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A mathematically reduced approach to predictive control of perishable inventory systems

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A MATHEMATICALLY REDUCED APPROACH TO PREDICTIVE CONTROL OF PERISHABLE INVENTORY SYSTEMS

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ABSTRACT

The design and optimisation of inventory replenishment systems has already been exhaustively studied by the operational research community. Many classical mathematical methods and simulation techniques have been developed and introduced in the literature. However, what can be observed is the fact that in a real case scenario the lead-time, deterioration of goods and demand for product are likely to be time-varying and uncertain, which traditionally have not necessarily been reflected in the model formulations. Therefore, in response to the dynamical nature of inventory systems, the potential of algorithms based on control theory to reduce the undesirable influences of system uncertainties on inventory level stability, have been investigated /proposed. Consequently, the mapping of the inventory problem into the control theory domain, for cost-benefit inventory trade-off achievement has been realised. Although, the application of control theory in inventory optimisation appears to be beneficial, there are certain reasons why the approach has gained yet little attention among the operational research community. One reason is that it cannot be adopted easily by researchers who are unfamiliar with control theory and another is due to a communication gap which exists between the control theory and operational research communities. Prompted by these observations, the thesis presents a novel, systematic mathematical approach for finding the optimal order quantities. The proposed approach has been mathematically demonstrated to be equivalent in study-sate to model-based predictive control, which is one of the more well-established productive control techniques with industrial application today. The mathematically reduced approach attempts to bridge the identified gap to fulfil the lacking dual perceptions of both communities. It enables the straightforward benefits afforded by predictive control without the necessity to become familiarised with principles of control theory. The method is shown to be applicable for both perishable and non-perishable inventory. Although the novel technique was inspired by MPC and noticing the MPC patterns in the mathematical description, the resulting proposal is no longer MPC. It is in fact a minimum variance approach, or dear beat controller, with an incorporated Smith predictor. Therefore using the adjective ‘predictive’ in the title of the thesis refers to both, the inspiration of MPC and the predictive nature of the minimum variance controller to accommodate lead time, being incorporated within an inherent Smith predictor. The developed approach is considered to be transferable to other applications, where similar model formulations may be applicable.

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TABLE OF ABBREVIATIONS

Abbreviation	Full Name
ARMA	Autoregressive Moving Average
ARX	Autoregressive with Exogenous Input
CT	Control Theory
EOQ	Economic Order Quantity
EPIOBPCS	Estimated Pipeline Inventory and Order Based Production Control System
FIR	Finite Impulse Response
GA	Genetic Algorithm
IC	Inventory Control
IMPC	Inventory Model Predictive Control
IOBPCS	Inventory and Order Based Production Control System
JIT	Just In Time
LS	Least Square
MPC	Model Predictive Control
OC	Operational Control
OR	Operational Research
PUrIC	Inventory Control with tuning parameter for perishable goods
RLS	Recursive Least Square
UrIC	Inventory Control with tuning parameter for non-perishable goods
WIP	Work in Progress

TABLE OF SYMBOLS

Symbol	Definition	Unit / domain
α		$\in\left(0,\frac{1}{I_{\max}}\right)$
a_1	Estimated value of $K(n+1)$ in UrIC controller	$\in\mathbb{R}$
a_2	Estimated value of $K(n)$ in UrIC controller	$\in\mathbb{R}$
$\boldsymbol{A}$	Open loop system state matrix	$\in\mathbb{R}^n\times\mathbb{R}^n$
$\boldsymbol{A_c}$	Closed loop system state matrix	$\in\mathbb{R}^{(n+1)\times(n+1)}$
β	Price discount	[£]
$\boldsymbol{b}$	Open loop system input matrix	$\in\mathbb{R}^n$
$\boldsymbol{b_c}$	Closed loop system input matrix	$\in\mathbb{R}^{n+1}$
$\boldsymbol{c}$	Open loop system output matrix	$\in\mathbb{R}^n$
$\boldsymbol{c_c}$	Closed loop system output matrix	$\in\mathbb{R}^{n+1}$
$C(s)$	Closed loop transfer function	-
$C_{SP}(s)$	Smith predictor transfer function	-
δ	Deterioration rate	$\in[0,1]$
d	Disturbance / demand	[items]
d_{Tot}	Total demand	[items]
$\boldsymbol{e}$	Error vector	$\in\mathbb{R}^n$
$\boldsymbol{F}$	MPC prediction vector	$\in\mathbb{R}^{N_p}$
$\boldsymbol{f}$	Dead-beat controller gain	$\in\mathbb{R}^n$
F_{prof}	Profit function	-
$G(s)$	Warehouse transfer function	-
H_c	Holding unit costs	[£]

$\boldsymbol{I}$	Identity matrix	$\mathbb{R}^n \times \mathbb{R}^n$
I	Current inventory level	[items]
I_L	Total number of backorders	[items]
$I_{\max}$	Maximum inventory level	$\in \mathbb{R}$
I_R	Reference inventory	[items]
I_u	Total number of stored goods	[items]
k	Time instance	$\in \mathbb{N}$
K	Positive constant	$\in \mathbb{R}_+$
$\boldsymbol{K}$	MPC and IC gain vector	$\in \mathbb{R}^{n+1}$
$\boldsymbol{K}_y$	Last element of MPC and IC gain vector	$\in \mathbb{R}$
$\boldsymbol{K}_x$	MPC and IC gain vector apart from last element of	$\in \mathbb{R}^n$
L	Lead time	[days]
m	Row number	$\in \mathbb{N}$
n	Lead time / system order	$\in \mathbb{N}$
N_c	Control horizon	$\in \mathbb{N}$
N_p	Prediction horizon	$\in \mathbb{N}$
$\boldsymbol{\Phi}$	MPC prediction matrix	$\in \mathbb{R}^{N_p - N_c} \times \mathbb{R}^{N_p}$
P	Price	[£]
p	exponent	$\in \mathbb{N}$
p_n	exponent	$\in \mathbb{N}$
$\boldsymbol{R}$	MPC tuning matrix	$\in \mathbb{R}^{N_p - N_c} \times \mathbb{R}^{N_p - N_c}$
r	Column number	$\in \mathbb{N}$
$\boldsymbol{Q}$	Demand incremental vector	$\in \mathbb{R}^{N_c}$
q	Demand incremental scalar	$\in \mathbb{R}$
Θ_m	m^{th} estimate of RLS algorithm	$\in \mathbb{R}^9$
u	Input signal / order size	[items]
$\Delta\boldsymbol{U}$	Control trajectory vector	$\in \mathbb{R}^{N_c}$
u_r	Tuning parameter	$\in \mathbb{R}_+$

u_{Tot}	Total number of orders	[items]
$\boldsymbol{x}$	Open loop state vector	$\in \mathbb{R}^{n}$
$\boldsymbol{x_c}$	Closed loop state vector	$\in \mathbb{R}^{n}$ or $\in \mathbb{R}^{n+1}$
x_r	Reference state vector	$\in \mathbb{R}^{n}$
$\boldsymbol{Y}$	Predicted inventory signal	$\in \mathbb{R}^{N_p}$
y	Current inventory level	[items]
$\boldsymbol{Y_R}$	Target inventory level signal	$\in \mathbb{R}^{N_p}$
$\overline{\boldsymbol{Y_R}}$	Empty vector	$\in \mathbb{R}^{N_p}$
y_R	Reference inventory level	[items]
Ψ	Plane points values matrix	$\in \mathbb{R}^{9\times 10}$

1 INTRODUCTION

1.1 Introduction to logistics

In the context of this thesis, logistics is understood as a discipline of science that develops strategies for appropriate motion and allocation of goods. It assures that all non-human resources are always in the right place at the right time. The origin of logistics science can be recognised in the times of World War Two when the need for appropriate allocation of military resources was crucial to accomplish military operations successfully (Rushton, Croucher and Baker, 2006). Nowadays the term logistics is usually identified with the distribution of goods within a supply chain. Starting from the raw materials needed for production, through to the components needed for assembly, up until the finished goods reaching the end customer. All the goods need to be properly distributed so that they reach the appropriate supply chain node at the right time. Although, logistics does not include production, the term logistics is often substituted by supply chain management. Logistic operations of any company are designed to achieve customer demand satisfaction. Each node within a supply chain becomes a customer for the previous one and the supplier for the next. A delay or failure might affect the whole supply network including the end customer.

Although logistics is a relatively new scope of science, techniques have already been established to reduce the risks of both not satisfying the customer demands and of not making the whole costs excessively expensive. The techniques can be differentiated between the managerial techniques (usually industrial approaches) and the engineering techniques (usually academic approaches). The current chapter focuses on the industrial approach to improve logistics operations. In order to acquaint the reader with the motivation for the research, Sections 1.1.1 presents several basic terms and logistics strategies.

1.1.1 Basic definitions and descriptions

Supply Chain Management: Several different definitions of supply chain management can be found in the logistics literature. According to Basu and Wright (2008) 'for every business transaction there is a supplier and a customer and there are activities, facilities and processes linking the supplier to the customer. The management process of balancing these links to deliver best value to the customer at minimum cost and effort for the supplier is supply chain management.'

According to Simchi-Levi et al. (2003, p. 1) 'Supply chain management is a set of approaches utilized to efficiently integrate suppliers, manufacturers, warehouses and stores, so that merchandise is produced and distributed in the right quantities, to the right locations, and at the right time, in order to minimize system-wide costs while satisfying service level requirements.' Therefore, supply chain management consists of the flow of materials as well as the reverse flow of information. The latter refers to the customer requirements management, in other words the demand. The goods and information flows for a typical supply chain are illustrated in Figure 1-1.

Figure 1-1: Supply chain management (source: Rushton, Croucher and Baker, 2010, p.5)

In general, the supply chain can be defined as a distribution network of goods, starting from the supply of raw materials to appropriate manufacturers, up to the finished goods being delivered to the retailer. ‘The key objective of supply chain management is to provide best value to the customer by measuring, planning and managing all the links in the chain’ (Basu and Wright, 2008). Therefore, the waste and cost reduction plays a vital role in a supply chain design.

Bullwhip effect (Forrester effect): ‘The bullwhip effect also known also as the Forrester effect, is the commonly used term for a dynamical phenomenon in supply chains. It refers to the tendency of the variability of order rates to increase as they pass through the echelons of a supply chain toward producers and raw materials suppliers’ (Disney and Lambrecht, 2008).

The bullwhip effect again describes the variations in order rates in a given supply chain echelon (customer) required to be supplied by a previous echelon (its supplier), which affects the ordering policies of the latter from the antecedent echelons (its suppliers) along the supply chain. If the end customer demand suddenly increases it causes the raising of the demands in all antecedent supply chain echelons by the same ratio (increased by the number of goods ordered to support a pre-defined safety stock policy). Figure 1-2 presents the fluctuations appearing in the inventory levels in each echelon due to customer demand change.

Figure 1-2: Bullwhip effect (source: Schniederjan, 1999)

Figure 1-3 presents the surge in demand (from end-product demand to raw-material demand) due to the bullwhip effect.

Figure 1-3: 'Bullwhip' or Forrester effect (source: Rushton, Oxley, Croucher. 2000:208)

Lead time: Lead time is the time between an order being placed and it being delivered. It involves the following activities: manufacturing or picking (whichever is relevant), packing, shipping, dispatching and delivering (Rushton, Oxley and Croucher, 2000, p.208).

Logistics operation: Logistics operation refers to any activity which is included in the overall supply chain management of a company. Several logistics operations are now well established. Typical logistics activities include:

- The movement of cargo in all transportation modes of air, land and water including planes, vehicles, trains, trucks, lorries and ships
- The storage of goods in warehouses or other storage rooms
- The ordering and purchasing of an appropriate number of raw materials by manufacturers
- The ordering of goods by each supply chain node to fulfil the expected demand for goods from their customers

- Planning the production in advance to enable an uncertain customer demands to be fulfilled

Logistics operations are activities involving goods only, so that labour and human resource management is not considered as logistics operation (Rushton, Croucher and Baker, 2010).

Value-added-activities: Value-added-activities are considered as the activities which increase the value of the item from the customer point of view. For instance, a finished product has a greater value for the customer than the sum of raw materials it was made from. As it can be noticed, although all of the above mentioned logistics operations are essential for a successful design of a supply chain network, none of them, in contrast to the process of production or assembling, can be considered as a value-added activity. Therefore they only generate costs related to holding goods in terms of time wastage, fuel wastage etc. Any possible cost reduction, which will not affect the customer demand satisfaction, is advantageous for the company. The aim of a successful supply chain is to find a way to balance logistics operations benefits against costs.

Economics of scale: Economics of scale is a management strategy which refers to cost reduction due to large batch production (or large batch transportation). For some industrial applications, the production set up cost (or empty transportation costs) outweigh the storage costs. In such situations the production quantity may be arbitrarily increased by manufacturers to reduce the production costs. Such a production quantity may significantly exceed the customer demand and may require additional inventory holding until the goods are actually demanded. The economics of scale is an opposite of the Just-In-Time (JIT) policy. JIT refers to very small batch production or even continuous production (or transportation) in respect of inventory reduction or elimination. Economics of scale implies a 'push' design of the supply chain. 'In the push system the stock is provided for the next stage of supply (e.g. buying items to sell or starting manufacture) without having the total production path clear' (Basu and Wright, 2008). It means that the products are produced and stored in the quantity which is economic in terms of designing a production system but not necessarily in the quantity which will be demanded by the customer (often significantly more than demanded). This is contrary to the 'pull' system.

Pull policy: The orders are placed by the supply chain nodes in response to the customer demand only. The flow of goods in the supply chain is pulled from the end customer side.

Push policy: The supply chain members tend to order, manufacture and store goods in order to be prepared to immediately satisfy the customer demand. The flow of goods is pushed by suppliers.

Replenishment policy: Replenishment is the act of refilling or complete again by supplying what has been used up or is lacking. In logistics it refers to placing new orders when the goods are used or sold. The replenishment policy can be decided for each individual node based on the purpose of the particular node in the supply chain and the overall company strategy (e.g. pull or push system). Exemplary replenishment policies are elaborated in Section 1.1.3.

1.1.2 Storage, warehousing & materials handling

There are several types of inventory, depending on the role of the supply chain node, the stock being held, and the role the inventory has for the node. For instance in some nodes the goods can be stored for several months, possibly kept in special environments to ensure they do not deteriorate. At other nodes goods are moved and processed very quickly from one place to another. Among other inventory types the most common are: raw materials, spare parts, work in progress, pipeline stock, finished products, and stock in warehouse or retail. Regardless of the inventory type the need for inventory holding is usually similar.

1.1.2.1 Need for stock holding

1. Enables the logistics operation to run smoothly by providing a buffer between suppliers and customers:
    - Compensates the negative consequences of production delay (such as break down, lack of raw materials etc.) or delivery delay (traffic issues, transportation mode break down etc.).
    - Compensates the uncertainties related to customer demand which, in most of the real world environments, is partially or highly uncertain. It enables immediate customer demand response even if the demand forecast was underestimated. Such kind of inventory is called the safety stock.

2. Enables reduction of costs of other activities:
    - Reduces the purchasing costs. It can be achieved as a result of economy of scale policy. Purchasing a large batch of products usually enables negotiation of satisfactory discounts with the suppliers. In the case of periodically fluctuating goods prices, the goods can be purchased at the moment of lowest price. This strategy is used if the stock holding is more cost efficient than regular small batches procurement at any time.
    - Reduces costs of production. 'Often it is costly to set up machines, so production needs to be run as long as possible to achieve low unit cost.' (Rushton, Oxley, Croucher, 2000:183). Reduction in the numbers of setting up of machines signify running the production without any break for longer than required, even if the produced goods are not demanded by the customer in such number. It requires the storage of finished goods after the production processes.

1.1.2.2 Disadvantages of stock holding:

From the previous section it could be concluded that keeping inventory is necessary for the company, although keeping inventory generates non-value-added costs, which has already been recognised as one of the higher logistic costs. Among the inventory related cost the most significant are as follows (Rushton, Oxley, Croucher 2000:191)

- Cost related to stock holding and maintaining - cost of space needed for storage (cost of building the warehouse and maintaining it in terms of electricity costs, taxes etc.), cost of manpower, cost of equipment such as appropriate racks, forklifts, cranes, picking devises etc.
- Cost associated with management of stock – the higher the inventory level, the higher the need for proper inventory management systems and people controlling it to avoid errors in delivery.
- Cost associated with insurance of the stock – the higher the inventory level, the risk of damage and the greater need for insurance.
- Risk cost – cost of possible theft, damage or deterioration of stock, which is more likely in the case of higher inventory level. Also it increases the risk of human accidents on the warehouse floor.

- Capital cost – loss of potential profits associated with investing capital, which is currently frozen in the inventory and reduces the cash flow in the company.

Therefore, high standard replenishment management requires achieving the appropriate balance between inventory holding benefits and expenses or equivalently the balance between processes organization and customer requirements. As a response to such a need, several different replenishment approaches have been established (Rushton, Oxley, Croucher, 2000). Some of them are reviewed in section 1.1.3.

1.1.3 Industrial approaches to replenishment policy

There are several approaches to inventory replenishment strategy commonly used in the industry. Some of them are listed below.

- Periodic review (shown in Figure 1-4): Stock level of product in the warehouse is reviewed at the end of each period of time. The order is placed if the inventory level is below the reorder level.

Figure 1-4: Periodic Review (source: Rushton, Oxley, Croucher. 2000:206)

- Fixed order quantity (illustrated in Figure 1-5): The order is placed when the quantity of product decreases below the reorder level, before it reaches the safety stock level.

  The disadvantage of both methods is the fluctuating inventory levels, the risk that during one period (in the case of the first method) or lead time (in the case of the second method) the inventory might decrease under the safety level, and might cause unnecessary costs related to inventory holding.

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Figure 1-5: Order quantity (source: Rushton, Oxley, Croucher. 2000:207)

- Economic order quantity (EOQ): 'The EOQ method is an attempt to estimate the best order quantity by balancing the conflicting costs between holding stock and placing replenishment orders (Rushton, Oxley, Croucher 2000:191). Figure 1-6 compares two exemplary replenishment policies. Policy A refers to less frequently replenishing, which reduces the transportation costs and order setting up costs. On the other hand it increases the inventory level significantly which increases the holding cost as well as the stock maintenance costs. The policy B, in turn, reduces the inventory costs and increases the transportation costs due to its delivery frequency.

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Figure 1-6: Order sizes (source: Rushton, Croucher and Baker, 2010)

To achieve a balance, the EOQ approach aims at minimising the total costs through appropriately designed objective function (cost function). Figure 1-7 presents the exemplary EOQ plan. The total cost is represented by the objective function. Its minimum value represents minimal costs for goods number of economic order quantity. The functions of ordering costs and holding costs (the two components of the objective function) are shown in Figure 1- 7 for comparison.

Figure 1-7: Economic order quantity (source: Shah, Gor and Soni, 2008:305)

Such an approach already seems more beneficial for the company than the previously presented ones. One has to remember, though, that the presented model does not consider system uncertainties such as varying customer demand and the lead time delay. It is still quite a basic approach. Chapter 2 is devoted to the discussion of more sophisticated replenishment decision support tools available in academic literature.

- Just in time – the zero inventory philosophy

The concept of just in time (JIT) refers to a type of replenishment system where the goods appear in the warehouse (or production line) exactly when they are needed, neither before nor after that time instance. Ideally it refers to zero inventory holding policy. The idea itself seems to be simple, nevertheless due to system uncertainties such as unknown demand or unexpected delays, it is not so straightforward in application. Adopting JIT requires either the patience of the customer (for instance car industry) or an accurate prior knowledge regarding future demand (for instance due to accurate forecasting) and/or a sophisticated material requirement planning system. Nevertheless in most of the real world cases the demand is uncertain, which justifies the importance of appropriate design replenishment decision systems.

Implementation of JIT is not just about reducing the inventory, but also reorganisation of processes in the manner that they become more efficient and the lead time is reduced. For instance, in terms of reduction of inventory held in the warehouse, the delivery frequency needs to be increased contrary to the economics of scale approach.

'Stock controllers have been sceptical about the efficiency of frequent deliveries of smaller batches without investigating all the options and potential. There are many instances where the accepted delivery quantity is now much smaller than it was a few years ago, and is destined to continue to be reduced' (Basu and Wright, 2008).

As it was mentioned in the definition of economics of scale in Section 1.1.1, the JIT policy is an opposite strategy to economics of scale and in this respect it refers to more frequent and smaller sizes of order quantity, ideally the continuous production system. Defining a scale of degree of 'pull/push' strategy, the extreme JIT (zero inventory policy) would be at the pull end of the scale, while the economics of scale, would be at the push end. The smaller production (or transportation) batches and the more frequent orders (deliveries), the more shifted to 'pull' the replenishment strategy is on the 'pull/push' scale.

From a mathematical modelling perspective, the JIT policy appears to have no constraints on order batch size, therefore the batch can be as small as needed.

'For some supply chains, the application of JIT is natural because of the nature of the products or the processes, e.g. in fast-cycle manufacture, where a stage in the production process takes very little time' (Basu and Wright, 2008). Nevertheless, in many industries the implementation of JIT is not straightforward and many companies do not adopt JIT simply because of the fear of decreasing service levels and disappointing customers by the lack of stock when required. There is a need for applying analytical techniques in replenishment planning so that the likelihood of lost business opportunity is reduced.

1.1.4 Lean Management

The Lean Management (or Toyota Production System) concept was pioneered in the Toyota production system and adopted by other Japanese manufacturers. Much later it was appreciated by Western manufacturers as well and recognised as an incredible source of cost reduction as well as improved customer response.

Lean Management is a management approach, realised by the implementation of established principles, theories and tools which, contrary to economy of scale approaches, practice economics of time, when designing logistics operations. Implementation of even a single one of the existing lean techniques is considered to be beneficial for a company (Bicheno, Holweg and Bicheno, 2009). The reduced cost is related to reduction or, if possible, the elimination of non-value-added activities (e.g. inventory holding) as well as the elimination or appropriate prevention of other identified wastes. Therefore, the lean approach aims at cost reduction via better organization of processes in the time line while maintaining high customer service and the rapid response to customer needs. The several lean tools listed below briefly explain concepts from the design of a lean inventory replenishment system point of view.

Lean theory is based on the five principles presented below. The conducted research especially refers to the flow (no 3) and pull (no 4).

The five lean principles (Bicheno, Holweg and Bicheno, 2009):

1. Specify the value from the customer point of view. The main stress should be put on the fact that the customer buys the result, not a product itself and the value should be assessed having this in mind.
2. Value Stream. It aims at identification of non-value-adding activities.
   In the context of efficient replenishment system design, inventory holding is not considered valuable in terms of both, principle no. 1 as well as principle no. 2. It does not add any value to the product.
3. Flow. According to lean philosophy, all goods should be kept in a flow. Any storage is considered a waste of money, any waiting for finishing of other components or batch to be delivered is a waste of time because it does not add any value to products.
4. Pull. In contrast to the commonly used ‘push’ approach, lean management proposes just in time supply and production. It means that the products should be delivered when needed but no later than required by the customer. It again refers to elimination of

stock. ‘With a pull system the first action in the chain is that the item is demanded. To satisfy this demand there is an item in stock. As soon as this stock is used up, another item is supplied, either from outside or from a production process’ (Basu and Wright, 2008). In a pull system the quantity of delivery is significantly smaller compared to the push system, thus the delivery takes a much higher frequency and refers to JIT policy.

5. Perfection. “Perfection does not only mean quality – it means producing exactly what the customer wants, exactly when (with no delay), at a fair price and with minimum waste” (Bicheno and Holweg, 2008:11). This principle summarises the previously presented lean principles and shows that the aim of lean management is to reorganise the processes in a way that overall time will be reduced and the quick response to customer requirements will be possible without any buffer inventory holding.

1.1.4.1 Lean Wastes

One of the more important lean concepts is identification and elimination of wastes. According to lean theory wastes can be differentiated into two categories: wastes from an organization’s perspective and the customer’s perspective. Both of them are presented as follows.

Service Wastes (customer point of view) (Bicheno, Holweg and Bicheno, 2009)

- Delay. In general it refers to waste of a customer’s time, especially in the form of delays in services, deliveries, waiting in queues etc.
- Duplication of activities which are required to be completed by the customer.
- Unnecessary customer movement whilst being served.
- Unclear communication.
- Incorrect inventory, such as lack of required stock in required time.
- Loss of business opportunity
- Errors in orders or damage of products

Seven Wastes (company point of view) (Bicheno, Holweg and Bicheno, 2009)

- Overproduction: Goods should be produced, delivered and stored in an appropriate quantity. Safety stock should be minimised as it contributes to wastage of time, money and storage space. It might also increase the likelihood of the deterioration of perishable goods.

- Waiting: This refers to delays from the company's point of view with regard to delivery or until a batch of products or components are finished. Attention should be paid to the fact that waiting also creates some bottlenecks in the system and might cause the bullwhip effect in the whole supply chain. In general it affects the goods flow which was previously presented as one of the lean principles.
- Unnecessary motion: This refers to the movement of products, people and machines.
- Waste of transportation: Besides the waste of energy it also refers to the increased likelihood of damage.
- Overprocessing: This refers to the unnecessary use of sophisticated and expensive software or machines. The designing of complex processes or the implementation of sophisticated software when not necessary is a waste of money as well as time devoted to change management.
- Unnecessary inventory: As previously mentioned the reduction of unnecessary inventory causes a cost reduction related to storage space or the number of employees needed to maintain the inventory etc. Implementation of a Just in Time (JIT) policy prevents unnecessary inventory holding. It also makes a company more flexible, simplifies inventory management and facilitates a quick response to customer requirements.
- Defects: Defects generate unnecessary costs to the company. Defective goods need to be repeated or ordered again. Defects are common in the case of a 'push' system, where mass production occurs due to economy of scale.

Lean management also suggests the reduction of suppliers in order to simplify the system design and the development of better relationships with those suppliers that are kept so that they become more reliable.

It should be noted that all wastes are related to each other, one causing another. High inventory can be identified as a factor that most affects other wastes. Unnecessary inventory might result in the increased likelihood of defects and deterioration, causing complexity while stock managing, which is considered over processing.

Defects might also involve cause the customer additional and unnecessary involvement (e.g. returns). Defects need to be repaired or the goods need to be ordered again, which causes overproduction as well as overprocessing. Overproduction causes unnecessary motion, transportation, delays, waiting or waste of transportation. This also causes additional inventory and reduces the ability to respond quickly to demand changes. Unnecessary transportation increases the probability of goods being damaged and increases the waiting time. This in turn, affects the smooth flow of goods within the supply chain. Both the defects as well as high inventory can cause errors in inventory, which could result in lost business opportunities. All of the above wastes bring about unnecessary costs to the system.

Designing of efficient replenishment systems with special focus on inventory control would directly and indirectly reduce costs significantly. It would enable a fair price to be set for the sold products without a decrease in quality caused by mass production and big-batch transportation. This, in turn, could result in increased customer demand. The current research recognises the importance of appropriate inventory control.

1.1.5 Fluctuating inventory level

As noted in Section 0 it can be identified that the surplus inventory holding affects other lean principles, unnecessary inventory holding generates hidden costs to the whole supply chain. Therefore, zero inventory policy seems ideal. This is the reason why so many academic researchers were devoted to this topic (see Section 2.2.5). Nevertheless, in most real life applications keeping some safety stock is still necessary (as explained in 1.1.2.1). The right quantity identification is a subject for academic research as well as industrial trial and error practice. Both the industrial and academic strategies tend to decrease the inventory as much as possible under the constraint of demand satisfaction. Due to the lack of accurate prediction of future demands in most real life replenishment systems, searching for inventory level balance is also often inaccurate.

As a response, in this section the author proposes an innovative way of thinking of replenishment design. It resembles JIT policy in respect of no constraint on order batch quantity. The batch can be as small as needed. However the introduced strategy involves a reduction of inventory fluctuations rather than inventory level itself. It has not necessarily been realised by many researchers and industrial managers but it is highlighted here that uncontrolled inventory level fluctuations can cause several hidden costs and surplus problems compared to stable inventory level holding such as:

- Bullwhip effect: as shown in Section 1.1.1 the sudden change of inventory level in one supply chain node causes inventory fluctuations in the others.
- Unnecessary deterioration of products: the shelf life management becomes more complex when the inventory level is varying. The short shelf life replenishment system is appropriate for low inventory levels when the stock is replenished often. Sometimes the supply chain node chooses goods with close end dates on purpose as they are usually less expensive and enable some purchasing cost reduction. In turn, long shelf life is necessary for high inventory levels where certain stock will be kept significantly longer. In the case of fluctuating inventory, keeping a balance between long and short shelf life purchases becomes complex, some unexpected product deterioration is likely to occur. Also in some storage areas of real life application the surplus inventory is stored on top of the regular stock, which for some product types increases the likelihood of deterioration (for instance due to reduced oxygen exposure of those at the bottom). In other cases, holding unexpected inventory may mean using emergency storage areas where conditions may possibly increase the likelihood of deterioration (for instance outdoor storage areas).
- Damage: the damage of goods is more likely to occur when the inventory level suddenly increases as it involves the additional movement of goods, which increases the likelihood of damage. It can involve the inventory being kept in inappropriate storage areas where damage to goods is more likely to occur.
- Storage capacity definition problem: either the storage room is not used, albeit still paid for, or exceeded.
- Fluctuations in storage cost: varying storage costs affect the cash flow of the company and financial planning ability.
- Difficulty in required employees planning: either the surplus labour is maintained and paid for, or the labour capacity is exceeded. A sudden increase of inventory requires more labour to be involved in logistics operations such as packing/repacking, labelling, moving to storage locations etc.

The thesis devotes its attention to model based control to reduce the inventory fluctuations and reduce hidden costs. No batch constraint on order quantity is considered in the thesis. The batch constraint in mathematical terms can refer to an order quantity smoothing parameter. For the purpose of this thesis the smoothing parameter is considered to be set to zero (see Equation (3.63)).

1.2 Research Problem

1.2.1 The need for system modelling

The purpose of modelling is to gain a better understanding of real systems at low-cost. Experimenting with the model itself instead of involving the real system is less time consuming and does not require reorganisation of processes for experimentation purposes and associated investments. As many as needed of different scenarios can be considered before implementation. Several mathematical and engineering techniques enable system modelling for optimisation purposes. They can be further used for optimal or suboptimal decision making.

1.2.2 Operational research approach

Operational research (OR) is a field of studies that deals with optimal or suboptimal decision making in real world, albeit mostly non-engineering, environments such as military operations, businesses processes, network design, scheduling, supply chain management, routing, optimal search, facility location etc. It uses analytical techniques for decision making involving mathematics for modelling and /or sophisticated software for scenarios analysis and simulation (Winston 1991).

1.2.3 Research description

Control theory (CT) is an interdisciplinary field of study that involves dynamical systems theory. Being applicable in any system where a feedback loop can be mathematically modelled, it has been developed for a broad range of techniques and algorithms to control the system input in order to obtain the desired output and desired system behaviour. As methods used in control theory have a quantitative nature, the natural consequence was to find a wide array of applications in engineering fields such as automotive, heating, electronics, robotics etc. Control engineering, then, regards application of control theory techniques to engineering disciplines. Nevertheless, the algorithms which are used for engineering application, being already recognised as powerful and well performing, can be benchmarked for a variety of problems such as inventory, manufacturing, network optimisation etc. Control theory, with its straightforward consideration of system dynamics, can potentially be recognised as powerful and efficient in the operational research field. The thesis focuses on the inventory problem studied from a control theory perspective. The approach brings the advantage of overcoming the problem of system uncertainties related to fluctuating customer demand (as illustrated in Figure 1-8), fluctuating lead times and deterioration of products by updating the current information in a feedback loop, and controlling system behaviour without having prior knowledge about the system dynamics.

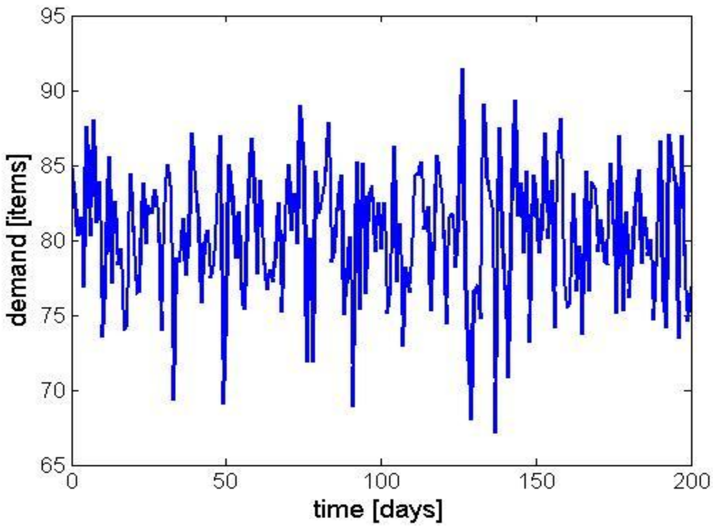


Figure 1-8: Exemplary varying demand showing how dynamically and randomly the demand can change

The final outcome presented in the thesis, which is an inventory level control methodology, aims to not only obtain good results compared to purely mathematical techniques, but rather at giving the reader the idea of the elegancy and power of control theory in such applications. It also highlights the potential benefits of collaboration between the control engineering and the operational research communities. Although such collaboration seems promising, there is still a gap in effective communication due to differences in perspective and understanding between these two groups. One of the reasons can be attributed to very little awareness of OR problems and its appropriate mapping into the control theory domain among the CT community. Another reason is the limited awareness of Control Theory science within the OR community

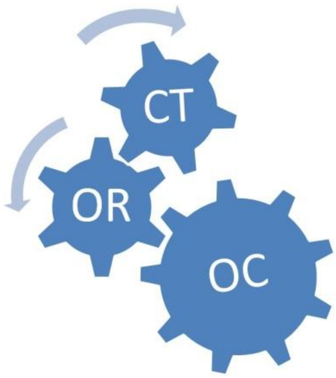


Figure 1-9: Operational Control

Recognising the above, this research aims at bridging the gap between the two by developing a new field of studies which, in the future, would enable the operational research community to apply the complex and efficient algorithms of control theory without any need to familiarise themselves with the deep and structured control theory. For the purposes of this thesis, the name operational control (OC) has been given to the control theory patterns and procedures being benchmarked in the operational research field.

To start with, one of the well-established optimisation algorithms, extensive in mathematical description and deeply ingrained in control theory science, known as model predictive control (MPC), has been benchmarked. Further, the algorithm, for the inventory model, has been mathematically reformulated to the form which is significantly simplified in description, and does not require the user to possess any initial control theory knowledge. As a result, the novel shortened optimisation procedure gives almost exactly the same results MPC, and both have been demonstrated to be mathematically equivalent.

The developed methodology appears to be transferable to other applications, for instance production systems. The reduced mathematical formulation was based on noticing patterns in the MPC formulation and followed a series of propositions and their demonstrations leading to the final approach. The final formulation is presented in the form of a proposition in Chapter 3, Section 3.3. The elaboration of its simplicity in application and the straightforward manual calculation of the next optimal order size on a daily basis, which maintains the inventory in balance, are shown in Chapter 4, Section 4.5. The mathematical demonstration of the equivalence of the initial and final formulation is shown in Chapter 5 and the final results are shown in Chapter 6.

1.3 Aim, Objectives Research Contribution and Outline of Thesis

1.3.1 Aim of the research

The aim of the research consists of two parts:

- Apply the control theory as an optimisation tool to develop a decision support system for replenishment and satisfactory performance
- Bridge the gap between the precision of control theory and the understanding of the operational research community

1.3.2 Objectives

In order to accomplish the aim of the research project, the following objectives have been established:

1. Identify the current practice of OR modelling of warehouse management.
2. Identify the state-of-the-art in application of control theory to inventory modelling and optimisation
3. Convert the inventory problem to control theory domain
4. Benchmark control as an optimisation technique for replenishment decision support system
5. Develop the simplified algorithm which can be easily adoptable by operational research community
6. Run a number of managerial scenarios in order to improve the current practice.

1.3.3 Research contributions

According to their importance the PhD research contributions can be listed as follows:

1. Making the model predictive control (MPC) technique available to the OR community via mathematical reduction – Chapter 3, Section 3.4

Based on literature review of control techniques applied in OR and presented in Section 2.3, it seems that many of the authors follow the same path. All of them apply various existing control techniques for a range of logistics models. Therefore, the accessibility of those techniques to OR is very limited, as it involves familiarity with control theory principles. In this thesis the author presents a totally different approach and shifts the research towards a different direction, developing the MPC inspired simplified method, which makes it instantly available to the OR community, but retaining the engineering precision of dynamical control. It is computationally significantly less consuming (as elaborated in Section 3.6) and, although initially developed for inventory control, it can be applicable in a wide range of other management/production/inventory problems.

2. Demonstration of the mathematical equivalency of the reduced approach to MPC for the inventory model (step-by-step process) – Chapter 5

It involves the presentation of a sequence of propositions and their demonstrations, comprising a whole chapter. Each proposition is a consequence of the previous one and the last one is the final version of the novel mathematically reduced technique. The demonstration is an illustration of the evolving thought process, which was inspired initially by MPC and subsequently applied for the inventory model.

3. Quantification of benefits of MPC with application to inventory model (and as a consequence the benefits of the novel reduced controller, as mathematically equivalent) – Chapter 4 (Section 4.4), Chapter 6

The quantification of benefits was an innovation in the OR perspective, with the reduction of inventory fluctuations elaborated in Section 1.1.5. The profitability of the MPC was assessed using a developed profit function (minor contribution).

4. Establishment of limitations of new method – Chapter 6 and Chapter 7

The method is limited to the applications where there is no constraint on order quantity. The limitation was recognised and justified in Chapter 6. In Chapter 7 the possible directions for overcoming the limitation was elaborated. It also showed that for the considered application the limitation becomes strength of the model (justified in Chapter 7).

5. Translation of inventory problem to control theory framework Chapter 3, Section 3.2,

the model is represented in a state space form, where the demand is treated as a disturbance, the system dimension refers to lead time and the deterioration of products is a time varying quantity. In contrast to typical control theory approaches, the disturbance (demand) is subtracted from the current inventory level signal.

1.3.4 Outline of thesis

The thesis is organised in the following manner. Chapter 1 introduces the reader with logistics definitions, which are necessary for the understanding of further chapters in the thesis. It also presents the fundamental concepts and problems of logistics that the thesis addresses. It enables an understanding of the need for mathematical modelling of supply chain and inventory problems. It also justifies the thesis goal of reducing the inventory fluctuations, which is the reference point for performance assessment later in the thesis.

Chapter 2 presents the literature review of a general OR approach and the more specific CT approach in logistics problems. It shows the variety of techniques and approaches to logistics systems modelling in the OR community. It highlights the gap in the OR approach and justifies how CT can effortlessly bridge the gap. Chapter 2 also allows the reader to see that the CT approaches found in the literature of inventory problem follows the same path initiated by Simon (1952). Therefore, in Chapter 2 the need for shifting towards a different direction is justified. The main contribution of the thesis (see point 1 Section 1.3.3), starts a new route which will hopefully be chosen by other researchers in the future.

Chapter 3 presents the translation of the inventory problem to the CT framework. It shows both, the continuous time and discrete time preliminary modelling. It also presents the first research attempts of the application of CT algorithms (Smith predictor in Section 3.2.3.1). Further, the model predictive control approach is shown in Section 3.3. Although the novel mathematically reduced controller is presented immediately after the model predictive control in Section 3.4, in the time scale of the research evolution the novel controller development (shown in Chapter 5) took place between these two milestones of Section 3.3 and Section 3.4. The novel algorithm, although inspired by model predictive control used for the inventory model, transpires to be a minimum variance approach, or dead beat controller with an incorporated Smith predictor. Although the reduced form was initially inspired by model predictive control, the final formulation which is essentially an optimal dead beat controller with Smith predictor appeared to be immensely beneficial. It’s perceived limitation (of lack of order size constraints) was in fact found to be the controller’s advantage, for which justification is shown in Chapter 7.

Chapter 4 is a natural consequence of Chapter 3. It shows the simulation results of each modelling stage shown in Chapter 3. It justifies the decision of the abandonment of the Smith predictor applied independently. It shows how the simulation results inform and redirect the research with new attempts to appropriate algorithm selection, to finally complete the cycle and obtain the dead beat controller, with incorporated Smith predictor. This is mathematically equivalent to the model predictive control applied for the inventory model when the tuning parameter is set to zero. Chapter 4 justifies the initial inspiration of model predictive control applied to the inventory problem and shows the equivalency in results between the original model predictive control and the developed/proposed novel controller.

Chapter 5 presents the process of recognising model predictive control patterns in mathematical descriptions and in obtaining the mathematically reduced equivalent formulation. As a sequence of propositions and their demonstrations the overall process of reaching the final mathematically reduced novel controller is shown.

Chapter 6 presents the results of the proposed method in respect to different numerical settings.

Although the lack of order constraints has been considered an advantage of the model in this particular application, Chapter 7 discusses possible future directions in order to overcome this limitation. Instead of finding the mathematically equivalent controller, the chapter proposes a good estimation in the case when the model predictive control tuning parameter is non-zero. It can be applied straightforwardly by the OR community and the generated results are very similar to the original model predictive control.

1.3.5 Publications

A list containing publications of the author is presented in Appendix II – List of publications and other academic activities. The publications were gradually reflecting every achieved milestone of the research. (Orzechowska, Burnham, and Petrovic, 2011a) was presented at an OR conference and the publication highlighted the advantage of the application of control theory to inventory modelling based on a literature review (which was particularly inspired by the Smith Predictor application of Ignaciuk and Bartoszewicz 2010b) and some preliminary experiments with continuous model predictive control combined with the Smith predictor. After the presentation, one question provided inspiration for the rest of the research. Being asked, if control engineering should be taught on OR courses. It was understood then, even better, that the OR community needs a tool which will maintain control engineering advantages but should be straightforward and understandable in OR terms. As a response (Orzechowska, Burnham, and Petrovic, 2011b) focused on convincing the control community to apply their method to inventory problems in a manner which would be acceptable by OR experts. The paper presented the application of continuous and discrete-time model predictive control with an incorporated Smith predictor. As consequence of this conference presentation a new international collaboration with Prof. Bartoszewicz, the author of the paper which significantly influenced the research in the beginning (Ignaciuk and Bartoszewicz, 2010b).

The next step was a journal review paper of control techniques applied to supply chain management. Having already been particularly interested in model predictive control at that stage, the main part of the review was devoted to that technique (Orzechowska, Bartoszewicz, Burnham and Petrovic, 2012a).

The first steps of the mathematical reduction process of discrete model predictive control can be found in (Orzechowska, Bartoszewicz, Burnham and Petrovic, 2012b). The accomplished mathematically reduced controller for non-perishable goods only was presented in Orzechowska, Bartoszewicz, Burnham and Petrovic (2012c). The first steps of the perishable goods in mathematical reduction form can be found in Orzechowska, Bartoszewicz, Burnham and Petrovic (2013a). The completed mathematically reduced controller, applicable for both, perishable and non-perishable goods was presented at an OR conference (Orzechowska, Bartoszewicz, Burnham and Petrovic 2013b) and a control theory conference (Orzechowska, Bartoszewicz, Burnham and Petrovic 2013c). The first one presented the research from an OR point of view, focusing on performance, results and simplicity. The second one focused on demonstrating the equivalency of the original model predictive control and the mathematically reduced model.

1.4 Summary

The current chapter presents an introduction to logistic science. It is aimed at familiarising the reader with relevant terms and management strategies necessary to understand the research application. It also highlights the problems that usually occur within logistics. Special focus is devoted to the inventory issues and corresponding management strategies as it relates to the research application.

As previously mentioned, there are several reasons for holding inventory within the distribution arms of supply chains. Among others, the inventory provides a buffer between the supply and customer demand in response to dynamic demands and lead times. It ensures a high service level, which is crucial for a company to survive in a competitive market. Although, the inventory holding is convenient for a company, it generates an expense, which has already been recognized as one of the highest logistics costs. Surplus inventory holding also causes problems also brings indirect costs as it affects the work flow in other areas. A replenishment system requires an appropriate balance between customer demand satisfaction and stock keeping expenses to be achieved with due consideration to system dynamics. Taking into account, only two such system uncertainties, including unpredictable demand and lead time delay, managing replenishment inventory without application of an engineering decision support framework becomes a complex process. This often results in uncontrolled inventory level fluctuations, which in turn lead to several other problems such as deterioration of products, variation in storage costs, extension of storage capacity and/or backorders.

As logistics is a relatively new field of science there is still a huge scope for further improvements. Due to the many of constraints in the real world scenario the developed theoretical and analytical techniques used in inventory replenishment design usually focus on a few selected aspects of the actual logistics systems and usually only achieve a trade-off between the theory and practice.

The conclusion that can be drawn is that the complexity of the cost-benefits balance achievement justifies the need for the development of decision support systems. Since the 1940s the logistics operations have elicited the attention of operational researchers using more sophisticated methods in designing more efficient logistics systems. The science of mathematics possesses a set of optimisation techniques which can be applied to improve the logistics performance decision making or/and cost reduction. It is sufficient to represent the existing problem in a mathematical language of variables and equations to enable the application of the techniques. The literature review, Chapter 2, is devoted to elaborating the history and the state-of-the art of logistics system design with special focus on inventory replenishment.

The aim of the research is to use mathematical techniques of control theory to solve inventory problem.

2 LITERATURE REVIEW

2.1 Introduction to an academic approach to supply chains

The expansion and globalization of companies can be widely observed in everyday life. Barely anything is produced locally in today's world. The productions sites are located in one country, management in another and the distribution network expands worldwide. What is more, the competition becomes stronger in a global market, as it develops dynamically and grows fast. Such a dynamic environment requires continuous improvement from the company. Lack of improvement in this context leads to recession and will eventually exclude the company from the market (Rushton, Croucher and Baker, 2006). This new situation compels the company managers to continuous rethinking of the company processes, policies, strategies, structures etc. The need to innovate ideas has become a matter of utmost importance not only within the company's core tasks but also in all processes enabling the completion of core tasks; logistics is one of them. Although logistics enables expansion of companies in terms of organisation and design of goods flow, it generates additional, not necessarily value-added expenses with respect to details as discussed in Chapter 1. The awareness of possible cost reduction in logistics processes, particularly in inventory holding, becomes crucial in times of market globalisation. The new logistics strategies are continuously developed by both industrial and academic communities, and find their application in global corporations as well as medium and small businesses. The industrial strategies have been introduced in the previous chapter (Chapter 1). The current chapter presents the academic approach to logistics systems improvement and optimisation with special focus on inventory management and replenishment. It is observed that inventory replenishment optimal policy design has gained increased attention in the field of operational research in recent times.

Though the origins of the application of regulation theory in the field of logistic systems control can be dated back to the 1950s there have been a relatively limited number of researchers dedicating their attention to this issue until recently. Therefore, the discussion of control methods in logistics, without mentioning the well-established techniques of the operational research field, would not give the reader a holistic view of the topic, disregarding the strengths and weaknesses of control methods. Thus, the chapter has been organised as follows. Firstly, an overview of OR techniques applied to overcome problems faced in logistics is presented. Secondly, the application of the control oriented techniques addressed to different logistics operations is discussed. Here, the main focus of attention is given to the inventory replenishment application, as this is relevant to the current research topic. Furthermore, the strengths and weaknesses of the discussed techniques in replenishment systems are listed with respect to model complexity and adequacy of the model to the case of real-world applications. The further sections of the chapter focus on control theory applications related to logistics, mostly inventory systems, and the gaps in knowledge are identified.

2.2 Operational Research Techniques

There are many techniques applied in the field of operational research to find optimal or quasi optimal solutions for a particular problem. The mathematical methods or simulation based methods or the combination of both can be found in the academic literature. Mathematical methods consist of mathematical optimisation (commonly mathematical programming) and other algorithms, among which the genetic algorithm plays a major role.

Mathematical programming, which includes linear programming, nonlinear programming, convex programming, geometric programming, integer programming, dynamic programming, quadratic programming and many others, refers to building a mathematical model with an objective function which minimises cost or maximises profit, and the equality and inequality constraints representing the physical constraints of the system. Since a real system has many uncertainties the mathematical description of the system usually requires some initial assumptions and simplifications.

A genetic algorithm (GA) is an iterative technique of searching for better solution(s) with close proximity to the optimal solution. In a similar manner to a natural selection process, a GA selects better solutions from populations of possible solutions and then combines them together with each other to breed a new generation of solutions. Just as in a natural selection process, the new generation of solutions becomes closer to the optimal value. The procedure is repeated until satisfactory results are obtained.

Simulation is another approach used for system design or process reengineering. Although it allows for relatively more complex model building than the pure mathematical methods, it is not typically meant to find the optimal solution. Rather than that, it enables deep system analysis, performance evaluation and experimenting with different scenarios for finding satisfactory results. For instance Hachicha et al. (2010) use simulation for performance evaluation in multi-stage, multi-product, multi-location and multi-resources within a production system. The objective is to find better lot sizes of each of the products being manufactured. Simulation can also be combined with different optimisation techniques such as a GA, applied for instance by Kochel and Nelander (2005) in a multi-echelon inventory problem or scatter search applied by Keskin, Melouk, Sharif and Ivan (2010) for an integrated vendor selection and inventory problem.

The reviewed models proposed in the operational research literature are briefly discussed in the current section under six categories: inventory modelling, supply chain management, routing/transportation problem, distribution network design, optimal packing, and manufacturing/scheduling problems. Therefore the presented papers are discussed with respect to their field of application as well as techniques applied. The considered techniques are: linear programming, other mathematical optimisation techniques (such as nonlinear programming, integer programming, convex programming, stochastic programming, dynamic programming and heuristic methods), simulation and genetic algorithm. All of the discussed papers are contained in Table 2-1 with respect to applied method and applications.

	Linear programming	**Other mathematical optimisation techniques**	**Genetic algorithm**	**Simulation**
Inventory	(Janssens and Ramaekers, 2011), (Amaya, Carvajal, and Castaño, 2013)	(Padmanabhan and Vrat, 1990) (Samadi, Mirzazadeh and Pedram, 2013)	(Zhang, et al. 2012), (Guchhait, Kumar, Maiti and Maiti, 2013), (Chun-Wei and Hsian-Jong, 2011)	(Kochel and Nelander, 2005), (Kochel and Nelander 2005), (Keskin,

				Melouk, Sharif and Ivan, 2010), (Healy and Schruben, 1991)
Supply chain management	(Bilgen, 2010), (Peidro et al., 2010).	(Paksoy, Özceylan and Weber, 2010), (Cintron, Ravindran and Ventura, 2010), (Poojari, Lucas and Mitra, 2008)	(Zhang, Zhang, Caiand Huang, 2011), (Seo, Jeong, Lee, Lee and Park, 2012), (Kaijun and Xiangjun, 2012)	(Kharazi and Jandaghi, 2011), (Siddiqui, Khan and Akhtar, 2008), (Long, Lin and Sun, 2011)
Routing	(Chien, Balakrishnan and Wong, 1989), (Adelman, 2003), (Adelman, 2004)	(Berman and Larson, 2001), (Lee, Moon, and Park, 2010), (Golden, Assad and Dahl, 1984)	(Jiang, Wang, and Ding, 2013), (Lin, and Yeh, 2013)	(Suzuki, 2011), (Lau et al. 2009)
Supply network design	(Bilgen, 2010), (Peidro et al. 2010)	(Cintron, Ravindran and Ventura, 2010), (Poojari, Lucas and Mitra, 2008)	(Lin, and Yeh, 2013), (Che, Chiang and Che, 2012), (Zhang, Cai and Huang, 2011)	(Long, Lin and Sun, 2011), (Turhan, Vayvay and Birgun, 2011)

Picking/packing	(Bidgoli, 2010), (Roodbergen, and Koster, 2001a), (Roodbergen, and Koster, 2001b), (Roodbergen, and Koster, 2001c)	(Roodbergen, Sharp, and Vis 2008), (Smolic-Rocak, et al., 2010), (Hall, 1993), (Gu, Goetschalckx, and McGinnis, 2007)	(Khanlarzade, Yegane and Nakhai, 2012)	(Serna and Pemberthy, 2010)
Manufacturing	(Bard and Nananukul, 2010), (Winston, 1998)	(Yong, Wang, Lai, 2009), (Lee and Yoon, 2010)	(Svancara, Kralova and Blaho, 2012), (Qiao, Ma, Li and Yu, 2013)	(Kumar and Sridharan, 2010), (Božičkovié et al., 2012)
Transportation	(Chien, Balakrishnan and Wong, 1989) (Adelman, 2003), (Adelman, 2004),	(Berman and Larson 2001), (Golden, Assad and Dahl, 1984) (Jung and Mathur, 2007)	(Jiang, Wang and Ding, 2013), (Lin and Yeh, 2013)	

Table 2-1: The OR techniques applied in logistics processes modelling

2.2.1 Transportation supporting models

The purpose of transportation in logistics is to link facilities within the logistics system and provide a flow of goods starting from the manufacturer to the end customer (Ghiani, Laporte and Musmanno, 2004). The movement of goods, regardless of the transportation mode being utilised, generates significant costs to the company. Motor vehicles (lorries, trucks, vans, cars) are regarded as the most commonly used transportation mode in a supply chain due to motion flexibility in the well-developed road network (Rushton, Croucher and Baker, 2006). As the most often used mode, it is also the most commonly targeted mode for a potential cost reduction in supply chain network design.

The most common approach towards transportation cost reduction is the Vehicle Routing Problem (VRP), which tends to optimise the total vehicle route length under the constraints and requirements of the distribution system: structure of the supply chain network, customer requirements, vehicle capacity etc. Several researchers (e.g. Chien, Balakrishnan and Wong (1989) and Adelman (2003, 2004)) have dedicated their work to the aforementioned problem, and designed routing models, employing linear programming as an optimisation technique. Berman and Larson (2001), in turn, applied dynamic and stochastic programming methods to find optimal vehicle paths, while Golden, Assad and Dahl (1984) and Jung and Mathur (2007) focused on heuristic analysis of the problem. Jiang, Wang and Ding (2013) and Lin and Yeh (2013) approached the routing problem by applying a genetic algorithm. The simulation based models optimising the vehicle routes, in turn, appeared in papers of Clarke and Wright (1964) and Schwart, Ward and Zhai (2006). A few papers presented the routing problem in combination with supply chain design (e.g. Lee, Moon and Park (2010)) or inventory management (e.g. Golden, Assad and Dahl (1984) and Jung and Mathur (2007)).

2.2.2 Supply chain management supporting models

The supply chain design involves both movement as well as storage of goods and therefore almost any single logistics aspect can be considered in this category (Mentzer, 2001). For this reason the problem can be considered as a multi-objective optimisation. Nevertheless, for the purposes of this section, the supply chain modelling and optimisation should be understood in terms of performance measures of supply chain networks. Allocation of facilities, inventory optimisation and routing of vehicles and integrated multi-objective models have been presented in several papers. Bilgen (2010) and Peidro et al. (2010) proposed fuzzy linear programming models for optimal supply chain design. Paksoy, Özceylan and Weber (2010), developed a mixed integer programming model to find the minimal inventory holding costs, transportation costs and the vacant capacity of a warehouse. While developing the model, several factors were taken as constraints: appropriate distribution centres assigned to appropriate customers, storage capacity, demands and throughput of suppliers. Also other mathematical programming techniques such as pure integer programming (as in the case of Cintron, Ravindran and Ventura 2010) and dynamical programming (for instance Poojari, Lucas and Mitra, 2008) can be commonly found in the literature. The supply network planning and design can be found in papers of Lin and Yeh (2013), Che, Chiang and Che, (2012) and Zhang, Zhang, Cai and Huang (2011) using the genetic algorithm as an optimisation method. In the papers of Bottani and Montana (2010) and Wangphanich, Kara and Kayis, (2010) the simulation based approach to this topic can be found.

2.2.3 Picking-packing supporting models

The picking and packing problem refers to several issues: the optimal or near optimal warehouse design, which would allow for quick stock picking based on pickers path reduction, pickers routing problem, order-picking policies, allocation of inventory, optimal fitting of the stock in the transportation vehicle, including pallet design and packing etc.

The picking costs have been recognized as one of the highest costs among warehousing costs, approximately 50-75% of the total (Coyle et al., 1996). It refers mostly to labour costs, therefore the optimisation of the picking time problem is often presented in the literature. Although the automation of the picking processes definitely brings significant cost reduction, the manual order-picking still plays a major role, due to diversity in the size and shape of warehouse stock (Bidgoli, 2010). Therefore several models have been developed in order to facilitate optimal order picking system design. Roodbergen, Sharp and Vis (2008), for instance, developed a linear programming model using statistical estimation of walking distances of pickers. Hall (1993) optimised the order-picking path within one warehouse block only. Smolic-Rocak et al. (2010), however, built the shortest path dynamic programming model for a given entire warehouse layout. In the work of Gu, Goetschalckx and McGinnis (2007) the optimisation of a picker’s travel routes was achieved by optimal assignment of products to particular storage locations.

Gagliardi, Renaud and Ruiz (2007) created a discrete event simulation-based model aimed at improvement of warehouse performance at picking and storage operations. It addresses the concerns of design of the storage areas and picking characteristics. The storage allocation influence on the warehouse's performance is discussed in the paper. Different scenarios are compared and evaluated to obtain the best results. The picker performance has significantly increased with optimal design of the storage area.

Petersen and Aase (2004) presented a simulation-based model of the existing picking area which was designed to improve the order-picking process with respect to both reduced cost and increased performance. The sensitivity analysis involved the experimentation with the shape of the warehouse, size of order, warehouse layout, pickers routing and the location of products in the storage area. In view of the former, the researchers presented several methods of picking to identify which of the examined methods brought the most improvement to the picking process. It concluded that batch-picking grossly affects the cost reduction in a given system.

2.2.4 Manufacturing supporting models

Production modelling refers to two issues here, namely the minimisation of total production cost based on inventory, delivery and planning as well as production scheduling. The first issue was addressed in the paper of Bard and Nananukul (2010). The linear programming model of a single product manufacturing system was built by considering the production capacity limits. Another linear programming model was introduced by Winston (1998). It considered a one-product inventory-production planning system. The demand was assumed to be known in advance, which made the model unrealistic and not applicable for most of real life scenarios. Yong, Wang and Lai (2009), in turn developed a convex programming production-inventory model for deteriorating products in order to obtain optimal scheduling of the production of goods being sold in multiple markets with different peak seasons. Lee and Yoon (2010) introduced an integer programming model to identify the optimal production schedule in addition to batch sizes to minimise the total work-in-progress inventory cost. The genetic algorithm approach to optimal manufacturing system design and scheduling can be found in Svancara, Kralova and Blaho (2012) and Qiao, Ma, Li and Yu (2013).

2.2.5 Inventory and replenishment supporting models

The inventory or replenishment problem is usually approached by computing the optimal order quantity or economic order quantity (EOQ). Regardless of the company strategy, usually the aim is to minimise the storage cost by the reduction of inventory held in the warehouse. This frequently results in more regular orders and deliveries. Different strategies are applied, depending on the type of product, to achieve the appropriate balance between maximising the chances of demand satisfaction and minimising the inventory. The linear programming (Janssens and Ramaekers 2011, Amaya, Carvajal, and Castaño, 2013) and other mathematical optimisation models - for instance the non-linear programming (Padmanabhan and Vrat, 1990) and fuzzy geometric programming (Samadi, Mirzazadeh and Pedram, 2013) - can be found in the literature. The reviewed models are discussed according to a few key criteria as listed under the bullet poits below. The more realistic the assumptions used while developing the model, the more complexity is added to the mathematical description. For this reason many authors have focused on less realistic assumptions, which is reasonable when considering the complexity of the mathematical description. The references presented in the categories below have been given additional local reference numbers in square parentheses '[ ]' for ease of presentation in Table 2-2.

- Nature of demand - whether it is deterministic, stochastic or totally random.

The models built under the assumption of having a prior knowledge about future demand are applicable only in a limited number of real world cases. Although, in real case scenarios the demand is usually not fully predictable, there might be some demand prediction done based on past data records, for instance with regard to seasonality of products. Though, the EOQ modelling for deterministic demand is rarely applicable in industry, it can be observed that there are relatively few academic papers, which present the demand as totally unpredictable and unknown. Deterministic demand can be found in models of Avinadav and Arponen (2009) [2], Chung and Liao (2009) [3], Hsu and Wen-Kai (2009) [4], Panda, Saha and Basu (2008) [5], Konstantaras and Skouri (2010) [8], Ghiani, Laporte and Musmanno (2004) [11] and Chungt (1998) [12]. The stochastic demand consideration can be found in papers of Madadi, Kurz and Ashayeri (2010) [6], Baumol and Vinod (1970) [7], Edwin, Cheng, and Wang (2007)

[9] Strack and Pochet (2010) [10], Xiong and Helo (2006) [13] and Ehrhardt (1997) [14]. The random demand is less common in the literature and can be found for instance in Dutta, Chakraborty and Roy (2005) [1].

- Nature of lead time – whether it is considered to be different than zero, fixed or variable.

The importance of considering the non-zero, albeit fixed lead time was appreciated in the literature by many researchers (Chakraborty and Roy (2005) [1], Avinadav and Arponen, 2009 [2], Chung and Liao, 2009 [3], Hsu and Wen-Kai, 2009 [4], Panda, Saha and Basu, 2008 [5], Konstantaras and Skouri, 2010 [8], Ghiani, Laporte and Musmanno, 2004 [11], Chungt, 1998 [12], Madadi, Kurz and Ashayeri, 2010 [6], Baumol and Vinod, 1970 [7], Edwin, Cheng, and Wang, 2007 [9], Strack and Pochet, 2010 [10] and Xiong and Helo, 2006 [13]). The papers published under the assumption that lead time is negligible and goods appear in the warehouse / factory the moment orders are placed, are still prominent in recent literature (Stadtler and Sahling, 2013). Such an approach seems suitable for some particular companies. Nevertheless the range of applications of models based upon such an assumption is narrower. In the most optimistic scenario the lead time is fixed and known in advance. Although having a fixed lead time often simplifies the mathematical description of the model the consideration of a varying lead time makes the model more realistic and easier to apply in the industry. Ehrhardt, 1997 [14], for instance, considered varying lead time in his model.

- Deterioration of products – whether not considered, fixed or increasing with time

Not every product is perishable, therefore there is not a need to consider the deterioration of products at all times. For instance the following authors developed the inventory models for non-perishable products: Chakraborty and Roy (2005) [1], Konstantaras and Skouri (2010) [8], Ghiani, Laporte and Musmanno (2004) [11], Chungt (1998) [12], Madadi, Kurz and Ashayeri (2010) [6], Baumol and Vinod (1970) [7], Edwin, Cheng, and Wang (2007) [9], Strack and Pochet (2010) [10], Xiong and Helo (2006) [13], Ehrhardt (1997) [14]. However, if one wants to make the model more universal and applicable in the perishable and non-perishable industry, the deterioration of goods should be taken into account. A fixed deterioration rate might be realistic for some products. For instance all yoghurts produced on one particular day will expire on one the same day. The appropriate models for such a scenario can be found in

Avinadav and Arponen (2009) [2], Chung and Liao (2009) [3], Hsu and Wen-Kai (2009) [4], Panda, Saha and Basu (2008) [5]. If the model allows for the deterioration rate to vary over time it makes the model more realistic for the same type of products (for e.g. fruits, which do not have a shelf-life). Such models become more frequently encountered in the literature: Sarkar and Sarkar (2013), Sett, Sarkar, and Goswami, (2012) or Sarkar, (2012).

- Number of products considered

The single-product models, though relatively popular in the literature, such as in the paper of Avinadav and Arponen (2009) [2], Chung and Liao (2009) [3], Chakraborty and Roy (2005) [1], Hsu and Wen-Kai (2009) [4], Panda, Saha and Basu (2008) [5], Madadi, Kurz and Ashayeri (2010) [6], Baumol and Vinod (1970) [7], Konstantaras and Skouri (2010) [8], Ghiani, Laporte and Musmanno (2004) [11], Chungt (1998) [12] and Ehrhardt (1997) [14], are sutable only for a limited number of industrial applications. The multi-product models, such as those presented by Li, Edwin, Cheng, and Wang (2007) [9] Strack and Pochet (2010) [10] or Xiong and Helo (2006) [13], bring out the advantage of broader applications in real life case studies.

- Static or dynamic model

The disadvantage of many models introduced in the literature is the fact that they conduct the optimisation for one period ahead only (Chakraborty and Roy (2005) [1], Avinadav and Arponen, 2009 [2], Chung and Liao, 2009 [3], Hsu and Wen-Kai, 2009 [4], Panda, Saha and Basu, 2008 [5], Konstantaras and Skouri, 2010 [8], Ghiani, Laporte and Musmanno, 2004 [11], Chungt, 1998 [12], Madadi, Kurz and Ashayeri, 2010 [6], Baumol and Vinod, 1970 [7], Edwin, Cheng, and Wang, 2007 [9], Strack and Pochet, 2010 [10] and Xiong and Helo, 2006 [13]). or find the optimal order quantities for several periods ahead based on initial information (Ehrhardt, 1997 [14]). The dynamic models bring the advantage of conducting the on-line optimisation by updating current information at every time instance. They are rarely found in mathematical optimisation literature: Hung, Chew, Lee and Liu (2012).

Also, additional aspects to those mentioned can be found in the EOQ based models. Several models are incorporated within the delivery costs (e.g. (Madadi, Kurz and Ashayeri, 2010 [6]) and (Baumol and Vinod, 1970 [7])). Konstantaras and Skouri (2010) [8], for instance, considered a recovery inventory remanufacturing system within the model. Li, Edwin, Cheng, and Wang (2007) [9], in turn, incorporated the postponement policy. Strack and Pochet (2010) [10] considered a limited storage capacity. Laporte and Musmanno (2004) [11], considered not only the purchasing cost, but also the order rate costs. Chungt (1998) [12], in turn, developed the permissible delay in payment model.

Considering the above listed criteria, the reality of the model can be assessed. For instance, the EOQ non-linear programming model developed by Avinadav and Arponen (2009) [2] can be assessed as follows. The inventory consists of one product type only but the deterioration of product is considered as a fixed shelf-life period, the lead time is fixed, the demand is deterministic, expressed as a polynomial function. The order is placed only when the need occurs, which means that the optimal order quantity might be equal to zero. It can be noticed that the range of applications in real life is decreased by the simplifications of deterministic demand and single product consideration.

A summary of the criteria in respect to the paper references can be found in Table 2-2.

The criteria have been split into two categories: namely basic and advanced assumptions. The term 'basic assumption' refers to those, which decrease the complexity of mathematical model description, hence simplify the reality, while the term 'advanced assumption' refers to more realistic modelling, which increases the model description complexity.

Basic assumption:	Paper index:	Advanced assumption:	Paper index:
One product model:	[1], [2], [3], [4], [5], [6], [7], [8], [11], [12], [14]	Multiple product model:	[9], [10], [13]
Deterministic demand:	[2], [3], [4], [5], [6], [7], [8], [9] [11], [12]	Stochastic demand:	[1], [10], [13], [14]
Deterministic lead time:	[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13],	Stochastic lead time:	[14]
Deterioration of products not considered:	[1], [6], [7], [8], [9], [10], [11], [12], [13]	Deterioration of products considered:	[2], [3], [4], [5] [14]

Static model:	[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] [14]	Dynamic model:	

Table 2-2: The accuracy of OR models applied in replenishment system optimisation

Based on the presented examples drawn from the literature, it can be concluded that the simplicity of the mathematical description prevails at the expense of the accuracy of the actual system representation.

2.3 Application of Control Theory to Logistics

The current chapter concentrates on those control theory techniques, which have been found in the literature of logistics in general and inventory in particular. It also justifies the applicability and advantages of control theory to logistics and production systems. Finally the model predictive control applications in replenishment systems only are discussed separately (as shown in Figure 2-1).

Although the fundamental concept in control theory of a feedback loop is a natural property of any real life system, it is not necessarily visible in non-engineering fields. Nevertheless, if attention is given, it can be realised that the feedback loop exists in almost every environment, starting from biological evolution, to human behaviour, to decision making to business development. It can be noticed that every improvement is based on taking a corrective action with respect to the current situation. The corrective actions (which can be defined as system inputs for control modelling purposes) are taken according to the obtained outcome (which can be defined as a system output) and are often affected by independent reasons (system disturbances). Therefore the current information is fed back to the system for future decision making / adjustment, which enables straightforward consideration of system dynamics. Just as in the case of pure engineering systems, where the error signal in the negative feedback loop is used to reduce the unstable fluctuation of the output signal, the same can be done in non-engineering disciplines as well. It is just a matter of finding an appropriate mathematical representation of the system in the control theory domain. Therefore the inventory or production system can be represented as a feedback loop too. Once this representation is obtained, the appropriate control can be applied to improve the overall system performance or profitability. Control theory, in fact, offers a sufficient range of mathematical techniques that facilitate modelling and control of inventory-production systems, which makes it worthwhile focus. Utilisation of control theory can be a major breakthrough for decision making within the dynamic nature of the manufacturing supply chain industry.

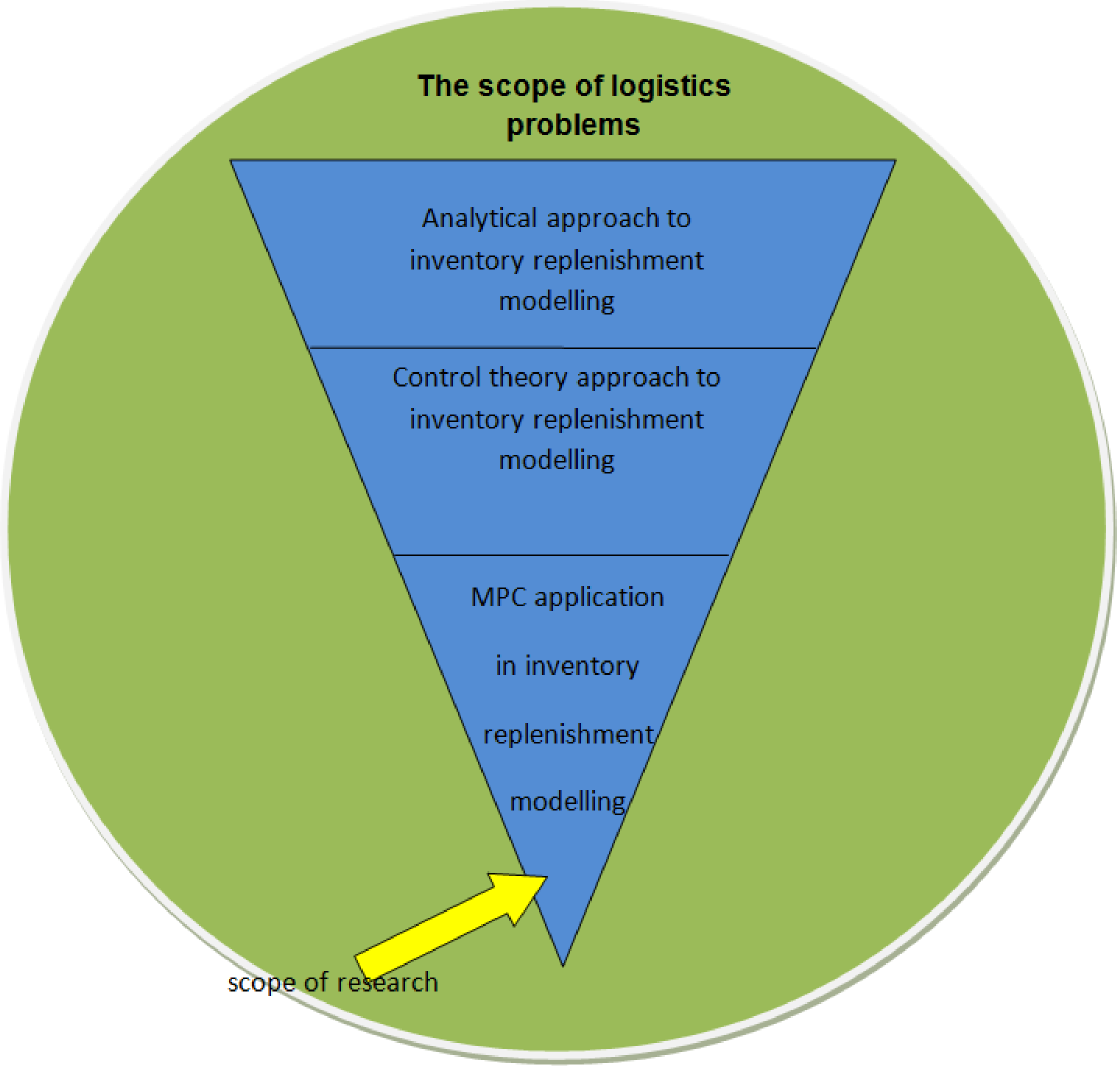


Figure 2-1: Narrowing the scope of the research

2.3.1 Historical review

The section herein provides a historical review of the first contributions made in the application of contol theory to logistics. The origins of application of regulation theory in the field of logistic systems control can be dated back to 1950's, when Simon (1952) applied the servomechanism algorithm to support the replenishment policy in a continuous single product inventory control system. The representation of inventory features, such as inventory level or order quantity, as system signals in the control theory domain brought an immediate substantial advantage in realistic inventory system modelling. Nevertheless the model was continuous, which makes its application somewhat limited. As a response, the discrete-time model with the application of servomechanism theory was presented by Vassian (1954) a few years later. The state space representation as well as the block diagram of an inventory system was introduced and control engineering attributes such as a transfer function, describing dynamics of the stock levels, reference inventory signal and feedback loop appeared in the above mentioned paper, to support the replenishment decision making process. The inventory system stability and transient response were determined. This showed the potential power of application of control theory techniques in inventory system modelling. Both authors, Simon and Vassian had applied the classic control approach to the inventory problem for the first time.

The next milestone in this field was achieved by Christen and Brogan (1971), who focused on optimisation of the simulation model of a production system as an early contribution to industrial control, based on matrix analysis and the states of a system. The analytical approach to a weighting function, defined to obtain satisfactory results in different scenarios, gained some attention in the literature. This approach was continued by several researchers and many techniques have been developed. Among others the main contributions were made by Mak, Bradshaw and Porter (1976) and Bradshaw and Porter (1974).

Forrester (1958) paid particular attention to fluctuating behaviour of inventory levels at a supply chain's nodes, later called the 'bullwhip effect', caused, among others, lead time lags and instabilities. He and his successors (Roberts, 1978; Coyle, 1997) tended to control the industrial dynamics based on simulation of equations of motion models. In his extensive study, Forrester (1958) showed adverse influences of lead time delay to the scale of inventory level fluctuation in the following periods and in different supply chain nodes. He discussed the dynamical behaviour of an industrial inventory system and suggested the application of feedback as a control. Since that time several papers have been published which have contributed to the field to allow one to predict, analyse, measure and avoid the bullwhip effect using a regulation or control theory approach. The first block diagram representation of on Inventory and Order Based Production Control System (IOBPCS) model and its dynamic analysis was proposed in the early 1980s by Towill (1982). The orders for the next period were assumed to be equal to the average orders plus a fraction of the shortage in inventory. In fact, Towill (1982) combined the transfer function approach with a suboptimal tuning parameter for the feedback gains. The demand was averaged as well, which was not necessarily practical. Since then the block diagram has been adopted by several researchers, see ((Wikner**,** Towill and Naim, 1991**)**, (Agrell and Wikner, 1996), (Grubbström and Wikner, 1996), (Samanta, and Al-Araimi., 2001), (Braun, et al 2003), (Gaalman, 2006), (Rodrigues and Boukas, 2006), (Hoberg, Bradley and Thonemann, 2007), (Zafra-Cabeza et al, 2007), (Venkateswaran, 2006), (Aggelogiannaki and Sarimveis, 2008), (Ignaciuk, and Bartoszewicz, 2011)) for further research and development. In the above works a target inventory level became to be modelled as an input to the system and treated as a reference signal, while the order quantity was taken as a manipulated variable. The actual inventory level was modelled as an output of the system and treated as a controlled variable to use for feedback. Subsequently such an approach to inventory system modelling has been applied by other researchers. Several tests were conducted for relevant design of inventory-production systems such as stability, tracking ability and/or noise rejection.

The IOBPCS model was subsequently extended and improved by adding more system components. Beside the already existing features, the new block diagram components inluded: target inventory level, feedback loop, the lead time delay, demand forecasting policy and work in progress (WIP). For the application of this model the forecast of demand is required, which is not always practical. Among IOBPCS contributions one can differentiate the continuous and periodic inventory level review approaches, see (Grubbström and Wikner, 1996). More details about the IOBPCS family and description of IOBPCS' components have been presented in (Lalwani, Disney and Towill, 2006).

Several different control theory attributes which have been used to model and analyse features of inventory-production systems can be found in the literature over the last decade. Important contributions to this scope of study have been done at Cardiff University, UK, with respect to control of the bullwhip effect ((Dejonckheere, et al., 2002, 2003, 2004), (Disney and Towill, 1996, 2002, 2003a, 2003b, 2006), (Gaalmanand and Disney, 2006), (Lalwani, Disney and Towill, 2006), (Potter et al., 2009), (Zhou, Disney and Towill, 2010)). In the considered papers, the researchers aimed at smoothing the ordering policies as well as inventory levels and presented the suitability of utilisation of control theory tools in terms of preventing the bullwhip system oscillations.

The examples of applications of control theory attributed to inventory systems which can be found in the literature are as follows. Autoregressive moving average (ARMA) system structure has been used by several researchers for different purposes. Gaalman (2006) and Gaalman and Disney (2006) used an ARMA system structure to model uncertain components of demand. Aggelogiannaki, Doganis and Sarimveis (2008), however, used an ARMA structure to model inventory position and a recursive least square (RLS) estimation in terms of demand forecasting. A state space representation, in turn, has been used for instance by Gaalman (2006), for demand modelling and Rodrigues and Boukas (2006) for stock accumulation representation. A differential equation model approach has been used by Ignaciuk and Bartoszewicz (2011, 2010b) as a stock balance equation.

The transfer function model has been commonly applied by different researchers. For instance Dejonckheereet. al. (2003, 2004) used it for order-up-to policy establishing with respect to the prevention of the bullwhip. Hoberg, Bradle and Thonemann (2007) formulated the transfer function of an inventory system for evaluation of the ordering signal stability with respect to different lead time delay values. Lin et al. (2004) presented a combined closed-loop transfer function representing material balance and information flow of a whole supply chain network.

The controllability and observability tests of supply chain systems can be found in a paper of Lalwani, Disney and Towill (2006). Dejonckheere et al (2003, 2004) introduced a damping factor for smoothing order policy and spectral analysis to obtain demand patterns and frequency response of a sinusoid demand. Gaalman and Disney (2006) applied a proportional controller in the inventory feedback loop and described the process of tuning it to prevent the bullwhip effect, while Grubbström and Wikner (1996) as well as Samanta and Al-Araimi (2001) applied a PID controller and combined it with fuzzy logic to maintain the stock at target level. Also estimation techniques can be identified in inventory management literature. The RLS method and Kalman filter, have been used by Aggelogiannaki and Sarimveis (2008) and Aggelogiannaki, Doganis and Sarimveis (2008) for lead time identification and by Gaalman (2006) and Gaalman and Disney (2006) for demand forecasting, respectively. A contribution to non-zero lead time modelling and compensation in periodic review systems has recently been reported by Ignaciuk and Bartoszewicz in (2010a, 2010c, and 2010d). In these papers the lead time delay has been taken into account in the *n*-th order state matrix and the optimal control action has been found by minimisation of a quadratic performance index.

2.3.2 Model predictive approach

Beside classical control techniques, which have been mentioned in the previous section the Model Predictive Control (MPC) has been also used as an optimisation tool by several researchers. The approach brings several advantages to inventory control. The MPC, being a moving horizon control theory technique, aims at finding the current and future control actions in the desired optimisation horizon, by on-line optimisation of the problem. The MPC approach then applies the first control action only. The system dynamics is updated at each sampling instant. It means that the feedback gains are updated according to the current situation and can be continually adjusted for non-linear models.

Perea-Lopez, Ydstie and Grossmann (2003) developed a dynamic decision framework for a multi-product, multi-echelon supply chain. The supply chain model includes plant, warehouses, distribution centres and retail levels. The MPC technique was applied to maximise the profit by reduction of the negative impact of unknown demand, considered as a system disturbance. The demand prediction is assumed to be known in advance and used by the model. Nevertheless, the demand error is regularly updated based upon past and current information. Upstream orders are inputs of a particular echelon, while shipments represent the echelon outputs. The orders are transferred from upstream to downstream echelons, while the shipment is moving the opposite direction. Nodes are assumed to handle as many products as the whole system is allowed to handle.

The model allows for consideration of many products by splitting each node to one product division. The received orders from the downstream nodes are accumulated during the day and shipped the day after, unless the inventory level is too low to satisfy the customer's requirements. Any kind of transportation, such as shipment from downstream to upstream nodes, delivery from external suppliers or shipment of goods to the end customers, is completed at the end of the day when the whole day's orders are accumulated. In the model it is represented as an additional term which usually is equal to zero beside the circumstances when the time is equal to the particular value representing the end of the day.

The transportation times and their costs between nodes are known with certainty. The authors consider two different raw material supply possibilities: the quick and costly response and the slow and economic response. The availability of raw sources is assumed to be infinite. The model objective function is related to the overall net profit and contains all cost and gross profits in the supply chain related to the production process, storage, transportation and sales. In the considered paper two optimal decision making approaches are examined, the centralised and the decentralised. The results showed that the centralised scenario leads the supply chain overall profit to be higher than in the case of the decentralised scenario.

Braunetal (2003) developed a decision support system for a single product, six-node and three-echelon production-inventory system. The discrete MPC aims at finding the optimal order quantities (system input) for reduced inventory levels. The demand (disturbance) prediction is assumed to be known in advance and it is updated based on a real demand pattern. The reference signal of the model is assumed to be equal to the predicted demand pattern increased by a safety stock level. The estimated order pattern (the predicted system input) is used as a predicted demand pattern (disturbance) for downstream echelon. The actual current value of disturbance is updated at each step. Goods posted to customers are understood as system outputs. The author proposed semi-decentralised decision making system. The separate model predictive controllers are used for each of the echelons so that they are shared between nodes included in each particular echelon. The forecast information is used by the downstream nodes. The transportation times between nodes are assumed to be known with certainty. The more time consuming routes are chosen only when necessary. It is achieved by setting the target order value to zero and applying different penalties for different ordering routes in case the orders placed for a particular route are greater than zero. The daily shipment capacities are constrained as well. The backorders are considered in the model. The time unit in the developed model is equal to one day, which enables modelling of lead time as a system delay. The model has been tested for different knowledge sharing strategies. It was shown that sharing the forecasted demand among all the nodes and suppressing the real demand pattern is beneficial to the company in order to achieve smoother order patterns, lower inventory levels and prevent inventory level fluctuations.

Wang, Rivera and Kempf (2007) presented the effectiveness of an MPC algorithm in strategic decision making in a semiconductor manufacturing process with respect to the system making sudden changes. In this case the MPC is only an element of a comprehensive decision making policy for a single product, single line and multi echelon manufacturing supply chain, where a fluid analogy is applied to illustrate the flow of materials. Here work in progress with respect to a manufacturing process is understood as flow of fluid in a pipe, while storage areas accumulating goods between manufacturing processes are treated as tanks. With respect to a control engineering representation of the developed model, the inventory levels are understood as system outputs and controlled variables, the orders are treated as first system input (manipulated variable) and demand is treated as a second input (disturbance signal). Several constraints, represented as linear equations, are taken into account for optimal decision making. These are: production and storage capacities, magnitude of starts, inventory levels, manipulated variable constraints, control variable constraints and work in progress capacity. The aim of MPC application in this case is to maintain the inventory level at a desired set point and satisfy the customers' requirements at the same time. Some uncertainties are taken into account in the model such as random breakdowns or mistakes of machines, which affect the lead time and storage levels. Therefore, the lead time is never known with certainty. To optimise the production scheduling, different speeds of assembling machines are considered and used in the model. The demand prediction is used and assumed to be similar to the average value of a real demand pattern. The actual demand is regularly updated.

Tzafestas, Kapsiotis and Kyriannakis (1997) presented a MPC application for production planning for multi-product manufacturing systems. The aim was to minimise the total cost of production and advertising so that the sales and inventory are maintained at desired levels. Selling prices are assumed to be fixed. In this case the production rate and advertising effort are manipulated variables (system inputs) while inventory and sales levels are system outputs (controlled variables). The demand (system disturbance) is controlled by an advertising parameter, which allows for demand prediction. The control variables have been constrained in the model. The paper presents the general idea of the developed model only. The applied case study is not explained in detail and the structure of studied system is not presented. The simulation results show that after some time the model tunes to achieve satisfying outcomes.

Li and Marlin (2009) presented the MPC decision framework to minimise the total supply chain costs with respect to storage costs, manufacturing costs, transportation costs and penalty backorders costs and at the same time to satisfy customers' requirements. The manufacturing rate, of semi-finished products, plant running time and transportation rates are model manipulated variables. Final product manufacturing rate, lead time and demand are assumed to be uncertain, which require a correlated uncertainty description. In simulation, two scenarios were examined: the case when demand, lead time and manufacturing rate are predicted correctly and the case when the prediction is not exact. The model performs very well in the first case while in the second case backorders occur. The reason for this is that the model tends to reduce inventory level to minimise costs and the inventory level is not always enough for the customers' requirements. Consideration of additional safety stock level increased total costs but prevented backorders at the same time.

Aggelogiannaki, Doganis and Sarimveis (2008) developed a MPC framework for optimisation of order quantities for production systems with consideration of system dynamics. Besides unknown demand, the model's unexpected behaviour was related to breakdowns of machines or running out of materials. The adaptive Finite Impulse Response (FIR) model has been applied to approximate the production system's dynamics. The output of the FIR system is production volume while the FIR system input is an order quantity. The RLS algorithm, as an on-line estimation technique has been used to estimate FIR model coefficients, which change over time. The inventory balance equation is represented by an autoregressive with exogenous input model (ARX), where demand represents system disturbance, inventory level represents system output and controlled variable and orders volume represent system input and manipulated variable. The MPC framework employs the objective function, which aims at maintaining a target inventory level. In the numerical example the authors compare the adaptive MPC framework with the Estimated Pipeline Inventory and Order Based Production Control System (EPIOBPCS) of Disney and Towill (2003b) and with non-adaptive MPC. It was noticed that the adaptive MPC is able to respond faster than EPIOBPCS and also avoids oscillations, which occur in case of non-adaptive MPC. Therefore the application of adaptive MPC has been justified as an advantage.

2.4 Summary and gap identification

The current chapter has familiarised the reader with the academic approach to the logistic problem modelling and optimisation. It has presented an overview of the methods applied in different logistic processes but focuses particularly on the inventory problem. From the literature review it can found that there are two ‘schools’ of inventory optimisation. The purely mathematical or business simulation based approach of the operational research community and the approach of the control theory community. The typical OR techniques often do not enable consideration of system dynamics due to increased complexity of the model. It indicates that the application of control theory to inventory is a reasonable path to be followed. It can be noticed that the application of control theory to inventory optimisation, though not a new idea, has not gained much attention yet, while the pure OR techniques have been exhaustively ‘battled to death’ in literature. The sophisticated techniques of control theory, though beneficial in application, might seem deterrent to be broadly applied by the OR community. As a response, this thesis aims firstly at informing both communities of the possible benefits of collaboration. Secondly it aims at bridging the gap and making the control tool of MPC available for the OR audience. The novel method can be easily adopted by other than control engineering researchers. Therefore, the gap has been identified and the research aims at exploring its richness and goes some way towards bridging this gap.

3 PRELIMINARY MODELLING AND SIMULATION MODELS

3.1 Introduction

The current research concerns an inventory replenishment system cost reduction, which was decided to be achieved by maintaining the inventory at a desired level (the benefits of keeping inventory level at a reference point and resulting cost reduction have been discussed in Section 0).

The current chapter presents the preliminary modelling of the inventory problem and the simulation models. It starts with elaborating the assumptions which have been used to build the inventory model (see Section 3.2.1). Then the process of translation of a conceptual model to the control theory domain is shown (see Section 3.2.2). A state space representation of the system (Section 3.2.3) facilitates consideration of the system dynamics (lead time delay, unknown demand and deterioration of products). The feedback loop enables an updating of the current inventory level on a sampled time instance basis and comparing it with the reference point.

Further, several control techniques have been applied for the established model (see Sections 1.1.1 -3.3). A dead beat controller enables the mathematical verification of the state space model accuracy with the actual system. A special focus, however, has been devoted to the model predictive control technique (Section 3.3) as it yield an inspiration to develop a novel method within this research. Finally, the novel technique of the inventory controller is presented in Section 3.4. Enabling the operational research (OR) specialists to use the method without the necessity of familiarising themselves with control theory principles is one of the main attribute of this research.

3.2 Inventory Modelling

3.2.1 Assumptions

In the considered inventory system (or distribution centre) the customer demand is assumed to be prior unknown. The goods that are needed to fulfil the customer’s demands are ordered from the remote supplier with a certain delay. Such a model is usually considered in inventory-production systems where the inventory model is a link between two other supply chain nodes such as a raw material supplier and factory or factory and wholesaler or wholesaler and retailer. The model allows for both scenarios - of consideration and for non-consideration of perishable goods. The warehouse is assumed to be initially empty, as it is a common practice in OR literature, and a single supplier case is considered.

3.2.2 Converting the inventory problem to the control domain

The current section presents a way of mapping of the exemplary inventory replenishment conceptual system into the control theory domain. To illustrate the transformation, Figure 3-1 shows an example of how the inventory (operational research) problem can be transformed into a control theory scheme. From Figure 3-1 it can be seen that the same system elements are labelled and understood in different ways within each domain.

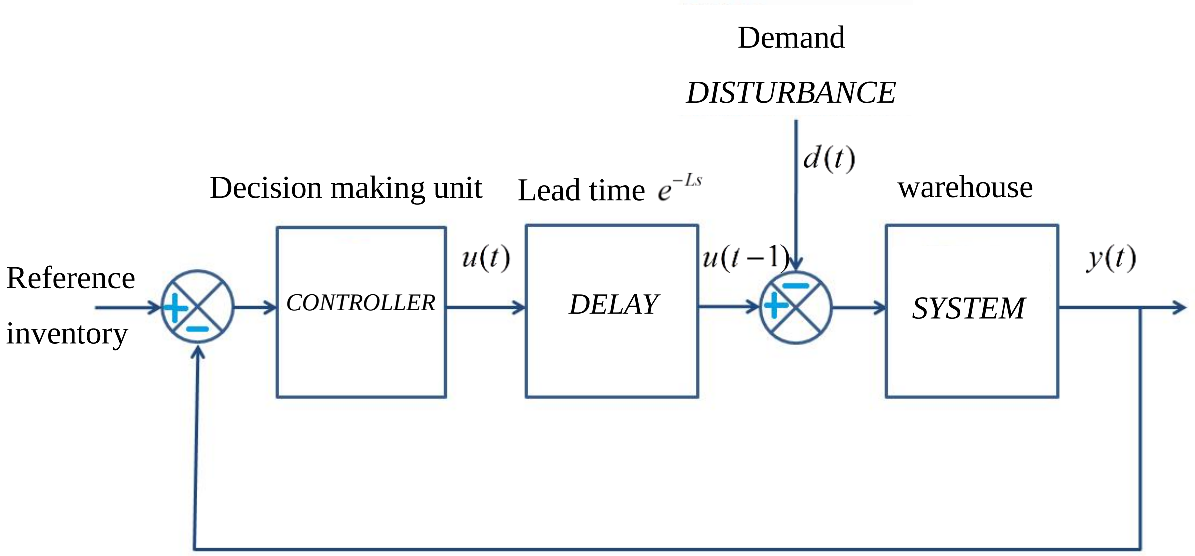


Figure 3-1: Transforming the inventory problem to a control scheme

Now, the initial problem of finding the optimal *order quantities* (u) to maintain an appropriate *inventory level* (y) with consideration of *lead time* (e^{-Ls}) and varying *demand* (d) turns into the following formulation: designing a controller to find an optimal *system input* (u) to maintain an appropriate *system output* (y) with consideration of *system time delays* (e^{-Ls}) and varying *disturbances* (d). Although in control theory literature, the disturbance is usually denoted as a positive signal, here it is negative, as it represents customer demand which due to the balance equation (3.1) must be subtracted from the inventory.

3.2.3 Inventory state space representation

The current section presents the initial model of the distribution centre. The continuous state space representation refers to a balance equation of the inventory level.

$$\begin{aligned} \dot{x}(t) &= \delta(t)x(t) + u(t-L) - d(t) \\ y(t) &= x(t) \end{aligned} \tag{3.1}$$

where

$x(t) \in \mathbb{R}$ is the current inventory level (system state)

$u(t) \in \mathbb{R}$ is the current order quantity (system input)

$d(t) \in \mathbb{R}$ is the current demand, backorders are allowed (system disturbance)

$y(t) \in \mathbb{R}$ is the current inventory level (system output)

$L \in \mathbb{N} \cup \{0\}$ is a lead time, the time needed for goods to be received in the warehouse after an order has been placed (system delay)

$\delta(t) \in (0,1]$ is the current goods survival rate and $1-\delta(t)$ refers to a time varying deterioration rate. The variation can be for instance dependent on the inventory level, such that $\delta(t) = \delta[y(t)]$. In any case it must satisfy $0 < \delta(t) \le 1$ and if backorders occur ($y(t) < 0$) then $\delta(t) = 1$. If $\delta(t)$ is set to 1, the model does not allow for consideration of perishable goods. The condition of $\delta(t)$ always being greater than zero assures the survival of some goods in the warehouse, where t denotes a time, and the first order is allowed to be placed at $t = 0$, $x(0) = 0$ and $y(0) = 0$.

Figure 3-2 shows the open loop system block diagram with the internal loop corresponding to stock accumulation.

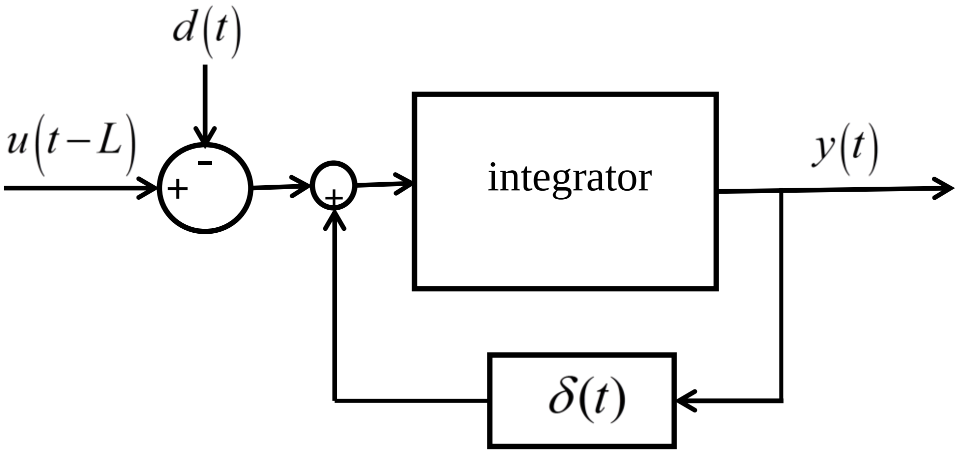


Figure 3-2: Open loop inventory system (warehouse) block diagram incuding stock deteriration

The next step refers to building the controller which will aim to maintain the inventory level at the reference level and prevent the system oscillations which are related to the lead time delay.

3.2.3.1 *Smith Predictor Decision Maker*

Because of the lead time delay, if the Smith predictor is not used, there would appear oscillations in the system, which in operational research terms corresponds to the bullwhip effect. The typical Smith predictor configuration is presented in Figure 3-3

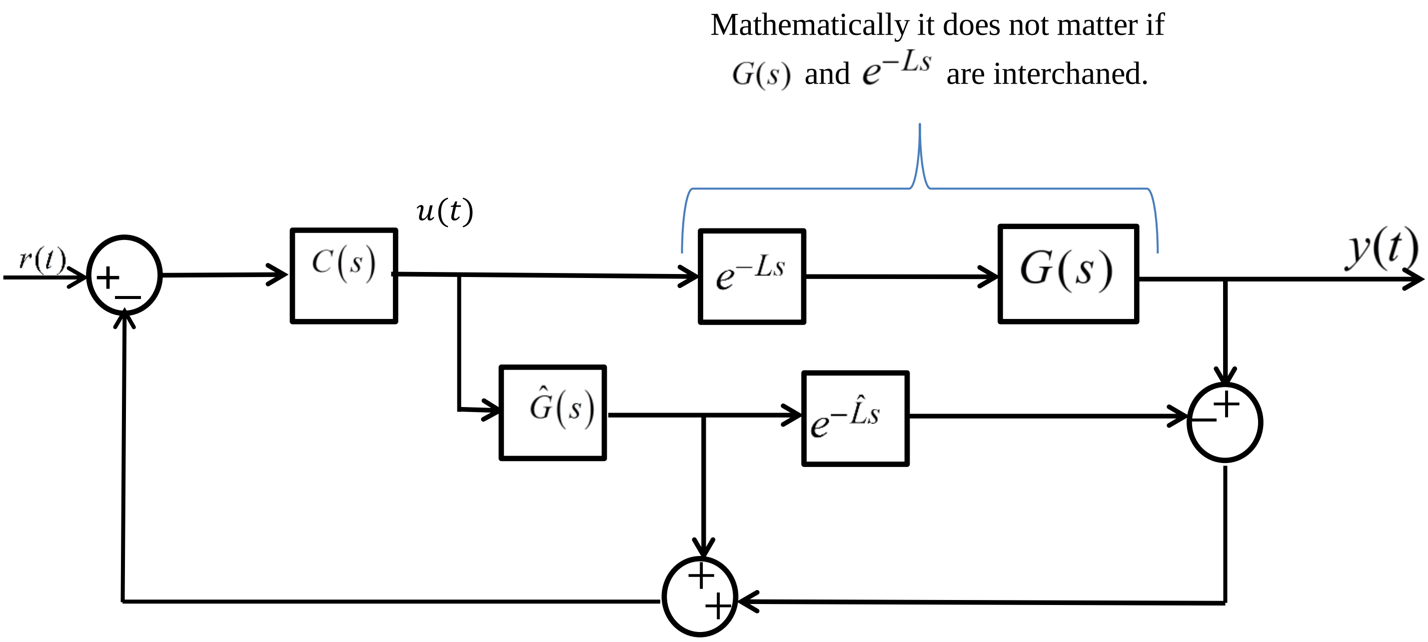


Figure 3-3: Typical Smith predictor configuration

In this thesis an alternative configuration of the Smith predictor (based on the non-perishable goods case, and developed in Ignaciuk and Bartoszewicz, 2010b) was applied in the form presented in the block diagram of Figure 3-4. Note that in the non-perishable good case $1-\delta(t)$ is zero, i.e. $\delta(t)=1$.

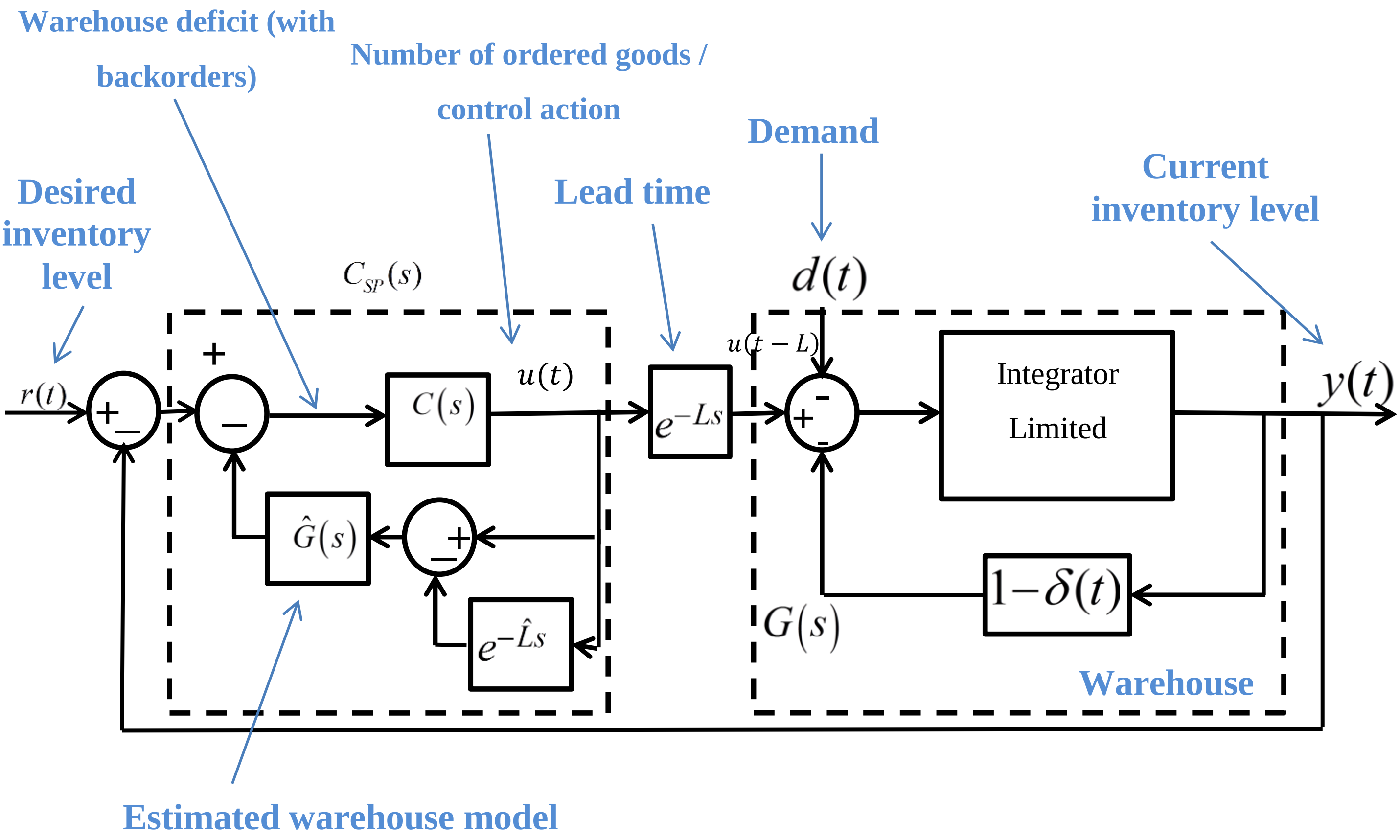


Figure 3-4: Smith predictor decision maker for inventory systems (block diagram)

The Figure 3-4 shows an alternative formulation of the Smith predictor, configured within a unity feedback closed loop system, where the feedback loop well is understood by the OR community and the Smith predictor is viewed as decision maker. The warehouse deficit is the 'Order signal' which feeds into to the decision maker. In the control engineering sense the control action, goods ordered, would be the output of the controller.

The transfer function C_{SP} has the form as follows

$$C_{SP}(s) = \frac{C(s)}{1 + C(s)G(s)\left(1 - e^{-Ls}\right)} \tag{3.2}$$

where $C(s)$ is a selected proportional controller (i.e. gain $C(s) = K$, where K is a positive constant) and $\delta(t)$ $G(s) = \dfrac{1}{s + (1-\delta)}$, with $0 < \delta(t) \leq 1$

In inventory control the transfer function $G(s)$ corresponds to the warehouse, where the future inventory level becomes equal to current inventory level decreased by the deterioration value δ increases by the number of delivered goods with a delay ($u(t-L)$) and decreased by number of demanded goods ($d(t)$).

The Smith predictor decision maker is included to show the complexity of the system, which should be ideally hidden from the OR community. It is elaborated here and in Chapter 4, Section 4.3, that the application of the Smith predictor as a decision maker is adequate but not ideal due to the perishable nature of the goods. The Smith predictor decision maker in this case is not tuned for the efficient handling of perishable goods (i.e. $\delta \neq 1$). Excluding the case $\delta \neq 1$ or including an integrator in $C(s)$ would improve the effectiveness of the method. However it would require a deep understanding of control the by the OR user, which is in opposition to the assumption driving the current research. As stated in this thesis the OR community requires simpler approaches to be developed, which does not need a deep knowledge of control to be applied.

3.2.4 Discrete state space representation

The above model assumes that the order quantity as well as the inventory levels are continuous, which is not a necessarily realistic assumption. The current section details the discrete state-space representation. In this formulation the lead time is taken into account and represented as a system delay within a state matrix. The inventory replenishment has been assumed to be carried out at regular intervals kT, where T is a review period and $k = 0,1,2\ldots$ Therefore, the lead time is assumed to be an integer multiple of the review period (sampling interval).

Consider the following state-space representation for the inventory system

$$\boldsymbol{x}\left[(k+1)T\right] = \boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{b}u(kT) + \boldsymbol{v}d(kT) \tag{3.3}$$

$$y(kT) = \boldsymbol{c}^T\boldsymbol{x}(kT) \tag{3.4}$$

Where

$L \equiv n$ is a system order and lead time delay

$\boldsymbol{x}(kT) \in \mathbb{R}^n$ is the state vector at time instance (kT)

$u(kT) \in \mathbb{R}$ is the system input, which represents the order quantity at time instance (kT)

$y(kT) \in \mathbb{R}^n$ is the system output, which represents the current inventory level at time instance (kT)

$d(kT) \in \mathbb{R}$ is a system disturbance, which represents demand (backorders are allowed) at time instance (kT)

$\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the system state transition matrix considering the lead time delay, and the time varying deterioration of goods denoted as $1-\delta(kT) \in [0,1)$ and $\boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{c}^T \in \mathbb{R}^n$, $\boldsymbol{v} \in \mathbb{R}^n$ are vectors, such that

$$\boldsymbol{A} = \begin{bmatrix} \delta(kT) & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{c}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.5}$$

Note that the matrix $\boldsymbol{A}$ is always the same structure and depends only on the deterioration rate value and its dimension is directly related to the lead time. Should lead time fluctuate, so the dimension will fluctuate.

The form of the vector v in (3.5) assures the subtraction of the demand from the current inventory level. It is assumed that the inventory level $y(kT)$ can be lower than zero, which corresponds to the allowance of backorders. The deterioration rate $\delta(kT)$ represents to the fraction of the total number of goods in the warehouse in the current period, which will be left (i.e., will survive) in the warehouses, until the next period. The remaining fraction $1-\delta(kT)$, then, is the number of goods which actually deteriorates within such a period. The deterioration rate is assumed to be time varying and for some applications it can be reasonable to be dependent on $y(kT)$, the current inventory level, such that $\delta(kT)=\delta\left[y(kT)\right]$. It makes the $\boldsymbol{A}$ matrix inventory level dependent. Defining $\delta\left[y(kT)\right]$ as a linear or non-linear monotonically decreasing or non-increasing function, it may be deduced that the goods are more likely to deteriorate in the warehouse if the number of goods increases. Regardless of the deterioration rate definition, it must be stressed that for any natural number k, for the deterioration rate such that $0<\delta(kT)\leq 1$ if backorders occur ($y(kT)<0$) then $\delta(kT)=1$. The latter condition prevents the goods to deteriorate when backorders occur.

Essentially, if $\delta(kT)$ is set to 1, the model does not allow for the consideration of perishable goods. The condition of $\delta(kT)$ always being greater than zero assures the survival of some goods in the warehouse.

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}[(k+1)T]=\begin{bmatrix} \delta(kT) & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}(kT)+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}u(kT)+\begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}d(kT) \tag{3.6}$$

$$y(kT)=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}(kT) \tag{3.7}$$

where

$x_i(kT)$, for $i = 1,2,3\ldots n$ represents the quantity being ordered at time instance $k-n+i-1$

Therefore

$$\begin{aligned}
x_1\left[(k+1)T\right] &= \delta(kT)x_1(kT)+x_2(kT)-d(kT) \\
x_2\left[(k+1)T\right] &= x_3(kT) \\
x_3\left[(k+1)T\right] &= x_4(kT) \\
&\vdots \\
x_{n-1}\left[(k+1)T\right] &= x_n(kT) \\
x_n\left[(k+1)T\right] &= u(kT)
\end{aligned} \tag{3.8}$$

This leads to the representation as follows, where the first line refers to the inventory balance equation of the form: the future inventory level is equal to current level increased by the number of goods arriving to the warehouse at the current time instance and decreased by the current sale/demand.

$$\begin{aligned}
x_1\left[(k+1)T\right] &= \delta(kT)y(kT)+u[(k-n+1)T]-d(kT) \\
x_2\left[(k+1)T\right] &= u\left[(k-n+2)T\right] \\
x_3\left[(k+1)T\right] &= u\left[(k-n+3)T\right] \\
&\vdots \\
x_{n-1}\left[(k+1)T\right] &= u\left[(k-1)T\right] \\
x_n\left[(k+1)T\right] &= u(kT)
\end{aligned} \tag{3.9}$$

From equations (3.9) it can be deduced that the orders reach the warehouse with a delay of $n-1$ periods.

The aim of this work is to maintain the inventory at the desired level with the assumption that the warehouse is initially empty ($\boldsymbol{x}(0)=0$). To do so, the reference inventory level signal, denoted $\boldsymbol{x_R}$, is defined such that

$$\boldsymbol{x_R} = \begin{bmatrix} x_r \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{3.10}$$

3.2.4.1 A stepping stone

The current section is shown to verify the correctness of discrete inventory model formulation. It verified by identifying what input signal must be applied to a system in order to eliminate the error $\boldsymbol{e} = \boldsymbol{x_R} - \boldsymbol{x}$ in the smallest number of time steps (which in fact refers to dead-beat controller). In this thesis the presentation of application of Dead-beat controller and its simulation results do not contribute to the leading thought. Therefore the current section is shown as a stepping stone to further mathematical formulations rather than to show preliminary formulation of one of tested algorithms.

Define the control vector

$\boldsymbol{f}^T = [f_1 \quad f_2 \quad \cdots \quad f_n]$, which will define the number of ordering items in period k such that $x_1 = x_r$

$$u(kT) = \boldsymbol{f}^T \boldsymbol{e} \tag{3.11}$$

It can be deduced that

$$u(kT) = [f_1 \quad f_2 \quad \cdots \quad f_n] \begin{bmatrix} x_r - x_1 \\ -x_2 \\ -x_3 \\ \vdots \\ -x_{n-1} \\ -x_n \end{bmatrix} \tag{3.12}$$

The solution is to apply feedback such that all poles of the system are at the origin of the complex z-plane.

Substituting (3.11) to (3.3) the closed loop system is obtained:

$$\begin{aligned} \boldsymbol{x}\left[(k+1)T\right] &= \boldsymbol{A}\boldsymbol{x}(kT)+\boldsymbol{b}\boldsymbol{f}^{T}\left[\boldsymbol{x}_{R}-\boldsymbol{x}(kT)\right]-\boldsymbol{v}d(kT)= \\ &=\left(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{f}^{T}\right)\boldsymbol{x}(kT)+\boldsymbol{b}\boldsymbol{f}^{T}\boldsymbol{x}_{R}-\boldsymbol{v}d(kT) \end{aligned} \tag{3.13}$$

An illustration of the closed loop system is given in Figure 3-5.

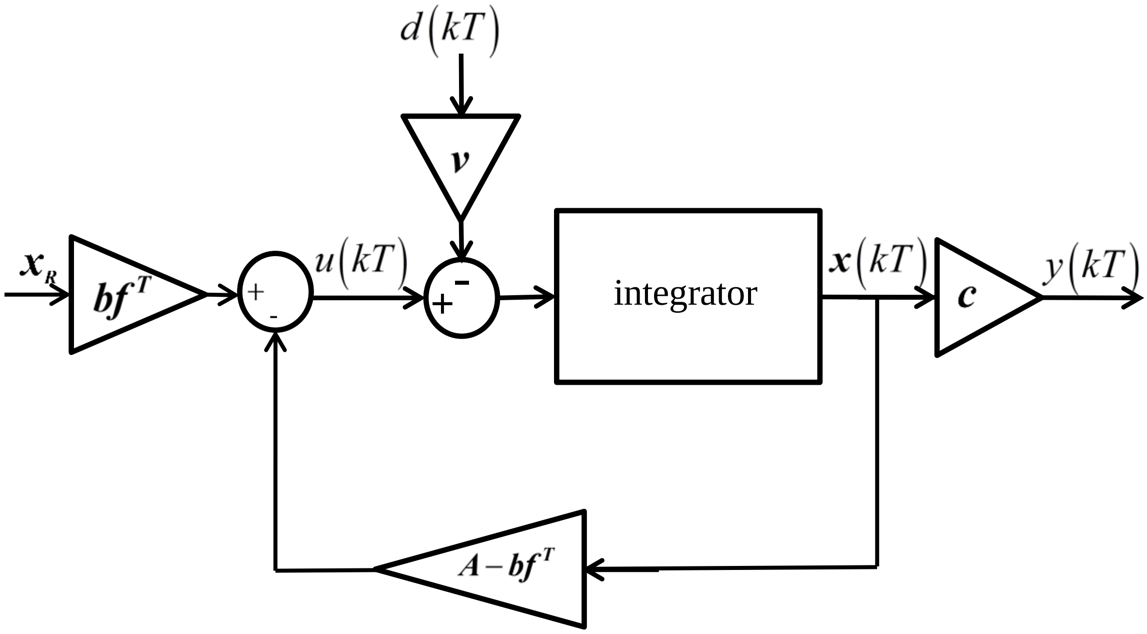


Figure 3-5: Dead-beat controller block diagram

Defining

$$\boldsymbol{A}_c = \boldsymbol{A}-\boldsymbol{b}\boldsymbol{f}^{T} \qquad \textit{and} \qquad \boldsymbol{b}_c = \boldsymbol{b}\boldsymbol{f}^{T} \tag{3.14}$$

it can be deduced that

$$\boldsymbol{A_c} = \begin{bmatrix} \delta(kT) & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -f_1 & -f_2 & -f_3 & -f_4 & \dots & -f_n \end{bmatrix} \tag{3.15}$$

$$\boldsymbol{b_c} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ -f_1 & -f_2 & -f_3 & -f_4 & \dots & -f_n \end{bmatrix} \tag{3.16}$$

So that

$$\boldsymbol{x}\left[(k+1)T\right] = \boldsymbol{A_c}\boldsymbol{x}(kT) + \boldsymbol{b_c}\boldsymbol{x_R} - \boldsymbol{v}d(kT) \tag{3.17}$$

To place all system poles at the origin of the *z*-plane, the eigenvalues of the matrix $\boldsymbol{A_c}$ must all be equal to zero.

To achieve this one considers the characteristic equation of the matrix:

$$det\left(\boldsymbol{A_c} - \lambda \boldsymbol{I}\right) = 0 \tag{3.18}$$

Therefore from the above and (3.15) it can be deduced that

$$det\left(\begin{bmatrix} \delta(kT)-\lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -f_1 & -f_2 & -f_3 & -f_4 & \dots & -f_n-\lambda \end{bmatrix}\right) = 0 \tag{3.19}$$

Expanding results in the expression

$$\left((-1)^n\right)\left(f_1 + \sum_{i=2}^{n} f_i \left(\lambda - \delta(kT)\right)\lambda^{i-2} + \left(\lambda - \delta(kT)\right)\lambda^{n-1} \right) = 0 \tag{3.20}$$

Further expansion of equation (3.20) it is possible to obtain the coefficients of the vector $\boldsymbol{f}^T$:

$$\left((-1)^n\right)\left(f_1+\sum_{i=2}^{n} f_i\left(\lambda-\delta\left(kT\right)\right)\lambda^{i-2}+\left(\lambda-\delta\left(kT\right)\right)\lambda^{n-1}\right)=$$

$$=\left((-1)^n\right)\left(f_1+\sum_{i=2}^{n}\left(f_i\lambda^{i-1}-\delta\left(kT\right) f_i\lambda^{i-2}\right)+\left(\lambda-\delta\left(kT\right)\right)\lambda^{n-1}\right)=$$

$$=\left((-1)^n\right)\left(f_1+f_2\lambda-\delta\left(kT\right) f_2+f_3\lambda^2-\delta\left(kT\right) f_3\lambda+f_4\lambda^3-\delta\left(kT\right) f_4\lambda^2\right)+\ldots$$

$$\ldots+\left((-1)^n\right)\left(f_n\lambda^{n-1}-\delta\left(kT\right) f_n\lambda^{n-2}+\left(\lambda-\delta\left(kT\right)\right)\lambda^{n-1}\right)$$

$$=\left((-1)^n\right)\left(f_1-\delta\left(kT\right) f_2+\lambda\left(f_2-\delta\left(kT\right) f_3\right)+\lambda^2\left(f_3-\delta\left(kT\right) f_4\right)\right)+\ldots$$

$$\ldots+\left((-1)^n\right)\left(\lambda^{n-2}\left(f_{n-1}-\delta\left(kT\right) f_n\right)+\lambda^{n-1}\left(f_n-\delta\left(kT\right)\right)+\lambda^n\right)=0 \qquad (3.21)$$

Since the steady state response is required, the poles must be settled at the origin, therefore the eigenvalues are assumed to be zeros.

Therefore

$$f_n = \delta\left(kT\right)$$

and

$$\begin{aligned} f_1 &= \delta\left(kT\right) f_2 \\ f_2 &= \delta\left(kT\right) f_3 \\ f_3 &= \delta\left(kT\right) f_4 \\ &\vdots \\ f_{n-1} &= \delta\left(kT\right) f_n \end{aligned} \qquad (3.22)$$

Therefore

$$
\begin{aligned}
f_n &= \delta(kT) \\
f_{n-1} &= \delta(kT)^2 \\
&\ldots \\
f_3 &= \delta(kT)^{n-2} \\
f_2 &= \delta(kT)^{n-1} \\
f_1 &= \delta(kT)^n
\end{aligned}
\tag{3.23}
$$

Finally

$$
\boldsymbol{f}^T = \left[\delta(kT)^n \quad \delta(kT)^{n-1} \quad \ldots \quad \delta(kT) \right] \tag{3.24}
$$

Therefore substituting (3.23) into (3.12) it can be deduced that

$$
u(kT) = \left[\delta(kT)^n \quad \delta(kT)^{n-1} \quad \ldots \quad \delta(kT) \right] \begin{bmatrix} x_r - x_1 \\ -x_2 \\ -x_3 \\ \vdots \\ -x_{n-1} \\ -x_n \end{bmatrix}
$$

which results in

$$
\begin{aligned}
u(kT) = \delta(kT)^n x_r - \delta(kT)^n x_1(kT) - \delta(kT)^{n-1} x_2(kT) + \\
- \delta(kT)^{n-2} x_3(kT) + \ldots - \delta(kT) x_n(kT)
\end{aligned}
\tag{3.25}
$$

From (3.9)

$$
\begin{aligned}
x_1(kT) &= \delta(kT)\,x_1\left[(k-1)T\right]+u\left[(k-n)T\right]-d\left[(k-1)T\right] \\
x_2(kT) &= u\left[(k-n+1)T\right] \\
x_3(kT) &= u\left[(k-n+2)T\right] \\
&\dots \\
x_{n-1}(kT) &= u\left[(k-2)T\right] \\
x_n(kT) &= u\left[(k-1)T\right]
\end{aligned}
\tag{3.26}
$$

From (3.7), and assuming that deterioration is dependent on inventory level such that $\delta(kT)=\delta\left[y(kT)\right]$ and substituting into (3.25) leads to

$$
\begin{aligned}
u(kT)=\delta\left[y(kT)\right]^n x_r-\delta\left[y(kT)\right]^{n+1} y\left[(k-1)T\right]-\delta\left[y(kT)\right]^n u\left[(k-n)T\right]+ \\
-\delta\left[y(kT)\right]^{n-1} u\left[(k-n+1)T\right]-\delta\left[y(kT)\right]^{n-2} u\left[(k-n+2)T\right]+\dots \\
\dots-\delta\left[y(kT)\right]u\left[(k-1)T\right]+\delta\left[y(kT)\right]^n d\left[(k-1)T\right]
\end{aligned}
\tag{3.27}
$$

Equation (3.27) enables a one by one consideration of the few initial order quantities to be obtained with the proposed approach mainly for verification that the results match the logical expectation. Therefore, it may be deduced as it will become clear in the following, that if the initial state space representation is formulated correctly for the application, the dead beat controller achieves its goal.

Initially the warehouse is assumed to be empty ($y\left[(k\le 0)T\right]=0$), no orders are placed for time instances $k\le 0$ ($u\left[(k\le 0)T\right]=0$) and no goods are demanded before the first order reaches the warehouse ($d\left[(k<n)T\right]=0$).

Considering the above assumptions from (3.27) the following may be deduced:

Let a and b denote two arbitrary time instances, then for $k=1$ it can be noticed that

$y\left[\left(k-1\right)T\right]=0$, and for any system dimension n $\forall b \leq n \quad \forall a \geq 1, a=n-b \quad u\left[\left(k-a)T\right]\right]=0$ and $d\left[\left(k-1\right)T\right]=0$, therefore, recalling the assumption that deterioration must be set to 1 for inventory less than or equal to zero

$$u\left(kT\right)=\delta\left[y\left(kT\right)\right]^{n} x_{r} \quad and \quad \delta\left[y\left(kT\right)=0\right]^{n}=1 \tag{3.28}$$

As if there are zero inventories, the goods cannot deteriorate, therefore from (3.28) eventually

$$u\left(kT\right)=x_{r}, \; for \quad k=1 \tag{3.29}$$

which is logical as the warehouse is assumed to be initially empty and the first order is equal to the inventory target level.

Further for $k=2$, if $n \geq 2$

$y\left[\left(k-1\right)T\right]=0$, since the ordered goods in time step $k=1$ have not yet reached the warehouse, $u\left[\left(k-1\right)T\right]=x_{r}$, from (3.29) and from the assumptions $d\left[\left(k-1\right)T\right]=0$ and $\forall b \leq n \quad \forall a \geq 1, \quad a=n-b \quad u\left[\left(k-a)T\right]\right]=0$, therefore

$$u\left(kT\right)=\delta\left[y\left(kT\right)\right]^{n} x_{r}-\delta\left[y\left(kT\right)\right]u\left[\left(k-1\right)T\right] \quad and \quad \delta\left[y\left(kT\right)\right]=1$$

$$\Downarrow \tag{3.30}$$

$$u\left(kT\right)=x_{r}-x_{r}$$

Therefore in the considered time instances

$$u\left(kT\right)=0 \tag{3.31}$$

which reflects the expectation, since until the first order reaches the warehouse, and demand is zero, there is no need to place new orders.

Analogously for any k, such that $1<k \leq n$

$y\left[(k-1)T\right]=0$, for such b that $k-b=1$ $u\left[(k-b)T\right]=x_r$, and $\forall a \neq k-1$ $u\left[(k-a)T\right]\right]=0$, $d\left[(k-1)T\right]=0$

Therefore for some

$$u(kT)=\delta\left[y(kT)\right]^n x_r-\delta(y)^b u\left[(k-b)T\right] \quad and \quad \delta\left[y(kT)\right]=1 \tag{3.32}$$

since the warehouse remains empty until the first order reaches the warehouse at the time instance $k=n$. Therefore

$$u(kT)=0 \tag{3.33}$$

which means that there is no order placed until the first order reaches the warehouse, which would appear to be logical again.

Further for $k=n+1$ it can be observed that
$y\left[(k-1)T\right]=0$, as the first order reached the warehouse in the current time instance k only

and $u\left[(k-n=1)T\right]=x_r$, $\forall a \neq n$ $u\left[(k-a)T\right]\right]=0$. Also a demand greater than or equal to zero appears.

Therefore,

$$u(kT)=\delta\left[y(kT)\right]^n x_r-\delta\left[y(kT)\right]^n y\left[(k-n)T\right]+\delta\left[y(kT)\right]^n d\left[(k-1)T\right] \tag{3.34}$$

If $\delta\left[y(kT)\right]=1$,

$$u(kT)=d\left[(k-1)T\right] \tag{3.35}$$

which means that the size of the orders placed is always equal to the amount of goods which have been sold one period backward.
Otherwise

$$u(kT)=\delta\left[y(kT)\right]^n d\left[(k-1)T\right] \tag{3.36}$$

which means that the number of goods chosen by the controller to be ordered is smaller than the number of goods demanded. This means that the demand might not be satisfied, depending on the chosen reference inventory level.

From the above reasoning, it can be noticed that, although, the dead-beat controller is chosen as an optimal order quantity finder, it may not be the most profitable in the case of system time delay. Nevertheless, through mathematical reformulation, its construction has highlighted, that the inventory state space model is designed correctly, which was indeed the purpose of the verification.

3.3 Model-based Predictive Control Approach

Defining $\Delta\boldsymbol{x}\left[(k+1)T\right]$ such that

$$\Delta\boldsymbol{x}\left[(k+1)T\right]=\boldsymbol{x}\left[(k+1)T\right]-\boldsymbol{x}(kT) \tag{3.37}$$

from (3.3) it follows that

$$\Delta\boldsymbol{x}\left[(k+1)T\right]=\boldsymbol{A}\left\{\boldsymbol{x}(kT)-\boldsymbol{x}\left[(k-1)T\right]\right\}+\boldsymbol{b}\left\{u(kT)-u\left[(k-1)T\right]\right\}+\boldsymbol{v}\boldsymbol{q}(kT) \tag{3.38}$$

where

$$\boldsymbol{q}(kT)=d(kT)-d\left[(k-1)T\right] \tag{3.39}$$

Defining $\Delta u(kT)$, such that

$$\Delta u(kT)=u(kT)-u\left[(k-1)T\right] \tag{3.40}$$

from (3.37), (3.39) and (3.40) it follows

$$\Delta\boldsymbol{x}\left[(k+1)T\right]=\boldsymbol{A}\Delta\boldsymbol{x}(kT)+\boldsymbol{b}\Delta u(kT)+\boldsymbol{v}\boldsymbol{q}(kT) \tag{3.41}$$

To create the closed loop system, $\boldsymbol{x_c}$ as a new state vector is defined such that

$$\boldsymbol{x_c}(kT)=\left[\Delta\boldsymbol{x}(kT)^T\ y(kT)\right]^T \tag{3.42}$$

From (3.4) and (3.41) it follows that

$$y\left[(k+1)T\right]-y(kT)=\boldsymbol{c}^T\Delta\boldsymbol{x}\left[(k+1)T\right]=\boldsymbol{c}^T\boldsymbol{A}\Delta\boldsymbol{x}(kT)+\boldsymbol{c}^T\boldsymbol{b}\Delta u(kT)+\boldsymbol{c}^T\boldsymbol{v}\boldsymbol{q}(kT) \quad (3.43)$$

so that (3.42) and (3.43) leads to new state equation

$$\boldsymbol{x_c}\left[(k+1)T\right]=\boldsymbol{A_c}\boldsymbol{x_c}(kT)+\boldsymbol{b_c}u(kT)+\boldsymbol{v_c}\boldsymbol{q}(kT) \quad (3.44)$$

$$y(kT)=\boldsymbol{c}^T{}_{\boldsymbol{c}}\boldsymbol{x_c}(kT) \quad (3.45)$$

where

$$\boldsymbol{A_c}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{O}_{1\times \boldsymbol{n}+1}\\ \boldsymbol{c}^T\boldsymbol{A} & 1\end{bmatrix}\in\mathbb{R}^{(n+1)\times(n+1)},\quad \boldsymbol{b_c}=\begin{bmatrix}\boldsymbol{b}\\ \boldsymbol{c}^T\boldsymbol{b}\end{bmatrix}\in\mathbb{R}^{n+1},\quad \boldsymbol{v_c}=\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{c}^T\boldsymbol{v}\end{bmatrix}\in\mathbb{R}^{n+1} \quad (3.46)$$

and

$$\boldsymbol{c}_c^T=\left[\boldsymbol{0}_{1\times n}\,\boldsymbol{I}\right]\in\mathbb{R}^{n+1} \quad (3.47)$$

where

$$\boldsymbol{0}_{1\times(n+1)}=\left[000\ldots0\right]\in\mathbb{R}^{n+1} \quad (3.48)$$

The matrices in (3.46) represent the augmented model. The prediction model uses the objective function of the quadratic form to optimise the future order quantities within the optimisation horizon so that the inventory achieves the reference level. The objective function has the form as follows

$$J=\left(\boldsymbol{Y_R}-\boldsymbol{Y}\right)^T\left(\boldsymbol{Y_R}-\boldsymbol{Y}\right)+\Delta\boldsymbol{U}^T\boldsymbol{R}\Delta\boldsymbol{U}\rightarrow min \quad (3.49)$$

where $\boldsymbol{Y_R}$ is a reference inventory level vector.

Defining N_c as the control horizon and N_p as the prediction horizon, such as $N_c \le N_p$,

$\boldsymbol{R}$ = $u_r\boldsymbol{I}_{N_c\times N_p}$ and $u_r \ge 0$ as a tuning parameter,

$\Delta\boldsymbol{U}$ is a control trajectory vector of future orders such that

$$\Delta\boldsymbol{U}=\left[\Delta u(kT)\quad \Delta u[(k+1)T]\quad \Delta u[(k+2)T]\quad \ldots\quad \Delta u[(k+N_c-1)T]\right]^T \quad (3.50)$$

where $u(k+i)$ for $i=0,1,\ldots N_c-1$, are future order control signal values to be optimised.

$\boldsymbol{Y}$ is a predicted inventory level signal represented by the vector

$$\boldsymbol{Y}=\left[y\left[(k+1|k)T\right] \quad y\left[(k+2|k)T\right] \quad y\left[(k+3|k)T\right] \quad \ldots \quad y\left[\left(k+N_p|k\right)T\right]\right]^T \tag{3.51}$$

where $y(k+i|k)$ is the *i*-step ahead prediction of the inventory level vector.

$\boldsymbol{Y_R}$ defines the target inventory level signal. Assuming that

$$\boldsymbol{Q}(kT)=\left[\boldsymbol{q}(kT) \quad \boldsymbol{q}\left[(k+1)T\right] \quad \boldsymbol{q}\left[(k+2)T\right] \quad \ldots \quad \boldsymbol{q}\left[(k+N_c-1)T\right]\right]^T \tag{3.52}$$

is a zero-mean white noise sequence, the predicted inventory level variables can be represented by the matrix form

$$\boldsymbol{Y}=\boldsymbol{F}\boldsymbol{x_c}(kT)+\boldsymbol{\Phi}\Delta\boldsymbol{U} \tag{3.53}$$

where

$$\boldsymbol{F}=\begin{bmatrix} \boldsymbol{c_c^T A_c} \\ \boldsymbol{c_c^T A_c^2} \\ \boldsymbol{c_c^T A_c^3} \\ \vdots \\ \boldsymbol{c_c^T A_c^{N_p}} \end{bmatrix}$$

$$\textit{and} \tag{3.54}$$

$$\boldsymbol{\Phi}=\begin{bmatrix} \boldsymbol{c_c^T b_c} & 0 & 0 & \ldots & 0 \\ \boldsymbol{c_c^T A_c b_c} & \boldsymbol{c_c^T b_c} & 0 & \ldots & 0 \\ \boldsymbol{c_c^T A_c^2 b_c} & \boldsymbol{c_c^T A_c b_c} & \boldsymbol{c_c^T b_c} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{c_c^T A_c^{N_p-1} b_c} & \boldsymbol{c_c^T A_c^{N_p-2} b_c} & \boldsymbol{c_c^T A_c^{N_p-3} b_c} & \ldots & \boldsymbol{c_c^T A_c^{N_p-N_c} b_c} \end{bmatrix}$$

From the above description of $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ it can be noticed that $\boldsymbol{Y}$ shown in (3.53) is already extensive in description and time consuming in calculation.

The MPC approach uses past and current information to predict the future inventory levels. The optimisation of order quantities is carried out over a fixed prediction horizon N_p. From (3.50) and (3.53) the control order quantities $\boldsymbol{\Delta U}$ can be derived such that

$$\Delta \boldsymbol{U} = \left(\boldsymbol{\Phi^T \Phi} + \boldsymbol{R}\right)^{-1}\left[\boldsymbol{\Phi^T Y_R} - \boldsymbol{\Phi}^T \boldsymbol{F x_c}\left(kT\right)\right] \tag{3.55}$$

Assuming that

$$\boldsymbol{Y}_R = \overbrace{\begin{bmatrix}1 & 1 & 1 & \dots & 1\end{bmatrix}}^{N_c}{}^{T} y_R\left(kT\right) = \overline{\boldsymbol{Y}}_R y_R\left(kT\right) \tag{3.56}$$

where y_R is a reference inventory level, this gives the optimal order quantity vector for the replenishment inventory. If not, attention is given to the size of orders during the optimisation process, and, as a result $u_r = 0$, then the optimal $\Delta \boldsymbol{U}$ for the control order signal is obtained as

$$\Delta \boldsymbol{U} = \left(\boldsymbol{\Phi^T \Phi}\right)^{-1}\left[\boldsymbol{\Phi^T \overline{Y}_R} y_R\left(kT\right) - \boldsymbol{\Phi^T F x_c}\left(kT\right)\right] \tag{3.57}$$

or

$$\Delta \boldsymbol{U} = \left(\boldsymbol{\Phi^T \Phi}\right)^{-1}\left[\boldsymbol{\Phi^T \overline{Y}_R} y_R\left(kT\right) - \boldsymbol{\Phi^T F x_c}\left(kT\right)\right] \tag{3.58}$$

$$\text{if } u_r \neq 0. \tag{3.59}$$

Because of the MPC principle the first element of $\Delta \boldsymbol{U}$ is applied only, such that

$$\Delta u\left(k\right) = \overbrace{\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}}^{N_c}{}^{T}\left(\boldsymbol{\Phi^T \Phi}\right)^{-1}\left(\boldsymbol{\Phi^T \overline{Y}_R} y_R\left(kT\right) - \boldsymbol{\Phi^T F}\left(kT\right)\right) \tag{3.60}$$

or

$$\Delta u\left(k\right) = \overbrace{\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}}^{N_c}{}^{T}\left(\boldsymbol{\Phi^T \Phi} + \boldsymbol{R}\right)^{-1}\left(\boldsymbol{\Phi^T \overline{Y}_R} y_R\left(kT\right) - \boldsymbol{\Phi^T F}\left(kT\right)\right) \tag{3.61}$$

is the actual value recommended to be ordered by the warehouse manager at time instance kT where

Denoting

$$\Delta u(k) \quad = \quad \boldsymbol{K_y} y_R(kT) - \boldsymbol{K} x_c(kT) \tag{3.62}$$

such that $\boldsymbol{K_y}$ is a first element of $\left(\boldsymbol{\Phi^T\Phi}\right)^{-1}\boldsymbol{\Phi^T Y_R}$ or $\left(\boldsymbol{\Phi^T\Phi}+\boldsymbol{R}\right)^{-1}\boldsymbol{\Phi^T Y_R}$

and if $u_r = 0$

$$\begin{aligned} \boldsymbol{K} \quad &= \quad [\overbrace{1 \quad 0 \quad 0 \quad \ldots \quad 0}^{N_p - N_c}]^T \left(\boldsymbol{\Phi^T\Phi}\right)^{-1}\boldsymbol{\Phi^T F} \\ & \qquad\qquad \textit{and} \\ \boldsymbol{K} \quad &= \quad \left[\boldsymbol{K}_x\boldsymbol{K_y}\right] \end{aligned} \tag{3.63}$$

or

$$\begin{aligned} \boldsymbol{K} \quad &= \quad [\overbrace{1 \quad 0 \quad 0 \quad \ldots \quad 0}^{N_p - N_c}]^T \left(\boldsymbol{\Phi^T\Phi}+\boldsymbol{R}\right)^{-1}\boldsymbol{\Phi^T F} \\ & \qquad\qquad \textit{and} \\ \boldsymbol{K} \quad &= \quad \left[\boldsymbol{K}_x\boldsymbol{K_y}\right] \end{aligned} \tag{3.64}$$

if $u_r \neq 0$.

It can be seen that calculation of the order size which should be ordered at time instance kT described in the manner shown in (3.37)-(3.64), requires an understanding of control theory principles and might not be easily adoptable by non-control-engineering researchers and practitioners.

In the thesis initially the tuning parameter is initially neglected, as the smoothness of order quantities is not in a concern of the research.

3.4 Novel Inventory Controller

The current section introduces a novel contribution to inventory control. The developed controller is non-model based yet adaptive and simple in implementation with low computational costs. As previously mentioned it can be applied with only a minimum of control theory knowledge. It is mathematically equivalent to MPC, applied for the defined inventory model as shown in equations (3.3)-(3.10). MPC applied for that particular inventory state space model will be termed the Inventory model predictive control (IMPC) in this thesis, while the developed novel technique will be termed the inventory controller (IC). Only the formulation is presented in this section, further chapters show the development process, the mathematical equivalency and benchmarking of the new method against IMPC.

Being straightforward in application, the IC attempts to bridge the gap between the precision afforded by methods of control theory and the expectations of the OR community. Finally, the developed model, being applicable with a minimum of control theory knowledge, appears to be applicable not only in inventory optimisation, but also in other non-engineering applications, such as production planning, plant or animal culture prediction, property insurance profit optimisation, extracurricular courses/classes planning, food restaurant management and many others.

The IC requires the same assumptions as IMPC, of lead time delay between the moment the order is placed and delivery. The inventory level is reviewed and orders are placed periodically, the technique is applicable for a single supplier case where demand is assumed to be unknown and the warehouse to be initially empty. The goods are assumed to deteriorate according to a time-varying deterioration rate.

The developed IC is defined in the form of the proposition, the origins of the IC proposition within IMPC are elaborated gradually in the following sections.

IC Proposition:

Denoting $\boldsymbol{x_c}$ as vector of the form

$$\boldsymbol{x_c}(k) = \big[I(kT) - I[(k-1)T] \quad u[(k-n)T] - u[(k-n+2)T] \\ u[(k-n+2)T] - u[(k-n+1)T] \quad \dots \quad u(kT) - u[(k-1)T] \quad I(kT) \big]^T \tag{3.65}$$

where kT represents the k^{th} time instance, $I(kT)$ represents the k^{th} time instance stock level and $u(kT)$ defines the k^{th} time instance order size, and denoting $I_R(kT)$ as the reference inventory level at time instance kT, $\delta(kT)$ as the stock deterioration rate in time instance kT and $n \in \mathbb{N}$ as the lead time delay, $\boldsymbol{K}$ can be defined as the transposed vector of $n+1$ dimension, where $r \in \mathbb{N}$ denotes the gain vector $\boldsymbol{K}$ column number such that $\boldsymbol{K} = \left[K(1) \ldots K(r) \ldots K(n+1) \right]$. Then the following can be formulated:

$$\forall_n \forall_{r \neq 1} \qquad K(r) = \sum_{i=0}^{n-r+1} \delta(kT)^i \quad \textit{and} \quad K(1) = \sum_{i=1}^{n} \delta(kT)^i \tag{3.66}$$

For such a formulation of vector $\boldsymbol{K}$, the current optimal order quantity can be found as follows:

$$u(kT) = \begin{cases} 0 & \textit{if} \quad u\left[(k-1)T\right] + I_R(kT) - \boldsymbol{K x_c}(kT) < 0 \\ u\left[(k-1)T\right] + I_R(kT) - \boldsymbol{K x_c}(kT) & \textit{otherwise} \end{cases} \tag{3.67}$$

which defines the proposed inventory controller (IC).

The above proposition uses the pre-defined $\boldsymbol{x_c}$ vector of past information. It proposes a method of calculating the gain vector $\boldsymbol{K}$ and indicates how to use the appropriate information in calculating non-negative optimal order quantities.

The advantage of the proposed method is the fact that the calculation is done on-line, separately for each time instance. Therefore the time-varying deterioration rate $\delta(kT)$ can be decided separately for each time instance, also the lead time can be set differently for each time instance. Moreover, the deterioration rate in such a case can be judged to be dependent on the current inventory level $\delta\left[I\left(kT\right)\right]$, which for some applications might be more realistic (for instance the higher the inventory level, the more chances of deterioration of products). $\delta\left[I\left(kT\right)\right] = 1-\alpha I(kT)$, where $\alpha \in (0,\frac{1}{I_{\max}}))$. Since backorders are allowed in the model, the $\delta\left[I\left(kT\right)\right]$ approach enables the deterioration rate to be set back to *1*, when the inventory level goes below zero. Therefore the non-realistic assumption, that the product can still deteriorate if their associated value is lower than zero, is avoided.

3.5 Novel Method Properties

It can be realised that although the novel technique was inspired by MPC and noticing the MPC patterns in the mathematical description, the resulting proposal is no longer MPC. It is in fact a minimum variance approach, or dear beat controller, with an incorporated Smith predictor. Note that using the adjective 'predictive' in the title of the thesis refers to both, the inspiration of MPC and the predictive nature of the minimum variance controller to accommodate lead time, being incorporated within an inherent Smith predictor.

It can be noticed that the vector $\boldsymbol{x_c}(k)$ of the form shown in (3.65) is the MPC augmented model state vector. In the OR perspective though, it is a past and current information vector of inventory levels and order quantities.

The control gain vector $\boldsymbol{K}$ of the form elaborated in (3.66) is a mathematically reduced form of the MPC gain. The mathematical reduction of this control vector is one of the main contributions of this thesis, and Chapter 5 is dedicated to the mathematical demonstration. Here, from an OR point of the view, vector $\boldsymbol{K}$ can be found according to the straightforward procedure described in (3.66) and does not need to be related with the MPC gain to be understood or applied.

The procedure for obtaining optimal order quantity $u\left(kT\right)$, shown in (3.67) is in fact the MPC system input or the incremental control input shown in (3.40) of the form as in (3.62) bounded by a saturation condition. The OR user does not need to be concerned by the above fact as long as the user follows the recipe described in *IC Proposition*.

Chapter 4, Section 4.5 and Chapter 6 present the model response to different demand patterns, different deterioration rates and different lead time delays.

3.6 Novel Methods vs. Smith Predictor, Dead Beat Controller and MPC

3.6.1 Novel method

The novel method can be used by anyone familiar with adding, subtracting and multiplying vectors, therefore it can be expected to be usable by the OR community. The novel algorithm is designed to be used on a discrete time basis. Moreover no parameter tuning is relevant in the method, which emphasise the simplify of the algorithm and eliminate the control theory knowledge requirement. The procedure of the proposed approach is presented below:

1. Define the lead time delay n, reference inventory level I_R and the current deterioration date $\delta(k)$
2. Substitute the current and past information of inventory levels and order quantities accordingly to the form of the vector $\boldsymbol{x_c}(k)$ (3.65). Such information is usually stored in the warehouse system and is easily accessible
3. Using the defined n and $\delta(k)$ values, build vector $\boldsymbol{K} = \left[K(1)\ldots K(r)\ldots K(n+1)\right]$ accordingly to the description given in (3.66).
4. Use the vectors $\boldsymbol{K}$ and $\boldsymbol{x_c}(k)$ as well as the reference inventory level I_R and the order quantity of the time instance one step before (i.e. yesterday, if one per day is the sampling time) and substitute in the equation (3.67) to obtain the current order quantity.

As it can be noticed the method does not require definition of state space representation. Therefore it does not require familiarity with difference equations as well as state space representations, nor transfer functions not even control engineering principles such as: input and output signals, system delay or disturbance.

The novel technique is sufficiently mathematically reduced to be calculable in using Excel, with no need for sophisticated software such as MATLAB.

Section 3.6.2, Section 3.6.3 and Section 3.6.4 present the description of the knowledge required as well as the list of main procedure for the Smith predictor, dead-beat controller and MPC respectively, to support the statement that the novel techniques of IC is both, the less computationally costly and the easier to apply by the non-control familiar OR expert.

3.6.2 Smith predictor

To apply the Smith predictor on its own of the form shown in Section 3.2.3.1, the following minimum knowledge is required: familiarity with differential equations, understanding of control theory principles such as: input and output signals, state space representation, system delay, transfer function, disturbance, open loop system, closed loop system, block diagram, ways of converting block diagrams to transfer function. The steps needed to be completed to apply the Smith predictor are listed as follows:

1. Define continuous time state space representation
2. Build an open loop system block diagram with consideration of the system delay
3. Close the loops of the system
4. Identify the closed loop system transfer function $C(s)$ of Figure 3-4
5. Apply the Smith predictor in the block diagram of Figure 3-4
6. Find the equivalent transfer function of the Smith predictor
7. Implement the new transfer function form in the block diagram

To use the method MATLAB and/or Simulink are recommended. To do so the familiarity with both of them as well as basics of coding are needed.

3.6.3 Dead-beat controller

To apply the dead-beat controller on its own of the form shown in Section 3.2.4.1**Error! Reference source not found.** the following minimum knowledge is required: familiarity with vector and matrix operations, difference equations, understanding of control theory principles such as: input and output signals, discrete state space representation, system delay, system error, disturbance, open loop system, closed loop system, z-plane, determinant and eigenvalues. The steps needed to be completed to apply the dead beat controller are listed as follows:

1. Define the lead time delay n, reference inventory level I_R and the current deterioration rate $\delta(k)$
2. Define the matrix form of state space representation, where the matrix dimension is dependent on the system delay
3. Define the open loop system
4. Define a control vector
5. Close the loop of the system
6. Identify the closed loop state space representation
7. Find the characteristic equation of the system matrix (3.18)
8. Place the system poles in the z-plane origin
9. Find the control vector

To apply the dead beat controller MATLAB and/or Simulink are recommended. To do so the familiarity with both of them as well as basics of coding are needed.

3.6.4 Model predictive control

To apply MPC of the form shown in Section 3.3 the following minimum knowledge is required: familiarity with vector and matrix operations, difference equations, objective function of the vector form definition, and derivatives of function of vector form calculation. Understanding of control theory principles such as: input and output signals, discrete state space representation, system delay, system error, disturbance, open loop system, closed loop system, augmented model and moving horizon concept. The steps needed to be completed to apply the model predictive control are listed as follows:

1. Define the lead time delay n, reference inventory level I_R and the current deterioration rate $\delta(k)$
2. Define the matrix form state space representation, where matrix dimension is dependent on the system delay
3. Define the open loop system

4. Find the incremental form of state vector, control variable system input and disturbance signal.
5. Find the closed loop system of the augmented form
6. Denote augmented model form of state space representations
7. Find the $i \in \left[1, N_p\right]$ step ahead predictions of inventory levels
8. Reformulate the predicted inventory level vector to the form of (3.53). From such description it can be noticed how computationally intensive the MPC algorithm is. The matrices described in (3.54) are themselves constructed from elements which are already multiplication of matrices of the closed loop system augmented model.
9. Define objective function of the vector form
10. Find the minimum of the function
11. Determine the future order quantities for the minimum value of the function
12. Use order quantities to determine the MPC gain vector form
13. Apply one time instance only of the prediction
14. Repeat the process

To apply the MPC MATLAB and/or Simulink are recommended. To do so the familiarity with both of them as well as basics of coding are needed.

3.6.5 Calculation efficiency of the novel method

The novel method was benchmarked against IMPC method using a 'tic toc' function in MATLAB to measure the elapsed time of both algorithms. The simulation was run 30 times and the lowest time was picked for both algorithms for the same simulation settings (200 time instances, 5 days lead time delay, 100 items reference inventory level, 0.9 deterioration rate). The elapsed time of novel technique appeared to be equal to 0.008137s which is 81.36% of the IMPC elapsed time. The IMPC elapsed time was equal to 0.019905s. Therefore the novel inventory controller appeared to be computationally less heavy that initial IMPC.

3.7 Summary

In the current chapter the preliminary modelling of the inventory system have been shown. The inventory state space representation has been used for the application of a number of control algorithms. The dead beat controller application enabled the verification of accuracy of the initial state space model through the mathematical reformulation and comparison of the results obtained as expected. The model has been verified to be correct as an inventory model.

Further, the MPC has been formulated to show the design process for the verified inventory model. The objective function of the predicted inventory levels has been constructed for obtaining the vector of optimal future order quantities within the defined control horizon. The moving horizon rule was used to make the optimisation dynamic and update inventory level at every time instance.

Further, the novel inventory controller has been proposed in the form of a proposition. The controller does not require any control theory understanding to be applied. It can be adopted by the operational research community to obtain the precision of MPC for the given model. The controller is mathematically equivalent to the MPC used for a given model and its mathematical formulation is significantly reduced when compared with state space and MPC formulation. Its computational cost is lower, which was shown by measuring elapsed time of each of both algorithms, it does not require control theory knowledge and its applicability is not limited to inventory control only.

4 VERIFICATION OF THE PROPOSED IC APPROACH

4.1 Introduction

The main goal of this chapter is a verification of the proposed IC approach, rather than a detailed discussion of the simulation results. Although, initially, the chapter shows a few results for the Smith predictor, there is not much attention devoted to this technique. The chapter shows that the Smith predictor applied in a presence of the perishable goods on its own does not generate results which fulfil the thesis aim. It does not meet the requirements of keeping the inventory at the set level, but it soothes them instead. Note that without the lead time accommodation of the Smith predictor there would be oscillations. On the other hand the dead-beat control, in isolation, would not generate stable inventory levels in case of system delay and deterioration of goods. As it was elaborated in the Chapter 3, it was mainly applied to mathematically verify the accuracy of the inventory state space model, rather to improve inventory performance. Although significantly more simulations were run to test these two methods, only a selection of results have been chosen to support the discussion. Once identified that they are not relevant for this research, there is no additional analysis and detailed presentation of more experimental results done.

The main aim here is to show how the proposed IC method is equivalent to IMPC in terms of its steady-state response.

The IMPC itself, as a model based approach, incorporates the Smith predictor, and, therefore, the presented results of IMPC and IC are more relevant to what was planned to achieve in this research, namely developed a control scheme which could be used/adopted by the OR community.

To confirm the equivalency of results between the proposed IC and the IMPC, several different tests were conducted. The first numerical example was run to illustrate the manual calculation of the IC approach. It was the shown that the manually obtained results were identical to at least the 4^{th} decimal place with the IMPC simulation results. The second test was run to show that both methods give very similar gains in respect to engineering accuracy regardless of the deterioration rate value. Therefore the inventory levels obtained for both methods are again very similar. The third test was run to confirm that the IC and IMPC approaches generate very similar gains for arbitrarily sinusoidally varying reference inventory levels. This test shows that both techniques generate very similar inventory levels regardless of the demand patterns, seasonal as well as random or mixed patterns.

Simulation settings are listed at the beginning of each section, to allow the reader to reproduce the results. The discretisation period $T = 1$ day at any test.

4.2 Simulation settings justification

In each test of the current chapter as well as Chapter 6 and Chapter 7 the initial simulation settings are similar. The current section elaborates the choice of particular numerical values.

- Simulations time: *200* time instances, which refers to *200* working days. The choice of *200* days for small (few days lead time) is both popular in the OR literature and in practice, as it gives a good time perspective of system behaviour.
- Deterioration rates: different values chosen for different simulations and different test purposes. The choice of 0.9 refers to 90% of the items remaining good from one time instance to another. In respect to the OR literature the deterioration of such a degree seems reasonable (especially for products such as flowers fresh juices or preservative free food such as organic vegetables). Deterioration of 1 corresponds to 100% of the items remaining good from one time instance to another. It means that the considered products are not perishable. It can refer to products such as car parts. For some tests the value of the deterioration rate was chosen to be lower than 0.9 (0.5 for instance, which refers to 50% of goods surviving from one time period to another). It is not a necessarily realistic assumption in real life problems, but used here to show system properties and the behaviour to such a system variable.
- Reference inventory level: Reference inventory level refers to safety stock. In real-life case scenarios the safety stock often exceeds the expected demand. Especially in the case of uncertain lead time, unreliable supplier or long lead times the safety stock kept can be very high. The choice of 100 items of safety stock for an expected demand between 0 and 70 items is not uncommon in reality as well as in OR literature. The more reliable the replenishment strategy, the lower the reference inventory level can be. Ideally, the reference inventory of zero items refers to Just-in-Time policy.
- Demand pattern: The choice of seasonal demand pattern is popular in the OR literature. It is also practical for seasonal products such as Christmas trees or Easter sweats, or New Year fireworks but also for products which can be affected by weather (for

instanced increase sale of soft drinks in the summer, or summer clothes) and other seasonal factors. The demand of seasonal patterns where each of all values is kept at constant level before the new change occurs is a popular approach in the OR literature but not necessarily the most realistic, as in most of real-life cases the demand varies. Therefore a random pattern is considered as well for some of the simulations. The choice of random pattern is relevant for products where there is no noticeable seasonality. They can be everyday products, such as cosmetics or cleaning products. The demand in that case is more or less stable and the variations of demand are not that drastic. There is also a mixture of both demand patterns considered in the thesis as the most realistic demand pattern where the seasonality is visible but the demand slightly varies on a daily bases. This is not a popular approach in the OR literature due to its complexity.

- Lead time delay: It has been chosen to be relatively low for most of tests (comparing to simulation period) to increase the visibility of results. In practice such a lead time is popular in real-life problems within a first tier local supply network and it is necessary if the perishable products are considered. Such a choice of small lead time values is common in the OR literature.
- Warehouse is initially empty: this is a very common assumption in the OR literature.

4.3 Smith Predictor

The current section presents the simulation results for the Smith predictor described in Section 3.2.3.1. It was used for the continuous state space inventory representation to present initial steps taken in the research. Also it shows the initial stage results on the way to find a satisfactory inventory level control technique.

The simulation was run for *200* time instances, which here refers to *200* working days. The proportional term was set to *1*. The deterioration of product was set to *0.9* (which means that 90% of the items remain good from one time instance to another). The reference inventory level was set to *100* items, which in respect to the OR literature represents the safety stock of a realistic level. The demand was arbitrarily assumed to be seasonal and changing periodically between four allowed values: 70 (for time instances 6-55), 20 (for time instances 56-105), 50 (for time instances 105-155) and zero (for the remaining time instances). The demand was set to be zero for the first 5 days to prevent unnecessary backorders for those cases where the lead time is set to 5 or less days (before the first order reaches the warehouse) and to show what happens for the cases where the lead time is greater than 5 days. In particular the lead times of 1, 5 and 10 days were chosen.

Therefore the controller of equation (3.2) has a form as follows

$$C_{SP}(s) = \frac{0.5}{1 + (0.5/(s+1-0.9))\left(1-e^{-ns}\right)} \tag{4.1}$$

where n is a system lead time delay. The inventory level, the order quantities and demand pattern are presented in Figure 4-1 for $t = 0...200$, and Figure 4-2 - Figure 4-5 (zoomed for specific time instances when the demand suddenly changes) with respect to different lead time delay values. The goal is to test the model response in general and in respect to changes in the lead time values. The simulation was run for several different values of lead time delay, nevertheless only three results, for $n = 1$, $n = 5$ and $n = 10$ are presented to support the conclusions.

For the specific simulation values the system model of equation (3.1) has the form as follows

$$\begin{aligned} \dot{x}(t) &= 0.9x(t) + u(t-n) - d(t) \\ y(t) &= x(t) \end{aligned} \tag{4.2}$$

for $n = 1$, $n = 5$ or $n = 10$ and $d(t)$ values shown in Figure 4-1 for $t = 0...200$.

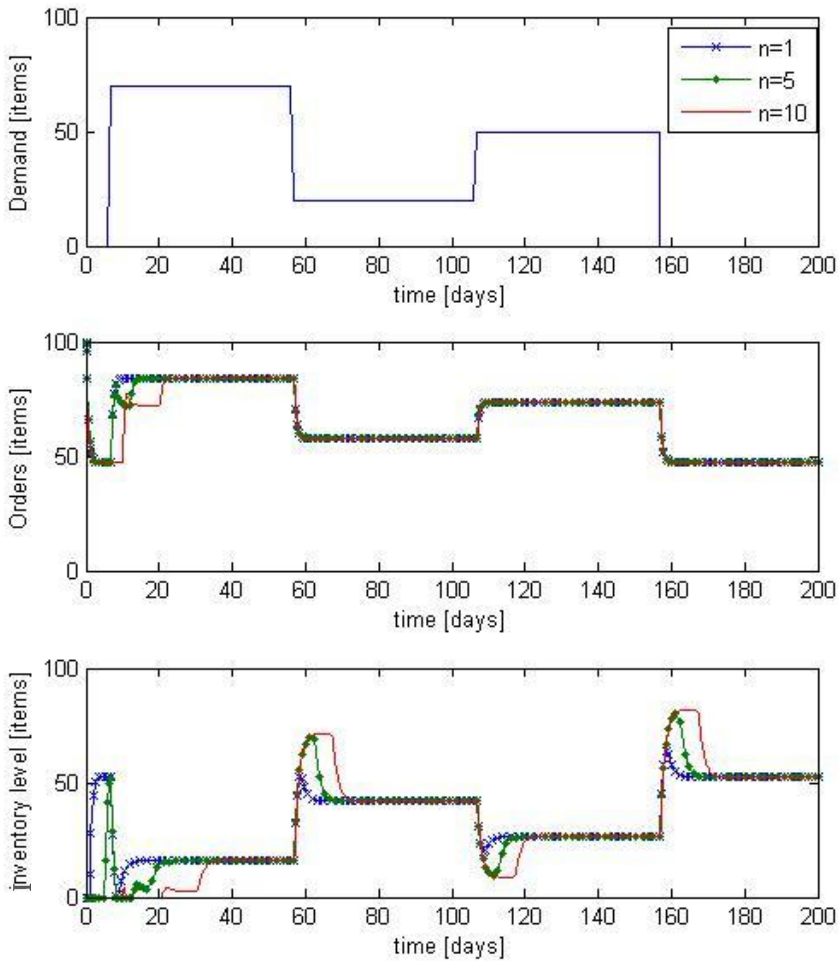


Figure 4-1: Smith Predictor in respect to different lead time values

The following representative observations have been carried out for the results shown in this section. This results are typical of those tested for other cases, but this are omitted here. The inventory levels fluctuate smoothly within the phases of demand pattern at relatively low levels: near *20* (for *70* items in demand), near 40-*50* items (for *20* demanded items), near *30* items (for *50* demanded items) and near *60* items (for 0 demanded items). At any time the inventory levels are not kept anywhere near to the reference inventory of *100* items, and always kept below that value. Therefore, the Smith predictor if applied in isolation does not generate satisfactory results in respect of the thesis goal (keeping inventory at a desired level). Instead the number of stored goods is smoothly fluctuating in response to changes in demand. However the higher lead time delay value, the less smooth the inventory levels and orders become. The fluctuations of inventory levels and order quantities can be observed to behave in a related manner. In control terms the order quantity is the control variable in a function of the feedback inventory level. The difference in results obtained for different lead times are mostly visible close to the time instances where the sudden demand fluctuation occurs. Figure 4-2 - Figure 4-5 show the zoomed parts of Figure 4-1 for specific demand values.

Figure 4-2 presents the zoomed demand, inventory levels and order quantities for time period of 0-30 days. It can be noticed that initially, for first 6 days, no items are demanded. As in general terms the demand increase means the subtraction of items from inventory levels (sale of items), consequently any increase of demand should be followed by the increase of orders (to replenish the sold goods), which indeed can be observed in Figure 4-2 for $n=1$ and $n=5$ at 6th time instance. For $n=10$ the increase of orders is observed in the 11th time instance only, as due to 10 days delay the previous orders have not reached the warehouse by 11th time instance.

As the warehouse is initially empty, the first order size (in the 1st time instance) is equal to reference inventory level, to supply the empty warehouse with the required number of goods. It can be noticed that within the time delay, some actions happens in inventory level after the first order is placed for the cases of $n=1$ and $n=5$, the results differ then for $n=10$, though.
For $n=1$ and $n=5$ the inventory level reaches the level of approximately 50 items and stays at that level until the demand suddenly changes on 6th day. Until that time the order quantities decreased to *50* items, to make sure no more goods appear in the inventory levels and no surplus inventory will be kept in the warehouse (apart from the reference inventory once the order reach as the warehouse with relevant delays).

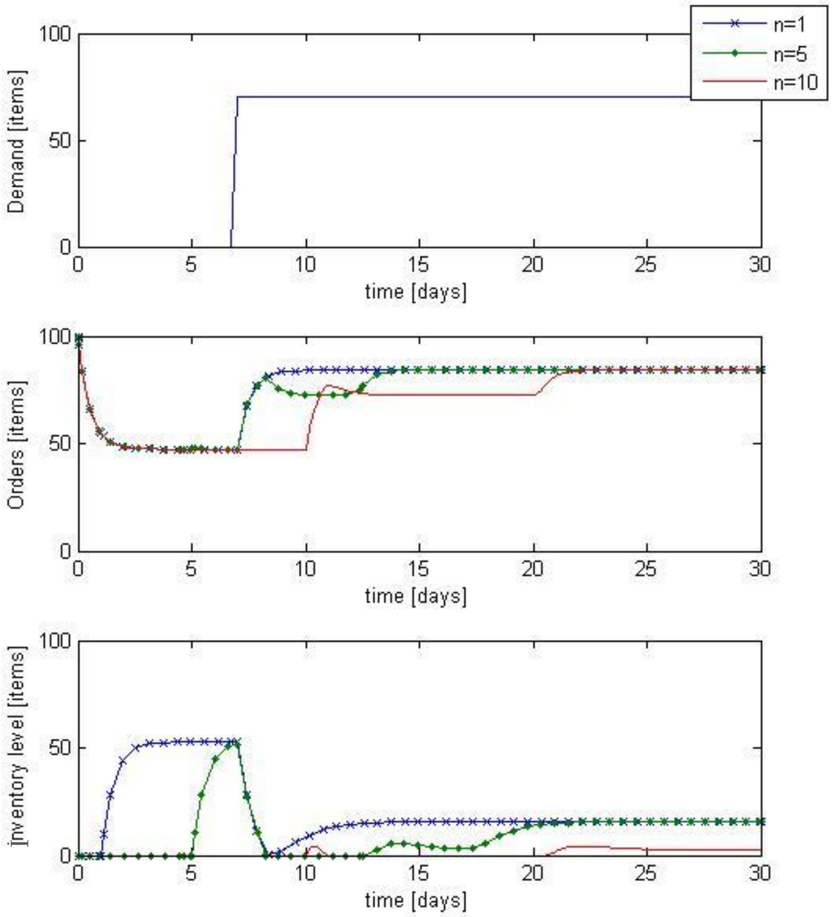


Figure 4-2: Smith predictor, zoomed result for 0-30 time instances

For $n=1$ and $n=5$ once *70* items the demand appears at the 6th day, the orders increase quickly to approximately *70-80* items, to make sure the demand will be satisfied. The decrease of inventory level to slightly zero is a result of demanding more products then available in the warehouse. The value never goes below zero as the Smith predictor integrator is limited bounds inventory from backorders. The $n=1$ response is faster, due to smaller delay, but eventually both $n=1$ and $n=5$ inventory level signals increase to some stable level (in this case of *20* items).

The behaviour of the system for $n=10$ is different due to the appearance of the demand (in the 6th time instance) before the first order reaches the empty warehouse. Also the delayed reaction of orders cannot quickly compensate the demanded items and the inventory level peaks above zero level in 10th time instance and goes almost immediately back to zero. It then increases around 21th time instance. Therefore the simulation produces the results as expected, with the case of the long delay ($n=10$) requiring more time to recover.

The converge to other inventory levels of $n=1$ and $n=5$ case is only noticeable in Figure 4-1 after 35th day of simulation.

Figure 4-3 presents the zoomed results for time instances between days 50-80. As the effect of the initial delay of delivery time (lead time) to the warehouse was overcome already and all inventory levels were stable at around the *20* items level. Now, when the demand (sale) suddenly decreased in the 56th time instance, the traces corresponding to the orders (for $n=1$, $n=5$ and $n=10$) decrease very smoothly and almost identically to a level of *55-60* items. The inventory levels slightly overshoot for the time of lead time and converges to the increased level of *40* items with respect to the time delay.

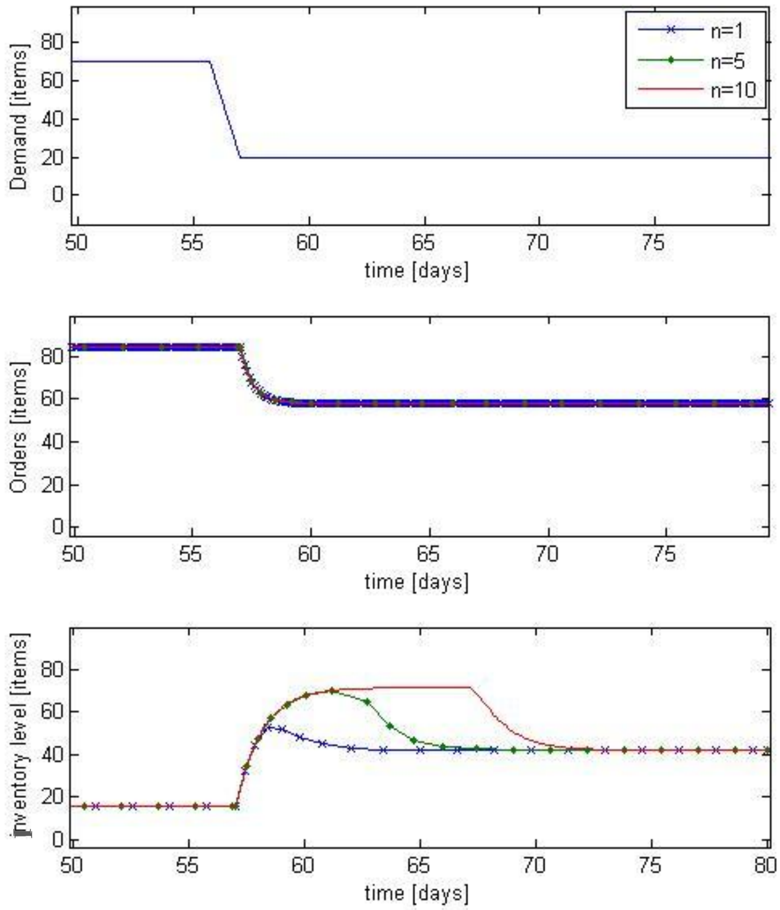


Figure 4-3: Smith Predictor, zoomed result for 50-80 time instances

In Figure 4-4 and Figure 4-5 the simulation is analogous to Figure 4-3. The orders increase or decrease smoothly and almost identically regardless of the delay (orders are not delayed by the lead time, but the inventory levels are). Then the inventory converges to a common level for all three delays, to a level of *40* items, within appropriate time delay. In particular, Figure 4-4 shows the zoomed results of Figure 4-1 for time instances between 100-120 days. The order signals (traces) regardless of the delay respond to the sudden change of demand (from *20* to *50* items) by increasing the order size to *70-80* items. The inventory level decreased immediately due to the sudden sale of items and with appropriate time delays to converge to a common level regardless of the delay to a value of about 30 items.

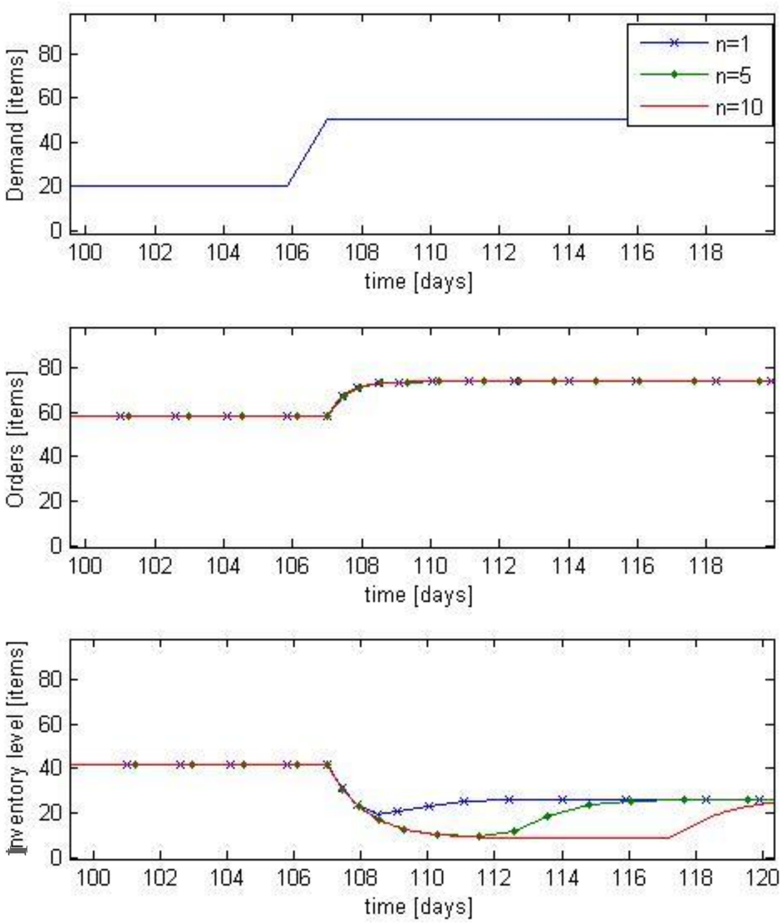


Figure 4-4: Smith Predictor, zoomed result for 100-120 time instances

Figure 4-5 presents the zoomed results for time instance between 150 and 170 days. The demand suddenly decreased in time 157th instance which results in smooth, no lead time affected, order sizes for all three cases of $n=1$, $n=5$ and $n=10$. For each case the inventory level initially slightly overshoots as the decreased order signal did not compensate the stock level kept in the warehouse for the duration of lead time delay. For all this cases the inventory level converges to a common level within the delay time.

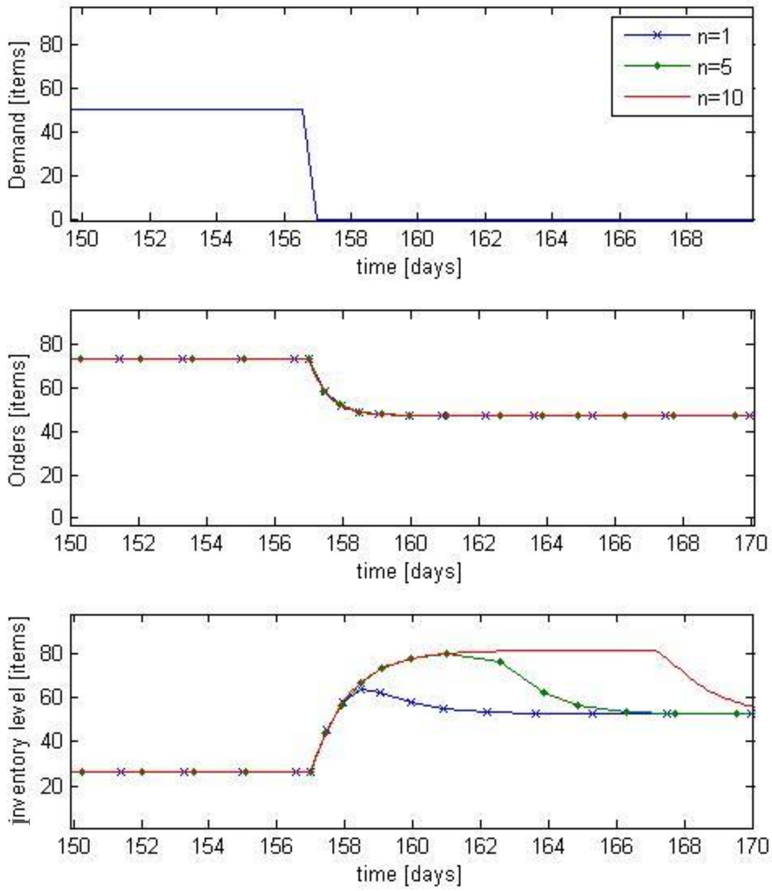


Figure 4-5: Smith Predictor, zoomed result for 150-170 time instances

Although the Smith predictor decision maker produces a response which keeps stock in the warehouse, it can be concluded that the 'blind' application of the Smith predictor is not sufficient for this particular application i.e. in the presence of perishable goods. It turns out that this in fact levels itself well to this application, which is further extended in the following. Analysis of the Smith predictor shows that under the 'ideal' case of non-perishable goods the configuration adopted here will keep the inventory at the desired level. From a control engineering perspective to maintain the inventory level in the perishable case would require the controller gain $\boldsymbol{K}$ to be replaced with a proportional plus integral term. However, since the aim is to develop schemes that are accessible by the OR community, attention is now forward to the optimal dead-beat controller.

4.4 MPC Approach

The purpose of this section is to present some initial results of inventory responses corresponding to the application of the MPC method. As it was mentioned before, the tuning parameter of equation (3.58) was set to zero (to be able to refer to the equivalent IC method, which does not require a tuning parameter neither). Therefore for all simulation settings here, the tuning parameter is arbitrarily set to zero, so that the result can be compared to IC in further sections.

The simulation was set as follows. The simulation was run for *200* time instances. The reference inventory level was set to *100* items. The prediction horizon was set to *14* and the control horizon was set to *8* time instances. The representative values of prediction and control horizon are producing typical system behaviour for most of tested numerical value of the prediction and control horizons. The lead time delay was set to *5* time instances. The deterioration rate was set to *1* (which effectively means no deterioration of products as $1-1=0$).

Therefore the inventory state space representation of equation (3.3) and (3.4) has a form

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u(kT)+\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}d(kT) \tag{4.3}$$

$$y(kT)=\begin{bmatrix}1&0&0&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT) \tag{4.4}$$

and the MPC gain from (3.63) has a form

$$\boldsymbol{K}=\begin{bmatrix}5&5&4&3&2&1\end{bmatrix} \tag{4.5}$$

Three different demand patterns were tested: the seasonal pattern of four different values - *70* items (days: 6-55), *20* items (days: 56-105), *50* items (days: 106-155) and *0* items (remaining days); random pattern: with *50* as a mean value and variance of *10* items; seasonal with some randomness allowed: the pattern of *70* items (days: 6-55), *20* items (days: 56-105), *50* items (days: 106-155) and *0* items (remaining days) plus one random pattern of zero items as a mean value with a variance of 5 items. Figure 4-6 shows the demand values of three patterns (the seasonal, random and mixed) and respective system responses in terms of order quantities and inventory levels. Therefore the sensitivity of the MPC inventories is tested here in respect to different demands.

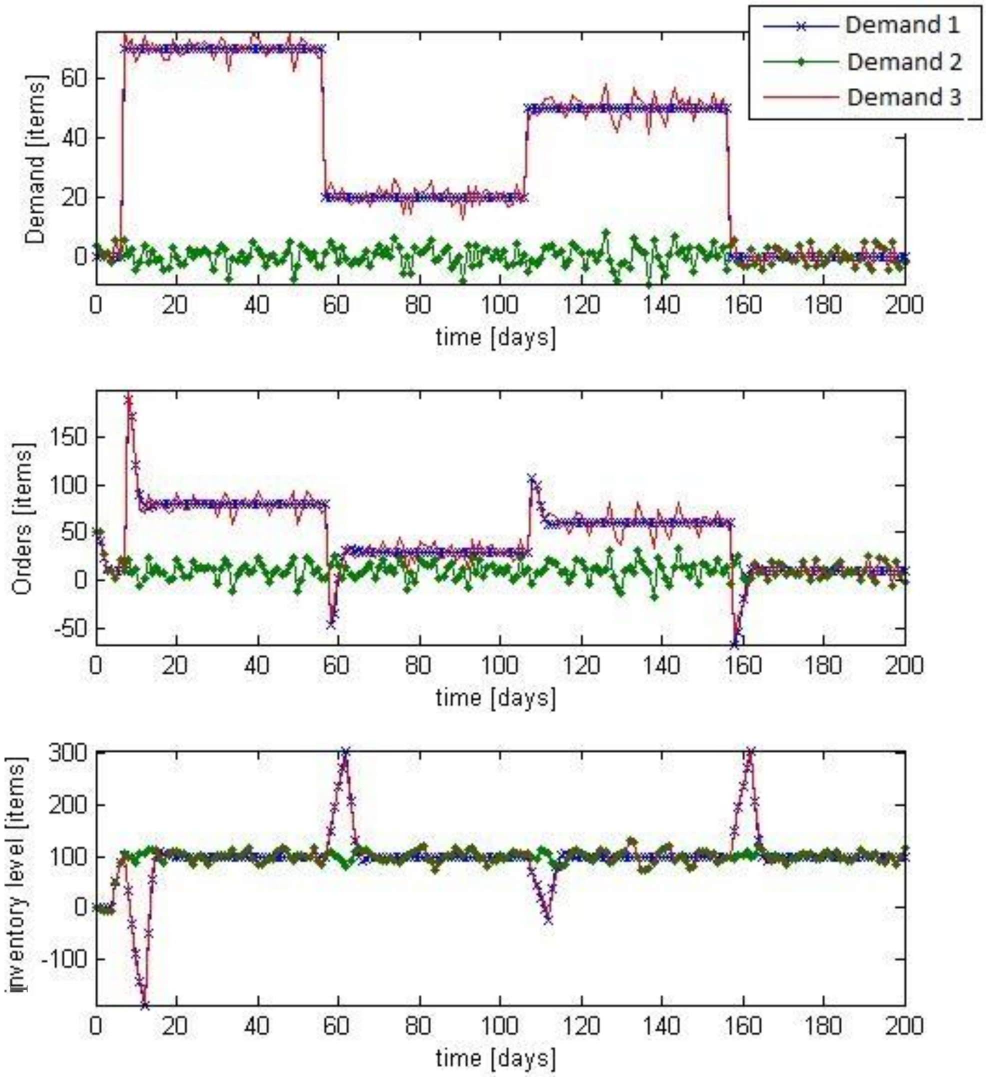


Figure 4-6: MPC in respect to 3 different demand patterns

The trace of demand 1 shown in Figure 4-6 presents the inventory level kept at the reference level of *100* items for most of the time (apart for the time instance when the demand changes suddenly). The sudden fluctuations themselves converge quickly (within the time delay) back to the reference level. Analogously, the order quantities fluctuate only when the demand suddenly change and converge back to expected mean values. Observing the trace of demand 2 it can be noticed that the inventory levels fluctuates within *+/-10* items from the reference inventory level of *100* items and the orders fluctuate in a similar manner near the demand mean value. Note that demand 2 is not representative of a practical case, but included to the test the algorithm. Finally, seeing trace of demand 3 it can be observed that the inventory remains near the desired value (with some oscillations in correlation to demand oscillations) for most of the time while it fluctuates when the demand suddenly changes too. Then, within the lead time delay the inventory level returns and converges back and remains with small oscillations near the reference level. The order quantities reflect the expected behaviour with the observation that it returns near to the demand mean trend.

Figure 4-7 - Figure 4-10 present the zoomed results shown in Figure 4-6 for selected time instances, where the demand suddenly changes and the corresponding changes in inventory levels and order quantities can be observed before the signals converge back to the reference inventory level. The zoomed figures increase the visibility of the results.

Figure 4-7 shows the system behaviour for the time period of *0 – 20* time instances. During these time instances the sudden increased from zero to 70 items in demand can be observed (for seasonal and mix demand patterns) at 6^{th} time instance.

As it can be noticed, the inventory levels and order quantities of demand 1 and demand 3 behave in similar way, with the difference that demand 3 adds some small fluctuations around the core values of the results of demand 1. For the first 5 time instances, the demand is zero (demand 1) or near zero (demand 3).

In Figure 4-7 it can be noticed that the order sizes start at the level of 50 items and then slowly decreased to zero (or around zero) for all cases regardless of the demand pattern. This way, the orders shift the inventory levels to the reference value. For demand 1 and demand 3 in respect to the demand sudden increase at the 6^{th} time instance the orders increase up to the level of 200 items for one time instance to quickly catch up with inventory level and slowly reduces back to the demand level to compensate the delay in inventory response. In Figure 4-7 it can be observed that the inventory initially drops after the 6^{th} time instance following the sudden demand change, as the previous, not decreased orders are still being delivered to the warehouse due to lead time delay of 5 days. Then the inventory starts building up again due to current decreased order deliveries (smaller quantities). Eventually the inventory reaches the reference level and order stabilizes as well.

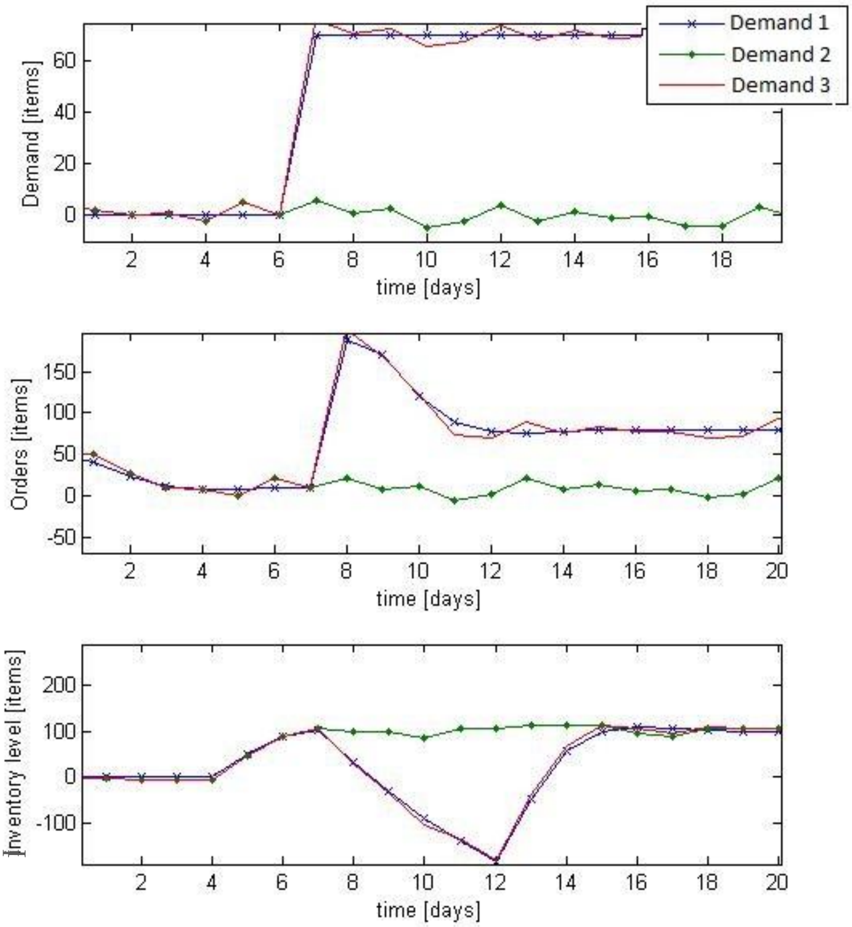


Figure 4-7: MPC in respect to 3 different demand patterns, zoomed values for 0-20 time instances

Figure 4-8 shows the system behaviour for the time period of *50 – 70* time instances. As it can be noticed, again for the demand 2, the orders lightly fluctuates to push the inventory levels builds up or down in respect to small demand fluctuations. The inventory levels and order quantities of demand 1 and demand 3 behave in similar way to each other, with a difference that demand 3 adds some small fluctuations around the core value of results of demand 1. For demand 1 and demand 3 in respect to demand sudden decrease at 56th time instance the orders decreases to about -50 items (which refers to returns) due to a sudden demand decrease. It should be noticed that returns are allowed in this chapter only for the purpose of testing the model behaviour rather than to use this in practice. In further chapters the returns are not considered.

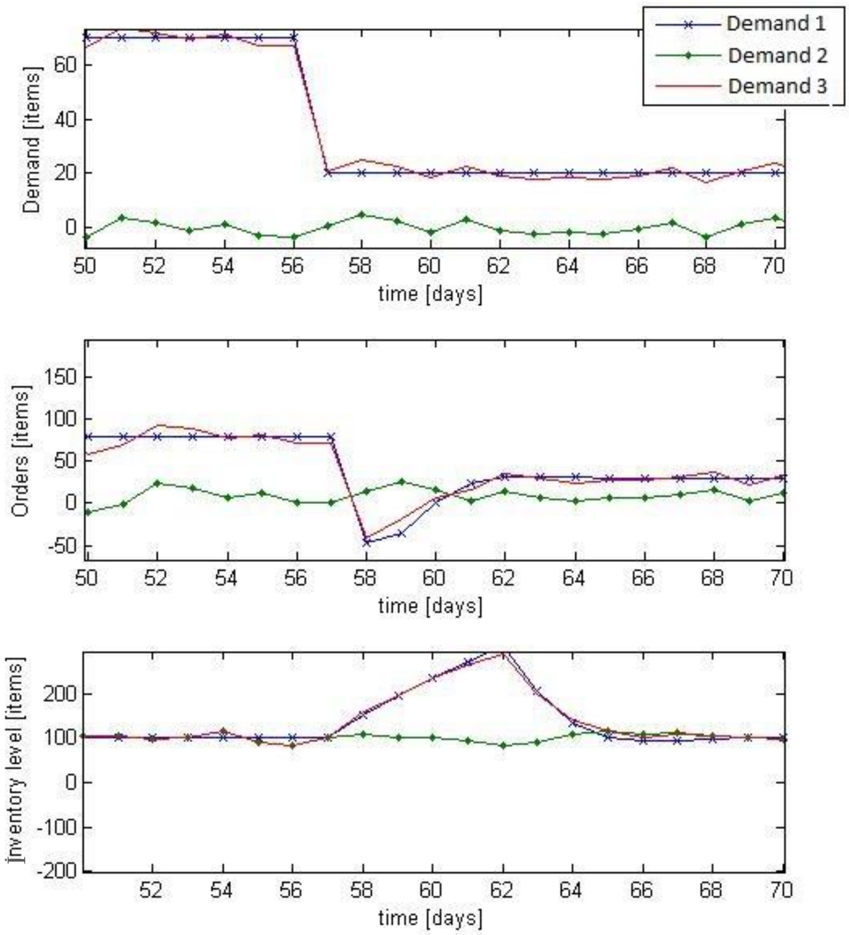


Figure 4-8: MPC in respect to 3 different demand patterns, zoomed values for 50-70 time instances

In Figure 4-8 it can be observed that the inventory initially slowly builds up, as the previous, not yet decreased orders are still being delivered to the warehouse due to the lead time delay. Then the inventory starts decreasing back due to new decreased order size deliveries after the time delay. Eventually the inventory reaches the reference level and order stabilizes as well.

Figure 4-9 shows the system behaviour for the time period of *100 – 120* time instances. It can be noticed, again for the demand 2, that the orders slightly fluctuate to push the inventory levels up or down in respect to small demand fluctuations. The inventory levels and order quantities of demand 1 and demand 3 behave in similar way to each other, with a difference that demand 3 adds some small fluctuations around the core value of results of demand 1. For demand 1 and demand 3 in respect to a sudden demand increase at the 106th time instant the orders increase to about 100 items due to sudden demand increase. The inventory initially slowly reduces, as the previous, not yet increased the orders are still being delivered to the warehouse due to the lead time delay. Then the inventory starts rising back due to the new increased order size deliveries after the time delay. Again, as was the case for Figure 4-8 the inventory reaches the reference level and the orders stabilize.

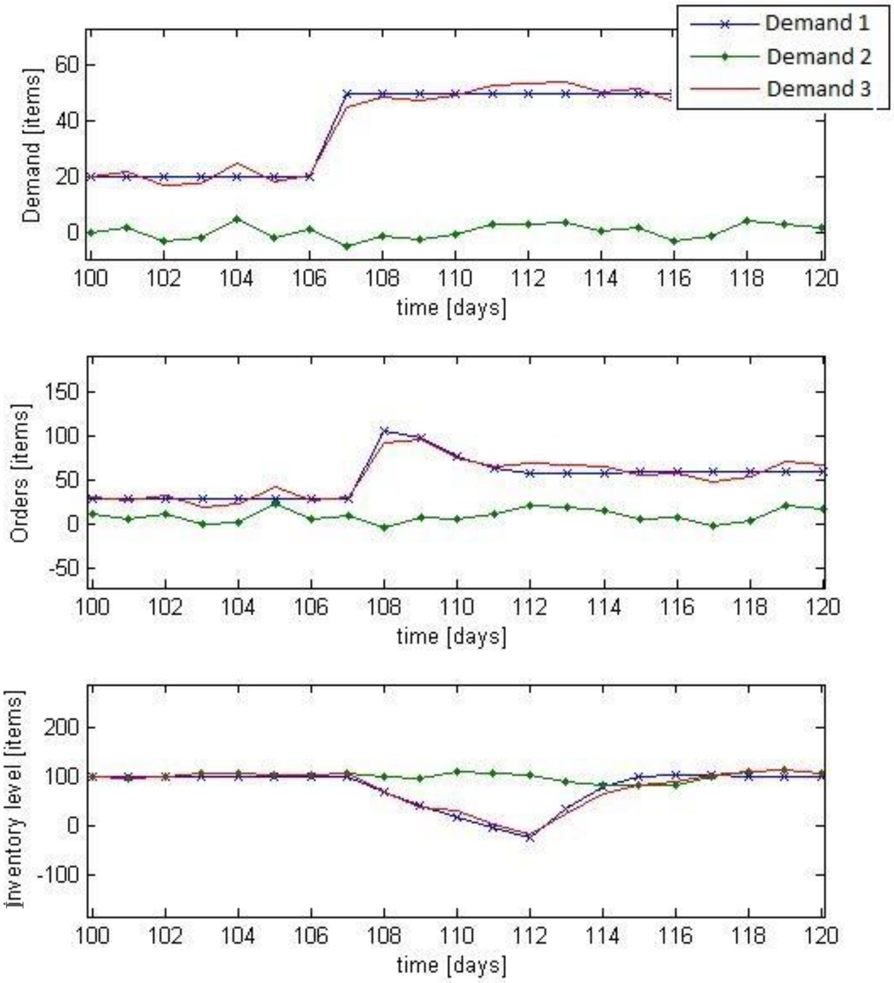


Figure 4-9: MPC in respect to 3 different demand patterns, zoomed values for 110-120 time instances

Figure 4-10 again shows the system behaviour for the time period of *150 – 170* time instances. Similar observations may be described around *156th* time instance as discussed for 56th time instance. Indeed the MPC approach is formed to provide a consistence performance.

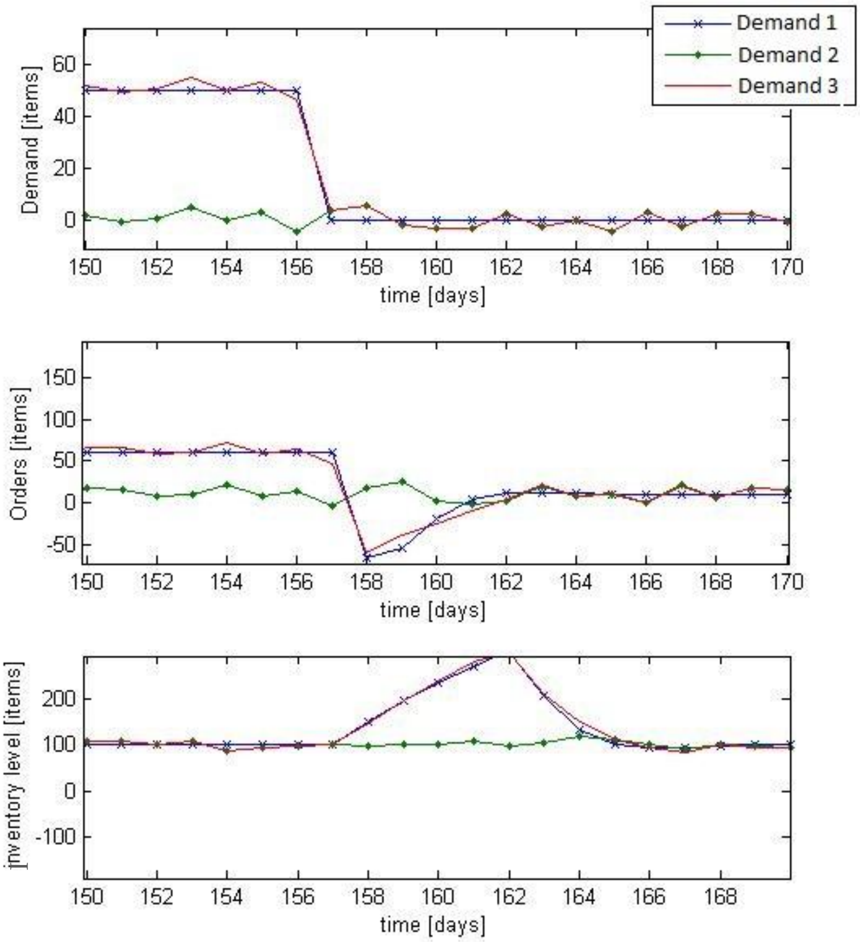


Figure 4-10: MPC in respect to 3 different demand patterns, zoomed values for 150-170 time instances

It can be concluded that for all the studied demand patterns the inventory level is maintained near to or at the reference inventory level for most of the simulation time. The inventory fluctuates (and so do the orders respectively) at the time instances of a sudden demand change. As there is a delay in the system, the system converges with a delay to or near the desired value. It can be noticed, that compared with the Smith predictor, the obtained results would be satisfactory for industrial purposes in the sense of the previously defined goal, namely to reduce inventory level fluctuations and keep it near the reference point (Section 1.1.5). Therefore, based on the presented results and other conducted simulations which are omitted from this section, the MPC as a technique can generate satisfactory results for inventory stability. As stated in the introduction, the MPC approach has the Smith predictor and potentially the dead-beat controller within its structure. Since MPC is considered as satisfactory for this research, the next subsections focus on a verification of the results obtained with the novel IC against the IMPC.

Since it is to be verified that the IC and IMPC generate the same results for different simulation settings, therefore it is worth noticing that the IC method presented in the *IC Proposition* (Section 3.4) does not include N_p and N_c in its definitions and therefore does not depend on them. Therefore, the current test was conducted for the IMPC method itself, to study the sensitivity of the IMPC to changes of the N_p and N_c values. Several different simulations were run for the following sets of pairs of the N_p and N_c values: $N_c = n-1 \ldots n+10$ and defining $N_o = N_p - N_c$, values of $N_o = 2..12$ were studied. The model appeared to be insensitive to changes of N_p and N_c when the tuning parameter is set to zero in equation (3.58).

Figure 4-12 shows the exemplary order quantities of the pair $N_p = 18$ and $N_c = 8$ (where $N_o = 10$) and Figure 4-13 of the pair $N_p = 14$ and $N_c = 11$ (where $N_o = 3$). The values were chosen in the manner that not only they differed from each other but also the difference between N_p and N_c (the N_o value) different too for each figure. Both simulations were run for 200 time instances and following settings: arbitrarily chosen three values of deterioration rate $\delta = 1$, $\delta = 0.8$ and $\delta = 0.5$, lead time delay of *5* days and the seasonal demand pattern of Figure 4-11.

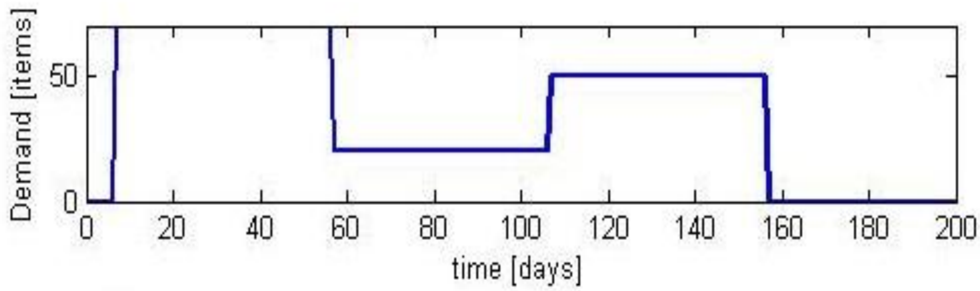


Figure 4-11: Demand pattern for IMPC vs. IC verification

Regardless of the N_p and N_c values the inventory state space models and IMPC gains have the same or very similar values (differences noticeable on or above 4th decimal place only).

1. For $\delta = 1$ the inventory state space representation of equations (3.3) and (3.4) has the form

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u(kT)+\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}d(kT) \tag{4.6}$$

$$y(kT)=\begin{bmatrix}1&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT) \tag{4.7}$$

and the MPC gain from (3.63) has the form

$$\begin{aligned}&\boldsymbol{K}=\begin{bmatrix}5.0000&5.0000&4.0000&3.0000&2.0000&1.0000\end{bmatrix}\quad \mathit{for}\quad N_p=18\quad \mathit{and}\quad N_c=8\\&\boldsymbol{K}=\begin{bmatrix}5.0000&5.0000&4.0000&3.0000&2.0000&1.0000\end{bmatrix}\quad \mathit{for}\quad N_p=14\quad \mathit{and}\quad N_c=11\end{aligned} \tag{4.8}$$

It can be noticed that the gains are identical up to the 4th decimal place.

2. For $\delta=0.8$ the inventory state space representation of equations (3.3) and (3.4) has the form

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}0.8&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u(kT)+\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}d(kT) \tag{4.9}$$

$$y(kT)=\begin{bmatrix}1&0&0&0&0\end{bmatrix}\boldsymbol{x}(kT) \tag{4.10}$$

and the MPC gain from (3.63) has the form

$$\begin{aligned}&\boldsymbol{K}=\begin{bmatrix}2.6893&3.3616&2.9520&2.4400&1.8000&1.000\end{bmatrix}\quad \mathit{for}\quad N_p=18\quad \mathit{and}\quad N_c=8\\&\boldsymbol{K}=\begin{bmatrix}2.6893&3.3616&2.9520&2.4400&1.8000&1.000\end{bmatrix}\quad \mathit{for}\quad N_p=14\quad \mathit{and}\quad N_c=11\end{aligned} \tag{4.11}$$

It can be noticed that the gains are identical up to the 4th decimal place.

3. For $\delta=0.5$ the inventory state space representation of equations (3.3) and (3.4) has the form

$$
\boldsymbol{x}\left[(k+1)T\right] = \begin{bmatrix} 0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(kT) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(kT) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d(kT) \tag{4.12}
$$

$$
y(kT) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(kT) \tag{4.13}
$$

and the MPC gain from (3.63) has the form

$$
\begin{aligned}
\boldsymbol{K} &= \begin{bmatrix} 0.9687 & 1.9375 & 1.8750 & 1.7500 & 1.5000 & 1.000 \end{bmatrix} \quad for \quad N_p = 18 \quad and \quad N_c = 8 \\
\boldsymbol{K} &= \begin{bmatrix} 0.9688 & 1.9375 & 1.8750 & 1.7500 & 1.5000 & 1.000 \end{bmatrix} \quad for \quad N_p = 14 \quad and \quad N_c = 11
\end{aligned} \tag{4.14}
$$

It can be noticed that the gains are very similar. All gain values are the same up to the 4th decimal places apart from the first element. The first element is the same up to the 3rd decimal place and different only on the 4th decimal place. Such two gains can control the system very identical.

Figure 4-12 and Figure 4-13 present the order quantities for all three of the above systems (deterioration values) for IMPC of $N_p = 18$ and $N_c = 8$ $N_p = 14$ and $N_c = 11$ respectively.

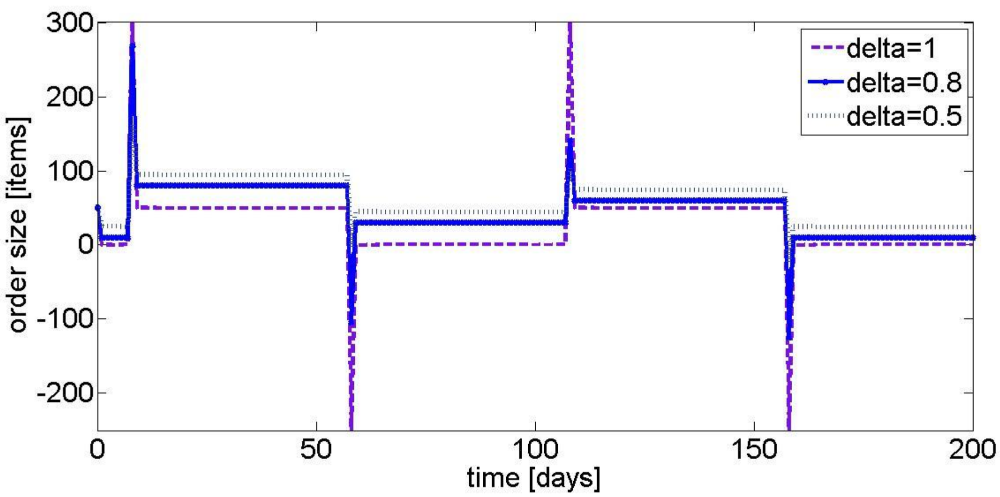


Figure 4-12: Order quantities for different deterioration rates when $N_p = 18$ and $N_c = 8$

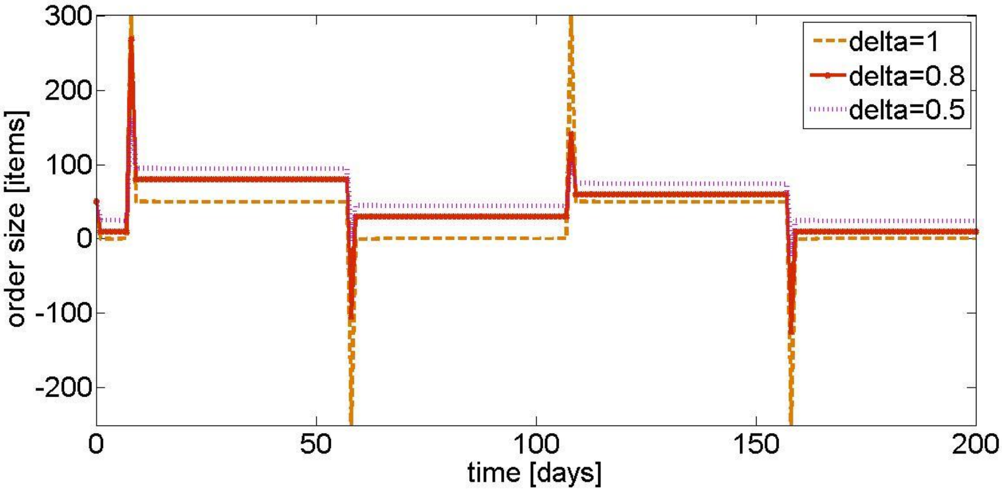


Figure 4-13: Order quantities for different deterioration rates when $N_p = 14$ and $N_c = 11$

From the results shown have and other simulations conducted which are omitted here, it can be concluded that the same values of order sizes were obtained regardless of the prediction and control horizon values for each of the values of deterioration rate. It is a result of the fact that the form of the IMPC gain for a given lead time is mainly dependent on the deterioration rate regardless of prediction and control horizon values. This appears to be further reinforced when the IMPC appears to be non-sensitive to changes in the control and prediction horizon vales when the tuning parameter is set to zero. Consequently the IC (which does not depend on N_p and N_c by definition) and IMPC can be now compared. The issue of non-zero tuning parameter approach of an alternative IC method is addressed in Chapter 7.

4.5 Alternative IC Method - Verification Against MPC

For the purpose of verification, it is assumed that negative orders are allowed (which correspond to returns). This way the pure model behavior can be tested. For further chapters, where the focus is devoted to optimization of inventory, rather than verification of the model, a saturation module preventing returns will be included in the model. Therefore from (3.67)

$$u(kT) \quad = \quad u\left[(k-1)T\right] + I_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT) \tag{4.15}$$

at any time instance. The assumption is far from the reality, as even if negative orders are considered as returns, they should be returned with a delay (in the same manner as they are delivered with a delay). Nevertheless, in the current section the purpose is to verify equivalency of the IC and IMPC results for different conditions and settings only. Saturation here could reduce the clarity of the analysis. Chapter 6 addresses the IC approach for the inventory application, therefore the saturation is considered there.

This way the results given by the IC can be directly compared with IMPC without any additional assumptions taken or constraint applied. The current section is devoted to verify that the IC behaves in the same manner as IMPC. Several numerical examples have been shown in the following sections to support the verification. The process of manual calculation of order quantities, which would result in satisfactory stable inventory levels, is shown in the Section 4.5.1. This serves to demonstrate how straightforward IC is in terms of computation with respect to the original MPC.

4.5.1 Test 1: To illustrate the manual calculation of IC

In terms of illustrating the simplicity of the developed control policy, the current numerical example shows the manual calculation process of finding the optimal order sizes. From the *IC Proposition* in Section 3.4, for $n=5$, $\delta=1$ and $y_R=100$, gain $\boldsymbol{K}$ was calculated and has exactly the same form, as was obtained from MPC, namely $\boldsymbol{K} = \begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix}$. The simplicity of determining the $\boldsymbol{K}$ values in this case is related to the fact that δ was set to 1. Although, the proposition was developed with the purpose of using calculation spreadsheets or simple (a few line of code) program in general, the current section enables the reader to gain a deeper understanding of the proposed method by following the calculation procedure step by step.

From the *IC Proposition* it can be noticed that the method enables the on-line optimisation based on above the current and past information vector of the inventory system (the inventory levels and sizes of placed orders) denoted as a current time instance vector $\boldsymbol{x_c}(kT)$. Using the mentioned data, the warehouse manager is able to calculate the next periods optimal order quantity as follows

$$\Delta u(kT) = 100 - \begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix} \boldsymbol{x_c}(kT) \tag{4.16}$$

As the warehouse is initially empty, $\boldsymbol{x}_c(1)$ shown in (4.17) is a zero vector of $n+1$ dimension

$$\boldsymbol{x}_c(1) = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T \tag{4.17}$$

Then,

$$\Delta u(1) = 100 - \begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T = 100 \tag{4.18}$$

From (3.40), knowing that there was no order before $u(1)$, the first order should be equal to

$$u(1) = 100+0 = 100 \tag{4.19}$$

While reaching the next time period, the warehouse manager is able to read the current state vector $\boldsymbol{x}_c(2) = \begin{bmatrix}0 & 0 & 0 & 0 & 100 & 0\end{bmatrix}^T$ and calculate the optimal orders of the second period as follows

$$\Delta u(2) = 100 - \begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 0 & 100 & 0\end{bmatrix}^T = -100 \tag{4.20}$$

Then

$$u(2) = -100+100 = 0 \tag{4.21}$$

Further, knowing $\boldsymbol{x}_c(3) = \begin{bmatrix}0 & 0 & 0 & 100 & -100 & 0\end{bmatrix}^T$

$$\Delta u(3) = 100 - \begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 100 & -100 & 0\end{bmatrix}^T = 0 \tag{4.22}$$

Then

$$u(3) = 0+0 = 0 \tag{4.23}$$

Similarly, based on monitoring the current value of $\boldsymbol{x}_c(kT)$, any $u(kT)$ can be calculated on-line. Continuing the calculations for the case when demand is assumed to be zero for any further time instance, the value $u(kT)$ is always obtained to be equal to zero. It is logical, since the 100 items ordered in the first step never get sold and therefore the inventory level remains at the reference level. Once the demand changes, the order size adjusts as well. The manual calculation was continued for the following demand pattern for *25* periods shown in Figure 4-14.

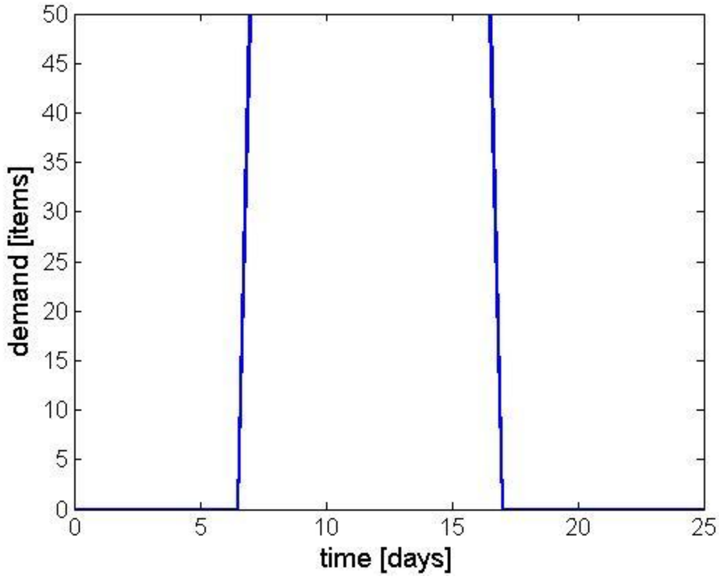


Figure 4-14: Demand pattern

The order sizes for this case were calculated and presented in Table 4-1 for the given time instances:

k	$u(kT)$
1	100
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	300
10	50
11	50
12	50
13	50
14	50
15	50
16	50
17	50
18	50

19	-250
20	0
21	0
22	0
23	0
24	0
25	0
26	0

Table 4-1: Manually obtained order quantities for each time instance by IC

Figure 4-15 illustrates the order sizes for the proposed IC from the Table 2-1 in graphical form. Figure 4-16 shows the order sizes obtained from the IMPC MATLAB simulation of the model and the gain of the form presented in (4.3)-(4.5), where control and prediction horizons were set as follows: $N_c = 14$ and $N_p = 8$.

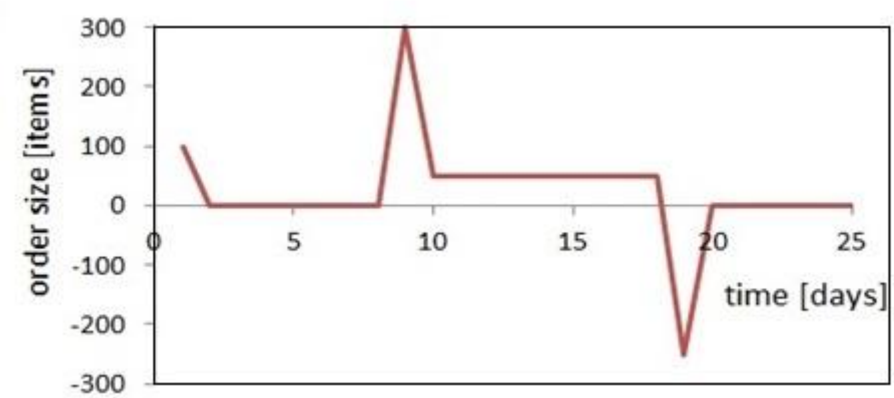


Figure 4-15: Manually obtained order quantities by IC

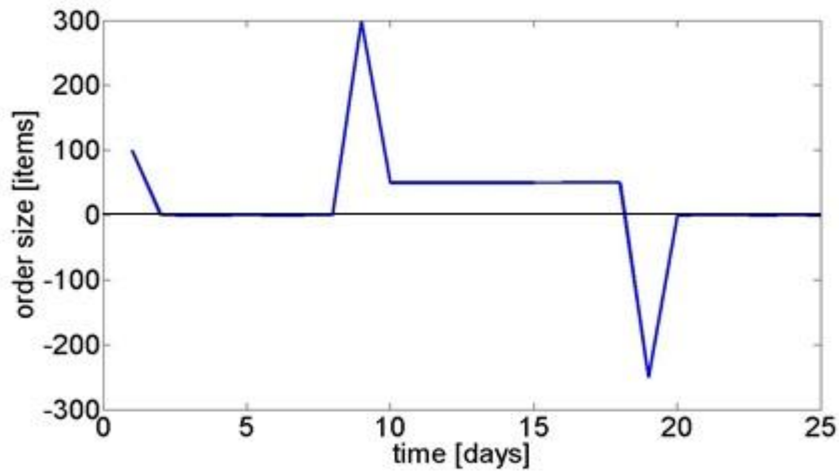


Figure 4-16: Simulated order quantities obtained by IMPC

As can be noticed, the results are the same or very similar for the given initial conditions, which concluded the main objective of this section.

4.5.2 Test 2: To show effect of varying deterioration rate

The second test aims at comparing the behavior of both methods regardless of the values of deterioration rate δ. Therefore, both, the IMPC and IC were used to obtain the vector $\boldsymbol{K}$ values and make sure that they are the same (or very similar).

In the considered scenario the time varying deterioration rate had an arbitrary form of $\delta(kT) = \frac{1}{2}\left(\sin(5kT)+1\right)$ (see Figure 4-17). The simulation was run for 33 periods. In this way 33 different values of $\delta \in \langle 0,1 \rangle$ are used for comparison purposes. The constant reference inventory level was set to $I_R = 100$. The lead-time delay was set to $n = 5$. The prediction and control horizons of the MPC method where set as follows: $N_p = 12$, $N_c = 8$

Table 4-2 and Table 4-3 show the values of the $\boldsymbol{K}$ vector obtained for every time instance $k = 1,2,3\ldots 33$ regarding the $\delta(kT)$ values.

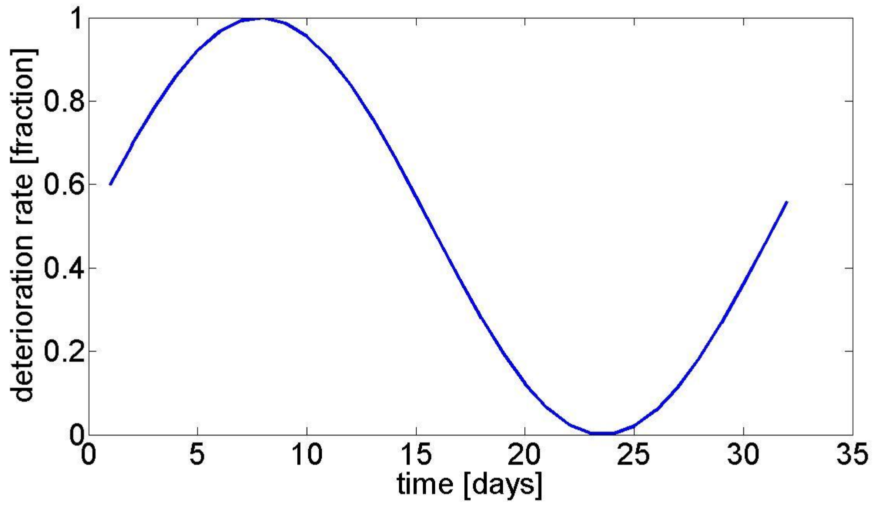


Figure 4-17: Varying deterioration rate

Table 4-2 shows the simulation results for the IMPC case and Table 4-3 for the IC case.

kT	$K(1)$	$K(2)$	$K(3)$	$K(4)$	$K(5)$	$K(6)$
1	1.3802	2.3028	2.1738	1.9585	1.5993	1.0000
2	1.9073	2.7455	2.5126	2.1773	1.6947	1.0000
3	2.5408	3.2477	2.8731	2.3943	1.7823	1.0000
4	3.2396	3.7728	3.2291	2.5960	1.8587	1.0000
5	3.9295	4.2677	3.5490	2.7685	1.9207	1.0000
6	4.5128	4.6715	3.8007	2.8992	1.9660	1.0000
7	4.8919	4.9278	3.9566	2.9782	1.9927	1.0000
8	4.9968	4.9979	3.9987	2.9994	1.9998	1.0000
9	4.8072	4.8709	3.9222	2.9609	1.9869	1.0000
10	4.3595	4.5666	3.7360	2.8660	1.9546	1.0000
11	3.7344	4.1299	3.4613	2.7219	1.9042	1.0000
12	3.0326	3.6200	3.1274	2.5395	1.8377	1.0000
13	2.3465	3.0967	2.7670	2.3319	1.7578	1.0000
14	1.7415	2.6090	2.4104	2.1130	1.6675	1.0000
15	1.2483	2.1878	2.0818	1.8961	1.5706	1.0000
16	0.8691	1.8460	1.7968	1.6925	1.4708	1.0000
17	0.5887	1.5816	1.5624	1.5108	1.3722	1.0000
18	0.3858	1.3841	1.3781	1.3564	1.2787	1.0000
19	0.2407	1.2405	1.2390	1.2317	1.1941	1.0000
20	0.1384	1.1384	1.1382	1.1364	1.1216	1.0000
21	0.0686	1.0686	1.0686	1.0683	1.0642	1.0000
22	0.0248	1.0248	1.0248	1.0248	1.0242	1.0000
23	0.0032	1.0032	1.0032	1.0032	1.0032	1.0000
24	0.0019	1.0019	1.0019	1.0019	1.0019	1.0000
25	0.0210	1.0210	1.0210	1.0210	1.0205	1.0000
26	0.0619	1.0619	1.0619	1.0617	1.0583	1.0000
27	0.1282	1.1282	1.1280	1.1265	1.1136	1.0000
28	0.2260	1.2258	1.2246	1.2184	1.1844	1.0000

29	0.3651	1.3637	1.3585	1.3394	1.2677	1.0000
30	0.5598	1.5537	1.5369	1.4901	1.3603	1.0000
31	0.8294	1.8092	1.7650	1.6686	1.4585	1.0000
32	1.1953	2.1411	2.0439	1.8699	1.5583	1.0000
33	1.6740	2.5527	2.3678	2.0858	1.6558	1.0000

Table 4-2: Values of K vector obtained with IMPC

kT	$K(1)$	$K(2)$	$K(3)$	$K(4)$	$K(5)$	$K(6)$
1	1.3802	2.3028	2.1738	1.9585	1.5993	1.0000
2	1.9073	2.7455	2.5126	2.1773	1.6947	1.0000
3	2.5408	3.2477	2.8731	2.3943	1.7823	1.0000
4	3.2396	3.7728	3.2291	2.5960	1.8587	1.0000
5	3.9295	4.2677	3.5490	2.7685	1.9207	1.0000
6	4.5128	4.6715	3.8007	2.8992	1.9660	1.0000
7	4.8919	4.9278	3.9566	2.9782	1.9927	1.0000
8	4.9968	4.9979	3.9987	2.9994	1.9998	1.0000
9	4.8072	4.8709	3.9222	2.9609	1.9869	1.0000
10	4.3595	4.5666	3.7360	2.8660	1.9546	1.0000
11	3.7344	4.1299	3.4613	2.7219	1.9042	1.0000
12	3.0326	3.6200	3.1274	2.5395	1.8377	1.0000
13	2.3465	3.0967	2.7670	2.3319	1.7578	1.0000
14	1.7415	2.6090	2.4104	2.1130	1.6675	1.0000
15	1.2483	2.1878	2.0818	1.8961	1.5706	1.0000
16	0.8691	1.8460	1.7968	1.6925	1.4708	1.0000
17	0.5887	1.5816	1.5624	1.5108	1.3722	1.0000
18	0.3858	1.3841	1.3781	1.3564	1.2787	1.0000
19	0.2407	1.2405	1.2390	1.2317	1.1941	1.0000
20	0.1384	1.1384	1.1382	1.1364	1.1216	1.0000
21	0.0686	1.0686	1.0686	1.0683	1.0642	1.0000
22	0.0248	1.0248	1.0248	1.0248	1.0242	1.0000
23	0.0032	1.0032	1.0032	1.0032	1.0032	1.0000

24	0.0019	1.0019	1.0019	1.0019	1.0019	1.0000
25	0.0210	1.0210	1.0210	1.0210	1.0205	1.0000
26	0.0619	1.0619	1.0619	1.0617	1.0583	1.0000
27	0.1282	1.1282	1.1280	1.1265	1.1136	1.0000
28	0.2260	1.2258	1.2246	1.2184	1.1844	1.0000
29	0.3651	1.3637	1.3585	1.3394	1.2677	1.0000
30	0.5598	1.5537	1.5369	1.4901	1.3603	1.0000
31	0.8294	1.8092	1.7650	1.6686	1.4585	1.0000
32	1.1953	2.1411	2.0439	1.8699	1.5583	1.0000
33	1.6740	2.5527	2.3678	2.0858	1.6558	1.0000

Table 4-3: Values of *K* vector obtained with IC

It can be noticed that the values of the vector $\boldsymbol{K}$ obtained in two different methods are identical to 4th decimal place for each time instance. In fact, the accuracy is identified to a higher number of decimal places, but this is not shown here. The inventory levels for IMPC (Figure 4-18) and for IC (Figure 4-19) are presented below for three chosen values of deterioration rate. Indeed, the results presented are very similar or identical results, thus the aim of this section has been demonstrated. The models used for IMPC for each deterioration rate are presented in (4.6), (4.7), (4.9), (4.10), (4.12) and (4.13). The vector $\boldsymbol{K}$ (4.8), (4.11) and (4.14) gains for both methods are given for the considered values of deterioration.

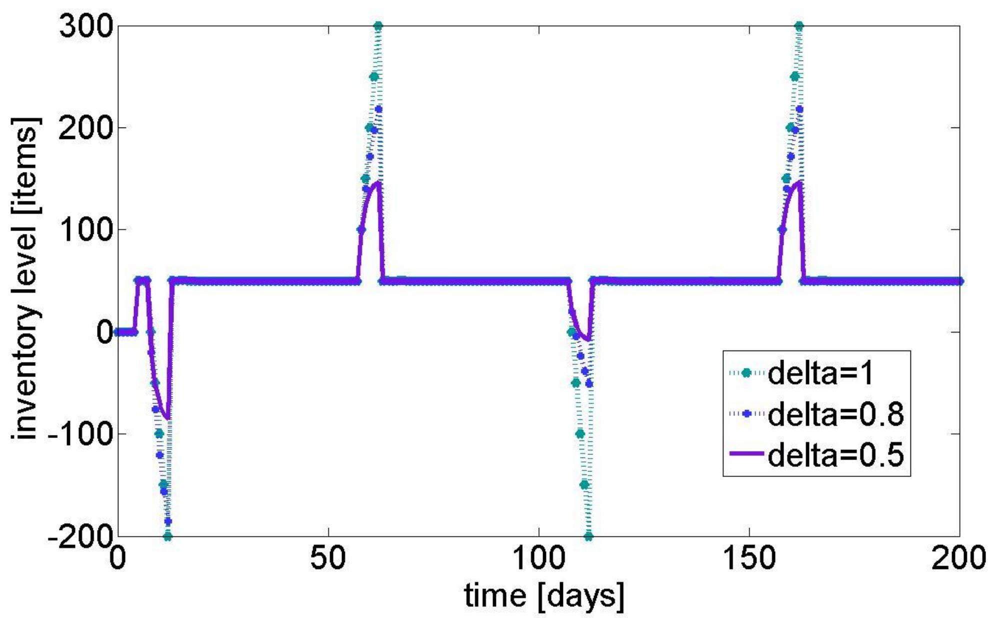


Figure 4-18: Inventory levels obtained with IMPC

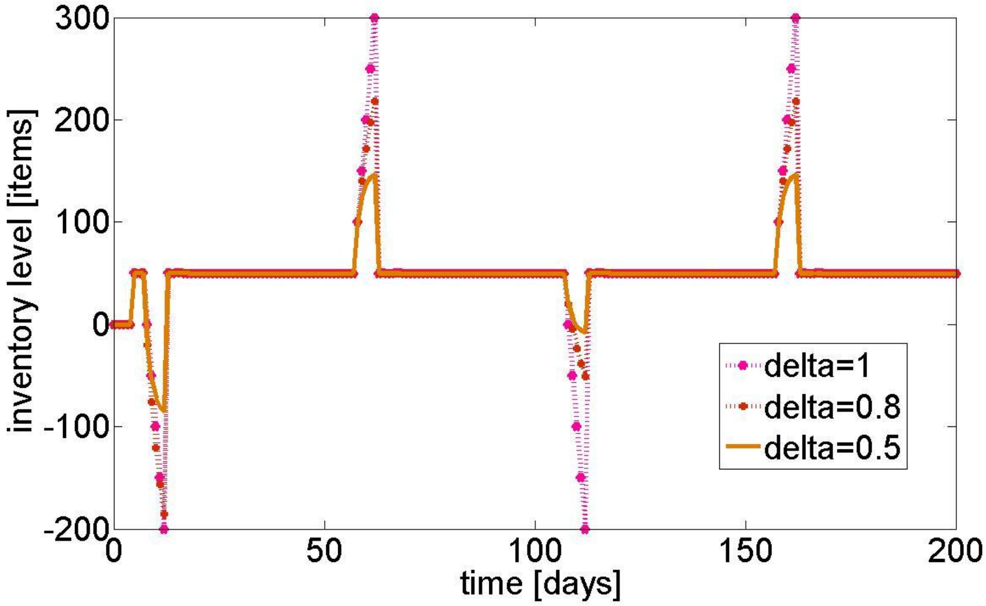


Figure 4-19: Inventory levels obtained with IC

4.5.3 Test 3: Sinusoidal reference inventory level

The next test was conducted for the constant deterioration rate $\delta(kT)=1$ to make the model time invariant. The delay was set $n=20$ and no demand disturbance was considered while conducting the test. The reference signal was set to be an arbitrary time varying value such that $I_R = \sin(10kT)$.

Therefore the IMPC model of equations (3.3) and (3.4) take the form as follows

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}1&1&0&0&\dots&0\\0&0&1&0&\dots&0\\0&0&0&1&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\dots&1\\0&0&0&0&\dots&0\end{bmatrix}_{20\times20}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}_{20\times1}u(kT)+\begin{bmatrix}-1\\0\\0\\\vdots\\0\\1\end{bmatrix}_{20\times1}d(kT) \tag{4.24}$$

$$y(kT)=\begin{bmatrix}1&0&0&\dots&0&0\end{bmatrix}_{1\times20}\boldsymbol{x}(kT) \tag{4.25}$$

The gain $\boldsymbol{K}$ for both methods has a form as follows (from (3.63) or (3.66))

$$\boldsymbol{K}=\begin{bmatrix}20&20&19&18&17&16&15&14&13&12&11&10&9&8&7&6&5&4&3&2&1\end{bmatrix}$$

(4.26)

The reference signal is shown in Figure 4-20:

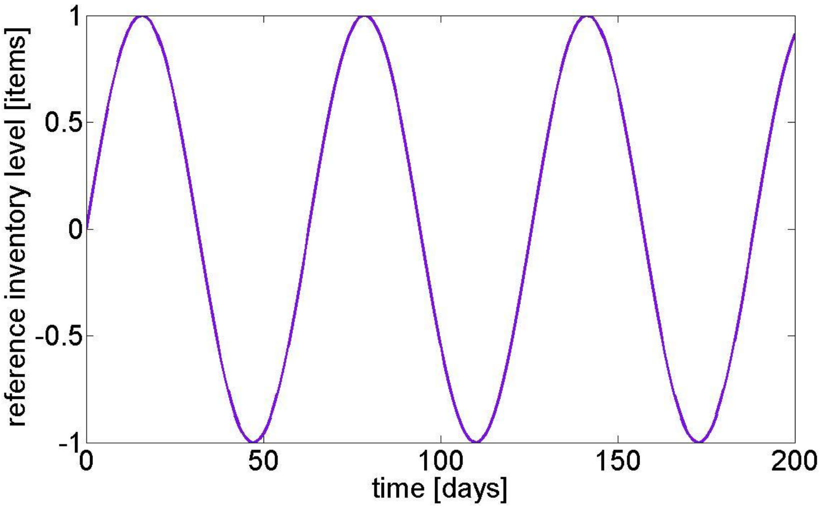


Figure 4-20: The sinusoidal reference signal

The MATLAB simulation for IMPC and Excel calculation for IC was run for 200 time instances. The aim of the test is to compare the system response using both methods. Figure 4-21 and Figure 4-22 show the inventory levels obtained due to simulation for IMPC and IC, respectively.

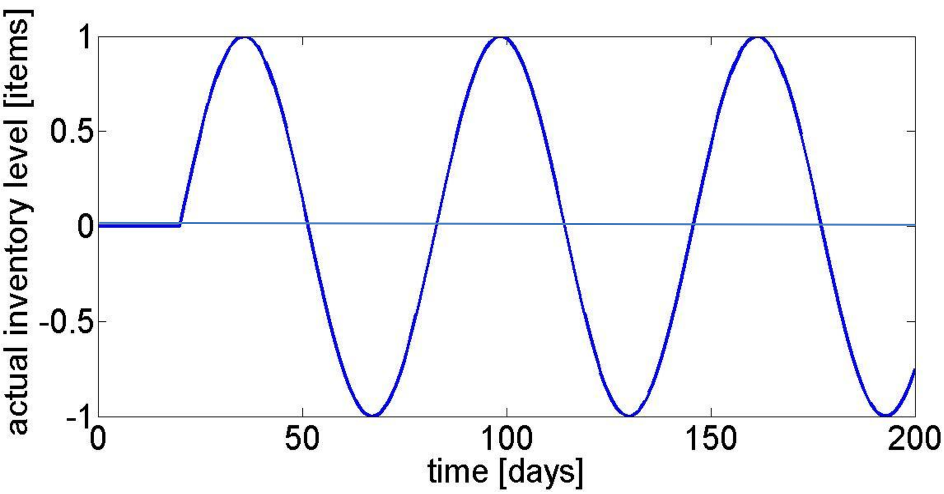


Figure 4-21: Inventory levels in response to sinusoidal reference signal by IMPC

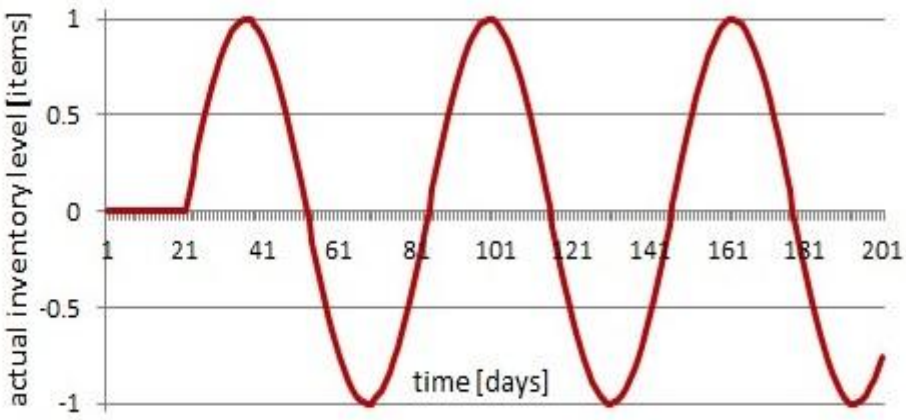


Figure 4-22: Inventory levels in response to sinusoidal reference signal by IC

As it can be noticed, both sets of results presented are exactly the same. Both track the reference inventory level with a fixed delay n. Similar tests were run for different values of deterioration rate and each time the same results were obtained for both approaches. The test aimed at showing the inventory level response, which is important from the system designer perspective. Although, for the future uses of the method, the simulation of inventory levels will not be necessary, as it involves the control theory perspective and therefore it is not included in the *IC Proposition.* The current and previous sections present the inventory results so that the reader can become convinced about the accuracy of formulation of IC method. In practice, the calculation of the optimal order quantities would only be of interest for the user of the IC method, which, as a result, would lead to satisfactory stable inventory levels. Chapter 6 focuses more on commentary of the results.

4.5.4 Test 4: Different demand patterns

The current test was conducted in order for comparing the results of both methods for different demand patents (seasonal, random, and mixture of seasonal with some randomness as described in section 4.4) and also to verify that they are very similar for both methods. The commentary on the results themselves is a concern of Chapter 6.

The simulation was run for *200* periods. The deterioration rate was set to *0.9*. The lead time delay was set to *5*, reference inventory level to *50* items.

The IMPC state space model of (3.3) and (3.4) has a form

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}0.9 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u(kT)+\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}d(kT) \tag{4.27}$$

$$y(kT)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}(kT) \tag{4.28}$$

and the gain $\boldsymbol{K}$ of (from (3.63) or (3.66)) for both methods has the form

$$\boldsymbol{K}=\begin{bmatrix}3.6856 & 4.0951 & 3.4390 & 2.4400 & 1.9000 & 1.000\end{bmatrix} \tag{4.29}$$

The result present the three tested demand patterns and the inventory levels and orders obtained for each of the demand patterns by the IMPC by in Figure 4-23 the IC approach in Figure 4-24.

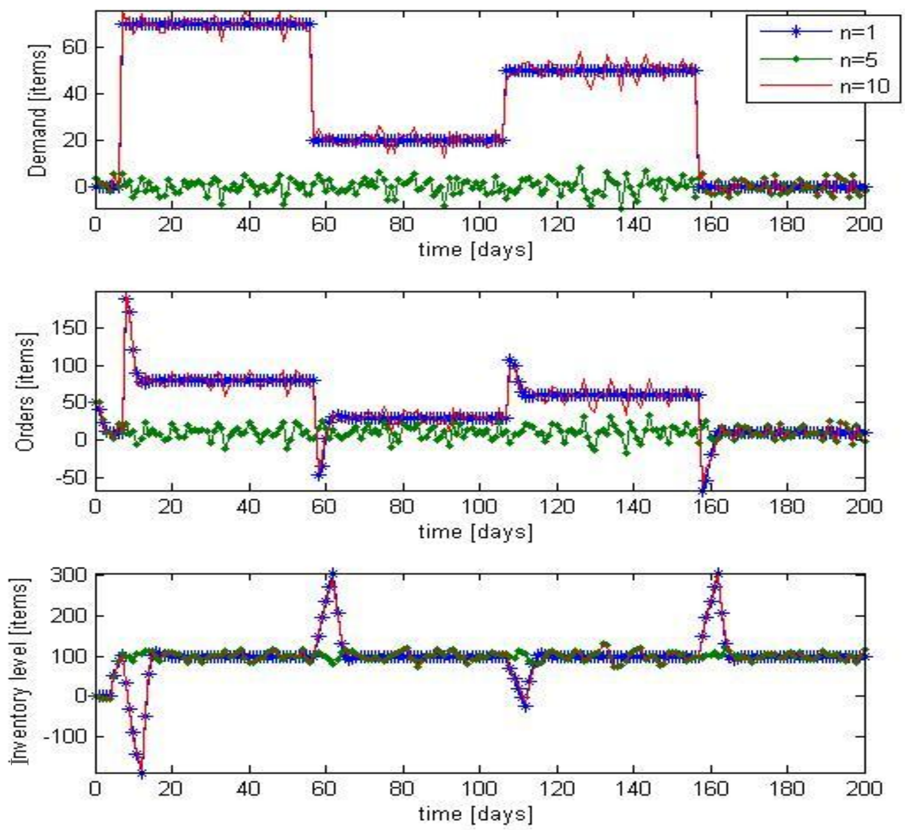


Figure 4-23: IMPC response to different demand pattern

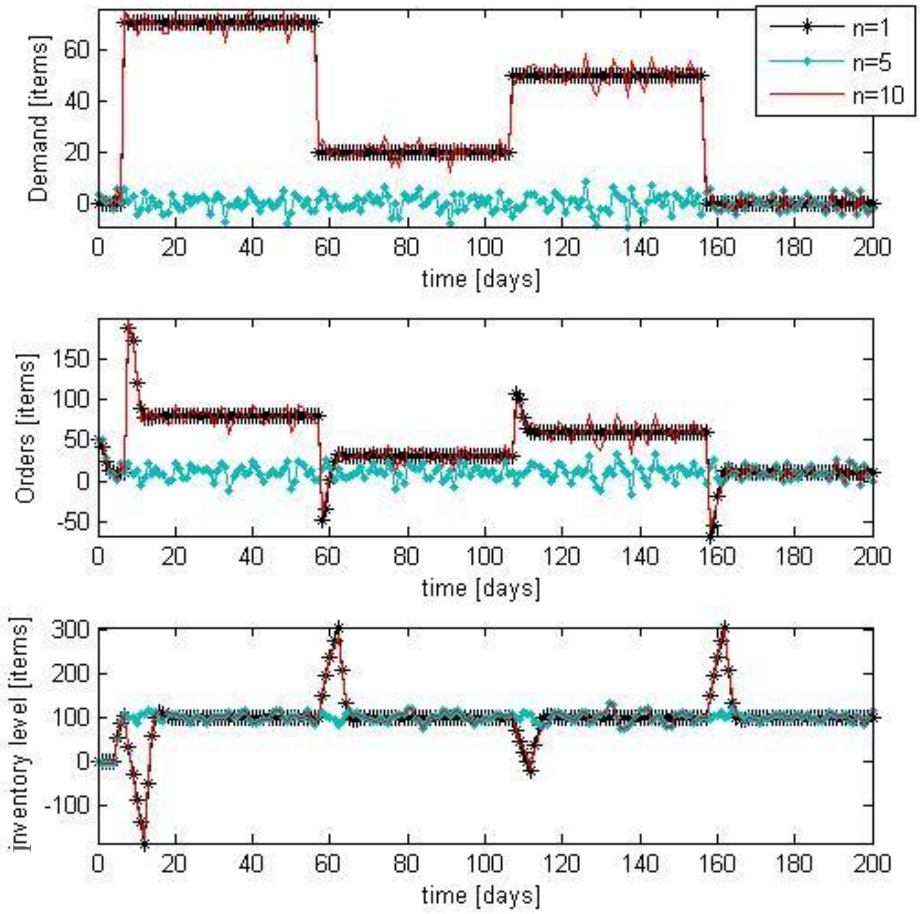


Figure 4-24: IC response to different demand patterns

Comparing of the results it reveals that the same values of the inventory levels were obtained regardless of the approach used for each demand pattern. Similar results were obtained for different demand patterns, which presentation (due to its repetitive character) was omitted here.

4.6 Summary

Initially, the simulation results of the Smith predictor method used for continuous systems were shown. Due to presence of deterioration the results are not considered as satisfactory for industrial purposes in the sense of this thesis (maintaining inventory at or near the reference point for storage capacity panning, manpower planning and bullwhip affect reduction purposes and maintaining control simplicity).

Further, the results of two methods, the IMPC (for $u_r = 0$) and IC (for no constraints on order size) were compared for a range of different test case scenarios. In each case both methods gave exactly the same results, which indicated that both of the methods might be mathematically equivalent.

Also it was noticed that the IMPC (for $u_r = 0$) and IC (for no constraints on order size) are sensitive to changes in demand, deterioration rate, lead time delay and reference inventory level. No change in IMPC results was observed for changes of the N_p and N_c values, as long as $N_p > N_c + 1$. The reason behind this is based on the particular definition of the inventory state space representation as well as the fact that the u_r was set to zero. As a result, the IMPC becomes non-predictive. In the Section 7.2, the non-zero values of the tuning parameter are considered for further investigation of the alternative IC approach.

5 DEMONSTRATION OF MATHEMATICAL EQUIVALENCY

The current section presents the mathematical demonstration of the equivalency of the novel IC method to the MPC technique applied for the inventory state space model. The demonstration process reflects the step by step development procedure, presenting how the IC was mathematically obtained from the initial MPC. The chapter presents the sequence of propositions and their demonstrations. Each of them has been developed based on recognition of some patterns of the MPC mathematical description. To familiarise the reader how the simplification procedure has evolved in time, first the process is shown for the non-perishable case and then, analogously, for the perishable case.

The non-perishable case refers to deterioration rate arbitrarily set to one. In such case, the metrics defined as an augmented model of MPC contain zeros and ones only.

Whether the non-perishable or perishable case considered, the augmented model mathematical description of the inventory system is relatively simple. The matrices defined in augmented model, when appropriately multiplied, build the elements of matrices $\boldsymbol{F}$ and $\boldsymbol{\Phi}$. The formulation of each element in matrices $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ was noticed to be dependent on its row and column indices, in respect to power index of the augmented state matrix. Therefore, both of matrices $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ could be formulated in general terms, depending on upon the row and column indices. The matrices $\boldsymbol{F}$ and $\boldsymbol{\Phi}$, in turn, are building elements of the MPC gain $\boldsymbol{K}$. Again some patterns were noticed there and the mathematical description was established in relation to row and column indices.

The initial propositions present the way of recognising the mentioned patterns in the $\boldsymbol{F}$ and $\boldsymbol{\Phi}$ matrices, the building elements used for construction of the gain in MPC. Their simplified, yet more general mathematical description is the subject of the propositions. The final proposition has the *IC Proposition* form and refers to the MPC gain $\boldsymbol{K}$ simplified description. The whole procedure is based on the fact, that the inventory model being used, has a particular state space form, as shown in inventory modelling chapter, equations (3.3) - (3.7). The content of the current chapter is the main originality and novelty of the research.

5.1 Non-perishable Case

If the elements of $\boldsymbol{\Phi}$ of equation (3.54) are considered, it can be noticed, that each of them, in general terms is either equal to zero or it has a form, which can denoted in general terms as $\boldsymbol{c}_c^T \boldsymbol{A}_c^p \boldsymbol{b}_c$, where p denotes the exponent, or power index.

Using the following propositions, the simplified form of elements of $\boldsymbol{\Phi}$ can be represented.

Proposition 1:

For $\boldsymbol{c}_c^T$ and $\boldsymbol{A}_c$ defined in (3.46), the general description of an $n+1$ dimentional vector $\boldsymbol{c}^T{}_c \boldsymbol{A}_c^p$ **,** can be formulated regardless of the values of the exponent $p \in \mathbb{N}$ and lead time delay $n \in \mathbb{N}$ in the following manner:

$$\begin{gathered}\forall_n \forall_{p>n-2} \forall_{1\le m\le n} \quad \boldsymbol{c}_c^T \boldsymbol{A}_c^p = \begin{bmatrix} p & p & p-1 & p-2 & \ldots & p-m+2 & \ldots & p-n+2 & 1 \end{bmatrix} \\ or \\ \forall_n \forall_{p\le n-2} \forall_{1\le m\le n} \quad \boldsymbol{c}_c^T \boldsymbol{A}_c^p = \begin{bmatrix} p & p & p-1 & p-2 & \ldots & p-m+2 & \ldots & 0 & \ldots & 1 \end{bmatrix}\end{gathered} \tag{5.1}$$

where any m^{th} element of the vector becomes zero only when $p-m+2 \le 0$

Remark 1:

Note that for a chosen p there is no zero element in the vector $\boldsymbol{c}^T{}_c \boldsymbol{A}_c^p$, if $p-n+2>0$. In other words, if the dimension of the vector $\boldsymbol{c}^T{}_c \boldsymbol{A}_c^p$ is smaller than $p+2$ for a chosen p there will be no zero element in the vector.

Demonstration:

From (3.46) it is known that $\boldsymbol{A}_c = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O}_{1\times(\boldsymbol{n}+1)} \\ \boldsymbol{c}^T \boldsymbol{A} & 1 \end{bmatrix}$ and $\boldsymbol{c}_c^T = \begin{bmatrix} 0_{1\times \boldsymbol{n}} \boldsymbol{I} \end{bmatrix}$.

Using the model described in (3.5) for $\delta(y)=1$ the above can be expanded as follows.

$$
\boldsymbol{A_c} = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \quad and \quad \boldsymbol{c}_c^T = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{5.2}
$$

Therefore, for $p = 1$ it can be shown that

$$
\boldsymbol{c}_c^T \boldsymbol{A_c} = \overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}}^{n+1} \tag{5.3}
$$

Further, assuming that for some $1 \le p \le n-2$

$$
\boldsymbol{c}^T{}_{\boldsymbol{c}} \boldsymbol{A}_{\boldsymbol{c}}^p = \begin{bmatrix} p & p & p-1 & p-2 & \dots & p-m+2 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \tag{5.4}
$$

is true, then,

$$
\begin{aligned}
\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p+1}} &= \boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}} \boldsymbol{A}_c \\
&= \begin{bmatrix} p & p & p-1 & p-2 & \dots & p-m+2 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \boldsymbol{A}_c \\
&= \begin{bmatrix} p+1 & p+1 & (p+1)-1 & (p+1)-2 & \dots & (p+1)-m+2 & \dots & 0 & \dots & 0 & 1 \end{bmatrix}
\end{aligned} \tag{5.5}
$$

Assumung now that for same $p > n-2$

$$
\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^p = \begin{bmatrix} p & p & p-1 & p-2 & \dots & p-m+2 & \dots & p-n+2 & 1 \end{bmatrix} \tag{5.6}
$$

is true, then,

$$
\begin{aligned}
\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p+1}} &= \boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}} \boldsymbol{A}_c \\
&= \begin{bmatrix} p & p & p-1 & p-2 & \dots & p-m+2 & \dots & p-n+2 & 1 \end{bmatrix} \boldsymbol{A}_c \\
&= \begin{bmatrix} p+1 & p+1 & (p+1)-1 & (p+1)-2 & \dots & (p+1)-m+2 & \dots & (p+1)-n+2 & 1 \end{bmatrix}
\end{aligned} \tag{5.7}
$$

which has demonstrated the proposition for both cases, when $p \le n-2$ and $p > n-2$.

Proposition 2:

For $\boldsymbol{c}_c^T$, $\boldsymbol{A}_c^p$ and $\boldsymbol{b}_c$ defined in (3.46) for the non-perishable case $\delta(y)=1$

$\forall_{n\in\mathbb{N}}\forall_{p\in\mathbb{N}}$ $\boldsymbol{c}_c^T\boldsymbol{A}_c^p\boldsymbol{b}_c = p-n+2$, when $p-n+2>0$ else $\boldsymbol{c}_c^T\boldsymbol{A}_c^p\boldsymbol{b}_c = 0$, where n is the order of the system described in (3.3) and also represents the lead time delay.

Demonstration:

From (3.46) it is known that $\boldsymbol{A}_c = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O}_{1\times(n+1)} \\ \boldsymbol{c}^T\boldsymbol{A} & 1 \end{bmatrix}$, $\boldsymbol{b}_c = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c}^T\boldsymbol{b} \end{bmatrix}$ and $\boldsymbol{c}^T{}_c = \begin{bmatrix} 0_{1\times n} & \boldsymbol{I} \end{bmatrix}$.

Using the model described in (3.5) for $\delta(y)=1$, the above can be expanded as follows.

$$\boldsymbol{A}_c = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{b}_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{c}_c^T = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{5.8}$$

From the formulation of $\boldsymbol{b}_c$ it can be noticed that $\boldsymbol{c}_c^T\boldsymbol{A}_c^p\boldsymbol{b}_c$ is the n^{th} (second last) element of the $\boldsymbol{c}_c^T\boldsymbol{A}_c^p$ vector, which was defined in *Porposition 1*. From *Proposition 1* it can be observed that depending on the dimension n of the original system the second last element is either $p-n+2$ or 0. Therefore

as $n, p \in \mathbb{N}$ then

if $p > n-2 \quad \Leftrightarrow \quad p-n+2>0$

$$\boldsymbol{c}_c^T\boldsymbol{A}_c^p\boldsymbol{b}_c = p-n+2 \tag{5.9}$$

else

$$\boldsymbol{c}_c^T\boldsymbol{A}_c^p\boldsymbol{b}_c = 0 \tag{5.10}$$

So that (5.9) and (5.10) provide the statement of the proposition.

Remark 2:

In the definition of $\boldsymbol{\Phi}$ in (3.54) it can be noticed that the exponent of $\boldsymbol{A_c}$ in any $\boldsymbol{c}_c^T \boldsymbol{A}_c^p \boldsymbol{b}_c$ element is dependent on the row and column denoted m and r respectively of the original $\boldsymbol{\Phi}$ matrix, , and it can be noticed that for each $\boldsymbol{\Phi}(m,r)$, the exponent p is equal to $|m-r|$.

Then from *Proposition 2* it can be deduced that the general formultion of $\boldsymbol{\Phi^T}$ is as follows.

$$\boldsymbol{\Phi^T} = \begin{bmatrix} \overbrace{0 \quad \dots \quad 0}^{n-1} & n-1-n+2 & n-n+2 & n+1-n+2 & \dots & N_p-1-n+2 \\ 0 \quad \dots \quad 0 & 0 & n-1-n+2 & n-n+2 & \dots & N_p-2-n+2 \\ 0 \quad \dots \quad 0 & 0 & 0 & n-1-n+2 & \dots & N_p-3-n+2 \\ \vdots \quad \vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \quad \dots \quad 0 & 0 & 0 & 0 & \dots & N_p-N_c+1-n+2 \\ 0 \quad \dots \quad 0 & 0 & 0 & 0 & \dots & N_p-N_c-n+2 \end{bmatrix} \tag{5.11}$$

Note that the equation (5.11) presents more generic from than one could expect, and might seem unnecessary (for instance the element given by $n-1-n+2$ could be simply denoted as 1). Nevertheless, retaining this notation enables this notation a regularity in the description is to be observed, which is later utilised to reduce the formulation of the whole matrix to a simple form. Using the particular numerical values here, would reduce the complexity of the description, but would reduce the clarity of the reasoning.

Further, as it was elaborated in Section 3.3, the MPC uses past and current information to predict the future inventory levels. The optimisation of order quantities is carried out over a fixed prediction horizon N_p. From (3.55) and (3.56) the control order quantities $\Delta \boldsymbol{U}$ can be derived such that

$$\Delta \boldsymbol{U} = \left(\boldsymbol{\Phi^T \Phi} + \boldsymbol{R}\right)^{-1} \left[\boldsymbol{\Phi^T Y_R} - \boldsymbol{\Phi}^T \boldsymbol{F x_c}(kT)\right] \tag{5.12}$$

For the time being assume $\boldsymbol{R}$ to be the null matrix such that $u_r = 0$. Assuming that

$$\boldsymbol{Y}_R = \overbrace{\begin{bmatrix}1 & 1 & 1 & \dots & 1\end{bmatrix}}^{N_p}{}^T y_R(kT) = \overline{\boldsymbol{Y}_R} y_R(kT) \tag{5.13}$$

where y_R is a reference inventory level, this gives as in (3.57)

$$\Delta \boldsymbol{U} = \left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1}\left[\boldsymbol{\Phi}^{T}\overline{\boldsymbol{Y_R}}y_R\left(kT\right) - \boldsymbol{\Phi}^{T}\boldsymbol{F}\boldsymbol{x_c}\left(kT\right)\right] \tag{5.14}$$

Because of the MPC principle the first element of $\Delta\boldsymbol{U}$ is applied

$$\begin{aligned}\Delta u\left(k\right) &= \boldsymbol{K}_y y_R\left(kT\right) - \boldsymbol{K}\boldsymbol{x_c}\left(kT\right)\\ &= \overbrace{\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}}^{N_c}{}^{T}\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi} + \boldsymbol{R}\right)^{-1}\left(\boldsymbol{\Phi}^{T}\overline{\boldsymbol{Y_R}}y_R\left(kT\right) - \boldsymbol{\Phi}^{T}\boldsymbol{F}\left(kT\right)\right)\end{aligned} \tag{5.15}$$

where

$\boldsymbol{K_y}$ is the first element of $\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{T}\boldsymbol{Y_R}$ and

$\boldsymbol{K} = \overbrace{\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}}^{N_p - N_c}{}^{T}\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{T}\boldsymbol{F}$ and

$\boldsymbol{K} = \left[\boldsymbol{K_x}\boldsymbol{K_y}\right]$ as in (3.63).

To find the new representation of the vector $\boldsymbol{K}$, the relevant representation of its components such as $\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^{T}\boldsymbol{F}$ need to be obtained first. From *Proposition 1* and the system description in (3.5), for the non-persisibile case $\delta(y) = 1$, the representation of $\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}$ can be simplified and the following proposition can be demonstrated.

Proposition 3:

$$\begin{gathered}\forall_n \forall_m \forall_r \boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\left(m,r\right) = \sum_{i=0}^{N_p-(g-1)-n+2}\left(\left|m-r\right| + i + 1\right)\left(i+1\right)\\ where \quad g = \begin{cases} m & when \quad m \ge r \\ r & when \quad m < r \end{cases}\end{gathered} \tag{5.16}$$

Demonstration:

Defining

$$L = n - 2 \tag{5.17}$$

from (5.11) $\boldsymbol{\Phi}^{T}$ has the form as follows

$$
\boldsymbol{\Phi}^{T} = \begin{array}{c}
\begin{array}{ccc} \overbrace{\hphantom{0 \quad \cdots \quad 0}}^{n-1} & \hphantom{n-1-L \quad n-L \quad n+1-L \quad \cdots \quad N_p-(N_p-N_c)+1-L} \end{array} \\
\begin{bmatrix}
0 & \cdots & 0 & n-1-L & n-L & n+1-L & \cdots & N_p-1-L \\
0 & \cdots & 0 & 0 & n-1-L & n-L & \cdots & N_p-2-L \\
0 & \cdots & 0 & 0 & 0 & n-1-L & \cdots & N_p-3-L \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & N_p-(N_p-N_c)+1-L \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & N_p-(N_p-N_c)-L
\end{bmatrix}
\end{array} \qquad (5.18)
$$

From the above, $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ can be obtained such that

$$\boldsymbol{\Phi}^T\boldsymbol{\Phi}=\begin{bmatrix} (n-1-L)(n-1-L)+\ldots+(N_p-1-L)(N_p-1-L) & (n-L)(n-1-L)+\ldots+(N_p-1-L)(N_p-2-L) & (n-1-L)(n-1-L+N_p-N_c)+\ldots+(N_c-L)(N_p-1-L) \\ (n-L)(n-1-L)+\ldots+(N_p-1-L)(N_p-2-L) & (n-1-L)(n-1-L)+\ldots+(N_p-2-L)(N_p-2-L) & (n-1-L)(n-L+N_p-N_c)+\ldots+(N_c-L)(N_p-2-L) \\ (n-1-L)(n+1-L)+\ldots+(N_p-3-L)(N_p-1-L) & (n-1-L)(n-L)+\ldots+(N_p-3-L)(N_p-2-L) & (n-1-L)(n+1-L+N_p-N_c)+\ldots+(N_c-L)(N_p-3-L) \\ \vdots & \vdots & \vdots \\ (n-1-L)(n-1-L+N_p-N_c)+\ldots+(N_c-L)(N_p-1-L) & (n-1-L)(n-L+N_p-N_c)+\ldots+(N_c-L)(N_p-2-L) & (n-1-L)(n-1-L)+\ldots+(N_c-L)(N_c-L) \end{bmatrix}$$

(5.19)

Therefore, colecting terms leads to

$$\boldsymbol{\Phi}^T\boldsymbol{\Phi}=\begin{bmatrix} \sum_{i=0}^{N_p-1-L}(n-1-L+i)(n-1-L+i) & \sum_{i=0}^{N_p-2-L}(n-1-L+i)(n-L+i) & \sum_{i=0}^{N_p-3-L}(n-1-L+i)(n+1-L+i) & \ldots & \sum_{i=0}^{N_p-(N_p-n+1)-n-1}\left(p_{N_p-1+i}-L\right)\left(p_{n-1+i}-L\right) \\ \sum_{i=0}^{N_p-2-L}(n-1-L+i)(n-L+i) & \sum_{i=0}^{N_p-2-L}(n-1-L+i)(n-1-L+i) & \sum_{i=0}^{N_p-3-L}(n-1-L+i)(n-L+i) & \ldots & \sum_{i=0}^{N_p-(N_p-n+1)-n-1}\left(p_{N_p-2+i}-L\right)\left(p_{n-1+i}-L\right) \\ \sum_{i=0}^{N_p-3-L}(n-1-L+i)(n+1-L+i) & \sum_{i=0}^{N_p-3-L}(n-1-L+i)(n-L+i) & \sum_{i=0}^{N_p-3-L}(n-1-L+i)(n-1-L+i) & \ldots & \sum_{i=0}^{N_p-(N_p-n+1)-n-1}\left(p_{N_p-3+i}-L\right)\left(p_{n-1+i}-L\right) \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \sum_{i=0}^{N_c-L}(n-1-L+i)(n+1-L+N_p-N_c+i) & \sum_{i=0}^{N_c-L}(n-1-L+i)(n-L+N_p-N_c+i) & \sum_{i=0}^{N_c-L}(n-1-L+i)(n+1-L+N_p-N_c+i) & \ldots & \sum_{i=0}^{N_c-L}(n-1-L+i)(n-1-L+i) \end{bmatrix}$$

(5.20)

It can now be observed that the formulation of each matrix element follows a pattern and relates to the row and column indices of a given element location, denoted m and r respectively. From the above, the general formulation of a given element of the $\boldsymbol{\Phi^T\Phi}$ matrix element can be derived such that:

$$\boldsymbol{\Phi^T\Phi}(m,r) = \sum_{i=0}^{N_p-(g-1)-L} \left(n-1+|m-r|-L+i\right)\left(n-1-L+i\right) \qquad \text{where} \quad g = \begin{cases} m & when \quad m \geq r \\ r & when \quad m < r \end{cases} \tag{5.21}$$

noticing that $L = n-2$ and from (5.17) it follows that

$$\begin{aligned} \boldsymbol{\Phi^T\Phi}(m,r) &= \sum_{i=0}^{N_p-(g-1)-n+2} \left(n-1+|m-r|-n+2+i\right)\left(n-1-n+2+i\right) = \\ &= \sum_{i=0}^{N_p-(g-1)-n+2} \left(|m-r|+i+1\right)\left(i+1\right) \\ & where \quad g = \begin{cases} m & when \quad m \geq r \\ r & when \quad m < r \end{cases} \end{aligned} \tag{5.22}$$

The above demonstrates the proposition.

Remark 3:

From *Proposition 3* it can be noticed that the elements of the matrix $\boldsymbol{\Phi^T\Phi}$ depend on their row and column index, denoted m and r respectively, the order of the original system n defined in (3.5) and the prediction horizon N_p.

Proposition 4:

The matrix $\boldsymbol{\Phi^T F}$ denoted hare as $\boldsymbol{M}$ is defined as follows

$$
\begin{aligned}
&a) \qquad \forall_m \forall_{1<r<n+1} \qquad \boldsymbol{M}(m,r) = \sum_{i=1}^{N_p-(m-1)-(n-1)} i\left[(n+m-1-r+2)+i-1\right] \\
&b) \qquad \forall_m \qquad \boldsymbol{M}(m,1) = \boldsymbol{M}(m,2) \\
&c) \qquad \forall_m \qquad \boldsymbol{M}(m,n+1) = \sum_{r=1}^{N_p} \boldsymbol{\Phi}^T(m,r)
\end{aligned}
\tag{5.23}
$$

where m and r, represent the rows and columns, respectively.

Demonstration:

Recalling the representation of the matrix $\boldsymbol{F}$ as shown in (3.54)

$$
\boldsymbol{F} = \begin{bmatrix} \boldsymbol{c}_c^T \boldsymbol{A}_c \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^2 \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^3 \\ \vdots \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^{N_p} \end{bmatrix}
$$

From (5.4) it can be observed that

$$
\boldsymbol{c}_c^T \boldsymbol{A}_c^p = \begin{bmatrix} p & p & p-1 & p-2 & \dots & p-n+2 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}
$$

Noticing that the exponent of the matrix $\boldsymbol{A}_c$ depends upon the row index of the matrix $\boldsymbol{F}$, it can be deduced that $p = m$, so that

$$
\boldsymbol{c}_c^T \boldsymbol{A}_c^p = \begin{bmatrix} m & m & m-1 & m-2 & \dots & m-n+2 & 0 & \dots & 0 & 1 \end{bmatrix} = \boldsymbol{F}(m) \tag{5.24}
$$

which is a general definition of a row in the matrix $\boldsymbol{F}$ in respect to its row index denoted as m. From the above, the matrix $\boldsymbol{F}$ can be presented in the following form

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 1 \\ 2 & 2 & 1 & 0 & 0 & \dots & 1 \\ 3 & 3 & 2 & 1 & 0 & \dots & 1 \\ 4 & 4 & 3 & 2 & 1 & \dots & 1 \\ 5 & 5 & 4 & 3 & 2 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ n & n & n-1 & n-2 & n-3 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ N_p & N_p & N_p-1 & N_p-2 & N_p-3 & \dots & 1 \end{bmatrix} \tag{5.25}$$

From the form of F , whether shown in (5.25) or (5.24), it can be observed that the 1st and 2nd columns of $\boldsymbol{\Phi}^T F$ must have the same form, which demonstrates part b) of the proposition.

From the form of F (where the last column contains ones only) it can be recognised that the last column of the matrix $\boldsymbol{\Phi}^T F$ must be a vector, in which any element denoted m is the sum of the elements of row index m of the matrix $\boldsymbol{\Phi}^T$ (or equivalently the sum of elements of column number r of the matrix $\boldsymbol{\Phi}$). This demonstrates part c) of the proposition.

Further, from the formulation of $\boldsymbol{\Phi}^T$ in (5.11), and noticing that $p_n = n-1$, the matrix $\boldsymbol{\Phi}^T$ can be obtained as follows

$$\boldsymbol{\Phi}^T = \begin{bmatrix} \overbrace{0 & \dots & 0}^{n-1} & 1 & 2 & 3 & 4 & \dots & (N_p)-n+1 \\ 0 & \dots & 0 & 0 & 1 & 2 & 3 & \dots & (N_p-1)-n+1 \\ 0 & \dots & 0 & 0 & 0 & 1 & 2 & \dots & (N_p-2)-n+1 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & (N_p-N_c)-n+1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & (N_p-N_c-1)-n+1 \end{bmatrix} \tag{5.26}$$

Further, the multiplication of $\boldsymbol{\Phi}^T$ in the formulation of (5.26) and F in the formulation (5.25), using the demonstrated parts b) and c) of (5.23), the following expression for the matrix M is obtained (note that due to its size the formulation continues over two lines).

$$\boldsymbol{M} = \left[\begin{array}{ccc} \boldsymbol{M}(1,2) & n+2(n+1)+\ldots+\left[N_p-(n-1)\right]N_p & (n-1)+2n+\ldots+\left[N_p-(n-1)\right](N_p-1) \\ \boldsymbol{M}(2,2) & (n+1)+2(n+2)+\ldots+\left[N_p-1-(n-1)\right]N_p & n+2(n+1)+\ldots+\left[N_p-1-(n-1)\right](N_p-1) \\ \boldsymbol{M}(3,2) & (n+2)+2(n+3)+\ldots+\left[N_p-2-(n-1)\right]N_p & (n+1)+2(n+2)+\ldots+\left[N_p-2-(n-1)\right](N_p-1) \\ \vdots & \vdots & \vdots \\ \boldsymbol{M}(N_p-N_c,2) & (n+N_p-N_c)+\ldots+\left[(N_p-N_c+1)-(n-1)\right]N_p & (n-1+N_p-N_c)+\ldots+(N_p-N_c+1)-(n-1)(N_p-1) \end{array}\right.$$

$$\left.\begin{array}{ccc} \ldots & [n-(n-2)]+2[n+1-(n-2)]+\ldots+(N_p-n+1)\left[N_p-n-(n-2)\right] & \sum_{r=1}^{N_p}\boldsymbol{\Phi}^T(1,r) \\ \ldots & [n-(n-3)]+2[n+1-(n-3)]+\ldots+(N_p-1-n+1)\left[N_p-n-(n-2)\right] & \sum_{r=1}^{N_p}\boldsymbol{\Phi}^T(2,r) \\ \ldots & [n-(n-4)]+2[n+1-(n-4)]+\ldots+(N_p-2-n+1)\left[N_p-n-(n-2)\right] & \sum_{r=1}^{N_p}\boldsymbol{\Phi}^T(3,r) \\ \vdots\vdots\vdots & \vdots & \vdots \\ \ldots & n+N_p-N_c+2(n+1+N_p-N_c)+\ldots+\left[(N_p-N_c-1)-n+1\right]\left[N_p-n-(n-2)\right] & \sum_{r=1}^{N_p}\boldsymbol{\Phi}^T(N_p-N_c,r) \end{array}\right] \quad (5.27)$$

From the above, it can be observed that for every $1 < m < n+1$

$$\begin{gathered} \boldsymbol{M}(m,r) = \sum_{i=1}^{i_{max}} i\left[\left(n+m-1-r+2\right)+i-1\right] \\ where \quad i_{max} = N_p - (m-1) - (n-1) \end{gathered} \tag{5.28}$$

The above ends the demonstration of part a) of (5.23) and the whole proposition.

Remark 4:

From the form of $\boldsymbol{\Phi}^{\boldsymbol{T}}$ presented in (5.26) it can be observed that in the case when $n > N_p$, zeros will always be generated in the $\boldsymbol{\Phi}^{\boldsymbol{T}}$ matrix, which will always result in a zero vector of the predicted order quantities. Therefore, n should always be assumed to be no greater than N_p, and the converse case will not be considered in the thesis.

The next step is to find the final representation of

$$\Delta u(k) = \boldsymbol{K}_y y_R(kT) - \boldsymbol{K}\boldsymbol{x}_n(kT) \tag{5.29}$$

which will allow the optimal order quantities to be found, but avoiding the extensive calculations of MPC at the same time.
In (3.63)

$$\begin{gathered} \boldsymbol{K} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}}^{N_p - N_c} \left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F} \\ and \\ \boldsymbol{K} = \left[\boldsymbol{K}_x \boldsymbol{K}_y\right] \end{gathered} \tag{5.30}$$

From the above it can be deduced that only the first row of the inverse matrix $\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1}$ is required to be obtained for further calculations. The following proposition shows the general form of the inverse matrix with respect to the values of n, N_p and N_c values.

Proposition 5:

$$\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1}=\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -2 & 5 & -4 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 6 & -4 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -4 & 6 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & b & c \end{bmatrix} \tag{5.31}$$

where

$$a=\frac{b\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S)\right)-\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-3,S-1)+4\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-2)}{\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-1)\right)}$$

$$b=\frac{1-c\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S,S-1)\right)-\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-2)}{\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-1)}$$

$$c=\frac{\left(1-\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-2,S-1)\right)\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-1)}{\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S,S)\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S-1,S-1)-\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S,S-1)\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}(S,S-1)}$$

where

$S = N_p - N_c$

Demonstration

With respect to (5.16) the demonstration serves to show that $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^T\boldsymbol{\Phi} = I$, where I is the identity matrix of appropriate dimension. The demonstration is shown in Appendix I.

Remark 5a:

From *Proposition 5,* it can be deduced that apart from four elements in the matrix $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}$ denoted a, b and c, none of the matrix elements depend on the values of N_p or N_c values for that particular state space model.

Remark 5b:

From the formulation in (5.31) it can be noticed that

$$\forall_n \forall_{N_p} \forall_{N_c < N_p - 2} \overbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}}^{N_p - N_c} \left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 \end{bmatrix} \tag{5.32}$$

Proposition 6:

$$\forall_{n\in\mathbb{N}} \forall_{r\in\mathbb{N}\setminus\{1\}} \boldsymbol{K}(r) = n - r + 2 \quad \textit{and} \quad \textit{for} \quad r = 1 \quad \boldsymbol{K}(r) = n \tag{5.33}$$

Demonstration:

From the form of $\overbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}}^{N_p - N_c}\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}$ in *Proposition 5,* it can be noticed that $\boldsymbol{K}$ will have the following form:

$$\overbrace{\begin{bmatrix} b_1 & b_1 & b_2 & \dots & b_{n-1} \end{bmatrix}}^{n} = \begin{bmatrix} a_{11} - 2a_{21} + a_{31} & a_{12} - 2a_{22} + a_{32} & \dots & a_{1n} - 2a_{2n} + a_{3n} \end{bmatrix} \tag{5.34}$$

where $a_{mr} = \boldsymbol{M}(m,r)$ denotes an element of the matrix $\boldsymbol{\Phi}^T\boldsymbol{F}$ corresponding to the m^{th} row and r^{th} column. From (5.23) part b) for the first element vector of the

$$b_1 = a_{12} - 2a_{22} + a_{32} = a_{11} - 2a_{21} + a_{31} \tag{5.35}$$

and for any other vector element $b_{r>1}$

$$b_r = a_{1r} - 2a_{2r} + a_{3r} \tag{5.36}$$

in respect to part a) and c) of (5.23).

Therefore, following the mathematical reformulation of such defined elements it can be deduced that

$$b_1 = \sum_{i=1}^{N_p-(1-1)-(n-1)} i\left[\left(n+1-1-2+2\right)+i-1\right] - 2\sum_{i=1}^{N_p-(2-1)-(n-1)} i\left[\left(n+2-1-2+2\right)+i-1\right] +$$

$$+\sum_{i=1}^{N_p-(3-1)-(n-1)} i\left[\left(n+3-1-2+2\right)+i-1\right]$$

$$= \sum_{i=1}^{N_p-n+1} i\left(n+i-1\right) - 2\sum_{i=1}^{N_p-n} i\left(n+i\right) + \sum_{i=1}^{N_p-n-1} i\left(n+i+1\right)$$

$$= \sum_{i=1}^{N_p-n-1} i\left(n+i\right) - \sum_{i=1}^{N_p-n-1} i + \left(N_p-n\right)\left(N_p-1\right) + \left(N_p-n-1\right)\left(N_p-1\right) +$$

$$-2\sum_{i=1}^{N_p-n-1} i\left(n+i\right) - 2\left(N_p-n\right)\left(N_p\right) + \sum_{i=1}^{N_p-n-1} i\left(n+i\right) + \sum_{i=1}^{N_p-n-1} i$$

$$= (N_p-n)\left(N_p-1\right) + \left(N_p-n+1\right)N_p - 2\left(N_p-n\right)N_p$$

$$= N_p^2 - N_p - nN_p + n + N_p^2 - nN_p + N_p - 2N_p^2 + 2nN_p$$

$$= n \tag{5.37}$$

and for any $1 < r < n+1$ according to part a) of (5.23)

$$b_r = \sum_{i=1}^{N_p-(1-1)-(n-1)} i\Big[\big(n+1-1-r+2\big)+i-1\Big]-2\sum_{i=1}^{N_p-(2-1)-(n-1)} i\Big[\big(n+2-1-r+2\big)+i-1\Big]+$$
$$+\sum_{i=1}^{N_p-(3-1)-(n-1)} i\Big[\big(n+3-1-r+2\big)+i-1\Big]$$

$$=\sum_{i=1}^{N_p-n+1} i\big(n+i-r+1\big)-2\sum_{i=1}^{N_p-n} i\big(n+i-r+2\big)+\sum_{i=1}^{N_p-n-1} i\big(n+i-r+3\big)$$

$$=\sum_{i=1}^{N_p-n-1} i\big(n+i-r+1\big)+\big(N_p-n\big)\big(n+N_p-n-r+1\big)+\big(N_p-n+1\big)\big(n+N_p-n+1-r+1\big)+$$

$$-2\sum_{i=1}^{N_p-n-1} i\big(n+i-r+2\big)-2\big(N_p-n\big)\big(n+N_p-n-r+2\big)+\sum_{i=1}^{N_p-n-1} i\big(n+i-r+3\big)$$

$$=\sum_{i=1}^{N_p-n-1} i\big(0\big)+\big(N_p-n\big)\big(N_p-r+1\big)+\big(N_p-n+1\big)\big(N_p-r+2\big)-2\big(N_p-n\big)\big(N_p-r+2\big)$$

$$=N_p^2-rN_p+N_p-nN_p+rn-n+N_p^2-rN_p+2N_p-nN_p+rn-2n+N_p-r+2+$$

$$-2N_p^2+2rN_p-rN_p+2nN_p-2rn+4n$$

$$=n-r+2 \tag{5.38}$$

and finally in respect to part c) of (5.23) and the form of $\boldsymbol{\Phi^T}$ shown in (5.26)

$$b_{n+1}=1+2+\ldots+N_p-n+1-2(1+2+2+\ldots+N_p-1-n+1)+1+2+3+\ldots+N_p-2-n+1$$

$$=1 \tag{5.39}$$

$$=n-(n+1)+2$$

which demonstrates the proposition.

Remark 6:

It can be noticed that the vector $\boldsymbol{K}$ defined in *Proposition 6* is a special case of the vector $\boldsymbol{K}$ shown in the *IC Proposition*. If the deterioration rate is not considered in the *IC Proposition*, so that $\delta(kT)=1$, the vector $\boldsymbol{K}$ defined in *Proposition 6* has the form

$$\forall_{n\in\mathbb{N}}\forall_{r\in\mathbb{N}\setminus\{1\}}\boldsymbol{K}(r)=\sum_{i=0}^{n-r+1}1^{i} \quad and \quad \boldsymbol{K}(1)=\sum_{i=1}^{n}1^{i} \tag{5.40}$$

From the above it can be noticed, that from now onwards the next order quantity can be immediately calculated by substituting the newly defined $\boldsymbol{K}$, the reference inventory level and current inventory levels into the equation (3.60)

$$\Delta u(k)=\boldsymbol{K}_{y}y_{R}(kT)-\boldsymbol{K}\boldsymbol{x}_{c}(kT)$$

Since $\Delta u(kT)$ is an incremental action defined in (3.40) such that

$\Delta u(kT)=u(kT)-u\left[(k-1)T\right]$, i.e.

Therefore

$$u(kT)=\boldsymbol{K}_{y}y_{R}(kT)-\boldsymbol{K}\boldsymbol{x}_{c}(kT)+u\left[(k-1)T\right] \tag{5.41}$$

Since the system output $y(kT)$ for an inventory model refers to the current inventory level it can now be denoted as $I(kT)$

$$I(kT)=y(kT) \tag{5.42}$$

Since the reference signal $y_{R}(kT)$ refers to the reference inventory level it can be denoted as $I_{R}(kT)$, and from (5.41) it can be deduced that

$$u(kT)=\boldsymbol{K}_{y}I_{R}(kT)-\boldsymbol{K}\boldsymbol{x}_{c}(kT)+u\left[(k-1)T\right] \tag{5.43}$$

From *Proposition 6* and knowing that $\boldsymbol{K}_{y}$ is the last element of the gain vector

$$\boldsymbol{K_y} = \boldsymbol{K}(n) = n - n + 1 = 1 \tag{5.44}$$

Therefore, eventually, it can be deduced that

$$u(kT) = I_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT) + u\left[(k-1)T\right] \tag{5.45}$$

which is the optimal order quantity (inventory system input) derived in the *IC Proposition*. Note that $\boldsymbol{x_c}$ defined in the *IC Proposition* is equivalent to that defined in (3.42), therefore

$$\begin{aligned} \boldsymbol{x_c}(kT) &= \left[\Delta\boldsymbol{x}(kT)^T \; y(kT)\right]^T \\ &= [\boldsymbol{x}_1(kT) - \boldsymbol{x}_1\left[(k-1)T\right] \boldsymbol{x}_2\left[(k-1)T\right] - \boldsymbol{x}_2(kT) \boldsymbol{x}_2\left[(k-1)T\right] - \boldsymbol{x}_2(kT) \ldots \\ &\quad \ldots \boldsymbol{x_n}\left[(k-1)T\right] - \boldsymbol{x_n}(kT)^T \; y(kT)] \end{aligned} \tag{5.46}$$

From (3.9) and (5.42) it can be finally deduced that

$$\begin{aligned} \boldsymbol{x_c} = \Big[& I(kT) - I\left[(k-1)T\right] \quad u\left[(k-n+1)T\right] - u\left[(k-n)T\right] \\ & u\left[(k-n+2)T\right] - u\left[(k-n+1)T\right] \quad \ldots \quad u\left[(k-1)T\right] - u(kT) \quad I(kT) \Big]^T \end{aligned} \tag{5.47}$$

As defined in the beginning of the *IC Proposition* for the non-perishable case.

5.2 Perishable Case

Proposition 7:

The general description of $\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}}$, can be formulated regardless of the values of p and n in the following way

$$\forall_n \forall_{p>n-2} \forall_{1\le m\le n}$$

$$\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}} = \left[\sum_{i=1}^{p} \delta^i(kT) \quad \sum_{i=0}^{p-1} \delta^i(kT) \quad \sum_{i=0}^{p-2} \delta^i(kT) \quad \sum_{i=0}^{p-3} \delta^i(kT) \quad \dots \quad \sum_{i=0}^{p-m+1} \delta^i(kT) \quad \dots \quad \delta^0(kT) \quad 1 \right]$$

$$or \tag{5.48}$$

$$\forall_n \forall_{p\le n-2} \forall_{1\le m\le n}$$

$$\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}} = \left[\sum_{i=1}^{p} \delta^i(kT) \quad \sum_{i=0}^{p-1} \delta^i(kT) \quad \sum_{i=0}^{p-2} \delta^i(kT) \quad \sum_{i=0}^{p-3} \delta^i(kT) \quad \dots \quad \sum_{i=0}^{p-m+1} \delta^i(kT) \quad \dots \quad 0 \quad 1 \right]$$

where any m^{th} element of a vector becomes zero only when $p-m+2\le 0$

Demonstration

From (3.46), and using (3.5) it can be deduced that

$$\boldsymbol{A}_{\boldsymbol{c}} = \begin{bmatrix} \delta(kT) & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \delta(kT) & 1 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \quad and \quad \boldsymbol{c}_c^T = [0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1] \tag{5.49}$$

Then, for $p=1$, multiplying $\boldsymbol{c}_c^T$ and $\boldsymbol{A}_{\boldsymbol{c}}$ the last row of the matrix $\boldsymbol{A}_{\boldsymbol{c}}$ is obtained as follows

$$\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}} = \overbrace{\left[\delta(kT) \quad 1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1 \right]}^{n+1} \tag{5.50}$$

which can also be represented as

$$\begin{bmatrix}\delta(kT) & 1 & 0 & 0 & \dots & 1\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{1}\delta^i(kT) & \sum_{i=0}^{1-1}\delta^i(kT) & 0 & 0 & \dots & 1\end{bmatrix} \tag{5.51}$$

which fullfils the poposition for the $p=1$ case.

Asummung, that the *Proposition 7* is true for some $1 \le p \le n-2$, then for $p+1$ it can be deduced that

$$\boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}+1} = \boldsymbol{c}_c^T \boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}} \boldsymbol{A}_{\boldsymbol{c}} =$$

$$= \begin{bmatrix}\sum_{i=1}^{p}\delta^i(kT) & \sum_{i=0}^{p-1}\delta^i(kT) & \sum_{i=0}^{p-2}\delta^i(kT) & \sum_{i=0}^{p-3}\delta^i(kT) & \dots & \sum_{i=0}^{p-m+1}\delta^i(kT) & \dots & 0 & 1\end{bmatrix}\boldsymbol{A}_{\boldsymbol{c}}$$

$$= \begin{bmatrix}1+\delta(kT)\sum_{i=1}^{p}\delta^i(kT) & \sum_{i=1}^{p}\delta^i(kT)+1 & \sum_{i=0}^{p-1}\delta^i(kT) & \sum_{i=0}^{p-2}\delta^i(kT) & \dots & \sum_{i=0}^{p-m+2}\delta^i(kT) & \dots & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\sum_{i=1}^{p+1}\delta^i(kT) & \sum_{i=0}^{(p-1)+1}\delta^i(kT) & \sum_{i=0}^{(p-2)+1}\delta^i(kT) & \sum_{i=0}^{(p-3)+1}\delta^i(kT) & \dots & \sum_{i=0}^{(p-m+1)+1}\delta^i(kT) & \dots & 0 & 1\end{bmatrix} \tag{5.52}$$

which satisfies the proposition for a given case.

Assuming now, that the *Proposition 7* is true for some $p > n-2$, then for $p+1$ it can be deduced that

$$\boldsymbol{c}^{T}{}_{\boldsymbol{c}}\boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}+1} = \boldsymbol{c}^{T}{}_{\boldsymbol{c}}\boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}}\boldsymbol{A}_{\boldsymbol{c}}$$

$$= \begin{bmatrix} \sum_{i=1}^{p}\delta^{i}(kT) & \sum_{i=0}^{p-1}\delta^{i}(kT) & \sum_{i=0}^{p-2}\delta^{i}(kT) & \sum_{i=0}^{p-3}\delta^{i}(kT) & \dots & \sum_{i=0}^{p-m+1}\delta^{i}(kT) & \dots & \delta^{0}(kT) & 1 \end{bmatrix}\boldsymbol{A}_{\boldsymbol{c}}$$

$$= \begin{bmatrix} 1+\delta(kT)\sum_{i=1}^{p}\delta^{i}(kT) & \sum_{i=1}^{p}\delta^{i}(kT)+1 & \sum_{i=0}^{p-1}\delta^{i}(kT) & \sum_{i=0}^{p-2}\delta^{i}(kT) & \dots & \delta^{0}(kT) & 1 \end{bmatrix} \tag{5.53}$$

$$= \begin{bmatrix} \sum_{i=1}^{p+1}\delta^{i}(kT) & \sum_{i=0}^{(p-1)+1}\delta^{i}(kT) & \sum_{i=0}^{(p-2)+1}\delta^{i}(kT) & \sum_{i=0}^{(p-3)+1}\delta^{i}(kT) & \dots & \delta^{0}(kT) & 1 \end{bmatrix}$$

which satisfies the proposition for a considered case and completes the demonstration the proposition.

Proposition 8:

For $\boldsymbol{c}_{\boldsymbol{c}}^{\boldsymbol{T}}$, $\boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}}$ and $\boldsymbol{b}_{\boldsymbol{c}}$ defined in (3.46)

$$\forall_{n\in\mathbb{N}}\forall_{p\in\mathbb{N}}\ \boldsymbol{c}_{\boldsymbol{c}}^{\boldsymbol{T}}\boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}}\boldsymbol{b}_{\boldsymbol{c}} = \sum_{i=0}^{p-n+1}\delta^{i}(kT) \quad \textit{when} \quad p-n+2>0, \quad \textit{else} \quad \boldsymbol{c}_{\boldsymbol{c}}^{\boldsymbol{T}}\boldsymbol{A}_{\boldsymbol{c}}^{\boldsymbol{p}}\boldsymbol{b}_{\boldsymbol{c}} = 0, \tag{5.54}$$

where n is the order of the system described in (3.5).

Demonstration:

From (3.46) and using (3.5) it can be deduced as follows.

$$\boldsymbol{A_c} = \begin{bmatrix} \delta(kT) & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \delta(kT) & 1 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{b_c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathit{and} \quad \boldsymbol{c}_c^T = [0 \quad 0 \quad 0 \quad \dots \quad 1] \qquad (5.55)$$

From the formulation of $\boldsymbol{b_c}$ it can be noticed that $\boldsymbol{c}_c^T \boldsymbol{A_c^p b_c}$ is the n^{th} (second last) element of the vector $\boldsymbol{c}_c^T \boldsymbol{A_c^p}$.

As $n, p \in \mathbb{N}$ then

$$\mathit{if} \qquad p > n-2 \quad \Leftrightarrow \quad p-n+2>0$$

$$\boldsymbol{c}_c^T \boldsymbol{A_c^p b_c} = \sum_{i=0}^{p-n+1} \delta^i (kT) \qquad (5.56)$$

$$\mathit{else}$$

$$\boldsymbol{c}_c^T \boldsymbol{A_c^p b_c} = 0$$

which provides the statement of the proposition.

Remark 7:

It can be noticed that *Proposition 1* and *Proposition 2* from section 5.1 are essentially special cases of *Propoistion 7* and *Proposition 8* when $\delta(kT) = 1$.

Remark 8:

From the form of the matrix $\boldsymbol{\Phi}$ shown in (5.11) it can be seen that $\boldsymbol{c}_c^T \boldsymbol{A_c^p b_c}$ are elements of the matrix $\boldsymbol{\Phi}$ and the exponent p depends on elements position in the matrix determined by the row (defined as m) and column (defined as r) indices, such that $p = |m-r|$. Therefore, from *Proposition 8* it can be concluded that

$$\forall_{n\in\mathbb{N}} \quad if \quad |m-r|>n-2 \quad then \quad \boldsymbol{\Phi}^T(m,r)=\sum_{i=0}^{|m-r|-n+1}\delta^i \quad else \quad \boldsymbol{\Phi}^T(m,r)=0 \tag{5.57}$$

where n is the order of the system described in (3.5).

From *Remark 8* and the system description in (3.5) the following proposition can be demonstrated.

Proposition 9:

$$\begin{gathered}\forall_n\forall_{N_p>N_c}\forall_{N_c\geq n-1}\forall_m\forall_r \boldsymbol{\Phi}^T\boldsymbol{\Phi}(m,r)=\sum_{j=0}^{N_p-n+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i(kT)\sum_{i=0}^{j}\delta^i(kT)\right)\\ where \quad g=\begin{cases} m & when \quad m\geq r \\ r & when \quad m<r\end{cases}\end{gathered} \tag{5.58}$$

where the matrix $\boldsymbol{\Phi}$ is defined in (3.54) and m and r denote the matrix row and column number respectively.

Remark 9:

From now onwards, for simplicity and ease of description, for any i, the $\delta^i(kT)$ will be denoted as δ^i, which should avoid confusion.

Demonstration:

From *Remark 8* the matrix $\boldsymbol{\Phi}^T$ can be expanded in the following way

$$
\boldsymbol{\Phi}^T = \begin{array}{l} \quad\overbrace{\hphantom{0 \quad \cdots \quad 0}}^{n-1} \\ \begin{bmatrix}
0 & \cdots & 0 & \sum_{i=0}^{0}\delta^i & \sum_{i=0}^{1}\delta^i & \sum_{i=0}^{2}\delta^i & \cdots & \sum_{i=0}^{m-1-n+1}\delta^i & \cdots & \sum_{i=0}^{N_p-1-n+1}\delta^i \\
0 & \cdots & 0 & 0 & \sum_{i=0}^{0}\delta^i & \sum_{i=0}^{1}\delta^i & \cdots & \sum_{i=0}^{m-2-n+1}\delta^i & \cdots & \sum_{i=0}^{N_p-2-n+1}\delta^i \\
0 & \cdots & 0 & 0 & 0 & \sum_{i=0}^{0}\delta^i & \cdots & \sum_{i=0}^{m-3-n+1}\delta^i & \cdots & \sum_{i=0}^{N_p-3-n+1}\delta^i \\
\vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sum_{i=0}^{N_p-(N_p-N_c+1)-n+1}\delta^i \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sum_{i=0}^{N_p-(N_p-N_c)-n+1}\delta^i
\end{bmatrix} \end{array} \tag{5.59}
$$

From the above, the matrix $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ can be obtained such that

$$\boldsymbol{\Phi}^T\boldsymbol{\Phi}=\begin{bmatrix}
\sum_{i=0}^{0}\delta^i\sum_{i=0}^{0}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{1}\delta^i+\ldots+\sum_{i=0}^{N_p-n}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{1}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{2}\delta^i+\ldots+\sum_{i=0}^{N_p-n-1}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{2}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{3}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \ldots & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-1}\delta^i+\ldots\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n}\delta^i \\
\sum_{i=0}^{0}\delta^i\sum_{i=0}^{1}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{2}\delta^i+\ldots+\sum_{i=0}^{N_p-n-1}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{0}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{1}\delta^i+\ldots+\sum_{i=0}^{N_p-n-1}\delta^i\sum_{i=0}^{N_p-n-1}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{1}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{2}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=0}^{N_p-n-1}\delta^i & \ldots & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-2}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n-1}\delta^i \\
\sum_{i=0}^{0}\delta^i\sum_{i=0}^{2}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{3}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{1}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{2}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=0}^{N_p-n-1}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{0}\delta^i+\sum_{i=0}^{1}\delta^i\sum_{i=0}^{1}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=0}^{N_p-n-2}\delta^i & \ldots & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-3}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n-2}\delta^i \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\
\sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-1}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-2}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n-1}\delta^i & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_p-N_c-3}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_p-n-2}\delta^i & \ldots & \sum_{i=0}^{0}\delta^i\sum_{i=0}^{N_c-N_c}\delta^i+\ldots+\sum_{i=0}^{N_c-n+1}\delta^i\sum_{i=0}^{N_c-n+2}\delta^i
\end{bmatrix}$$

(5.60)

Therefore

$$\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}=\begin{bmatrix}
\sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) & \sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+m-r}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-1}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-1}\delta^i\right) \\
\sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) & \sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+m-r}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) \\
\sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+m-r}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-3}\delta^i\right) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\sum_{j=0}^{N_p-n-r+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+r-m}\delta^i\right) & \sum_{j=0}^{N_p-n-r+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+r-m}\delta^i\right) & \sum_{j=0}^{N_p-n-r+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+r-m}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c+1-r}\delta^i\right) & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-r}\delta^i\right) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-1}\delta^i\right) & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c+1-m}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-(N_p-N_c)}\delta^i\right) & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-(N_p-N_c)+1}\delta^i\right) \\
\sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-1}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-2}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-3}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-m}\delta^i\right) & \cdots & \sum_{j=0}^{N_c-n+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-(N_p-N_c)+1}\delta^i\right) & \sum_{j=0}^{N_c-n+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+N_p-N_c-(N_p-N_c)}\delta^i\right)
\end{bmatrix} \tag{5.61}$$

From the above the general formulation of $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}$ a matrix element can be reduced to:

$$\forall_n \forall_{N_p} \forall_{n-1\le N_c<N_p} \forall_m \forall_r \boldsymbol{\Phi}^T\boldsymbol{\Phi}(m,r) = \sum_{j=0}^{N_p-n+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i \sum_{i=0}^{j}\delta^i\right) \qquad (5.62)$$
$$where \quad g = \begin{cases} m\, when\, m > r \\ r\, when\, m < r \end{cases}$$

The above demonstrates the proposition.

Remark 10a:

From *Proposition 9* it can be noticed that the elements of the matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}$ depend on their row and column indices, m and r, respectively, the order of the original system n defined in (3.5), the prediction horizon N_p and the deterioration rate δ.

Remark 10b:

It can be noticed that for $N_c \le n-1$ the matrix definition in (5.58) does not hold and that case will not be considered in the thesis.

Proposition 10:

From matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F}$ denoted as $\boldsymbol{M} = \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F}$ the following hold

$$\text{a)}\quad \forall_m \forall_r \forall_n \forall_{N_p} \forall_{n-1\le N_c<N_p} \quad \boldsymbol{M}(m,r) = \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+m-r}\delta^i\right)$$

$$\text{b)}\quad \forall_m \forall_n \forall_{N_p} \forall_{n-1\le N_c<N_p} \quad \boldsymbol{M}(m,1) = \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i \sum_{i=1}^{j+n+(m-1)}\delta^i\right) \qquad (5.63)$$

$$\text{c)}\quad \forall_m \forall_n \forall_{N_p} \forall_{n-1\le N_c<N_p} \quad \boldsymbol{M}(m,n+1) = \sum_{r=1}^{N_p}\boldsymbol{\Phi}^{\boldsymbol{T}}(m,r)$$

where the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{F}$ are defined in (3.54) and m and r denote matrix row and column index respectively.

Demonstration:

Recalling the representation of the matrix $\boldsymbol{F}$ shown in (3.54)

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{c}_c^T \boldsymbol{A}_c \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^2 \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^3 \\ \vdots \\ \boldsymbol{c}_c^T \boldsymbol{A}_c^{N_p} \end{bmatrix}$$

from (5.48) it can be noticed that

$$\boldsymbol{F}(m) = \boldsymbol{c}_c^T \boldsymbol{A}_c^m$$

where m is a matrix row index, it can be observed that

$$\boldsymbol{F} = \begin{bmatrix} \sum_{i=1}^{1}\delta^i & \sum_{i=0}^{0}\delta^i & 0 & 0 & \cdots & 0 & 1 \\ \sum_{i=1}^{2}\delta^i & \sum_{i=0}^{1}\delta^i & \sum_{i=0}^{0}\delta^i & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ \sum_{i=1}^{n-2}\delta^i & \sum_{i=0}^{n-3}\delta^i & \sum_{i=0}^{n-4}\delta^i & \sum_{i=0}^{n-5}\delta^i & \cdots & \sum_{i=0}^{0}\delta^i & 1 \\ \sum_{i=1}^{n-1}\delta^i & \sum_{i=0}^{n-2}\delta^i & \sum_{i=0}^{n-3}\delta^i & \sum_{i=0}^{n-4}\delta^i & \cdots & \sum_{i=0}^{1}\delta^i & 1 \\ \sum_{i=1}^{n}\delta^i & \sum_{i=0}^{n-1}\delta^i & \sum_{i=0}^{n-2}\delta^i & \sum_{i=0}^{n-3}\delta^i & \cdots & \sum_{i=0}^{2}\delta^i & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ \sum_{i=1}^{N_p}\delta^i & \sum_{i=0}^{N_p-1}\delta^i & \sum_{i=0}^{N_p-2}\delta^i & \sum_{i=0}^{N_p-3}\delta^i & \cdots & \sum_{i=0}^{N_p-(n-1)}\delta^i & 1 \end{bmatrix} \tag{5.64}$$

It can be noticed that for row indices $m \geq n-2$ there are no zero elements in the rows.

From the form of the matrix $\boldsymbol{F}$ shown in (5.64) (the last column contains ones only) it can be recognised that the last column of the matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F}$ must be a vector of elements which are the sum of the elements of the *m-th* row of the matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}$ (elements of the *r-th* column of the matrix $\boldsymbol{\Phi}$, equivalently). This demonstrates part c) of the proposition.

Further multiplying the matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}$ (formultion shown in (5.59)) by the first column of the matrix $\boldsymbol{F}$ (formulation shown in (5.64)) it may be deduced that

$$\boldsymbol{M}(m,1)=\begin{bmatrix} \sum_{i=0}^{0}\delta^i\sum_{i=1}^{n}\delta^i+\ldots+\sum_{i=0}^{N_p-n}\delta^i\sum_{i=1}^{N_p}\delta^i \\ \sum_{i=0}^{0}\delta^i\sum_{i=1}^{n+1}\delta^i+\ldots+\sum_{i=0}^{N_p-n-1}\delta^i\sum_{i=1}^{N_p}\delta^i \\ \sum_{i=0}^{0}\delta^i\sum_{i=1}^{n+2}\delta^i+\ldots+\sum_{i=0}^{N_p-n-2}\delta^i\sum_{i=1}^{N_p}\delta^i \\ \vdots \\ \sum_{i=0}^{0}\delta^i\sum_{i=1}^{n+N_p-N_c-1}\delta^i+\ldots+\sum_{i=0}^{N_p-\left(N_p-N_c-1\right)}\delta^i\sum_{i=1}^{N_p}\delta^i \end{bmatrix} \tag{5.65}$$

So that

$$\boldsymbol{M}(m,1)=\begin{bmatrix} \sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=1}^{j+n}\delta^i\right) \\ \sum_{j=0}^{N_p-(n+1)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=1}^{j+n+1}\delta^i\right) \\ \sum_{j=0}^{N_p-(n+2)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=1}^{j+n+2}\delta^i\right) \\ \vdots \\ \sum_{j=0}^{N_p-(N_p-N_c-1)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=1}^{j+n+N_p-N_c-1}\delta^i\right) \end{bmatrix} \tag{5.66}$$

Therefore
$$\boldsymbol{M}(m,1)=\sum_{j=0}^{N_p-n-(r+1)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=1}^{j+n+(r-1)}\delta^i\right) \tag{5.67}$$

This demonstrates part b) of the proposition.

Further, multiplying (5.59) by the remaining columns of the matrix $\boldsymbol{F}$ shown in (5.64) it can be observed that for $r>1$

$$
\boldsymbol{M}\left(m,r\right)=\begin{bmatrix}
\sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)}\delta^i\right) & \sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-2)}\delta^i\right) & \sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-3)}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+(1-n)}\delta^i\right) \\
\sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n}\delta^i\right) & \sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)}\delta^i\right) & \sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-2)}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+(2-n)}\delta^i\right) \\
\sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n+1)}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n}\delta^i\right) & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+(3-n)}\delta^i\right) \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\
\sum_{j=0}^{N_p-n-\left(N_p-N_c-1\right)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+\left(N_p-N_c-1\right)}\delta^i\right) & \sum_{j=0}^{N_p-n-\left(N_p-N_c-1\right)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+\left(N_p-N_c-2\right)}\delta^i\right) & \sum_{j=0}^{N_p-n-\left(N_p-N_c-1\right)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+\left(N_p-N_c-3\right)}\delta^i\right) & \cdots & \sum_{j=0}^{N_p-n-\left(N_p-N_c-1\right)}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+(n-1)+\left(N_p-N_c-n\right)}\delta^i\right)
\end{bmatrix}
\tag{5.68}
$$

Therefore

$$\boldsymbol{M}(m,r) = \sum_{j=0}^{N_p-n-m+1}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+m-r}\delta^i\right) \tag{5.69}$$

This demonstrates part a) of the proposition and ends the demonstration of *Proposition 10*.

The last step is to obtain the final representation of

$$\Delta u(k) = K_y y_R(kT) - K x_n(kT)$$

using $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^T\boldsymbol{F}$ in the simplified descriptions shown in *Proposition 9* and *Proposition 10.* This will allow the optimal order quantities to be found, avoiding at the same time the extensive calculations related to the multiplication of extensive matrices $\boldsymbol{\Phi}$ and $\boldsymbol{F}$ described in (3.54) for the system desribed in (3.5).
As in (3.63)

$$\begin{aligned}\boldsymbol{K} &= \overbrace{\begin{bmatrix}1 & 0 & 0 & \ldots & 0\end{bmatrix}}^{N_p-N_c}{}^T \left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^T\boldsymbol{F} \\ & \quad\quad\quad \textit{and} \\ \boldsymbol{K} &= \left[\boldsymbol{K}_x\boldsymbol{K}_y\right]\end{aligned}$$

It can be noticed that only the first row of the inverse matrix $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}$ is required to be obtained. The following proposition shows the general form of the inverse matrix in respect to the δ, n, N_p and N_c values.

Proposition 11:

The inverse of the matrix $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ has dimension $N_p - N_c$ and has the following form

$$
\left(\boldsymbol{\Phi}^{T}\boldsymbol{\Phi}\right)^{-1} =
$$

$$
\begin{bmatrix}
1 & -1-\delta & \delta & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
-1-\delta & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \delta & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \delta & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \delta & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 2+2\delta+\delta^{2} & \ldots & 0 & 0 & 0 & 0 & 0 & 0. \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & -\left(1+2\delta+\delta^{2}\right) & \delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & \delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \delta & -\left(1+2\delta+\delta^{2}\right) & 2+2\delta+\delta^{2} & -\left(1+2\delta+\delta^{2}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \delta & -\left(1+2\delta+\delta^{2}\right) & a & b \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \delta & b & c
\end{bmatrix}
\tag{5.70}
$$

where

$$a = \frac{b\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S)\right) - \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-3,S-1) + 4\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-2)}{\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-1)\right)}$$

$$b = \frac{1 - c\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S-1)\right) - \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-2)}{\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-1)}$$

$$c = \frac{\left(1 - \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-2,S-1)\right)\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-1)}{\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S)\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-1) - \boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S-1)\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S-1)}$$

are scalars and
where

$$S = N_p - N_c$$

Demonstration:

Demonstration of the proposition makes reference to the product of $\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}\right)\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}\right)^{-1}$ to ensure that the identity matrix is obtained. The process is shown in Appendix I.

Remark 11:

$$\forall_n \forall_{N_p} \forall_{n-1 \le N_c < N_p - 2} \overbrace{\left[100\ldots0\right]}^{N_p - N_c}\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}\right)^{-1} = \begin{bmatrix} 1 & -1-\delta & \delta & 0 & \ldots & 0 \end{bmatrix} \tag{5.71}$$

the matrix $\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F}$ is defined in (3.54) and m and r denote matrix row and column number respectively, and for a system desribed in (3.5), using *Proposition 11,* and *Remark 11,* the general form of the vector $\boldsymbol{K} = \overbrace{\left[100\ldots0\right]}^{N_p - N_c}\left(\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{F}$ was obtained and shown in *Proposition 12*.

Proposition 12:

$$\forall_{n\in\mathbb{N}}\forall_{1<r\leq n+1}\quad \boldsymbol{K}\left(r\right)=\sum_{i=0}^{n-r+1}\delta^{i}\quad and\quad \boldsymbol{K}\left(1\right)=\sum_{i=1}^{n}\delta^{i} \tag{5.72}$$

Demonstration:

Knowing the tripartite structure of the matrix $\boldsymbol{\Phi}^{T}\boldsymbol{F}$ shown in *Proposition 10,* to demonstrate the current proposition, the following separate cases need to be considered:

- for $r=1$, covering the second part of the current proposition ($\boldsymbol{K}\left(1\right)=\sum_{i=1}^{n}\delta^{i}$) with reference to part b) of *Proposition 10*
- for $1<r<n+1$ and for $r=n+1$ covering the first part of the current proposition ($\forall_{1<r\leq n+1}\boldsymbol{K}\left(r\right)=\sum_{i=0}^{n-r+1}\delta^{i}$), with reference to part a) and c) of *Proposition 10* where $n+1$ is the column index of the matrix $\boldsymbol{\Phi}^{T}\boldsymbol{F}$.

Based on *Remark 11*, starting from the second part of the proposition, $\boldsymbol{K}\left(1\right)=\sum_{i=1}^{n}\delta^{i}$ and recalling the matrix $\boldsymbol{M}(m,1)$, representing the first column of the matrix $\boldsymbol{\Phi}^{T}\boldsymbol{F}$ shown in (5.63), it needs to be shown that

$$\boldsymbol{K}\left(1\right)\;=\;\left[1\quad -1-\delta\quad \delta\quad 0\quad \ldots\quad 0\right]\boldsymbol{M}(m,1)\;=\;\sum_{i=1}^{n}\delta^{i} \tag{5.73}$$

Therefore, from (5.66) it needs to be shown that

$$\boldsymbol{K}\left(1\right)=\sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n}\delta^{i}\right)+\left(-1-\delta\right)\sum_{j=0}^{N_p-(n+1)}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+1}\delta^{i}\right)+\delta\sum_{j=0}^{N_p-(n+2)}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+2}\delta^{i}\right)=\sum_{i=1}^{n}\delta^{i} \tag{5.74}$$

Defining $z=N_p-n$, the principle of mathematical induction is now used to show (5.74) above. As a first stage, consider $z=2$, then

$$\boldsymbol{K}(1)=\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n}\delta^{i}\right)+(-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+1}\delta^{i}\right)+\delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+2}\delta^{i}\right)$$

$$=\sum_{i=1}^{n}\delta^{i}+\sum_{i=0}^{1}\delta^{i}\sum_{i=1}^{n+1}\delta^{i}+\sum_{i=0}^{2}\delta^{i}\sum_{i=1}^{n+2}\delta^{i}+(-1-\delta)\left[\sum_{i=1}^{n+1}\delta^{i}+\sum_{i=0}^{1}\delta^{i}\sum_{i=1}^{n+2}\delta^{i}\right]+\delta\sum_{i=1}^{n+2}\delta^{i}$$

$$=\sum_{i=1}^{n}\delta^{i}+(1+\delta)\sum_{i=1}^{n+1}\delta^{i}+\left(1+\delta+\delta^{2}\right)\sum_{i=1}^{n+2}\delta^{i}+(-1-\delta)\left[\sum_{i=1}^{n+1}\delta^{i}+(1+\delta)\sum_{i=1}^{n+2}\delta^{i}\right]+\delta\sum_{i=1}^{n+2}\delta^{i} \qquad (5.75)$$

$$=\sum_{i=1}^{n}\delta^{i}+(1+\delta)\sum_{i=1}^{n+1}\delta^{i}+\left(1+\delta+\delta^{2}\right)\sum_{i=1}^{n+2}\delta^{i}+(-1-\delta)\sum_{i=1}^{n+1}\delta^{i}-\left(1+2\delta+\delta^{2}\right)\sum_{i=1}^{n+2}\delta^{i}+\delta\sum_{i=1}^{n+2}\delta^{i}$$

$$=\sum_{i=1}^{n}\delta^{i}$$

Assuming now that for some $z>2$

$$\boldsymbol{K}(1)=\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n}\delta^{i}\right)+(-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+1}\delta^{i}\right)+\delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+2}\delta^{i}\right)=\sum_{i=1}^{n}\delta^{i} \qquad (5.76)$$

then for $z+1$ it can be deduced that:

$$\boldsymbol{K}(1)=\sum_{j=0}^{z+1}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n}\delta^{i}\right)+(-1-\delta)\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+1}\delta^{i}\right)+\delta\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+2}\delta^{i}\right)$$

$$=\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n}\delta^{i}\right)+(-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+1}\delta^{i}\right)+\delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^{i}\sum_{i=1}^{j+n+2}\delta^{i}\right)+$$

$$+\sum_{i=0}^{z+1}\delta^{i}\sum_{i=1}^{z+n+1}\delta^{i}+(-1-\delta)\left(\sum_{i=0}^{z}\delta^{i}\sum_{i=1}^{z+n+1}\delta^{i}\right)+\delta\left(\sum_{i=0}^{z-1}\delta^{i}\sum_{i=1}^{z+n+1}\delta^{i}\right) \qquad (5.77)$$

From the assumption in (5.76), continuing (5.77), the following is observed

$$\boldsymbol{K}(1)=\sum_{i=1}^{n}\delta^i+\sum_{i=0}^{z+1}\delta^i\sum_{i=1}^{z+n+1}\delta^i+(-1-\delta)\left(\sum_{i=0}^{z}\delta^i\sum_{i=1}^{z+n+1}\delta^i\right)+\delta\left(\sum_{i=0}^{z-1}\delta^i\sum_{i=1}^{z+n+1}\delta^i\right)$$

$$=\sum_{i=1}^{n}\delta^i+\sum_{i=1}^{z+n+1}\delta^i\left[\sum_{i=0}^{z+1}\delta^i+(-1-\delta)\sum_{i=0}^{z}\delta^i+\delta\sum_{i=0}^{z-1}\delta^i\right]$$

$$=\sum_{i=1}^{n}\delta^i+\sum_{i=1}^{z+n+1}\delta^i\left[\sum_{i=0}^{z+1}\delta^i-\sum_{i=0}^{z}\delta^i+\sum_{i=1}^{z}\delta^i-\sum_{i=1}^{z+1}\delta^i\right]$$

$$=\sum_{i=1}^{n}\delta^i+\sum_{i=1}^{z+n+1}\delta^i\left[\delta^{z+1}-\delta^{z+1}\right]$$

$$=\sum_{i=1}^{n}\delta^i+0$$

$$=\sum_{i=1}^{n}\delta^i \tag{5.78}$$

which, in terms of proof by induction, demonstrates that for every $z\geq 2$, $\boldsymbol{K}(1)=\sum_{i=1}^{n}\delta^i$.

Further, based on *Remark 11*, focusing on first part of the proposition and the case when $1<r<n+1$ and recalling the matrix $\forall_{1<r<n+1}\boldsymbol{M}(m,r)$ in (5.63), it needs to be shown that

$$\boldsymbol{K}(r) \;=\; \begin{bmatrix}1 & -1-\delta & \delta & 0 & \dots & 0\end{bmatrix}\boldsymbol{M}(m,r) \;=\; \sum_{i=0}^{n-r+1}\delta^i \tag{5.79}$$

Therefore, from (5.68), it needs to be shown that

$$\boldsymbol{K}(r)=\sum_{j=0}^{N_p-n-1+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+1-r}\delta^i\right)+(-1-\delta)\sum_{j=0}^{N_p-n-2+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+2-r}\delta^i\right)+\delta\sum_{j=0}^{N_p-n-3+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+3-r}\delta^i\right)$$

$$=\sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+1-r}\delta^i\right)+(-1-\delta)\sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+2-r}\delta^i\right)+\delta\sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+3-r}\delta^i\right) \tag{5.80}$$

$$=\sum_{i=0}^{n-r+1}\delta^i$$

Defining $z = N_p - n$ the principle of mathematical induction is used to show the above. As a first stage consider $z = 2$, then

$$\boldsymbol{K}(r) = \sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+1-r}\delta^i\right) + (-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+2-r}\delta^i\right) + \delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+3-r}\delta^i\right)$$

$$= \sum_{i=0}^{n+1-r}\delta^i + \sum_{i=0}^{1}\delta^i \sum_{i=0}^{n+2-r}\delta^i + \sum_{i=0}^{2}\delta^i \sum_{i=0}^{n+3-r}\delta^i + (-1-\delta)\left[\sum_{i=0}^{n+2-r}\delta^i + \sum_{i=0}^{1}\delta^i \sum_{i=0}^{n+3-r}\delta^i\right] + \delta \sum_{i=0}^{n+3-r}\delta^i$$

$$= \sum_{i=0}^{n-r+1}\delta^i + (1+\delta)\sum_{i=0}^{n-r+2}\delta^i + \left(1+\delta+\delta^2\right)\sum_{i=0}^{n-r+3}\delta^i +$$

$$+(-1-\delta)\left[\sum_{i=0}^{n-r+2}\delta^i + (1+\delta)\sum_{i=0}^{n-r+3}\delta^i\right] + \delta\sum_{i=0}^{n-r+3}\delta^i$$

$$= \sum_{i=0}^{n-r+1}\delta^i + (1+\delta)\sum_{i=0}^{n-r+2}\delta^i + \left(1+\delta+\delta^2\right)\sum_{i=1}^{n-r+3}\delta^i +$$

$$+(-1-\delta)\sum_{i=0}^{n-r+2}\delta^i - \left(1+2\delta+\delta^2\right)\sum_{i=0}^{n-r+3}\delta^i + \delta\sum_{i=0}^{n-r+3}\delta^i \tag{5.81}$$

$$= \sum_{i=0}^{n-r+1}\delta^i$$

Assuming now that for some $z > 2$

$$\boldsymbol{K}(r) = \sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+1-r}\delta^i\right) + (-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+2-r}\delta^i\right) + \delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^i \sum_{i=0}^{j+n+3-r}\delta^i\right)$$
$$= \sum_{i=1}^{n-r+1}\delta^i \tag{5.82}$$

then for $z+1$ it can be obtained:

$$\boldsymbol{K}(r)=\sum_{j=0}^{z+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+1-r}\delta^i\right)+(-1-\delta)\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+2-r}\delta^i\right)+\delta\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+3-r}\delta^i\right)$$

$$=\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+1-r}\delta^i\right)+(-1-\delta)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+2-r}\delta^i\right)+\delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+n+3-r}\delta^i\right)+ \tag{5.83}$$

$$+\sum_{i=0}^{z+1}\delta^i\sum_{i=1}^{z+n+2-r}\delta^i+(-1-\delta)\left(\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+n+2-r}\delta^i\right)+\delta\left(\sum_{i=0}^{z-1}\delta^i\sum_{i=1}^{z+n+2-r}\delta^i\right)$$

From the assumption in (5.82) and continuing (5.83) the following is observed

$$\begin{aligned}
\boldsymbol{K}(r)&=\sum_{i=1}^{n-r+1}\delta^i+\sum_{i=0}^{z+1}\delta^i\sum_{i=1}^{z+n+2-r}\delta^i+(-1-\delta)\left(\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+n+2-r}\delta^i\right)+\delta\left(\sum_{i=0}^{z-1}\delta^i\sum_{i=1}^{z+n+2-r}\delta^i\right)\\
&=\sum_{i=1}^{n-r+1}\delta^i+\sum_{i=1}^{z+n+2-r}\delta^i\left[\sum_{i=0}^{z+1}\delta^i+(-1-\delta)\sum_{i=0}^{z}\delta^i+\delta\sum_{i=0}^{z-1}\delta^i\right]\\
&=\sum_{i=1}^{n-r+1}\delta^i+\sum_{i=1}^{z+n+2-r}\delta^i\left[\sum_{i=0}^{z+1}\delta^i-\sum_{i=0}^{z}\delta^i+\sum_{i=1}^{z}\delta^i-\sum_{i=i}^{z+1}\delta^i\right]\\
&=\sum_{i=1}^{n-r+1}\delta^i+\sum_{i=1}^{z+n-r+2}\delta^i\left[\delta^0-\delta^0\right]\\
&=\sum_{i=1}^{n-r+1}\delta^i+0\\
&=\sum_{i=0}^{n-r+1}\delta^i
\end{aligned} \tag{5.84}$$

which, in terms of proof by induction, demonstrates that for every $z\geq 2$, $\boldsymbol{K}(r)=\sum_{i=0}^{n-r+1}\delta^i$.

The last case which needs to be considered is $r=n+1$. It needs to be shown that for such defined r, $\boldsymbol{K}(r)=\sum_{i=0}^{n-r+1}\delta^i$. Recalling the matrix $\boldsymbol{M}(m,n+1)$ defined in (5.63), representing the last column of the matrix $\boldsymbol{\Phi}^T\boldsymbol{F}$ (column number $n+1$) it needs to be shown that for every n

$$\boldsymbol{K}\left(r=n+1\right) \;=\; \left[1 \quad -1-\delta \quad \delta \quad 0 \quad \dots \quad 0\right]\boldsymbol{M}(m,n+1) \;=\; \sum_{i=0}^{n-(n+1)+1}\delta^i = \sum_{i=0}^{0}\delta^i = 1 \qquad (5.85)$$

Therefore, from (5.59)

$$\begin{aligned}\boldsymbol{K}\left(n+1\right) &= \sum_{r=1}^{N_p-N_c}\boldsymbol{\Phi}^{\boldsymbol{T}}\left(2,r\right)+\left(-1-\delta\right)\sum_{r=1}^{N_p-N_c}\boldsymbol{\Phi}^{\boldsymbol{T}}\left(2,r\right)+\delta\sum_{r=1}^{N_p-N_c}\boldsymbol{\Phi}^{\boldsymbol{T}}\left(3,r\right)\\ &= \sum_{j=0}^{N_p-n}\sum_{i=0}^{j}\delta^i+\left(-1-\delta\right)\sum_{j=0}^{N_p-n-1}\sum_{i=0}^{j}\delta^i+\delta\sum_{j=0}^{N_p-n-2}\sum_{i=0}^{j}\delta^i\end{aligned} \qquad (5.86)$$

Defining $z = N_p - n$, the principle mathematical induction is used to show the above.

As a first stage consider $z = 2$, then

$$\begin{aligned}\boldsymbol{K}\left(n+1\right) &= \sum_{j=0}^{z}\sum_{i=0}^{j}\delta^i+\left(-1-\delta\right)\sum_{j=0}^{z-1}\sum_{i=0}^{j}\delta^i+\delta\sum_{j=0}^{z-2}\sum_{i=0}^{j}\delta^i\\ &= \sum_{j=0}^{2}\sum_{i=0}^{j}\delta^i+\left(-1-\delta\right)\sum_{j=0}^{1}\sum_{i=0}^{j}\delta^i+\delta\sum_{j=0}^{0}\sum_{i=0}^{j}\delta^i\\ &= \sum_{i=0}^{0}\delta^i+\sum_{i=0}^{1}\delta^i+\sum_{i=0}^{2}\delta^i+\left(-1-\delta\right)\left(\sum_{i=0}^{0}\delta^i+\sum_{i=0}^{1}\delta^i\right)+\delta\sum_{i=0}^{0}\delta^i\\ &= 1+1+\delta+1+\delta+\delta^2+\left(-1-\delta\right)\left(1+1+\delta\right)+\delta\\ & 3+3\delta+\delta^2-2-3\delta-\delta^2\\ &= 1\\ &= \sum_{i=0}^{0}\delta^i\end{aligned} \qquad (5.87)$$

Assuming now that for some $z > 2$

$$\boldsymbol{K}(n+1)=\sum_{j=0}^{z}\sum_{i=0}^{j}\delta^{i}+(-1-\delta)\sum_{j=0}^{z-1}\sum_{i=0}^{j}\delta^{i}+\delta\sum_{j=0}^{z-2}\sum_{i=0}^{j}\delta^{i}=\sum_{i=0}^{0}\delta^{i} \tag{5.88}$$

then for $z+1$ it can be obtained:

$$\begin{aligned}\boldsymbol{K}(n+1)&=\sum_{j=0}^{z+1}\sum_{i=0}^{j}\delta^{i}+(-1-\delta)\sum_{j=0}^{z}\sum_{i=0}^{j}\delta^{i}+\delta\sum_{j=0}^{z-1}\sum_{i=0}^{j}\delta^{i}\\ &=\sum_{j=0}^{z}\sum_{i=0}^{j}\delta^{i}+(-1-\delta)\sum_{j=0}^{z-1}\sum_{i=0}^{j}\delta^{i}+\delta\sum_{j=0}^{z-2}\sum_{i=0}^{j}\delta^{i}+\sum_{i=0}^{z+1}\delta^{i}+(-1-\delta)\sum_{i=0}^{z}\delta^{i}+\delta\sum_{i=0}^{z-1}\delta^{i}\end{aligned} \tag{5.89}$$

From the assumption in (5.88) and continuing (5.89) the following is obtained

$$\begin{aligned}\boldsymbol{K}(n+1)&=\sum_{i=0}^{0}\delta^{i}+\sum_{i=0}^{z+1}\delta^{i}+(-1-\delta)\sum_{i=0}^{z}\delta^{i}+\delta\sum_{i=0}^{z-1}\delta^{i}\\ &=\sum_{i=0}^{0}\delta^{i}+\sum_{i=0}^{z+1}\delta^{i}-\sum_{i=0}^{z}\delta^{i}-\sum_{i=1}^{z+1}\delta^{i}+\sum_{i=1}^{z}\delta^{i}\\ &=\sum_{i=0}^{0}\delta^{i}+\delta^{z+1}-\delta^{z+1}\\ &=\sum_{i=0}^{0}\delta^{i}\end{aligned} \tag{5.90}$$

which ends the demonstration of the proposition.

Remark 12

From (3.63) $\boldsymbol{K}=\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{K}_y\end{bmatrix}$, where $\boldsymbol{K}_y$ is the last element of the vector $\boldsymbol{K}$, it can be noticed that as the dimension of $\boldsymbol{K}$ is equal to $n+1$, $\boldsymbol{K}_y=\boldsymbol{K}(n+1)$. Then recalling *Proposition 12*

$$\boldsymbol{K}(n+1)=\sum_{i=0}^{n-(n+1)+1}\delta^{i}=\delta^{0}=1 \tag{5.91}$$

Therefore

$$\forall_{n\in\mathbb{N}} \quad \boldsymbol{K_y} = 1 \tag{5.92}$$

Remark 13:

From *Proposition 12* it can be noticed, that from now onwards the next order quantity can be immediately calculated by substitution of the simplified description of the vector $\boldsymbol{K}$, the reference inventory level and the current inventory levels into the equation shown in (3.60) such that:

$$\Delta u(k) = \boldsymbol{K}_y y_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT)$$

Since $\Delta u(kT)$ is an incremental quantity defined in (3.40) such that

$$\Delta u(kT) = u(kT) - u\left[(k-1)T\right]$$

Therefore $\quad u(kT) = \boldsymbol{K}_y y_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT) + u\left[(k-1)T\right] \quad$ (5.93)

Since the system output $y(kT)$ for the inventory model refers to the current inventory level it can now be denoted as $I(kT)$, such that

$$I(kT) = y(kT) \tag{5.94}$$

Since the reference signal $y_R(kT)$ refers to the reference inventory level it can be denoted as $I_R(kT)$, from (5.93) it can be deduced that

$$u(kT) = \boldsymbol{K_y} I_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT) + u\left[(k-1)T\right] \tag{5.95}$$

From equation (5.92) it can be eventually observed that

$$u(kT) = I_R(kT) - \boldsymbol{K}\boldsymbol{x_c}(kT) + u\left[(k-1)T\right] \tag{5.96}$$

which is the optimal order quantity (inventory system input) shown in the *IC Proposition* for the perishable case.

5.3 Summary

The demonstration of the mathematical equivalency between the IC and IMPC methods has been systematically shown. To illustrate the development of the IC method, the mathematical demonstration has been presented initially for the non-perishable case where $\delta(kT)=1$ and then the perishable case, where $\delta(kT)\in(0\ ,\ 1]$, separately. In a logical sequence the simplification procedure of the initial MPC method for the inventory model has been deduced in a step by step manner, firstly for non-perishable and then for perishable conditions. Although, eventually, the non-perishable case was elaborated to be a special case of the wider perishable case, such an organisation of the material of the chapter has enabled the reader to follow how the proposed IC method has been developed. Therefore, all propositions established for demonstration of mathematical equivalency of the non-perishable case appeared to be the special cases of the propositions established for the perishable case. Eventually the final proposition has demonstrated the equivalency of the gain vector $\boldsymbol{K}$ from the *IC Proposition* with gain vector $\boldsymbol{K}$ of IMPC. The procedure of obtaining the optimal order size for the current time instance kT was elaborated. The proposed IC method can be used equivalently to IMPC, and can therefore significantly decrease the computational cost and complexity and enable the wider accessibility of the OR community. The findings presented in this chapter constitute the main originality and novel work of the thesis.

6 RESULTS OF PROPOSED METHOD

6.1 Introduction

In this chapter, simulation results of the proposed IC method are presented to justify the efficiency of the proposed method for the inventory application. In Section 4.4 and Section 4.5 the low-bound saturation of order levels were not considered (the orders were allowed to be lower than zero, returns). The order quantities in the current chapter are prevented from dropping below zero for the purpose of obtaining realistic simulation results of given inventory control.

Defining the profit function to be of the form:

$$F_{prof} = P\left(d_{Tot} - I_L\right) + \left(P - \beta\right) I_L - H_c I_u - P_c u_{Tot} \tag{6.1}$$

where:

d_{Tot} denotes the total number of goods sold in N periods

I_u denotes the total number of goods stored in the warehouse in N periods (the inventory level greater or equal than zero)

I_L denotes the total number of backorders in N periods (the inventory level lower than zero)

u_{Tot} denotes the total number of goods ordered in N periods

P denotes the selling price per unit

$P - \beta$ denotes the discounted selling price per unit, where β is a discount rate, applicable when the demand is not satisfied immediately and the customer has to wait for a delivery (backorders)

H_c denotes the holding cost per kept unit in an unit time

P_c denotes the purchasing cost per unit.

Using the profit function (6.1), it can be analysed to show how the simulation settings affect the profit. Based on the observations, the suggestions for improvement can be made.

The current chapter presents the simulation settings first, where, among others, the time varying and inventory level dependent deterioration rate are shown. The deterioration is assumed to proportionally increase when the inventory level increases. Indeed, this is true for some real life applications, where the higher the inventory level, the greater chances the stock will deteriorate.

The first simulation test was run to show that the reference inventory level affects the system response characteristics. Then the defined profit function in equation (6.1) is used to test the influence of the reference inventory levels to profitability (for some chosen numerical values of costs and prices of the profit function). The following test is run for the most profitable inventory level accordingly to the test results. It shows the system behaviour with respect to different demand patterns to see if the IC control performs according to expectations. A more detailed discussion of the system behaviour is presented in the following test for inventory levels, order sizes and deterioration rate separately.

6.2 Simulation Settings

In the current numerical example, the simulation parameters were set as follows:
The lead time delay was set to $n = 5$. The inventory was assumed to be no greater than 1000 items at any time (to assure the deterioration rate is never negative), the deterioration rate was assumed to be dependent on current inventory level in the following manner

$$\delta\left[I\left(kT\right)\right]=\begin{cases}1-0.001\cdot I(kT) & if \quad I(kT)\geq 0\\ 1 & otherwise\end{cases} \tag{6.2}$$

where the current value is dynamically substituted to obtain the control vector of *IC Proposition* of (3.66) such that

$$\boldsymbol{K}=\left[\sum_{i=1}^{5}\delta\left(kT\right)^{i} \quad \sum_{i=0}^{4}\delta\left(kT\right)^{i} \quad \sum_{i=0}^{3}\delta\left(kT\right)^{i} \quad \sum_{i=0}^{2}\delta\left(kT\right)^{i} \quad \sum_{i=0}^{1}\delta\left(kT\right)^{i} \quad 1\right] \tag{6.3}$$

The unit costs and prices of profit function (6.1) were set as follows: $P = 2$, $\beta = 0.2$, $H_c = 0.3$, $P_c = 0.5$ and the simulation was run for 200 periods. The different demand patterns used in the following tests have the same, or approximately the same (in the case of randomness) mean values of *50* items (if the first *5* null values time instances are not considered, while the mean calculation) or *48.75* in general (considering the null values into mean calculation), as the demand in each case is set to zero for the first $n = 5$ periods. Since the mean value of demand is always the same regardless of demand pattern, once the profit function in (6.1) is used, the assessment of the profit can be considered. This way, the differences in profits obtained and results in general depends only on different demand trends rather than total number of products sold within the simulation period (as it is exactly the same for all demand patterns shown). It accounts for the fact that the warehouse does not sell any items until the first order reaches the warehouse.

The five different demand patterns are presented in Figure 6-1 (all of them on the same scale) and in Figure 6-2 separately to increase the visibility, respectively, as follows: constant demand, seasonal demand pattern with two possible values only, seasonal pattern with more values allowed, random demand oscillating around the mean value and finally the seasonal pattern with some randomness allowed.

The demand, apart from the initial *5* time instances, is constant and equal to *50* items. Later in the chapter it is referred to as Demand 1. Demand 1 is used in the chapter for simulation of the system to show that the response of the inventory level is smooth, stable and the costs related to inventory overshoots are very limited.

The seasonal demand of two allowed values is equal to *100* items (for *6-45* and *100-154* time instances) and *0* (for the remaining time instances). It is later referred to as Demand 2. The chosen values of demand differ extremely (comparing to other tested demand patterns) therefore they enable the reader to see that the more sudden the changes in demand, the more cost generated to the warehouse in terms of inventory overshoots (storage costs) as well as order overshoots (purchasing costs).

The seasonal demand pattern with more values allowed is equal to *50* (for *6-60* and *151-190* time instances), and *100* items (for *61-110*). It is later referred to as Demand 3. The chosen values are less extreme than in the case of Demand 2. Therefore, they enable to see the difference in profit increase in comparison to Demand 2. It enables to notice that the sudden increase of inventory is smoother than in case of the Demand 2. Therefore, the less the storage costs are generated. Also, comparing with Demand 2, the purchase costs are smaller, due to the less sudden increase of order quantities, when they occur.

The random demand pattern is set to zero for the first *5* time instances and then oscillates around a mean value of *50* with a variance of *5* items. It is later referred to as Demand 4. Demand 4 has a relatively smooth pattern as the randomness allowed is relatively small. Therefore, it enables to see that the profit is relatively high, as in case of constant demand. The inventory levels and orders are relatively smooth, which does reduce storage and purchasing costs. The seasonal pattern with some randomness allowed contains the seasonal pattern of *50* (for *6-60* and *151-190* time instances), *100* items (for *61-110*) plus randomness of *0* as a mean value and variance of *5* items. It is later referred to as a Demand 5. It tends to highlight the difference in results obtained with comparison to Demand 3 – an identical pattern, but the randomness or lack of randomness constitutes the difference.

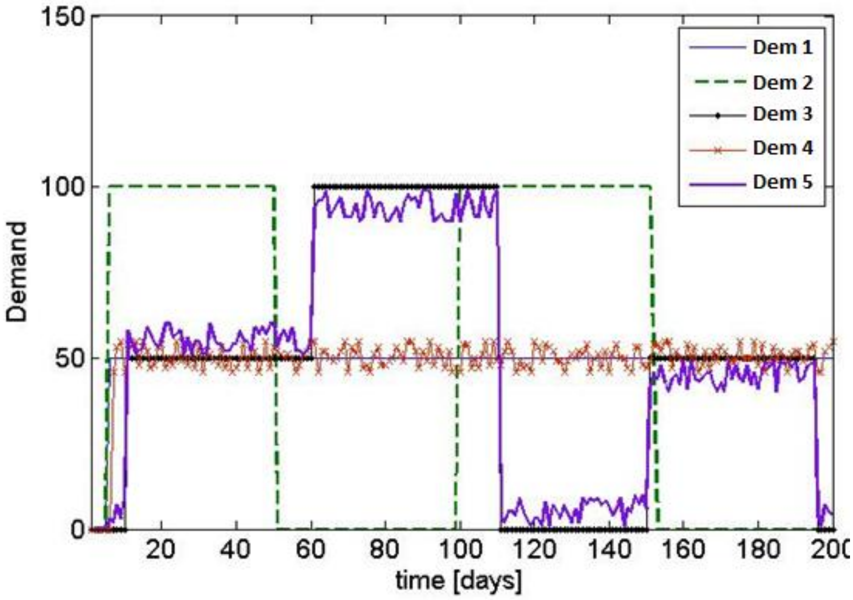


Figure 6-1: 5 Demand patterns shown on one scale

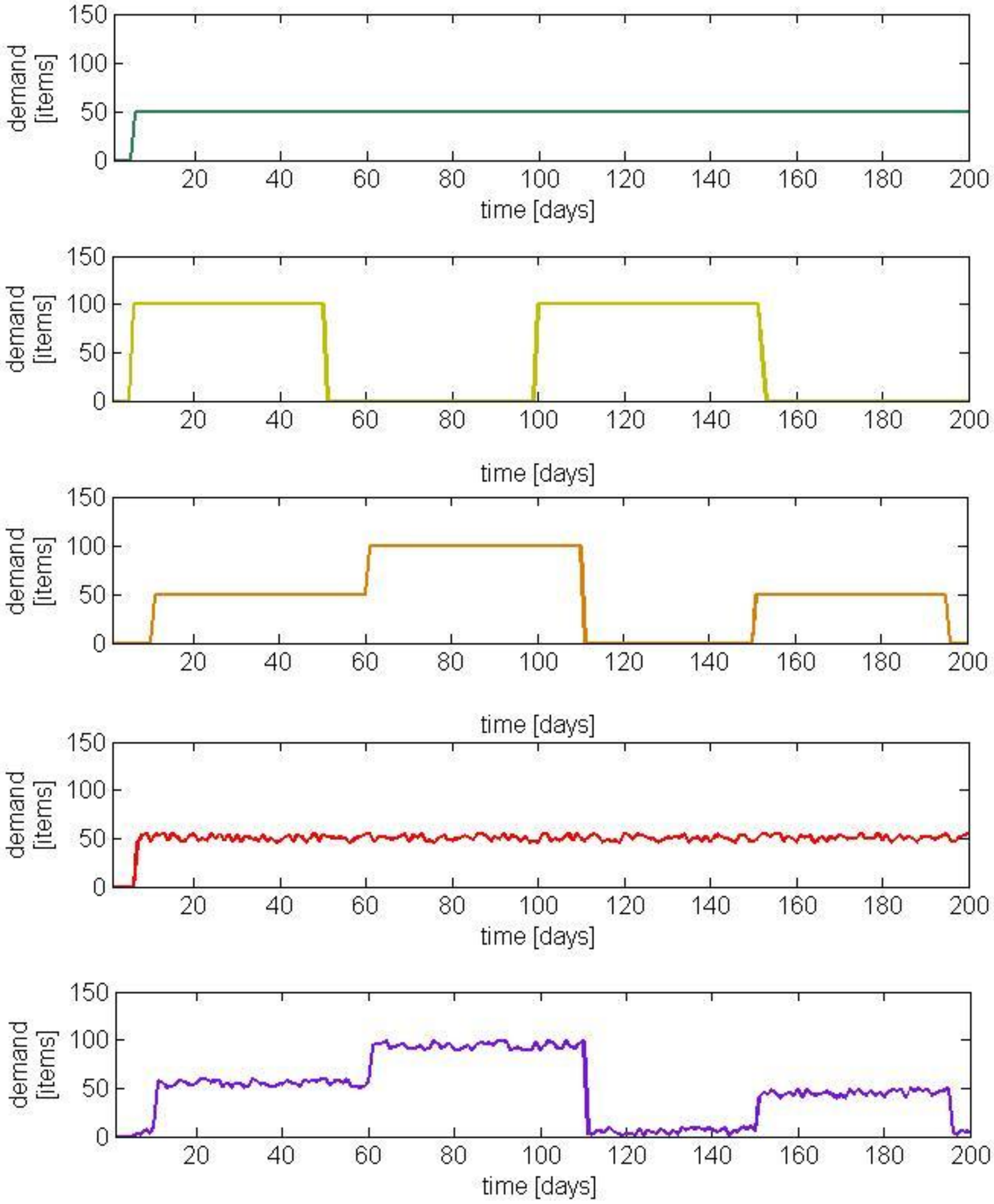


Figure 6-2: 5 Demand patterns shown on separate scales

6.3 Test 1: To illustrate how the reference inventory affects the system behaviour

From the profit function in (6.1) it can be noticed that the positive values of the inventory level (the actual number of stored goods) generate the storage costs which, perceptibly, decreases the profit. Therefore, it can be assumed that the lower the reference inventory level, the greater the profit for the organisation. On the other hand, the negative values of inventory levels (the backorders), are more likely to compensate for the low value of reference inventory level, which generates additional cost for the organisation. There are three different reference inventory levels tested: $y_r = 0$, $y_r = 100$ and $y_r = 200$ for the last (the fifth) demand pattern of Figure 6-2. Figure 6-3 shows the system response for each of them on the same graph. Figure 6-4 and Figure 6-6 illustrate the inventory system response for each of $y_r = 0$, $y_r = 100$ and $y_r = 200$ separately to increase the visibility. First the discussion of each reference inventory level is conducted separately based on analysis of Figure 6-4 and Figure 6-6, later all of the results are compared with each other with reference to Figure 6-3.

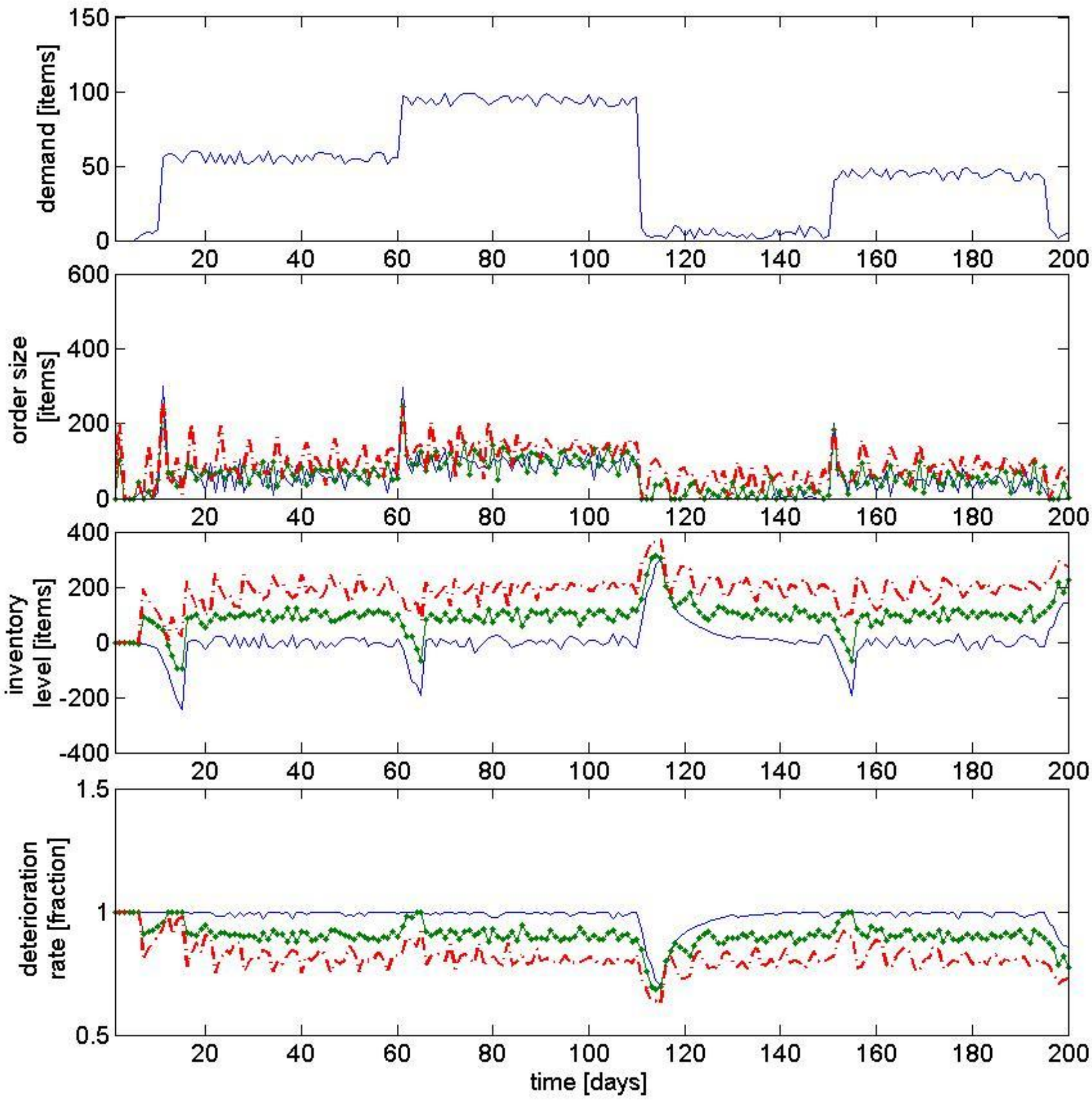


Figure 6-3: System response for all three reference inventory levels

Figure 6-4 shows the system response to the reference inventory level set to zero (empty warehouse). As a result of demand a sudden increase at *10*th time instance is noticed where the order size suddenly overshoots to up to *300* items (due to increased sale, more items must be replenished) around the *10*th day, to compensate for the backorders of *220* items (negative inventory) obtained in the inventory levels (sold/demanded items) at the same time instance. Then the orders converge quickly (within 5 days) to approximately *50* items, which correspond to the demand value (*50* items too) and which drives the inventory levels back to near reference value (zero) within lead time delay. For the times instances between approximately *15* to *60* days, the order sizes oscillate slightly around value of *50* items. They do so in noticeable correlation to demand behaviour at the respective time instances. When the demand increases slightly, the orders increase respectively to compensate for the inventory level reduction (due to goods sale), and when the demand slightly decreases, the orders decrease respectively as less goods were sold and the current inventory increases slightly.

When the demand suddenly increases to *100* items at the *60*th time instance, the orders overshoot again to the level of *300* items, to converge quickly (within *5* days – lead time delay) to a level of approximately *100* items (which corresponds to the demand value of *100* items too). Again it happens due to the inventory levels compensation, which, as it can be seen in Figure 6-4 drops to *200* items of backorders when the demand suddenly increases. Further, in correlation to demand, at time instances between *65* and *110*, the orders slightly oscillate near the value of *100* items (directly proportionally to demand) and inventories slightly oscillate near the reference value of zero items (inverse proportionally to demand).

Then, as the demand suddenly decreases to just above zero, the orders suddenly decrease to just above zero too, to reduce the surplus inventory after the real time delay. If returns were allowed, the orders would have compensated for the surplus inventory after the real time period of *5* days completely by dropping significantly below zero. At the same time instance, the inventory suddenly rises as fewer items were sold and due to the system delay the previous order quantities were still delivered to the warehouse. As returns are not allowed in the model, the orders could not go below zero to compensate for the surplus inventory and thus the inventory returns to the desired level of about zero at a slower rate than before.

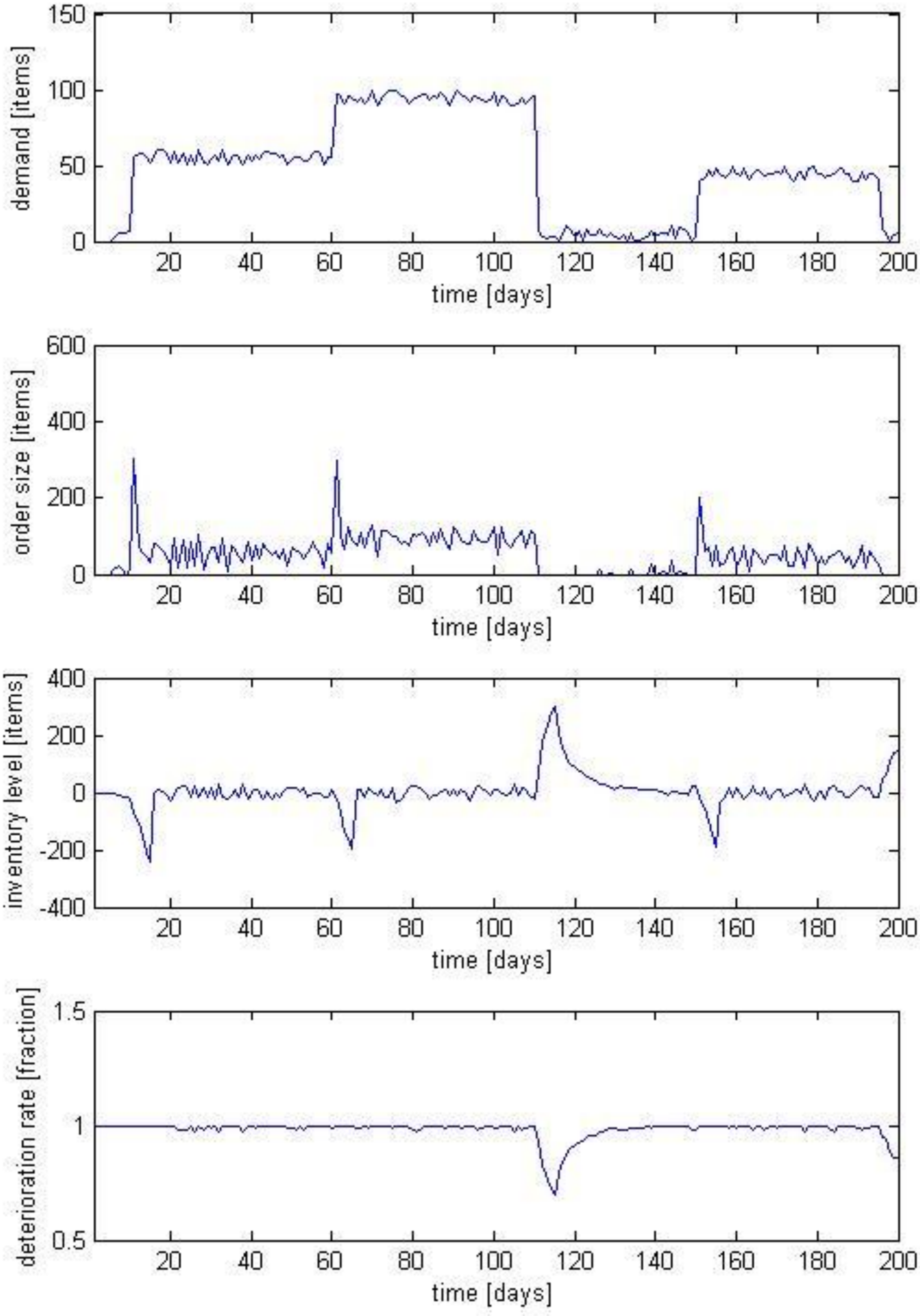


Figure 6-4: System response to reference inventory of value zero items

For the next season of 40 days for *110-150* time instances (the orders stay at or slightly above zero) none or very little items are ordered to the warehouse. The inventory levels take about half of that period to come back to around the reference zero inventory level. Then, on the *150*th day, the orders overshoot to a value of *200* items due to a sudden increase in demand to around *50* items. In this way they drive the inventory level (within time delay) back to around reference level faster, as they dropped to *200* items of backorders due to the sudden demand change. Then the orders converge within *5* days to the level near to *50* items, which corresponds to the level of demand (*50* items too). Subsequently, the orders keep oscillating near the value of *50* items. The inventory is maintained around the reference level after the lead time delay. The demand changes once more for the last *5* time instances by dropping to just above zero. This causes the reduction of orders to zero and increases the inventory to around 180 items.

In general it can be noticed that the system responds in an expected manner: the order quantity controls the inventory level through quick responses to demand changes and drives them back to the reference value usually with respect to a system delay.

Figure 6-4 also presents the deterioration rate to confirm that it behaves according to expectations. Indeed it never exceeds one (as designed) and drops below one inversely proportionally to the inventory level. The significant drop below one (to about 0.7) occurs in the *110-120* time instances, when the inventory overshoots to *200* items.

Figure 6-5 shows the system response to the reference inventory level set to 100 items. With respect to changes in demand, the order size suddenly overshoot to approximately *250* items at approximately the *10*th day (due to the increased sale, more items must be replenished) in order to compensate for the backorders of about *150* items obtained in inventory levels (sold/demanded items) at the same time instance. It can be noticed that the overshoot is less than in the case when $y_r = 0$. The orders then converge quickly (within *5* days) to the level around *50* items, which correspond to the demand value (*50* items too) and which drives the inventory levels back to near the reference value (*100* items) within the lead time delay. For the time instances between approximately *15* to *60* days the order sizes oscillate around value of *50* items. The oscillations are slightly greater than in the case of $y_r = 0$. Orders oscillate in noticeable correlation to demand behaviour at the respective time instances. When the demand increases slightly, the orders increase respectively to compensate for the inventory level reduction (due to goods sale), and when the demand slightly decreases, the orders decrease respectively as less goods are sold and the current inventory increases slightly.

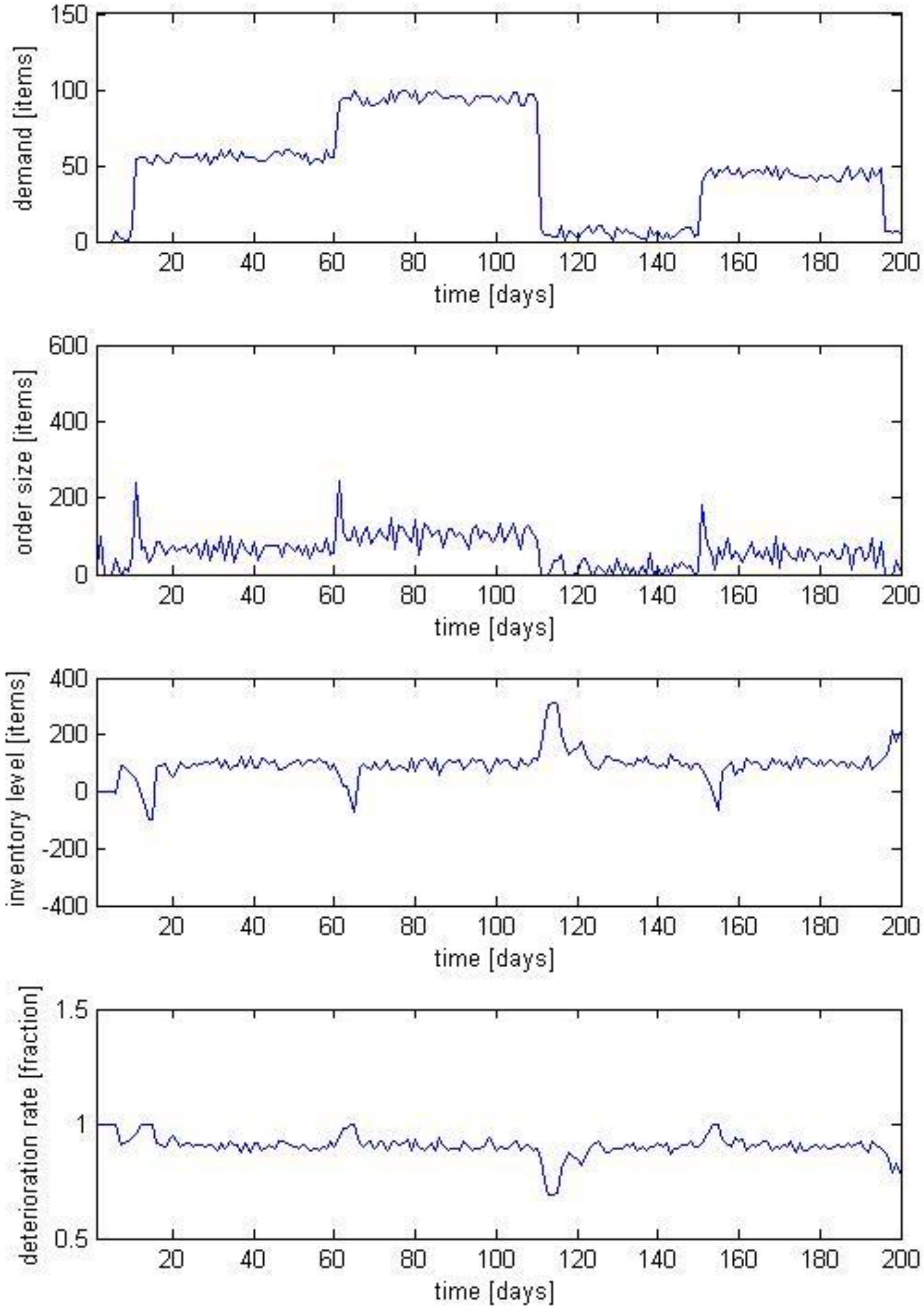


Figure 6-5: System response to reference inventory of value 100 items

When the demand suddenly increases to *100* items at the *60*th time instance, the orders overshoot again to the level of *220* items (the order overshoot here is smaller than in the case of $y_r = 0$), to converge quickly (within *5* days – lead time delay) to a level of approximately *100* items (which corresponds to the demand value of *100* items too). Again it happens due to the compensation in the inventory levels, which, as it can be seen in Figure 6-5 drops to *200* items of backorders when the demand suddenly increases. Further, in correlation to the demand at time instances between *65* and *110*, the orders oscillate near the value of *100* items (directly proportionally to demand) and inventories slightly oscillate near the reference value of zero items (inverse proportionally to demand). The oscillations of both, orders and inventories are greater than in the case of $y_r = 0$.

As the demand suddenly decreases to just above zero, the orders suddenly decrease to just above zero too, to reduce the surplus inventory after the real time delay. If the returns were allowed, the orders would have compensated for the surplus inventory after the real time period of *5* days completely by dropping significantly below zero. At the same time instance the inventory would suddenly rise up to around *250* items as less items are sold and due to the system delay the previous order quantities are still delivered to the warehouse. As returns are not allowed in the model, the orders could not go below zero to compensate for the surplus inventory, and thus the inventory returns to the desired level of about *100* items slower than before. However the period of inventory level recovery is smaller than in the case of $y_r = 0$.

For the next season of 40 days for *110-150* time instances (the orders oscillate just above zero) none or very little items are ordered to the warehouse). The oscillations are significant here and are greater than in case of $y_r = 0$. Then, on the *150*th day, the orders overshoot to a value of *200* items due to a sudden increase in demand to around *50* items. This way they drive the inventory level faster (within the time delay) back to around reference level, as they dropped to *120* items of backorders due to a sudden demand change. The orders then converge within *5* days to the level near to *50* items, which correspond to the level of demand (*50* items too). Subsequently, the orders keep oscillating near the value of *50* items and the oscillations are significant. The inventory is maintained around the reference level after the lead time delay with some oscillations which are greater than in the case of $y_r = 0$. The demand changes once more for the last *5* time instances by dropping to just above zero. This causes the reduction of orders to just above zero and increases the inventory to around 180 items.

In general it can be noticed that the system responds in an expected manner where the order quantity controls the inventory level through quick responses to demand changes and drives them back to the reference value usually with respect to system delay. The oscillations are always more significant here than in the case of $y_r = 0$, which can be reflected in the deterioration rates.

Figure 6-4 also presents the deterioration rate to confirm that it behaves according to expectations. Indeed it never exceeds one (as designed) and if it does drop below one it is inversely proportionally to the inventory level. The significant drop below one (to about 0.7) happens in the *110-120* time instances, when the inventory overshoots to *200* items. The clearly visible oscillations just below one or near 0.9 are visible all over the simulation period, which, in comparison to $y_r = 0$, shows even more clearly that the inventory levels oscillate here more.

Figure 6-6 shows the system response to the reference inventory level set to 200 items. The relation between the behaviour of order sizes and the demand pattern is not clearly noticeable for the current case. The order sizes fluctuate significantly comparing to the previous cases. The inventory level oscillations reach almost 400 items, but the back orders never occur. The deterioration rate oscillates between one and 0.75 which reflects the significant oscillations in inventory levels.

As a conclusion it can be noticed that different reference inventory levels generate different responses of the system. Combining all the results together in Figure 6-3 it can be noticed that the inventory level goes below zero the least (actually zero times) for $y_r = 200$ and the most often for $y_r = 0$. Therefore backorders do not appear at all for $y_r = 200$, while they occur relatively often for the case where $y_r = 0$. Nevertheless, the inventory level becomes less stable for $y_r = 200$, while it is the most stable for $y_r = 0$ (which in terms of the thesis objective is an advantage). Also, respectively, the stock is the greatest for the case of $y_r = 200$ and the lowest for the case of $y_r = 0$. The aim of this thesis is to keep the inventory at a stable level therefore in this case the $y_r = 0$ reference level can be recommended for the industrial purposes. In any case the profit function described in (6.1) enables numerical analysis and assessment of the most cost efficient reference inventory level for a given unit cost for a given set of numerical values.

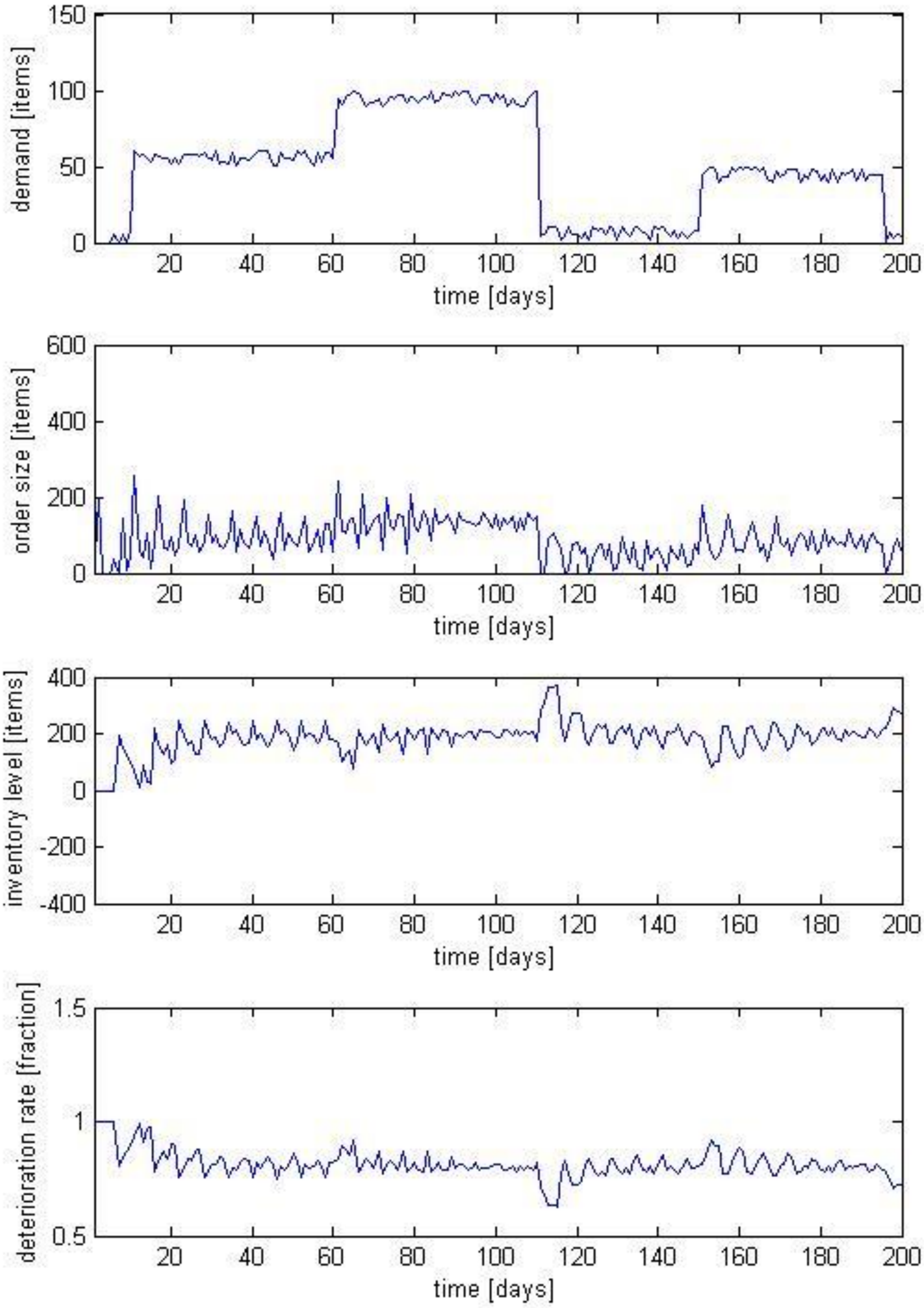


Figure 6-6: System response to reference inventory of value 200 items

6.4 Test 2: To find the most profitable reference inventory level

Table 6-1 represents the profit values obtained for a given numerical example. The profits were calculated separately for each of the five given demand patterns shown in Figure 6-2 and for eleven different reference inventory level values as given in Table 6-1. For each reference inventory level, the mean profit value of the five respective demand patterns is given in Table 6-1 in order to simplify the analysis.

reference inventory level y_r	Profit values					mean
	Demand 1	Demand 2	Demand 3	Demand 4	Demand 5	
100	7766.17	6483.50	7336.78	7856.11	7475.33	7383.58
90	8505.67	7156.72	7989.80	8720.04	8217.24	8117.89
80	9227.42	7806.11	8618.55	9361.03	8718.64	8746.35
70	9931.29	8434.09	9228.76	10218.50	9496.86	9461.90
60	10617.18	9041.40	9817.43	10850.02	10209.68	10107.14
50	11284.98	9626.18	10385.70	11412.98	10776.19	10697.21
40	11934.62	10164.93	10915.41	12048.36	11216.19	11255.90
30	12566.01	10639.83	11407.74	12742.25	11652.67	11801.70
20	13179.07	10999.25	11843.51	13245.13	12229.72	12299.34
10	13773.76	11287.35	12264.98	13540.19	12493.58	12671.97
0	14350.00	11567.60	12672.20	13829.35	12665.78	13016.98

Table 6-1: Profit values

It can be observed that for each reference inventory level, the profit differs with respect to the demand trend. It can be noticed that for most cases of the reference inventory level, the best profits are obtained for Demand 4, (which is the constant demand with some randomness allowed), and Demand 1, (which is the constant demand). For the very low reference inventory levels the demand 1 is slightly more profitable than Demand 4, for higher inventory levels the Demand 4 generates slightly better profit then Demand 1 (see Table 6-2). It is related to the fact that both of these demand patterns are smooth and do not cause significant inventory overshoots or backorders (which both would generate extra costs for the warehouse). It can be also noticed that Demand 3 and Demand 5 generate similar profits, smaller than Demand 4 and Demand 1, but greater than Demand 2. Their sudden changes of value are smaller than in the case of Demand 2, but greater than in case of demand 4 and Demand 1. Also, it can be noticed that regardless of the reference inventory level value, the lowest profits were obtained for Demand 2 (see Table 6-2), which is the demand pattern with two extremely different values. Based on the results obtained, it can be concluded that the more sudden and extreme changes in demand pattern, the more cost is generated due to overshoots in inventory levels (more goods stored) and order quantities (more goods bought). Table 6-2 presents the demand patterns in the order of profitability with respect to reference inventory levels.

reference inventory level y_r	**Profit 4**	**Profit 2**	**Profit 3**	**Profit 4**	**Profit 5**
100	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
90	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
80	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
70	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
60	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
50	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
40	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
30	Demand 4	Demand 1	Demand 5	Demand 3	Demand 2
20	Demand 1	Demand 1	Demand 5	Demand 3	Demand 2
10	Demand 1	Demand 4	Demand 5	Demand 3	Demand 2
0	Demand 1	Demand 4	Demand 5	Demand 3	Demand 2

Table 6-2: Profit vs. Demand

Figure 6-7 represents the monotonically decreasing function of a relationship of the mean profit to reference inventory level (the last column of Table 6-1). Therefore, it can be seen that for the considered numerical example, the backorder associated costs, do not affect the profit as much as the holding associated costs and $y_r = 0$ generates the highest profit for the organisation.

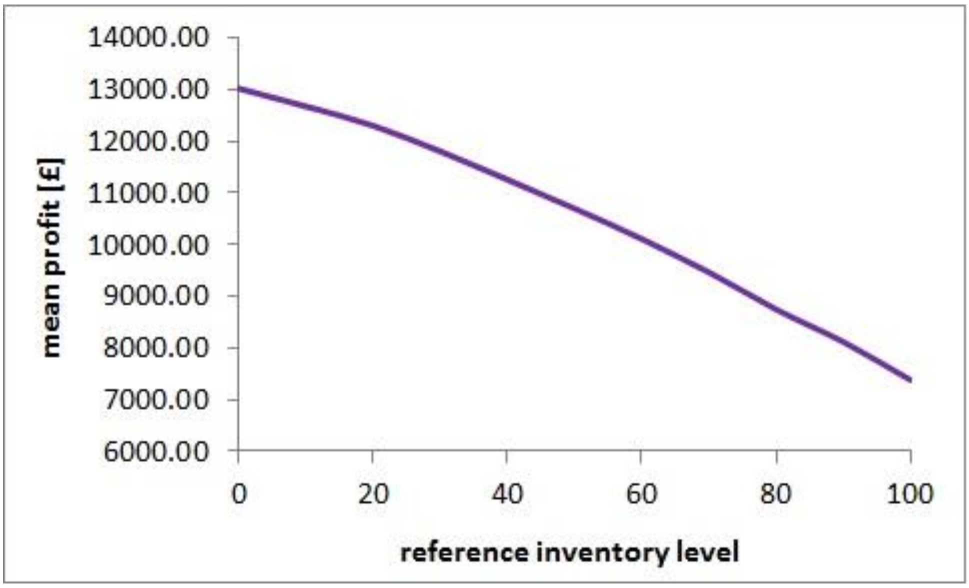


Figure 6-7: Mean profit in respect to reference inventory level

Therefore, in general, the profit function (6.1) can be used by the system designer to decide which reference inventory level is the most profitable for the organization for real unit costs and prices of a case scenario, or can help to decide the unit costs and price adjustment for an arbitrarily selected inventory reference level. In the considered numerical example, setting the reference value to zero is recommended. Therefore further experiments are conducted for that particular value only.

6.5 Test 3: To illustrate how demand pattern affects system behaviour

The simulation was run for the recommended inventory reference level of zero items. The current section presents results of optimal order quantities, inventory levels and deterioration rate values, respectively, to each demand pattern. Each of the mentioned results are presented twice, one on a common scale for all demand patterns and the second time on separate scales for each demand pattern.

Figure 6-8 represents the optimal order quantities for all considered demand patterns on one scale, while Figure 6-9 represents the same results on separate scales for each demand to increase the visibility of the results.

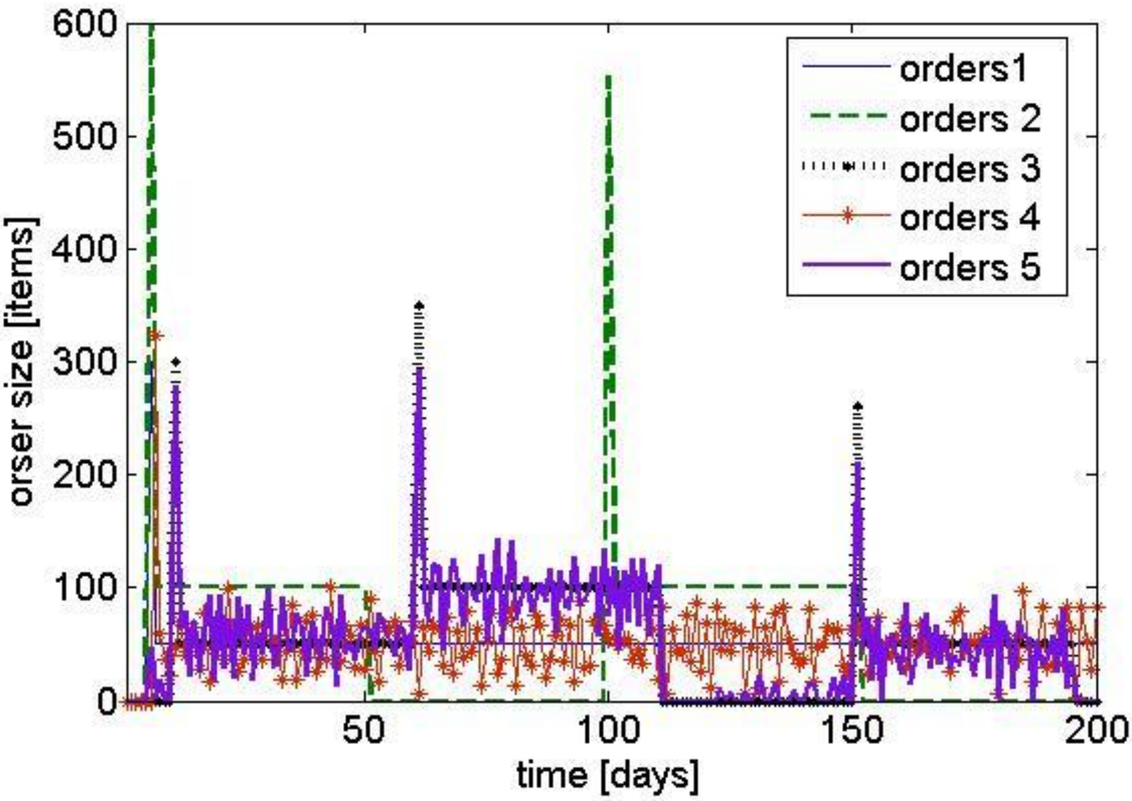


Figure 6-8: Order quantities for 5 demands on one scale

Observing Figure 6-8 and Figure 6-9 it can be noticed that order quantities are always above zero (accordingly to the model design) regardless of the demand pattern. It can be observed that the orders overshoot when the demand suddenly changes from one value to another and the more the sudden changes, the higher the overshoots. For instance, Demand 2 generates higher overshoots (at 5th and 100th time instances) in orders sizes (up to 600 items) than Demand 3 (up to 300 items, at 10th, 60th and 150th time instance). It happens because the difference in demand level at Demand 2 (see Figure 6-1or Figure 6-2) between time instances no 5 and no 100 is 100 more (*100* items) than at demand 3 for time instances no 10, 60 and 150 (*50* items only). It explains partially the reason of Demand 2 being less profitable than demand 3, as discussed in Section 6.4. In Figure 6-8 and Figure 6-9 it can also be observed that the overshoots of Demand 5 are slightly smaller (about 250 items) at time instances: 10, 60 and 150 than Demand 3 (about 300 items) for the same time instances. Although the seasonal factor of the demand pattern in both of them are identical, the random factor is the factor which differs the two demands from each other (see Figure 6-1or Figure 6-2). It can be concluded that the randomness slightly smoothens the overshoots. It would explain the observation that Demand 5 generates slightly better profit than Demand 3 in Section 6.4.

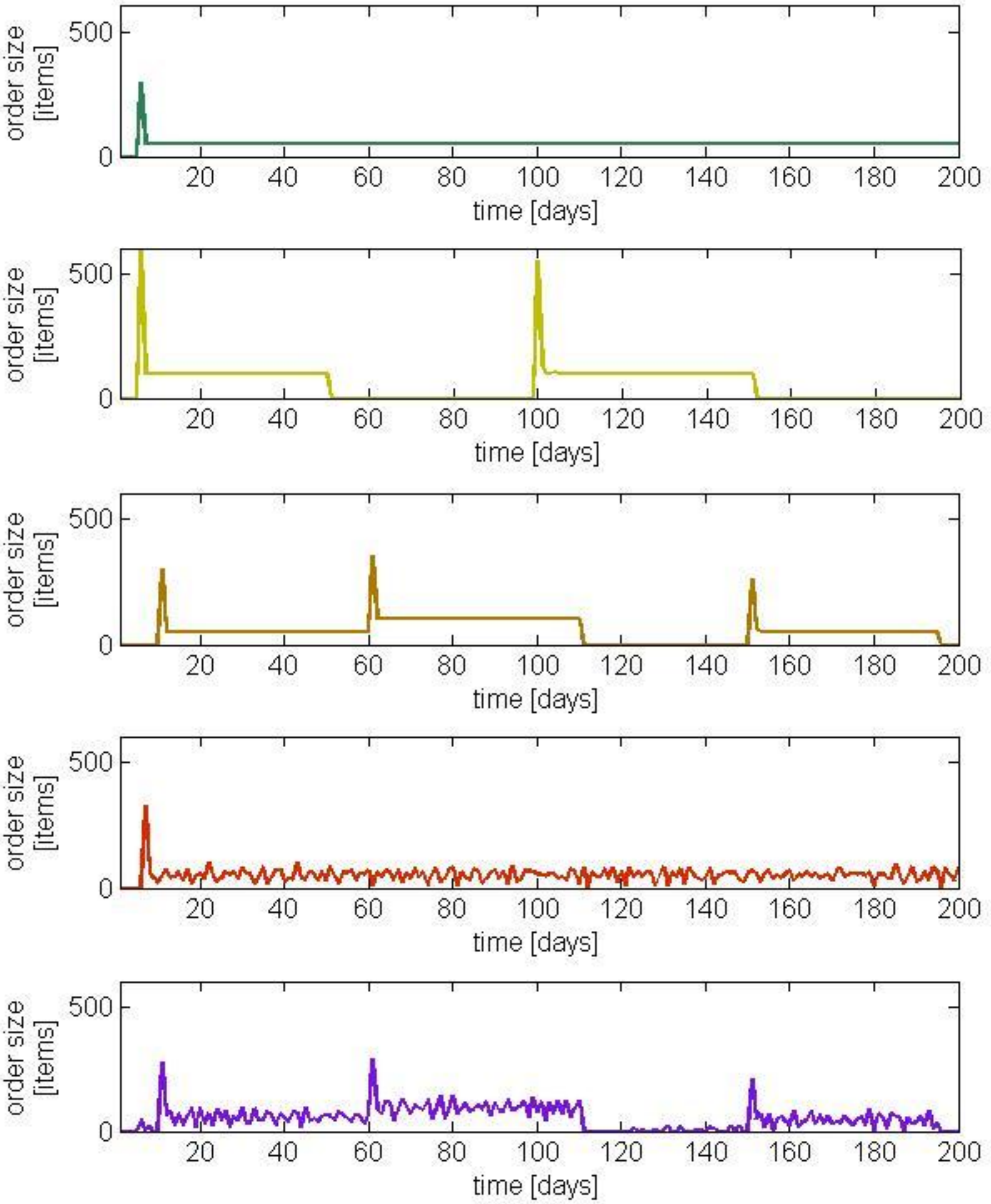


Figure 6-9: Orders for 5 different demand patterns on separate scales

Nevertheless, the opposite conclusion can be drawn in the case of comparing Demand 1 and Demand 4 in Figure 6-8 and Figure 6-9. The overshoot of the 5^{th} time instance is slightly greater for the demand Pattern 4, compared to Demand 1, which differs from one another by the randomness added only (see Figure 6-1 or Figure 6-2). It again can be related with observations in Section 6.4., where it can be seen that for low inventory levels as a reference signal, the Demand 4 is slightly less profitable than Demand 1. In general in that case, the orders are reasonably low for most of the time, regardless of the demand pattern (as they remain at or near the demand trend line for most of the time). They do overshoot at the times instances when the demand suddenly fluctuates, but they stabilise and recover within the system delay to (when there is no randomness in the demand pattern) or within +/-5 items near (in the case of the randomness considered) the demand value.

Figure 6-10 represents the inventory levels for all the considered demand patterns on one scale, while Figure 6-11 represents the same results on separate scales for each demand to increase the visibility of the results. It can be observed that the backorders as well as overshoots occur only in the case of sudden demand changes (see Figure 6-1or Figure 6-2). Also the backorders occur at the same time instances when the overshoots of orders occur and the overshoots occur when the orders go to zero level (see Figure 6-8 and Figure 6-9). If the demand suddenly increases, as in the case of, for instance, Demand 1 at the 5^{th} time instance, then the backorders occur for the same time instance in inventory levels. It is due to an insufficient number of goods stored in the warehouse with respect to the demanded number. At the same time instance the orders overshoot to (see Figure 6-8 and Figure 6-9) supply the warehouse the demanded items, which within the lead time delay drives the inventory back to the reference level of zero items. This behaviour is observed for all demand patterns.

On the other hand if the demand suddenly decreases, as in the case of, for instance, Demand 2 on the 50th day, then the overshoot of inventory levels occur for the same time instance in the respective inventory level. The overshoots stay in the system longer than the lead time delay only due to the fact that the order quantities (see Figure 6-8 and Figure 6-9) are not allowed to go below zero, therefore they can drive the inventory back to the zero level slower than in the case of backorders. Nevertheless this is observable for all demand patterns.

In Figure 6-10 and Figure 6-11 it can be also observed that overshoots and backorders are greater for larger demand changes. For instance, the backorders, related to Demand 5 at the 5th time instance, are larger (about 400 items) than in the case of Demand 3 at the same time instance (around 220 items). Both the observations correlate with the fact that Demand 2 generates the lowest profit for the company, shown in Section 6.4. In Figure 6-10 and Figure 6-11 it can be seen that the inventory levels are kept on (in the case of lack of demand randomness) or near +/-5 items (in the case of considered randomness of a demand pattern) at the reference level for most of the simulation time. In the case of sudden demand fluctuations, they stabilise and converge quickly to or near to the reference level.

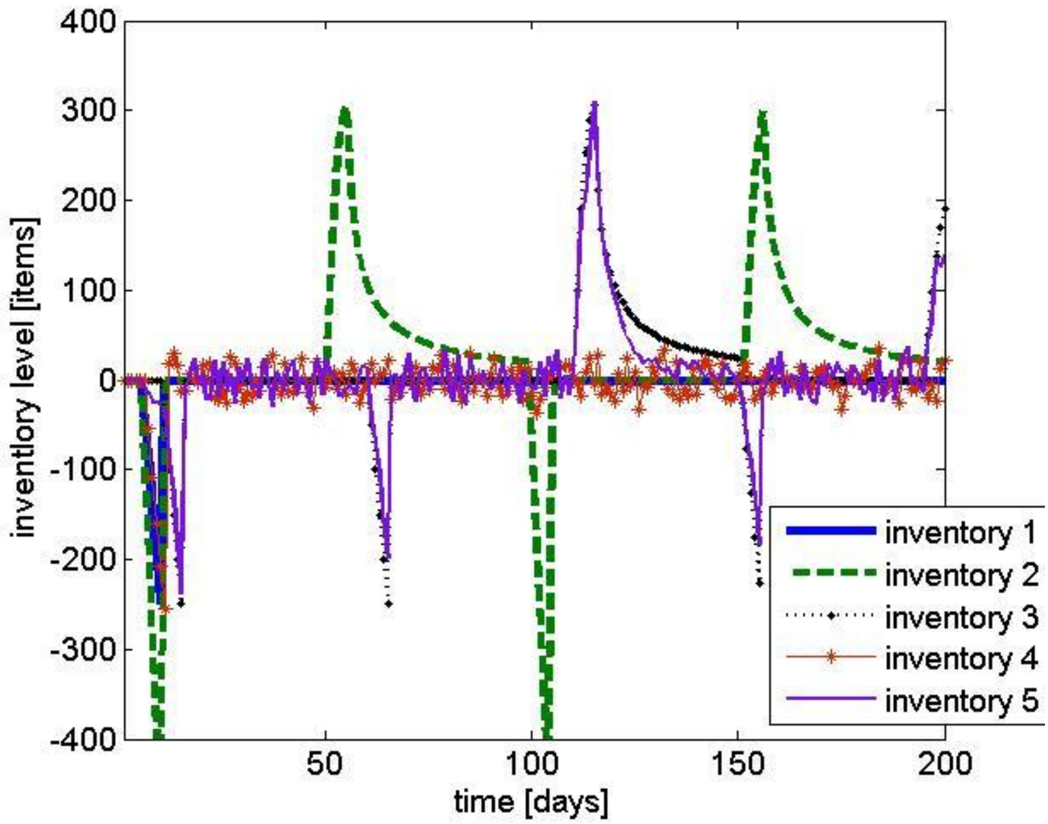


Figure 6-10: Inventory levels for 5 demand patterns on one scale

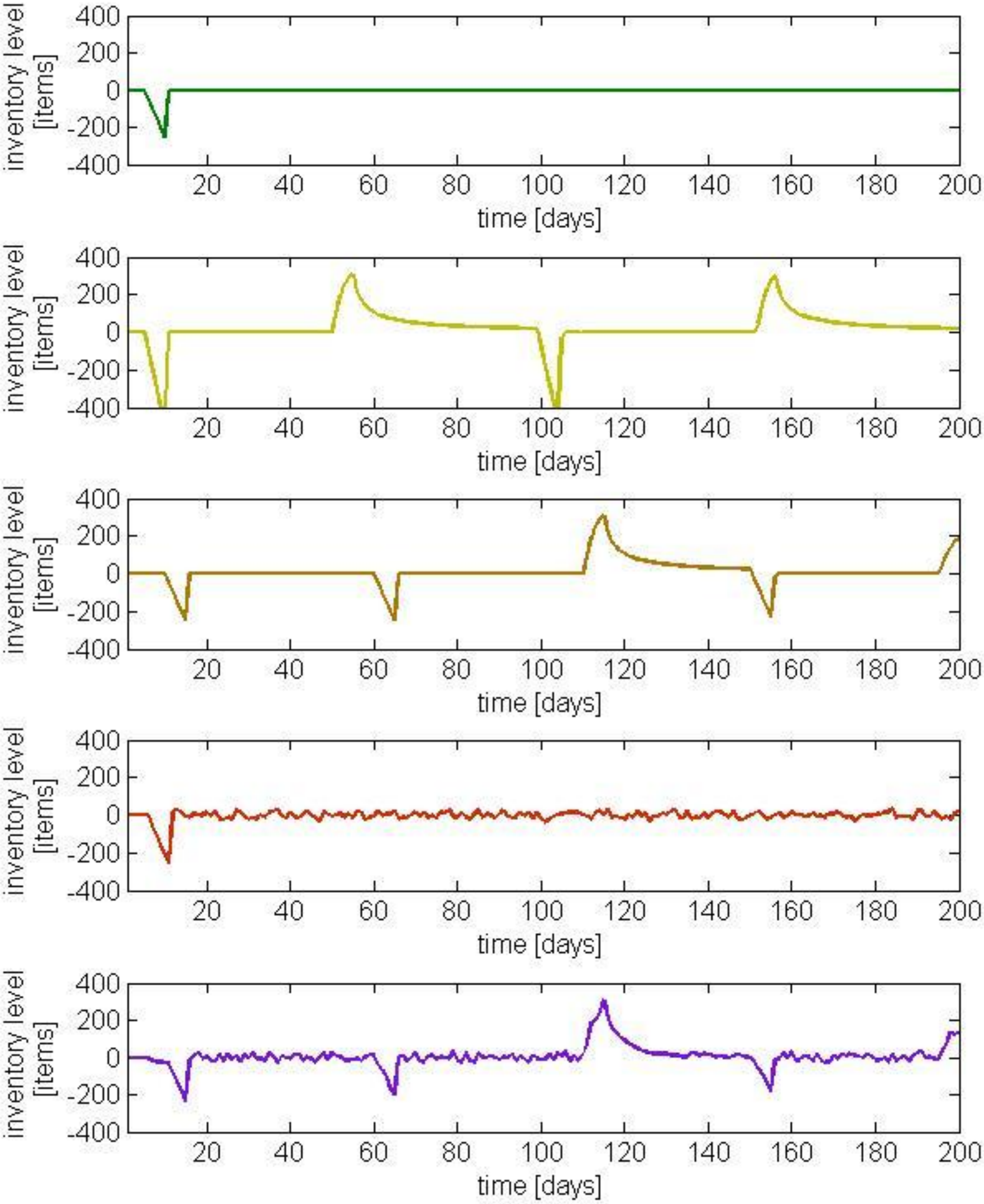


Figure 6-11: Inventory levels for 5 demand patterns on separate scales

Figure 6-12 represents the deterioration rates for all the considered demand patterns on one scale, while Figure 6-13 presents the same results on separate scales for each demand to increase the visibility of the results. The changes of the deterioration rate are representative (i.e. related) to the inventory level variations (see Figure 6-10 and Figure 6-11) over time regardless of demand pattern of Figure 6-1 or Figure 6-2 and are represented in Figure 6-12 and Figure 6-13. They are consistent with the time varying deterioration rate mathematical description in (6.2). The deterioration never goes above one or below zero. It stays at value of one when there is no inventory stored at the current time instance. It decreases proportionally with the increase of inventory. The detailed relation between demand fluctuations and inventory levels, order quantities and deterioration rate values are discussed in Section 6.6 for the most profitable reference inventory level of zero items (although this theoretical level may be impractical in reality).

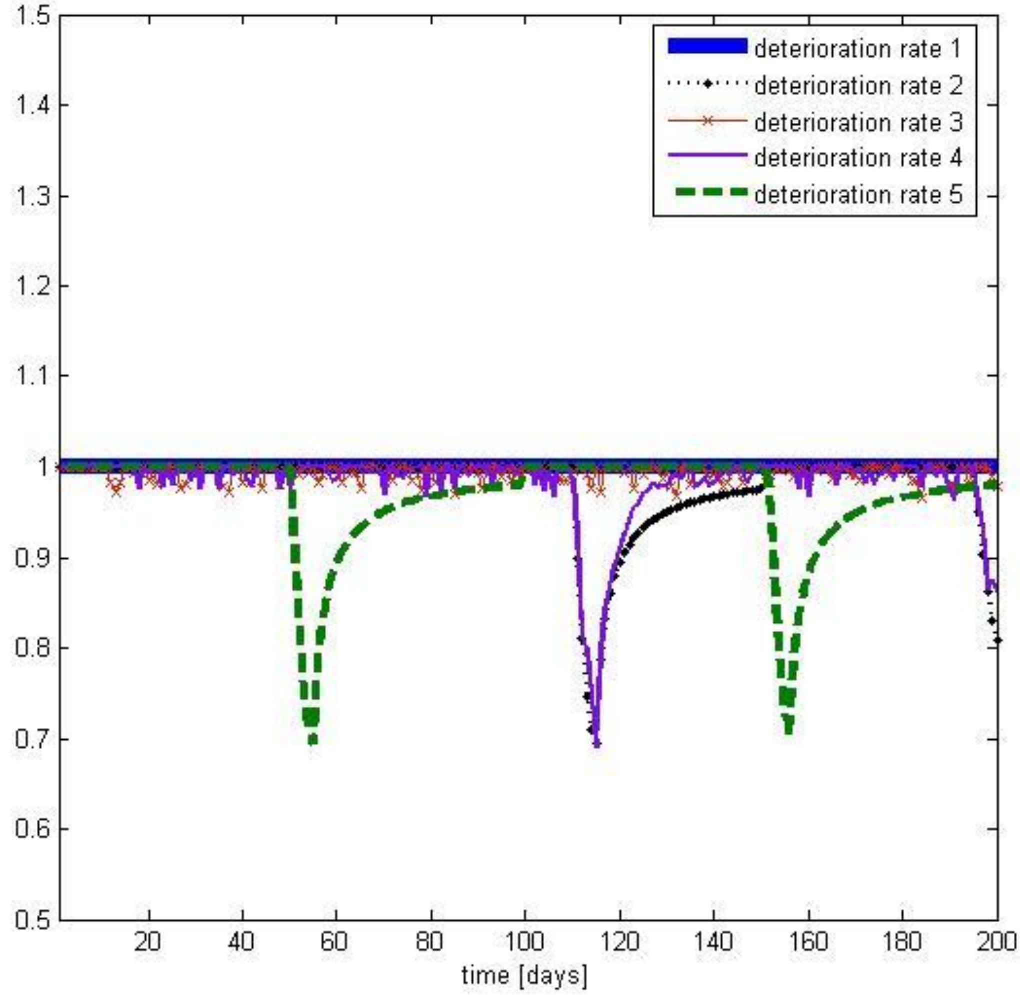


Figure 6-12: Deterioration rates for 5 demand patterns on one scale

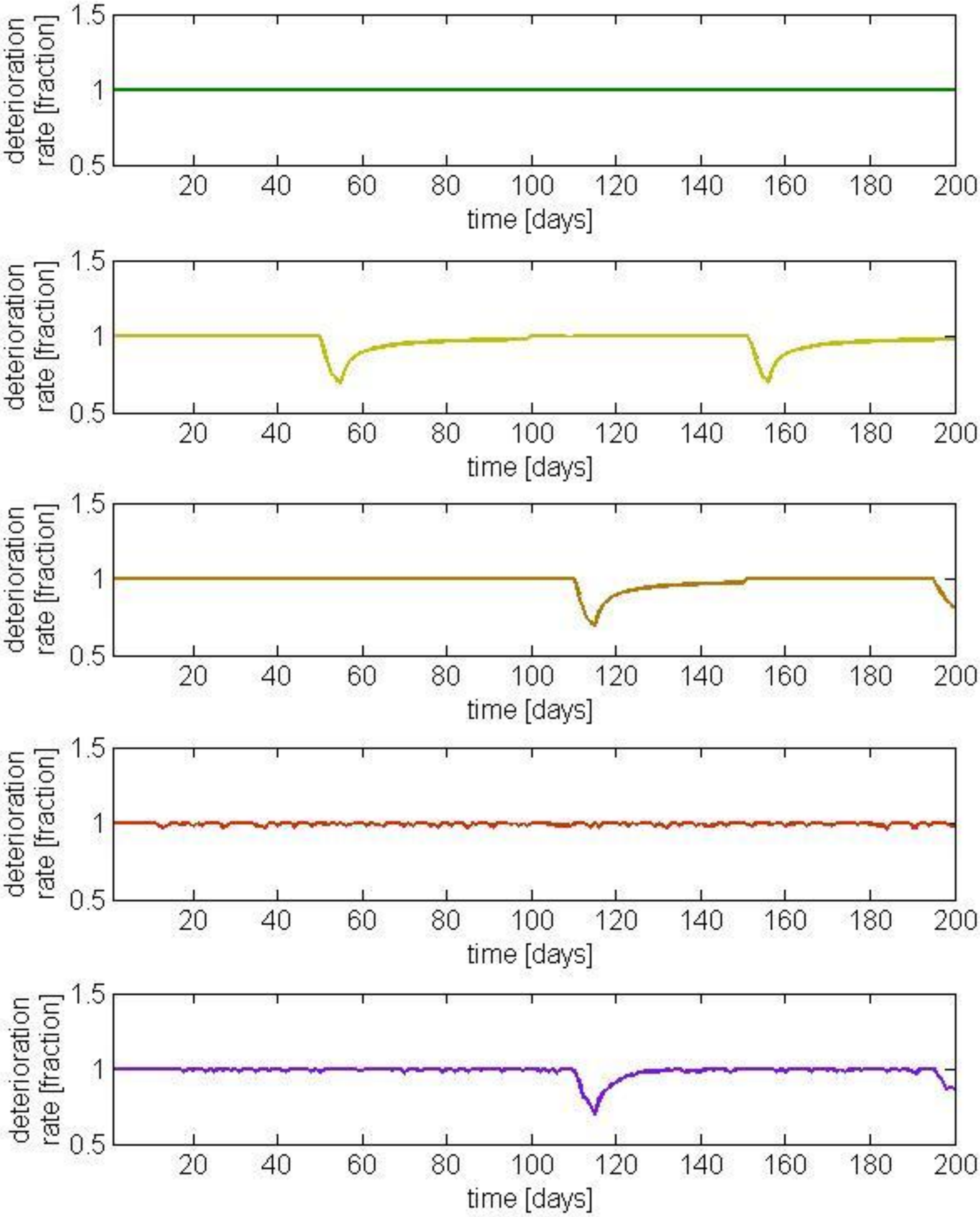


Figure 6-13: Deterioration rates for 5 demand patterns on separate scales

6.6 Test 4: To illustrate how demand pattern affects system behaviour continued

This section discusses results for the same numerical example as in Section 6.5, but the results are represented here in a different perspective. Figure 6-14 - Figure 6-18 show how the IC method deals with different demand patterns with respect to inventory, orders and deterioration rate on the same figure. It also deals with how the demand pattern influences the optimal order quantities, which in turn control the inventory levels and deterioration, which is dependent on inventory levels.

In Figure 6-14 the warehouse is initially empty and the orders are zero as the reference inventory level is zero, to keep the warehouse empty till the first demand occurs. The demand is zero too for first 4 time instances. The demand increases from 0 to 50 items at the 5th time instance and then remains constant. As a response, the backorders of approximately 220 items initially occur in the warehouse (at 5th time instance), since the warehouse was initially empty and no order was placed until now. The respective order quantities increase to 300 items at the 5th time instance to compensate for the backorders in the inventory levels. Then the order size quickly stabilizes and converges to 50 items (the demand level), within the lead time delay. In this way the inventory level is driven back to the reference zero items (empty warehouse) by appropriate replenishment. The inventory level remains at the reference level of zero items for the rest of the simulated periods due to the fact that the order size compensates for the constant demand of 50 items for the rest of the simulation. The deterioration rate of products remains at a value of one, since the warehouse inventory level never goes above zero. In practice the empty warehouse refers to a cross-docking center (whatever is purchased is immediately dispatched to the customer) (Rushton, Croucher and Baker, 2010).

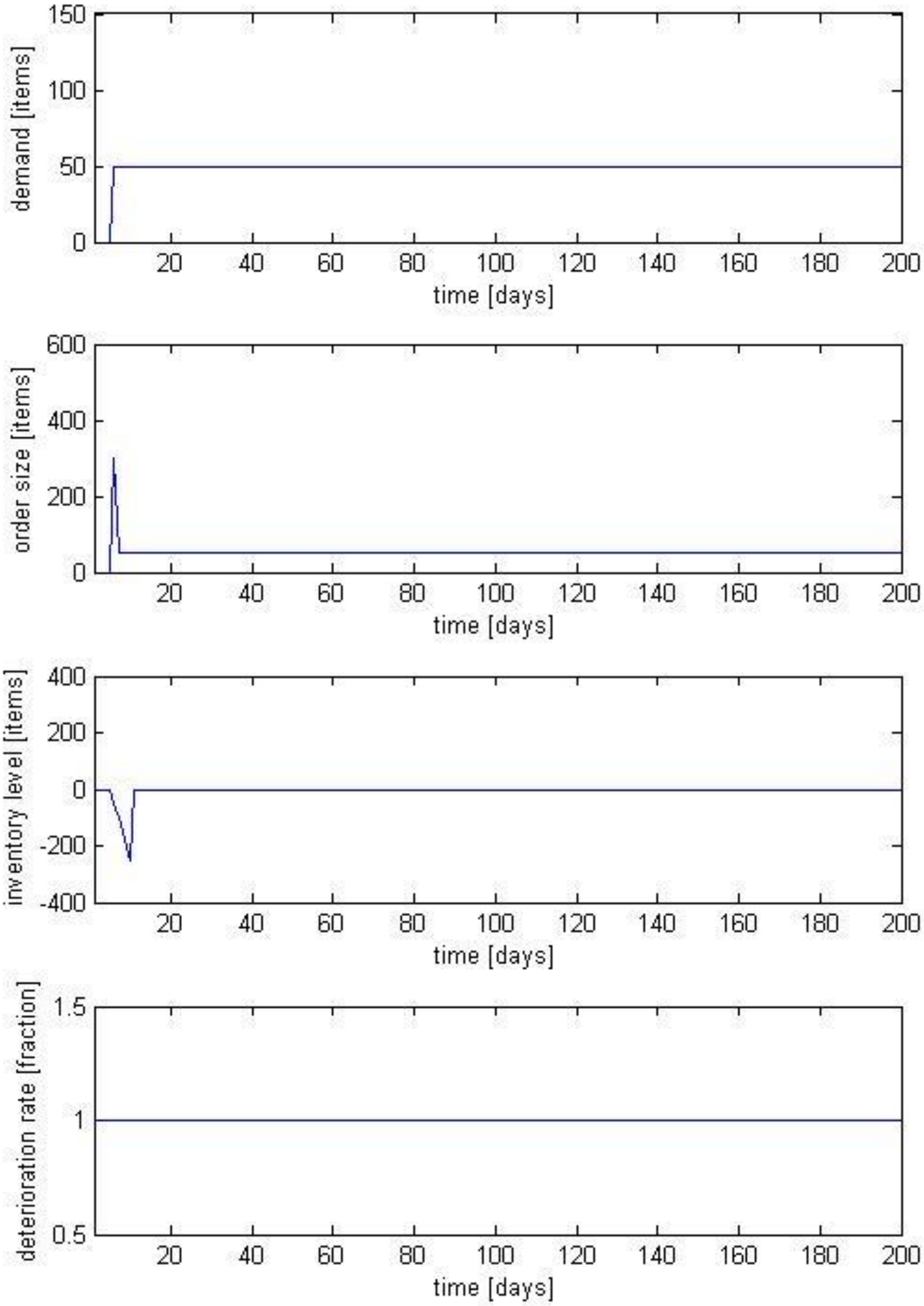


Figure 6-14: Demand 1 and respective system response

In Figure 6-15 the warehouse is initially empty and the orders initially are zero as the reference inventory level is zero too, to keep the warehouse empty until the first demand occurs. The considered demand pattern is seasonal with only two values allowed. No items are initially demanded for the first 4 time instances. The demand increases drastically from 0 to 100 items at the 5th time instance. As a response, the backorders of 400 items occur in inventory levels, as no goods are stored in the warehouse and 100 items are suddenly demanded. The orders suddenly overshoot to the level of 600 items to compensate for the backorders. The orders then stabilize and converge quickly (within the time delay) to the level of 100, which is the demand value, at that time instance. In this way once the inventory level has converged back to the reference inventory level within the lead time delay thanks to order size compensation, the orders can be equal to demanded number of goods to maintain the desired state. No difference is noticed in deterioration rate at that time instance.

Then, at the 50th time instance, the demand decreases to zero. As a response, the inventory builds up quickly (it overshoots to a value of 300 items) and the orders suddenly decrease to zero to compensate for the increased inventory. The orders are not allowed to go below zero therefore the process of driving the inventory back to the reference level takes more than the lead time delay. Eventually the inventory smoothly converges back to zero. The almost symmetric shape of deterioration rate line to inventory overshoots can be noticed at respective times as according to deterioration value design it decreases below zero only if inventory goes above zero (above reference level).

As a result of the repetitive character of Demand 2, the whole cycle starts over again at the time instance 100.

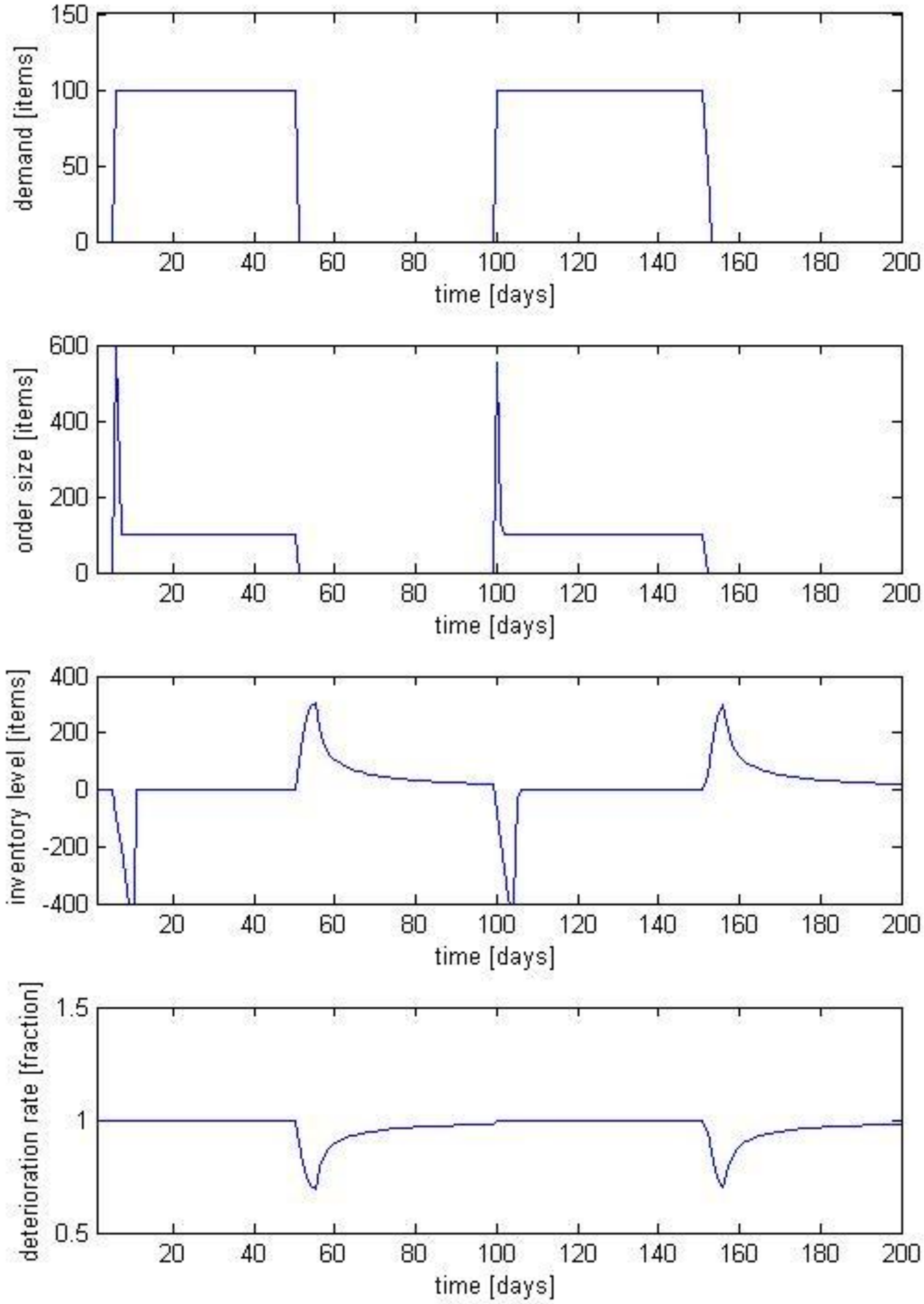


Figure 6-15: Demand 2 and respective system response

Figure 6-16 shows the simulation results for the seasonal demand pattern when more values are allowed. Initially the warehouse is empty and the orders are zero as no goods are demanded at the beginning of the simulation. The demand changes are less extreme than in the case of Figure 6-15 (over 50 items most of the time, not 100 as it was with demand 2). The first non-zero demand (of 50 items) occurs at the 10^{th} time instance. It can be observed that as a response, the backorders of about 220 items occur in inventories in the respective time, as there were no goods in the warehouse to satisfy the demand. The order quantities overshoot on the 10^{th} day (up to 300 items) to compensate for the backorders and future demand. As the demand remains at the level of 50 till the 60^{th} time instance, the orders coverage to the demand value (50 items) to replenish the warehouse in what was sold. Through appropriate order size the inventory levels are driven back to the zero reference inventory level within the lead time delay. There is no difference observed in deterioration rate level at that time unit.

Then, the sudden increase of demand from 50 to 100 items causes the backorders in inventories of approximately 220 items and overshoots in orders to the level of approximately 350 items in order to control the inventory fluctuations. Again no difference in deterioration rate values is observed. The order quantity again converges to the demand value to replenish the warehouse on a daily basis with the sold number of goods. This way the inventory is driven back to zero items.

Then the drop to zero of the demand at the 110^{th} time instance causes overshoots to the level of 300 items in inventories, as the inventory quickly builds up due to a system delay (back orders manage to take over control). The reduction of optimal orders to zero is observed in the respective time. The inventory converges smoothly and slowly to zero, but some extra time (apart from lead time delay) is needed as the orders cannot go below zero to speed up the inventory reduction process. It causes an analogous drop of the deterioration rate (below the constant level of one). The deterioration rate drops below one only when the surplus inventory appears in the warehouse.

The demand increases back to 50 items at the 150^{th} time instance which causes the same system response as at the 5^{th} time instance. The demand drop to zero at the 195^{th} time instance increases the inventories and zero orders when the simulation time finishes at the 200^{th} time instance.

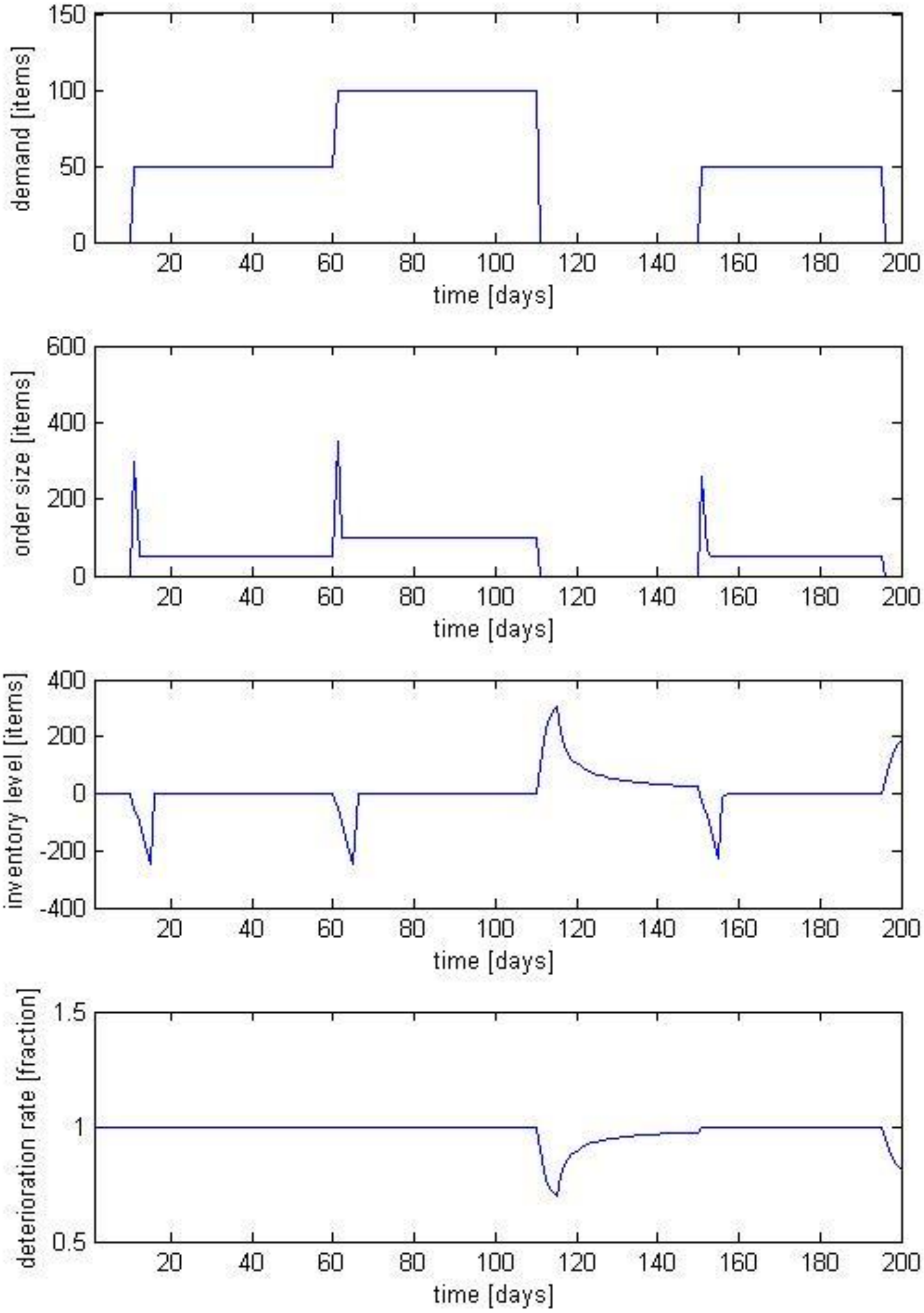


Figure 6-16: Demand 3 and respective system response

Figure 6-17 shows the simulation results for the random demand pattern. Initially the warehouse is empty and as a result no order is placed till the first non-zero demand occurs. The demand is equal to zero for the initial time instances, then increases to the level of 50 items and then remains at similar level oscillating +/- 5 items. The system response is similar to that of Demand 1 with some more fluctuations of results added.

For the sudden increase of demand at the 5th time instance, the backorders of about 220 items occur in the inventory levels, as there were no goods in the warehouse to satisfy the demand. The orders overshoot (to a level of approximately 350 items) to take control over the backorders at the respective time. This causes the inventory levels to converge to the reference level of zero items. The order size converge to the level of demand in order to sustain the replenishment of inventories at a similar level. Due to the demand variation of around a value of 50 items from the 6th time instance onwards, the oscillations can be observed in inventory levels too. They oscillate near to zero items for the remainder of the simulation. It can be noticed that the order sizes tend to push the inventory level towards the reference point at any time by replenishing the warehouse on a daily basis by the amount which oscillates around the demanded number of goods. The oscillations are greater than +/-5 items due to lead time delay.

The slight deviations of the deterioration rate from the value of one are noticeable at any time instance the inventory goes above zero. As the inventory oscillates around zero on a daily basis, the frequency of inventory going slightly above zero is high as well, which can be observed as fluctuation in deterioration rate.

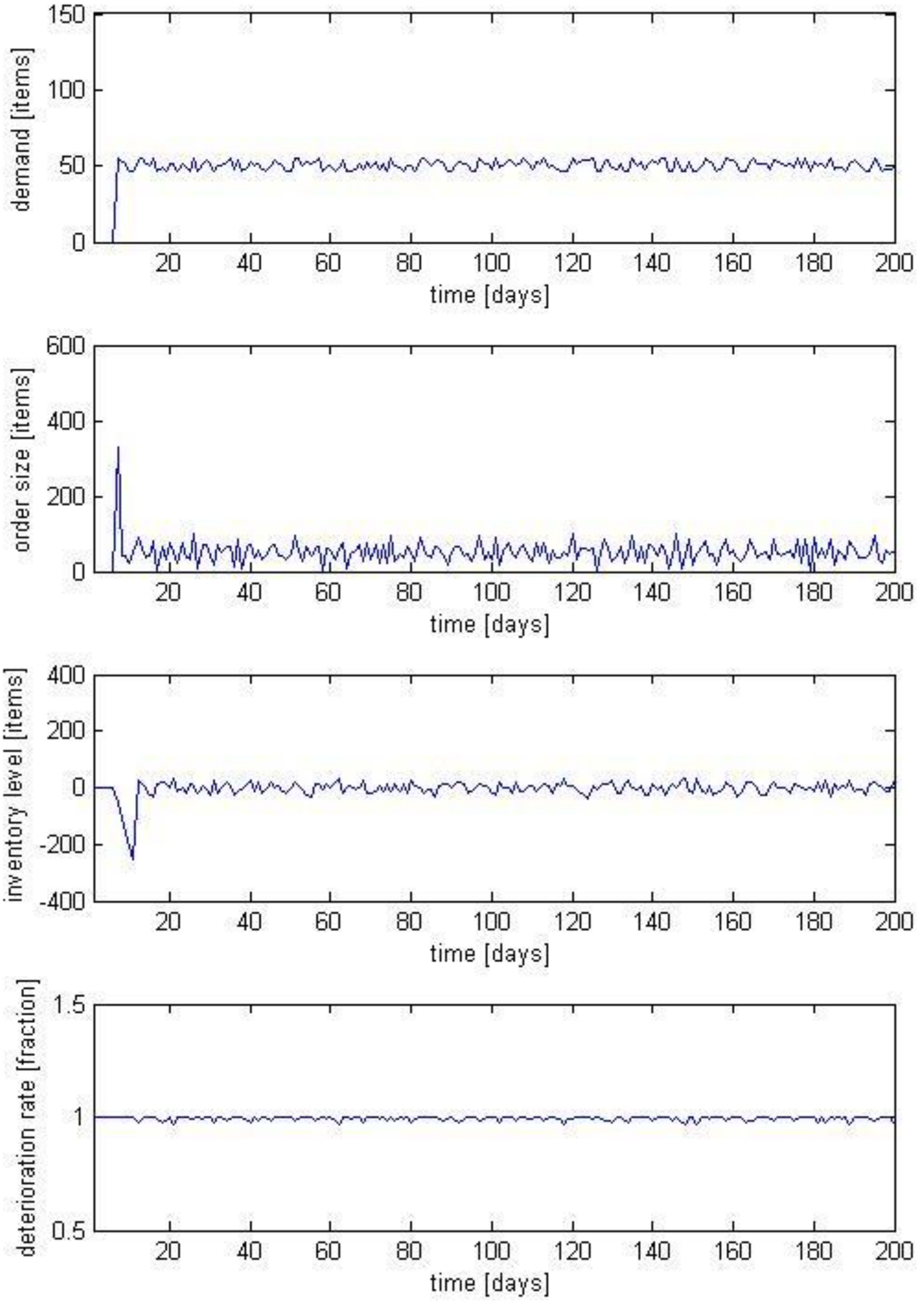


Figure 6-17: Demand 4 and respective system response

Figure 6-18 shows the results for a seasonal demand (of more than 2 values allowed) with some randomness allowed. Initially the warehouse is empty and as a result no order is placed until the first non-zero demand occurs. The demand is equal to zero (no sale requested) for the first 5 time instances. Then the first non-zero demand occurs at the 5th time instance at a value of 5 items and just after, at the 10th time instance when the demand suddenly increases (from 0 to 50). These two demand changes cause backorders in inventory levels. The order size at the 5th time instance tends to drive the inventory back to the reference level of zero items, which is eventually not take place due to the new demand change (the one at 10th time instance) before the first orders goods manage to reach the warehouse due to the lead time delay. Therefore only the order overshoot (of around 300 items at 10th time instance) manages to drive the inventory around the reference point within the lead time.

The inventory remains near to the reference level due to constant inventory level correction corresponding to the appropriate order size (fluctuation around demand current value). The fluctuation of demand is greater than 5 items due to the lead time delay. The order quantities sustain inventory levels near the current demand value.

The first order size overshoot of 300 items refers to the sudden increase of demand from 0 to 50 items at the 5th time instance. The backorders of about 210 items occur in the inventory levels in a respective time instance which converge quickly to the reference inventory level. There is no difference noticed in the respective deterioration rate values. Then the next overshoot of orders of about 250 items and backorders in inventory of about 200 items refer to the demand increase of around 100 at the 60th time instance. They both then converge to the demand and the reference inventory levels, where they remain oscillating within the band of +/-50 items. Then the next difference in behavior can be noticed for the time instance of 110. In this time the order sizes drop very close to zero and inventory overshoots appear which reach almost 250 items. Here also the respective drop of deterioration rate to inventory level increase can be observed. All of them converge back to the regular levels. In general the deterioration rate falls below one proportionally to the inventory level in the current time instance. The correction on a daily basis of inventory level through order size are noticeable through all simulation time in the form of order fluctuations.

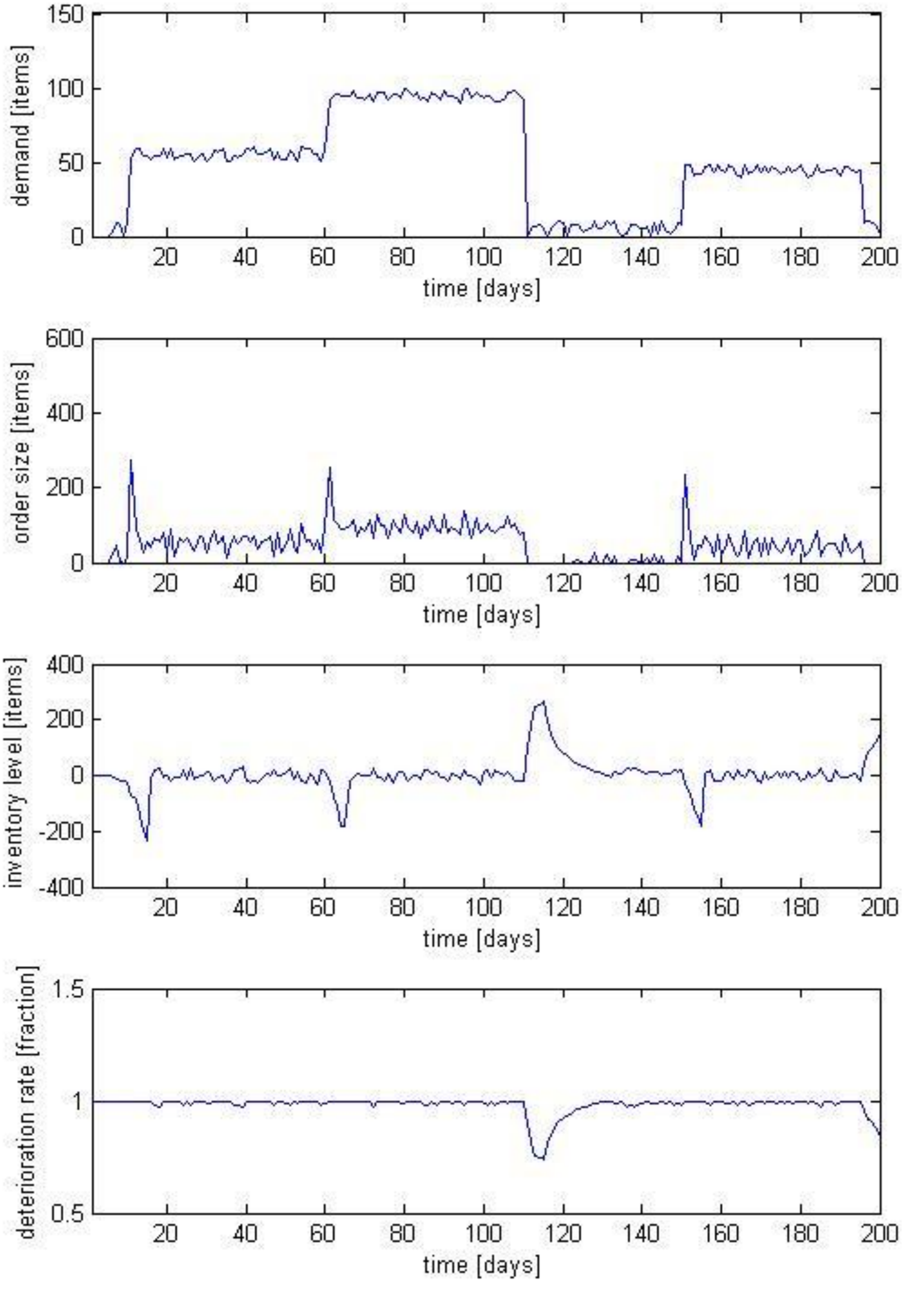


Figure 6-18: Demand 5 and respective system response

From this designed experiment it can be concluded that for each demand pattern, the inventory behaviour is acceptably stable and would appear to be beneficial for practical industrial applications. Therefore, it is considered that the optimal order quantities obtained are with respect to the given inventory levels and are satisfactory for efficient cost management.

6.7 Summary

In this chapter the simulation experiments were designed for a case scenario as realistic as the model would allow. The order quantities were prevented from going below zero items at any point in time. The mathematical description of the deterioration rate assured that the products were not allowed to deteriorate when the inventory was equal to or less than zero.

From the simulation results it can be deduced that the inventory and order sizes respond according to the demand patters as expected. Fluctuations are proportional to demand changes and converge to the expected level with the system time delay (the inventory converges to the reference inventory level and the order quantities to demand current value) are reasonable. The deterioration rate behaves according to its mathematical definition and falls below one only when the inventory is above zero.

A realistic profit function has been constructed, which considers the unit costs of storage and purchase, as well as normal and discounted unit prices (in the case of backorders, when a customer has to wait for a delivery). The profit function enables an assessment of what reference inventory level is more profitable for the organisation. This includes the high inventory levels, increasing the total holding costs or the low inventory levels increasing costs related to backorders. For the arbitrarily selected numerical unit costs and process values, the simulation was run for different demand patterns and the inventory level of zero items appeared to be the most profitable, as one would expect. Further experiments were then conducted for that value only.

For each demand pattern, whether constant, seasonal, random or mixed, the inventory level converges quickly and stays at or near zero. It can be considered as a satisfactory result for industrial applications in the sense of both, inventory level reduction as well as reduction in inventory fluctuations.

7 FUTURE MODELLING AND SIMULATION RESULTS

In Chapter 5 and Chapter 4, Section 4.5 both the verification of behaviour and the mathematical equivalency of IC against IMPC in the case where the tuning parameter was equal to zero $(u_r = 0)$ was presented. Nevertheless from Section 4.4 it can be noticed that the MPC for the given inventory model becomes insensitive to changes in the N_p and N_c values when $u_r = 0$. Therefore, the IMPC approach essentially becomes non predictive, rather it corresponds to a minimum variance approach or dead-beat controller with an inherent Smith predictor. This does not mean that the lack of model predictability is a disadvantage as long as the research aim is achieved (to design the inventory optimisation tool accessible for the OR community, which would keep the inventory close to the reference point and consider the system dynamics). Nevertheless, the current section examines the case of $u_r \neq 0$ scenario and presents the results for IMPC for different values of the tuning parameter. Based on the results it is then assessed whether a development of a mathematically equivalent formulation (hence alternative formulation) would be advantageous, and whether the potential improvement for $u_r \neq 0$ would balance the inconvenience with respect to the increased complexity of the resulting IC model. It is found that the increased complexity of the IC model would be disproportionally high with respect to the difference in results obtained. The chapter justifies the decision of relaxing the need for a mathematically equivalent model to IMPC with $u_r \neq 0$. Instead, it presents the simplified methods which generate similar results to IMPC with $u_r \neq 0$, but are not strictly mathematically equivalent, but gives a fairly good approximation. It also satisfies the aim of making the method accessible to the OR community. The non-perishable and perishable cases are shown separately. Nevertheless, for many real case scenarios the initial IC model presented in Chapter 3 and its performance discussed in Chapter 6 would be sufficient, as the results do not differ much whether $u_r \neq 0$ or $u_r = 0$ for the inventory application.

The approach developed here is systematic, although repetitive. The development aims to allow the reader to gain an insight into the way the approach has naturally evolved.

7.1 IMPC Results for $u_r \neq 0$

The current section presents a comparison of IMPC for $u_r \neq 0$ with IC, which enables only the case of the $u_r = 0$ scenario from definition, as a special case. As in the previous development, the non-perishable and perishable cases are presented separately. In each case, a simulation is designed to run for 200 time instances and $n=5$, $N_p = 18$, $N_c = 14$ and the demand pattern shown in Figure 7-1 (*50* items for time instances between *6-55* and zero for the remaining time instances).

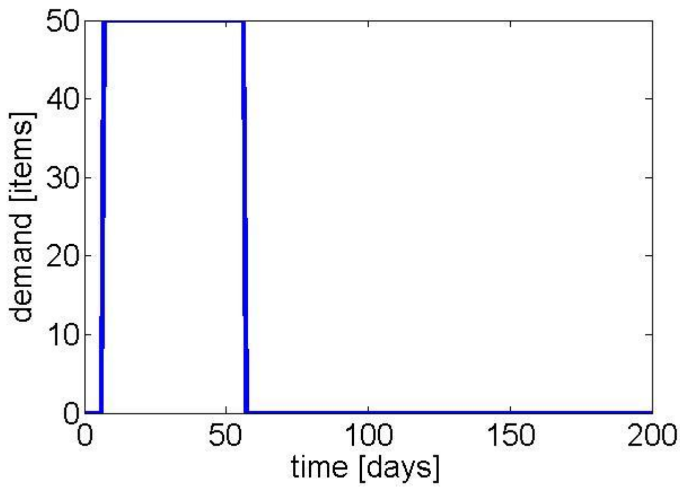


Figure 7-1: Demand pattern

7.1.1 Non-perishable case

Figure 7-2 presents the inventory levels of IC and IMPC for $u_r = 0$, upper plot and of IMPC for two different values of the tuning parameter ($u_r = 0.5$ and $u_r = 0.9$), middle and lower plot. Here the reference inventory level is set to *50* and deterioration rate to 1 as the non-perishable case is considered.

The IMPC state space model and IMPC and IC gains have the forms as follows.

Form (3.3) and (3.4)

$$\boldsymbol{x}\left[(k+1)T\right] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(kT) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(kT) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d(kT) \tag{7.1}$$

$$y(kT)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}\mathbf{x}(kT) \tag{7.2}$$

From (3.64) or (3.66)

for $u_r = 0$ $\quad \boldsymbol{K}=\begin{bmatrix}5 & 5 & 4 & 3 & 2 & 1\end{bmatrix}$ (7.3)

for $u_r = 0.5$ $\quad \boldsymbol{K}=\begin{bmatrix}3.1716 & 3.1716 & 2.5858 & 2.0000 & 1.4142 & 0.5858\end{bmatrix}$ (7.4)

for $u_r = 0.9$ $\quad \boldsymbol{K}=\begin{bmatrix}2.7631 & 2.7631 & 2.2670 & 1.7708 & 1.2747 & 0.4961\end{bmatrix}$ (7.5)

In Figure 7-2 it can be noticed that the obtained inventory level looks the same.

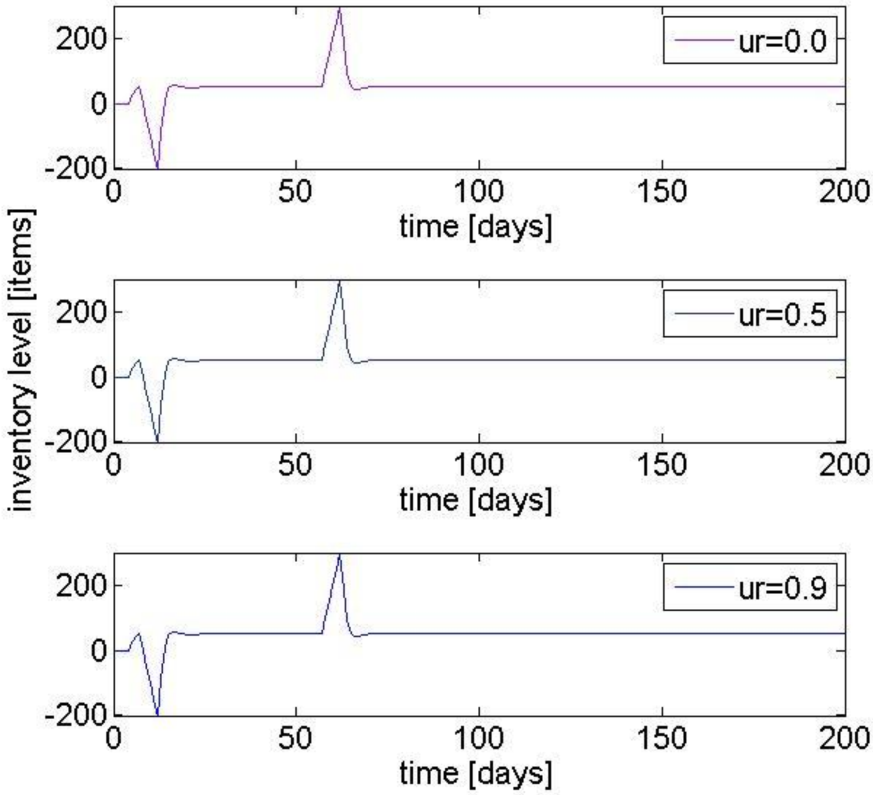
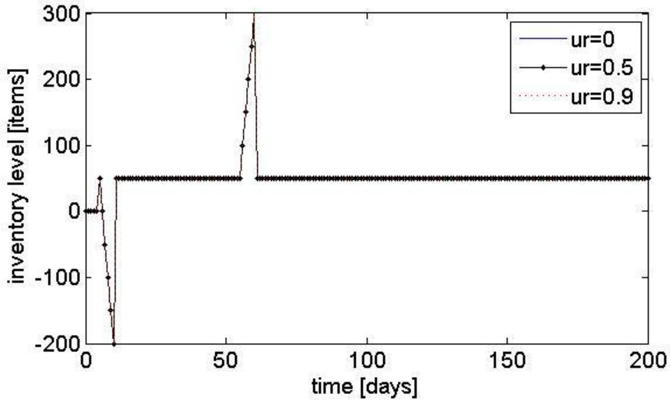
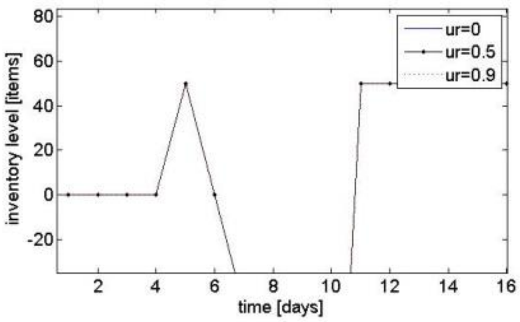


Figure 7-2: Inventory levels obtained with respect to u_r - non-perishable, separate plots

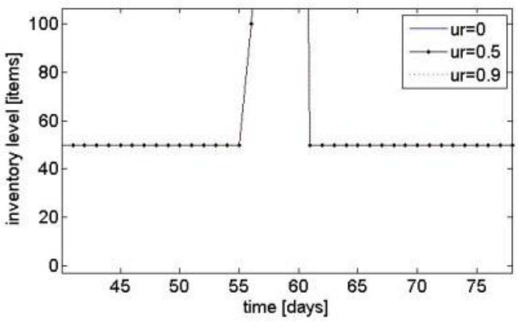
To confirm whether the results are very similar Figure 7-3 shows the same results on one plot (upper) as well as (first from the left) with zoomed results of time instance 0-16 (second form the left) and 40-80 (third from the left) in the two lower plots.



All signals on one plot



Zoomed values of 0-16 time instanes



Zoomed values of 40-80 time instanes

Figure 7-3: Inventory levels obtained with respect to u_r - non-perishable, one plot

In both sets of results, Figure 7-2 and Figure 7-3 it is noted that, the differences obtained are barely noticeable even near the regions of sudden demand changes. Similar conclusions can be drawn from simulation runs for different settings. This justifies neglecting the importance of the tuning parameter for non-perishable inventory level control. Therefore the IC can be considered to be both simple and sufficient for the given inventory application.

7.1.2 Perishable case

Figure 7-4 presents the inventory levels of IC ($u_r = 0$), upper plot and of IMPC for two different values of the tuning parameter ($u_r = 0.5$ and $u_r = 0.9$), middle and lower plots. The deterioration rate is set to 0.7.

The IMPC state space model, IMPC and IC gains have the forms as follows.

Form (3.3) and (3.4)

$$\boldsymbol{x}\left[(k+1)T\right]=\begin{bmatrix}0.7 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}(kT)+\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}u(kT)+\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}d(kT) \tag{7.6}$$

$$y(kT)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x}(kT) \tag{7.7}$$

From (3.64) or (3.66)

for $u_r=0$ $\quad \boldsymbol{K}=\begin{bmatrix}1.9412 & 2.7731 & 2.5330 & 2.1900 & 1.7000 & 1.0000\end{bmatrix}$ (7.8)

for $u_r=0.5$ $\quad \boldsymbol{K}=\begin{bmatrix}1.2563 & 1.7947 & 1.6600 & 1.4675 & 1.1926 & 0.6327\end{bmatrix}$ (7.9)

for $u_r=0.9$ $\quad \boldsymbol{K}=\begin{bmatrix}1.0835 & 1.5479 & 1.4380 & 1.2810 & 1.0567 & 0.5413\end{bmatrix}$ (7.10)

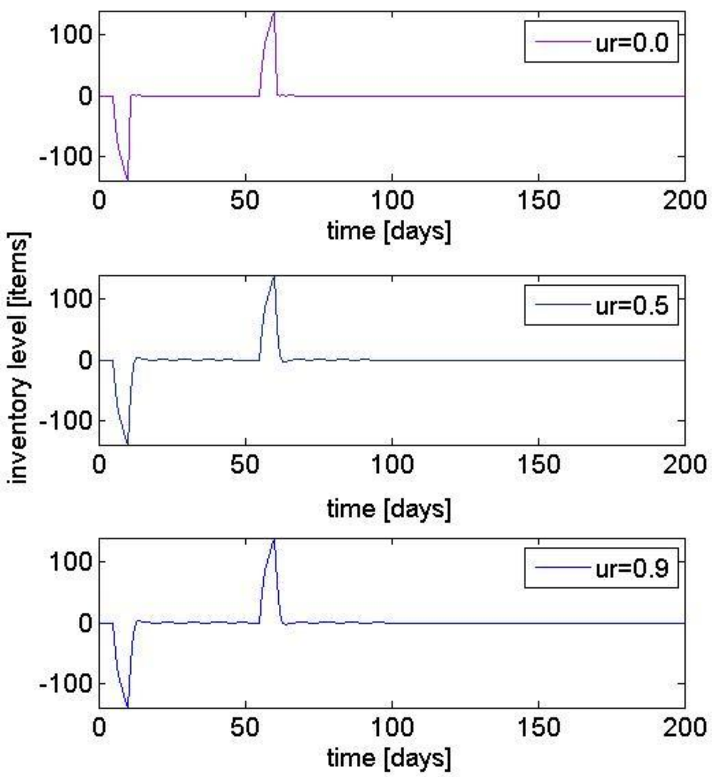
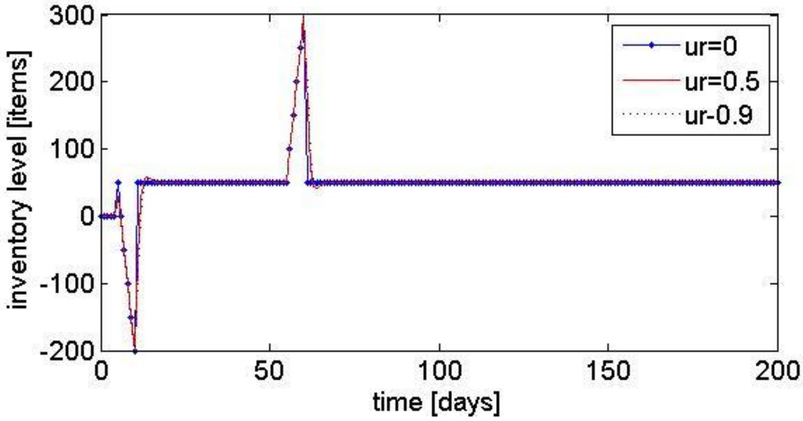
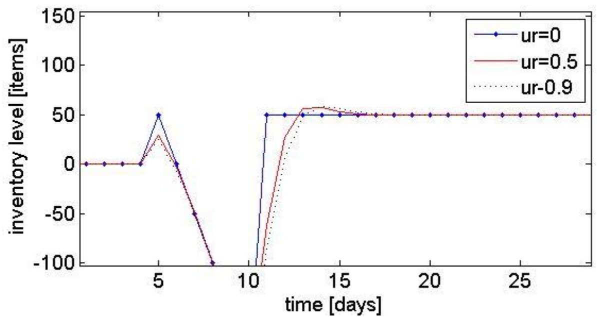


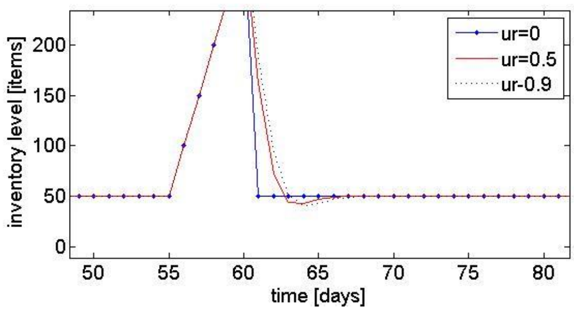
Figure 7-4: Inventory levels obtained with respect to u_r - perishable, separate plots

In Figure 7-4 it can be noticed that the differences between results of different tuning parameters look relatively small. To increase the clarity of the results they have been printed again on one plot and shown in Figure 7-5, analogous to Figure 7-3 for the non-perishable case.



All signals on one plot





Zoomed reluts for time instances between 0-30 days	Zoomed reluts for time instances between 40-80 days

Figure 7-5: Inventory levels obtained with respect to u_r - perishable, one plot

The differences in results obtained are mainly noticeable near the time instances where the demand suddenly changes (around 5 and 55 time instances). The result of $u_r = 0.5$ and $u_r = 0.9$ are almost identical and both of them still do not differ much from the $u_r = 0.0$ signal. The signals of $u_r = 0.5$ and $u_r = 0.9$ are slightly smoother around 5$^{\text{th}}$, 10$^{\text{th}}$, 55$^{\text{th}}$ and 62$^{\text{nd}}$ time instances than for $u_r = 0.0$ (less sharp) but it does not affect the overall supply chain performance. The levels of overshoots is the same for all u_r and the resilience (recovery from the disturbance) is almost the same (2 days more at most for $u_r = 0.5$ and $u_r = 0.9$ then

$u_r = 0.0$). It can be deduced that the differences in the results obtained (the minor fluctuations) are not significant for a practical inventory application. Similar conclusions can be drawn from simulation runs for different settings. Again it justifies neglecting the importance of the tuning parameter for non-perishable inventory level control. Therefore, as can be seen from the above, the IC can be considered to be both simple and sufficient for a given application.

7.2 Attempt to Develop the Simplified Model for $u_r \neq 0$

It was already stated that enabling $u_r \neq 0$ in the model would not bring about significant differences in the results for a given application. Nevertheless, if different applications of the same model are considered, where the precision of results are more vital, then it may be worth taking the case of $u_r \neq 0$ in the IC construction into consideration.

An initial attempt was to analogously repeat the mathematical reformulation as presented in Chapter 5. The attempt had to be modified, however, due to an irregularity of the matrix $\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{R}$ in (3.61). It is found that the tuning parameter when differing from zero disrupts the consistency in the description identified in the form of $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ in (5.58). Therefore, the complexity of the general description of the inverse matrix $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{R}\right)^{-1}$ disproportionally increases. This, in turn, increases the complexity of the description of the gain vector $\boldsymbol{K}$ and disables the construction of an equivalent method, which is 'simplified' with respect to IMPC. On the contrary, the alternative equivalent description would be significantly more complex.

A further approach comprises the formulation of the simplified methods of the IC type, which is not necessarily mathematically equivalent, but generates similar results to the IMPC with $u_r \neq 0$. The model of the non-perishable case is presented in the form of a *UrIC Proposition* and is termed here as the UrIC with reference to an inventory controller allowing $u_r \neq 0$.

7.2.1 Non-perishable case

UrIC Proposition

Denoting $\boldsymbol{x_c}$ as a vector of the form

$$\boldsymbol{x_c} = \left[I[(k-1)T]\text{-}I(kT) \quad u[(k-n+1)T]\text{-}u[(k-n+2)T] \right.$$
$$\left. u[(k-n+2)T]\text{-}u[(k-n+3)T] \quad \ldots \quad u(kT)\text{-}u[(k-1)T] \quad I(kT) \right]^T \tag{7.11}$$

where kT represents the current time instance, $I\left(kT\right)$ represents the current stock level and $u\left(kT\right)$ defines the current order size, and denoting the $I_R\left(kT\right)$ as reference inventory level in time instance kT and n as the lead time delay, the gain vector $\boldsymbol{K}$ as the transposed vector of dimension $n+1$ and where m denotes the vector $\boldsymbol{K}$ column index such that $\boldsymbol{K}=\left[K(1)\ldots K(m)\ldots K(n+1)\right]$ can be defined. Then the estimated values of the $\boldsymbol{K}$ vector can be found as follows:

$$\begin{gathered} K\left(n+1\right) \approx a_1 \\ K\left(n\right) \approx a_2 \\ \forall n \quad \forall 1<m<n \quad K\left(m\right)=K\left(n\right)+(n-m)K\left(n+1\right) \\ K\left(1\right)=K\left(2\right) \end{gathered} \tag{7.12}$$

For such a formulation of the vector, $\boldsymbol{K}$ the current optimal order quantity can be found as follows:

$$u\left(kT\right)=\begin{cases} 0 \quad \mathit{if} & u\left[\left(k-1\right)T\right]+I_R\left(kT\right)-\boldsymbol{Kx_c}\left(kT\right)<0 \\ u\left[\left(k-1\right)T\right]+I_R\left(kT\right)-\boldsymbol{Kx_c}\left(kT\right) & \mathit{otherwise} \end{cases} \tag{7.13}$$

where a_1 and a_2 are arbitrarily defined and are dependent on the values of N_p, N_c and n.

It should be highlighted here that the purpose of this section is to find a solution for a non-control familiar OR specialist. In such a case the values of N_p and N_c can be arbitrarily decided here, based on the sensitivity of the results to the particular application parameter settings, and provide the system user with certain values of a_1 and a_2.

7.2.1.1 UrIC results

The current numerical example presents the comparison of the simulation results of IMPC, for $u_r \neq 0$ and $\delta \neq 1$ and UrIC, where δ is not considered for the definition and $u_r \neq 0$. The aim of the numerical example is to verify if the UrIC method is sufficiently accurate, yet of a simple formulation.

The simulation was conducted for $n=5$, $N_p=18$, $N_c=14$, $I_r=50$ and 200 time instances. For the UrIC, the values of a_1 and a_2 were estimated respectively:

$$0.4962u_r^{-0.227} \quad and \quad 1.2801u_r^{-1.141} \tag{7.14}$$

Figure 7-6 and Figure 7-7 present the actual values of $\boldsymbol{K}(n+1)$ and $\boldsymbol{K}(n)$ (solid blue lines) as well as its estimations of a_1 and a_2 (red dashed lines) with respect to the u_r values.

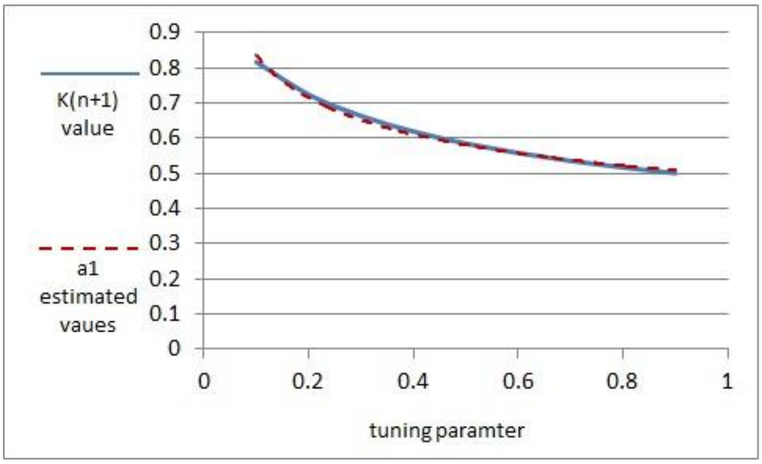


Figure 7-6: $\boldsymbol{K}(n+1)$ *and* a_1

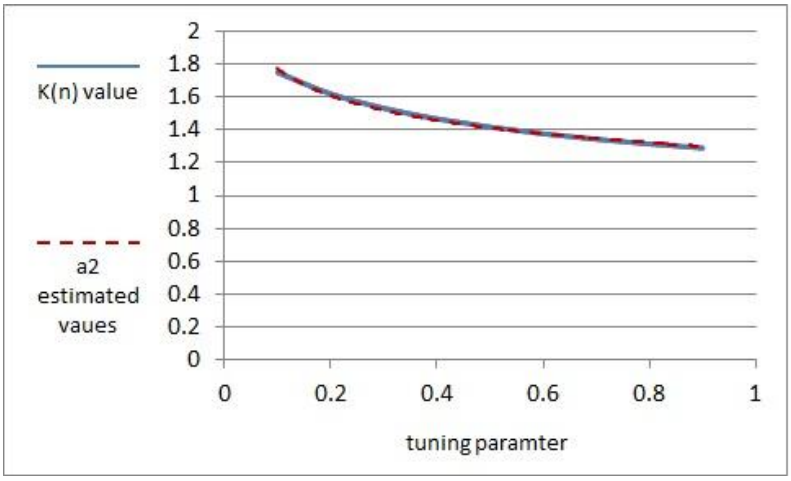


Figure 7-7: $K(n)$ *and* a_2

As can be observed from Figure 7-6 and Figure 7-7, the estimation would appear to be satisfactorily accurate for the given values of n, I_r, N_p and N_c.

Table 7-1 and Table 7-2 compare the values obtained for the vector $\boldsymbol{K}$ for the IMPC and UrIC methods. It can be noticed that the values obtained from the UrIC method are quite close to the IMPC values.

u_r	$K(1)$	$K(2)$	$K(3)$	$K(4)$	$K(5)$	$K(6)$
0.1	4.1996	4.1996	3.3830	2.5665	1.7499	0.8166
0.2	3.7912	3.7912	3.0672	2.3432	1.6192	0.7240
0.3	3.5217	3.5217	2.8584	2.1952	1.5319	0.6633
0.4	3.3250	3.3250	2.7059	2.0867	1.4675	0.6192
0.5	3.1730	3.1730	2.5878	2.0026	1.4174	0.5852
0.6	3.0509	3.0509	2.4929	1.9349	1.3768	0.5580
0.7	2.9500	2.9500	2.4143	1.8787	1.3431	0.5356
0.8	2.8646	2.8646	2.3479	1.8311	1.3143	0.5168
0.9	2.7911	2.7911	2.2906	1.7900	1.2894	0.5006

Table 7-1: $\boldsymbol{K}$ **values obtained by IMPC**

u_r	$K(1)$	$K(2)$	$K(3)$	$K(4)$	$K(5)$	$K(6)$
0.1	4.2817	4.2817	3.4448	2.6080	1.7711	0.8369
0.2	3.7513	3.7513	3.0363	2.3212	1.6062	0.7150
0.3	3.4734	3.4734	2.8213	2.1691	1.5169	0.6522
0.4	3.2894	3.2894	2.6785	2.0676	1.4566	0.6109
0.5	3.1538	3.1538	2.5730	1.9923	1.4115	0.5808
0.6	3.0473	3.0473	2.4901	1.9329	1.3757	0.5572
0.7	2.9603	2.9603	2.4222	1.8842	1.3461	0.5380
0.8	2.8870	2.8870	2.3650	1.8430	1.3210	0.5220
0.9	2.8239	2.8239	2.3157	1.8075	1.2993	0.5082

Table 7-2: $\boldsymbol{K}$ values obtained by UrIC

From the above comparison it can be concluded that the UrIC method enables quite an accurate estimation of the IMPC gain $\boldsymbol{K}$ regardless of the values of n, I_r, N_p and N_c. Therefore the simulation results for both methods are expected to be similar.

The estimation was also carried out for other values of n, I_r, N_p and N_c. In all cases the estimations obtained for a_1 and a_2 were similar to those presented above. The $\boldsymbol{K}$ values obtained based on the a_1 and a_2 estimation was also sufficiently accurate regardless of the simulation settings.

Figure 7-9 and Figure 7-10 show the simulation results for the chosen values of the tuning parameter respectively: $u_r = 0.1$, $u_r = 0.5$ and $u_r = 0.9$ for both IMPC and UrIC methods for the demand pattern shown in Figure 7-8, where the IMPC state space model has a form as in (7.1) and (7.2) and the IMPC and UrIC gains for give tuning parameter values are contained in Table 7-1 and Table 7-2.

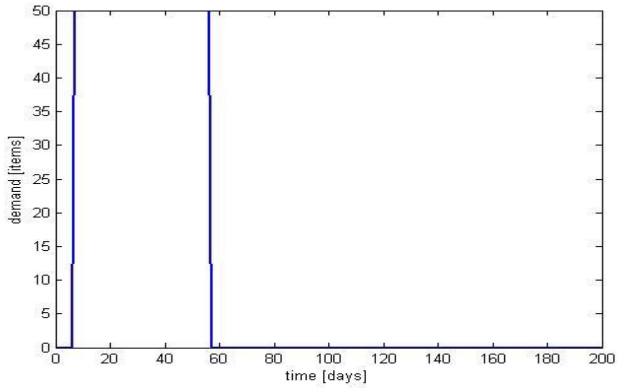


Figure 7-8: Demand

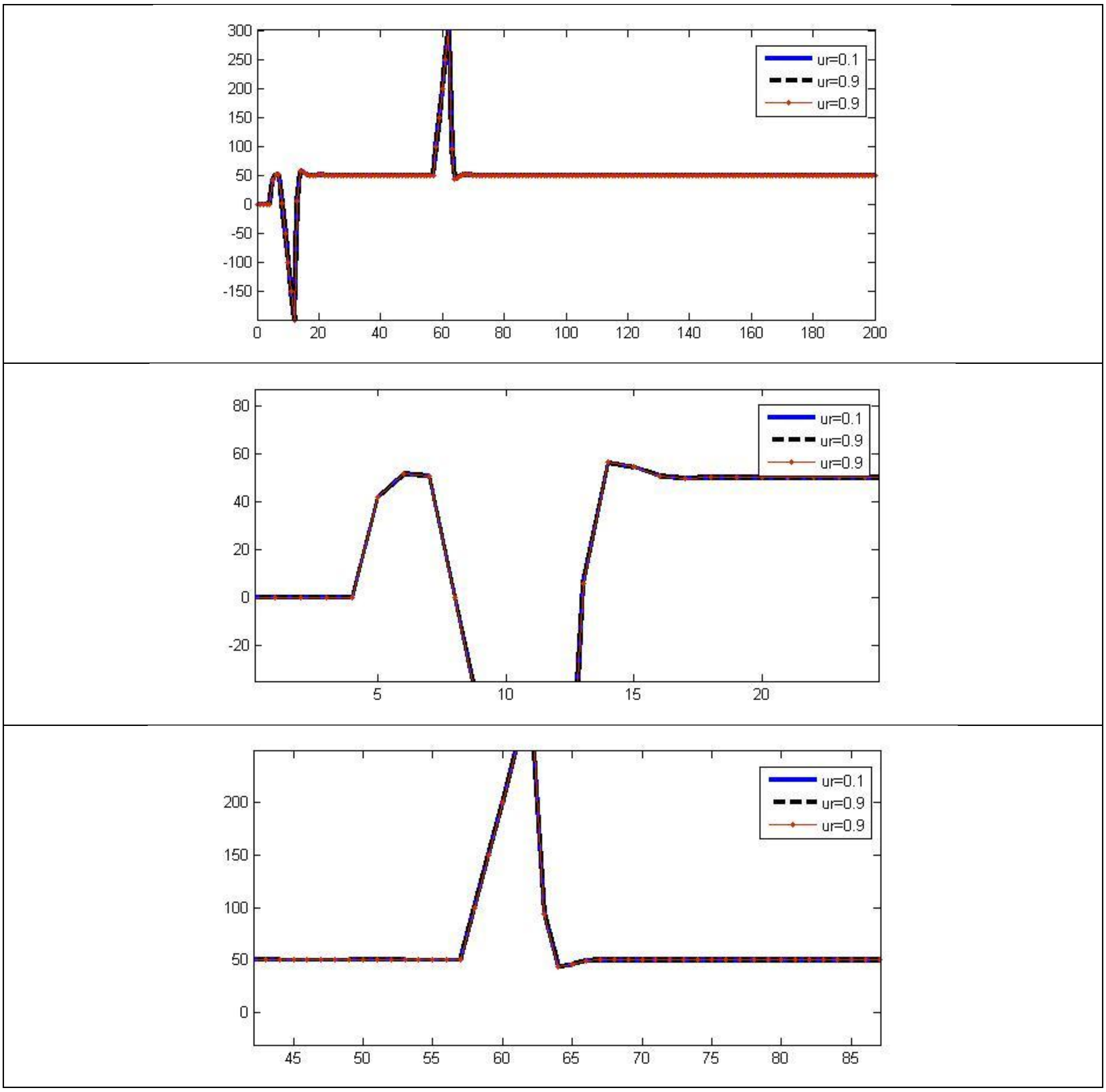


Figure 7-9: Results of IMPC in respect to tuning parameter

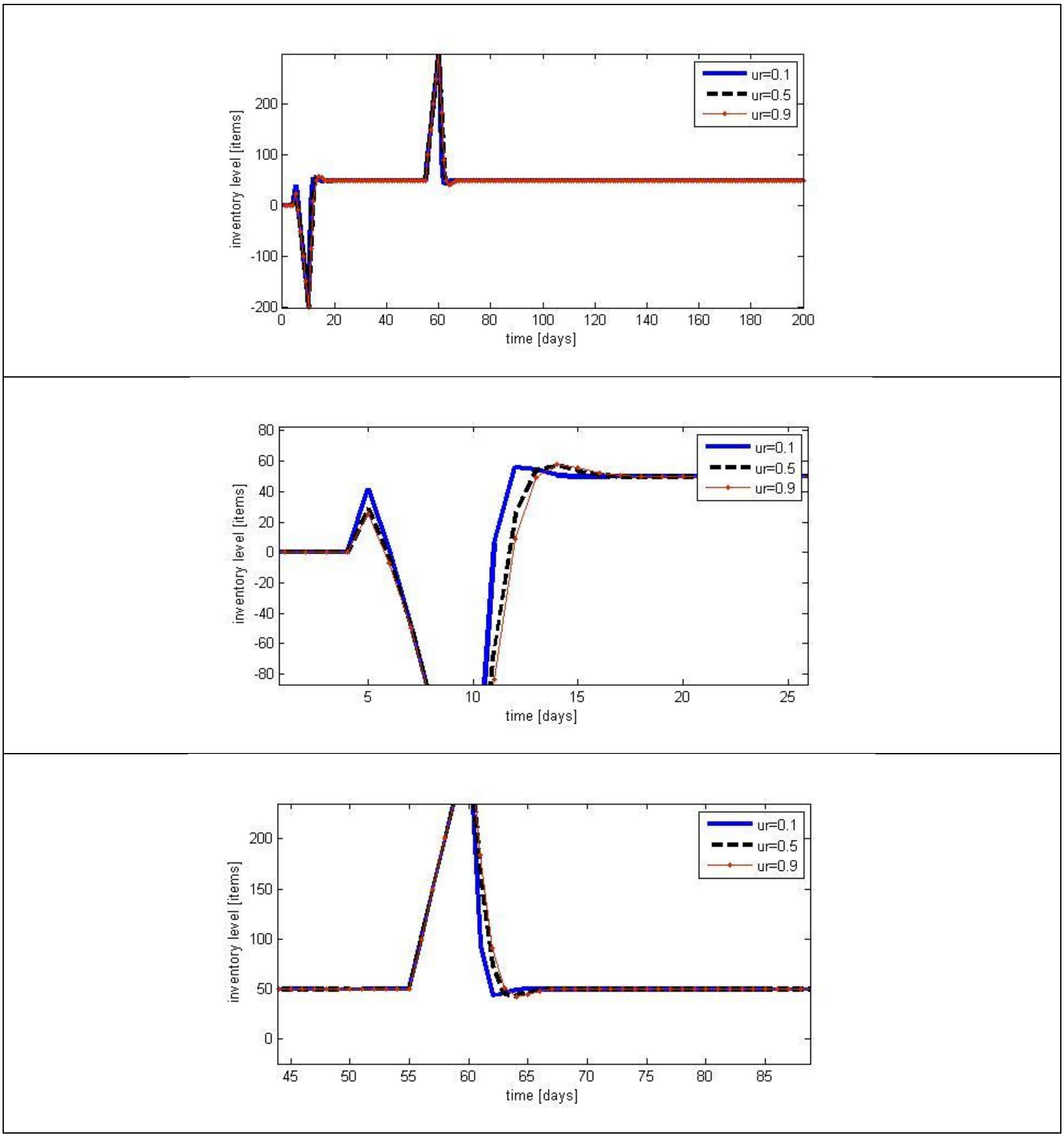


Figure 7-10: Results of UrIC in respect to tuning parameter

From Figure 7-9 and Figure 7-10 it can be noticed that the inventory levels obtained using the UrIC method are almost the same as for the IMPC method. The difference between IMPC and UrIC results can be mainly noticed just near the time instances where the demand changes suddenly (5th -10th and 55th - 62 nd). Similar simulations were run for different values of N_c, N_p and n and in all cases the results were similar for both methods. Therefore, it can be concluded that UrIC has been verified against IMPC and it gives satisfactory and similar results. Hence, if $u_r \neq 0$ is needed for specific applications, the UrIC can be used by the OR community for non-perishable applications, with an equally good performance to IMPC.

7.2.2 Perishable case

The current section presents an attempt at finding the ‘simplified’, hence not equivalent IMPC method for the perishable case. It can be immediately noticed that the method is no longer that simple and not as elegant in its description (LS). Nevertheless it can still be used by a non-control specialist. The least squares method was used here for finding the estimated values of the gain vector $\boldsymbol{K}$. Once the values of $\boldsymbol{K}$ are estimated, they can be used in the form of the following procedure:

PUrIC Proposition

Denote $\boldsymbol{x}_c$ as a vector of the form

$$\begin{aligned}\boldsymbol{x}_c = \big[&I[(k-1)T]-I(kT) \quad u[(k-n+1)T]-u[(k-n+2)T] \quad u[(k-n+2)T]-u[(k-n+3)T] \quad \ldots \\ &\ldots \quad u(kT)-u[(k-1)T] \quad I(kT)\big]^T\end{aligned} \tag{7.15}$$

where kT represents current time instance, $I(kT)$ represents the current stock level and $u(kT)$ defines the current order size and denoting $I_R(kT)$ as the reference inventory level in time instance kT and n as the lead time delay, defining $\boldsymbol{K}$ as a transposed vector of $n+1$ dimension where m denotes the $\boldsymbol{K}$ vector column index such that $\boldsymbol{K} = \left[K(1)\ldots K(m)\ldots K(n+1)\right]$, where the each $\boldsymbol{K}(m)$ can be denoted by the function

$$\forall 1 \le m \le n+1 \ \forall 0 < \delta \le 1 \ \forall 0 \le u_r < 1 \ \boldsymbol{K}(m) = \begin{bmatrix} 1 & \delta & u_r & \delta u_r & \delta^2 & u_r^2 & \delta^2 u_r^2 & \delta^3 & u_r^3 \end{bmatrix} \cdot \hat{\Theta}_m$$

(7.16)

where vector Θ_m is defined arbitrarily for each $K(m)$.

Then the optimal order quantities can be found as follows

$$u(kT) = \begin{cases} 0 \quad if & u\left[(k-1)T\right] + I_R(kT) - \boldsymbol{Kx_c}(kT) < 0 \\ u\left[(k-1)T\right] + I_R(kT) - \boldsymbol{Kx_c}(kT) & otherwise \end{cases} \quad (7.17)$$

It can be noticed that the user is provided with arbitrarily defined values for $\hat{\Theta}_m$. This makes the model less flexible, as the user cannot manipulate neither N_p and N_c nor n, but enables the non-control specialist, or OR familiar person to use the method.

7.2.2.1 Estimation of parameters

The values of Θ_m are arbitrarily defined for the OR user, however in general they are obtained from the LS method in the following manner.

For each $\boldsymbol{K}(m)$ the least squares (LS) estimation of the IMPC (non-zero tuning parameter case) is used to define a separate plane of points of values of $\boldsymbol{K}(m)$ for different deterioration rates δ and tuning parameters u_r in $\mathbb{R}^3$. The values for the $n=5$ case are contained in Table 7-3, Table 7-5, Table 7-7, Table 7-9, Table 7-11 and Table 7-13. The plane shapes are shown in Figure 7-11, Figure 7-13, Figure 7-15, Figure 7-17, Figure 7-19 and Figure 7-21, respectively. Based on observations of the plane shapes and several numerical experiments, it was decided to represent the plane as a set of quadratic curves. It was observed that the front and side views of the planes can be fairly estimated as quadratic functions. Therefore the 3rd order assumption was made in 3 dimensional surfaces hence the 3×3 equation parameters resulted in a 9×1 vector of the following form.

$$\boldsymbol{K}(m,i,j)=\begin{bmatrix}1 & \delta(i) & u_r(j) & \delta(i)u_r(j) & \delta^2(i) & u_r^2(j) & \delta^2(i)u_r^2(j) & \delta^3(i) & u_r^3(j)\end{bmatrix}\begin{bmatrix}\alpha_0(m)\\ \alpha_1(m)\\ \alpha_2(m)\\ \alpha_3(m)\\ \alpha_4(m)\\ \alpha_5(m)\\ \alpha_6(m)\\ \alpha_7(m)\\ \alpha_8(m)\end{bmatrix} \quad (7.18)$$

where $\delta(i)=0.1,\quad 0.2,\quad 0.3,\quad \ldots\quad 1.0$ and $u_r(j)=0,\quad 0.1,\quad 0.2,\quad \ldots\quad 0.9$ when $i,j=1,2...10$. The values of $\delta(i)$ and $u_r(j)$ are taken from any of the following tables: Table 7-3, Table 7-5, Table 7-7, Table 7-9, Table 7-11 or Table 7-13 for $i-th$ table row and $j-th$ table column.

Defining

$$\hat{\Theta}_m=\begin{bmatrix}\alpha_0(m)\\ \alpha_1(m)\\ \alpha_2(m)\\ \alpha_3(m)\\ \alpha_4(m)\\ \alpha_5(m)\\ \alpha_6(m)\\ \alpha_7(m)\\ \alpha_8(m)\end{bmatrix} \quad (7.19)$$

and having the actual values of $\boldsymbol{K}(m,i,j)$ from the tables: Table 7-3, Table 7-5, Table 7-7, Table 7-9, Table 7-11 and Table 7-13 for appropriate $m=1,2...n+1$ the LS method can be used to estimate the parameter vector $\hat{\Theta}_m$ in the following manner.

Define the matrix $\Psi(m)$ such that

$$\Psi(m)=\begin{bmatrix}\Psi(m,1,1)\\ \Psi(m,1,2)\\ \Psi(m,2,1)\\ \vdots\\ \Psi(m,10,10)\end{bmatrix}=$$

$$=\begin{bmatrix}
1 & \delta(1) & u_r(1) & \delta(1)u_r(1) & \delta^2(1) & u_r^2(1) & \delta^2(1)u_r^2(1) & \delta^3(1) & u_r^3(1)\\
1 & \delta(1) & u_r(2) & \delta(1)u_r(2) & \delta^2(1) & u_r^2(2) & \delta^2(1)u_r^2(2) & \delta^3(1) & u_r^3(2)\\
1 & \delta(1) & u_r(3) & \delta(1)u_r(3) & \delta^2(1) & u_r^2(3) & \delta^2(1)u_r^2(3) & \delta^3(1) & u_r^3(3)\\
 & & & & \vdots & & & & \\
1 & \delta(1) & u_r(10) & \delta(1)u_r(10) & \delta^2(1) & u_r^2(10) & \delta^2(1)u_r^2(10) & \delta^3(1) & u_r^3(10)\\
1 & \delta(2) & u_r(1) & \delta(2)u_r(1) & \delta^2(2) & u_r^2(1) & \delta^2(2)u_r^2(1) & \delta^3(2) & u_r^3(1)\\
1 & \delta(2) & u_r(2) & \delta(2)u_r(2) & \delta^2(2) & u_r^2(2) & \delta^2(2)u_r^2(2) & \delta^3(2) & u_r^3(2)\\
1 & \delta(2) & u_r(3) & \delta(2)u_r(3) & \delta^2(2) & u_r^2(3) & \delta^2(2)u_r^2(3) & \delta^3(2) & u_r^3(3)\\
 & & & & \vdots & & & & \\
1 & \delta(2) & u_r(10) & \delta(2)u_r(10) & \delta^2(2) & u_r^2(10) & \delta^2(2)u_r^2(10) & \delta^3(2) & u_r^3(10)\\
 & & & & \vdots & & & & \\
1 & \delta(10) & u_r(10) & \delta(10)u_r(10) & \delta^2(10) & u_r^2(10) & \delta^2(10)u_r^2(10) & \delta^3(10) & u_r^3(10)
\end{bmatrix}$$

Therefore $\Psi(m)$ is a matrix of 100×9 dimension such that

$$K(m)=\Psi(m)\Theta$$

The values of the vector Θ are estimated as follows:

$$\hat{\Theta}_m=\left[\psi^T(m)\psi(m)\right]^{-1}\psi^T(m)K(m)$$

which is the optimal estimate in the sense of least square error.

The estimated vector $\hat{\Theta}_m$ is used in the *PUrIC Proposition* as arbitrarily defined for straightforward OR user application.

7.2.2.2 PUrIC results

The current section presents the estimated values of gain vector $\boldsymbol{K}$. The aim of this section is to decide if the PUrIC controller can provide similar performance to IMPC for $u_r\in[0,1)$ and $\delta\in(0,1]$

The simulation parameters were set as follows: $n=5$, $N_p=14$, $N_c=8$ for the IMPC method and Θ_m values have been estimated as follows

$$\hat{\Theta}_1=\begin{bmatrix}-0.2103\\3.0438\\-0.8833\\-2.7451\\-2.9463\\3.0051\\0.5449\\4.5145\\-1.6431\end{bmatrix},\quad \hat{\Theta}_2=\begin{bmatrix}0.8793\\2.2343\\-1.8630\\-2.6455\\-0.8958\\4.0334\\0.8014\\2.3343\\-2.1949\end{bmatrix},\quad \hat{\Theta}_3=\begin{bmatrix}0.9399\\1.6850\\-1.7009\\-2.1466\\0.2664\\3.4325\\0.7893\\0.8268\\-1.8818\end{bmatrix},$$

$$\hat{\Theta}_4=\begin{bmatrix}0.9824\\1.3644\\-1.5291\\-1.5108\\0.4166\\2.7591\\0.6721\\0.0907\\-1.5117\end{bmatrix},\quad \hat{\Theta}_5=\begin{bmatrix}1.0191\\0.9877\\-1.3110\\-0.6997\\-0.0663\\1.9586\\0.3974\\0.0114\\-1.0473\end{bmatrix},\quad \hat{\Theta}_6=\begin{bmatrix}1.0048\\0.0400\\-1.1394\\-0.3003\\-0.1048\\1.5783\\0.2003\\0.0194\\-0.8252\end{bmatrix}$$

for the PUrIC method.

Firstly, the IMPC method was used for identifying the value of each element of gain $\boldsymbol{K}$ for different values for the deterioration rate δ and tuning parameter u_r. Then the respective values of gain $\boldsymbol{K}$ were obtained using the PUrIC method. Both the IMPC and PUrIC values for $\boldsymbol{K}(m)$ were completed and stored in the tables as a set of three dimensional point dependent variables (based on the two variables δ and u_r). Therefore they can be represented as planes in three dimensional space for comparison purposes and repetitive observations made for each of the planes separately.

Table 7-3 shows the $K(1)$ values obtained by IMPC with respect to δ and u_r. Table 7-4 shows the $K(1)$ values obtained by PUrIC with respect to δ and u_r. Figure 7-11 represents the shape of the plane of the points represented in Table 7-3 (solid line) and Table 7-4 (starred line). Figure 7-12 presents the difference between the IMPC and PUrIC values (note the difference in axes scales between the last two figures).

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.1111	0.1010	0.0938	0.0882	0.0838	0.0800	0.0768	0.0740	0.0716	0.0694
0.2	0.2499	0.2252	0.2082	0.1954	0.1852	0.1767	0.1696	0.1633	0.1579	0.1530
0.3	0.4275	0.3816	0.3513	0.3288	0.3111	0.2966	0.2843	0.2738	0.2645	0.2563
0.4	0.6598	0.5830	0.5342	0.4988	0.4712	0.4488	0.4300	0.4138	0.3998	0.3873
0.5	0.9688	0.8472	0.7731	0.7202	0.6795	0.6466	0.6192	0.5957	0.5754	0.5574
0.6	1.3834	1.1979	1.0890	1.0127	0.9545	0.9079	0.8691	0.8361	0.8075	0.7823
0.7	1.9412	1.6655	1.5096	1.4022	1.3210	1.2563	1.2028	1.1573	1.1180	1.0835
0.8	2.6893	2.2884	2.0700	1.9218	1.8106	1.7225	1.6498	1.5883	1.5352	1.4887
0.9	3.6856	3.1140	2.8138	2.6127	2.4631	2.3450	2.2479	2.1660	2.0954	2.0336
1	5.0000	4.1998	3.7940	3.5257	3.3274	3.1716	3.0439	2.9364	2.8439	2.7630

Table 7-3: K(1) values obtained by IMPC in respect to δ and u_r

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.0691	0.0182	0.0552	0.0517	0.0175	0.0374	0.1033	0.1702	0.2283	0.2678
0.2	0.3167	0.2021	0.1381	0.1150	0.1229	0.1518	0.1920	0.2336	0.2667	0.2815
0.3	0.5595	0.4177	0.3272	0.2780	0.2603	0.2642	0.2800	0.2977	0.3074	0.2994
0.4	0.8247	0.6558	0.5390	0.4642	0.4217	0.4017	0.3942	0.3894	0.3774	0.3484
0.5	1.1393	0.9435	0.8006	0.7009	0.6344	0.5913	0.5617	0.5358	0.5038	0.4556
0.6	1.5304	1.3077	1.1392	1.0150	0.9253	0.8601	0.8097	0.7641	0.7136	0.6482
0.7	2.0251	1.7757	1.5818	1.4337	1.3215	1.2353	1.1652	1.1014	1.0340	0.9533
0.8	2.6505	2.3744	2.1556	1.9841	1.8502	1.7438	1.6553	1.5747	1.4921	1.3978
0.9	3.4336	3.1311	2.8876	2.6933	2.5383	2.4129	2.3071	2.2111	2.1149	2.0089
1	4.4016	4.0726	3.8048	3.5882	3.4131	3.2695	3.1477	3.0376	2.9296	2.8137

Table 7-4: K(1) values obtained by PUrIC in respect to δ and u_r

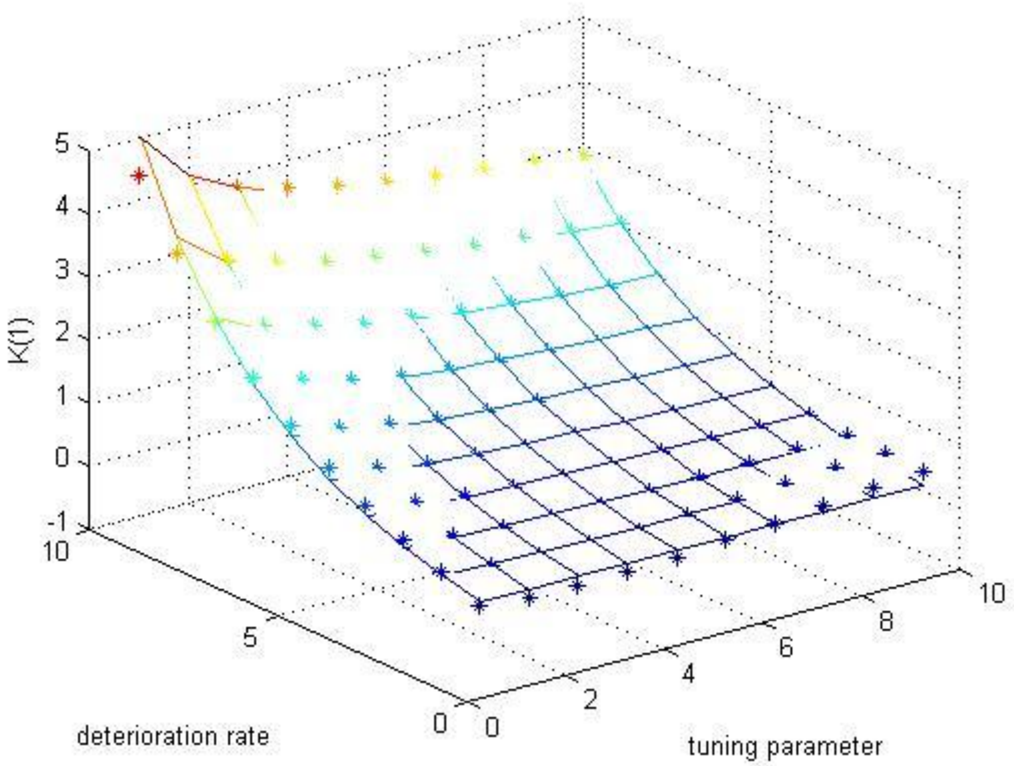


Figure 7-11: IMPC points – solid line and PUrIC points – starred line for K(1)

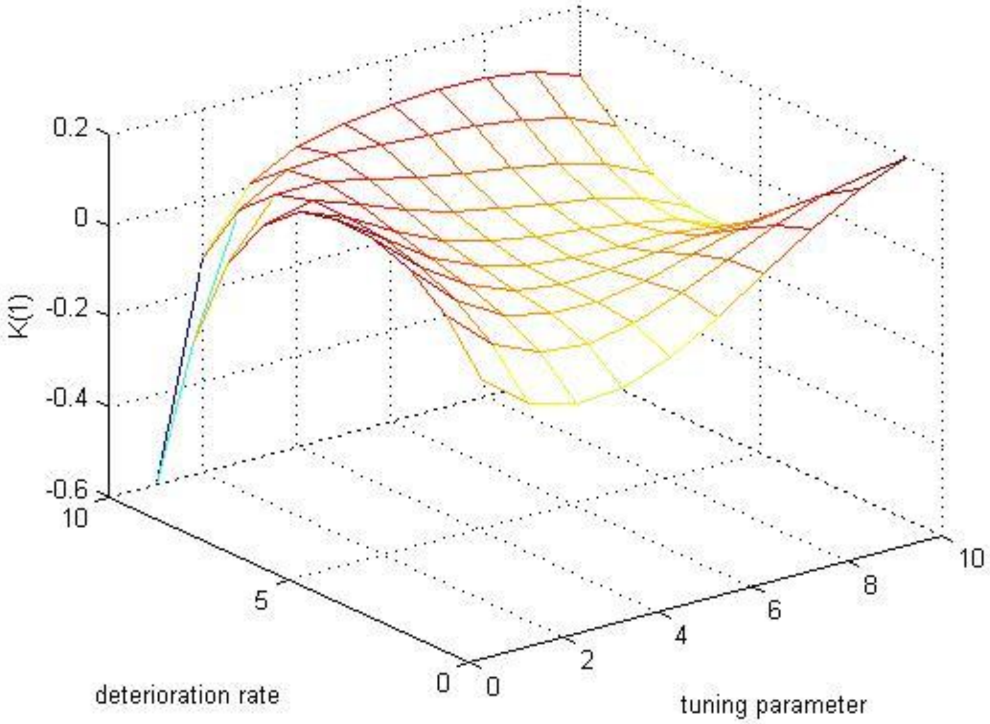


Figure 7-12: Difference between IMPC and PUrIC for K(1)

As can be observed, the estimated values are more accurate for middle values of deterioration rate and tuning parameter.

Table 7-5 shows the $K(2)$ values obtained by IMPC in respect to δ and u_r. Table 7-6 shows the $K(2)$ values obtained by PUrIC in respect to δ and u_r. The Figure 7-13 represents the shape of the plane of the points presented in Table 7-5 (solid line) and Table 7-6 (starred line). The Figure 7-14 represents the difference between the IMPC and PUrIC values (note the difference in scales between last two figures).

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1111	1.0101	0.9380	0.8824	0.8376	0.8001	0.7682	0.7404	0.7159	0.6940
0.2	1.2496	1.1261	1.0412	0.9771	0.9260	0.8837	0.8478	0.8167	0.7895	0.7652
0.3	1.4251	1.2720	1.1708	1.0959	1.0369	0.9885	0.9477	0.9125	0.8818	0.8545
0.4	1.6496	1.4576	1.3356	1.2470	1.1781	1.1220	1.0749	1.0346	0.9994	0.9683
0.5	1.9375	1.6945	1.5461	1.4404	1.3590	1.2932	1.2383	1.1915	1.1507	1.1148
0.6	2.3056	1.9965	1.8149	1.6878	1.5909	1.5131	1.4485	1.3936	1.3459	1.3039
0.7	2.7731	2.3792	2.1566	2.0031	1.8872	1.7947	1.7183	1.6534	1.5972	1.5478
0.8	3.3616	2.8605	2.5875	2.4022	2.2633	2.1531	2.0623	1.9854	1.9191	1.8608
0.9	4.0951	3.4600	3.1264	2.9030	2.7368	2.6055	2.4977	2.4067	2.3282	2.2595
1	5.0000	4.1998	3.7940	3.5257	3.3274	3.1716	3.0439	2.9364	2.8439	2.7630

Table 7-5: K(2) values obtained by IMPC in respect to δ and u_r

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.0961	0.9216	0.8147	0.7623	0.7513	0.7683	0.8004	0.8343	0.8568	0.8548
0.2	1.3090	1.1083	0.9756	0.8980	0.8622	0.8550	0.8632	0.8738	0.8735	0.8491
0.3	1.5320	1.3052	1.1473	1.0452	0.9858	0.9557	0.9419	0.9312	0.9105	0.8665
0.4	1.7791	1.5264	1.3438	1.2180	1.1360	1.0846	1.0505	1.0206	0.9818	0.9209
0.5	2.0643	1.7859	1.5789	1.4304	1.3269	1.2555	1.2029	1.1560	1.1016	1.0264
0.6	2.4016	2.0976	1.8669	1.6962	1.5725	1.4826	1.4132	1.3513	1.2837	1.1971
0.7	2.8050	2.4756	2.2216	2.0297	1.8868	1.7798	1.6955	1.6206	1.5421	1.4468
0.8	3.2886	2.9339	2.6570	2.4447	2.2838	2.1611	2.0636	1.9779	1.8910	1.7896
0.9	3.8663	3.4865	3.1872	2.9553	2.7775	2.6406	2.5316	2.4372	2.3442	2.2396
1	4.5521	4.1474	3.8262	3.5754	3.3818	3.2322	3.1135	3.0124	2.9159	2.8107

Table 7-6: K(2) values obtained by PUrIC in respect to δ and u_r

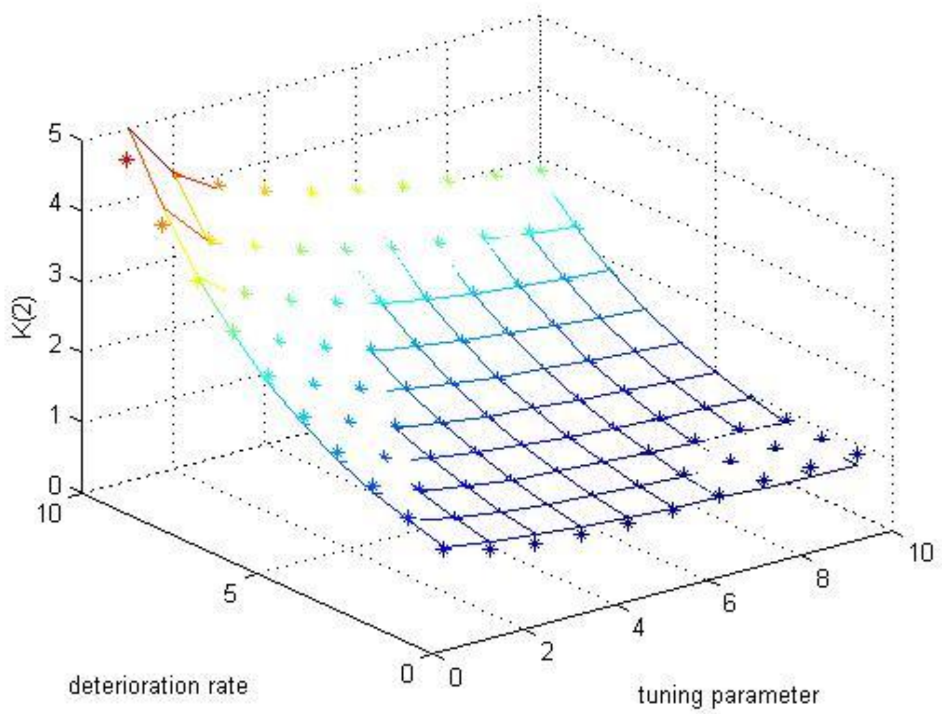


Figure 7-13: IMPC points – solid line and PUrIC points – starred line for K(2)

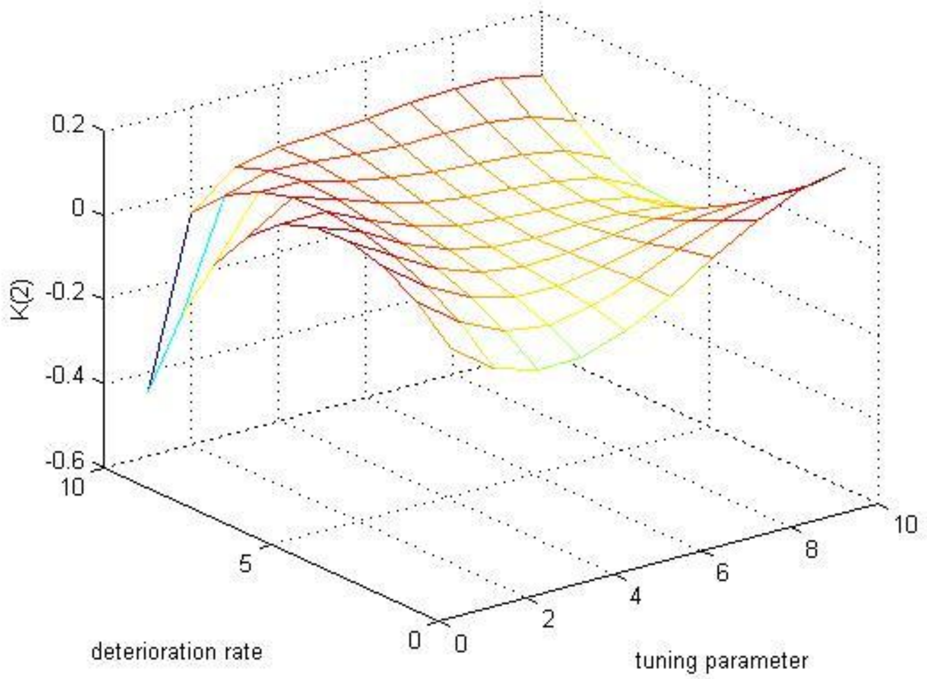


Figure 7-14: Difference between IMPC and PUrIC for K(2)

As can be observed from tables and figures, the estimated values are more accurate for the middle range of the deterioration rate and tuning parameter.

Table 7-7 shows the $K(3)$ values obtained by IMPC in respect to δ and u_r. Table 7-8 shows the $K(3)$ values obtained by PUrIC in respect to δ and u_r. Figure 7-15 presents the shape of the plane of the points presented in Table 7-7 (solid line) and Table 7-8 (starred line). Figure 7-16 presents the difference between the IMPC and PUrIC values (note the difference in axes scales between last two figures).

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1110	1.0100	0.9379	0.8824	0.8375	0.8001	0.7681	0.7403	0.7158	0.6940
0.2	1.2480	1.1248	1.0401	0.9761	0.9251	0.8828	0.8470	0.8160	0.7888	0.7646
0.3	1.4170	1.2653	1.1650	1.0907	1.0322	0.9842	0.9436	0.9087	0.8782	0.8511
0.4	1.6240	1.4365	1.3172	1.2306	1.1631	1.1081	1.0620	1.0224	0.9879	0.9574
0.5	1.8750	1.6431	1.5013	1.4001	1.3220	1.2590	1.2063	1.1613	1.1221	1.0875
0.6	2.1760	1.8902	1.7219	1.6039	1.5137	1.4413	1.3811	1.3298	1.2853	1.2461
0.7	2.5330	2.1827	1.9840	1.8468	1.7430	1.6600	1.5913	1.5330	1.4824	1.4380
0.8	2.9520	2.5256	2.2925	2.1339	2.0148	1.9201	1.8420	1.7758	1.7186	1.6683
0.9	3.4390	2.9240	2.6525	2.4702	2.3343	2.2269	2.1385	2.0637	1.9992	1.9427
1	4.0000	3.3832	3.0693	2.8611	2.7071	2.5858	2.4863	2.4024	2.3301	2.2668

Table 7-7: K(3) values obtained by IMPC in respect to δ and u_r

u_r / δ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1119	0.9529	0.8514	0.7961	0.7757	0.7790	0.7946	0.8113	0.8178	0.8028
0.2	1.2942	1.1139	0.9917	0.9161	0.8759	0.8599	0.8566	0.8550	0.8435	0.8110
0.3	1.4917	1.2904	1.1478	1.0528	0.9939	0.9599	0.9396	0.9216	0.8946	0.8473
0.4	1.7095	1.4872	1.3248	1.2111	1.1346	1.0842	1.0484	1.0161	0.9759	0.9166
0.5	1.9524	1.7094	1.5277	1.3960	1.3030	1.2375	1.1881	1.1436	1.0926	1.0239
0.6	2.2254	1.9618	1.7612	1.6124	1.5041	1.4249	1.3636	1.3089	1.2495	1.1740
0.7	2.5335	2.2495	2.0305	1.8654	1.7428	1.6514	1.5799	1.5170	1.4515	1.3721
0.8	2.8817	2.5774	2.3405	2.1598	2.0240	1.9218	1.8419	1.7730	1.7038	1.6230
0.9	3.2749	2.9505	2.6962	2.5007	2.3528	2.2412	2.1546	2.0817	2.0111	1.9317
1	3.7181	3.3737	3.1024	2.8930	2.7341	2.6146	2.5230	2.4480	2.3785	2.3031

Table 7-8: K(3) values obtained by PUrIC in respect to δ and u_r

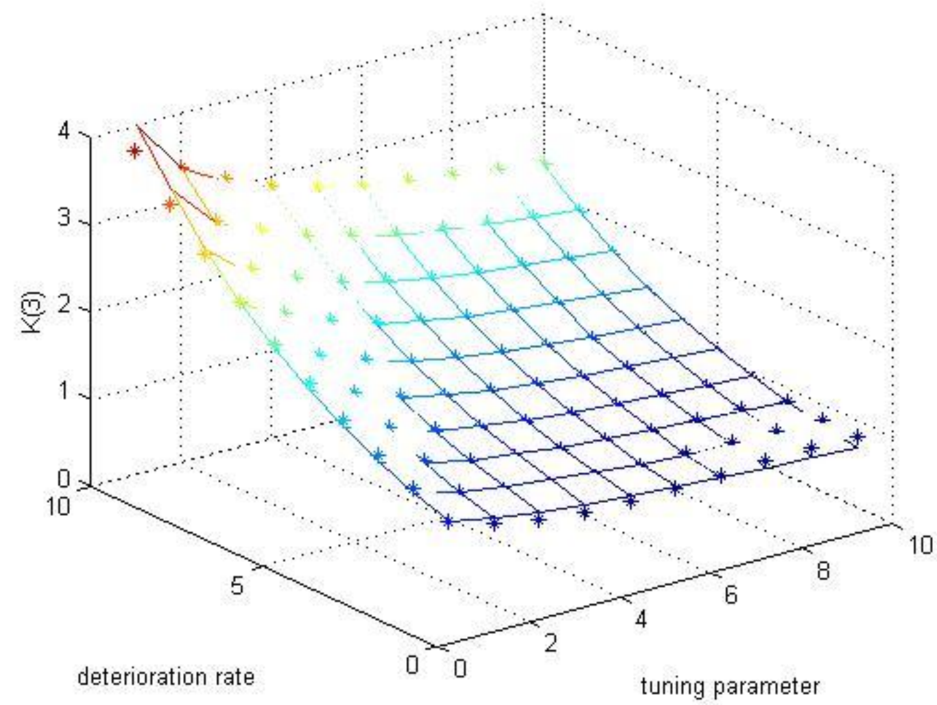


Figure 7-15: IMPC points – solid line and PUrIC points – starred line for K(3)

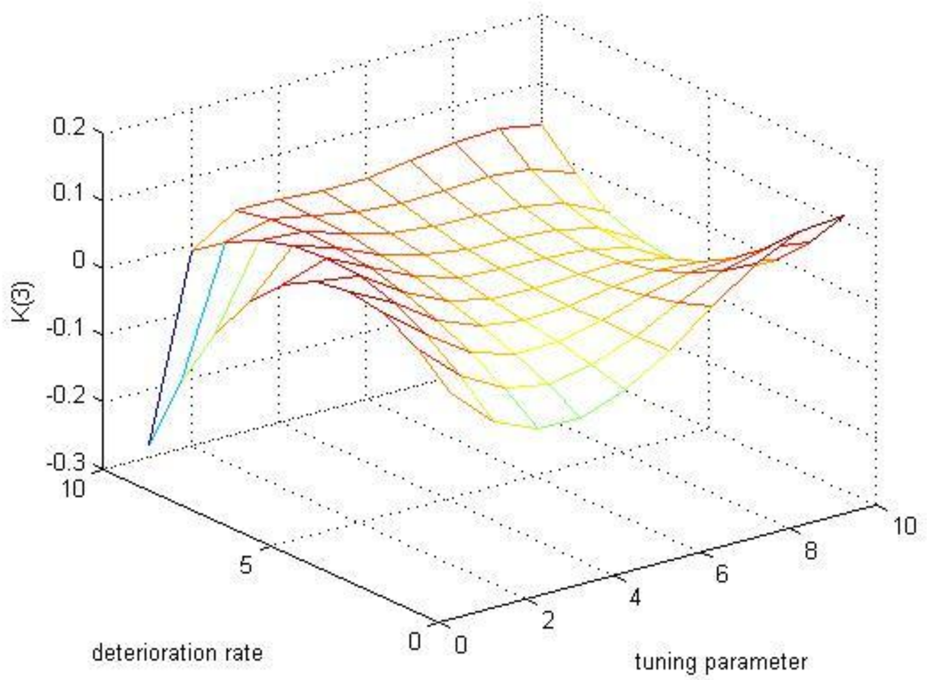


Figure 7-16: Difference between IMPC and PUrIC for K(3)

As can be observed from tables and figures, the estimated values are more accurate for the middle values of deterioration rate and tuning parameter.

Table 7-9 shows the $K(4)$ values obtained by IMPC in respect to δ and u_r. Table 7-10 shows the $K(4)$ values obtained by PUrIC in respect to δ and u_r. Figure 7-17 presents the shape of the plane of the points presented in Table 7-9 (solid line) and Table 7-10 (starred line). Figure 7-18 presents the difference between the IMPC and PUrIC values (note the difference in axes scales between last two figures).

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1100	1.0092	0.9372	0.8817	0.8369	0.7995	0.7676	0.7399	0.7154	0.6936
0.2	1.2400	1.1182	1.0343	0.9710	0.9204	0.8786	0.8430	0.8123	0.7853	0.7613
0.3	1.3900	1.2430	1.1456	1.0734	1.0164	0.9696	0.9301	0.8961	0.8663	0.8398
0.4	1.5600	1.3837	1.2713	1.1894	1.1255	1.0734	1.0296	0.9920	0.9592	0.9301
0.5	1.7500	1.5404	1.4116	1.3194	1.2482	1.1905	1.1423	1.1009	1.0649	1.0331
0.6	1.9600	1.7130	1.5669	1.4639	1.3851	1.3217	1.2688	1.2236	1.1844	1.1498
0.7	2.1900	1.9018	1.7375	1.6235	1.5369	1.4675	1.4100	1.3610	1.3184	1.2810
0.8	2.4400	2.1069	1.9237	1.7984	1.7041	1.6288	1.5666	1.5137	1.4679	1.4277
0.9	2.7100	2.3284	2.1259	1.9893	1.8872	1.8061	1.7393	1.6826	1.6337	1.5907
1	3.0000	2.5665	2.3445	2.1966	2.0868	2.0000	1.9287	1.8684	1.8164	1.7707

Table 7-9: K(4) values obtained by IMPC in respect to δ and u_r

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1231	0.9812	0.8856	0.8271	0.7968	0.7854	0.7841	0.7836	0.7750	0.7491
0.2	1.2726	1.1159	1.0057	0.9332	0.8891	0.8645	0.8503	0.8373	0.8166	0.7791
0.3	1.4316	1.2601	1.1359	1.0499	0.9931	0.9564	0.9307	0.9070	0.8763	0.8293
0.4	1.6006	1.4144	1.2765	1.1777	1.1091	1.0616	1.0260	0.9933	0.9545	0.9004
0.5	1.7801	1.5794	1.4282	1.3173	1.2378	1.1806	1.1366	1.0967	1.0518	0.9929
0.6	1.9706	1.7555	1.5914	1.4692	1.3798	1.3141	1.2631	1.2176	1.1688	1.1073
0.7	2.1727	1.9434	1.7668	1.6338	1.5354	1.4625	1.4060	1.3568	1.3059	1.2442
0.8	2.3870	2.1436	1.9549	1.8118	1.7054	1.6264	1.5659	1.5147	1.4638	1.4042
0.9	2.6139	2.3565	2.1562	2.0038	1.8902	1.8064	1.7433	1.6919	1.6431	1.5877
1	2.8541	2.5829	2.3712	2.2101	2.0903	2.0029	1.9388	1.8889	1.8441	1.7954

Table 7-10: K(4) values obtained by PUrIC in respect to δ and u_r

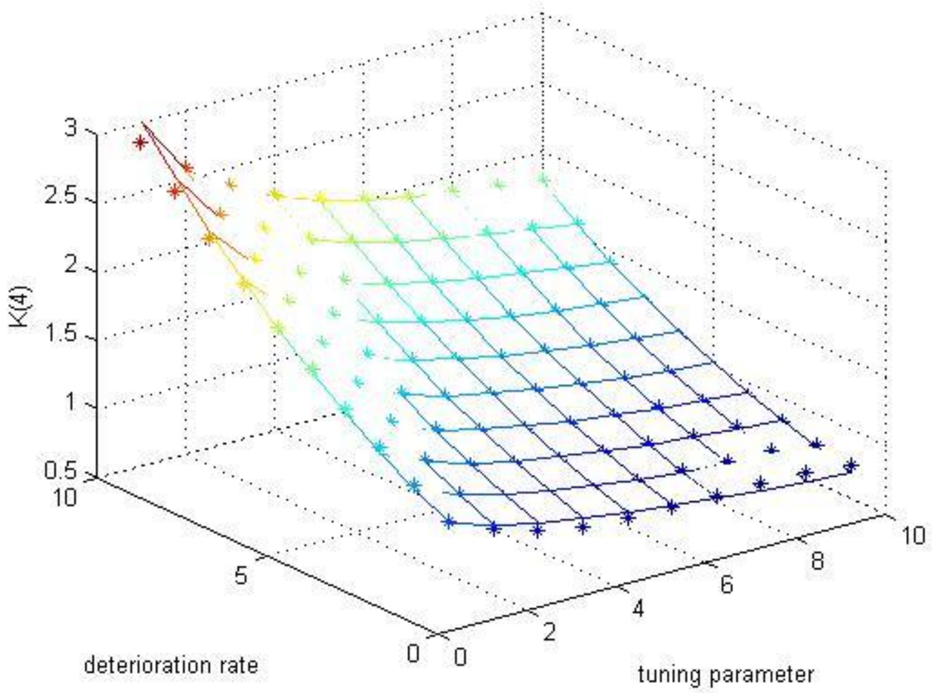


Figure 7-17: IMPC points – solid line and PUrIC points – starred line for K(4)

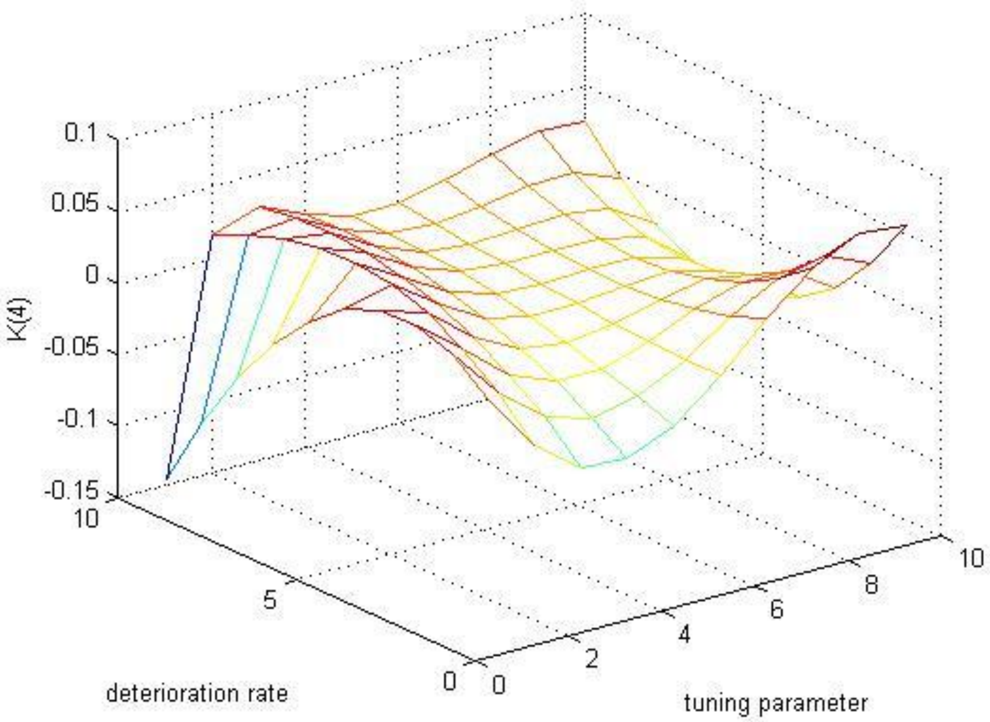


Figure 7-18: Difference between IMPC and PUrIC for K(4)

As can be observed from tables and figures, the estimated values are more accurate for the middle range of deterioration rate and tuning parameter.

Table 7-11 shows the $K(5)$ values obtained by IMPC in respect to δ and u_r. Table 7-12 shows the $K(5)$ values obtained by PUrIC with respect to δ and u_r. Figure 7-19 presents the shape of the plane of the points presented in Table 7-11 (solid line) and Table 7-12 (starred line). Figure 7-20 presents the difference between the IMPC and PUrIC values (note the difference in axes scales between last two figures).

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1000	1.0008	0.9300	0.8753	0.8311	0.7942	0.7627	0.7353	0.7111	0.6895
0.2	1.2000	1.0849	1.0055	0.9452	0.8971	0.8571	0.8232	0.7938	0.7679	0.7448
0.3	1.3000	1.1685	1.0809	1.0155	0.9638	0.9212	0.8851	0.8539	0.8266	0.8022
0.4	1.4000	1.2518	1.1564	1.0864	1.0316	0.9866	0.9487	0.9160	0.8874	0.8620
0.5	1.5000	1.3348	1.2322	1.1581	1.1005	1.0536	1.0141	0.9802	0.9506	0.9243
0.6	1.6000	1.4177	1.3084	1.2308	1.1708	1.1222	1.0815	1.0466	1.0162	0.9892
0.7	1.7000	1.5006	1.3853	1.3044	1.2425	1.1926	1.1509	1.1153	1.0842	1.0567
0.8	1.8000	1.5836	1.4627	1.3791	1.3157	1.2647	1224	1.1862	1.1547	1.1268
0.9	1.9000	1.6667	1.5408	1.4550	1.3903	1.3386	1.2957	1.2592	1.2275	1.1995
1	2.0000	1.7499	1.6197	1.5321	1.4664	1.4142	1.3711	1.3344	1.3026	1.2746

Table 7-11: K(5) values obtained by IMPC in respect to δ and u_r

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.1173	0.9977	0.9112	0.8513	0.8119	0.7865	0.7690	0.7530	0.7323	0.7006
0.2	1.2141	1.0877	0.9945	0.9283	0.8826	0.8514	0.8282	0.8067	0.7808	0.7441
0.3	1.3098	1.1766	1.0770	1.0047	0.9535	0.9170	0.8890	0.8632	0.8332	0.7929
0.4	1.4044	1.2645	1.1587	1.0808	1.0245	0.9836	0.9516	0.9224	0.8896	0.8470
0.5	1.4979	1.3513	1.2396	1.1565	1.0958	1.0510	1.0160	0.9844	0.9500	0.9065
0.6	1.5904	1.4373	1.3199	1.2320	1.1673	1.1195	1.0823	1.0494	1.0146	0.9715
0.7	1.6820	1.5224	1.3996	1.3073	1.2392	1.1890	1.1505	1.1173	1.0833	1.0420
0.8	1.7727	1.6068	1.4787	1.3824	1.3115	1.2597	1.2207	1.1883	1.1562	1.1181
0.9	1.8627	1.6904	1.5574	1.4575	1.3843	1.3316	1.2931	1.2624	1.2334	1.1998
1	1.9520	1.7734	1.6357	1.5325	1.4576	1.4047	1.3675	1.3397	1.3151	1.2873

Table 7-12: K(5) values obtained by PUrIC in respect to δ and u_r

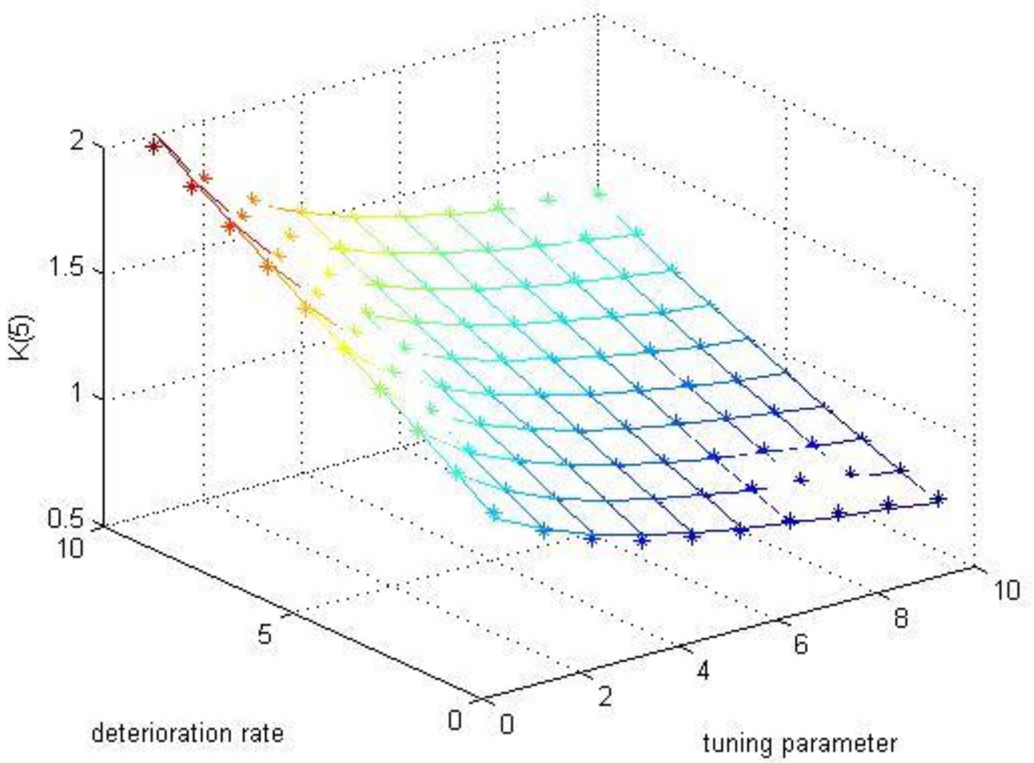


Figure 7-19: IMPC points – solid line and PUrIC points – starred line for K(5)

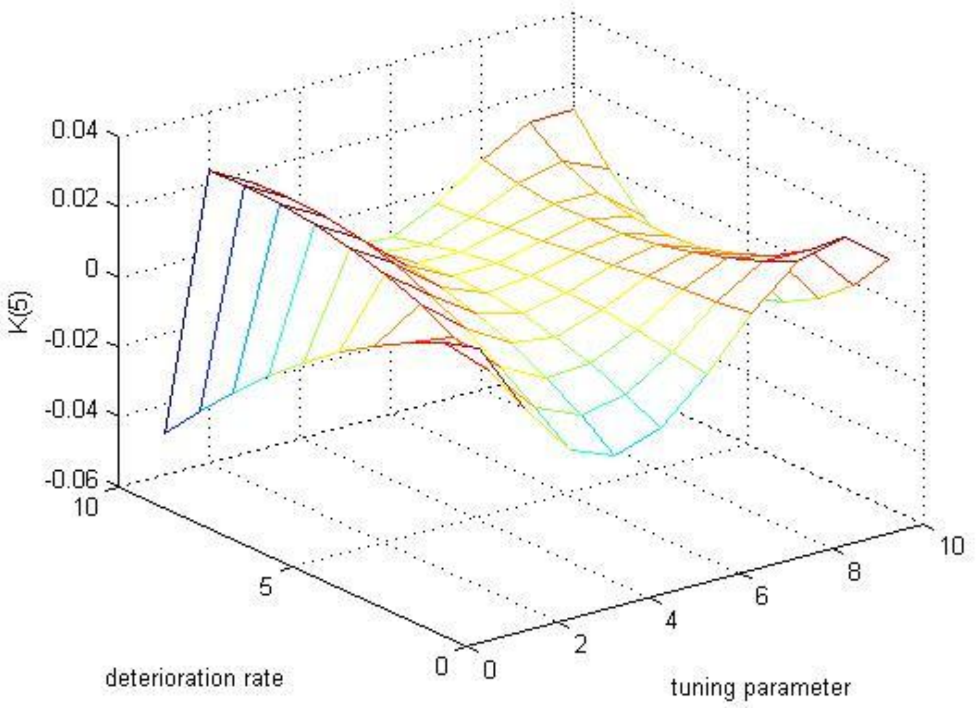


Figure 7-20: Difference between IMPC and PUrIC for K(5)

As can be observed from tables and figures, the estimated values are more accurate for the middle range of deterioration rate and tuning parameter.

Table 7-13 shows the $K(6)$ values obtained by IMPC in respect to δ and u_r. Table 7-14 shows the $K(6)$ values obtained by PUrIC with respect to δ and u_r. Figure 7-21 presents the shape of the plane of the points presented in Table 7-13 (solid line) and Table 7-14 (starred line). Figure 7-22 presents the difference between the IMPC and PUrIC values.

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.0000	0.9091	0.8442	0.7942	0.7538	0.7201	0.6913	0.6663	0.6443	0.6246
0.2	1.0000	0.9012	0.8332	0.7819	0.7410	0.7071	0.6784	0.6535	0.6317	0.6123
0.3	1.0000	0.8924	0.8213	0.7687	0.7273	0.6933	0.6646	0.6399	0.6183	0.5992
0.4	1.0000	0.8830	0.8087	0.7548	0.7129	0.6787	0.6501	0.6256	0.6042	0.5853
0.5	1.0000	0.8729	0.7955	0.7403	0.6979	0.6637	0.6352	0.6108	0.5896	0.5710
0.6	1.0000	0.8624	0.7818	0.7255	0.6826	0.6483	0.6199	0.5957	0.5747	0.5563
0.7	1.0000	0.8514	0.7678	0.7104	0.6671	0.6327	0.6043	0.5803	0.5595	0.5413
0.8	1.0000	0.8400	0.7535	0.6951	0.6515	0.6170	0.5887	0.5648	0.5442	0.5262
0.9	1.0000	0.8284	0.7392	0.6798	0.6359	0.6014	0.5731	0.5493	0.5289	0.5111
1	1.0000	0.8166	0.7248	0.6645	0.6203	0.5858	0.5576	0.5340	0.5138	0.4961

Table 7-13: K(6) values obtained by IMPC in respect to δ and u_r

δ \ u_r	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.0078	0.9058	0.8305	0.7769	0.7400	0.7150	0.6968	0.6805	0.6611	0.6338
0.2	1.0087	0.9038	0.8257	0.7694	0.7300	0.7025	0.6819	0.6634	0.6420	0.6126
0.3	1.0079	0.9001	0.8192	0.7604	0.7187	0.6891	0.6666	0.6464	0.6235	0.5928
0.4	1.0053	0.8946	0.8112	0.7501	0.7063	0.6750	0.6511	0.6297	0.6058	0.5746
0.5	1.0010	0.8875	0.8016	0.7384	0.6929	0.6602	0.6353	0.6132	0.5891	0.5579
0.6	0.9952	0.8790	0.7908	0.7256	0.6787	0.6449	0.6194	0.5972	0.5734	0.5430
0.7	0.9881	0.8691	0.7786	0.7118	0.6637	0.6293	0.6036	0.5818	0.5589	0.5299
0.8	0.9796	0.8579	0.7654	0.6971	0.6480	0.6133	0.5880	0.5671	0.5457	0.5187
0.9	0.9700	0.8457	0.7511	0.6815	0.6319	0.5972	0.5726	0.5532	0.5338	0.5097
1	0.9594	0.8324	0.7360	0.6653	0.6153	0.5811	0.5577	0.5401	0.5235	0.5029

Table 7-14: K(6) values obtained by PUrIC in respect to δ and u_r

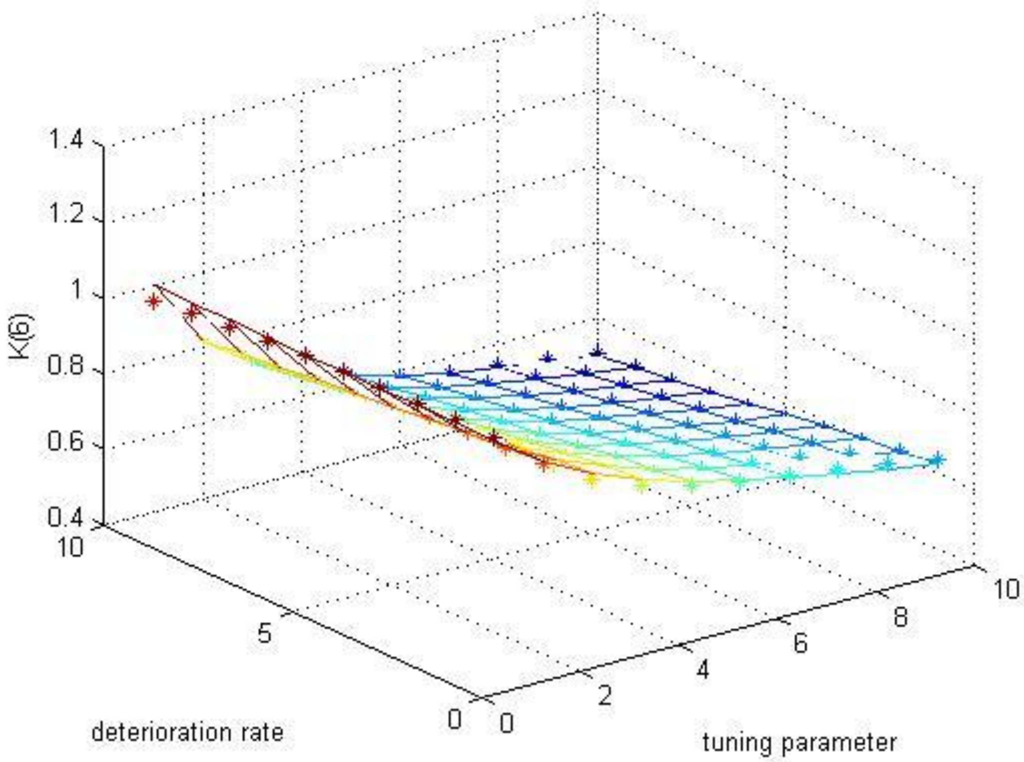


Figure 7-21: IMPC points – solid line and PUrIC points – starred line for K(6)

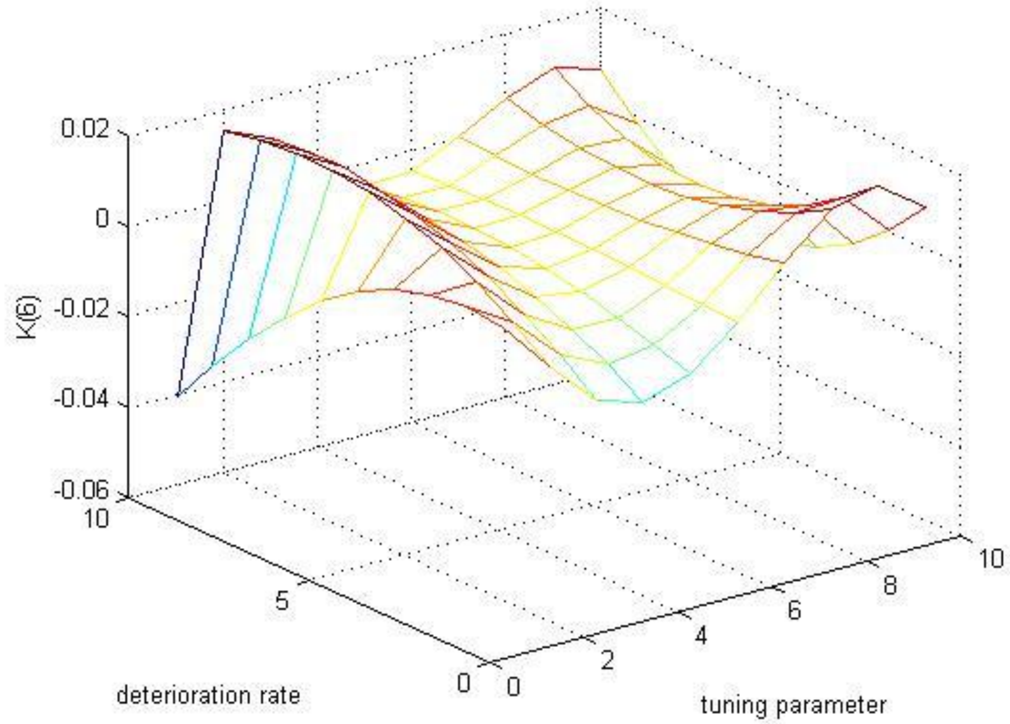


Figure 7-22: Difference between IMPC and PUrIC for K(6)

As can be observed from tables and figures, the estimated values are more accurate for the middle range of deterioration prate and tuning parameter.

The simulation of inventory levels was run for the demand pattern presented in Figure 7-1 and chosen values of δ and u_r. The Figure 7-23 presents the results for $\delta = 0.7$ and $u_r = 0.1$, $u_r = 0.5$ and $u_r = 0.9$. As it can be observed in Figure 7-23, the results obtained do not significantly differ between each other and would seem to be a good estimation of IMPC results shown in Figure 7-9. Similar tests were conducted for different simulation settings and the conclusion was always similar. Therefore the estimation can be used for inventory control.

The IMPC state space model and IMPC gains have the forms as follows.
From (3.3) and (3.4)

$$\boldsymbol{x}\left[(k+1)T\right] = \begin{bmatrix} 0.7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(kT) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(kT) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d(kT) \qquad (7.20)$$

$$y(kT) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(kT) \qquad (7.21)$$

From (3.64) or (3.66)

for $u_r = 0.1$ $\quad \boldsymbol{K} = \begin{bmatrix} 1.6655 & 2.3792 & 2.1827 & 1.9018 & 1.5006 & 0.8514 \end{bmatrix}$

(7.22)

for $u_r = 0.5$ $\quad \boldsymbol{K} = \begin{bmatrix} 1.2563 & 1.7947 & 1.6600 & 1.4675 & 1.1926 & 0.6327 \end{bmatrix}$

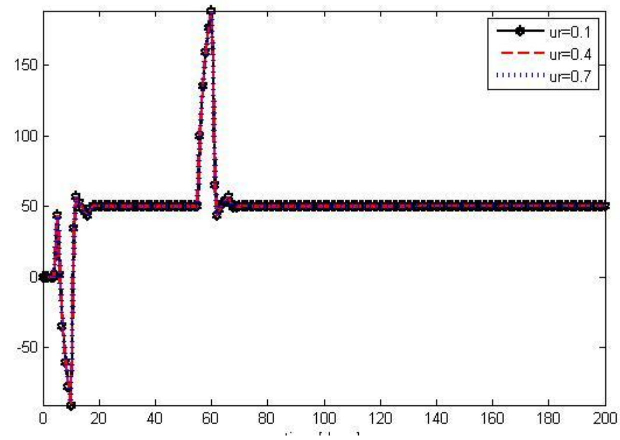
(7.23)

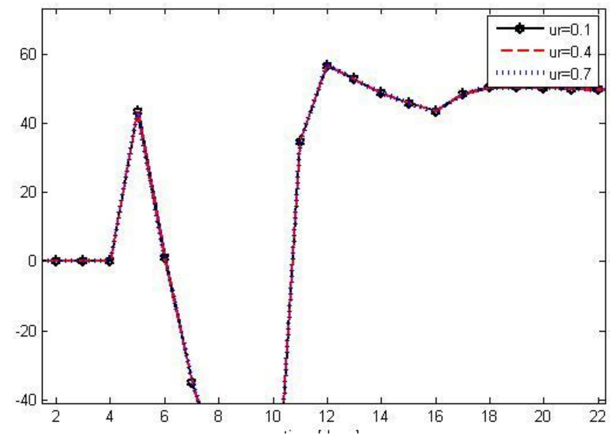
for $u_r = 0.9$ $\quad \boldsymbol{K} = \begin{bmatrix} 1.0835 & 1.5479 & 1.4380 & 1.2810 & 1.0567 & 0.5413 \end{bmatrix}$ (7.24)

Gains of PUrIC method were obtained from (7.12) and (7.14) as follows

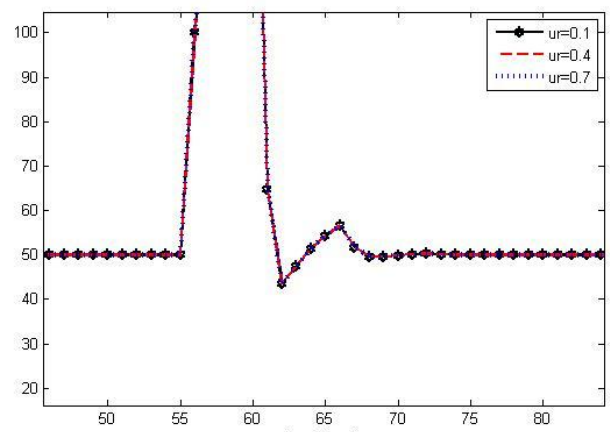
for $u_r = 0.1$ $\quad \boldsymbol{K} = \begin{bmatrix} 1.7757 & 2.4756 & 2.2494 & 1.9434 & 1.5224 & 0.8691 \end{bmatrix}$ (7.25) for $u_r = 0.5$ $\quad \boldsymbol{K} = \begin{bmatrix} 1.2353 & 1.7798 & 1.6514 & 1.4625 & 1.1890 & 0.6293 \end{bmatrix}$ (7.26)

for $u_r = 0.9$ $\quad \boldsymbol{K} = \begin{bmatrix} 0.9533 & 1.4468 & 1.3721 & 1.2442 & 1.0420 & 0.5299 \end{bmatrix}$ (7.27)





Zoomed results for time instances of 0-22 days



Zoomed results for time instances of 45-90 days

Figure 7-23: Inventory levels obtained with PUrIC

7.3 Summary

The chapter has investigated the possible directions, which can be taken for making the model sensitive to the tuning parameter being different to zero for various order sizes. Two separate paths have been chosen for finding simplified approaches requiring little knowledge of control and aimed at the OR familiar specialist: for non-perishable and perishable products. Neither of them are mathematically equivalent to the original IMPC with the tuning parameter being different from zero, but they do provide sufficiently accurate estimations and would appear to perform sufficiently similarly to IMPC with the tuning parameter different to zero. The mathematical equivalent formulation was not developed due to the much reduced regularity of the description, hence lack of complex mathematical description. Therefore the initial idea of searching for a mathematically equivalent form which is actually simpler in description than the original IMPC method with tuning parameter was relaxed.

What is also an important conclusion of the current chapter is the fact that the difference in the results obtained between the IMPC with and without the tuning parameter different to zero is usually insignificant for the inventory application considered. Therefore, there was not a strong need to use the alternative methods for the non-control familiar practitioners. Nevertheless, whether applied or not, the proposed methods of the current chapter can become an inspiration for future extensions of the IC model.

8 CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE DIRECTIONS

8.1 Conclusions

The thesis has presented a novel control approach to inventory management. The common approaches to inventory modelling and optimisation, which are widely used by the operational research (OR) community usually involve discrete event simulation, and classical mathematical methods of optimisation. Discrete event simulation approaches enable consideration of many perspectives influencing the system at the same time. Nevertheless, it is a quasi-optimisation approach with the optimisation only through comparison of different scenarios coupled with sensitivity analyses. The results obtained can improve the performance of the system but are not necessarily optimal in practice. The classical mathematical methods, in turn, enable optimal results to be achieved but the consideration of system dynamics becomes extremely complex in terms of a mathematical description. Therefore, such models are usually constructed under unrealistic assumptions, such as, for example, that the customer demand is at least partially known in advance. In practice such models are applicable only in a very few and narrow group of industries, where the business structure requires pre-paid transactions, such as e.g. in the case of the car manufacturing industry. In most of the real-life case scenarios, the demand is uncertain and unpredictable, and the development of a model, which could deal with such an unknown demand rather than not accurately predicting it, would be beneficial for many industrial organisations. Furthermore, the mathematical optimisation models often over-simplify the real case problem through unrealistic assumptions such as zero lead time and/or constant deterioration rate of a product.

In this thesis, a state space representation of an inventory system has been constructed, which enabled a straightforward accommodation/design of the lead time as well as inventory deterioration (both are included in the system state matrix), for a discrete-time system or as a system delay and deterioration coefficient for a continuous-time system. The developed approach also enabled the modelling of the unknown system demand in the form of a system disturbance.

Appropriate selection of the closed loop system, where the current inventory is on-line compared on-line with a target inventory level, enabled the consideration of system dynamics related to varying and previously unknown demand. The feedback loop itself enables the updating of the current information and the appropriate choice of control algorithm enables on-line optimisation based on the current inventory level. In this way, instead of predicting the demand and then conducting an optimisation based on the potential inaccurate prediction, the system deals with the actual demand changes (modelled as a system disturbance) and adapts to them at every subsequent time instance. The approach described is that of applied control theory, and this forms the pivotal step in this thesis.

A popular approach within the OR community usually relates to obtaining an optimal inventory level (for inventory keeping benefits and cost balance) based on the estimation of demand. The presence of inventory fluctuations are usually not normally of any concern to the OR researcher. The control theoretic approach, on the other hand, aims at keeping the inventory as close as possible to the reference point. In this way the inventory level fluctuations are automatically reduced and several problems related to varying inventory levels can thus be avoided. Fluctuating inventory levels are accompanied by the attendant problem of storage capacity. Either the storage space becomes unused and, as a consequence, unnecessarily maintained, or the storage capacity becomes exhausted. Moreover, the problem of planning human resources becomes a complex issue with fluctuating inventory. This leads to there being either too little or too much manpower in the warehouse. The same concerns apply to capital equipment and machinery. The issue of product deterioration also becomes a controllable problem if the inventory is fluctuating. Additionally, backorders occur more often, which also generate additional cost. All of these unnecessary costs might be mitigated, if systematic approaches of control engineering could be deployed. Moreover, inventory level fluctuations of one supply chain can easily affect many others in the form of a bullwhip instability effect. It is commonly understood that the higher the fluctuations the more the cost in terms of lost revenue in a given particular supply chain node as well as in those connected. Ideally, the supply chain, even if not owned by the same organisation, should collaborate for the purpose of the common goal. Therefore, the reduction of varying inventory in one supply chain node will in practice bring about cost savings for all connected supply chain members. Indeed, it is against this background that the motivation for conducting the work in this thesis has been based.

In the thesis a number of novel algorithms have been developed for inventory control purposes. Initially, based on the principles of model based control, the Smith predictor was a first attempt for a continuous system scenario and was aimed at reducing the negative influence of the lead time in an inventory system, which is referred to here as a system delay. The continuous system design considered here was straightforward. It referred to the well documented and established case in the OR literature of the balance equation. This is where the future inventory level is equal to the current, but decreased by the number of deteriorated products, plus the number of goods which have been ordered in the past and delivered to the warehouse, minus the number of goods being sold. In this scenario, the inventory levels are changing (fluctuating) smoothly, but were never kept particularly close to an inventory reference level. Moreover, the approach considers deliveries and demands as continuous variables, which is usually an unrealistic assumption in practice. Unless some specific application of a continuous inventory control system can be imagined, the inventory level control systems are more truthfully, at least in the case of periodic review, more of a discrete-time sampled data nature, being controlled at every time instance, e.g. each day.

The next control algorithm considered was the dead-beat model based controller, whose mathematical formulation enabled the verification of the logic of the periodic review inventory state space representation. This way the discrete-time inventory model was demonstrated to be correctly constructed which made it a valuable stepping stone in the development and evaluation of other model based control techniques and applications. Attention was then given to model predictive control (MPC), which by definition potentially incorporates both the dead-beat and Smith predictor actions as inherent constituent parts. It was applied to the previously verified discrete inventory system. This approach enabled efficient control of inventory levels with respect to specification of the reference set point, and a satisfactory cost reduction for industrial purposes, has been elaborated.

Although the application of control theory to inventory control seems to be extremely beneficial with respect to on-line optimisation, keeping inventory at the target level and consideration of system dynamics (varying demand, lead time delay and varying deterioration rate) it is surprising that such an approach does not appear to have gained significant attention. A possible reason for this could be due to the fact that the OR community, which dedicates much of its work and attention to inventory performance improvement, is typically unfamiliar with control theory. Control theory is in itself an extensive branch of science / engineering and as such does not appear in curriculum of taught OR courses. This identified gap prompts the need to build a communication bridge between the OR and control theory communities; thus making the techniques of control theory accessible to researchers in OR who are non-control specialists, via a transformation, providing a link and a better understanding of a common mathematical framework. In response to this need, the thesis has presented what could be described as the first step in this bridge building process.

The developed method was obtained via a mathematical reformulation of the MPC approach used for the specific inventory state space model, which was subsequently termed IMPC in the thesis, for the case when the tuning parameter within the MPC was set equal to zero. The proposed approach which has been developed in this thesis is termed the inventory controller (IC). It has been evaluated and mathematically demonstrated to be equivalent to the IMPC approach. Particular attention was paid to the mathematical simplification which specifically relates to the inventory problem. It has been shown that the technique is applicable to problems, where the state space model has a particular form, regardless whether it is an inventory problem or indeed something else. The IMPC results for the case when the tuning parameter differs from zero, have also been presented and shown to provide a near, but not exact, mathematical equivalence. In the development of the mathematically reduced approach, the MPC algorithm displays an independence towards changes in the prediction and control horizons, effectively being a non-predictive scheme. Nevertheless, the developed simplified method of IC, being equivalent to IMPC for the non-tuning parameter case, brings the advantage of straightforward application for the non-control familiar OR society. It generates satisfactory results from an industrial warehouse point of view as well as bridging the gap between the mathematical precision of control theory and the expectation of OR practitioners in industry.

The IC method represents a first step towards converting the MPC to a non-control engineering form, and only the basic case, when the tuning parameter was set to zero, has been considered here. For those, who would be interested to continue obtaining simplified methods for the non-zero tuning parameter case, several findings and guidelines have been presented in the thesis. There is probably little to be gained in the case of the current application (the inventory control problem), as the difference in the results was found to be insignificant when the tuning parameter was considered. Nevertheless, if there is an interest in applying the technique for systems which are analogous to the state space representation, consideration of the tuning parameter might appear beneficial.

It is concluded, therefore, that the developed IC technique can be applied to other problems of OR within industrial applications, provided the state space model can be represented in a similar way. Several possible further applications and methods of modelling the system variables are proposed below.

8.2 Recommendations for future directions

Consider a fish breeding pond. The system output would refer here to the number of adult fish, the deterioration factor could refer to the number of individuals, which have died. The pond is periodically replenished / supplied by new born entities of an optimal quantity (the control variable), which become adults following a constant delay. The disturbance here could refer to the number of individuals which have been sold. The benefits of the application of such a model can be understood as follows. Maintaining too many fish in the pond can create unnecessary costs related to feeding as well as a higher risk of fish becoming unhealthy due to over population. Too little, on the other hand, can bring the risk of not satisfying demand.

Consider a greenhouse as another application. The system output would represent the number of fully grown plants which are ready for sale, deterioration rate would represent the number of plants, which have reached maturity and/or are of poor quality. The control variable would refer to optimal number of small plants, or seeds, which could be sown to grow and be ready for sale after a certain delay. The disturbance could refer to the number of plants which have been harvested for sale.

Finally, consider a single product manufacturing, assembling, or packing line, which consumes a certain amount of time (system delay) to complete the process. The system output would refer to number of finished goods, deterioration rate would refer to the number of goods which are faulty and are not exploitable. The control variable could refer to the optimal amount of raw materials, which should be delivered to the production line to maintain the finished goods inventory level. The disturbance could represent the number of goods being sold by the manufacturing plant.

The above examples enlighten the applicability rang of the developed control approach to the traditional OR inventory problems.

8.3 Overall Summary

The main contribution of the thesis has been that of realising that the MPC formulation itself has an exploitable form and a special pattern based upon which a mathematical reduction can be deduced. In the case of a prior known state space model, the mathematical reformulation can be carried out based on an utilisation of the model parameter values. The demonstration of the mathematical equivalency between the IMPC and IC methods was shown in the thesis in a systematic step-by-step form. Starting from recognition the key patterns in the mathematical description, each of the IMPC gain components have been developed and separately demonstrated via a sequence of mathematical propositions. The simplified forms of the IMPC gain components are then combined to give the final alternative form of the IMPC. If there is further interest in conducting similar reformulations of MPC in other areas of OR, the sequence of propositions and their demonstrations presented in the thesis should provide a set of guidelines for further research and development in bridging the gap between the OR and control communities.

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LIST OF TABLES:

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Appendix I – Guidelines for demonstration of *Proposition 5* and *Proposition 11*

Proposition 5 is a special case of *Proposition 11* in this case of $\delta^i = 1$, therefore demonstrating *Proposition 11* is sufficient for demonstrating both propositions. The demonstration of *Proposition 11* is extremely extensive in description, hence it refers to the multiplication of $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}$ only, to ensure that the identity matrix is obtained as a result of the multiplication. Here, the guidelines for demonstrating the procedure are elaborated.

Firstly, define a matrix such that

$$\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,(N_p-N_c)} \\ a_{21} & a_{22} & \cdots & a_{2,(N_p-N_c)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{(N_p-N_c),1} & a_{(N_p-N_c),2} & \cdots & a_{(N_p-N_c)(N_p-N_c)} \end{bmatrix}$$

As both matrices, i.e. $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)$ and $\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}$ are symmetric, their product is symmetric too and it can be denoted as follows

$$\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)\left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,(N_p-N_c)} \\ a_{12} & a_{22} & \cdots & a_{2,(N_p-N_c)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,(N_p-N_c)} & a_{2,(N_p-N_c)} & \cdots & a_{(N_p-N_c)(N_p-N_c)} \end{bmatrix}$$

From (5.70) in *Proposition 11* and (5.61) it mey be deduced that

$$\alpha_{11} = \sum_{j=0}^{N_p-n}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) +$$

$$+(-1-\delta)\sum_{j=0}^{N_p-n-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) + \delta\sum_{j=0}^{N_p-n-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right)$$

It must now be shown that a_{11} as well as every other element on the diagonal are equal to 1. It must alo be shown that any element which is not located in the diagonal is qual to zero.

Firstly, the demonstration must be conducted separately for elements a_{11}, a_{12} and a_{22} in an analogous way to the following:

Define $z = N_p - n$. Assuming $z \geq 2$

$$\alpha_{11} = \sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) + \left(-1-\delta\right)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) + \delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right)$$

Assume that $z = 2$, then

$$\alpha_{11} = \sum_{j=0}^{2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) + \left(-1-\delta\right)\sum_{j=0}^{1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) + \delta\sum_{j=0}^{0}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right) =$$

$$= 1 + \sum_{i=0}^{1}\delta^i\sum_{i=0}^{1}\delta^i + \sum_{i=0}^{2}\delta^i\sum_{i=0}^{2}\delta^i + \left(-1-\delta\right)\left(\sum_{i=0}^{1}\delta^i + \sum_{i=0}^{1}\delta^i\sum_{i=0}^{2}\delta^i\right) + \delta\sum_{i=0}^{2}\delta^i =$$

$$= 1 + \left(1+\delta\right)\left(1+\delta\right) + \left(1+\delta+\delta^2\right)\left(1+\delta+\delta^2\right) + \left(-1-\delta\right)\left[\left(1+\delta\right) + \left(1+\delta\right)\left(1+\delta+\delta^2\right)\right] + \delta\left(1+\delta+\delta^2\right)$$

$$= 3 + 5\delta + 5\delta^2 + 3\delta^3 + \delta^4 - 2 - 5\delta - 5\delta^2 - 3\delta^3 - \delta^4 = 1$$

Assume now that for some $z \geq 2$, that $\alpha_{11} = 1$. Then for $z+1$ it can be deduced that

$$\alpha_{11} = \sum_{j=0}^{z+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right) +$$

$$+ \left(-1-\delta\right)\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right) + \delta\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right) =$$

$$=\sum_{j=0}^{z}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right)+$$

$$+\left(-1-\delta\right)\sum_{j=0}^{z-1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right)+\delta\sum_{j=0}^{z-2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right)+$$

$$\sum_{i=0}^{z+1}\delta^i\sum_{i=0}^{z+1}\delta^i+\left(-1-\delta\right)\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+1}\delta^i+\delta\sum_{i=0}^{z-1}\delta^i\sum_{i=0}^{z+1}\delta^i$$

Using the assumption

$$\alpha_{11}=1+\sum_{i=0}^{z+1}\delta^i\sum_{i=0}^{z+1}\delta^i+\left(-1-\delta\right)\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+1}\delta^i+\delta\sum_{i=0}^{z-1}\delta^i\sum_{i=0}^{z+1}\delta^i=$$

$$=1+\sum_{i=0}^{z+1}\delta^i\sum_{i=0}^{z+1}\delta^i-\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+1}\delta^i+\delta\left(-\sum_{i=0}^{z}\delta^i\sum_{i=0}^{z+1}\delta^i+\sum_{i=0}^{z-1}\delta^i\sum_{i=0}^{z+1}\delta^i\right)=$$

$$=1+\delta^{z+1}\sum_{i=0}^{z+1}\delta^i+\delta\left[\sum_{i=0}^{z+1}\delta^i\left(\sum_{i=0}^{z-1}\delta^i-\sum_{i=0}^{z}\delta^i\right)\right]=$$

$$1+\delta^{z+1}\sum_{i=0}^{z+1}\delta^i-\delta\left(\delta^z\sum_{i=0}^{z+1}\delta^i\right)=$$

$$=1+\delta^{z+1}\sum_{i=0}^{z+1}\delta^i-\left(\delta^{z+1}\sum_{i=0}^{z+1}\delta^i\right)=$$

$$=1$$

Therefore $a_{11}=1$, regardless of the values of the prediction and control horizons or lead time delay.

Then, for $m,r\in[3,N_p-N_c-3]$ the generic formulation of the a_{mr} element, where m and r denote row and column indices, can be deduced from (5.70) in *Proposition 11* and (5.61).

The generic formulation of the diagonal elements is deduced as follows:

$$a_{mm}=\delta\sum_{j=0}^{z-m}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right)-\left(1+2\delta+\delta^2\right)\sum_{j=0}^{z-m}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right)+$$

$$+(2+2\delta+2\delta^2)\sum_{j=0}^{z-m}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j}\delta^i\right)-\left(1+2\delta+\delta^2\right)\sum_{j=0}^{z-m+1}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+1}\delta^i\right)+\delta\sum_{j=0}^{z-m+2}\left(\sum_{i=0}^{j}\delta^i\sum_{i=0}^{j+2}\delta^i\right)$$

Then, it must be demonstrated that $a_{mm}=1$.

The inductive demonstration can be conducted for the assumption of $z=m+2$ and $z\geq m+2$ separately.

Then, the generic formulation of non-diagonal elements is deduced as follows:

$$a_{mr}=\delta\sum_{j=0}^{z+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i\sum_{i=0}^{j}\delta^i\right)-\left(1+2\delta+\delta^2\right)\sum_{j=0}^{z+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i\sum_{i=0}^{j}\delta^i\right)+$$
$$+\left(2+2\delta+2\delta^2\right)\sum_{j=0}^{z+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i\sum_{i=0}^{j}\delta^i\right)-\left(1+2\delta+\delta^2\right)\sum_{j=0}^{z+1-g}\left(\sum_{i=0}^{j+|m-r|}\delta^i\sum_{i=0}^{j}\delta^i\right)$$
$$where \quad g=\begin{cases} m\, when\, m>r \\ r\, when\, m<r \end{cases}$$

Then, it must be demonstrated that for $m\neq r$ the elements $a_{mr}=0$.

This inductive demonstration can be conducted for the assumption of $z=g+1$ and $z\geq g+1$ separately.

Finally, it should be demonstrated that $a_{(N_p-N_c)(N_p-N_c)}=1$, $a_{(N_p-N_c-1)(N_p-N_c-1)}=1$, $a_{(N_p-N_c-1)(N_p-N_c)}=0$ in the following manner:

$$a_{(N_p-N_c)(N_p-N_c)}=\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-2,S)+b\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S)+\mathrm{c}\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S)$$

$$a_{(N_p-N_c-1)(N_p-N_c-1)}=\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-2,S-1)+\mathrm{b}\cdot\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S-1)+\mathrm{c}\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S)$$

$$a_{(N_p-N_c-1)(N_p-N_c)}=\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-3,S-1)+(-4)\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-2,S)+a\cdot\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S-1,S)+b\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{\Phi}(S,S)$$

where

a, b and c are scalars defined in *Proposition 11* and $S=N_p-N_c$.

Appendix II – List of publications and other academic activities

List of publications:

1. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D., (2013). Inventory Replenishemet Control: A Predicitve Approach. *14th International Carpathian Control Conference*, 2013, 274-279, Sinaia, Romania

2. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D., (2013a). Predictive Control of Perishable Inventory, 55thCanadian Operational Research Society Conference, 2013, Vancover, Canada

3. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D., (2013b). A model Predictive Control Framework for Perishable Inventory Management. *OR55 2013, The OR Society OR55 Conference,* Exeter, UK

4. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D. (2012a). Control Theory Applications In Logistics – MPC And Other Approaches, *Logistyka*, 12(3), 1769 – 1774

5. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D., (2012b). Inventory modelling and management: considerations of a predictive approach. *International Conference on System Engineering2012*, Coventry, UK

6. Orzechowska, J. E., Bartoszewicz, A., Burnham, K. J. and Petrovic, D., (2012c). A Predictive Approach to Inventory Replenishment Decision Support. *The 10thEuropean Workshop on Advanced Control and Diagnosis 2012*, Kopenhagen, Denmark

7. Shen, Z. Burnham, K.J. Orzechowska, J. E. Smalov, L. (2012). Towards Formulatinga Business Process Simulation Model for a Brewery Production System: Preliminary steps. *International Conference on System Engineering2012*, Coventry, UK

8. Orzechowska, J. E., Burnham, K. J. and Petrovic, D., (2011a). An Application of Control Theory to Inventory Systems: A Predictive Approach. *OR53 2011, The OR Society OR53 Conference*, Nottingham, UK

9. Orzechowska, J. E., Burnham, K. J. and Petrovic, D., (2011b). Control Theory Based Approach to Inventory Replenishment Management. *The 9th European Workshop on Advanced Control and Diagnosis 2011*, Budapest, Hungary

Other academic inveolvment:

1. Charing the session of the conference in the 55th Canadian Operational Research Society Conference, 2013

2. Reviewing a paper of Bartoszewicz, A. and Lesniewski, P. titled: ‘Reaching Law Approach to the Sliding Mode Control of Periodic Review Inventory Systems’ for the acceptance for *IEEE Transactions on Automation Science and Engineering 2014*.

3. Reviewing a paper of Lie, G., Zejian, R., Pingshu G.E. and Jing C., titled: ‘Advanced Emergency Braking Controller Design for Pedestrian Protection Oriented Automotive Collision Avoidance System’ for the acceptance in *The Scientific World Journal*.