Identification and Classification of Off-Vertex Critical Points for Contour Tree Construction on Unstructured Meshes of Hexahedra

Marius K. Koch, Paul H. J. Kelly, Peter E. Vincent

Abstract—The topology of isosurfaces changes at isovalues of critical points, making such points an important feature when building contour trees or Morse-Smale complexes. Hexahedral elements with linear interpolants can contain additional off-vertex critical points in element bodies and on element faces. Moreover, a point on the face of a hexahedron which is critical in the element-local context is not necessarily critical in the global context. In "Exploring Scalar Fields Using Critical Isovalues" [1], Weber *et al.* introduce a method to determine whether critical points on faces are also critical in the global context, based on the gradient of the asymptotic decider [2] in each element that shares the face. However, as defined, the method of Weber *et al.* contains an error, and can lead to incorrect results. In this work we correct the error.

Index Terms—Isosurface, critical points, hexahedra, contour tree.

1 Introduction

A level-set of a C^0 continuous scalar function in $\mathbb{R}^n$ at some isovalue consists of zero, one, or more connected components, the so-called contours. In 3D these contours are isosurfaces. The topology of such isosurfaces changes at isovalues of critical points, making such points an important feature when building contour trees or Morse-Smale complexes. For data descretised on meshes of simplex elements with linear interpolants all critical points occur at element vertices. However, hexahedral elements with linear interpolants F can contain additional off-vertex critical points in their bodies and on their faces. Moreover, a point on the face of a hexahedron which is critical in the element-local context is not necessarily critical in the global context.

In Section 2 we describe the original method of Weber *et al.* [1] for identifying critical points in the global context on faces of hexahedra, and explain how it can also be used to classify their type; as either 1-saddles or 2-saddles. We proceed to show that the approach contains an error, and then present a correction to this error in Section 3. Finally, in Section 4 we apply the original method of Weber *et al.* and the new method to identify whether, under various scenarios, critical points exist in the global context on a face between two unit cubes, and if so to then classify their type. The new method produces correct results for all scenarios, where as the original method of Weber *et al.* does not.

2 Original Method

Without loss of generality we restrict the discussion to a saddle point P on the face T_{ABCD}, which is shared by two unit cubes as shown in Figure 1, with face local maxima at A and C. The original method by Weber *et al.* is based on the AsD, which returns the function value at a face's saddle without computing its position. AsD was introduced by Nielson and Hamann [2] to resolve ambiguities in the marching cubes algorithm [3]. On T_{ABCD} it is defined as:

$$AsD = \frac{AC - BC}{A + C - B - D}. \quad (1)$$

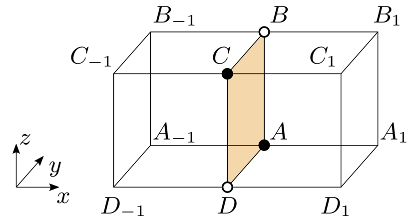

Fig. 1. Vertex numbering scheme according to [1]. The face T_{ABCD} shared by the two elements is shaded, with the black vertices A and C marking the face's maxima and the white vertices B and D marking its minima.

Motivated by Chernyaev [4], Weber *et al.* analyse the connectivity between corner vertices on a 'pseudo-face' $\mathcal{T}$, parallel to T_{ABCD}, as it moves a face-normal distance $x_{\mathcal{T}}$ into a given element. The function values at vertices of $\mathcal{T}$ assume the local values of the interpolant F within the element, thus changing linearly with $x_{\mathcal{T}}$. Specifically, defining the face-normal gradient of AsD in the element-inwards direction on T_{ABCD} as

$$AsD_{x_{\mathcal{T}}} = \left. \frac{\partial AsD}{\partial x_{\mathcal{T}}} \right|_{x_{\mathcal{T}}=0} \quad (2)$$

Weber *et al.* show in the context of a single element that if $AsD_{x_{\mathcal{T}}} < 0$ the two *maxima* of $\mathcal{T}$ will be separated by the intersection of $\mathcal{T}$ with the isosurface associated with P (this implicitly classifies P in the local 3D context as a 2-saddle), but if $AsD_{x_{\mathcal{T}}} > 0$ the two *minima* of $\mathcal{T}$ will be separated by the intersection of $\mathcal{T}$ with the isosurface associated with P (this implicitly classifies P in the local 3D context as a 1-saddle). Subsequently, it is reasoned that if $\text{sign}(AsD_{x_{\mathcal{T}}})$ is the same in the context of each element sharing the face, then P is critical in the global context, but if $\text{sign}(AsD_{x_{\mathcal{T}}})$ is different in the context of each element sharing the face, then P is regular in the global context. We note that P is a 1-saddle in the global context if $\text{sign}(AsD_{x_{\mathcal{T}}})$ is positive in both elements, and P is a 2-saddle in the global context if $\text{sign}(AsD_{x_{\mathcal{T}}})$ is negative in both elements.

However, Weber *et al.* then make the erroneous assumption that AsD, and thus its numerator

$$\nu = AC - BC, \quad (3)$$

are equal to zero on T_{ABCD}, when in fact AsD returns the function value at P, which is not necessarily zero. Combined with the fact that the denominator of AsD

$$\xi = A + C - B - D \quad (4)$$

is always positive on T_{ABCD}, since A and C are maxima, this leads to the erroneous conclusion that

$$\text{sign}(AsD_{x_{\mathcal{T}}}) = \text{sign}(\nu_{x_{\mathcal{T}}}), \quad (5)$$

and thus erroneous use of $\text{sign}(\nu_{x_{\mathcal{T}}})$ as a surrogate for $\text{sign}(AsD_{x_{\mathcal{T}}})$ to determine the existence of global critical points on faces as per the above logic. This can lead to those points being wrongly identified, missed, or correctly identified but mis-classified. We note that in [5] the function value at P is set to zero by subtracting the saddle value from all vertices of the associated hexahedra. The following section presents a correction that does not require such a manipulation of the function values at the vertices.

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3 Correction of the Original Method

The method of Weber *et al.* can be corrected by directly considering the sign of the face-normal gradient of AsD

$$\mathrm{sign}(AsD_{x_\mathcal{T}}) = \mathrm{sign}\left(\frac{\nu_{x_\mathcal{T}}\xi - \nu\xi_{x_\mathcal{T}}}{\xi^2}\right), \tag{6}$$

rather than using $\mathrm{sign}(\nu_{x_\mathcal{T}})$ as a surrogate. In the right element, the derivatives of ν and ξ are:

$$\begin{aligned}\nu_{x_\mathcal{T}} &= (AC - BD)_{x_\mathcal{T}} \\ &= F_{x_\mathcal{T}}|_A\, C + A\, F_{x_\mathcal{T}}|_C - F_{x_\mathcal{T}}|_B\, D - B\, F_{x_\mathcal{T}}|_D\,,\end{aligned} \tag{7}$$

$$\begin{aligned}\xi_{x_\mathcal{T}} &= (A - B + C - D)_{x_\mathcal{T}} \\ &= F_{x_\mathcal{T}}|_A - F_{x_\mathcal{T}}|_B + F_{x_\mathcal{T}}|_C - F_{x_\mathcal{T}}|_D\,,\end{aligned} \tag{8}$$

with the derivatives at the vertices being:

$$\begin{aligned} F_{x_\mathcal{T}}|_A &= A_1 - A, \qquad & F_{x_\mathcal{T}}|_B &= B_1 - B, \\ F_{x_\mathcal{T}}|_C &= C_1 - C, \qquad & F_{x_\mathcal{T}}|_D &= D_1 - D. \end{aligned} \tag{9}$$

By inserting Eq. (3), Eq. (4), and Eqs. (7 - 9) into Eq. (6), $AsD_{x_\mathcal{T}}$ can be expressed in terms of scalar values at vertices:

$$\begin{aligned} AsD_{x_\mathcal{T}} = &\frac{1}{(A - B + C - D)^2}[(AC - BD) \\ &(A - B + C - D - A_1 + B_1 - C_1 + D_1) \\ &+(A - B + C - D)[-A(C - C_1) \\ &+B(D - D_1) - C(A - A_1) + D(B - B_1)]]. \end{aligned} \tag{10}$$

To obtain $AsD_{x_\mathcal{T}}$ in the left element the terms with subscript 1 need to be replaced with the analogous terms with subscript -1.

4 Two-Element Test Cases

Two-element test cases were used to verify the new approach and demonstrate its utility. They consisted of two unit cubes as shown in Figure 1. Specifically, 15 individual test cases were constructed by selecting scalar values at vertices as per Table 1. For all test cases, there exists a critical point P on the face T_{ABCD}.

Table 1 identifies whether P is a critical point in the global context for each test case, and if it is, provides a classification of its type. There are three possibilities:

- RP: P is a regular point in the global context, e.g. test case V, see Figure 2 (a).
- 1-S: P is a 1-saddle in the global context, e.g. test case I, see Figure 2 (b).
- 2-S: P is a 2-saddle in the global context, e.g. test case XI, see Figure 2 (c).

A ground truth determined by visual inspection is provided, along with results obtained using the original method of Weber *et al.* and the new method. We note that the new method correctly identifies whether a global critical point exists on a face and classifies it correctly for all test cases, whereas the original method misses critical points, e.g. case XI, misidentifies regular points as critical points, e.g. case V, and misclassifies critical points, e.g. case VI.

5 Conclusion

We have corrected an error in the method of Weber *et al.* for identifying critical points in the global context on faces of hexahedra with linear interpolants. The new method, which is based on $\mathrm{sign}(AsD_{x_\mathcal{T}})$, correctly identifies if critical points exist in the global context for a range of test cases — and if they are determined to exist correctly classifies their type. However, the original method of Weber *et al.* is not able to do this.

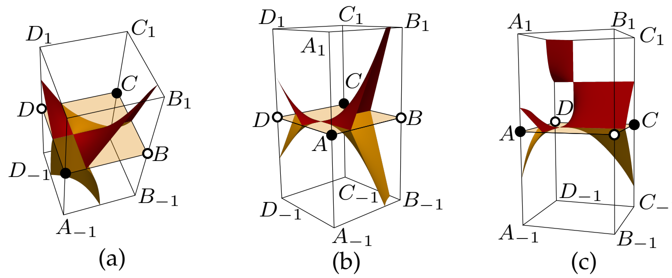

Fig. 2. Isosurfaces through P for two-element test cases V (a), I (b), and XI (c). Minima and maxima on T_{ABCD} are marked with white and black circles, respectively.

TABLE 1
Scalar values at vertices for the test cases and identification of whether P is critical in the global context for each test case, and if it is provision of a classification. There are three possibilities for P: regular point (RP), 1-saddle (1-S), and 2-saddle (2-S). A ground truth (GT) determined by visual inspection is provided, along with results obtained using the original method of Weber *et al.* (WM) and the new method (NM). For each test case $*$ indicates that the method of Weber *et al.* correctly identified a RP via incorrect classification of the local face saddles in both elements.

Test case	A_{-1}	B_{-1}	C_{-1}	D_{-1}	A	B	C	D	A_1	B_1	C_1	D_1	GT	WM	NM
I	4	2	5	3	3	1	4	2	4	2	5	3	1-S	1-S ✓	1-S ✓
II	4	2	0	3	3	1	4	2	4	2	5	3	1-S	RP ×	1-S ✓
III	1	2	1	0	3	1	4	2	4	2	5	3	RP	RP ✓	RP ✓
IV	2	−3	2	−2	3	1	4	2	4	2	5	3	RP	1-S ×	RP ✓
V	5	0	2	1	3	1	4	2	4	2	5	3	RP	1-S ×	RP ✓
VI	4	2	0	3	3	1	4	2	4	2	0	3	1-S	2-S ×	1-S ✓
VII	1	2	1	0	3	1	4	2	4	2	0	3	RP	2-S ×	RP ✓
VIII	2	−3	2	−2	3	1	4	2	4	2	0	3	RP	RP ✓*	RP ✓
IX	5	0	2	1	3	1	4	2	4	2	0	3	RP	RP ✓*	RP ✓
X	1	2	1	0	3	1	4	2	1	2	1	0	2-S	2-S ✓	2-S ✓
XI	2	−3	2	−2	3	1	4	2	1	2	1	0	2-S	RP ×	2-S ✓
XII	5	0	2	1	3	1	4	2	1	2	1	0	2-S	RP ×	2-S ✓
XIII	2	−3	2	−2	3	1	4	2	2	−3	2	−2	2-S	1-S ×	2-S ✓
XIV	5	0	2	1	3	1	4	2	2	−3	2	−2	2-S	1-S ×	2-S ✓
XV	5	0	2	1	3	1	4	2	5	0	2	1	2-S	1-S ×	2-S ✓

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