t) - A_2 \int_{-h}^0 \dot{x}(t + \theta) d\theta + Bu(t) + f(t, x(t), x(t - x)) \\ + \int_{-\infty}^0 G(t, x_s) ds, \end{aligned} \quad (6.9)$$

for  $x(t) = \phi(t)$ ,  $t \in [-2h, 0]$ ,  $h > 0$ .

Let

$$\sigma = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \quad (6.10)$$

$$\delta = \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}. \quad (6.11)$$

Let  $P$ , and  $Q$  be symmetric positive definite matrices involved in the following Lyapunov equation

$$(A_1 + A_2)^T P + P(A_1 + A_2) = -Q, \quad (6.12)$$

where  $A_1 + A_2$  is a Hurwitz stable matrix.

Associated with (6.3) is the quadratic cost function given by

$$\mathcal{J} = \int_0^{\infty} (x^T(t)Q_1x(t) + u^T(t)Q_2u(t))dt, \quad (6.13)$$

where  $Q_1 \in E^{n \times n}$  and  $Q_2 \in E^{m \times m}$  are positive definite matrices.

Now, define a state feedback controller  $u(t)$  for (6.3) as

$$u(t) = -B^T P x(t), \quad (6.14)$$

where  $P \in E^{n \times n}$  is a symmetric positive definite matrix to be designated.

The closed-loop design for system (6.3), using (6.13), (6.14) and the transformed equation (6.9) is defined by

$$\begin{aligned} \dot{x}(t) - A_0 \dot{x}(t-h) \\ = (A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^0 \dot{x}(t+\theta) d\theta + f(t, x(t), x(t-h)) \\ + \int_{-\infty}^0 G(t, x_s) ds. \end{aligned} \quad (6.15)$$

The task now is to ensure that system (6.15) is asymptotically stable and the closed loop value of (6.13) satisfies  $J \leq J^*$ , where  $J^*$  is the guaranteed cost for the output feedback control.

### **Definition 6.6**

For the system (6.3) and cost function (6.13), if there exists a control law  $u^*(t)$  and a positive  $J^*$  such that for all admissible uncertainty, the closed-loop (6.15) is asymptotically stable and the closed loop value of the cost function (6.13) is less than or equal to  $J^*$ , then  $J^*$  is a guaranteed cost function and  $u^*$  is the guaranteed cost control law of the system (6.3) and the cost function (6.13) (Park 2003).

#### **6.4.1. Designing a guaranteed cost controller**

The main result for this section will be derived by developing appropriate conditions and utilizing the Lyapunov matrix equation, and the Razumikhin approach for stabilisation of the

closed-loop system (6.15). The method of selecting a guaranteed cost controller that would ensure the minimization of  $J^*$  for the neutral system (6.3) will also be given.

**Theorem 6.4**

Let the difference system  $x(t) - A_0x(t - h) = 0$  be uniformly stable. Given  $Q_1 > 0$  and  $Q_2 > 0$ ,  $u(t) = -B^T Px(t)$  is a robust guaranteed cost controller for (6.3), if there exist positive-definite matrices  $P$  and  $Q$  satisfying (6.12) such that  $A_1 + A_2$  is Hurwitz stable matrix satisfying

$$\frac{\sigma - \|BB^T P\| - \gamma_1 - \gamma_0}{\delta \left[ \frac{\|(A_1 + BB^T P)^T A_0\| + \|A_0^T\|(\gamma_1 + \gamma_0) + \|A_2\| \times (\|A_1 + BB^T P\| + \|A_2\| + \gamma_1 + \gamma_2 + \gamma_0) + \delta \|A_0^T\|(\|A_2\| + \gamma_2) + \gamma_2}{\delta} \right]} > 0. \quad (6.16)$$

Then, (6.3) with (6.13) is uniformly asymptotically stable i.e. the system can tolerate perturbation for any constant time delay  $0 \leq h < h^*$ , and the guaranteed cost is given by  $J^* = x^T(0)Px(0)$ , where  $P = X^{-1}$ .

**Proof:** Consider (6.12) given by  $(A_1 + A_2)^T P + P(A_1 + A_2) = -Q$ , and let the following positive definite function be the Lyapunov function

$$V(x(t)) = [x(t) - A_0x(t - h)]^T P [x(t) - A_0x(t - h)]. \quad (6.17)$$

Thus, taking the derivative of  $V$  in (6.17) along the solutions of (6.15) gives the following

$$\begin{aligned}
\dot{V}(x(t)) &= \left[ (A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^0 \dot{x}(t + \theta) d\theta + f(t, x(t), x(t - h)) \right. \\
&\quad \left. + \int_{-\infty}^0 G(t, x_s) ds \right]^T P[x(t) - A_0 x(t - h)] \\
&\quad + [x(t) - A_0 x(t - h)]^T P \left[ (A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^0 \dot{x}(t + \theta) d\theta \right. \\
&\quad \left. + f(t, x(t), x(t - h)) + \int_{-\infty}^0 G(t, x_s) ds \right] \\
&= 2[x(t) - A_0 x(t - h)]^T P \left[ (A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^0 \dot{x}(t + \theta) d\theta \right. \\
&\quad \left. + f(t, x(t), x(t - h)) + \int_{-\infty}^0 G(t, x_s) ds \right] \\
&= 2x^T(t)P \left[ (A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^0 \dot{x}(t + \theta) d\theta + f(t, x(t), x(t - h)) \right. \\
&\quad \left. + \int_{-\infty}^0 G(t, x_s) ds \right] \\
&\quad - 2x^T(t - h)A_0^T P \left[ (A_1 - BB^T P)x(t) + A_2 x(t - h) + f(t, x(t), x(t - h)) \right. \\
&\quad \left. + \int_{-\infty}^0 G(t, x_s) ds \right]
\end{aligned}$$

$$\begin{aligned}
&= x^T(t)[(A_1 + A_2)^T P + P(A_1 + A_2) - 2PBB^T P]x(t) - 2x^T(t)PA_2 \int_{-h}^0 \dot{x}(t + \theta)d\theta \\
&\quad + 2x^T(t)Pf(t, x(t), x(t - h)) + 2x^T(t)P \int_{-\infty}^0 G(t, x_s)ds \\
&\quad - 2x^T(t)(A_1 - BB^T P)^T PA_0 x(t - h) - 2x^T(t - h)A_0^T PA_2 x(t - h) \\
&\quad - 2x^T(t - h)A_0^T Pf(t, x(t), x(t - h)) \\
&\quad - 2x^T(t - h)A_0^T P \int_{-\infty}^0 G(t, x_s)ds. \tag{6.18}
\end{aligned}$$

The following terms in (6.18) are further simplified such that:

$$2x^T(t)Pf(t, x(t), x(t - h)) \leq 2x^T(t)P\gamma_1 x(t) + 2x^T(t)P\gamma_2 x(t - h).$$

$$\begin{aligned}
&2x^T(t - h)A_0^T Pf(t, x(t), x(t - h)) \\
&\leq 2x^T(t - h)A_0^T P\gamma_1 x(t) + 2x^T(t - h)A_0^T P\gamma_2 x(t - h).
\end{aligned}$$

$$2x^T(t)P \int_{-\infty}^0 G(t, x_s)ds \leq 2x^T(t)P\gamma_0 x(t). \tag{6.19}$$

Observe that because of the substituting of (6.14) in (6.9), the model transformation technique used and the assumption on Lemma 4.1 of Section 4.2 in Chapter 4, the expression

$2x^T(t)PA_2 \int_{-h}^0 \dot{x}(t + \theta)d\theta$  can be estimated by

$$\begin{aligned}
&2x^T(t)PA_2 \int_{-h}^0 \dot{x}(t + \theta)d\theta \\
&\leq 2x^T(t)PA_2 \int_{-h}^0 [(A_1 - BB^T P)x(t + \theta) + A_2 x(t - h + \theta) \\
&\quad + (\gamma_1 x(t + \theta) + \gamma_2 x(t - h + \theta)) + \gamma_0 x(t + \theta)]d\theta \tag{6.20}
\end{aligned}$$

The overall derivative of  $V$  along the solution of (6.15) can now be expressed as follows

$$\begin{aligned}
\dot{V}(x(t)) &\leq x^T(t)[(A_1 + A_2)^T P + P(A_1 + A_2) - 2PBB^T P]x(t) \\
&\quad - 2x^T(t)PA_2 \int_{-h}^0 [(A_1 - BB^T P)x(t + \theta) + A_2 x(t - h + \theta) \\
&\quad + (\gamma_1 x(t + \theta) + \gamma_2 x(t - h + \theta)) + \gamma_0 x(t + \theta)]d\theta \\
&\quad + [2x^T(t)P\gamma_1 x(t) + 2x^T(t)P\gamma_2 x(t - h)] + 2x^T(t)P\gamma_0 x(t) \\
&\quad - 2x^T(t)(A_1 - BB^T P)^T PA_0 x(t - h) - 2x^T(t - h)A_0^T PA_2 x(t - h) \\
&\quad - [2x^T(t - h)A_0^T P\gamma_1 x(t) + 2x^T(t - h)A_0^T P\gamma_2 x(t - h)] \\
&\quad - 2x^T(t - h)A_0^T P\gamma_0 x(t) \\
&\leq x^T(t)[(A_1 + A_2)^T P + P(A_1 + A_2)]x(t) - 2x^T(t)PBB^T Px(t) \\
&\quad - 2x^T(t)PA_2 \int_{-h}^0 [(A_1 - BB^T P)x(t + \theta) + A_2 x(t - h + \theta) \\
&\quad + (\gamma_1 x(t + \theta) + \gamma_2 x(t - h + \theta)) + \gamma_0 x(t + \theta)]d\theta \\
&\quad + [2x^T(t)P\gamma_1 x(t) + 2x^T(t)P\gamma_2 x(t - h)] + 2x^T(t)P\gamma_0 x(t) \\
&\quad - 2x^T(t)(A_1 - BB^T P)^T PA_0 x(t - h) - 2x^T(t - h)A_0^T PA_2 x(t - h) \\
&\quad - [2x^T(t - h)A_0^T P\gamma_1 x(t) + 2x^T(t - h)A_0^T P\gamma_2 x(t - h)] \\
&\quad - 2x^T(t - h)A_0^T P\gamma_0 x(t) - x(t)(Q_1 + PBQ_2 B^T P)x(t). \tag{6.21}
\end{aligned}$$

Now, using the Razumikhin type theorem, assume for any nonnegative number  $q > 1$ , the following inequality holds:

$$V(x(\xi)) < q^2 V(x(t)), \quad t - 2h \leq \xi \leq t. \tag{6.22}$$

Hence,

$$\|x(\xi)\| < q\delta \|x(t)\|. \tag{6.23}$$

Substituting equation (6.23) into (6.21) gives the following inequality

$$\dot{V}(x(t)) \leq -\hat{w}\|x(t)\|^2, \quad (6.24)$$

where  $\hat{w} = w + (Q_1 + PBQ_2B^TP)$ , and

$$\begin{aligned} w = & \lambda_{\min}(Q) - 2\{\|BB^TP\| + \gamma_1 + \gamma_0 \\ & + q\delta h[\|A_2\|(\|A_1 + BB^TP\| + \|A_2\| + \gamma_1 + \gamma_2 + \gamma_0) + \|(A_1 + BB^TP)^TA_0\| \\ & + \|A_0^T\|(\gamma_1 + \gamma_0) + q\delta\|A_0^T\|(\|A_2\| + \gamma_2) + \gamma_2]\lambda_{\max}(P)\}. \end{aligned}$$

Here, the conditions (6.22), (6.23) and (6.24) by the Razumikhin theory implies that

$$\dot{V}(x(t)) \leq -x^T(t)(Q_1 + PBQ_2B^TP)x(t) \leq 0, \quad (6.25)$$

and since  $Q_1 > 0$  and  $Q_2 > 0$ , if condition (6.16) of Theorem 6.4 is satisfied, then a sufficiently small  $q > 1$  exists such that  $w > 0$ , which implies  $\hat{w} > 0$ . Thus, by the Razumikhin Theorem (see Hale and Verduyn Lunel 1993), (6.15) is uniformly asymptotically stable since  $\dot{V}(x(t)) < 0$ ,  $\hat{w} > 0$  based on the above proof for Theorem 6.4. Furthermore, integrating (6.25) from 0 to  $\infty$ , gives

$$\begin{aligned} \int_0^\infty \dot{V}(x(t)) dt & \leq V(x(0)) - V(x(\infty)) \\ & \leq \int_0^\infty x^T(t)(Q_1 + PBQ_2B^TP)x(t) dt + \int_0^\infty x^T(t)wx(t) dt \\ & \leq \int_0^\infty x^T(t)(Q_1 + PBQ_2B^TP)x(t) dt. \end{aligned}$$

Considering that (6.15) is asymptotically stable leads to  $x(\infty) \rightarrow 0$ , and hence,

$$\int_0^\infty x^T(t)(Q_1 + PBQ_2B^TP)x(t) dt \leq V(x(0)) \leq x^T(0)Px(0) \stackrel{\text{def}}{=} J^*. \quad (6.26)$$

**Remark 6.1**

The selection in (6.12) guarantees  $Q > 0$  when  $P = I$ , and maximizes  $\delta$  when  $P = I$ . The maximum bound for the time delay becomes

$$\begin{aligned} h^* = & \sigma + \|BB^T P\| + \gamma_1 + \gamma_0 \\ & + \delta \left( \|(A_1 + BB^T P)^T A_0\| + \|A_0^T\|(\gamma_1 + \gamma_0) + \delta \|A_0^T\|(\|A_2\| + \gamma_2) + \gamma_2 \right. \\ & \left. + \|A_2\|(\|A_2\|\|A_1 + BB^T P\| + \gamma_1 + \gamma_2 + \gamma_0) \right). \end{aligned} \quad (6.27)$$

for  $0 \leq h < h^*$ .

**6.4.2. Designing optimal robust controller that minimizes the guaranteed cost****Theorem 6.5**

Consider system (6.15) and (6.13). Suppose the following optimization problem

$$\min_{X>0, h^*>0} h^*. \quad (6.28)$$

Subject to;

- (i) inequality (6.16), such that  $w > 0$
- (ii)  $x^T(0)X^{-1}x(0) < h^*$ ,

has a solution with  $X > 0$ ,  $h^* > 0$ , then (6.14) is an optimal robust guaranteed cost control law which ensures the minimization of (6.26) for the system (6.15).

**Proof.** The proof of (i) in (6.28) is clear by Theorem 6.4. Also, by Lemma 4.6 of Chapter 4, the inequality (6.28) (ii) can be expressed equivalent as  $\begin{pmatrix} -h^* & x^T(0) \\ x(0) & -X \end{pmatrix} < 0$ . It therefore follows from (6.26) that  $J^* \leq h^*$ . Thus, the minimization for (6.26) follows from the minimization of  $J^* \leq h^*$  and the proof is complete.  $\square$

**6.5. Examples on optimal control for neutral systems with infinite delays**

Here, numerical examples will be given as an illustration to the methods proposed.



### 6.5.1. Example on time optimal control

Consider the neutral control system with infinite delays

$$\begin{aligned} (d/dt)(x - A_0x(t-h)) &= A_1x(t) + A_2x(t-h) + C_0 \int_{-\infty}^0 \exp(v\theta)x(t+\theta)d\theta, \\ &+ Bu(t) \end{aligned} \quad (6.29)$$

and its linear control base system

$$(d/dt)(x - A_0x(t-h)) = A_1x(t) + A_2x(t-h) + Bu(t). \quad (6.30)$$

where,

$$A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The uniform stability of the system  $D(t)x_t = x(t) - A_0x(t-h)$  for  $h > 0$  has been computed in Example 4.5.1 of Chapter 4. The uniform asymptotic stability of (6.29) with  $u = 0$  has also been calculated in Example 5.5.1 of Chapter 5.

Next is to deduce the optimal control law by determining the fundamental matrix,  $X(t)$ , as follows.

First evaluate the eigenvalues of  $A_1$  as  $-2.6180$  and  $-0.3820$  and obtain the associated matrix of eigenvectors as

$$X(t) = \begin{pmatrix} 1.6180 \exp(-2.6180 t) & 0.6180 \exp(-0.3820 t) \\ -2.6180 \exp(-2.6180 t) & 0.3820 \exp(-0.3820 t) \end{pmatrix}.$$

Verify also that the matrix inverse is given by

$$X^{-1}(t) = \begin{pmatrix} 0.1708 \exp(2.6180 t) & -0.2764 \exp(2.6180 t) \\ 1.1708 \exp(0.3820 t) & 0.7236 \exp(0.3820 t) \end{pmatrix},$$

and at  $t = 0$

$$X^{-1}(0) = \begin{pmatrix} 0.1708 & -0.2764 \\ 1.1708 & 0.7236 \end{pmatrix}.$$

Therefore, the principal fundamental matrix on  $[0, h]$  is given by

$$\begin{aligned} X(t) &= \exp(A_1 t) \\ &= \begin{pmatrix} 0.2764e^{-2.6180t} + 0.7236e^{-0.3820t} & 0.4472e^{-0.3820t} - 0.4472e^{-2.6180t} \\ 0.4473e^{-0.3820t} - 0.4472e^{-2.6180t} & 0.7236e^{-2.6180t} + 0.2764e^{-0.3820t} \end{pmatrix}. \end{aligned}$$

Now, selecting  $c^T X(t - s)$ ,  $t - s \in [0, h]$  gives

$$\begin{aligned} (c_1, c_2)X(t - s) &= \left( c_1(0.2764e^{-2.6180(t-s)} + 0.7236e^{-0.3820(t-s)}) \right. \\ &\quad + c_2(0.4473e^{-0.3820(t-s)} - 0.4472e^{-2.6180(t-s)}), \quad c_1(0.4472e^{-0.3820(t-s)} \\ &\quad \left. - 0.4472e^{-2.6180(t-s)}) + c_2(0.7236e^{-2.6180(t-s)} + 0.2764e^{-0.3820(t-s)}) \right), \end{aligned}$$

and

$$\begin{aligned} (c_1, c_2)X(t - s)B &= c_1(0.4472e^{-0.3820(t-s)} - 0.4472e^{-2.6180(t-s)}) \\ &\quad + c_2(0.7236e^{-2.6180(t-s)} + 0.2764e^{-0.3820(t-s)}), \quad t - s \in [0, h]. \end{aligned}$$

Therefore  $(u_1, u_2) = \text{sgn}((c_1, c_2)X(t - s)B)$ , where

$$\begin{aligned} u_2 &= \text{sgn} \left( c_1(0.4472e^{-0.3820(t-s)} - 0.4472e^{-2.6180(t-s)}) \right. \\ &\quad \left. + c_2(0.7236e^{-2.6180(t-s)} + 0.2764e^{-0.3820(t-s)}) \right), \end{aligned}$$

is the time optimal control which drives the system to target.

### 6.5.2. Example on robust guaranteed cost control problem

Consider the perturbed neutral control system given by

$$\begin{aligned} \dot{x}(t) - A_0 \dot{x}(t-h) = & A_1 x(t) + A_2 x(t-h) + Bu(t) + \int_{-\infty}^0 G(t, x_s) ds \\ & + f(t, x(t), x(t-h)), \end{aligned} \quad (6.31)$$

where,

$$\begin{aligned} A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \\ f(t, x(t), x(t-h)) = \begin{pmatrix} 0 \\ 0.1 \times \sin(x(t) + x(t-h)) \cdot (x(t) + x(t-h)) \end{pmatrix}, \\ G(s, x_s) = \begin{pmatrix} 0 \\ \exp(t-3) \cdot x(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

and let  $x(t) = [\exp(t) - \exp(2t)]^T, -0.2 \leq t \leq 0$ .

The uniform stability of the system  $D(t)x_t = x(t) - A_0 x(t-h)$  for  $h > 0$  has been computed in Example 4.5.1 of Chapter 4 and note that the function  $G(t, x_t)$  satisfies its conditions with  $M(t) = \exp(t-3)$ , where  $\int_{-\infty}^0 M(t)dt = \exp(-3) = 0.0498$ . Also,  $|f(t, x(t), x(t-h))| \leq 0.1\|x(t)\| + 0.1\|x(t-h)\|$ . Associated with (6.31) is the cost function (6.13) with  $Q_1 = I$  and  $Q_2 = 0.1I$ . The aim now is to find a maximum delay bound to guarantee that the resulting closed-loop subsystem design from the controller  $u(t)$  for system (6.31) and the cost function (6.13) is uniformly asymptotically stable. Normally, when a control input is not forced to (6.31), for example when  $u(t) = 0$ , the system becomes unstable within some delay limits when the states of the system approach infinity.

Set  $Q = I$  and observe from (6.12) that, the matrix  $(A_1 + A_2)$  is Hurwitz stable with,

$$P = \begin{pmatrix} 0.1833 & -0.0500 \\ -0.0500 & 0.2500 \end{pmatrix}.$$

Therefore,

$$\lambda_{\min}(P) = 0.1566, \lambda_{\max}(P) = 0.2768, \lambda_{\min}(Q) = 1,$$

$$\sigma = 1.8064, \delta = 0.7522, \gamma_0 = 0.0498, \gamma_1 = \gamma_2 = 0.1, \text{ and}$$

$$\|(A_1 + BB^T P)^T A_0\| = 1.0004, \|A_0^T\|(\gamma_1 + \gamma_0) = 0.0749,$$

$$\|A_2\|(\|A_1 + BB^T P\| + \|A_2\| + \gamma_1 + \gamma_2 + \gamma_0) = 6.2595,$$

$$\|A_0^T\|(\|A_2\| + \gamma_2) = 0.8590, \gamma_2 = 0.1,$$

$$(\sigma - \|BB^T P\| - \gamma_1 - \gamma_0) = 1.4016.$$

Using (6.16) gives  $h = 0.2307$  and a maximum bound  $0 \leq h < h^* = 7.4795$ . Now, setting

$$P = I, \text{ using (6.12) gives } Q = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}. \text{ Therefore, } \lambda_{\min}(P) = 1, \lambda_{\max}(P) = 1, \delta = 1,$$

$$\lambda_{\min}(Q) = 3.5858, \gamma_0 = 0.0498, \gamma_1 = \gamma_2 = 0.1, \text{ and } \sigma = 1.7929, \text{ which gives } h = 0.0776 \text{ and } h^* = 8.9353$$

Note, that if  $A_0 = f = G = B = 0$ , the results obtained are equivalent to that of Su and Huang (1992) when the linear parameter uncertainties  $\Delta A = \Delta A_1 = 0$ .

The stabilizing optimal control law  $u(t)$  for (6.31) when  $Q = I$  is

$$u(t) = -B^T P x(t) = -B^T X^{-1} x(t) = -[1.1541 \quad 4.2308]x(t)$$

and the optimal guaranteed cost of the closed-loop system is given by  $J^* \leq h^* = 7.6930$ .

Simulation results of this section are given under the heading ‘effects of delay on simulation’.

### 6.5.3. Comparative results with other examples

Consider the neutral system investigated in example 2 (Table 1) of Liu (2005). Let

$f = G = B = 0$  for (6.31) with

$$A_0 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad |c| < 1,$$

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

Table 6.1 shows a comparison of this result and others using the above assumptions. The comparison shows that the result obtained by using this proposed method is less conservative.

Table 6.1: Maximum delay bound for  $c$  when  $f = G = B = 0$

$ c $	0.90	0.70	0.50	0.30	0.10	0
Han (2002)	0.99	2.73	3.62	4.10	4.33	4.56
Liu (2005)	3.37	3.66	3.96	4.26	4.56	4.70
This thesis result	8.14	7.67	7.20	6.73	6.27	6.03

Also, example 3 of Liu (2005), where

$$A_0 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad |c| < 1, \quad A_1 = \begin{pmatrix} -2 & -0.6 \\ -0.5 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0.2 \\ 0.5 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 6 & 0.4 \\ 0.4 & 6 \end{pmatrix},$$

with  $f = G = B = 0$  in (6.31),  $c = 0.2$  and  $P = I$ . The maximum bound of the delay is obtained using the conditions of Theorem 6.4 and Remark 6.1, as

$h = 0.4585 < h^* = 8.9071$ , and therefore (6.31) under these assumptions is uniformly asymptotic stable.

Now, using the same conditions and system matrices above, the asymptotic stability condition in Remark 2, Liu (2005) for the criterion given by:  $\|A_0\| + h\|A_2\| < 1$  and

$$\mu(A_1 + A_2) + \|A_0\| \|A_1 + A_2\| + h \|(A_1 + A_2)^T A_2\| < 0$$

does not satisfy the maximum bound for  $c = 0.2$ . i.e.  $\|A_0\| + h\|A_2\| = 12.3242 > 1$  and

$$\mu(A_1 + A_2) + \|A_0\| \|A_1 + A_2\| + h \|(A_1 + A_2)^T A_2\| = 35.5183 > 0.$$

Table 6.2 shows a comparison of the result obtained in this thesis and that given in example 1 of El Haoussi and Tissir (2010), where

$$A_0 = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}, \quad Q = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

Here, the maximum bound of the delay is obtained on the assumption that  $f = G = B = 0$  in (6.31), with  $P = I$ ,  $\delta = 1$ . Under these assumptions (6.31) is uniformly asymptotically stable using the conditions of Theorem 6.4 and Remark 6.1.

Table 6.2: Maximum delay bound ( $h^*$ ) comparison

Methods	$h^*$
Chen (2006)	1.5497
El Haoussi and Tissir (2010)	1.7191
This thesis result	5.0512

These comparative studies have shown the criterion developed in this paper to be less conservative, robust and easy to compute.

#### 6.5.4. Effects of delay on simulation

The simulation of the above example for different values of delays within the bound and outside the bound has been examined and the effects of the time delay on the optimal

performance analyzed for both the controlled and uncontrolled system. For computer simulation purposes, the perturbation function was chosen as

$$f = 0.1 x(t) \sin(t) + 0.1x(t - h) \sin(t),$$

with frequency of 1Hz and amplitude equal to 1 on Simulink setting.

The Simulations were carried out in Simulink with the default parameter setting. Figure 6.1 and Figure 6.2 depict the simulation of the system within the delay bounds. i.e.

$h = 0.1690$  and  $h = 0.2306$  respectively, while Figure 6.3 shows when the delay is outside the range  $h^* = 7.4795$ . Figure 6.4 shows the control law for  $h = 0.1690$  and  $h = 0.2306$  resulting in the stabilisation of controlled states in Figure 6.1 and Figure 6.2. As shown in the simulated output results below; settling time is faster when the delay  $h \geq 0.2306$  see Figure 6.2. Oscillations are observed on the uncontrolled system when the time delay is at  $h = 0.1690$ , see Figure 6.1. The states are approaching zero as time increases.

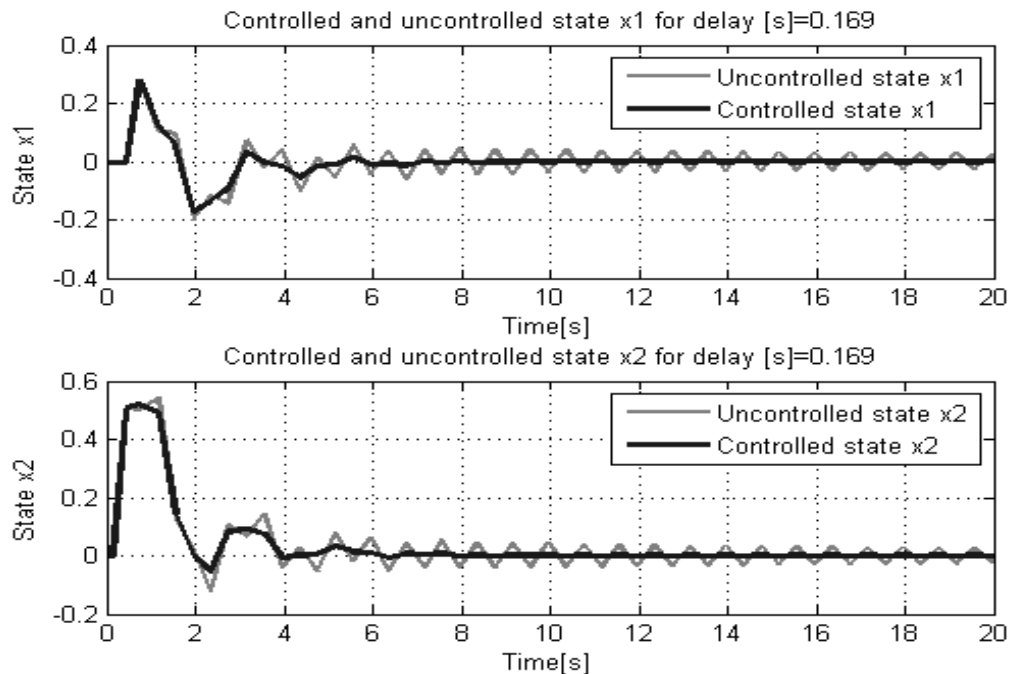


Figure 6.1: Simulation example for  $h = 0.1690$

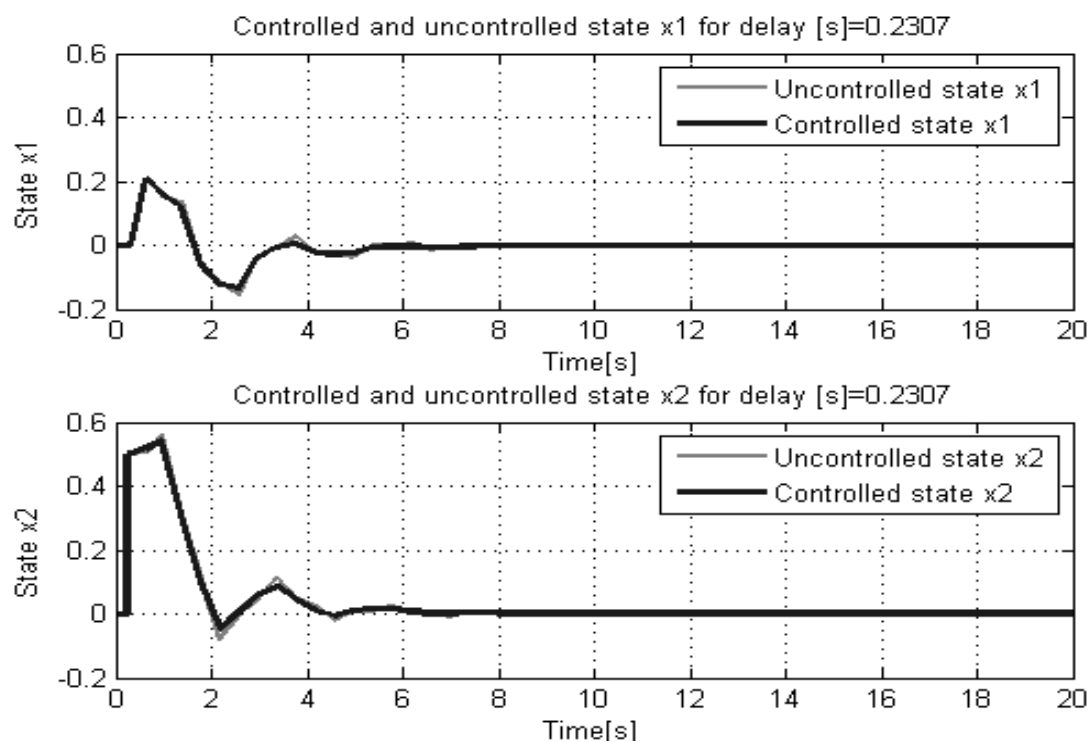


Figure 6.2: Simulation example for  $h = 0.2306$

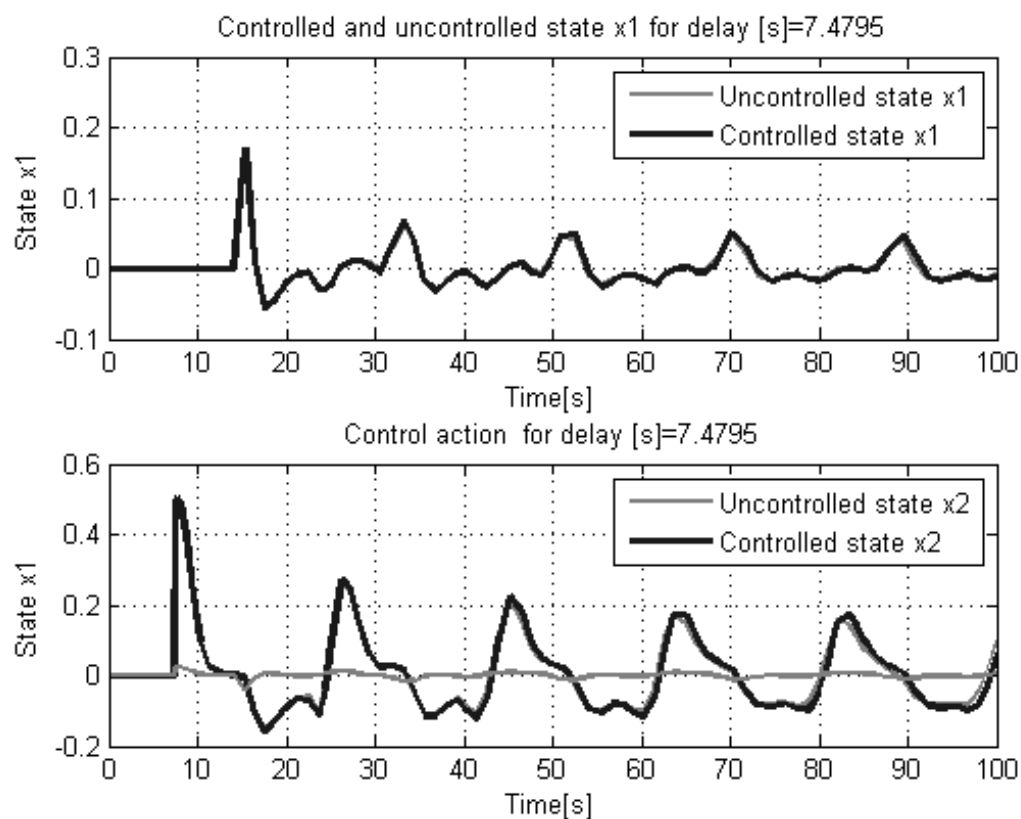


Figure 6.3: Simulation example for  $h = 7.4795$



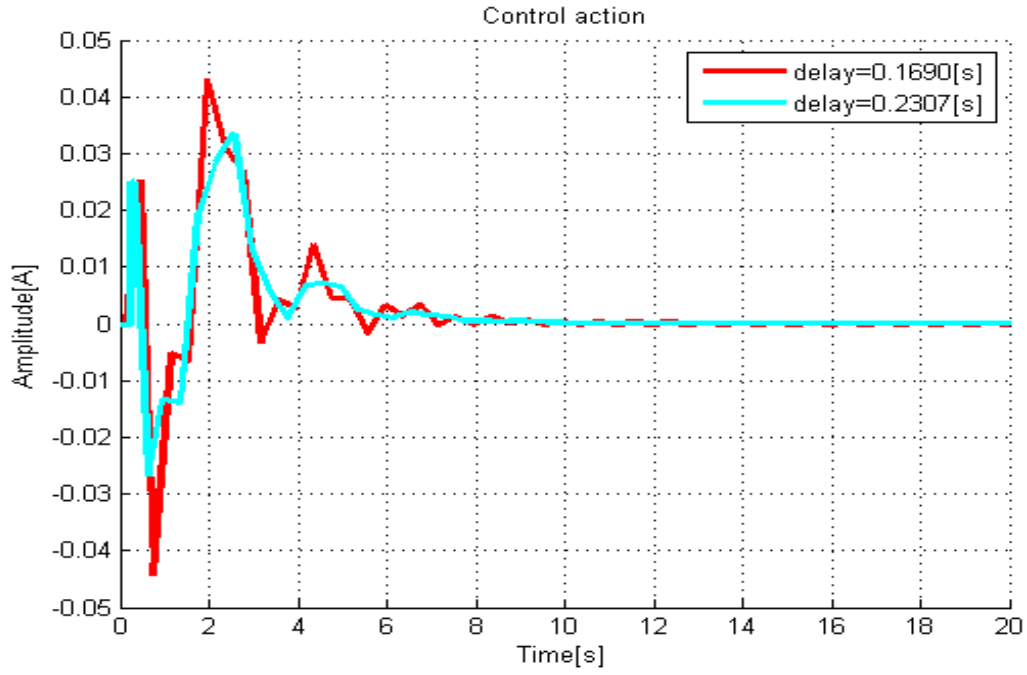


Figure 6.4: Control law for the system

## 6.6. Concluding remarks

The chapter presented results of investigations covering the time optimal control problem for neutral functional differential control systems with infinite delays; necessary and sufficient conditions for normality and complete controllability conditions of the system were deduced. The bang-bang form of optimal control has been given for zero targets. Easily computable criteria for the system to be normal and completely controllable were developed. Also proved is the condition for the system to be null controllable. Methods for obtaining an optimal robust guaranteed cost control problem via state feedback control laws for the system were presented. A new robust guaranteed cost control result has been obtained with a transformation technique combined with the Lyapunov matrix equation and the Razumikhin approach. A guaranteed cost control gain was obtained by solving an optimization problem. The checking of the stabilisation criterion is simple and the example illustrates the robustness of the methods.

Furthermore, an optimal robust guaranteed cost control problem was obtained using state feedback control laws for a class of nonlinear neutral systems having infinite delays. A new result has been obtained using a transformation technique combined with the Lyapunov matrix equation and the Razumikhin approach. A guaranteed cost control gain for the system was also obtained by solving an optimization problem.

The checking of the conditions developed in this chapter is simple and the example with simulated state outputs illustrates the robustness of the method.

## Chapter 7

# Application of results to lossless transmission line

### 7.1. Introduction

This chapter presents the application of the theoretical work carried out in this thesis to lossless transmission lines, a special case of the general system investigated in this project. The stability and control of voltage with current fluctuations are key issues for system planners in transmission lines (see Chukwu 2001). Transmission lines have previously been modelled as a single neutral functional differential equation (Nagumo and Shimura 1961, Brayton 1967, Shimura 1967, Slemrod 1971, Lopes 1976, Wu and Xia 1996, Zhihong et al. 2012, Angelov 2012, and Angelov 2014) to analyse the phenomena of the existence of periodic solutions (Wu and Xia 1996), self-oscillations (Nagumo and Shimura 1961), and synchronization and asynchronous quenching (Shimura 1967) arising from lossless transmission lines terminated by nonlinear lumped circuits. To date, no attention has been given to robust guaranteed control of transmission lines terminated with nonlinear lumped circuits such as those used as basic elements in the design of digital computers using these methods. This chapter will therefore bridge such a gap by analysing the robust guaranteed control for such systems with and without the nonlinearity.

The novel approach is to first formulate the mixed boundary conditions in terms of voltage and current changes for the system using Kirchoff's law and then use d'Alembert's solution to reduce the mixed problem to an initial value problem for NFDSID.

The first section of this chapter reviews existing operational conditions in transmission lines. The second section describes the application to lossless transmission lines through a mathematical derivation of NFDSID from the model. The third section presents robust guaranteed control results which is essentially an application of the results developed in Chapter 6 to the transmission line models.

## **7.2. Operational conditions in transmission lines**

Transient instability in power systems and in particular transmission lines terminated by nonlinear lumped circuit in parallel with capacitance, and a series combination of inductance and resistance in power systems is the focus of this chapter.

Decreasing of power system stability margins beyond a certain operational condition can lead to frequent power system collapses if power system control measures are not put in place. Kundur et al. (2004) have observed that power system stability is similar to the stability of any other dynamical system, and has some fundamental mathematical underpinnings. There are several definitions of power system stability aimed at encompassing all practical scenarios see (Fouad and Vittal 1991, Ernst et al. 2004, and Kundur et al. 2004). For example, power system stability is defined by Ernst et al. (2004) as the property of a power system that enables it to remain in a state of equilibrium under normal operating conditions and to regain an acceptable state of equilibrium after a disturbance, while Kundur et al. (2004) in a joint task force of IEEE/CIGRE on stability terms and definition defined power system stability “as the ability of an electronic power system, for a given initial operating condition, to regain a state of operating equilibrium after being subjected to a physical disturbance with most system variables bounded so that practically the entire system remains intact”. However, the study of power system stability under transient conditions is a complex task because the mathematical underpinnings that

describe even the simplest systems that are often modelled into differential or partial differential equations are nonlinear in nature (Gless 1966). Moreover, as power systems continue to experience growing interconnections through the use of new technologies, highly diversified operations with devices interacting with the power system in stressed conditions has led to the emergence of different forms of power system instability including voltage stability (Kundur et al. 2004, and Althowibi and Mustafa 2013).

#### **7.2.1. Voltage stability**

Voltage stability is “the ability of a power system to maintain steady voltages on all buses in the system after being subjected to a disturbance from a given initial operating condition” (Kundur et al. 2004). Voltage stability may be short-term or long-term based on their classification in Kundur et al. (2004) and are often studied using static or dynamic analysis approaches (Althowibi and Mustafa 2013). Voltage instability occurs more often when power systems are operated close to the transmission line full capacity and this has been seen as a serious threat to power system stability and operations. Common causes of voltage instability include unexpected load increase, insufficient active and reactive power supply of the transmission line network, progressive drop in bus voltages, overvoltage, and self-excitation (Kundur et al. 2004, and Althowibi and Mustafa 2013). Voltage instability in transmission lines can be limited by operating within designed operational guidelines (Althowibi and Mustafa 2013).

#### **7.3. Application to lossless transmission lines**

Power is a practical problem that confronts industrial activities, requiring optimal control for effective use of power. Stability and control of voltages with current fluctuations are key issues for system planners in transmission lines. The natural models for these voltages and current fluctuations arising in transmission lines are mathematical models for neutral

functional differential equations. The act of driving fluctuations of voltages to their stable equilibrium state as rapidly as possible has been termed the time optimal control problem (Chukwu 2001).

### 7.3.1. Network of flip – flop circuit

Some dynamical systems possess multiple equilibria and are used as a memory device in the design of digital computers; the flip-flop circuit has such dynamics and serves as the basic element in a digital computer (see Chukwu 2001 and references therein for details). A standard model is given below in Figure 7.1, while an interconnection of these models is given in Figure 7.2. The portion between  $\xi = 0$  and  $\xi = l$  is a lossless transmission line with inductance per unit length  $L$  and capacitance per unit length  $C$ . The current flowing through the  $k$ th line at time  $t$  and distance  $\xi$  is denoted by  $i_k$ , while  $v_k$  is the voltage across it at both  $\xi$  and  $t$ . The function  $g(v_k)$  is a nonlinear function of  $v_k$  and gives the current in the indicated box in the direction shown.

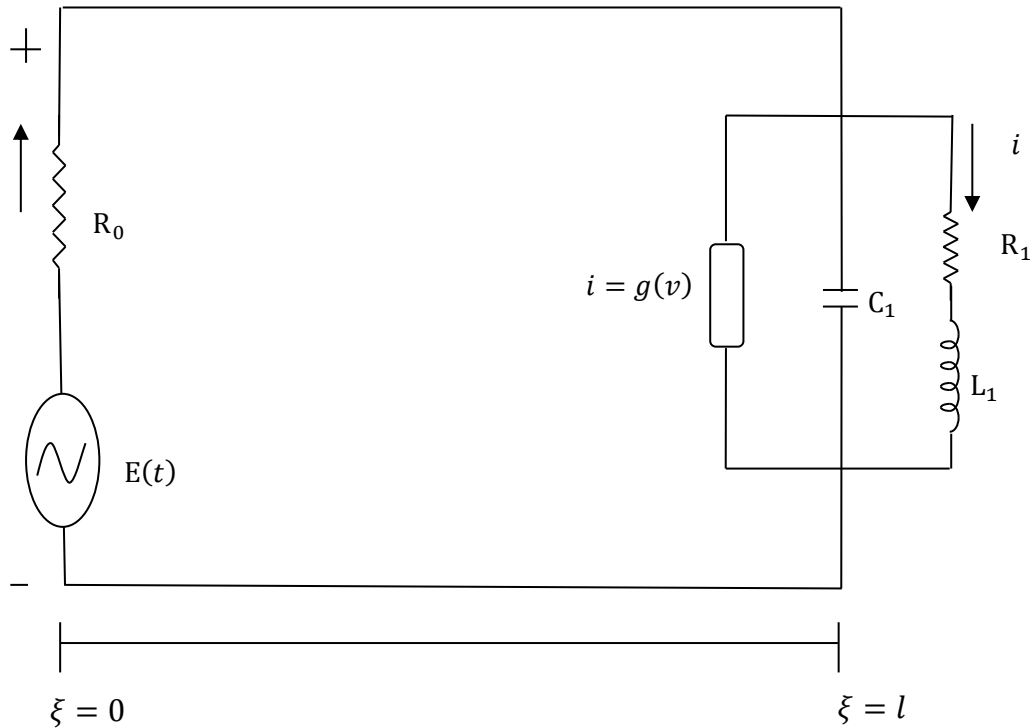


Figure 7.1: Fundamental diagram of a flip-flop circuit

The focus of this Section is to derive a novel state space equation in terms of current and voltage changes for a NFDSID associated with a network of  $N$  mutually connected lossless transmission lines which are interconnected in a decentralized form. The aim of this derivation is to analyse the stability behaviour of the systems through simulation output studies in state space form. One advantage of this new form of analysis is that the state space form will also enable application of the optimal robust control results developed in Chapter 6. The reason for studying multi-connected systems is that previously studied single transmission line circuits are assumed not to be affected by changes in the electrical dynamics of other lines. However, in real life a voltage is induced by fields of the first line and current from the second line when a second transmission line is placed near the first line. The magnetic field generated by the closeness of the networks produces inductive coupling without necessarily being physically connected. Further, the electric field lines that start from one end and terminate on the other produce capacitive coupling even when they may not be electrically connected. The following assumptions are therefore made for ease of applicability.

- The transmission lines considered are lossless.
- All coupled lossless transmission line networks are identical and each of them is a uniformly distributed lossless transmission line terminated with a nonlinear function in parallel with capacitance, resistance and inductance.
- All lossless transmission line networks are resistively coupled and all other forms of coupling are negligible on the system.

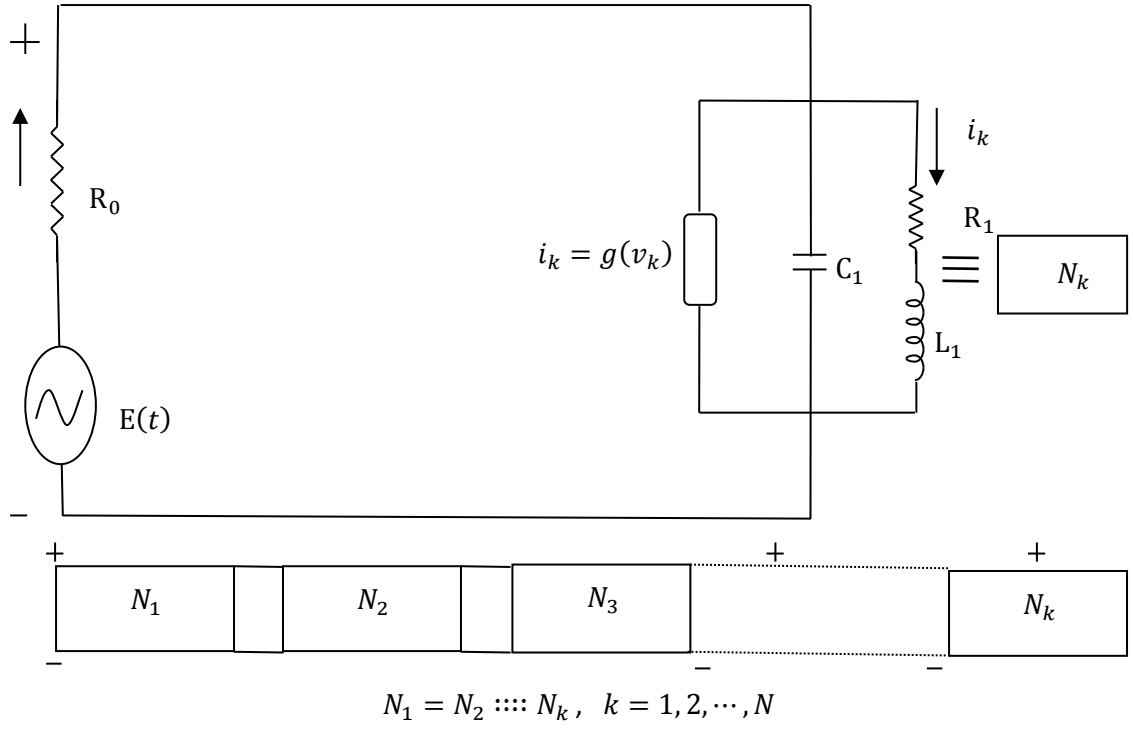


Figure 7.2:  $N$  mutually connected transmission line network

It is a well-known fact that the relation of the voltage  $v_k$  and the current  $i_k$  in a transmission line obeys the following Telegraphers' equations

$$\frac{\partial v_k}{\partial x} = -L \frac{\partial i_k}{\partial t}, \quad \frac{\partial i_k}{\partial x} = -C \frac{\partial v_k}{\partial t}, \quad (7.1)$$

where  $0 < \xi < l$ ,  $t > 0$ ,  $k = 1, 2, \dots, N$ . In (7.1),  $i_k$  or  $v_k$  can be eliminated accordingly to give

$$\frac{\partial^2 v_k}{\partial x^2} = LC \frac{\partial^2 v_k}{\partial t^2}, \quad \frac{\partial^2 i_k}{\partial x^2} = LC \frac{\partial^2 i_k}{\partial t^2}. \quad (7.2)$$

Now, when  $N$  networks are interconnected the middle lines in each circuit are influenced by an interacting term, this interaction term will be represented by  $G_k(\cdot)$ . The boundary conditions of the circuit at the ends  $\xi = 0$  and  $\xi = l$  are given by

- $H_0: E(t) = v_k(0, t) + R_0 i_k(0, t),$
- $H_1: i_k(l, t) = C_1 \frac{dv_k(l, t)}{dt} + g(v_k(l, t)) + I_k + i_k(t),$



- $H_2: v_k(l, t) = L_1 \frac{di_k(t)}{dt} + R_1 i_k(t)$

where  $E(t)$  is an external source of ac voltage,  $I_k$  is term for the network coupling current so that  $v_{k+1}(l, t) - v_k(l, t) = RI_k$ . The current-voltage characteristics of the nonlinear function are given by  $g(v_k(l, t))$ . The empirical characteristics of the chosen nonlinear function  $g(v_k(l, t))$  at the  $k^{th}$  circuit as shown in Figure 7.3 satisfies that in Slemrod (1971) and is such that  $g(0) = 0$ , and has a very steep maximum afterwards, which is followed by a slanting positive minimum and after which the function increases. The system (7.1) with the boundary conditions  $H_0$  and  $H_1$  may possess one or multiple equilibrium points if the term with capacitor  $C$  in  $H_1$  is set to zero as shown in Figure 7.3 below.

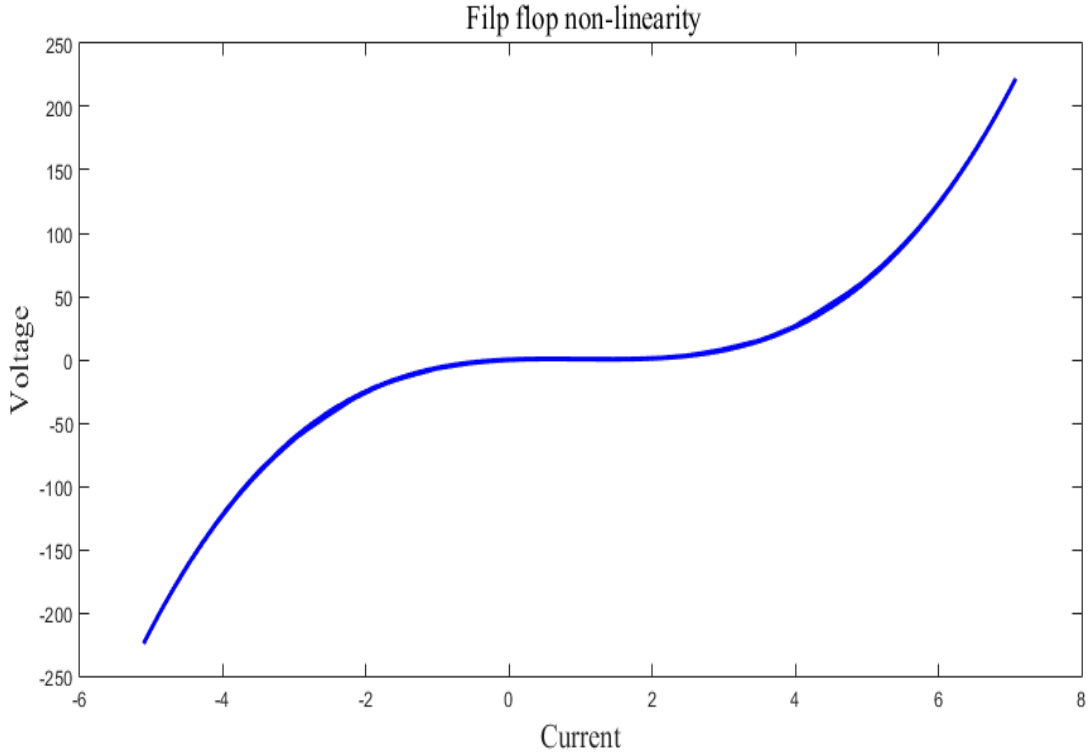


Figure 7.3: Current-voltage characteristics of the nonlinear function

The system (7.1) with boundary conditions  $H_0$ ,  $H_1$  and  $H_2$  will now be converted into a NFDSID as part of the contributions of the thesis in this chapter as follows.

First, note that there exists a unique general solution (D'Alembert solution) for  $i_k(\xi, t)$  and  $v_k(\xi, t)$  which are given by

$$\begin{cases} v_k(\xi, t) = \phi_k(\xi - bt) + \psi_k(\xi + bt), \\ i_k(\xi, t) = \frac{1}{Z}[\phi_k(\xi - bt) - \psi_k(\xi + bt)], \end{cases} \quad (7.3)$$

where  $b = 1/\sqrt{LC}$  is the propagation velocity of waves and  $Z = \sqrt{L/C}$  is the characteristic impedance of the line. The equation (7.3) can be expressed equivalently as

$$\begin{cases} 2\phi_k(\xi - bt) = v_k(\xi, t) + Zi_k(\xi, t), \\ 2\psi_k(\xi + bt) = v_k(\xi, t) - Zi_k(\xi, t), \end{cases} \quad (7.4)$$

This implies that by setting  $\xi = 0$  and replacing  $t$  by  $t - l/b$  and using (7.4) with the boundary conditions gives

$$\begin{cases} 2\phi_k(-bt) = v_k\left(l, t + \frac{l}{b}\right) + Zi_k\left(l, t + \frac{l}{b}\right), \\ 2\psi_k(bt) = v_k\left(l, t - \frac{l}{b}\right) - Zi_k\left(l, t - \frac{l}{b}\right). \end{cases} \quad (7.5)$$

Now using the boundary condition  $H_0$  and these expressions in the general solution (7.3), an equation for the current can be derived as follows. Substitute  $v_k(0, t)$  and  $i_k(0, t)$  into  $H_0$

$$E(t) = \phi_k(-bt) + \psi_k(bt) + \frac{R_0}{Z}\phi_k(-bt) - \frac{R_0}{Z}\psi_k(bt),$$

which gives

$$E(t) = \frac{Z + R_0}{Z}\phi_k(-bt) + \frac{Z - R_0}{Z}\psi_k(bt),$$

So that,

$$\phi_k(-bt) = \frac{ZE(t)}{Z + R_0} - \frac{Z - R_0}{Z + R_0}\psi_k(bt). \quad (7.6)$$

This implies that,

$$\phi_k(l - bt) = \frac{ZE\left(t - \frac{l}{b}\right)}{Z + R_0} - \frac{Z - R_0}{Z + R_0} \psi_k(bt - l). \quad (7.7)$$

Substituting (7.7) into (7.4) at  $\xi = l$ , and replacing  $t$  by  $t - l/b$  gives

$$v_k(l, t) + Zi_k(l, t) = \frac{ZE\left(t - \frac{l}{b}\right)}{Z + R_0} - \frac{Z - R_0}{Z + R_0} \times 2\psi_k(bt - l). \quad (7.8)$$

Observe from the definition of  $2\psi_k(bt)$  that,  $2\psi_k(bt - l)$  is equivalent to  $2\psi_k(b(t - l/b))$

for  $t = t - l/b$  and can be defined by,

$$2\psi_k(bt - l) = v_k\left(l, t - \frac{2l}{b}\right) - zi_k\left(l, t - \frac{2l}{b}\right).$$

Substituting  $2\psi_k(bt - l)$  into equation (7.8) gives

$$v_k(l, t) + Zi_k(l, t) = \frac{ZE\left(t - \frac{l}{b}\right)}{Z + R_0} - \frac{Z - R_0}{Z + R_0} \left\{ v_k\left(l, t - \frac{2l}{b}\right) - Zi_k\left(l, t - \frac{2l}{b}\right) \right\},$$

Expanding and rearranging the above equation gives

$$\begin{aligned} i_k(l, t) - \frac{Z - R_0}{Z + R_0} i_k\left(l, t - \frac{2l}{b}\right) \\ = \frac{ZE\left(t - \frac{l}{b}\right)}{Z + R_0} - \frac{v_k(l, t)}{Z} - \frac{Z - R_0}{Z + R_0} \times \frac{v_k\left(l, t - \frac{2l}{b}\right)}{Z}, \end{aligned} \quad (7.9)$$

Now letting,

$$\sigma(t) = \frac{ZE\left(t - \frac{l}{b}\right)}{Z + R_0}, \quad r = \frac{Z - R_0}{Z + R_0}, \quad h = \frac{2l}{b},$$

and substituting  $\sigma$ ,  $r$  and  $h$  into (7.9) gives

$$i_k(l, t) - r i_k(l, t - h) = \sigma(t) - \frac{v_k(l, t)}{Z} - \frac{r v_k(l, t - h)}{Z}, \quad (7.10)$$

Note that using the boundary condition  $H_0$  and the expressions (7.4) and (7.5) in the general solution (7.3) gives the following equations

$$\begin{cases} v_k(l, t) + Zi_k(l, t) = \sigma(t) - r \times 2\psi_k(bt - l) \\ v_k(l, t - h) - Zi_k(l, t - h) = 2\psi_k(bt - l) \end{cases} \quad (7.11)$$

Now using (7.11) and the second boundary condition  $H_1$  gives

$$\begin{aligned} v_k(l, t) + Z \left[ C_1 \frac{dv_k(l, t)}{dt} + g(v_k(l, t)) + \frac{1}{R} (v_{k+1}(l, t) - v_k(l, t)) + i_k(t) \right] \\ = \sigma(t) - r[v_k(l, t - h) - Zi_k(l, t - h)]. \end{aligned}$$

Expressing  $i_k(\cdot)$  in terms of  $v_k(\cdot)$  in the above equation gives,

$$\begin{aligned} v_k(l, t) + Z \left[ C_1 \frac{dv_k(l, t)}{dt} + g_k(v_k(l, t)) + \frac{1}{R} (v_{k+1}(l, t) - v_k(l, t)) + i_k(t) \right] \\ = \sigma(t) - r v_k(l, t - h) \\ + Zr \left[ C_1 \frac{dv_k(l, t - h)}{dt} + g_k(v_k(l, t - h)) \right. \\ \left. + \frac{1}{R} (v_{k+1}(l, t - h) - v_k(l, t - h)) \right]. \end{aligned}$$

Expanding and rearranging the above equation now gives the following,

$$\begin{aligned} ZC_1 \frac{dv_k(l, t)}{dt} - ZC_1 r \frac{dv_k(l, t - h)}{dt} \\ = \sigma(t) - v_k(l, t) - r v_k(l, t - h) - Zi_k(t) - Zg_k(v_k(l, t)) \\ + Zr g_k(v_k(l, t - h)) - \frac{Z}{R} (v_{k+1}(l, t) - v_k(l, t)) \\ + \frac{Zr}{R} (v_{k+1}(l, t - h) - v_k(l, t - h)) \end{aligned}$$

Dividing through by  $Z$  and setting  $x_k(t) = v_k(l, t)$ , with  $E(t) = 0$  at the initial condition gives

$$\begin{aligned} C_1 \left[ \frac{dx_k(t)}{dt} - r \frac{dx_k(t-h)}{dt} \right] \\ = -\frac{x_k(t)}{Z} - \frac{rx_k(t-h)}{Z} - i_k(t) - g_k(x_k(t)) + rg_k(x_k(t-h)) \\ - \frac{1}{R} (v_{k+1}(l, t) - v_k(l, t)) + \frac{r}{R} (v_{k+1}(l, t-h) - v_k(l, t-h)) + u(t), \end{aligned} \quad (7.12)$$

where  $u(t)$  is a control function generated by the control device at  $x = 0$  and is related to  $E(t)$  Chukwu (2001). Note that the coupling term can be rearranged as

$\frac{1}{R} (-v_{k+1}(l, t) - rv_k(l, t-h) + rv_{k+1}(l, t-h) + v_k(l, t))$  and since the resistances at the connections are very small  $v_k(l, t)$  and  $v_{k+1}(l, t-h)$  refer to different points.

Again, using the boundary condition  $H_2$  and (7.11) gives

$$L_1 \frac{di_k(t)}{dt} + R_1 i_k(t) + Zi_k(l, t) = \sigma(t) - rv_k(l, t-h) + Zi_k(l, t-h),$$

Rearranging the above equation gives the following,

$$L_1 \frac{di_k(t)}{dt} + R_1 i_k(t) = \sigma(t) - Zi_k(l, t) + Zi_k(l, t-h) - rv_k(l, t-h),$$

Now, setting  $x_k(t) = v_k(l, t)$  and using (7.10), noting that  $E(t) = 0$  and  $v_k(l, t-h)$  due to  $i_k(t)$  will be zero at  $\xi = l$  gives

$$L_1 \frac{di_k(t)}{dt} = -R_1 i_k(t) + x_k(t) - rx_k(t-h). \quad (7.13)$$

Equations (7.12) and (7.13) can be put in the vector or state space form to get

$$\begin{aligned}
& \frac{d}{dt}[x_k(t) - A_{0k}x_k(t-h)] \\
& = A_{1k}x_k(t) + A_{2k}x_k(t-h) + f_k(t, x_k(t), x_k(t-h)) \\
& + \int_{-\infty}^0 G_k(x_k(s), x_k(s-h)) ds + Bu(t)
\end{aligned} \tag{7.14}$$

where  $x_k(t) = (v_k(t), i_k(t))^T$

$$A_{0k} = \sum \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{1k} = \sum \begin{pmatrix} -\frac{1}{ZC_1} & -\frac{1}{C_1} \\ \frac{1}{L_1} & -\frac{R_1}{L_1} \end{pmatrix}, \quad A_{2k} = \sum \begin{pmatrix} -\frac{r}{ZC_1} & 0 \\ -\frac{r}{L_1} & 0 \end{pmatrix},$$

$$f_k(t, x(t), x(t-h)) = \sum \begin{pmatrix} -g_k(x_k(t)) + rg_k(x_k(t-h)) \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$G_k(x_k, x_k(t-h)) = \frac{1}{R} (v_k(l, t) - rv_k(l, t-h)).$$

#### 7.4. Simulation output studies for the wave patterns

In this section, a transmission line system without the nonlinear function is modelled in MATLAB/SIMULINK to compare with the simulation study of the nonlinear model. The behaviour of the system to changes in resistance and capacitance of the line is investigated for a single and then an interconnected system in line with the stability studies in Chapter 4. The simulation output studies in this section and Section 7.5 are part of the contributions of the thesis in this chapter. The Simulink models of this set-up are given in Appendix III and IV.

##### 7.4.1. Changing resistance $R_0$ of the systems

Consider a supply source of 220 V lossless transmission line sent from the plant to a receiving station. The wave pattern for the distributed network models is shown in Figures

7.1 and 7.2. Such lines have been observed with an inductance  $L = 1.75mH$  and capacitance  $C = 0.01F$  per unit length of line in metres. The resistance at the beginning of the line is given by  $R_0 = 10\Omega$ . The receiving end parameters are a nonlinear function  $g(v) = -0.5v + (v - 1)^3 + 1$  in parallel with capacitor  $C_1 = 10pF$ , resistance  $R_1 = 100\Omega$  and inductance  $L_1 = 1pH$ .

The simulation output studies Figures 7.4-7.11 was done with different values  $R_0 = 10$  and  $R_0 = 0.001$  for both systems in order to observe its effect on the transmission lines. The simulation output studies shows that the current and voltage waveforms changes with change in  $R_0$ . The oscillation patterns are the same for the single distributed system and the interconnected systems. It is observed that the waveform of the oscillation is distorted for both the nonlinear single and distributed systems. The amplitude of oscillation for the systems without nonlinearity increases as  $R_0$  is reduced and assumes a constant pattern when  $R_0$  is further reduced below  $Z$ . However, reducing  $R_0$  increases the amplitude and reduces the wavelength for the system with nonlinear function and its interconnection but becomes unstable when  $R_0$  is further reduced to a value less than  $Z$ . The amplitude of oscillations is in general higher with the systems without the nonlinear function.

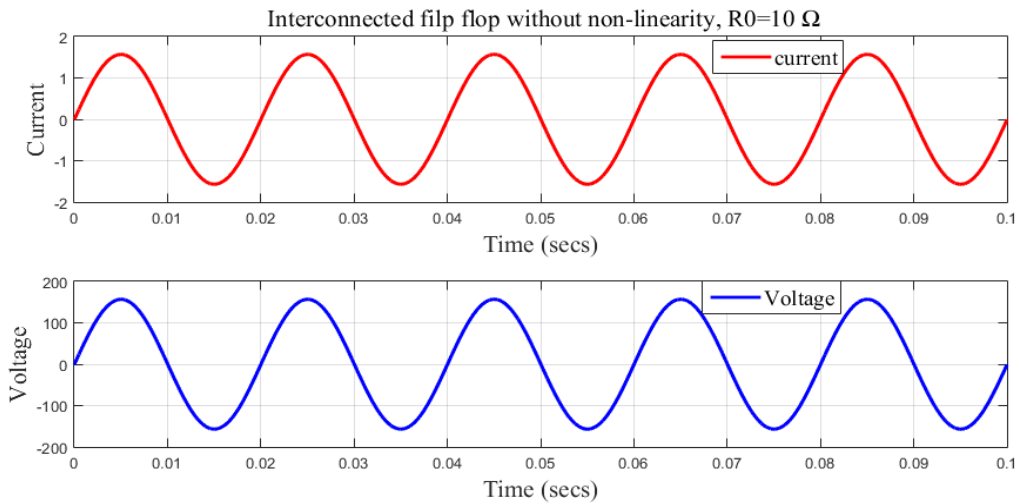


Figure 7.4: Current and voltage waveforms for interconnected system without nonlinearity,  $R_0 = 10$

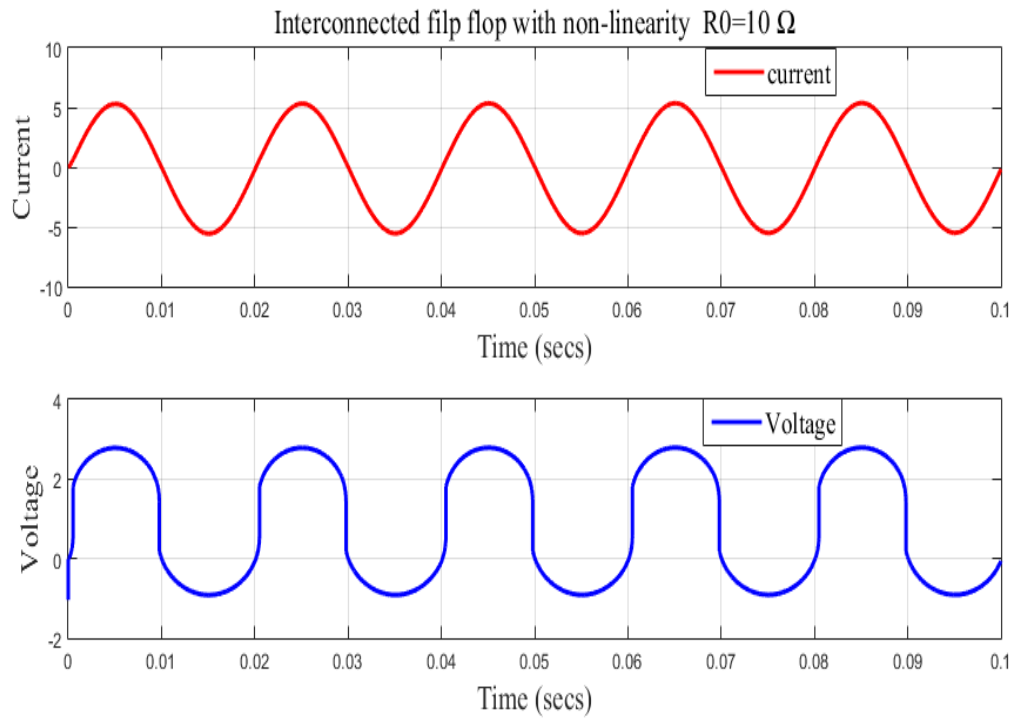


Figure 7.5: Current and voltage waveforms for interconnected system with nonlinearity  $R_0 = 10$

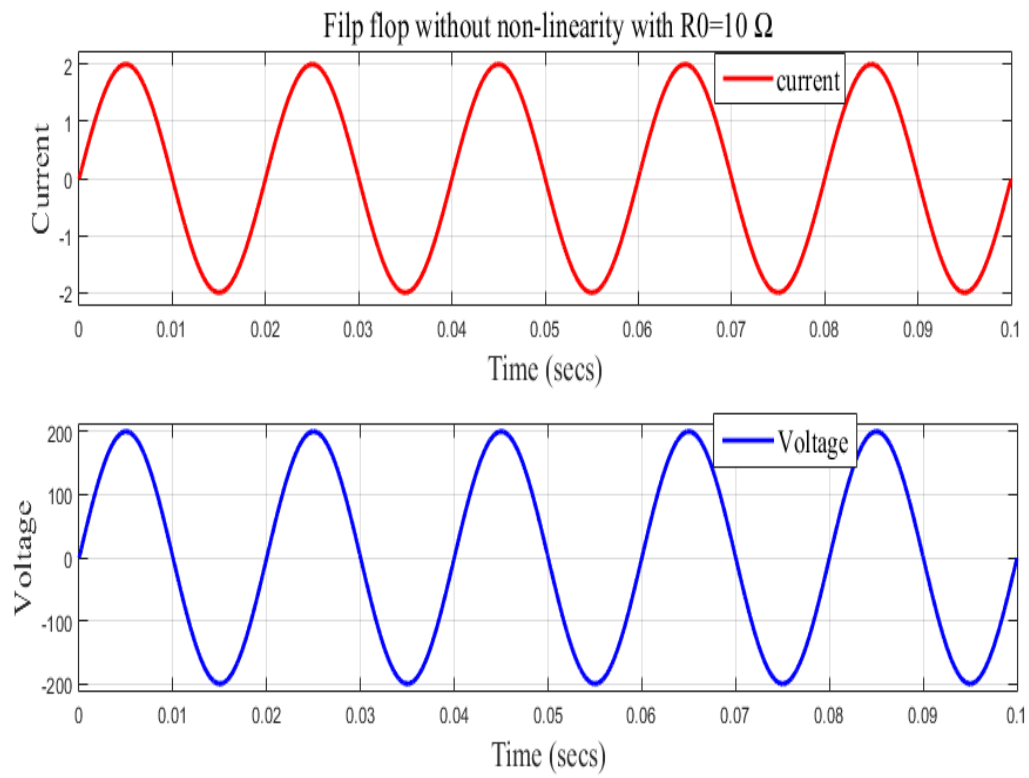


Figure 7.6: Current and voltage waveforms for system without nonlinearity,  $R_0 = 10$



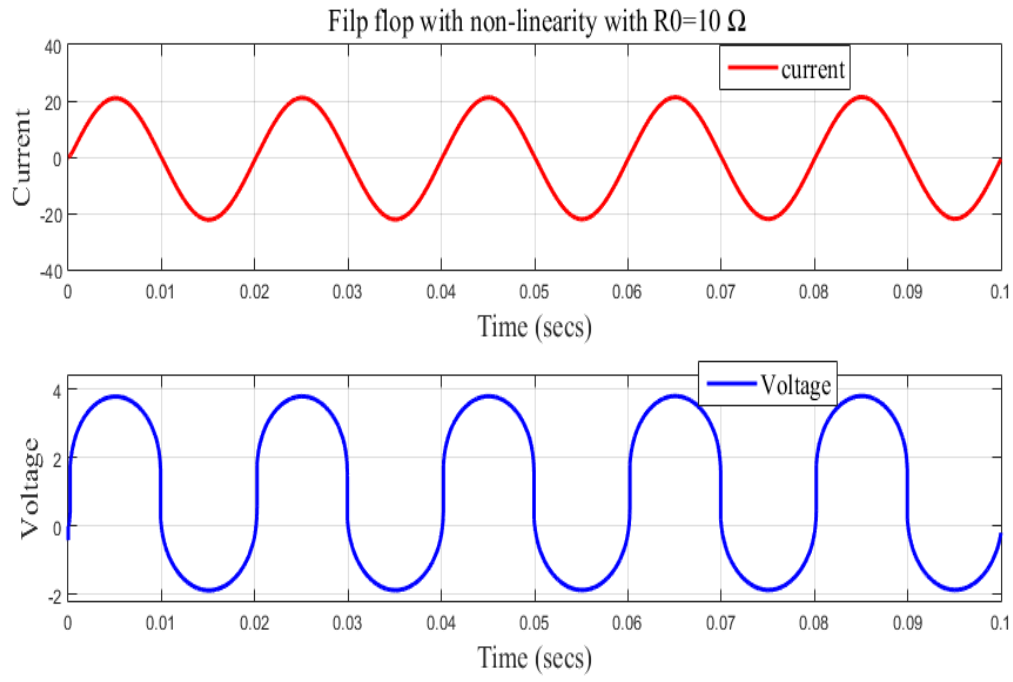


Figure 7.7: Current and voltage waveforms for the nonlinear system with  $R_0 = 10$

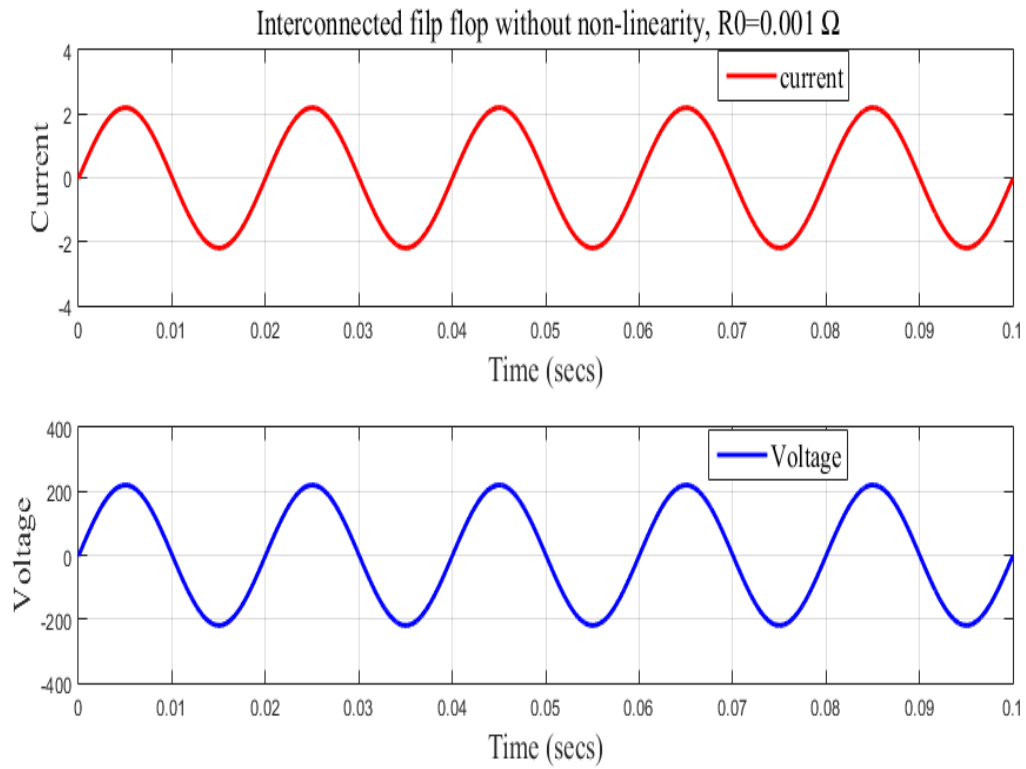


Figure 7.8: Current and voltage waveforms for interconnected system without nonlinearity,  $R_0 = 0.001$

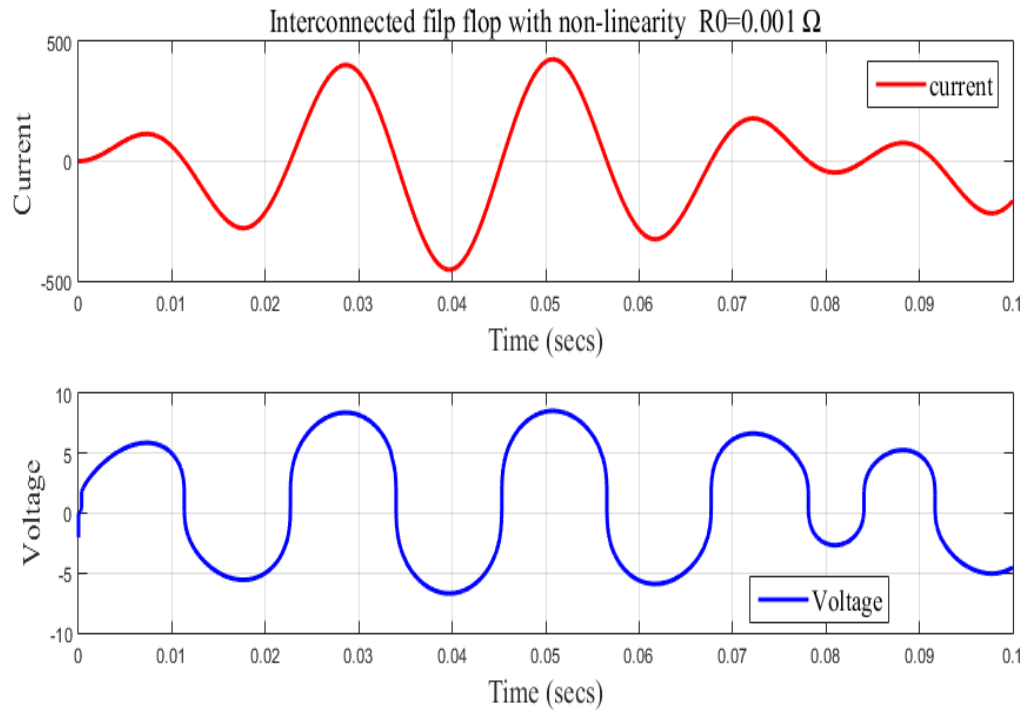


Figure 7.9: Current and voltage waveforms for interconnected system with nonlinearity,  $R_0 = 0.001$

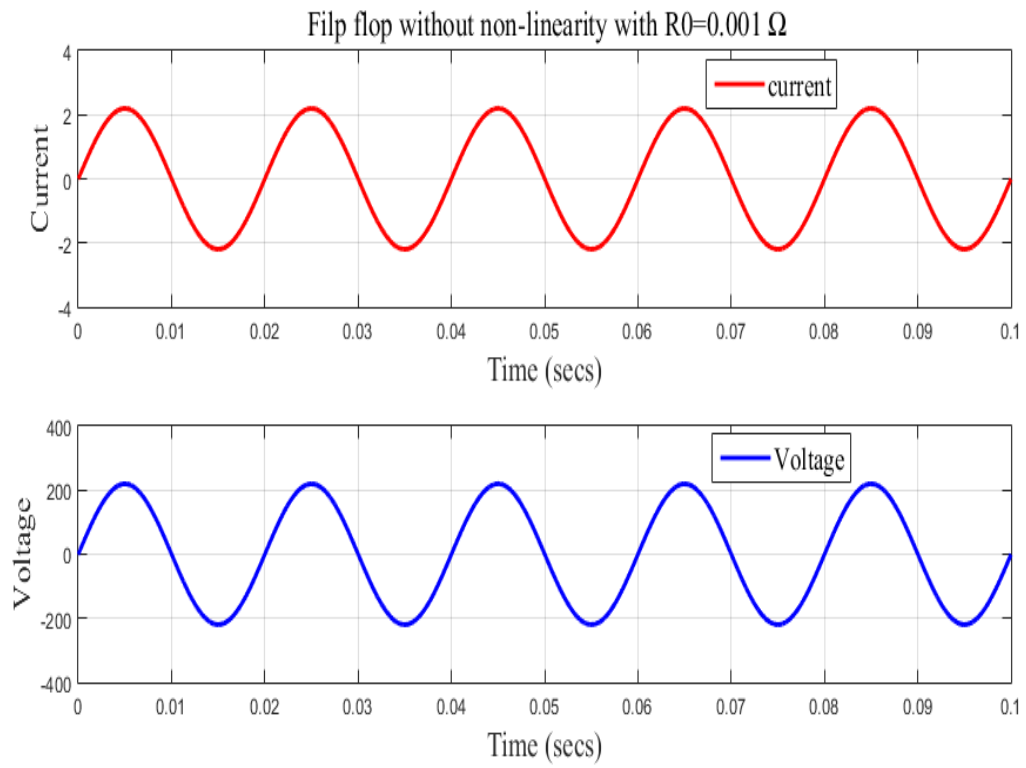


Figure 7.10: Current and voltage waveforms for system without nonlinearity,  $R_0 = 0.001$

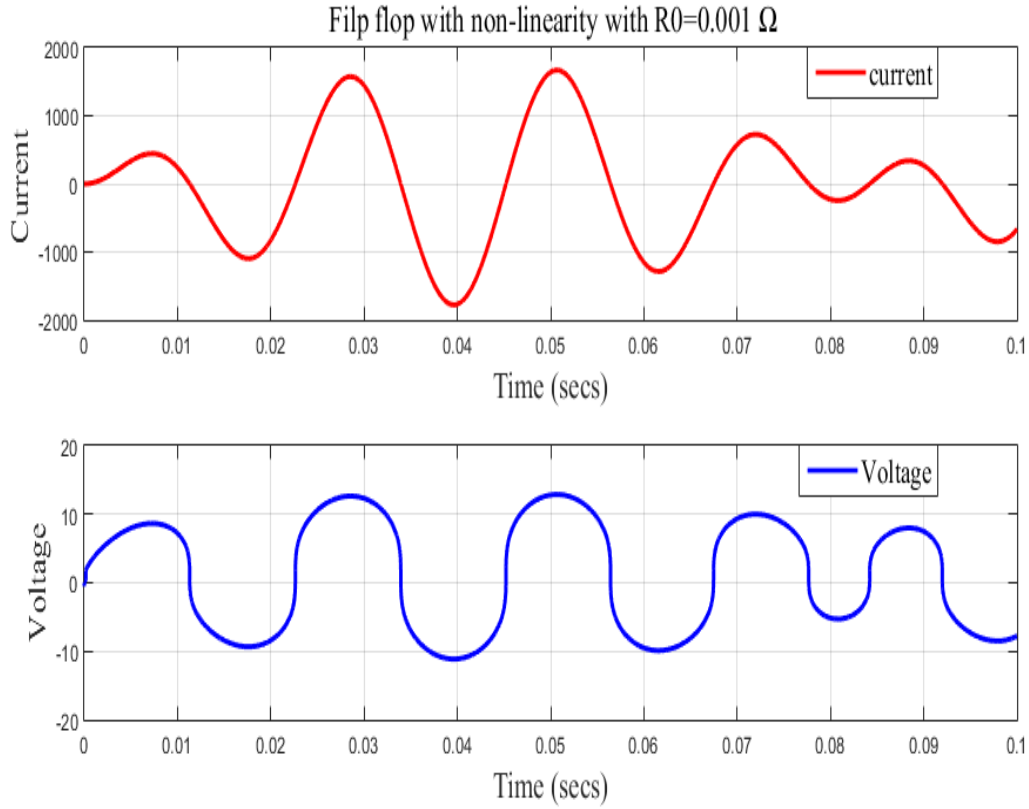


Figure 7.11: Current and voltage waveforms for the nonlinear system with  $R_0 = 0.001$

These results are expected, and conform to the theoretical analysis in Chapter 4 and the discussion on Section 2.3.4 about the role of the difference differential operator for a neutral system. That is, stability of the system depends on the functional difference operator  $D$  and that the uniform stability of the system is possible when  $\|A_0\| < 1$  (Lemma 4.1 of Chapter 4). Also note from the transmission line derivations that the value of  $r$  depends on  $R_0$  and hence the oscillation patterns observed are in agreement.

#### 7.4.2. Changing capacitance $C$ of the systems

The simulation output studies of Figures 7.12-7.19 shows that the current and voltage waveforms changes with changes in  $C$  for  $C = 100, 10^{-4}$  F with  $R_0 = 10 \Omega$ . The waveforms of the oscillation are the same for the single distributed system without nonlinearity and its interconnected systems. It is however, observed that the waveforms of the oscillations gets an oval shape for the system with nonlinear function and its interconnection. The shape

continues to get diminished and the oscillation disappears as  $C$  is further reduced. The behaviour is also expected as  $C$  indirectly determines the value of  $r$  which depends on  $Z$  and is the key element in  $A_0$ . This behaviour can also be obtained by integrating and analysing the neutral integro-differential control system derived. The stable oscillations observed in this section are in agreement with the mathematical analysis for existence of stable and periodic oscillations, see for example Nagumo and Shimura (1961), Wu and Xia (1996), Angelov (2013), and Angelov (2014). The simulation output analysis in this section also conforms to the mathematical observations of Brayton (1967).

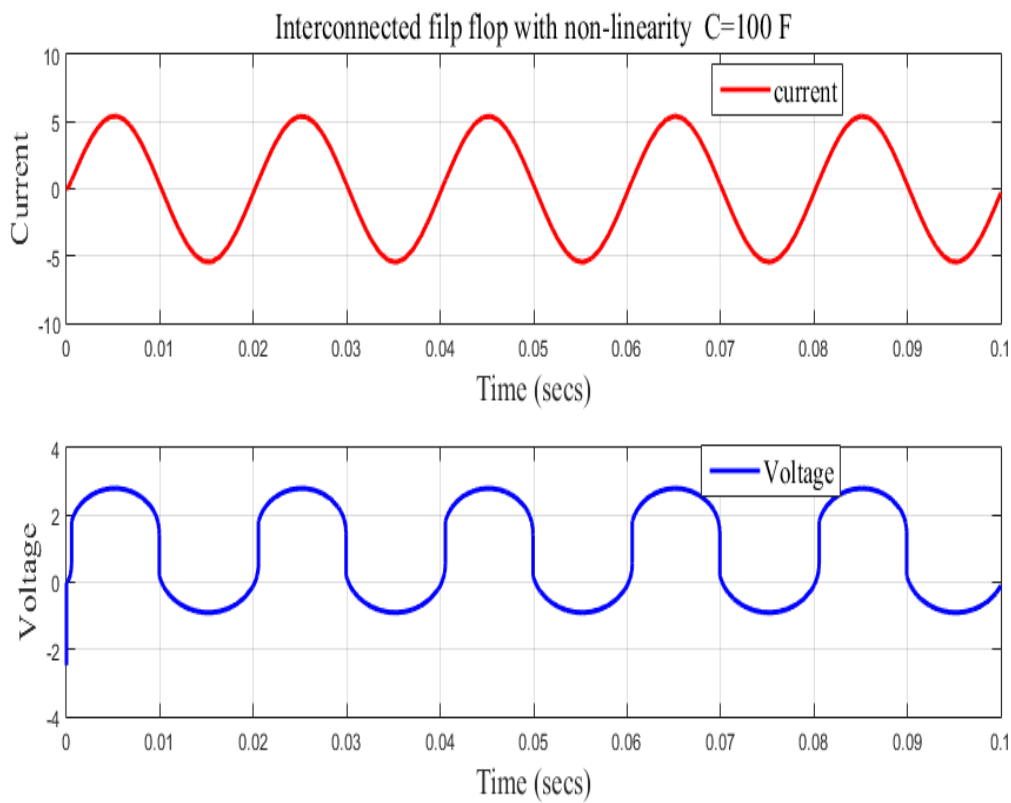


Figure 7.12: Current and voltage waveforms for interconnected system with nonlinearity,  $C = 100$

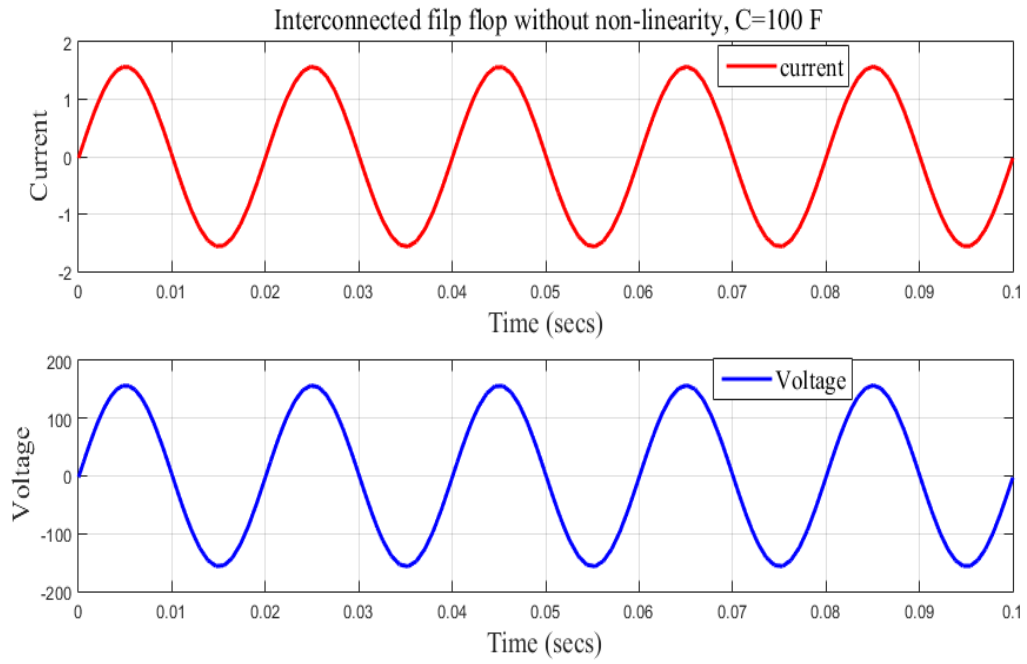


Figure 7.13: Current and voltage waveforms for interconnected system without nonlinearity,  $C = 100$

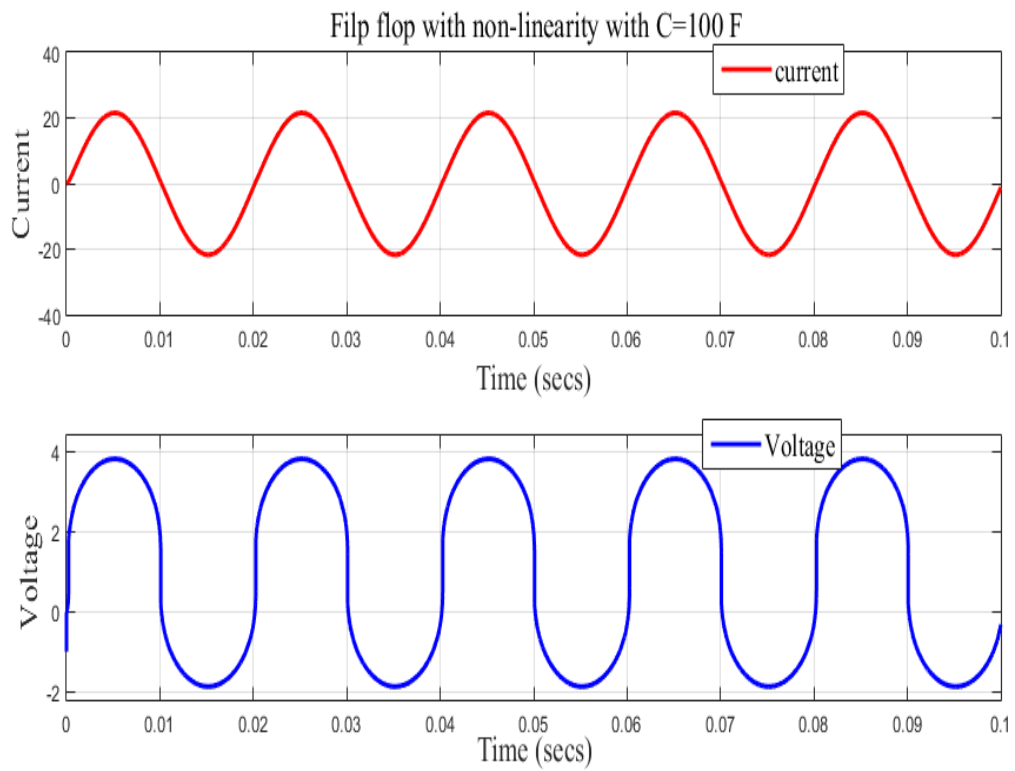


Figure 7.14: Current and voltage waveforms for system with nonlinearity,  $C = 100$

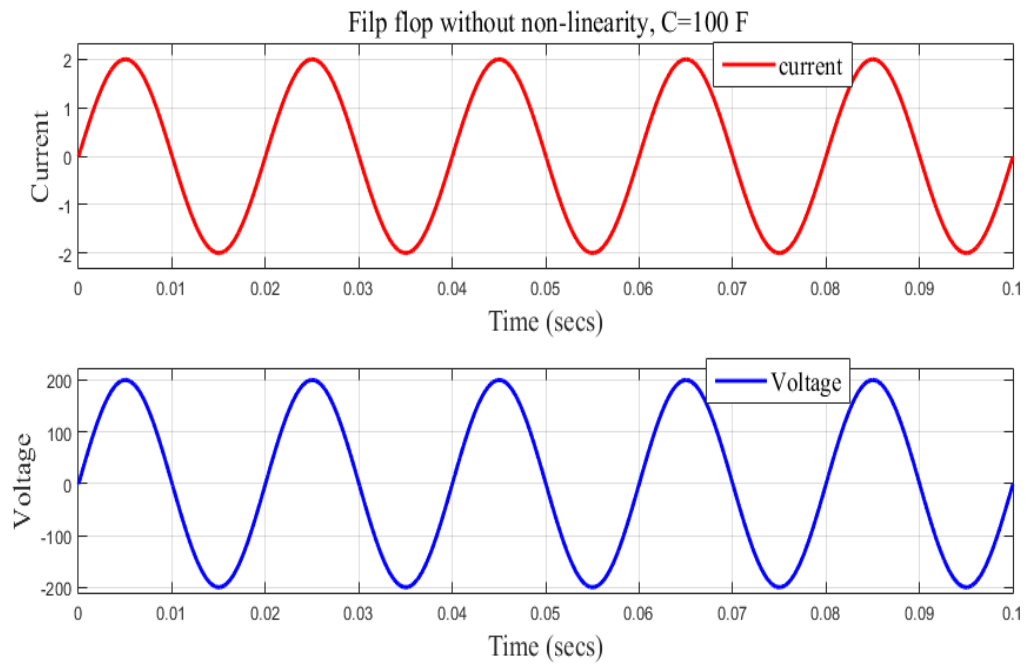


Figure 7.15: Current and voltage waveforms for system without nonlinearity,  $C = 100$

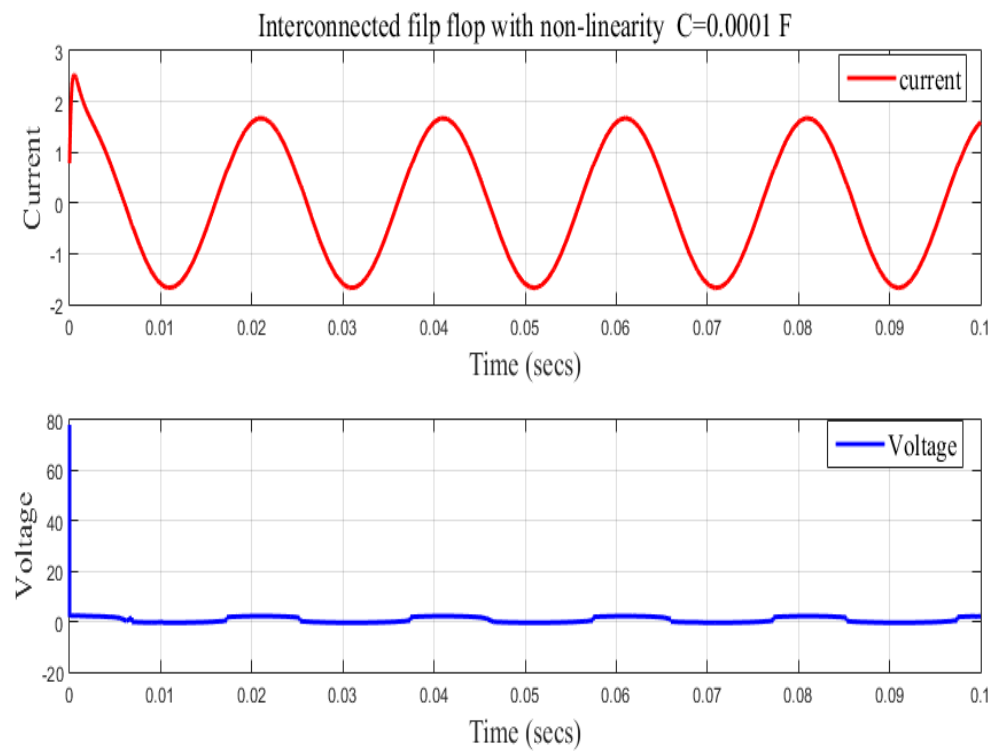


Figure 7.16: Current and voltage waveforms for interconnected system with nonlinearity,  $C = 0.0001$

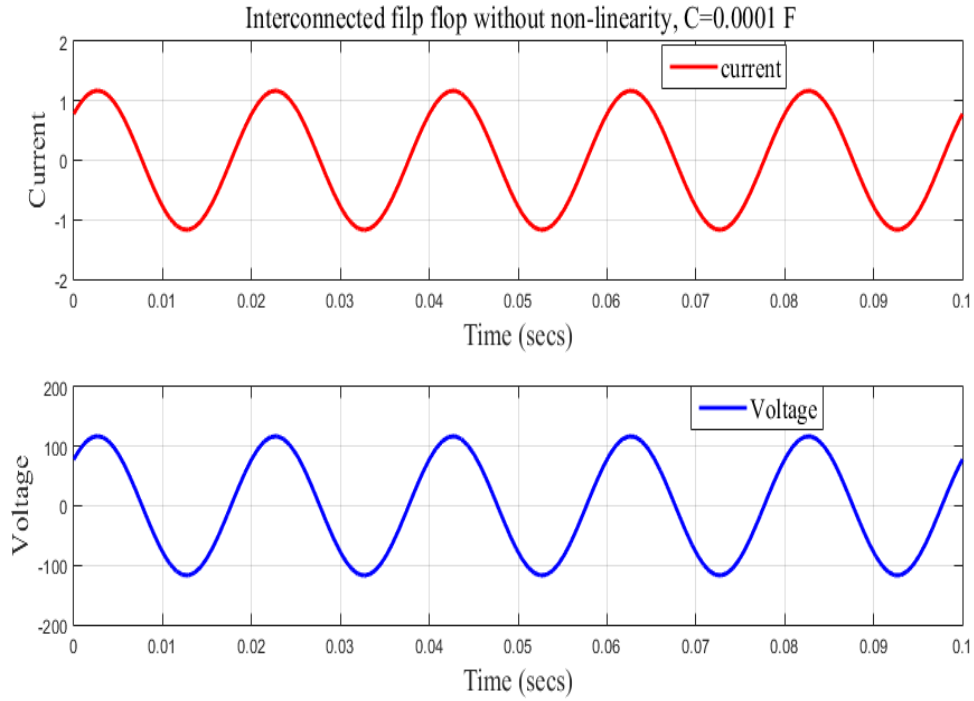


Figure 7.17: Current and voltage waveforms for interconnected system without nonlinearity,  $C = 0.0001$

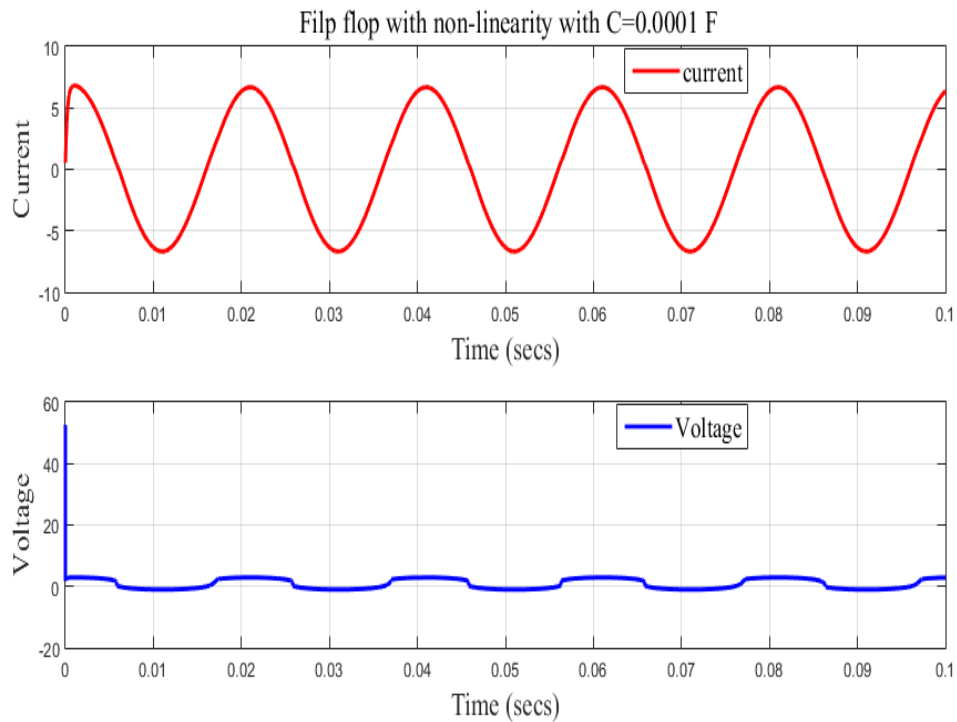


Figure 7.18: Current and voltage waveforms for system with nonlinearity,  $C = 0.0001$

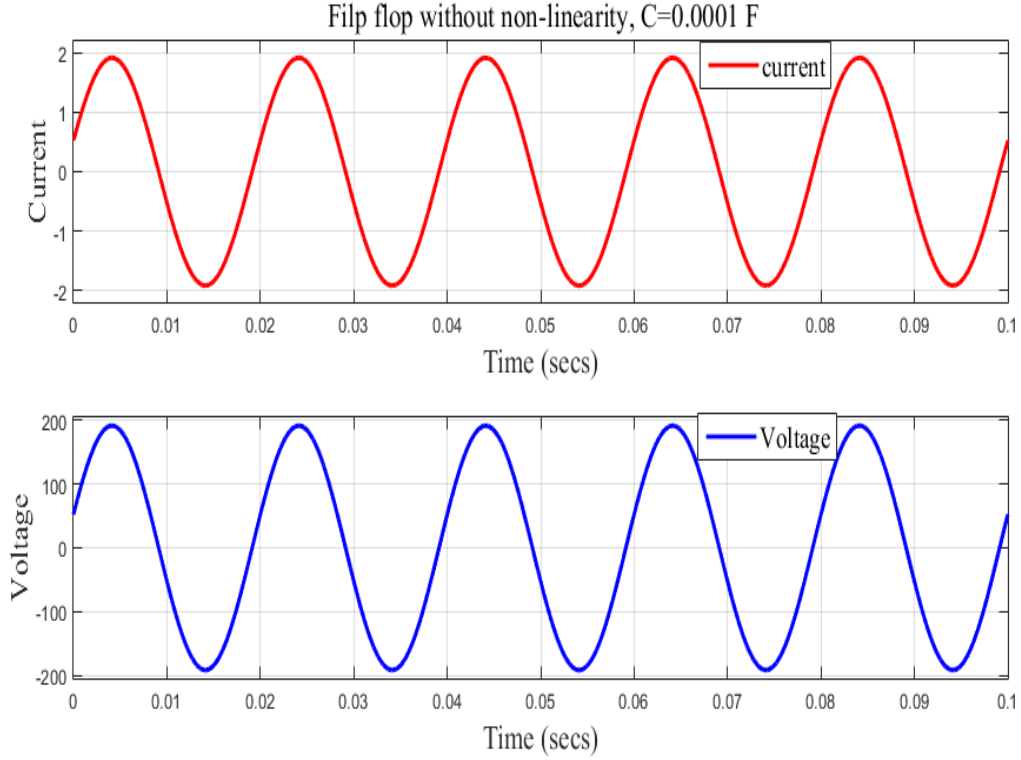


Figure 7.19: Current and voltage waveforms for system without nonlinearity,  $C = 0.0001$

### 7.5. Optimal robust control for transmission line systems

The distributed network model shown in Figure 7.1 is considered to have a supply source of 220 volts with 2000m lossless transmission line from the plant to a receiving station. The network has an inductance  $L = 1.75mH$  and capacitance  $C = 9.5\mu F$  per unit length of line in metres. The resistance at the beginning of line is given by  $R_0 = 10\Omega$ . The receiving end parameters are a nonlinear function  $g(v) = -0.5v + (v - 1)^3 + 1$  in parallel with capacitor  $C_1 = 37mF$ , resistance  $R_1 = 5\Omega$  and inductance  $L_1 = 95mH$ . The parameter setups for this chapter are obtained following Zhihong et al. (2012). The propagative velocity of the waves and the characteristic impedance of the line can be calculated as follows

$$b = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1.75 \times 10^{-3} \times 9.5 \times 10^{-6}}} = 7.76 \times 10^3 \text{ m/s}.$$



$$Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.75 \times 10^{-3}}{9.5 \times 10^{-6}}} = 13.57 \Omega.$$

The time taken to reach a unit length in metres can be obtained as

$$t = \frac{l}{b} = \frac{2000 \text{ m}}{7.76 \times 10^3 \text{ m/s}} = 0.26 \text{ s}.$$

The delayed time ( $h$ ) and reflective coefficient of voltage at the receiving end ( $r$ ) for the transmission line can be calculated as follows

$$h = \frac{2l}{b} = \frac{2 \times 2000 \text{ m}}{7.76 \times 10^3 \text{ m/s}} = 0.52 \text{ s}.$$

$$r = \frac{Z - R_0}{Z + R_0} = \frac{13.57 - 10}{13.57 + 10} = 0.15.$$

Having obtained the transmission line parameters, the parameters in the state space form of the transmission line equation (7.14) in Figure 7.1 can be obtained with

$$ZC_1 = 13.57 \times 0.037 = 5.02 \times 10^{-1}$$

$$\frac{R_1}{L_1} = \frac{5}{0.095} = 52.63$$

$$\frac{r}{L_1} = \frac{0.15}{0.095} = 1.58$$

$$\frac{r}{ZC_1} = \frac{0.15}{13.57 \times 0.037} = 0.30.$$

where,

$$A_0 = \begin{pmatrix} 0.15 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1.99 & -27.03 \\ 10.53 & -52.63 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.30 & 0 \\ -1.58 & 0 \end{pmatrix},$$

To ensure that the closed loop design for (7.14) is uniformly asymptotically stable, obtain  $f(x(t), x(t-h))$  as defined in (7.14) from the nonlinear system  $g(v) = -0.5v + (v-1)^3 + 1$  so that  $rg(v-h) = -0.075(v-0.52) + 0.15(v-1.52)^3 + 0.15$ . Now, finding the roots of  $g(v)$ , and  $rg(v-h)$  respectively and choosing values less than  $\delta$  (see assumption  $H_3$  in Section 6.2 of Chapter 6) in each gives

$$f_k(t, x(t), x(t-h)) = -g_k(x_k(t)) + rg_k(x_k(t-h)) \leq 0.2877\|x(t-h)\|.$$

Now set  $Q = I$  and observe from (7.14) that matrices  $(A_1 + A_1)$  are Hurwitz stable with

$$P = \begin{pmatrix} 0.0970 & 0.0103 \\ 0.0103 & 0.0112 \end{pmatrix}, \lambda_{\min}(P) = 0.0100, \lambda_{\max}(P) = 0.0982, \lambda_{\min}(Q) = 1,$$

$$\sigma = 5.0917, \delta = 0.3196, \gamma_0 = 0, \gamma_1 = 0, \gamma_2 = 0.2877, \text{ and}$$

$$\|(A_1 + BB^T P)^T A_0\| = 4.0629, \|A_0^T\|(\gamma_1 + \gamma_0) = 0,$$

$$\|A_2\|(\|A_1 + BB^T P\| + \|A_2\| + \gamma_1 + \gamma_2 + \gamma_0) = 99.1823,$$

$$\|A_0^T\|(\|A_2\| + \gamma_2) = 0.2844, (\sigma - \|BB^T P\| - \gamma_1 - \gamma_0) = 4.9916.$$

The performance of a single system in (7.14) with  $G_k(\cdot) = 0$  has been examined using (6.15) and the values above for simulation within delay bounds  $h = 0.0481$  and  $0 \leq h < h^* = 5.0295$ . The perturbation function is chosen as  $f = 0.2877x(t-h)\sin(t)$  for simulation purposes with frequency 2 Hz, and amplitude of 2. Figures 7.20 and 7.21 show the simulated outputs for a single system of (7.14) with  $G_k(\cdot) = 0$ .

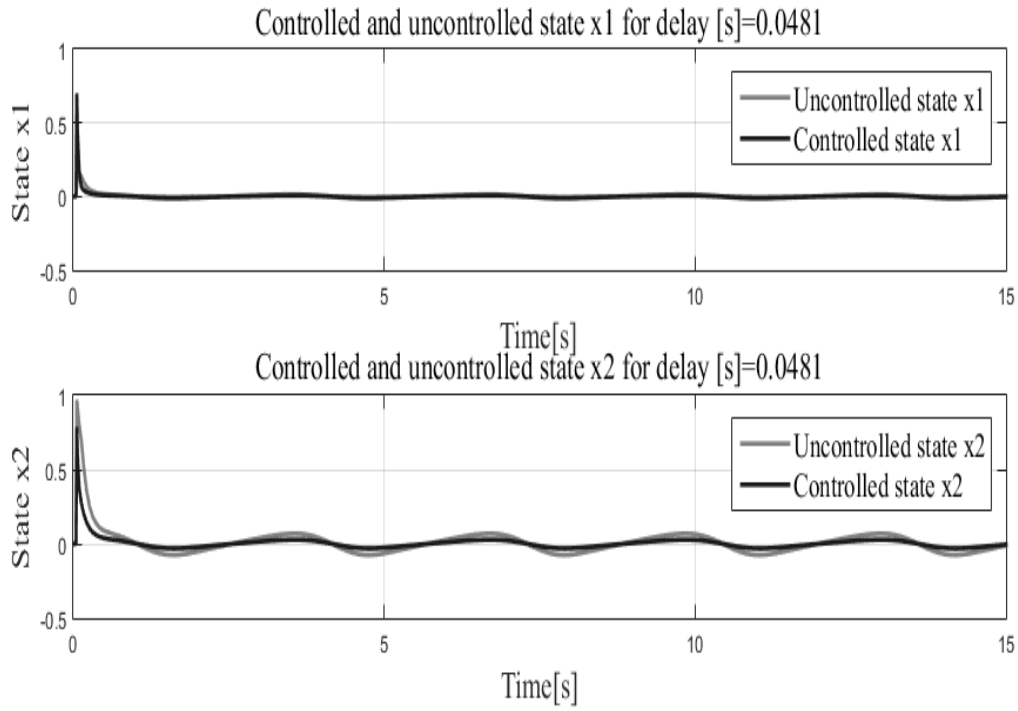


Figure 7.20: Robust stability performance for single transmission line with  $h = 0.0481$

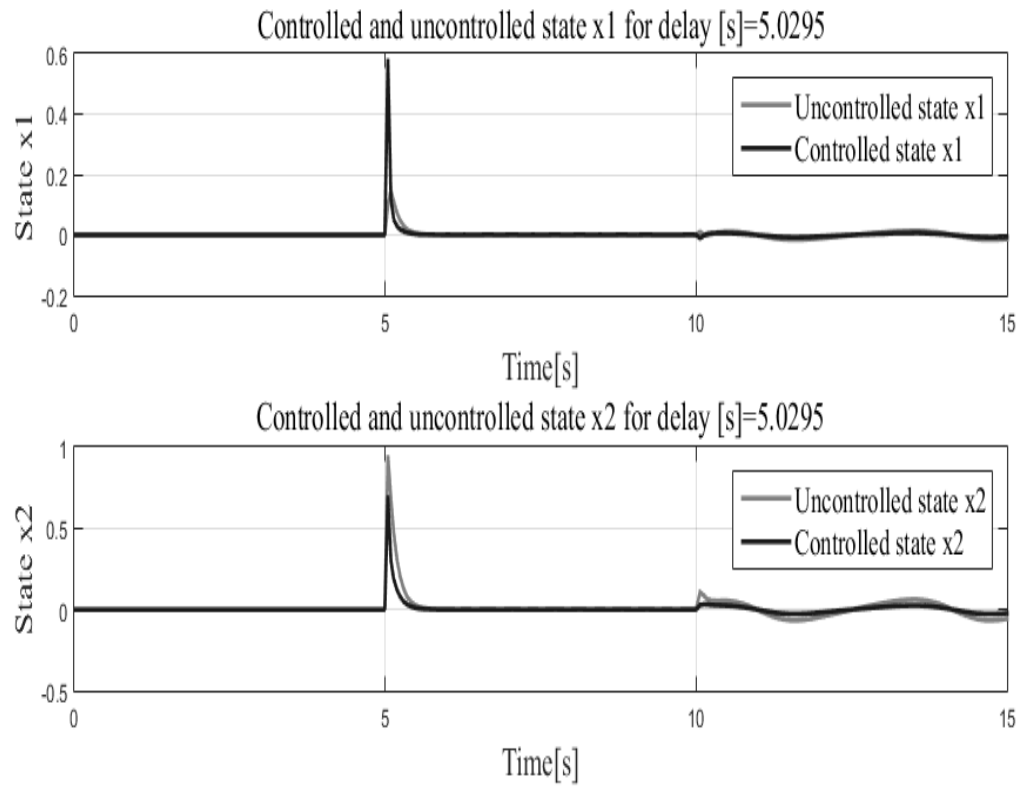


Figure 7.21: Robust stability performance for single transmission line with  $h = 5.0295$

The performance for (7.14) with four interconnected systems, see Figure 7.2, with  $N = 4$  has also been examined (see Figure 7.22 and Figure 7.23), with the following parameter values

$$A_0 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -7.96 & -108.12 \\ 42.12 & -210.52 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1.2 & 0 \\ -6.32 & 0 \end{pmatrix}, \quad \gamma_0 = 0.1998, \\ \gamma_1 = 0, \quad \gamma_2 = 0.2877.$$

where  $\gamma_0$  is obtained from the definition of  $G_k(\cdot)$  using Lemma 4.4 and assumption (iii) in Section 4.2 of Chapter 4 (see Appendix V for details). Setting  $Q = I$  in this case gives,  $P = \begin{pmatrix} 0.0243 & 0.0026 \\ 0.0026 & 0.0028 \end{pmatrix}$ ,  $\lambda_{\min}(P) = 0.0025$ ,  $\lambda_{\max}(P) = 0.0246$ ,  $\lambda_{\min}(Q) = 1$ ,  $\sigma = 20.3566$ ,  $\delta = 0.3196$ , and

$$\|(A_1 + BB^T P)^T A_0\| = 65.0450, \quad \|A_0^T\|(\gamma_1 + \gamma_0) = 0.1199,$$

$$\|A_2\|(\|A_1 + BB^T P\| + \|A_2\| + \gamma_1 + \gamma_2 + \gamma_0) = 1582.6,$$

$$\|A_0^T\|(\|A_2\| + \gamma_2) = 4.0324, \quad (\sigma - \|BB^T P\| - \gamma_1 - \gamma_0) = 20.1324.$$

The simulated results (Figure 7.20 and 7.22) have shown that the settling time for states  $x_1$  and  $x_2$  which represent current and voltage respectively is faster when the delay is minimal for  $h = 0.0481s$ . Similar to that in Section 7.4, the amplitude for the state  $x_1$  is observed to be lower than that for state  $x_2$  because of the actions of the nonlinear function which is a function of the voltage. See also Figure 7.3 for its characteristic. As expected, higher oscillations are observed for the uncontrolled state  $x_2$  than for  $x_1$ . The observations in the simulated outputs in this section can be interpreted in terms of robust stability and control with parametric effects making reference to Section 7.4. The controlled state  $x_1$  is observed to be uniformly asymptotically stable because its oscillations are uniformly stable. There are no distortions in shapes as can also be seen in Section 7.4. Its dynamics were not distorted when compared with the system without nonlinearity; this shows that current passing through the systems can be

used or stored but its overall output not deformed. The effort of the applied control is to bring the oscillations effect due to the nonlinear function and its parallel capacitance in the uncontrolled state  $x_2$  to zero steady state as time increases. Meanwhile, repeated spikes observed in Figures (7.21 and 7.23) when operating outside the minimum delay  $h = 4.1363$  could be due to the actions of the nonlinear function and its parallel capacitance. At some points in transmission, when the transient times becomes very fast, the capacitor acts like a short circuit making reflected wave magnitude to be equal to incident wave and of opposite polarity causing the voltage to drop to zero, when the capacitor starts charging reflection subsides until the transmission is normalized. The amplitudes of the spikes are higher with the interconnected system because of their increased magnitude. Also, the observation at the beginning of Figures 7.20 and 7.22 follows similar action from the nonlinear function and inductor in parallel. In this case the current cannot change instantaneously so the reflected wave takes the same magnitude and polarity as the incident wave causing voltage increase at that point. The reflected wave subsides when the current through the inductor increases and the transmission becomes normalized.

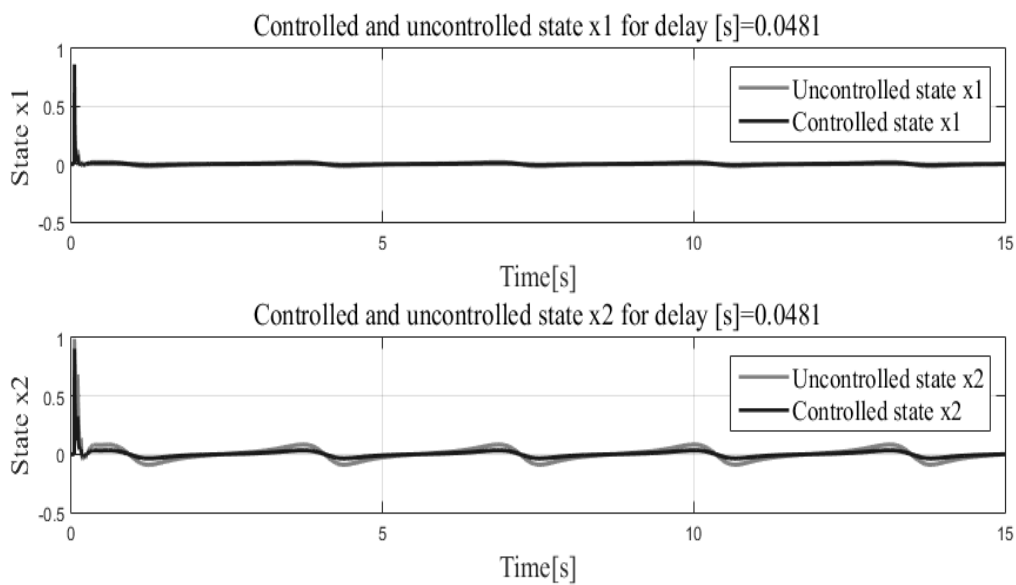


Figure 7.22: Robust stability performance for interconnected transmission line,  $h = 0.0481$

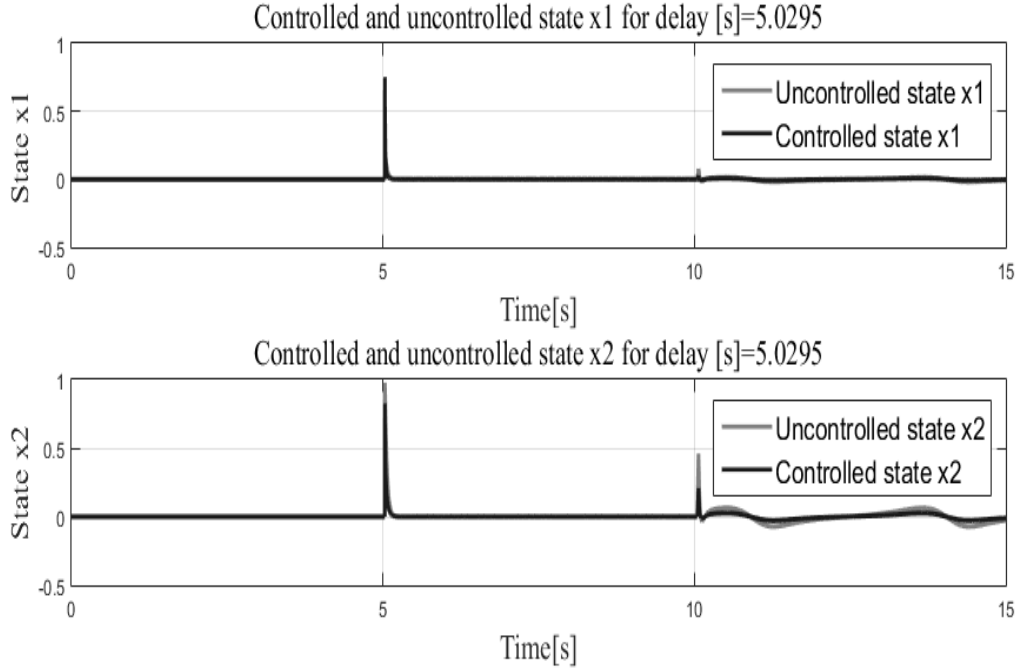


Figure 7.23: Robust stability performance for interconnected transmission line,  $h = 5.0295$

## 7.6. Concluding remarks

It was shown that a lossless transmission line network with  $N$  mutually interconnected lossless transmission lines terminated with a nonlinear function in parallel with capacitance, resistance and inductance gives rise to NFDSID. Stability of the oscillation and corresponding amplitude of the nonlinear interconnected lossless transmission line system was investigated using output simulation studies. The telegrapher's equation was first used to reduce the equation describing this transmission line system to a NFDSID. This was made possible by deriving some boundary conditions in terms of voltage and current changes through Kirchoff's law to formulate a mixed boundary problem. The NFDSID was then obtained by reducing the mixed boundary problem using D'Alembert's solution for wave equations and boundary conditions at the terminals. The operational conditions in transmission lines were reviewed and the behaviour of the transmission line system to changes in resistance and capacitance of the line was observed for single and then

interconnected systems in line with existing stability studies which were found to be effective.

The optimal robust control problem via state feedback law for NFDSID was also obtained by applying the results from Chapter 5. Specific conditions for the stability behaviour of the states were analysed using simulation output studies and the applicability of the theory analysis to lossless transmission line was demonstrated in this chapter through derivations and several simulation examples. The guaranteed robust control applied to coupling phenomenon in this chapter will a new avenue in the treatment of modern high-speed integrated circuits used in the design of digital computers. More so, the repeating spikes observed while operating outside the required delay limit can motivate a new circuit study for a pseudo-pulse generator if the rise and fall times are known.

## Chapter 8

# Conclusions and further-work

### 8.1. Conclusion

This chapter describes a conclusion of the work based on all the theoretical and simulated results obtained, prior to giving suggestion for a further work as indicated in the title of the chapter.

First, the thesis investigated the stability of NFDSID by extending fundamental stability results of functional differential equation to neutral functional integro-differential system with infinite delays. Lyapunov methods in general are known to be the cornerstones in time domain analysis of neutral functional differential systems. New results on total asymptotic stability results was presented using the uniform stability properties of the functional difference operator for neutral systems, Razumikhin stability theories, and the uniqueness property of the eigenvalues. The Lyapunov-Razumikhin technique is adopted for this investigation rather than resort to Lyapunov functional method which has a practical difficulty associated with the construction of functional. It is also considered more scalable for optimal robust guaranteed cost control design since in practice, continuous model parameters are often obtained from measured data. It was easier to embed the uncertainties into norm bounded elements by reason of measurements and obtain stability results by the Razumikhin method. It is important to note that investigation into existence and uniqueness studies for solutions of NFDSID was bypassed as they have previously been added by (Hale and Verduyn Lunel 1993, Cruz and Hale 1970).



The next investigation was on the realization that, when designing robust controls for NFDSID which consider uncertainties explicitly, their stability and controllability properties are key issues to be analysed. Controllability results in this thesis were established with the controls assumed to be restrained and null controllability result obtained using Schauder's fixed point theorem. The novel results obtained in this thesis have shown that, the controllability of NFDSID can be computed without the knowledge of the controllability matrix. Unlike the conditions in Dauer et al. (1998), the controllability conditions introduced in this Thesis were explicit and computationally more effective. In this Thesis, the computation of the controllability matrix is not required since it is obtained by an equivalent rank condition. It generalises to neutral systems the rank condition in Davies (2006). Indeed, applying the rank condition from (Davies 2006) to systems (5.3) with  $A_1 = A_2 = 0$  on  $[0, \tau]$ ,  $u \in L_2([0, \tau], E^m)$  satisfying  $x(t) = 0$ ,  $-h \leq t \leq 0$  would result in  $\text{rank } B = n$  which limits the results to retarded systems. This thesis has introduced a different rank condition, see (Corollary 5.1, (i)) based on Rivera Rodas and Langenhop (1978). The rank condition alone is not considered to be necessary and sufficient; the algebraic requirement in (Corollary 5.1, (ii)) makes it sufficient as well as necessary for controllability. Therefore the controllability condition in this Thesis generalizes the results of null controllability to neutral systems with infinite delays and yields a less conservative result.

The condition (ii), which relates to the initial condition or structure of the information for the system considered, and (iii) of Theorem 5.3 imposed in this Thesis, ensure that the error signals are contained within the neighbourhood of the origin as time increases and not asymptotically tend to zero. Conditions (ii) and (iii) are required for null controllability when the controls are restrained. They make the system less conservative being able to handle internal and external disturbances that may prevent signals from converging asymptotically to zero.

Perturbations cause conservatism as they do not vanish in some cases when the state approximates the origin. It makes uniform asymptotic stability impossible for such systems that have non-vanishing perturbations (Kofman 2005). Therefore, condition (v) of Theorem 5.3 was imposed in this thesis to preserve system properties and directional consideration. Ignoring or altering condition (v) may affect the system's properties differently and even lead to conservatism.

By using the stability and controllability results, new easily computable criteria for the time optimal control for the neutral functional integro-differential systems with infinite delays were formulated and proved. Furthermore, a novel method for obtaining an optimal robust guaranteed cost control problem via memoryless state feedback control laws for the system was presented using a transformation technique, combined with the Lyapunov matrix equation and the Razumikhin approach. The utilization of memoryless feedback controllers was based on the fact that the delay does not need to be explicitly known for simulation purposes. Information that may not be readily available and, consequently using memory-less feedback control, may have a structure that will be more implementable in real life control. A guaranteed cost control gain for the system was also presented through an optimization problem. A particular advantage of using the Razumikhin technique based optimal robust control strategy was that it provided a direct method of assuring uniform asymptotic stability. The checking of the conditions developed in using this strategy for the design of the optimal robust guaranteed cost control for the system was also simple and easily verifiable.

Furthermore, the well-known Lyapunov functional method was considered for the investigation of the system as a comparison with the Razumikhin's approach by constructing a Lyapunov functional along the solution path. A new delay-independent condition that was sufficient to make the system asymptotically stable was obtained through LMI expressions.

To assess the potential for practical application of the theoretical work, Chukwu's (2001) statement that "it is possible that some dynamical systems possess multiple equilibria and are therefore suited to be used as a memory device in the design of a digital computer, the flip-flop circuit has such dynamics and serves as the basic element in a digital computer" was critically evaluated in terms of transmission lines. An integrated lossless transmission line network terminated with a nonlinear function in parallel with capacitance, resistance and inductance was modelled and investigated. The equation obtained by reducing the transmission line model to a neutral system with an infinite delay serves as a special case of the general NFDSID considered in the thesis. It was found that a natural model for these voltages and current fluctuations arising in the network of the integrated circuits were a mathematical model for NFDSID and its perturbation. The act of driving those fluctuations of voltages to its stable equilibrium state as rapidly as possible was termed the time optimal control problem for the NFDSID.

NFDSID is considered to appear in variety of real life applications and, in this thesis it has originated out from theoretical study of  $N$  mutually interconnected lossless transmission line network terminated with a nonlinear function. The Razumikhin technique which was the basic concept in the theoretical investigation was explored in developing optimal robust control strategy for this interconnected transmission line evaluation. The mathematical approaches of analysis to the problems in this thesis were quite new and the salient features in these strategies and results presented are:

- (i) The simplicity in checking their stability results using the unique properties of eigenvalues and the difference differential operator for the system.
- (ii) The simplicity in obtaining controllability and null controllability results by a rank condition which generalizes to neutral systems.

(iii) Their clear insight about the systems application and to the optimal robust control strategy for the systems and their perturbations.

Though there are previous studies in this area and there have been great interest in the study of these systems in recent years, to the best of the researchers knowledge none have derived NFDSID from studying interconnected transmission lines. The new results and methods of analysis expounded in this Thesis are therefore more explicit, computationally more effective than existing ones and will serve as a working document for the present and future generations in the comity of researchers and industries alike.

## 8.2. Further-work

This research work can be extended to study conditions that would preserve the controllability and null controllability results when both system and input matrices undergo some parameter uncertainties. The problem of controllability of linear parameter uncertainty in systems has received considerable research effort from control audience because of its significance in theory and its applications. These uncertainties in control systems analysis and design can be structured if the uncertain parameter is an elemental part of the system and input matrices, or unstructured if the parameter uncertainties are contained in the systems matrices only. If  $\alpha_i$  are introduced into system (5.2) as uncertain parameters, and the constant  $n \times n$ ,  $n \times 1$  matrices  $A_i$  and  $B_i$ , ( $i = 1, 2, \dots, m$ ) respectively, depends linearly on  $\alpha_i$  for information, then (5.2) will be of the form

$$\begin{aligned}
 (d/dt)(x(t) - A_0x(t-h)) \\
 &= (A_1 + \Delta A_1)x(t) + (A_2 + \Delta A_2)x(t-h) + (B + \Delta B)u(t) \\
 &+ \int_{-\infty}^0 A(\theta)x(t+\theta)d\theta + f(t, x_t, u(t)),
 \end{aligned} \tag{8.1}$$

where  $\Delta A_1 = \sum_{i=1}^m \alpha_i A_{1i}$ ,  $\Delta A_2 = \sum_{i=1}^m \alpha_i A_{2i}$ ,  $\Delta B = \sum_{i=1}^m \alpha_i B_i$ , ( $i = 1, 2, \dots, m$ ).

The question of null controllability for (8.1) on  $[0, \tau]$  would then depend on the stability of (8.1) when  $u = 0$ ; which can be estimated by using matrix norm or spectral radius (Tai et al. 2009), and the controllability of the system which is relative to the controllable base for (8.1) amongst other assumptions on Theorem 5.3. That is, suppose the control base for (8.1) is controllable on the interval  $[0, \tau]$ , then the uncertain system (8.1) is controllable on  $[0, \tau]$  for all  $h > 0$  sufficiently small if it satisfies Corollary 5.1 and the conditions that  $\text{rank } P_n[A_0, B_i] = n$ , and  $\det[G_{n-1}\alpha_i, G_{n-2}\alpha_i, \dots, G_1\alpha_i, G_0\alpha_i] \neq 0$ , ( $i = 1, 2, \dots, m$ ). This criterion can be easily proved by introducing an  $n(n-1) \times n(n-1)$  matrix  $G$ , with an  $n^2 \times n^2$  matrix  $F$ ; establishing their dependence and exploring their algebraic properties. This will involve a more extensive investigation in terms of these uncertainties to develop a potentially accurate and a more dependable optimal robust control design for system (8.1).

A further possibility of extending this research work is to carry out investigations in discrete form based on the systems assumptions. This may provide some benefit in terms of model accuracy in simulation time and a potentially more improved control for the system. In this case, the optimal robust control development and design using Razhuminkhin technique can be usefully extended.

With regards to the stabilisation design for neutral systems with infinite delays, this thesis only implicitly treated it as a comparative method using the Lyapunov-Krasovskii approach which often leads to LMIs. A short coming with the method is that the resultant matrix evaluation is usually not an LMI because of the presence of the distributed delays and the perturbation function. It may therefore be considered useful to investigate other methods of converting the resultant matrices obtained using this method into an LMI or develop other means of solving these matrices in order to get novel results that would stabilise the system

asymptotically. A delay-dependant approach in such investigation would give better results as they are known to be more effective than delay-independent criteria.

Moreover, there is room for improvement in the application of the theoretical results to transmission lines in Chapter 7. In Chapter 7, the stability and control of the system was examined and discussed in terms of voltage with current fluctuations through simulation studies for an interconnected network of a distributed transmission lines, which are each terminated by a nonlinear function in parallel with capacitance, resistance and an inductance. By reducing the distributed equations describing the systems, a nonlinear neutral differential system with an infinite delay was obtained. In this consideration, the network of transmission lines was considered lossless; this was because losses are required to be kept as low as possible in transmission systems to minimise their impact in the overall signal propagated. However, it will be of interest for system planners and power users if this work could be extended to allow the parameterization for the effect of losses in the behaviour of the system parameters, that would determine the final solution to a lossy transmission line network. A novel result could be achieved by deriving the resulting neutral equations to the network, analysing the oscillatory behaviour and making a comparative simulation studies.

Another possibility for extension of this thesis work is in the area of designing model-based controllers, process-monitoring and regulation and in filtering and fault detection. All the state variables for the system considered in this thesis may rarely be available for direct online measurements. In most cases, for process-monitoring purposes, there is a need to design an observer for NFDSID that would reliably estimate the variables. Various methods used in observer design includes algebraic, geometric, inversion approaches, generalized inverse, singular-valued decomposition, input-output representation of systems and Kronecker canonical form techniques see Dong et al. (2014), Busawon (2014) and references therein for details. However, novel state observer design can easily be proposed for the

system with perturbation considered in this thesis using the transformation technique and Lyapunov functional method. The observer for the system can be obtained in the form of an adaptive control and the stability in form of LMI. The observer can be designed so that it depends on the feasible solutions of the LMI which can easily be solved using MATLAB's LMI Toolbox.

.

# References

- Ahmadi, A.A. and Parrilo, P.A. (eds. ) (2008) ‘Non-Monotonic Lyapunov Functions for Stability of Discrete Time Nonlinear and Switched Systems.’ In *Proceedings of the 47th IEEE Conference on Decision and Control*. Held 2008 at Cancun: Mexico, 614–621
- Akhmet, M.U. (2003) ‘On the General Problem of Stability for Impulsive Differential Equations.’ *Journal of mathematical analysis and applications* 288 (1), 182–196
- Althowibi, F.A. and Mustafa, M.W. (2013) ‘Power System Voltage Stability: Indications, Allocations and Voltage Collapse Predictions.’ *International Journal of Advanced Research in Electrical and Instrumentation Engineering* 2 (7), 3138–3152
- Anderson, B.D.O. and Moore, J.B. (1990) *Optimal Control: Linear Quadratic Methods*. New Jersey: Prentice Hall
- Angelov, V.G. (2012) ‘Oscillations in Lossy Transmission Lines Terminated by in Series Connected Nonlinear RCL-Loads.’ *International Journal of Theoretical and Mathematical Physics* 2 (5), 143–162
- Angelov, V.G. (2013) ‘Lossless Transmission Lines Terminated by in-Series Connected RL - Loads Parallel to C -Load.’ *International Journal of Theoretical and Mathematical Physics* 3 (1), 10–25
- Angelov, V.G. (2014) ‘Note on Lossless Transmission Lines Terminated by Triode Generator Circuits.’ *Nonlinear Dynamics of Electronic Systems, Springer International Publishing* 438, 84–91
- Angelov, V.G. and Hristov, M. (2011) ‘Distortionless Lossy Transmission Lines Terminated by in Series Connected RCL-Loads.’ *Circuits and Systems* 02 (04), 297–310
- Baker, C.T.H., Bocharov, G.A., and Rihan, F.A. (2008) ‘Neutral Delay Differential Equations in the Modelling of Cell Growth.’ *J. Egypt. Math. Soc.* 16 (2), 133–160
- Balachandran, K. and Anandhi, E.R. (2003) ‘Neutral Functional Integro-Differential Control Systems in Banach Spaces.’ *Kybernetika* 39 (3), 359–367
- Balachandran, K. and Balasubramaniam, P. (1994) ‘Controllability of Nonlinear Neutral Volterra Integrodifferential Systems.’ *Journal of the Australian Mathematical Society-Series B* 36 (1), 107–116
- Balachandran, K. and Dauer, J.P. (1996) ‘Null Controllability of Nonlinear Infinite Delay Systems with Time Varying Multiple Delays in Control.’ *Applied Mathematics Letters* 9 (3), 115–121



- Balachandran, K., Dauer, J.P., and Balasubramaniam, P. (1997) 'Asymptotic Null Controllability of Nonlinear Neutral VolterraIntegro-Differential Systems in Banach Spaces.' *Journal of Mathematical Systems, Estimation, and Control* 7 (2), 247–250
- Banks, H.T. and Kent, G. (1972) 'Control of Functional Differential Equations of Retarded and Neutral Type to Target Sets in Function Space.' *SIAM J. Control* 10 (4), 567–593
- Bellman, R., Glicksberg, I., and Gross, O. (1956) 'On the Bang-Bang Control Problem.' *Q. Appl. Math.* 14, 11–18
- Bhattacharyya, S.P., Chapellat, H., and Keel, L.H. (1995) *Robust Control: The Parametric Approach* [online] New Jersey: Upper Saddle River. Available from <<http://www.eolss.net/sample-chapters/c18/e6-43-14-00.pdf>>
- Boucekkine, R., Fabbri, G., and Pintus, P. (2012) 'On the Optimal Control of a Linear Neutral Differential Equation Arising in Economics.' *Optimal Control Applications and Methods* 33 (5), 511–530
- Bouzahir, H. (2006) 'On Controllability of Neutral Functional Differential Equations with Infinite Delay.' *Monografias del Seminario Matemático Garcia de Galdeano* 33, 75–81
- Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994) *Linear Matrix Inequalities in System and Control Theory*. Vol. 15. Philadelphia, Pennsylvania: Society for Industrial and Applied Mathematics
- Brayton, R.K. (1966) 'Bifurcation of Periodic Solutions in a Nonlinear Difference-Differential Equation of Neutral.' *Quarterly of Applied Mathematics* 24, 215–224
- Brayton, R.K. (1967) 'Nonlinear Oscillations in a Distributed Network.' *Quarterly of Applied Mathematics* 24 (4), 289–301
- Briat, C. (ed. ) (2011) 'Robust Stability Analysis in the  $\infty$ -Norm and Lyapunov-Razumikhin Functions for the Stability Analysis of Time-Delay Systems.' In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*. Held 2011 at Orlando, FL: USA.
- Burghes, D.N. and Graham, A. (1980) *Introduction to Control Theory, Including Optimal Control*. Chichester, United Kingdom: Ellis Horwood
- Burnham, K.J. and Ersanilli, V.E. (2011) *Mathematics and Computing for Control*. Wroclaw: Wroclaw University of Technology
- Burton, T.A. (1983) *Volterra Integral and Differential Equations*. New York: Academic Press
- Busawon, K. (2014) 'Output Observer Design for Linear Systems : Application to Filtering and Fault Detection.' In *Proc. of the European Control Conference (ECC), 2014*. Held at Strasbourg, France. 318–323
- Carroll, T.L. and Pecora, L.M. (1991) 'Synchronizing Chaotic Circuits.' *IEEE transactions*

*on circuits and systems* 38 (4), 453–456

- Chen, J-D. (2006) ‘Improved Results for Neutral Systems with Nonlinear Perturbations.’ *Journal of the Chinese Institute of Engineers* 29 (1), 165–171
- Chukwu, E.N. (1979) ‘Euclidean Controllability of Linear Delay Systems with Limited Controls.’ *IEEE Transactions on Automatic Control* AC-20 (5), 798–800
- Chukwu, E.N. (1981) ‘Global Asymptotic Behaviour of Functional Differential Equations of the Neutral Type.’ *Nonlinear Analysis, Theory, Methods & Applications* 5 (8), 853–872
- Chukwu, E.N. (1988) ‘The Time Control Theory of Linear Differential Equations of Neutral Type.’ *Computers & Mathematics with Applications* 16 (10), 851–866
- Chukwu, E.N. (2001) *Stability and Time-Optimal Control of Hereditary Systems: With Application to the Economic Dynamics of the US*. Singapore: World Scientific Publishing Co. Pte. Ltd
- Connor, M. (1974) ‘Time Optimal Control of Neutral Systems with Control Integral Constraint.’ *Journal of Mathematical Analysis and Applications* 45, 91–95
- Cruz, M.A. and Hale, J.K. (1970) ‘Stability of Functional Differential Equations of Neutral Type.’ *Journal of Differential Equations* 7, 334–355
- Dacka, C. (1982) ‘Relative Controllability of Perturbed Nonlinear Systems with Delay in Control.’ *IEEE Transactions on Automatic Control* AC-27 (1), 268–270
- Dauer, J.P., Balachandran, K., and Anthoni, S. (1998) ‘Null Controllability of Nonlinear Infinite Neutral Systems with Delays in Control.’ *Computers & Mathematics with Applications* 36 (2), 39–50
- Dauer, J.P. and Gahl, R.D. (1977) ‘Controllability of Nonlinear Delay Systems.’ *Journal of Optimization Theory and Applications* 21 (1), 59–70
- Davies, I. (2006) ‘Euclidean Null Controllability of Infinite Neutral Differential Systems.’ *Anziam Journal* 48 (2), 285–293
- Davies, I. and Haas, O.C.L. (2013) ‘Optimal Control of Functional Differential Systems.’ In *United Kingdom Automatic Control Council (PhD Presentations) Conference*. Held 2013 at Westminster, London. Institution of Mechanical Engineers
- Davies, I. and Haas, O.C.L. (2015a) ‘Delay-Independent Closed-Loop Stabilization of Neutral System with Infinite Delays.’ *World Academy of Science, Engineering and Technology International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering* 9 (9), 496–500
- Davies, I. and Haas, O.C.L. (2015b) ‘Delay-Independent Closed-Loop Stabilization of Neutral Systems with Infinite Delays.’ In *International Conference on Applied Mathematics and Computation (ICAMC)*. Held 2015 at Rome, Italy

- Davies, I. and Haas, O.C.L. (2015c) ‘Null Controllability of Neutral System with Infinite Delays.’ *European Journal of Control* 26, 28–34
- Davies, I. and Haas, O.C.L. (2015d) ‘Stability of Neutral Systems with Infinite Delays.’ in *International Conference on Systems Engineering (ICSE)*. Held 2015 at Coventry, United Kingdom
- Davies, I. and Haas, O.L.C. (2015e) ‘Robust Guaranteed Cost Control for a Nonlinear Neutral System with Infinite Delay.’ In *European Control Conference*,. Held 2015 at Linz, Austria. IEEE Xplore Digital Library, 1255–1260
- Deng, H., Krstić, M., and Williams, R.J. (2001) ‘Stabilization of Stochastic Nonlinear Systems Driven by Noise of Unknown Covariance.’ *IEEE Transactions on Automatic Control* 46 (8), 1237–1253
- Dhaene, T. and De Zutter, D. (1992) ‘Selection of Lumped Element Models for Coupled Lossy Transmission Lines.’ *IEEE Transactions on Computer-Aided Design* 11 (7), 805–815
- Djordjevic, A.R., Sarkar, T.K., and Harrington, R.F. (1986) ‘Analysis of Lossy Transmission Lines with Arbitrary Nonlinear Terminal Networks.’ *IEEE Transactions on Microwave Theory and Techniques* 34 (6), 660–666
- Dong, Y., Li, T., and Zhang, X. (2014) ‘Stability Analysis of Nonlinear Observer for Neutral Uncertain Time-Delay Systems.’ *Advances in Difference Equations* 2014 (133), 1–17
- Dullerud, G.E. and Paganini, F. (2013) *A Course in Robust Control Theory: A Convex Approach* [online] Springer Science & Business Media. Available from <<https://books.google.com/books?hl=en&lr=&id=KGbTBwAAQBAJ&pgis=1>>
- Ernst, D., Glavic, M., and Wehenkel, L. (2004) ‘Power Systems Stability Control: Reinforcement Learning Framework.’ *IEEE Transactions on Power Systems* 19 (1), 427–435
- Feng, G. (2002) ‘Stability Analysis of Piecewise Discrete-Time Linear Systems.’ *IEEE Transactions on Automatic Control* 47 (7), 1108–1112
- Fernando, T.L., Phat, V.N., and Trinh, H.M. (2013) ‘Output Feedback Guaranteed Cost Control of Uncertain Linear Discrete Systems with Interval Time-Varying Delays.’ *Applied Mathematical Modelling* 37 (3), 1580–1589
- Ferreira, J.M. (1986) ‘On the Stability of a Distributed Network.’ *SIAM Journal on Mathematical Analysis* 17 (1), 38–45
- Fouad, A.-A. and Vittal, V. (1991) *Power System Transient Stability Analysis Using the Transient Energy Function Method*. Upper Saddle River, New Jersey: Pearson Education
- Frommer, A. and Hashemi, B. (2013) ‘Verified Stability Analysis Using the Lyapunov

- Matrix Equation.’ *Electronic Transactions on Numerical Analysis* 40, 187–203
- Gago-Ribas, E. (eds. ) and Carril-Campa, M. (2012) ‘Complex Parameterization of the Lossy Transmission Line Theory.’ *Progress In Electromagnetics Research Symposium Proceedings*. Held 2012 at KL, MALAYSIA, 1781–1786
- Gago-Ribas, E., Dehesa-Martinez, C., and Gonzalez-Morales, M.J. (2006) ‘Complex Analysis of the Lossy-Transmission Line Theory: A Generalized Smith Chart.’ *Turkish Journal of Electrical Engineering* 14 (1), 173–194
- Gahinet, P., Nemirovski, A., Laub, A.J., and Chilali, M. (eds. ) (1995) *LMI Control Toolbox User’s Guide - For Use With MATLAB*. Natick, MA: The MathWorks
- Gahl, R.D. (1978) ‘Controllability of Nonlinear Systems of Neutral Type.’ *Journal of Mathematical Analysis and Applications* 63, 33–42
- Gless, G.E. (1966) ‘Direct Method of Liapunov Applied to Transient Power System Stability.’ *IEEE Transactions on Power Apparatus and Systems* PAS-85 (2), 159–168
- Gopalsamy, K. (1992) *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Dordrecht, Boston: Kluwer Academic Publishers
- Gu, K., Chen, J., and Kharitonov, V.L. (2003) *Stability of Time-Delay Systems*. Boston: Birkhäuser
- Haddad, W.M. and Bernstein, D.S. (1994) ‘Parameter-Dependent Lyapunov Functions and The Discrete-Time Popov Criterion for Robust Analysis.’ *Automatica* 30 (6), 1015–1021
- Haddock, J., Krisztin, T., Terjéki, J., and Wu, J. (1994) ‘An Invariance Principle of Lyapunov-Razumikhin Type for Neutral Functional Differential Equations.’ *Journal of Differential Equations* 107 (2), 395–417
- Hale, J.K. (1974) ‘Functional Differential Equations with Infinite Delays.’ *Journal of Mathematical Analysis and Applications* 48 (1), 276–283
- Hale, J.K. (1977) *Theory of Functional Differential Equations*. New York: Springer-Verlag
- Hale, J.K. and Cruz, M.A. (1969) ‘Asymptotic Behavior of Neutral Functional Differential Equations.’ *Archive for Rational Mechanics and Analysis* 34 (5), 331–353
- Hale, J.K. and Verduyn Lunel, S.M. (1993) *Introduction to Functional Differential Equations*. New York: Springer-Verlag
- Han, Q. (2002) ‘Robust Stability of Uncertain Delay-Differential Systems of Neutral Type.’ *Automatica* 38 (4), 719–723
- El Haoussi, F. and Tissir, E.H. (2010) ‘Delay and Its Time-Derivative Dependent Robust Stability of Uncertain Neutral Systems with Saturating Actuators.’ *International Journal of Automation and Computing* 7 (4), 455–462
- Heemels, W.P.M. and Camlibel, M.K. (2007) ‘Controllability of Linear Systems with Input

- and State Constraints.’ in *Proceeding of the 46th IEEE Conference on Decision and Control*. held 2007 at USA: New Orleans. 536–541
- Hu, G-D. and Hu, G-D. (1996) ‘Some Simple Criteria for Stability of Neutral Delay-Differential Systems.’ *Applied Mathematics and Computation* 80 (2), 257–271
- Hu, G-D., Hu, G-D., and Zou, X. (2004) ‘Stability of Linear Neutral Systems with Multiple Delays: Boundary Criteria.’ *Applied Mathematics and Computation* 148, 707–715
- Ionescu, T.C. and Stefan, R. (2009) ‘Stability Analysis of Neutral Systems : A Delay-Dependent Criterion.’ *Lecture Notes in Control and Information Science* 388, 15–24
- Izé, A.F. and Freiria, A.A. (1981) ‘Total Stability for Neutral Functional Differential Equations.’ *Proc. Amer. Math. Soc* 81, 437–442
- Jacobs, M.Q. and Langenhop, C.E. (1976) ‘Criteria for Function Space Controllability of Linear Neutral Systems.’ *SIAM J. Control and Optimization* 14 (6), 1009–1048
- Jeetendra, R. and Vivin, J. V (2012) ‘Stability Analysis of Uncertain Stochastic Systems with Interval Time-Varying Delays and Nonlinear Uncertainties via Augmented Lyapunov Functional.’ *Filomat* 26 (6), 1179–1188
- Jiang, Z.P. and Wang, Y. (2002) ‘A Converse Lyapunov Theorem for Discrete-Time Systems with Disturbances.’ *Systems and Control Letters* 45 (1), 49–58
- Kalman, R.E., Ho, Y.C., and Narendra, K.S. (1963) ‘Controllability of Linear Dynamical Systems.’ *Contr. Diff. Eqs.* 1, 189–213
- Kantorovich, L. V and Akilov, G.P. (1982) *Functional Analysis*. Oxford: Pergamon Press
- Kent, G.A. (1971) *Optimal Control of Functional Differential Equations of Neutral Type*. Brown University: USA
- Khargonekar, P.P., Petersen, I.R., and Zhou, K. (1990) ‘Robust Stabilization of Uncertain Linear Systems: Quadratic Stabilizability and  $H_\infty$  Control Theory.’ *IEEE Transactions on Automatic Control* 35 (3), 356–361
- Kharitonov, V.L. (1979) ‘Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations.’ *Differentsial’nie Uravneniya*, 14 (1978), pp. 2086–2088. Translation in *Differential Equations* 14, 1483–1485
- Khartovskii, V.E. (2012) ‘Complete Controllability Problem and Its Generalization for Linear Autonomous Systems of Neutral Type.’ *Journal of Computer and Systems Sciences International* 51 (6), 755–769
- Khartovskii, V.E. and Pavlovskaya, A.T. (2013) ‘Complete Controllability and Controllability for Linear Autonomous Systems of Neutral Type.’ *Automation and Remote Control* 74 (5), 769–784
- Kirk, D.E. (1970) *Optimal Control Theory: An Introduction*. USA: Prince-Hall, Inc.

- Klamka, J. (1999) 'Controllability of Semilinear Systems with Delays in Control.' In *IEEE International Conference on Systems, Man, and Cybernetics*. IEEE SMC'99 Conference Proceedings, 89–94
- Klamka, J. (2007) 'Stochastic Controllability of Linear Systems with Delay in Control.' *Bulletin of the Polish Academy of Sciences Technical Sciences* 55 (1), 23–29
- Knospe, C.R. and Roozbehani, M. (2006) 'Stability of Linear Systems With Interval Time Delays Excluding Zero.' *IEEE Transactions on Automatic Control* 51 (8), 1271–1288
- Knospe, C.R. and Roozbehani, M. (eds.) (2003) 'Stability of Linear Systems with Interval Time-Delay.' In *Proceedings of the 2003 American Control Conference*, 1458–1463
- Kofman, E. (2005) 'Non-Conservative Ultimate Bound Estimation in LTI Perturbed Systems.' *Automatica* 41, 1835–1838
- Kofman, E., Seron, M.M., and Haimovich, H. (2008) 'Control Design with Guaranteed Ultimate Bound for Perturbed Systems.' *Automatica* 44 (7), 1815–1821
- Kolmanovskii, V.B. and Myshkis, A. (1992) *Applied Theory of Functional Differential Equations*. Dordrecht, Boston: Kluwer Academic Publishers
- Kolmanovskii, V.B. and Nosov, V. (1986) *Stability of Functional Differential Equations*. Orlando, FL: Academic Press
- Korobov, A., Pavlichkov, S., and Schmidt, W. (2001) 'The Controllability Problem for Certain Nonlinear Integro-Differential Volterra Systems.' *Optimization* 50 (3-4), 155–186
- Krawcewicz, W., Wu, J., and Xia, H. (1993) 'Global Hopf Bifurcation Theory for Condensing Fields and Neutral Equations with Applications to Lossless Transmission Problems.' *Canadian Appl. Math. Quart.* 1 (2), 167–219
- Kuang, Y. (1993) *Delay Differential Equations: With Applications in Population Dynamics*. United Kingdom: Academic Press Limited
- Kundur, P., Paserba, J., Ajarapu, V., Andersson, G., Bose, A., Van Cutsem, T., Canizares, C., Hatziargyriou, N., Hill, D., Vittal, V., Stankovic, A., and Taylor, C. (2004) 'Definition and Classification of Power System Stability IEEE/CIGRE Joint Task Force on Stability Terms and Definitions.' *IEEE Transactions on Power Systems* 19 (3), 1387–1401
- Lewis, F.L. (1986) *Optimal Estimation: With an Introduction to Stochastic Control Theory*. New York, USA: John Wiley and Sons
- Li, H., Zhong, S., and Li, H. (2007) 'Some New Simple Stability Criteria of Linear Neutral Systems with a Single Delay.' *Journal of Computational and Applied Mathematics* 200, 441 – 447
- Li, L-M. (1988) 'Stability of Linear Neutral Delay-Differential Systems.' *Bull. Austral.Math.*

- Liao, X.X. and Mao, X. (2000) ‘Exponential Stability of Stochastic Delay Interval Systems.’ *Systems & Control Letters* 40 (3), 171–181
- Lien, C. (2006) ‘Guaranteed Cost Output Control for Uncertain Neutral Systems with Time-Varying Delays Via LMI.’ *International Journal of Systems Science* 37 (10), 723–734
- Lien, C.H., Yu, K.W., Chang, H.C., Chung, L.Y., and Chen, J.D. (2015) ‘Robust Reliable Guaranteed Cost Control for Uncertain T-S Fuzzy Neutral Systems with Interval Time-varying Delay and Linear Fractional Perturbations.’ *Optimal Control Applications and Methods* 36 (1), 121–137
- Liu, B. and Hill, D.J. (2009) ‘Input-to-State Stability for Discrete Time-Delay Systems via the Razumikhin Technique.’ *Systems & Control Letters* 58 (8), 567–575
- Liu, B., Liu, X., Teo, K.L., and Wang, Q. (2006) ‘Razumikhin-Type Theorems on Exponential Stability of Impulsive Delay Systems.’ *IMA Journal of Applied Mathematics* 71 (1), 47–61
- Liu, B. and Marquez, H.J. (2007) ‘Razumikhin-Type Stability Theorems for Discrete Delay Systems.’ *Automatica* 43 (7), 1219–1225
- Liu, J., Liu, X., and Xie, W.C. (2009) ‘Exponential Stability of Switched Stochastic Delay Systems with Non-Linear Uncertainties.’ *International Journal of Systems Science* 40 (6), 637–648
- Liu, P.-L. (2005) ‘On the Stability of Neutral – Type Uncertain Systems With Multiple Time Delays.’ *Int. J. Appl. Math. Comput. Sci* 15 (2), 221–229
- Liu, X. and Ballinger, G. (2001) ‘Uniform Asymptotic Stability of Impulsive Delay Differential Equations.’ *Computers & Mathematics with Applications* 41 (7), 903–915
- Liu, X. and Wang, Q. (2007) ‘The Method of Lyapunov Functionals and Exponential Stability of Impulsive Systems with Time Delay.’ *Nonlinear Analysis, Theory, Methods and Applications* 66 (7), 1465–1484
- Liu, X., Wu, M., Martin, R., and Tang, M. (2007) ‘Stability Analysis for Neutral Systems with Mixed Delays.’ *Journal of Computational and Applied Mathematics* 202 (2), 478–497
- Liu, X. and Ye, J. (2011) ‘Heuristic Algorithm Based on the Principle of Minimum Total Potential Energy (HAPE): A New Algorithm for Nesting Problems.’ *Journal of Zhejiang University Science A* 12 (11), 860–872
- Liu, X., Zhong, S., and Ding, X. (2012) ‘Robust Exponential Stability of Nonlinear Impulsive Switched Systems with Time-Varying Delays.’ *Nonlinear Analysis Modelling and Control* 17 (2), 210–222
- Lopes, O. (1975) ‘Forced Oscillations in Nonlinear Neutral Differential Equations.’ *SIAM J.*

*Appl. Math.* 29 (1), 196–207

- Lopes, O. (1976) ‘Stability and Forced Oscillations.’ *Journal of Mathematical Analysis and Applications* 55, 686–698
- Lu, S. and Ge, W. (2004) ‘Existence of Positive Periodic Solutions for Neutral Population Model with Multiple Delays.’ *Journal of Computational and Applied Mathematics and Computation* 166, 371–383
- Lyapunov, A.M. (1967) *Collected Works of Academician* [online] Volume 1. Translation Division, Foreign Technology Division. Available from  
<[https://books.google.co.uk/books/about/Collected\\_works\\_of\\_Academican\\_A\\_M\\_Lyapun.html?id=Gb9UAAAAYAAJ&pgis=1](https://books.google.co.uk/books/about/Collected_works_of_Academican_A_M_Lyapun.html?id=Gb9UAAAAYAAJ&pgis=1)> [18 December 2015]
- Mao, X. (1996) ‘Razumikhin-Type Theorems on Exponential Stability of Stochastic Functional Differential Equations.’ *Stochastic Processes and their Applications* 65 (2), 233–250
- Mao, X., Rodkina, A., and Koroleva, N. (1998) ‘Razumikhin-Type Theorems for Neutral Stochastic Functional Differential Equations.’ *Functional Differential Equations* 5 (1-2), 195–211
- Metel’skii, A. V and Minyuk, S.A. (2007) ‘Complete Controllability and Complete Constructive Identifiability of Completely Regular Differential-Algebraic Delay Systems.’ *Journal of Differential Equations* 43 (3), 311–327
- Metel’skii, A. V and Minyuk, S.A. (2008) ‘Almost Complete Controllability of Linear Autonomous Difference-Differential Systems of Neutral Type.’ *Journal of Differential Equations* 44 (11), 1605–1617
- Mordukhovich, B.S. and Wang, L. (2004) ‘Optimal Control of Neutral Functional-Differential Inclusions Linear in Velocities.’ *TEMA Tend. Mat. Apl. Comput.* 5 (1), 1–15
- Morozov, A. and Petrovskii, S. (2009) ‘Excitable Population Dynamics, Biological Control Failure, and Spatiotemporal Pattern Formation in a Model Ecosystem.’ *Bulletin of Mathematical Biology* 71 (4), 863–887
- Murakami, S. (1984) ‘Stability in Functional-Differential Equations with Infinite Delay.’ *Tohoku Mathematical Journal* 36, 561–570
- Myshkis, A.D. (1995) ‘Razumikhin’s Method in the Qualitative Theory of Processes with Delay.’ *Journal of Applied Mathematics and Stochastic Analysis* 8 (3), 233–247
- Naghshabrizi, P., Hespanha, J.P., and Teel, A.R. (2008) ‘Exponential Stability of Impulsive Systems with Application to Uncertain Sampled-Data Systems.’ *Systems & Control Letters* 57 (5), 378–385
- Nagumo, J. and Shimura, M. (1961) ‘Self-Oscillation in a Transmission Line with a Tunnel Diode.’ *Proceedings of the IRE* (12), 1281–1291



- Neustadt, L.W. (1961) 'Time Optimal Control Systems with Position and Integral Limits.' *Journal of Mathematical Analysis and Applications* 3 (3), 406–427
- Neustadt, L.W. (1969) 'A General Theory of Extremals.' *Journal of Computer and System Sciences* 3, 57–92
- Niculescu, S-I. (2001) *Delay Effects on Stability: A Robust Control Approach*. Berlin: Springer Science & Business Media
- Onwuatu, J.U. (1984) 'On the Null-Controllability in Function Space of Nonlinear Systems of Neutral Functional Differential Equations with Limited Controls.' *Journal of Optimization Theory and Applications* 42 (3), 397–420
- Onwuatu, J.U. (1993) 'Null Controllability of Nonlinear Infinite Neutral System.' *Kybernetika* 29 (4), 325–336
- Orta, R. (2012) *Lecture Notes on Transmission Line Theory*. 1–140. Available from <<http://personal.delen.polito.it/Renato.Orta/PassOpticalComp/TransmissionLinesLectureNotesNov2012.pdf>>
- Park, J.H. (2003) 'Robust Guaranteed Cost Control for Uncertain Linear Differential Systems of Neutral Type.' *Applied Mathematics and Computation* 140, 523–535
- Park, J.H. and Won, S. (2000) 'Stability Analysis for Neutral Delay-Differential Systems.' *Journal of the Franklin Institute* 337, 1–9
- Park, J.Y. and Kang, Y.H. (2001) 'Optimal Problem of Cost Function for the Linear Neutral Systems.' *International Journal of Mathematics and Mathematical Sciences* 25 (12), 777–785
- Payam, N.S. and Mansour, V. (2014) 'Positive Periodic Solution for a Nonlinear Neutral Delay Population Equation with Feedback Control.' *Journal of Nonlinear Science and Applications* 7, 218–228
- Pecora, L.M. and Carroll, T.L. (1990) 'Synchronization in Chaotic Systems.' *Physical review letters* 64 (8), 821–824
- Pecora, L.M. and Carroll, T.L. (1991) 'Driving Systems with Chaotic Signals.' *Physical Review A* 44 (4), 2374–2383
- Rivera Rodas, H. and Langenhop, C.E. (1978) 'A Sufficient Condition for Function Space Controllability of a Linear Neutral System.' *SIAM Journal on Control and Optimization* 16 (3), 429–435
- Sakthivel, R., Mathiyalagan, K., and Anthoni, S.M. (2012) 'Robust Stability and Control for Uncertain Neutral Time Delay Systems.' *International Journal of Control* 85 (4), 373–383
- Salamon, D. (1983) 'Observability of Nontrivial Small Solutions for Neutral Systems.' *Systems & Control Letters* 3 (2), 91–99

- Saldivar Marquez, B.M., Boussaada, I., Mounier, H., and Niculescu, S-I. (2015) *Analysis and Control of Oilwell Drilling Vibrations: A Time-Delay Systems Approach*. Switzerland: Springer International Publishing
- Shen, J. and Yan, J. (1998) 'Razumikhin Type Stability Theorems for Impulsive Functional Differential Equations.' *Nonlinear Analysis* 33 (5), 519–531
- Shimura, M. (1967) 'Analysis of Some Nonlinear Phenomena in a Transmission Line.' *IEEE Transactions on Circuit Theory* 14 (1), 60–68
- Sikora, B. (2003) 'On the Constrained Controllability of Dynamical Systems with Multiple Delays in the State.' *International Journal of Applied Mathematics and Computer Science* 13 (4), 469–480
- Sinha, A.S.C. (1985) 'Null-Controllability of Non-Linear Infinite Delay Systems with Restrained Controls.' *International Journal of Control* 42 (3), 735–741
- Sinha, A.S.C. and Yokomoto, C. (1980) 'Null Controllability of a Nonlinear System with Variable Time Delay.' *IEEE Transactions on Automatic Control* 25 (6), 1234–1236
- Sivasundaram, S. and Uvah, J. (2008) 'Controllability of Impulsive Hybrid Integro-Differential Systems.' *Nonlinear Analysis: Hybrid Systems* 2 (4), 1003–1009
- Slemrod, M. (1971) 'Nonexistence of Oscillations in a Nonlinear Distributed Network.' *Journal of Mathematical Analysis and Applications* 36, 22–40
- Soliman, H.M., Dabroum, A., Mahmoud, M.S., and Soliman, M. (2011) 'Guaranteed-Cost Reliable Control with Regional Pole Placement of a Power System.' *Journal of the Franklin Institute* 348 (5), 884–898
- Stamova, I.M. and Stamov, G.T. (2001) 'Lyapunov – Razumikhin Method for Impulsive Functional Differential Equations and Applications to the Population Dynamics.' *Journal of Computational and Applied Mathematics* 130, 163–171
- Su, T-J. and Huang, C-G. (1992) 'Robust Stability of Delay Dependence for Linear Uncertain Systems.' *IEEE Transactions on Automatic Control* 37 (10), 1656–1659
- Sun, X., Yan, L., and Cui, J. (2013) 'Controllability of Fractional Neutral Stochastic Integro-Differential Systems with Infinite Delay.' *Mathematical Problems in Engineering* 2013, 1–10
- Tai, Z., Wang, X., and Wang, D. (2009) 'Robust Stability Analysis of Uncertain Neutral Functional Systems.' *International Journal of Innovative Computing, Information and Control* 5 (10(A)), 3013–3024
- Tang, M.L. and Tang, X.H. (2012) 'Positive Almost Periodic Solutions for Neural Multi-Delay Logarithmic Population Model.' *Journal of Inequalities and Applications* 2012 (1), 1–9
- Tsung-Lieh, H. and Chien-Hua, L. (1995) 'Exponential Stability of Discrete Time Uncertain

- Systems with Time-Varying Delay.’ *Journal of the Franklin Institute* 332 (4), 479–489
- Underwood, G.R. and Chukwu, E.N. (1988) ‘Null Controllability Neutral Differential of Nonlinear Equations.’ *Journal of Mathematical Analysis and Applications* 129 (2), 326–345
- Vemuri, V. (1978) *Modeling of Complex Systems: An Introduction*. New York, USA: Academic Press
- Wei, G., Wang, Z., Shu, H., and Fang, J. (2007) ‘Delay-Dependent Stabilization of Stochastic Interval Delay Systems with Nonlinear Disturbances.’ *Systems and Control Letters* 56, 623–633
- Williams, N. (2008) *Robust Control An Entry for the New Palgrave* [online] 2nd Editio. New York: Palgrave Macmillan. Available from <<http://www.ulb.tu-darmstadt.de/tocs/103070079.pdf>>
- Winful, H. and Rahman, L. (1990) ‘Synchronized Chaos and Spatiotemporal Chaos in Arrays of Coupled Lasers.’ *Physical Review Letters* 65 (13), 1575–1578
- Wu, J. and Xia, H. (1996) ‘Self-Sustained Oscillations in a Ring Array of Coupled Lossless Transmission Lines.’ *Journal of Differential Equations* 124 (1), 247–278
- Wu, J. and Xia, H. (1999) ‘Rotating Waves in Neutral Partial Functional Differential Equations.’ *Journal of Dynamics and Differential Equations* 11 (2), 209–238
- Xu, S., Lam, J., Yang, C., and Verriest, E.I. (2003) ‘An LMI Approach to Guaranteed Cost Control for Uncertain Linear Neutral Delay Systems.’ *International Journal of Robust and Nonlinear Control* 13 (1), 35–53
- Yu, Z. (2013) ‘Razumikhin-Type Theorem and Mean Square Asymptotic Behavior of the Backward Euler Method for Neutral Stochastic Pantograph Equations.’ *Journal of Inequalities and Applications* 299, 1–15
- Zames, G. (1981) ‘Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses.’ *IEEE Transactions on Automatic Control* 26 (2), 301–320
- Zhang, H.R., Knospe, C.R., and Tsiotras, P. (2003) ‘New Results for the Analysis of Linear Systems with Time-Invariant Delays.’ *International Journal Of Robust And Nonlinear Control* 13 (12), 1149–1175
- Zhang, J., Knopse, C.R., and Tsiotras, P. (2001) ‘Stability of Time-Delay Systems: Equivalence between Lyapunov and Scaled Small-Gain Conditions.’ *IEEE Transactions on Automatic Control* 46 (3), 482–486
- Zhang, S. and Chen, M. (1998) ‘A New Razumikhin Theorem for Delay Difference Equations.’ *Computers & Mathematics with Applications* 36 (10), 405–412
- Zhihong, H., Jiankun, L., and Yan, R. (2012) ‘The Application of Delay Control Theory in

Electric Power Transmission System.’ In *IEEE 2012, 24th Chinese Control and Decision Conference (CCDC)*. 2339–2341

# Appendix I

Code for example using Lyapunov's approach in Section 4.5.2 of Chapter 4

```
%% MATRIX VALUES
```

```
A=[-1,0;0,-1]
```

```
C=[0,0.8448;0.8448,0]
```

```
D=[0,0.4;0.4,0]
```

```
I=[1,0;0,1]
```

```
L=-0.02489
```

```
%% INITIALIZING
```

```
setlmis([]);
```

```
X=lmivar(1,[2 0]);
```

```
R=lmivar(1,[2 0]);
```

```
U=lmivar(1,[1 0]);
```

```
V=lmivar(1,[1 0]);
```

```
W=lmivar(1,[1 0]);
```

```
%% Main Coding
```

```
lmiterm([1 1 1 X],1,A,'s');
```

```
lmiterm([1 1 1 X],2*L,-1);
```

```
lmiterm([1 2 1 X],A,1);
```

```
lmiterm([1 2 2 0],-I);
```

```
lmiterm([1 3 1 X],1,L);
```

```
lmiterm([1 3 3 0],-I);
```

```
lmiterm([1 4 1 U],1,A*X);
```

```
lmiterm([1 4 4 U],1,-I);
```

```
lmiterm([1 5 1 X],1,L);
```

```
lmiterm([1 5 5 U],1,-I);
```

```
lmiterm([1 6 1 X],1,L);
```

```
lmiterm([1 6 6 V],1,-I);
```

```
lmiterm([1 7 1 X],1,L);
```

```
lmiterm([1 7 7 W],1,-I);
```

```
lmiterm([1 8 1 -R],1,X);
```

```
lmiterm([1 8 8 R],1,-1);
```

```
lmiterm([1 9 1 X],A*C',1);
```

```
lmiterm([1 9 1 0],C');
```

```
lmiterm([1 9 9 V],1,C*C);
```

```
lmiterm([1 9 9 R],1,-1);
```

```
lmiterm([1 9 9 0],C*C);
```

```
lmiterm([1 10 1 0],D'+A*D');
```

```
lmiterm([1 10 9 0],C*D');
```

```
lmiterm([1 10 10 W],1,D'*D);
```

```
lmiterm([1 10 10 0],D*D-I);
```

```
% LMI #1: X*A'+A*X
```

```
% LMI #1: -2*L*X (NON SYMMETRIC?)
```

```
% LMI #1: A*X
```

```
% LMI #1: -I
```

```
% LMI #1: X*L
```

```
% LMI #1: -I
```

```
% LMI #1: U*A*X
```

```
% LMI #1: -U*I (NON SYMMETRIC?)
```

```
% LMI #1: X*L
```

```
% LMI #1: -U*I (NON SYMMETRIC?)
```

```
% LMI #1: X*L
```

```
% LMI #1: -V*I (NON SYMMETRIC?)
```

```
% LMI #1: X*L
```

```
% LMI #1: -W*I (NON SYMMETRIC?)
```

```
% LMI #1: R'*X
```

```
% LMI #1: -R
```

```
% LMI #1: A*C'*X
```

```
% LMI #1: C'
```

```
% LMI #1: V*C'*C (NON SYMMETRIC?)
```

```
% LMI #1: -R
```

```
% LMI #1: C'*C
```

```
% LMI #1: D'+A*D'
```

```
% LMI #1: C*D'
```

```
% LMI #1: W*D'*D (NON SYMMETRIC?)
```

```
% LMI #1: D*D-I
```

```

lmiterm([-2 1 1 X],1,1);          % LMI #2: X

lmiterm([-3 1 1 R],1,1);          % LMI #3: R

lmiterm([-4 1 1 U],1,1);          % LMI #4: U

lmiterm([-5 1 1 V],1,1);          % LMI #5: V

lmiterm([-6 1 1 W],1,1);          % LMI #6: W

RRR=getlmis;

%% TEST FEASIBILITY
[tmin,Xfeas]=feasp(RRR)
P=dec2mat(RRR,Xfeas,X)
R=dec2mat(RRR,Xfeas,R)
r0=dec2mat(RRR,Xfeas,U)
r1=dec2mat(RRR,Xfeas,V)
r2=dec2mat(RRR,Xfeas,W)

%% OUTPUT of LMI

Solver for LMI feasibility problems  $L(x) < R(x)$ 
This solver minimizes  $t$  subject to  $L(x) < R(x) + t*I$ 
The best value of  $t$  should be negative for feasibility
Iteration : Best value of  $t$  so far

1          0.669788
2          0.240455
3          0.240455
4          0.240455
5          0.079776
6          0.079776
***        new lower bound: -0.167671
7          0.020053
***        new lower bound: -0.074947
8          0.020053
9          6.207959e-03
***        new lower bound: -0.042371
10         6.207959e-03
***        new lower bound: -0.012742
11         4.829127e-03
12         2.399568e-03
13         1.344639e-03
***        new lower bound: -0.010126
14         9.667752e-04
***        new lower bound: -7.973276e-03
15         4.087294e-04

```

```

***      new lower bound: -6.243150e-03
  16      4.087294e-04
***      new lower bound: -4.863177e-03
  17      4.087294e-04
***      new lower bound: -1.303494e-03
  18      1.850116e-04
  19      1.850116e-04
***      new lower bound: -3.688659e-04
  20      9.026033e-05
  21      6.525857e-05
***      new lower bound: -8.434159e-05
  22      2.507302e-05
  23      1.047373e-05
***      new lower bound: -2.201073e-05
  24      4.971172e-06
  25      4.971172e-06
***      new lower bound: -1.964643e-05
  26      2.370210e-06
***      new lower bound: -5.500865e-06
  27      -9.162452e-07

```

Result: best value of t: -9.162452e-07  
 f-radius saturation: 0.000% of R = 1.00e+09

tmin = -9.1625e-07

Xfeas =

```

1.0000
0.8746
0.0249
0.0294
0.1500

```

P =

```

1.0000    0
0    1.0000

```

R =

```

0.8746    0
0    0.8746

```

r0 = 0.0249

r1 = 0.0294

r2 = 0.1500

# Appendix II

Code for example on stabilisation in Section 5.5.2 of Chapter 5

```
%% MATRIX VALUES
A=[-1,0;0,-2]
C=[-1,0;1,-2]
D=[0.5,0;0,0.5]
I=[1,0;0,1]
L=-0.02489
B=[1,0;0,1]
%% INITIALIZING
setlmis([]);
X=lmivar(1,[2 0]);
R=lmivar(1,[2 0]);
U=lmivar(1,[1 0]);
V=lmivar(1,[1 0]);
W=lmivar(1,[1 0]);

%% Main Coding
lmiterm([1 1 1 X],1,A,'s');           % LMI #1:  $X^*A+A^*X$ 
lmiterm([1 1 1 X],L,-1,'s');          % LMI #1:  $-2^*L^*X$  (NON SYMMETRIC?)
lmiterm([1 1 1 X],1,-A'*B*B','s');    % LMI #1:  $-2^*X^*A^*B^*B'$  (NON SYMMETRIC?)
lmiterm([1 1 1 X],-B*B'*L,1,'s');    % LMI #1:  $2^*B^*B^*L^*X$  (NON SYMMETRIC?)
lmiterm([1 1 1 0],-2*B*B','s');       % LMI #1:  $-2^*B^*B'$ 
lmiterm([1 2 1 X],A,1);               % LMI #1:  $A^*X$ 
lmiterm([1 2 2 0],-I);                % LMI #1:  $-I$ 
lmiterm([1 3 1 0],B*B');              % LMI #1:  $B^*B'$ 
lmiterm([1 3 3 0],-I);                % LMI #1:  $-I$ 
lmiterm([1 4 1 U],1,A*X);              % LMI #1:  $U^*A^*X$ 
lmiterm([1 4 4 U],1,-I);              % LMI #1:  $-U^*I$  (NON SYMMETRIC?)
lmiterm([1 5 1 X],R,1);               % LMI #1:  $R^*X$ : a PLACE OF DIFFERENCE
lmiterm([1 5 5 R],1,-1);              % LMI #1:  $-R$ 
lmiterm([1 6 1 X],1,L);               % LMI #1:  $X^*L$ 
lmiterm([1 6 6 0],-I);                % LMI #1:  $-I$ 
lmiterm([1 7 1 X],1,L);               % LMI #1:  $X^*L$ 
lmiterm([1 7 7 U],1,-I);              % LMI #1:  $-U^*I$  (NON SYMMETRIC?)
lmiterm([1 8 1 X],1,L);               % LMI #1:  $X^*L$ 
lmiterm([1 8 8 V],1,-I);              % LMI #1:  $-V^*I$  (NON SYMMETRIC?)
lmiterm([1 9 1 X],1,L);               % LMI #1:  $X^*L$ 
lmiterm([1 9 9 W],1,-I);              % LMI #1:  $-W^*I$  (NON SYMMETRIC?)
lmiterm([1 10 1 X],C*A,1);             % LMI #1:  $C^*A^*X$ 
lmiterm([1 10 1 0],C'-C'*B*B');       % LMI #1:  $C'-C^*B^*B'$ 
lmiterm([1 10 10 V],1,C'*C);          % LMI #1:  $V^*C^*C$  (NON SYMMETRIC?)
lmiterm([1 10 10 R],1,-1);            % LMI #1:  $-R$ 
```



```

lmiterm([1 10 10 0],C'*C);           % LMI #1: C'*C
lmiterm([1 11 1 X],D'*A,1);          % LMI #1: D'*A*X
lmiterm([1 11 1 0],D'-D'*B*B');      % LMI #1: D'-D'*B*B'
lmiterm([1 11 10 0],D'*C);           % LMI #1: D'*C
lmiterm([1 11 11 W],1,D'*D);          % LMI #1: W*D'*D (NON SYMMETRIC?)
lmiterm([1 11 11 0],D'*D-I);         % LMI #1: D'*D-I

lmiterm([-2 1 1 X],1,1);              % LMI #2: X

lmiterm([-3 1 1 R],1,1);              % LMI #3: R

lmiterm([-4 1 1 U],1,1);              % LMI #4: U

lmiterm([-5 1 1 V],1,1);              % LMI #5: V

lmiterm([-6 1 1 W],1,1);              % LMI #6: W

RRR=getlmis;

%% TEST FEASIBILITY
[tmin,Xfeas]=feasp(RRR)
X=dec2mat(RRR,Xfeas,X)
R=dec2mat(RRR,Xfeas,R)
r4=dec2mat(RRR,Xfeas,U)
r5=dec2mat(RRR,Xfeas,V)
r6=dec2mat(RRR,Xfeas,W)

%% OUTPUT of LMI

Solver for LMI feasibility problems  $L(x) < R(x)$ 
This solver minimizes  $t$  subject to  $L(x) < R(x) + t*I$ 
The best value of  $t$  should be negative for feasibility

Iteration : Best value of  $t$  so far

1          0.981564
2          0.782327
3          0.350122
4          0.145377
5          0.145377
6          0.052453
7          0.052453
8          -0.012032

Result: best value of  $t$ : -0.012032
f-radius saturation: 0.000% of  $R = 1.00e+09$ 

tmin = -0.0120

```

Xfeas =

0.1449  
383.8617  
0.1849  
54.6079  
1.6422

X =

0.1449      0  
0   0.1449

R =

383.8617      0  
0 383.8617

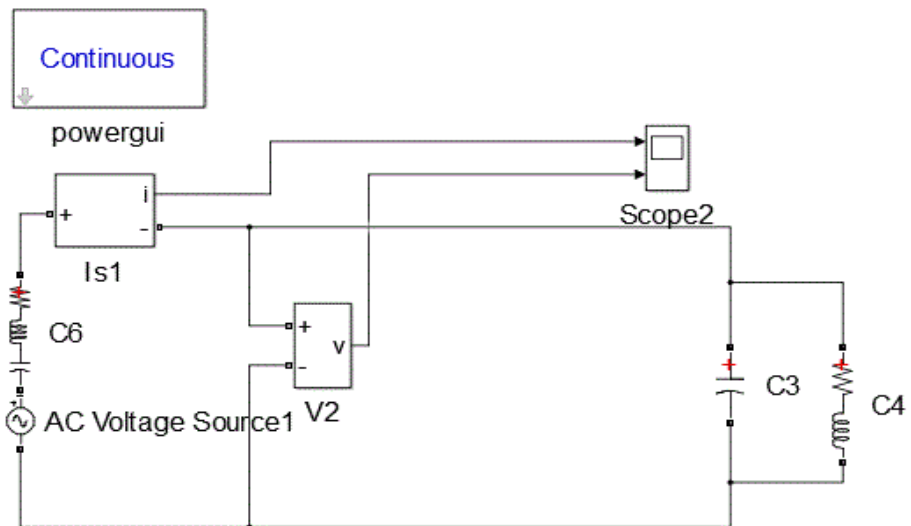
r4 = 0.1849

r5 = 54.6079

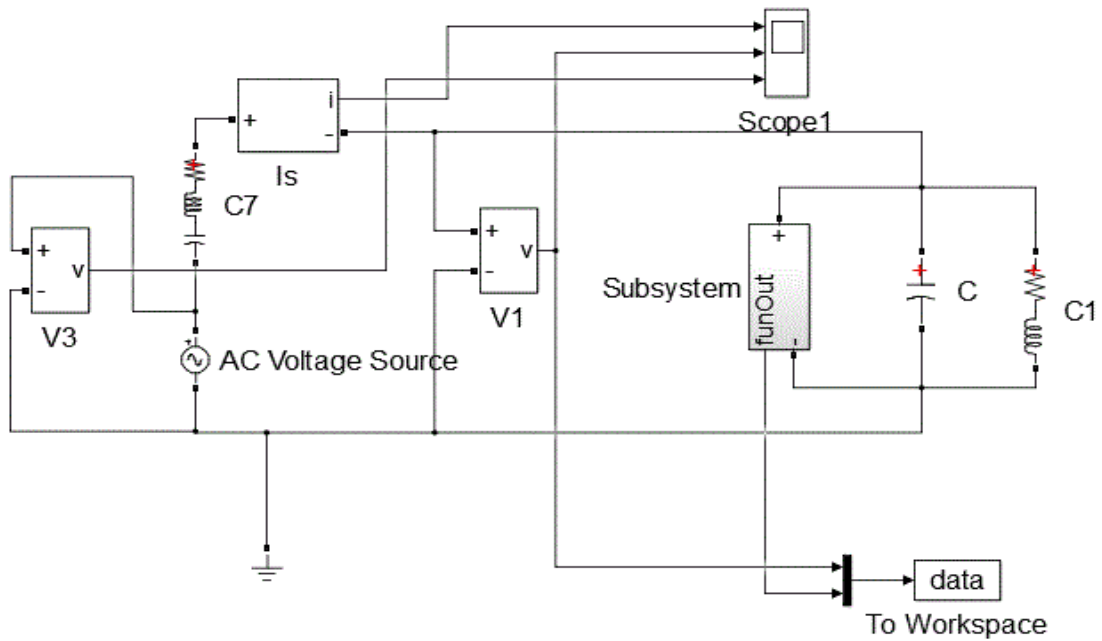
r6 = 1.6422

# Appendix III

Simulink model set-up for single transmissions discussed in Section 7.4 of Chapter 7



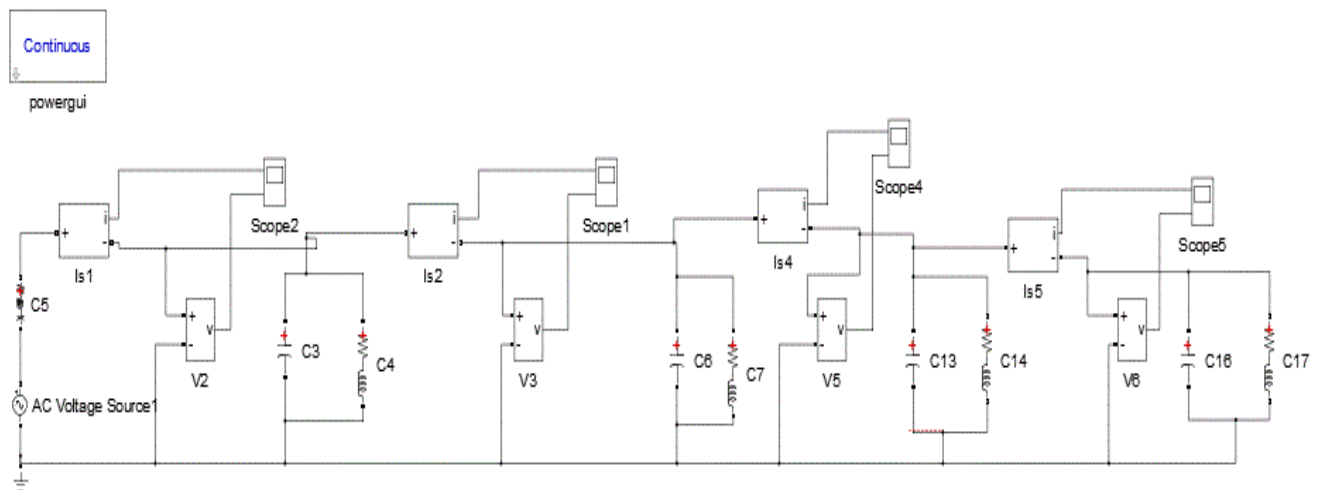
III1: Simulink model for single transmission line without nonlinear function



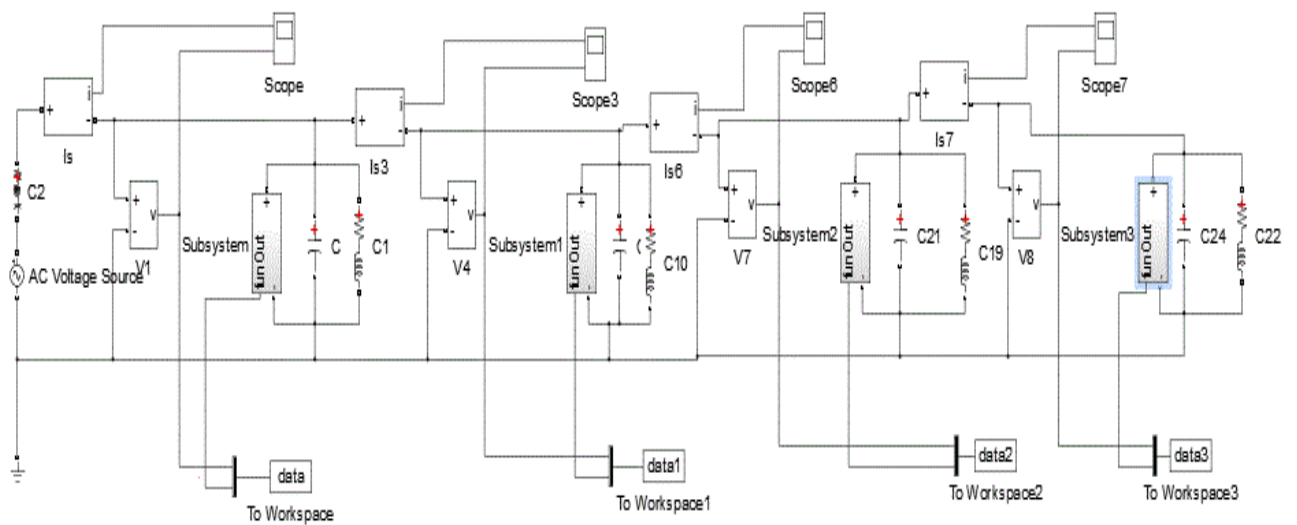
III 2: Simulink model for single transmission line with nonlinear function

# Appendix IV

Simulink model set-up for interconnected transmission lines discussed in Section 7.4 of Chapter 7



IV1: Simulink model for interconnected transmission lines without nonlinear functions



IV2: Simulink model for interconnected transmission lines with nonlinear functions

# Appendix V

Obtaining  $\gamma_0$  from the definition of  $G_k(\cdot)$  using Lemma 4.4 in Section 4.2 of Chapter 4, and the resistance of the interconnection  $R = 0.01$  with the following system matrices

$$A_0 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -7.96 & -108.12 \\ 42.12 & -210.52 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1.2 & 0 \\ -6.32 & 0 \end{pmatrix},$$

gives

$$\det \Delta(\lambda) = \begin{pmatrix} \lambda - 0.6\lambda \exp(-\lambda h) + 7.96 + 1.2 \exp(-\lambda h) & 108.12 \\ -42.12 + 6.32 \exp(-\lambda h) & \lambda + 210.52 \end{pmatrix} - 48 = 0.$$

Since oscillations appears at the imaginary axis when an eigenvalue from the characteristic equation crosses it, by setting  $\lambda = iw$  as the condition on  $\Delta(\lambda)$  under which an imaginary eigen-vector exists and substituting this into  $\Delta(\lambda)$  above gives

$$0.4 w^4 - 93.368iw^3 - 5799.0592w^2 - 48 = 0.$$

Using MATLAB in solving the above polynomial gives

$$w_1 = -0.09i; \quad w_2 = 29.60 + 116.71i; \quad w_3 = -29.60 + 116.71i; \quad w_4 = 0.09i.$$

Now, since only one pair of roots can cross the imaginary axis for the first time before oscillation begins, choosing  $w_2$  as a critical value and finding its eigen-vector gives

$$x = \begin{pmatrix} 11870 + 678.43i \\ 4130.2 - 1059.7i \end{pmatrix}.$$

Then the general solution for the system is obtained as

$$\begin{aligned}
v = & C_1 \exp(29.60t) \begin{pmatrix} 11870 \cos(116.71t) - 678.43 \sin(116.71t) \\ 4130.2 \cos(116.71t) + 1059.7 \sin(116.71t) \end{pmatrix} \\
& + C_2 \exp(29.60t) \begin{pmatrix} 11870 \sin(116.71t) + 678.43 \cos(116.71t) \\ 4130.2 \sin(116.71t) - 1059.7 \cos(116.71t) \end{pmatrix},
\end{aligned}$$

and the constants are obtained as  $C_1 = 0.0591$  and  $C_2 = 0.0252$ . Substituting this in  $G_k(\cdot)$  using assumption (iii) in Section 4.2 gives

$$\left( \frac{C_1}{R} + rC_2 \right) \int_{-\infty}^0 \exp(29.60s) ds < \infty,$$

so that  $\gamma_0 = 0.1998$ .