On the formal justification of informal proofs

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Abstract: When modeling the informal proofs of Euclid's *Elements* using the sound logical system *E*, we go from proofs seen as somewhat nonrigorous – even having gaps to be filled – to rigorous proofs. According to the 'standard view', the correctness of an informal proof is underwritten by the existence of a corresponding formal derivation. However, metalogic grounds the soundness of the logical system *E*, and proofs in metalogic are not like formal proofs and look suspiciously like informal proofs. In our view, they are informal proofs. This brings about what we are calling here the groundedness problem: how can we show with certainty that our metalogical proofs are correct and sustain our logical system? According to the 'standard view', we cannot. In this way, we would have to doubt the soundness of the formal system *E*. This in turn might lead us to doubt the justification of Euclidean informal proofs in terms of the corresponding formal proofs in *E*.

1. Introduction: on the relationship between informal and formal proofs

The correctness of informal proofs seems to depend on corresponding formal proofs. They are even seen as a sort of sketches that point to the underlying formal proof (Avigad 2020, p. 5). In this way, "an informal mathematical statement is a theorem if and only if its formal counterpart has a formal derivation" (Avigad 2020, p. 3). This position has been called the 'standard view':

> According to the standard view, a mathematical statement is a theorem if and only if there is a formal derivation of that statement, or, more precisely, a suitable formal rendering thereof. When a mathematical referee certifies a mathematical result […] the correctness of the judgement stands or falls with the existence of such a formal derivation. (Avigad 2020, p. 5)

In Avigad's view, "the standard view is essentially correct" (Avigad 2020, p. 4).[1] There is a sense in which this view seems natural. Let us consider the proofs in planar geometry in Euclid's *Elements*. Avigad, Dean, and Mumma (2009) proposed a logical system called *E* that intends to formalize the proofs in planar geometry in Euclid's *Elements*. In their analysis of Euclidean proofs, they found what they called logical gaps: they mention inferential gaps due to some assumptions being implicit (in particular when Euclid assumes that the geometric configuration is nondegenerate) (Avigad et al. 2009, p. 708); there are assumptions used with theorems without being proved (Avigad et al. 2009, p. 735); in what should be proofs by cases, Euclid sometimes only considers one case (Avigad et al. 2009, p. 738); and, sometimes, a proof is not detailed enough and lacks an argument (Avigad et al. 2009, p. 740). However, they consider that, in general, a line-by-line comparison of the proofs in *E* with that of the *Elements*, renders their formal proofs close to Euclid's. Because of this, they concluded that "the proofs in the *Elements* are more rigorous than is usually claimed" (Avigad et al. 2009, p. 760).

This does not contradict the above view on the relationship between informal and formal proofs. It is only after we have the formal counterpart of the informal proof that we can be sure that a particular informal proof is not as sketchy as it might seem at first. For this to be the case, the formal proof must be faithful to the informal proof. According to Avigad, Dean, and Mumma, a model in *E* is faithful to the Euclidean proof when it reproduces line-by-line the 'argumentative structure' of the proof (Avigad et al. 2009, p. 760). According to the creators of *E*, the departures of *E* from the Euclidean proofs are minor or can be reduced by adopting appropriate definitional extensions (Avigad et al. 2009, p. 734). In this way, they consider that *E* provides faithful models of the proofs in Euclid's *Elements* (Avigad et al. 2009, pp. 731-8).

[1] For analysis and dismissal of arguments against the 'standard view', see Hamami (2019).

Granted that we have formal proofs faithful to the Euclidean informal proofs, what does this bring to us? We can rely on all the rigor inbuilt in the formal system. In particular, Avigad, Dean, and Mumma showed that the logical system *E* is sound and complete (Avigad et al. 2009, pp. 743-57). Everything that we derive in *E* is correct, and everything that is entailed can be derived. We have a powerful logical system that models faithfully the Euclidean proofs. We could even say that an informal proof in planar geometry in Euclid's *Elements* is correct in the sense that its corresponding formal proof is correct. Even if Avigad, Dean, and Mumma recognize that the less rigorous aspects of Euclidean proofs did not lead mathematicians to question the correctness of Euclidean proofs (Avigad et al. 2009, p. 701), the existence of a sound formal system that models faithfully the Euclidean proofs would show why the mathematicians' attitude was the right one.

Things do not seem to be as simple as this. Most of the rigor and inferential possibilities of logical systems come from general properties of these, in particular their soundness and completeness. However, we do not prove that a logical system is sound and complete from within the logical system. To prove results about a logical system we need a metalogic. But here we face a problem. The metalogical proofs seem very much like informal proofs. It seems that we are applying a formal apparatus that is grounded on metatheorems that are proved informally. Being this the case, there is a contradiction inbuilt in the 'standard view'. We cannot trust informal proofs without a formal counterpart but, ultimately, the formal counterpart relies on informal proofs.

If we are relying, ultimately, on an informal proof we face what we are calling the groundedness problem: how can we show with certainty that our metalogical proofs are correct? According to the 'standard view', the correctness of an informal proof depends on a suitable formal rendering of it. If we take the metalogical proof to be an informal proof, we have no way, when adopting the 'standard view', to show its correctness. While we can conceive of a formal counterpart to a Euclidean informal proof (and proofs in *E* are an example of that), this is not the case with a proof in metalogic (there is no meta-metalogic).[2]

In this work, we will make the case that the 'standard view' faces the groundedness problem. For that, we will consider, initially, an informal proof from the *Elements* (section 2), and afterward a metalogical proof of soundness (section 3). We will see that it is an informal proof, like those of the *Elements*. Adopting the 'standard view', we would have to say that the soundness of the formal system *E* relies on a potentially nonrigorous informal proof. As such, we would have to say that it is not a theorem since "a mathematical statement is a theorem if and only if there is a formal derivation of that statement" (Avigad 2020, p. 5).

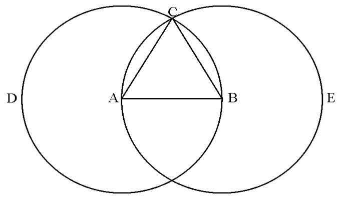
2. Informal proofs in Euclid's *Elements*

Instead of a general overview of the proofs in the *Elements*, the approach adopted here is to look into one particular proof that we find representative as an example of Euclidean proofs. It is the proof of proposition 1 of book 1 (I.1). In the standard English version still at use, it is the following:

On a given finite straight line to construct an equilateral triangle.

Let *AB* be the given finite straight line.
Thus it is required to construct an equilateral triangle on the straight line *AB*.

[2] Even if we imagine a meta-metalogic of some sort this would lead to an infinite regress (the proof in meta-metalogic, being based on a meta-metalanguage, would be informal, and so on).



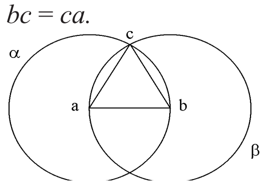
With center *A* and distance *AB* let the circle *BCD* be described; [Post. 3]
again, with center *B* and distance *BA* let the circle *ACE* be described; [Post. 3]
and from the point C, in which the circles cut one another, to the points *A, B*
let the straight lines *CA, CB* be joined. [Post. I]
Now, since the point *A* is the center of the circle *CDB, AC* is equal to *AB.* [Def. I5]
Again, since the point *B* is the center of the circle *CAE, BC* is equal to *BA.* [Def. I5]
But *CA* was also proved equal to *AB;* therefore each of the straight lines *CA, CB* is equal to *AB.*
And things which are equal to the same thing are also equal to one another; [*C. N.* I]
therefore *CA* is also equal to *CB.*
Therefore the three straight lines *CA, AB, BC* are equal to one another.
Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB.*
(Being) what it was required to do. (Euclid 1956, pp. 241-2)[3]

The purpose of this proposition is to construct an equilateral triangle *proving* that it is, in fact, equilateral.[4] To enable the proof one needs the triangle to be constructed according to a well-established prescription that will make possible the proof to be made. In the initial lines of the proof, we find the construction of the triangle. What is strictly the construction part ends when constructing the lines *CA* and *CB*. Then we have the proof proper. Let us look into it, line-by-line:[5]

1) Now, since the point *A* is the center of the circle *CDB, AC* is equal to *AB.* [Def. I5]

The 'comment' inside the bracket gives us an indication of how one concludes that *AC* is equal to *AB*. Definition 15 is the definition of a circle. This definition is complemented by the definition of the center of a circle (definition 16).[6] From these definitions, we know that the line segments connecting

[3] The model in *E* of this proof is as follows: *Assume a and b are distinct points. Construct point c such that ab = bc and bc = ca.*



Proof.
Let α be the circle with center *a* passing through *b*.
Let β be the circle with center *b* passing through *a*.
Let *c* be a point on the intersection of α and β.
Have *ab* = *ac* [since they are radii of α].
Have *ba* = *bc* [since they are radii of β].
Hence *ab* = *bc* and *bc* = *ca*.
Q.E.F. (Avigad et al. 2009, p. 734).

[4] To be more exact, instead of "construct" one might say "instantiate". One is not constructing a triangle in the sense of, e.g., drawing a triangle. One is instantiating a geometrical object in the Euclidean plane (on this see, e.g., Mueller 1981, 208; Taisbak 2003, 28-9). In the present work, we do not need to address this type of issue.

[5] In what follows, we give our presentation and interpretation of the proof. There are many views regarding Euclidean proofs, in particular on the subject of the role of diagrams in the proofs. We cannot mention here all the relevant references. We will only point to a few that we find particularly interesting: Manders (1995), Netz (1999), Macbeth (2010).

[6] According to definition 15, a circle is a plane figure contained by one line [the circumference] such that all the straight lines falling upon it from one point among those lying within the figure are equal (Euclid 1956, p. 153). This point is called, according to definition 16, the center of the circle (Euclid 1956, p. 154).

the center of a circle to the points of the circumference are equal. We call these line segments radii of the circle. Let us recall that previous to this line of the proof we have just constructed the line segment *CA*, connecting the points *C* and *A*, and the line segment *CB*, connecting the points *C* and *B*. To make the inference using definition 15, we must notice first that these line segments are radii of the circle *CDB*. We simply see that in the diagram. Noticing that *AC* and A*B* are not just line segments but are line segments that are radii of the circle *CDB*, enables us to make the inference that *AC* is equal to *AB*. This is a direct consequence of the definition of circle. The argumentation might be something like the following: since *A* is the center of the circle, and *B* and *C* are points of the circumference, the line segments *AC* and *AB* are radii; as such, they are of equal length.

From our reasoning, we have concluded that these line segments are equal. While it is still implicit at this point of the proof, at the conclusion we return to focus on *AC* and *AB* as just line segments.

2) Again, since the point *B* is the center of the circle *CAE, BC* is equal to *BA.* [Def. I5]

We have, *mutatis mutandis*, the same situation as that of the previous line. This is indicated by the term "again" (in the same way). We conclude that the line segments *BC* and *BA* are equal.

3) But *CA* was also proved equal to *AB;* therefore each of the straight lines *CA, CB* is equal to *AB.*

The term "but" indicates that a new premise is being 'added' to the previous result. More exactly, we are recovering a previous conclusion (*AC* is equal to *AB*) and taking it as a premise to be considered together with the previous result, also taken as a premise (*BC* is equal to *BA*). The term "therefore" indicates that a new conclusion is reached. The conclusion is simply a blending of the two premises into one proposition. It can be expressed also as follows: *CA* is equal to *AB* and *CB* is equal to *AB*.

4) And things which are equal to the same thing are also equal to one another; [*C. N.* I]
therefore *CA* is also equal to *CB.*

Here, the so-called common notion 1 is part of the argument. It is used to conclude, by taking into account the conclusion of the previous line, that *CA* is equal to *CB*. In this part, it is made evident that in 3) and 4) we are considering *CA, CB*, and *AB* as line segments. That they can be radii is not relevant in this part. In this case, the 'things' in the application of common notion 1 are line segments.

5) Therefore the three straight lines *CA, AB, BC* are equal to one another.

Here, a new conclusion is inferred from previous conclusions. We have: *CA* is equal to *AB* and *CB* is equal to *AB*; and we also have: *CA* is also equal to *CB*. From these conclusions taken as premises, we conclude that the line segments *CA, AB,* and *BC* are equal.

6) Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB.*

A new conclusion arises from the previous one. The diagram is crucial in this part. There was no mention of triangles until this last line of the proof. In the previous line, we concluded that the line segments *CA, AB,* and *BC*, as line segments, are equal. Now, we see in the diagram that these line segments form a figure with three sides; and these three sides were proved to be equal. Taking into account the definition of a triangle (Euclid 1956, p. 154), *CA, AB,* and *BC* are the sides of an equilateral triangle. In this way, we have proved that the adopted construction leads to an equilateral triangle.

The Euclidean proofs are made adopting a regimented language; what we might consider a part of natural language in which the terms have a very specific meaning and use (Netz 1999, pp. 89-167). For example, "therefore" and "so that" are used to introduce conclusions; "and" and "but" are used to introduce assertions to be considered together with previous assertions, in this way enabling to infer a new assertion; "for" and "since" are used to introduce an assertion that supports a previous assertion; "since" is also used to start or restart an argument (Netz 1999, pp. 115-6).

Another aspect of the language adopted in the *Elements* is the adoption of letters to symbolize, in particular, geometrical objects that are represented in the accompanying diagram. Through these letters, and also by explicit mention, different mathematical notions are taken into account in the proof, like, e.g., that of point and circle. Finally, the diagram itself is used in making inferences like in the case when we see line segments as radii or line segments as the sides of a geometrical object.

In simple terms, the proofs in the *Elements* are made adopting a regimented natural language, mathematical notions (in particular from geometry and about magnitudes), and diagrams. These informal proofs are modeled in the sound formal system *E*. We can say that these proofs are correct in the sense that we take there to be faithful models of them in *E*.

3. Proofs in metalogic

How do we know that the logical system *E* is sound? Avigad, Dean, and Mumma showed that *E* can be expressed in terms of an axiomatization of planar geometry made by adopting first-order predicate logic (PL). More precisely, we can have a translation between proofs in *E* and proofs in a fragment of this axiomatic geometry that is valid for so-called 'ruler-and-compass' constructions (which are the ones modeled in *E*) (Avigad et al. 2009, p. 744). And the soundness of this system, where does it come from? Ultimately, from the soundness of PL. Here, we are back where we started. And the soundness of PL, where does it come from? That PL is sound is proved with metalogic.

When proving a property of a logical system – in particular soundness and completeness – we have to engage with the logical system from something more general that supersedes it. We do that using our natural language. And how do we go from addressing a logical system in terms of our natural language to say that we are proving theorems (metatheorems) using a metalogic?

We will see with an example that we use the natural language in the proofs in a rather specific way. We can say that we have a regimented use of language, like in the case of the use of language in the proofs in the *Elements*. Like Euclid that adopts letters as symbols, in particular, of geometric objects, in our use of natural language as a metalanguage we will adopt letters or even new symbols to symbolize first-order logic concepts (e.g., terms, predicates, sentences, interpretations, derivations, etc.). We will also have symbols to include the mathematics of natural numbers and set theory. In this way, our natural language as a metalanguage has as resources the logic system itself, arithmetic, and set theory (Yaqub 2015, pp. 87-104). This is not that different from how language is adopted in Euclid's planar geometry.[7] In that case, we have a regimented use of language, and this language is supplemented by concepts and resources from geometry, the mathematics of magnitudes, and diagrams.

In the case of logic, we have a formal language and a deductive apparatus. With metalogic, we have a similar situation. Our regimented language (supplemented by special symbolism) has associated with it also a deductive apparatus. On one side, we have the resources of the logical system itself. This does not lead to any problematic circularity (Yaqub 2015, p. 89; Gensler 2010, pp. 339-40). On the other side, we have deductive resources arising from the mathematics that are taken into account in the language. The most relevant for metalogical proofs is mathematical induction (Krantz 2002, pp. 107-20; Hunter 1971).

In this work, we are interested in seeing metalanguage at work in proofs about logical systems. In particular, we want to see how different these proofs are from the so-called informal proofs, like the proof of I.1. The first problem we face is to choose the proof we will use as a 'case-study'. Taking

[7] That this is the case is one of the points we want to make in this work. We will make our case simply by showing part of a proof in metalogic and seeing how the language is used and how the available resources are used, just like in the case of the proof of I.1.

into account the importance of soundness (e.g., to know that Euclidean proofs are being modeled using a sound formal system *E*), it will be a proof of soundness of PL. There are many proofs available (see, e.g., Teller 1989, pp. 193-9; Smith 2012, pp. 361-3; Bergmann, Moor, and Nelson 2014, pp. 244-50; Barker-Plummer, Barwise, and Etchemendy 2011, pp. 525-8; Yaqub 2015, pp. 139-44). How do we choose between these proofs?

One of the most important characteristics of formal proofs is that, as Avigad puts it, "a formal derivation with even a single error is simply not a derivation" (Avigad 2020, p. 8). There is no gradation of rigor in formal proofs. All the inferential steps must be correct. How do we know this, regarding the different metalogical proofs we can choose from? If they are somehow 'formal proofs' then they are all correct – step by step. If they are informal proofs, then, according to the 'standard view', we cannot commit ourselves to their correctness without having a formal counterpart, which we do not have. At this point, we have to choose one of the proofs without knowing if it is formal or informal. If it turns out that we determine that it is an informal proof, then we know that all the other metalogical proofs are informal (they are all based on the same metalogic). If it is formal, then all the others are formal too (unless some have errors, and we take for granted that this is not the case). Not having a clear criterion to choose a proof, we simply choose the proof that we consider that will enable us to show more easily if the proof can be considered formal or informal. We will consider the proof in Yaqub (2015, pp. 139-44). The statement of the soundness theorem is as follows:

For every set Γ of PL sentences and every sentence X of PL, if X is a theorem of Γ, then X is a logical consequence of Γ. Symbolically, if Γ |– X, then Γ |= X. (Yaqub 2015, p. 43)

This proof is made by resorting to mathematical induction. The general pattern of a proof by mathematical induction is as follows:

1. Enumeration. We assign to each object (in this case each line of the derivation) a number 1, 2, 3, …
2. Base step. We prove that property *P* holds for the object numbered 1 (in this case, we prove that Γ |= X for the first line of the derivation).
3. Induction step. We prove that *P* holds for the object(s) numbered *n* + 1, assuming that it holds for all objects numbered 1 through *n* (in this case, we prove that Γ |= X for line *n* + 1of the derivation, assuming that it holds for the previous lines). (Smith 2012, p. 359)

Since to make our point, we do not need to look into the whole proof, we will only consider the first part of the proof – the proof of the base step:

Let n = 1. The first line of any derivation has no antecedents. So, the sentence $\mathbf{Z}_1$ is either a premise, an assumption of a hypothetical rule, or an identity statement of the form **s** = **s**, which is introduced by the MDS rule Identity. If $\mathbf{Z}_1$ is a premise or an assumption, then $\Sigma_1 = \{Z_1\}$. It is obvious that, in this case, $\Sigma_1 \models \mathbf{Z}_1$. If $\mathbf{Z}_1$ is of the form **s** = **s** (where **s** is any PL singular term), then Σ_1 is empty. But s = s is a valid sentence, that is, it is true on all the PL interpretations that are relevant to it. By definition, all PL interpretations are models of the empty set, since an interpretation would fail to be a model of the empty set only if the empty set were to contain a sentence that is false on that interpretation; given that the empty set contains no sentences, no PL interpretation fails to be a model of it. Hence **s** = **s** is true on every model of the empty set that is relevant to it. In other words, **s** = **s** is a logical consequence of the empty set. Therefore, in all cases, $\Sigma_1 \models \mathbf{Z}_1$. This establishes the Base Step. (Yaqub 2015, p. 140)

A brief overview of the above extract already reveals many aspects of what a proof is in metalogic. The reference to arithmetic is already present in the choice of a proof using mathematical induction. We also have in this part of the proof arithmetic symbolism (n = 1). The reference to set theory is present, e.g., with the set $\Sigma_1 = \{Z_1\}$ and the mention to the empty set. However, the overwhelming majority of symbols and technical terms are related to PL. For example, we have the symbol |= that corresponds to the notion of logical consequence. In natural language, $\Sigma_1 \models \mathbf{Z}_1$ means that $\mathbf{Z}_1$ is a logical consequence of Σ_1. If we know first-order predicate logic, the meaning of the terms is clear. When this is the case, the argumentative structure of the proof becomes clear also.

Let us look at each successive part of the proof of the base step:

1) Let n = 1. The first line of any derivation has no antecedents.

We start at the first line of the derivation; as such, it has no antecedents. From or knowledge of PL, we know that the sentence of line 1 – $\mathbf{Z}_1$ – can only be one of three possibilities: a premise, an assumption of a hypothetical rule, or an identity statement:

2) So, the sentence $\mathbf{Z}_1$ is either a premise, an assumption of a hypothetical rule, or an identity statement of the form **s** = **s**, which is introduced by the MDS rule Identity.

From this, we move into a proof by cases, since there are two cases that need a different treatment: a) $\mathbf{Z}_1$ is a premise or an assumption; b) $\mathbf{Z}_1$ is an identity statement. We start with a):

3) If $\mathbf{Z}_1$ is a premise or an assumption, then $\Sigma_1 = \{Z_1\}$. It is obvious that, in this case, $\Sigma_1 \models \mathbf{Z}_1$.

Σ_n is the set of all the premises and 'active' assumptions that are members of Γ and appear at line n or before. In the case of line 1, there are no previous lines. In this case, we only have $\mathbf{Z}_1$. In this way, $\Sigma_1 = \{Z_1\}$. From the definition of logical consequence, it is evident that $\Sigma_1 \models \mathbf{Z}_1$. We have finalized our proof of the base step for case a).

We are going now to consider case b). This part is self-explanatory. It consists of a detailed argumentation where the aspects of PL that are necessary to take into account are made explicitly in it:

If $\mathbf{Z}_1$ is of the form **s** = **s** (where **s** is any PL singular term), then Σ_1 is empty. But s = s is a valid sentence, that is, it is true on all the PL interpretations that are relevant to it. By definition, all PL interpretations are models of the empty set, since an interpretation would fail to be a model of the empty set only if the empty set were to contain a sentence that is false on that interpretation; given that the empty set contains no sentences, no PL interpretation fails to be a model of it. Hence **s** = **s** is true on every model of the empty set that is relevant to it. In other words, **s** = **s** is a logical consequence of the empty set.

With this argumentation in metalanguage, which we might call a 'meaningful discourse', we finalize our proof of the base step for case b). Having proved that $\Sigma_1 \models \mathbf{Z}_1$ for both cases we can then conclude this part of the proof:

Therefore, in all cases, $\Sigma_1 \models \mathbf{Z}_1$. This establishes the Base Step.

The crucial aspect of this proof – shared with all metalogical proofs – is the following: "A proof about a formal system is a piece of meaningful discourse, expressed in the metalanguage, justifying a true statement about the system" (Hunter 1971, p. 11).

Now, the proofs in Euclid's *Elements* are also examples of pieces of 'meaningful discourse'. There are evident differences. In the *Elements*, we have diagrams; here, we do not. In the *Elements*, the adopted regimented language is supplemented, in particular, by special symbolism regarding geometry and the mathematics of magnitudes; here, we take into account natural number theory, set theory, and first-order logic. However, this proof using metalanguage seems much closer to the proofs in the *Elements* than to a formal proof, like that of I.1 made using *E* (see footnote 3).

How do we find this proof convincing? We do not have any formal means to show it; we only rely on our reasoning expressed in natural language to address the text and conclude this. Is this proof more rigorous than Euclid's proofs? That is very unlikely. In both cases, the argumentation is made with a regimented natural language. We would have to argue, e.g., that relying on diagrams is less rigorous than relying on number theory and set theory. But if we are to trust Avigad, Dean, and Mumma's formal system *E*, diagrams do not bring any lack of rigor into the proofs. We might try to make the case that, nevertheless, number theory and set theory bring more rigor into the reasoning. But we would be playing with the idea of degrees of 'rigorousness' due to the mathematics that we put in the mix. But this is contrary to the idea of formal proof. As we have seen, "a formal derivation with even a single error is simply not a derivation" (Avigad 2020, p. 8). There is no gradation of rigor in formal proofs. All this is very far from the rigor of formal proofs. In fact, we do not have formal means to determine if this proof is 'correct' or rigorous 'enough' (or even to determine if asking these questions makes any sense). To understand the proof and evaluate its 'correctness' we have to look into the 'discourse' like we did in the case of the proof of I.1. In our view, metalogical proofs are informal proofs.

Here, we face what we call the groundedness problem: how can we show with certainty that our metalogical proof is correct? According to the 'standard view', we need a formal rendering of the metalogical proof, which we do not have. In this way, when adopting the 'standard view', we have no way to show that the proof is correct. This would imply that we have not determined rigorously the soundness of the formal system *E*.

What would it mean to use *E* without knowing that it is a sound formal system? We would not know if a derivation arrives at a false conclusion, or, even if this was not the case, we would not know for sure if there was not some small gap in the deductive structure of the proof. Our trust in *E* comes from knowing that it was proved that it is a sound system. But now we see that what is sustaining a formal system are a series of metatheorems that are proved using informal proofs. If we endorse the 'standard view' we would have to doubt the soundness of the formal system *E*.

4. Conclusion

Informal proofs like the ones in Euclid's *Elements* might seem to be untrustworthy; they might not be rigorous enough – there is always lurking the suspicion of gaps in the reasoning. We can make amends. We adopt proofs made using a formal system *E* that are faithful to the Euclidean proofs. This is, according to the 'standard view', the procedure that must be adopted. Only if we have a formal counterpart of our informal proof can we trust it. We might say that our informal proofs are correct because of this. But where does the soundness of the formal system *E* come from? From a proof made using metalogic. The proof in metalogic is made using a regimented language (like the Euclidean informal proofs we started with). It is a 'meaningful discourse' that, it is fair to say, is convincing. However, it is not a formal proof; it is an informal proof. This, gives rise to what we called the groundedness problem: how can we show with certainty that our metalogical proofs are correct? According to the 'standard view', we need a formal rendering of the informal proof. But we do not have it in the case of a metalogical proof. This presents a difficulty to the 'standard view'. Even if we have a formal counterpart to an informal proof, like the proof of I.1, we seem to have no rigorous way to prove the soundness of the formal system adopted in the formal proof.

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