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# Non-unique topological sofic entropy and a von Neumann algebra multiplicative ergodic theorem

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# Non-unique topological sofic entropy and a von Neumann algebra multiplicative ergodic theorem

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**Abstract**

# Non-unique topological sofic entropy and a von Neumann algebra multiplicative ergodic theorem

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A sofic approximation to a countable group is a sequence of partial actions on finite sets that asymptotically approximates the action of the group on itself by left-translations. A group is sofic if it admits a sofic approximation. Sofic entropy theory is a generalization of classical entropy theory in dynamics to actions by sofic groups. However, the sofic entropy of an action may depend on a choice of sofic approximation. All previously known examples showing this dependence rely on degenerate behavior. In joint work with D. Airey and L. Bowen an explicit example is exhibited of a mixing subshift of finite type with two different positive sofic entropies. The example is inspired by statistical physics literature on 2-colorings of random hyper-graphs. Also, in joint work with L. Bowen and B. Hayes, the classical Multiplicative Ergodic Theorem (MET) of Oseledets is generalized to cocycles taking values in a type II von Neumann algebra. This appears to be the first MET involving operators with continuous spectrum.

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# Chapter 1

# Introduction

## 1.1 General introduction to "A topological dynamical system with two different positive sofic entropies"

### Shannon entropy

Suppose we have a random variable $X$ taking values in a finite set $A$, and suppose that we seek a function giving a quantitative representation of how much information one would obtain, or how "surprised" one would be, to learn of a particular outcome in $A$. Suppose we want this function to depend only on the probability of an outcome. Naturally, the rarer the outcome the more surprising its occurrence would be. On the other hand if an outcome has probability 1, then its occurrence holds no information or surprise. Another reasonable assumption is that one learns the same amount of information from the simultaneous occurrence of two independent events as from the separate occurrence of these events. This leads us to a function $I : (0, 1] \to [0, \infty)$ such that $I$ is monotone decreasing, $I(1) = 0$, and $I(xy) = I(x) + I(y)$. A natural candidate is then $I(x) = -\log(x)$. The above is a simplified account of the approach that Claude Shannon took when he introduced information entropy in [Sha48]. The motivation for Shannon's work was in data communication - he showed that entropy is a theoretical limit to how efficiently data can be coded without losing any information. This **Shannon entropy** $H(X)$ is defined to be the expected value of the information one learns from $X$, i.e. -

$$H(X) = -\sum_{a \in A} p_a \log p_a$$

where $p_a$ is the probability that $X$ has outcome $a \in A$. By continuity we let $0 \log 0 = 0$. It can be deduced that $H$ is maximized when all outcomes are equally likely.

The history of entropy can be traced back to thermodynamics in 1854, when Rudolf Clausius ([Cla56]) gave the first mathematical formulation of entropy. In the 1870s Boltzman and Gibbs introduced a statistical mechanical view of entropy. We give a brief illustration found in Petersen ([Pet89]) of the connection to Shannon entropy. Suppose there are $n$ molecules and states $s_1, ..., s_k$, and we have a system where $p_i$ proportion of the particles are in state $i$. Then the number of ways for the particles to be in such a distribution is $\binom{n}{p_1 n, p_2 n, ..., p_k n}$. Using Stirling's formula, this number is approximately $e^{nH(p)}$, where $H(p) = -\sum p_i \log p_i$ as seen earlier. Thus higher entropy is related to a larger number of possibilities, which can be interpreted as higher uncertainty or randomness. This statistical mechanical view of counting the exponential growth rate of "microstates" turns out to be the approach taken in sofic entropy.

**Entropy rate**

Suppose now that we have a stationary stochastic process $X = (X_1, X_2, ...)$ and we want to quantify the amount of information per unit of time that we learn from this process. Let $h(X) = \lim_{n\to\infty} \frac{1}{n} H(X_1, ..., X_n)$. This is the notion of **entropy rate**. A stochastic process is stationary if the law of the shifted process $(X_2, X_3, ...)$ is the same as the law of $X$. Assuming stationarity, all random variables in the sequence have the same law, and therefore the same Shannon entropy. The entropy rate of the process is in general less than or equal to the Shannon entropy of any single random variable $X_i$ since knowledge of the past values allows one to make better predictions of the future.

For example, Shannon ([Sha51]) tried to estimate the entropy of the English language. Using standard tables the Shannon entropy of a single letter, chosen uniformly at random from amongst pieces of typical English prose, assuming a twenty-seven letter alphabet (including spaces), is about 4.03 bits (i.e. if log base 2 is used in the formula for entropy). On the other hand, by asking English speakers to predict the letters of an unfamiliar English passage, Shannon estimated the entropy rate of English prose to be about 1 bit per letter. This is because of the many redundancies formed by many rules, at various scales from the spelling of a single word to sentence and paragraph and composition structure, of English prose. Thus theoretically on average a very long piece of English writing could be compressed to a quarter of its length and still retain all of its information content. On the other hand, a mathematics paper can be thought of as having high entropy - even a single equation may contain a lot of information, but an error at a single symbol can also easily confuse the reader.

Intuitively, if after observing the process for a finite period of time one can almost surely predict the value of the next outcome, then the process can be thought of as **deterministic** and has entropy rate zero. On the other extreme, an independent and identically distributed process (iid) always outputs completely new information, so the entropy rate is the same as the Shannon entropy of each individual random variable in the process. For example, let $X = (X_1, X_2, ...)$ represent a sequence of iid fair coin tosses. Then $(X_1, ..., X_n)$ is uniformly distributed amongst $2^n$ outcomes, so that $H(X_1, ..., X_n) = n \log 2$, and so $h(X) = \log 2$.

**Kolmogorov-Sinai entropy**

For a measure-preserving dynamical system $(X, T, \mu)$, the entropy rate (or **Kolmogorov-Sinai entropy**) $h(X, T, \mu)$ was carried into ergodic theory by Kolmogorov and Sinai ([Kol59], [Sin59]) about 10 years after Shannon's work. Kolmogorov-Sinai entropy is important for classification purposes in ergodic theory. Let $\mathcal{B}(p_1, ..., p_k)$ denote the iid process (also called **Bernoulli shift**) where each random variable has outcome $k$ with probability $p_k$. Before Kolmogorov and Sinai, it was unknown whether $\mathcal{B}(1/2, 1/2)$ is isomorphic (in the category of measure-preserving dynamical systems) to $\mathcal{B}(1/3, 1/3, 1/3)$. However the above question has been resolved in the negative because Kolmogorov-Sinai entropy is an isomorphism invariant and $H(1/2, 1/2) = \log 2$ while $H(1/3, 1/3, 1/3) = \log 3$. Important theorems in this area include Sinai's Factor theorem, Shannon-McMillan-Breiman theorem, and Kreiger Generator theorem. Another celebrated achievement in this area is Ornstein's isomorphism theorem ([Orn70]), which states that entropy is in fact a complete isomorphism invariant for Bernoulli shifts.

**Topological entropy and symbolic dynamics**

A topological dynamical system is a pair $(X, T)$ where $X$ is a compact metrizable Hausdorff space and $T : X \to X$ a homeomorphism. **Topological entropy**, $h(X, T)$, first defined in [AKM65] is an isomorphism invariant of topological dynamical systems. It is related to measure-theoretic entropy via the **variational principle**, which states that $h(X, T) = \sup_\mu h(X, T, \mu)$ where $\mu$ varies over all probability measures that make $(X, T, \mu)$ measure-preserving. Note that the Krylov-Bogolyubov theorem guarantees that invariant measures always exist for topological dynamical systems.

A special case of topological dynamics is **symbolic dynamics**, where $X \subset A^{\mathbb{Z}}$ for a finite set $A$ with the product topology on $A^{\mathbb{Z}}$ and $T = \sigma$ is the shift map $\sigma(x)_n = x_{n+1}$. Usually we think of $A = 0, 1$ so that $X$ consists of a bi-infinite sequence of 0s and 1s. Assume that $X$ is closed and $\sigma$-invariant (a **subshift**). In this setting topological entropy has a particular simple description - $h(X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log W_n(X)$ where $W_n(X)$ is the number of blocks of length $n$ that can appear in $X$. Consider the following examples:

- $h(A^{\mathbb{Z}}, \sigma) = \log |A|$. This is sometimes called the full $|A|$-shift over $\mathbb{Z}$
- If $X_1 \subset \{0, 1\}^{\mathbb{Z}}$ is defined by the rule that a 0 must follow any 1, then it can be shown that $h(X_1, \sigma) = \log \frac{1+\sqrt{5}}{2}$
- If $X_2 \subset \{0, 1\}^{\mathbb{Z}}$ is defined by the rule that any $x \in X_2$ can have at most ten 0s, then $h(X_2, \sigma) = 0$. This is because $W_n(X_2)$ is only growing polynomially but entropy only measures exponential growth rate.

**Entropy for general group actions**

Entropy theory was extended to actions of countable amenable groups by [Kie75], [MO85], and [OW80]. There are many equivalent definitions of amenability; we mention one of them commonly used in ergodic theory - a group $\Gamma$ is **amenable** if there exists a sequence of finite sets $F_n \subset \Gamma$ with the **Folner property** - for every $g \in \Gamma$, $\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$. Such a sequence is called a **Folner sequence**. For example the sequence $F_n = \{1, 2, ..., n\}$ is a Folner sequence for $\mathbb{Z}$. A Folner sequence can also be thought of as having small boundary compared to volume, allowing for one to replace averaging over $\{1, ..., n\}$ by averaging over Folner sequences and still obtain an isomorphism invariant in both the topological and measure-theoretic setting. Important theorems such as Ornstein isomorphism theorem, Sinai's factor theorem, and the variational principle continue to hold in the amenable setting.

*Remark* 1. As an aside, amenable groups were introduced by von Neumann in response to the Banach-Tarski paradox. Indeed, another characterization of amenability involves the property of non-paradoxicality - that amenable groups are those that cannot be used to produce something akin to the Banach-Tarski paradox.

In the case of symbolic dynamics, given an amenable group $\Gamma$ with Folner sequence $F_n$, $A^\Gamma$ with the $\Gamma$- shift action $\sigma_g(x)_h = x_{g^{-1}h}$ for every $g \in \Gamma$, and $X \subset A^\Gamma$ closed and invariant under $\sigma_g$ for every $g$, one can define the topological entropy $h(X, \sigma) = \lim_{n\to\infty} \frac{1}{|F_n|} \log W_{F_n}(X)$, where $W_{F_n}(X)$ is now the number of patterns with coordinates in $F_n$ that can appear in $X$. It can be shown that such a definition does not depend on the choice of Folner sequence. On the other hand, much less is understood for an analogous question in the generalization of entropy theory to actions of countable sofic groups, which will be discussed soon and is the motivation behind our work.

The class of amenable groups includes $Z^d$, solvable groups, virtually nilpotent groups, and all groups of subexponential growth. However, $\mathbb{F}_n$ for $n \geq 2$ is nonamenable, where $\mathbb{F}_n$ is the free group on $n$ generators. In contrast to averaging over Folner sequences, a naive definition of entropy by averaging over any sequence of finite sets of a nonamenable group does not lead to an isomorphism invariant. Furthermore, an example by Ornstein-Weiss in [OW87] showed that the full 2-shift over $\mathbb{F}_2$ contains the full 4-shift over $\mathbb{F}_2$ as a factor, whereas in classical entropy theory, entropy could only decrease under factor maps, suggesting another obstruction to generalizing entropy theory to actions of nonamenable groups.

Nevertheless, the work of L. Bowen in [Bow10] in fact began the extension of entropy theory to **sofic** group (see following remark) actions. Given $\Gamma$ a sofic group with sofic approximation $\Sigma$ and $\Gamma \curvearrowright (X, \mu)$ a probability-measure-preserving group action, a value $h_\Sigma(\Gamma \curvearrowright (X, \mu)) \in -\infty \cup [0, \infty]$ was defined, now commonly known as **sofic measure entropy**, such that

- It is also an isomorphism invariant of measure-preserving $\Gamma$-dynamical systems.
- It agrees with classical entropy when $\Gamma$ is $\mathbb{Z}$ or amenable.

- The entropy of a Bernoulli shift over $\Gamma$ is also equal to the Shannon entropy of its base.

*Remark* 2. Informally, a **sofic approximation** $\Sigma$ to a countable group $\Gamma$ is a sequence of approximate $\Gamma$-actions on finite graphs that approximate $\Gamma$ acting on its Cayley graph. A group is sofic if it has a sofic approximation, but in general there could be many sofic approximations to a given group. All amenable groups and residually finite groups are sofic, including nonabelian free groups and countable linear groups. In fact whether all countable groups are sofic is a significant open question . Thus sofic entropy indeed extends classical entropy to a much larger class of systems.

Topological sofic entropy was also defined in [KL11]. As before symbolic dynamics is the setting where the definition of topological sofic entropy is relatively simple - count the exponential growth rate of configurations on the vertices of the finite graphs that approximate valid configurations in the subshift well.

Versions of some properties in classical entropy theory continue to hold for sofic entropy (e.g. Ornstein's isomorphism theorem ([Bow12],[Sew18a]), Sinai's factor theorem ([Sew18b]), and the variational principle ([KL11], while others no longer hold (including, as shown previously by Ornstein-Weiss, entropy no longer being monotonically decreasing under factor maps). Our contribution is in the investigation of the dependence of sofic entropy on sofic approximation.

If $\Gamma$ is amenable, then sofic entropy does not depend on the choice of sofic approximation (we will informally call this non-dependence "**unique sofic entropy**"). When $\Gamma$ is nonamenable, certain dynamical systems with strong properties have been shown to have unique sofic entropy (e.g. Bernoulli shifts in [Bow10], certain algebraic dynamical systems in [Hay16], and Gibbs measures with strong spatial mixing in [AP18]).

However, in both the measure and topological case, in general it could be that $h_{\Sigma_1}(\Gamma \curvearrowright X) \neq h_{\Sigma_2}(\Gamma \curvearrowright X)$ where everything is held constant except for the two sofic approximations $\Sigma_1$, $\Sigma_2$ to the same group $\Gamma$ . But until now all known examples of nonunique sofic entropy are based on actions where $h_{\Sigma_1}(\Gamma \curvearrowright X) = -\infty$ (which we consider to be degenerate) and $h_{\Sigma_2}(\Gamma \curvearrowright X) = 0$.

Using a random hypergraph 2-coloring model adapted from statistical physics (in particular from the work of [COZ11] and [AM06]), we construct an explicit example in the topological (more specifically, symbolic) case:

**Theorem 2.1.1** *There exists a sofic group* $\Gamma$*,* $X$ *a mixing subshift of finite type, and sofic approximations* $\Sigma_1, \Sigma_2$ *such that* $0 < h_{\Sigma_1}(\Gamma \curvearrowright X) < h_{\Sigma_2}(\Gamma \curvearrowright X) < \infty$*.*

We will discuss how [COZ11] and [AM06] relates to our work after introducing random hypergraph 2-coloring.

**Random hypergraph 2-coloring**

Given a graph, can one color each the vertex with one of two colors so that no edge of $G$ is monochromatic? This property, called **2-colorability**, is equivalent to asking whether the graph is bipartite, which has a fast (linear) algorithmic solution.

A much harder problem is to replace graphs with hypergraphs, in which edges are replaced by hyperedges, which can be subsets of the vertices of any size. This problem was introduced by Bernstein in 1908 ([Ber07]) and studied and popularized by Erdos, who gave the property of being 2-colorable the name "Property B", in the 1960s. A common hypothesis is that the hypergraph is **$k$-uniform** - all hyperedges have size $k$. Dinur-Regev-Smyth showed that even for 3-uniform 2-colorable hypergraphs, finding such a coloring is an NP-hard problem.

The motivation behind [COZ11] is to study **random** k-uniform hypergraph 2-coloring. In this setting the random hypergraph $H_k(n,m)$ has $n$ vertices and $m$ hyperedges each of size $k$, but all $\binom{\binom{n}{k}}{m}$ possible choices of edge sets are equally likely to occur. We informally call this the "uniform" model. Let $r = m/n$ be the edge density ratio. $H_k(n,m)$ is said to have a property $P$ **with high probability** (w.h.p.) if the probability that $H_k(n,m)$ has $P$ converges to 1 as $n \to \infty$. The question that [COZ11], [AM06], and various others have studied is "for what values of $r$ is $H_k(n, \lceil rn \rceil)$ 2-colorable w.h.p?"

Hypergraph coloring can be viewed as part of the class of **constraint satisfaction problems**, which includes problems such as $k$-SAT, $k$-NAESAT, $k$-XORSAT. These problems have been studied not only in combinatorics and computer science but also in statistical physics ([MZ97],[KMRT[+]07]) as "diluted mean field models". Here (non-rigorous) frameworks such as "condensation transition" and "replica symmetry breaking" often provide intuition and predictions.

As is a feature of many properties of random graph models, a conjecture is that there exists a sharp threshold for $r_{sat}$ such that $H_k(n, \lceil rn \rceil)$ is 2-colorable w.h.p. for $r <$

$r_{sat}$ and not 2-colorable w.h.p. if $r > r_{sat}$. Common methods of bounding $r_{sat}$ include the **first moment method**, which uses Markov's inequality on the expected number of monochromatic (or **proper**) colorings, $\mathbb{E}(Z)$, to provide an upper bound $r_{first} \geq r_{sat}$, and the **second moment method** (first used by [AM06]), which uses the Paley-Zygmund inequality to provide a lower bound $r_{second} \leq r_{sat}$.

Unfortunately, there is a strict gap between $r_{first}$ and $r_{second}$. In particular, for $r > r_{second}$, the Paley-Zygmund inequality breaks down because $\mathbb{E}(Z^2) = exp(\Omega(n))E(Z)^2$. A computation also shows that this implies that there is another random hypergraph model called the "planted model" such that $\mathbb{E}^{planted}(Z) = exp(\Omega(n))E(Z)$. One of the contributions of [COZ11] is to improve the lower bound from $r_{second}$ to $r_{cond} \approx r_{second} + 0.153$ by using an "enhanced second moment method".

How our work uses the above ideas and results is the following:

- Fix large integers $d$ and $k$, which correspond to the vertex degree and hyperedge size, respectively. Let $\Gamma$ be the $d$-fold free product of $\mathbb{Z}/k\mathbb{Z}$.

- We modify versions of the "uniform" and "planted" random hypergraph models from the literature to obtain "random sofic approximations", from which we eventually extract deterministic sofic approximations $\Sigma_1$ and $\Sigma_2$ to $\Gamma$.

- $\Gamma$ acts on the subshift $X \subset \{0,1\}^\Gamma$, where $X$ is the set of all "proper colorings" in the Cayley hypergraph of $\Gamma$. Entropy is then related to the exponential growth rate of the number of proper 2-colorings in the finite approximations.

- Just as in the literature, first and second moment calculations show that the exponential growth rate of the expected number of proper colorings is positive for both uniform and planted models, and strictly larger for the platned model for edge densities (which is also equal to $d/k$) between $r_{second}$ and $r_{first}$.

- In a similar fashion as in the literature, an enhanced second moment method shows concentration about the expectation in the uniform model for edge densities less than $r_{cond}$. Thus between $r_{second}$ and $r_{cond}$, we are able to conclude non-unique sofic entropy.

**Further directions**

Two limitations of note exist with our example:

1. It is not uniquely ergodic, so it does not give as a corollary a nondegenerate example of nonunique measure-theoretic sofic entropy.

2. It is not minimal, so it is still possible that our example contains a closed subsystem exhibiting degenerate nonunique sofic entropy, albeit not in any obvious fashion.

So the following two questions could be asked:

*Question* 1. Does there exist a measure-preserving system $\Gamma \curvearrowright (X, \mu)$ and $\Sigma_1, \Sigma_2$ sofic approximations such that $0 < h_{\Sigma_1}(\Gamma \curvearrowright (X, \mu)) < h_{\Sigma_2}(\Gamma \curvearrowright (X, \mu) < \infty$?

Regarding the above question, the next natural class of measures after Bernoulli shifts to study is invariant Markov chains (over say the free group). To this end, our above work inspires us to study the $f$-invariant over $m$-fold self joinings of a Markov measure, which can be shown to be equivalent to the growth rate of the mth moment of the number of "good models". We may be able to draw inspiration from other models and methods from statistical physics, for example the property of replica symmetry breaking may imply relevant information about various moments. Also, [DSS16] on independent sets of random regular graphs seems to contain relevant ideas, such as the "frozen model" for encoding clusters.

*Question* 2. Does there exist a minimal topological dynamical system $\Gamma \curvearrowright X$ and $\Sigma_1, \Sigma_2$ sofic approximations such that $0 < h_{\Sigma_1}(\Gamma \curvearrowright X) < h_{\Sigma_2}(\Gamma \curvearrowright X) < \infty$?

To begin, it may be useful to study B. Weiss's construction of a universal minimal $\Gamma$ action for every countable group $\Gamma$ ([Wei12]).

Our example is also highly specific. So one might investigate the flexibility of our methods under perturbations.

*Question* 3. Can we still obtain a result if $\Gamma =< s_1, ..., s_d | s_i^{k(i)} = 1 >$, with $k$ a function of $i$? What if we perturb $X$ to require colorings with a general range of color distributions for each edge, or we allow more than two colorings? Studies of hypergraph q-coloring exist in the literature (e.g. [ACOG15]).

More generally, perhaps one can define the notion of a topological tail and conjecture that any subshift of finite type over a (virtually) free group $\Gamma$ with trivial topological tail has unique sofic entropy.

**Statement of contribution**

The dissertator has participated substantially in discussions, development, writing, or revisions in the following sections: 3, part of 4, 6.1, 6.2, 8.2, 9, Appendix A, Appendix B. This paper has recently received positive reviews from the Transactions of the AMS and is undergoing revisions.

## 1.2 General introduction to "A The multiplicative ergodic theorem for von Neumann algebra valued cocycles"

**Overview** Oseledets' multiplicative ergodic theorem can be viewed in multiple ways: as a dynamical or random version of the Jordan normal form, as a noncommutative or matrix version of Birkhoff's pointwise ergodic theorem (which itself can be viewed as a more general version of the strong law of large numbers). It is applicable to nonlinear perturbations of linear differential equations.

Suppose we are given a sequence of $d \times d$ random matrices $A_1, A_2, ....$ Let $B_n = A_n ... A_1$ and we want to describe the properties of $(B_n)^{\frac{1}{n}}$ for large $n$. This geometric mean of a random product can be compared to the better known problem of taking the arithmetic mean of a random sum.

In the case $d = 1$ with each $A_i$ positive and independent and identically distributed (iid), by taking logarithms the problem reduces to the strong law of large numbers.

However, in higher dimensions more interesting phenomenon arise. Consider an example of Furstenberg-Kesten([FK60]). Let each $A_i$ be iid with $\mathbb{P}(A_i = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = 1/2$ and $\mathbb{P}(A_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) = 1/2$

Then $B_n$ oscillates between $\left(\begin{smallmatrix} 2^a & 0 \\ 0 & 2^b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 2^d \\ 2^c & 0 \end{smallmatrix}\right)$ for some a,b,c,d.

In particular the coefficients of $B_n$ do not converge.

Early works studying this problem include those of Bellman ([Bel54]) and Furstenberg-Kesten . They worked in the setting of stationary stochastic processes. Assuming all entries of each $A_i$ are positive, with certain additional conditions they obtained asymptotic results for individual entries of $B_n$ - i.e. that $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log(B_n)_{i,j})$ exists. Another result of Furstenberg-Kesten, which will be built upon by later researchers in proving the multiplicative ergodic theorem, is the existence of $\lim_{n\to\infty} \frac{1}{n} \log \|B_n\|$. Furstenberg-Kesten also obtained a central limit theorem type of result which is beyond our scope here.

**Classical MET**

Perhaps a more interesting question would be to ask about the spectral properties of $B_n$. The major classical result, in the setting of dynamical systems, is the **multiplicative ergodic theorem**, first proven by Oseledets in ([Ose68]), a version of which is given below.

Let $(X, \mu, f)$ be a measure-preserving system. Let $A : X \to GL(d, \mathbb{R})$ be a measurable map. Let $B_n(x) = A(f^{n-1}x)A(f^{n-2}x)...A(x)$. Suppose

$$\int \log^+ \|A(x)\| \; d\mu(x) < \infty,$$

Then

$$\lim_{n\to\infty} [B_n(x)^* B_n(x)]^{\frac{1}{2n}} = \Lambda(x)$$

exists, where $\Lambda(x)$ is positive definite and symmetric. Furthermore,

- Let $\lambda_1 > ... > \lambda_k$ be distinct eigenvalues of $\Lambda$. For each $i$ let $\chi_i = \log \lambda_i$. The $\chi_i$ are known as **Lyapunov exponents**.

- Let $U_1, ..., U_k$ be eigenspaces of dimension corresponding to $\lambda_1, ..., \lambda_k$. Let $V_i = \bigoplus_{j\geq i} U_j$. The $V_i$ are known as **Oseledets subspaces** or the **Lyapunov filtration**.

- **Growth rates**: For each $i$, and any $v \in V_i \setminus V_{i+1}$, $\lim_{n\to\infty} \frac{1}{n} \log \|B_n(x)v\| = \chi_i$

- **Equivariance of subspaces**: For each $i$, $B(x)V_i(x) = V_i(fx)$.

- **Invariance of Lyapunov exponents**: For each $i$, $\chi_i(fx) = \chi_i(x)$.

- **Regularity**: $\lim_{n\to\infty} \frac{1}{n} \log |\det(B_n(x))| = \sum_{i=1}^{k} m_i\chi_i$ where $m_i$ is the dimension of $U_i$.

*Remark* 3. There is also a version of the above MET for measure preserving flows, which can be found in Oseledets' original paper or in [Rue79].

*Remark* 4. There are various versions of the MET depending on whether one assumes that the dynamics ($f : X \to X$) is invertible and whether the random matrices are invertible. The assumptions used here of **noninvertible dynamics and invertible matrices** are the ones most suitable for our contribution to a new MET, which will be discussed later.

*Remark* 5. If $f$ is the identity map, i.e. if there are no dynamics, then $B_n(x) = (A(x))^n$, and the conclusions of the MET can be obtained by using the Jordan normal form.

**Motivation**

We now provide some motivation for the MET, mostly following Barreira and Pesin ([BP02]). The origin of the attribution "Lyapunov" exponents comes from Lyapunov's 1892 work ([Lya92]) on stability theory of solutions of ordinary differential equations. Consider a differential equation of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)(x)$, where $x \in \mathbb{R}^d$, and $A(t)$ is a matrix. Let $v_y(t)$ be a solution of the differential equation with initial condition $y \in \mathbb{R}^d$. Suppose now that $\chi^+(y) := \limsup_{t\to\infty} \frac{1}{t} \log \|v_y(t)\| < 0$ for all $y$ (it turns out that $\chi^+$ only attains finitely many distinct values). Then every solution $v_y(t)$ converges to 0 at an exponential rate as $t \to \infty$ - i.e. the trivial solution $v_0 \equiv 0$ is exponentially stable.

Now suppose there is a small nonlinear perturbation of $\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)(x)$- i.e. the equation $\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t,u)$. This equation is often more representative of problems encountered in the real world. The question arises of whether the linearized system, which is simpler to analyze, can be used to provide information about any small enough nonlinear perturbation. In particular, does $\chi^+ < 0$ from the linear system imply that every solution of the nonlinear system with initial conditions close to 0 converges to 0 at an exponential rate as $t \to \infty$? The answer turns out to be negative in general ([Per30]). However, Lyapunov introduced the notion of **regularity**; under such an additional assumption one can guarantee exponential stability even in nonlinear perturbations. Regularity is often difficult to verify directly, but it turns out to be equivalent to the regularity that is guaranteed almost everywhere by the MET.

Building upon Lyapunov stability theory, Pesin's theory of (nonuniformly) hyperbolic systems involves a measure-preserving diffeomorphism or flow (including those generated by an ordinary differential equation) on a smooth Riemannian manifold. This is a vast area of research for which ([BP07]) is a detailed reference. The following is a brief description in the discrete case: let $f : M \to M$ be a diffeomorphism of a Riemannian manifold $M$ with $f-$invariant measure $\mu$. Let $A(x) = D_x f$. Then $B_n(x) = D_x f^n$, and the MET applies to show that the asymptotic infinitesimal data at almost every point on the manifold is well-defined and in some sense "regular."

From this infinitesimal data, the analogous stability theory developed is known as local stable and unstable manifolds. Assuming that Lyapunov exponents are nonzero, along with some other weak (in the sense of being widely satisfiable) conditions, nonuniform hyperbolicity theory further deduces even global data such as the existence of strong topological and ergodicity properties and a formula for the measure-theoretic Kolmogorov Sinai entropy of the system based on the positive Lyapunov exponents for systems preserving a Sinai-Ruelle-Bowen measure (a physically important class of measures). The results of this theory has seen connections to various other areas of research such as geometry, rigidity theory, and partial differential equations.

**Proof methods of the classical MET**

Many approaches to proving the MET exist in the literature. Our work takes the geometric approach of Kaimanovich and Karlsson Margulis.

Oseledets' original proof of the MET involves, via conjugation, reducing the matrices to lower triangular ones. A more popular proof approach, used in many later infinite-dimensional generalizations of the MET, originated from Ragunathan in [Rag79]. This proof involves showing that the eigenvalues of $|B_n(x)| := (B_n(x)^* B_n(x))^{1/2}$ have a well defined asymptotic exponential growth rate, and then that the eigenspaces of $|B_n(x)|$ also converge in some sense. The convergence of eigenvalues uses multilinear algebra and Kingman's subadditive ergodic theorem while the convergence of eigenspaces involves highly technical linear algebra.

A third major approach, originating from Kaimanovich ([Kau87]) and later generalized by Karlsson-Margulis ([KM99]) and others, involves the geometric idea of viewing $GL(d, \mathbb{R})$ as acting by isometries on $P(d, \mathbb{R})$, the space of positive definite symmetric matrices given a suitable metric making it a non-positively curved (in fact CAT(0)) space. The theorem in this setting is that (for almost every $x \in X$) starting from any point $y \in P(d, \mathbb{R})$ (in particular we can take $y = I$), the result of the action of $B_n(x)$ on $y$ is asymptotically close to a certain geodesic ray $\gamma(\cdot, x)$ starting from $y$. This geodesic ray in fact contains the information to recover the limit $\Lambda(x)$ described in the MET. Our work takes this geometric approach.

**Generalizations**

Generalizations of the MET to bounded operators on Hilbert and Banach spaces have appeared in the literature ([Rue82, Mn83, Blu16, LL10, Thi87, GTQ15, Sch91]). Correspondingly there have also been some results generalizing nonuniform hyperbolicity theory to infinite dimensions ([Rue82, LY12]). However, all of the above results are assume that the operators $B_n(x)$ satisfy some sort of quasi-compactness condition and so limit operators have discrete spectrum.

We now elaborate on generalizing the geometric approach to the MET. The metric on $P(d, \mathbb{R})$ making it into a non-positively curved space is given by $d(a, b) = \| \log(b^{-1/2} a b^{-1/2}) \|_2$, where for a matrix $A$, $\|A\|_2 = \sqrt{tr(A^*A)}$. This is known as the Frobenius or Hilbert-Schimdt norm on matrices. This norm can also be defined for bounded operators on a separable Hilbert space $\mathcal{H}$ with the standard trace $tr(A^*A) = \sum_i \|Ae_i\|^2$ (with $e_i$ a orthonormal basis for $\mathcal{H}$) whenever it is finite (i.e. on the set of Hilbert-Schmidt operators). In fact just as in the finite-dimensional setting this allows one to show that there is an associated non-positively curved associated space of positive operators on which the geometric approach can be used to obtain an MET.

The discussion above leads one to consider whether there are more general settings where a notion of trace allows one to obtain the objects and properties necessary to apply the geometric approach. Thus we are led to the setting of von Neumann algebras, viewed as a *-subalgebra of bounded operators on a Hilbert space closed under the strong operator topology, equipped with a faithful normal semifinite trace. This trace is a positive linear functional with additional properties, which are satisfied by the standard trace on a separable Hilbert space but also allow for more general phenomenon. For example, whereas the standard trace only takes a discrete values on projection operators, The space $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$ for some (nonatomic) measure space $(X, \mu)$ is a von Neumann algebra with trace $\tau(f) = \int_X f \mathrm{d}\mu$ which can take on continuum values on indicator functions. The trace of a projection operator can also be interpreted as giving a notion of **dimension** of the subspace associated with the projection. In general von Neumann algebras are noncommutative, but having a faithful normal trace allows one to define many notions and perform analysis analogous to classical measure theory such as $L^p$ spaces and convergence in measure.

The space of positive operators of a von Neumann algebra with a faithful normal finite trace has already been studied by Andruchow-Larontoda ([AL06]) and shown to be a nonpositively curved incomplete metric space. However, the theorem of Karlsson-Margulis that we eventually use includes in its hypothesis that the metric space be complete. Completing the metric space in order to apply the Karlsson-Margulis theorem is a major part of our work, requiring us to deal with unbounded operators affiliated with the von Neumman algebra. The space of such operators is suggestively called "$L^0$".

The results we obtain include a limit operator (Theorem 3.1.1), versions of Lyapunov distribution and equivariance of subspaces (Theorem 3.1.2), growth rates (Theorem 3.1.4, and regularity (Theorem 3.1.3). Many of our results are weaker than in the classical setting because of complications arising in infinite dimensions such as continuous spectrum and the nonequivalence of the many topologies on operators, taken for granted to be equivalent in the finite-dimensional setting, that one works with in the course of proving the MET. These include the operator norm topology, the $L^2$ topology defined by the trace, the strong and weak operator topologies, and the convergence in measure topology.

**Further directions**

If clearer definitions and stronger results could be obtained on such notions as growth rates and regularity, perhaps one could embark on a project of extending the theory of infinite-dimensional nonuniform hyperbolicity to operators with continuous spectrum.

Our work can be compared to results of Haagerup-Schultz ([HS09]), where their motivation is to study the invariant subspace problem. They obtain a similar convergence result for powers of a single operator (i.e. when the dynamics is trivial) in a $\mathrm{II}_1$ factor, but in the strong operator topology and not assuming invertibility of the operators. Studying their work and also the work of Dykema and others ([DNZ18]) on norm convergence may provide further insights in our setting.

Another direction of study would be to generalize results in the literature concerning non-invertibile and semi-invertible versions of the MET (see [GTQ15]) and stability under perturbations of the cocycle (see [FGTQ19] and [BNV10]).

**Statement of contribution**

The dissertator's main contribution to this project is in generalizing the results, obtained by L. Bowen and B. Hayes in the setting of von Neumann algebras with finite trace, to the setting of von Neumann algebras with semifinite trace.

# Chapter 2

# A topological dynamical system with two different positive sofic entropies

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## 2.1 Introduction

The topological entropy of a homeomorphism $T : X \to X$ of a compact Hausdorff space $X$ was introduced in [AKM65]. It was generalized to actions of amenable groups via Følner sequences in the 1970s [MO85] and to certain non-amenable groups via sofic approximations more recently [KL11]. It plays a major role in the classification and structure theory of topological dynamical systems.

To explain further, suppose $\Gamma$ is a countable group with identity $1_\Gamma$ and $\sigma : \Gamma \to \mathrm{sym}(V)$ is a map where $V$ is a finite set and $\mathrm{sym}(V)$ is the group of permutations of $V$. It is not required that $\sigma$ is a homomorphism. Let $D \Subset \Gamma$ be finite and $\delta > 0$. Then $\sigma$ is called

- $(D, \delta)$-**multiplicative** if

$$\#\{v \in V : \ \sigma(gh)v = \sigma(g)\sigma(h)v \ \forall g, h \in D\} > (1 - \delta)|V|,$$

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- $(D,\delta)$**-trace preserving** if

$$\#\{v \in V : \sigma(f)v \neq v \ \forall f \in D \setminus \{1_\Gamma\}\} > (1-\delta)|V|,$$

- $(D,\delta)$**-sofic** if it is both $(D,\delta)$-multiplicative and $(D,\delta)$-trace preserving.

A **sofic approximation** to $\Gamma$ consists of a sequence $\Sigma = \{\sigma_i\}_{i\in\mathbb{N}}$ of maps $\sigma_i : \Gamma \to \mathrm{sym}(V_i)$ such that for all finite $D \subset \Gamma$, $\delta > 0$ and all but finitely many $i$, $\sigma_i$ is $(D,\delta)$-sofic. A group is **sofic** if it admits a sofic approximation. In this paper we will usually assume $|V_i| = i$.

If $\Gamma$ acts by homeomorphisms on a compact Hausdorff space $X$ and a sofic approximation $\Sigma$ to $\Gamma$ is given then the $\Sigma$**-entropy** of the action is a topological conjugacy invariant, denoted by $h_\Sigma(\Gamma \curvearrowright X) \in \{-\infty\} \cup [0,\infty]$. It is also called **sofic entropy** if $\Sigma$ is understood. It was first defined in [KL11] where the authors obtain a variational principle connecting it with the previously introduced notion of sofic measure entropy [Bow10]. It is monotone under embeddings and additive under direct products but not monotone under factor maps. See [Bow17] for a survey.

A curious feature of this new entropy is that it may depend on the choice of sofic approximation. This is not always the case; for example, if $\Gamma$ is amenable then sofic entropy and classical entropy always agree. However, there are examples of actions $\Gamma \curvearrowright X$ by non-amenable groups $\Gamma$ with sofic approximations $\Sigma_1, \Sigma_2$ satisfying

$$h_{\Sigma_1}(\Gamma \curvearrowright X) = -\infty < h_{\Sigma_2}(\Gamma \curvearrowright X).$$

See [Bow17, Theorem 4.1]. The case $h_{\Sigma_1}(\Gamma \curvearrowright X) = -\infty$ is considered degenerate: it implies that there are no good models for the action with respect to the given sofic approximation. Until this paper, it was an open problem whether a mixing action could have two different *non-negative* values of sofic entropy. Our main result is:

**Theorem 2.1.1.** *There exists a countable group $\Gamma$, a mixing action $\Gamma \curvearrowright X$ by homeomorphisms on a compact metrizable space $X$ and two sofic approximations $\Sigma_1, \Sigma_2$ to $\Gamma$ such that*

$$0 < h_{\Sigma_1}(\Gamma \curvearrowright X) < h_{\Sigma_2}(\Gamma \curvearrowright X) < \infty.$$

*Remark* 6. The **range of sofic entropies** for an action $\Gamma \curvearrowright X$ is the set of all non-negative numbers of the form $h_\Sigma(\Gamma \curvearrowright X)$ as $\Sigma$ varies over all sofic approximations to $\Gamma$. By taking disjoint unions of copies of sofic approximations, it is possible to show the range of sofic entropies is an interval (which may be empty or a singleton). So for the example of Theorem 2.1.1, the range of sofic entropies is uncountable.

*Remark* 7. It remains an open problem whether there is a measure-preserving action $\Gamma \curvearrowright (X, \mu)$ with two different non-negative sofic entropies. Theorem 2.1.1 does not settle this problem because it is entirely possible that any invariant measure $\mu$ on $X$ with $h_{\Sigma_2}(\Gamma \curvearrowright (X, \mu)) > h_{\Sigma_1}(\Gamma \curvearrowright X)$ satisfies $h_{\Sigma_1}(\Gamma \curvearrowright (X, \mu)) = -\infty$.

In this paper we often assume $V_n = [n] := \{1, 2, ..., n\}$

### 2.1.1 Random sofic approximations

We do not know of any explicit sofic approximations to $\Gamma$ which are amenable to analysis. Instead, we study **random sofic approximations**. For the purposes of this paper, these are sequences $\{\mathbb{P}_n\}_n$ of probability measures $\mathbb{P}_n$ on spaces of homomorphisms $\mathrm{Hom}(\Gamma, \mathrm{sym}(n))$ such that, for any finite $D \subset \Gamma$ and $\delta > 0$ there is an $\epsilon > 0$ such that

$$\mathbb{P}_n(\sigma \text{ is } (D, \delta)\text{-sofic}) > 1 - n^{-\epsilon n}$$

for all sufficiently large $n$. Because $n^{-\epsilon n}$ decays super-exponentially, if $\Omega_n \subset \mathrm{Hom}(\Gamma, \mathrm{sym}(n))$ is any sequence with an exponential lower bound of the form $\mathbb{P}_n(\Omega_n) > e^{-cn}$ (for some constant $c > 0$) then there exists a sofic approximation $\Sigma = \{\sigma_n\}$ with $\sigma_n \in \Omega_n$ for all $n$.

It is this non-constructive existence result that enables us to use random sofic approximations to prove Theorem 2.1.1.

### 2.1.2 Proper colorings of random hyper-graphs from a statistical physics viewpoint

The idea for main construction comes from studies of proper colorings of random hyper-graphs. Although these studies have very different motivations than those that inspired this paper, the examples that they provide are roughly the same as the examples used to prove Theorem 2.1.1. The relevant literature and an outline is presented next.

A **hyper-graph** is a pair $G = (V, E)$ where $E$ is a collection of subsets of $V$. Elements of $E$ are called **hyper-edges** but we will call them **edges** for brevity's sake. $G$ is $k$**-uniform** if every edge $e \in E$ has cardinality $k$.

A **2-coloring** of $G$ is a map $\chi : V \to \{0, 1\}$. An edge $e \in E$ is **monochromatic** for $\chi$ if $|\chi(e)| = 1$. A coloring is **proper** if it has no monochromatic edges.

Let $H_k(n, m)$ denote a hyper-graph chosen uniformly among all $\binom{\binom{n}{k}}{m}$ $k$-uniform hyper-graphs with $n$ vertices and $m$ edges. We will consider the number of proper 2-colorings of $H_k(n, m)$ when $k$ is large but fixed, and the ratio of edges to vertices $r := m/n$ is bounded above and below by constants.

This random hyper-graph model was studied in [AM06, COZ11, COZ12]. These works are motivated by the *satisfiability conjecture*. To explain, the **lower satisfiability threshold** $r^-_{\text{sat}} = r^-_{\text{sat}}(k)$ is the supremum over all $r$ such that

$$\lim_{n\to\infty} \text{Prob}[H_k(n, \lceil rn\rceil) \text{ is properly 2-colorable}] = 1.$$

The **upper satisfiability threshold** $r^+_{\text{sat}} = r^+_{\text{sat}}(k)$ is the infimum over all $r$ such that

$$\lim_{n\to\infty} \text{Prob}[H_k(n, \lceil rn\rceil) \text{ is properly 2-colorable}] = 0.$$

The satisfiability conjecture posits that $r^-_{\text{sat}} = r^+_{\text{sat}}$. It is still open.

Bounds on these thresholds were first obtained in [AM06] as follows. Let $Z(G)$ be the number of proper 2-colorings of a hyper-graph $G$. A first moment computation shows that

$$f_k(r) = \lim_{n\to\infty} n^{-1} \log \mathbb{E}[Z(H_k(n, \lceil rn\rceil))]$$

where $f_k(r) := \log(2) + r\log(1 - 2^{1-k})$. Let $r_{\text{first}} = r_{\text{first}}(k)$ be such that $f_k(r_{\text{first}}) = 0$. If $r > r_{\text{first}}$ then $f_k(r) < 0$. Therefore $r^+_{\text{sat}} \le r_{\text{first}}$.

Let $r_{\text{second}}$ be the supremum over numbers $r \ge 0$ such that the second moment $\mathbb{E}[Z(H_k(n, \lceil rn\rceil))^2]$ is equal to $\mathbb{E}[Z(H_k(n, \lceil rn\rceil))]^2$ up to sub-exponential factors. The Paley-Zygmund inequality gives the bound $r_{\text{second}} \le r^-_{\text{sat}}$.

In [AM06], it is shown that

$$\begin{aligned} r_{\text{first}} &= \frac{\log(2)}{2}2^k - \frac{\log(2)}{2} + O(2^{-k}), \\ r_{\text{second}} &= \frac{\log(2)}{2}2^k - \frac{\log(2)+1}{2} + O(2^{-k}). \end{aligned}$$

So there is a constant-sized gap between the two thresholds.

A more detailed view of the second moment is illuminating. But before explaining, we need some terminology. Let $[n]$ be the set of natural numbers $\{1, 2, .., n\}$. A coloring $\chi$ of $[n]$ is **equitable** if $|\chi^{-1}(0)| = |\chi^{-1}(1)|$. We will assume from now on that $n$ is even so that equitable colorings of $[n]$ exist. Let $Z_e(G)$ be the number of equitable proper colorings of a hyper-graph $G$. A computation shows that $\mathbb{E}[Z(H_k(n, \lceil rn \rceil))]$ equals $\mathbb{E}[Z_e(H_k(n, \lceil rn \rceil))]$ up to sub-exponential factors. This enables us to work with equitable proper colorings in place of all proper colorings. This reduces the computations because there is only one equitable coloring up to the action of the symmetric group $\mathrm{sym}(n)$.

A computation shows that the second moment factorizes as

$$\mathbb{E}[Z_e(H_k(n, m))^2] = \mathbb{E}[Z_e(H_k(n, m))]\mathbb{E}[Z_e(H_k(n, m))|\chi \text{ is proper}]$$

where $\chi : [n] \to \{0, 1\}$ is any equitable 2-coloring. Let $H_k^\chi(n, m)$ be the random hyper-graph chosen by conditioning $H_k(n, m)$ on the event that $\chi$ is a proper 2-coloring. This is called the **planted model** and $\chi$ is the **planted coloring**. So computing the second moment of $Z_e(H_k(n, m))$ reduces to computing the first moment of $Z_e(H_k^\chi(n, m))$.

The **normalized Hamming distance** between colorings $\chi, \chi' : [n] \to \{0, 1\}$ is

$$d_n(\chi, \chi') = n^{-1}\#\{v \in [n] : \ \chi(v) \neq \chi'(v)\}.$$

Let $Z^\chi(\delta)$ be the number of equitable proper colorings $\chi'$ with $d_n(\chi, \chi') = \delta$. Then

$$Z_e(H_k^\chi(n, m)) = \sum_\delta Z^\chi(\delta).$$

In [AM06], it is shown that $\mathbb{E}[Z^\chi(\delta)|\chi \text{ is proper}]$ is equal to $\exp(n\psi(\delta))$ (up to sub-exponential factors) where $\psi$ is an explicit function.

Note that $\psi(\delta) = \psi(1-\delta)$ (since if $\chi'$ is a proper equitable coloring then so is $1-\chi'$ and $d_n(\chi, 1-\chi') = 1 - d_n(\chi, \chi')$). A computation shows $\psi(1/2) = f_k(r)$. If $r < r_{\text{second}}$ then $\psi(\delta)$ is uniquely maximized at $\delta = 1/2$. However, if $r > r_{\text{second}}$ then the maximum of $\psi$ is attained in the interval $\delta \in [0, 2^{-k/2}]$. In fact, $\psi(\delta)$ is negative for $\delta \in [2^{-k/2}, 1/2 - 2^{-k/2}]$. So with high probability, there are no proper equitable colorings $\chi'$ with $d_n(\chi, \chi') \in [2^{-k/2}, 1/2 - 2^{-k/2}]$. This motivates defining the **local cluster**, denoted $\mathcal{C}(\chi)$, to be the set of all proper equitable 2-colorings $\chi'$ with $d_n(\chi, \chi') \leq 2^{-k/2}$.

The papers [COZ11, COZ12] obtain a stronger lower bound on the lower satisfiability threshold using an argument they call the *enhanced second moment method*. To explain, we need some terminology. We say a proper equitable coloring $\chi$ is **good** if the size of the local cluster $|\mathcal{C}(\chi)|$ is bounded by $\mathbb{E}[Z_e(H_k(n,m))]$. One of the main results of [COZ11, COZ12] is that $\Pr[\chi \text{ is good} | \chi \text{ is proper}]$ tends to 1 as $n \to \infty$ with $m = rn + O(1)$ and $r < r_{\text{second}} + \frac{1-\log(2)}{2} + o_k(1)$. An application of the Paley-Zygmund inequality to the number of good colorings yields the improved lower bound

$$r_{\text{second}} + \frac{1-\log(2)}{2} + o_k(1) \leq r_{\text{sat}}^-.$$

The argument showing $\Pr[\chi \text{ is good} | \chi \text{ is proper}] \to 1$ is combinatorial. It is shown that (with high probability) there is a set $R \subset [n]$ with cardinality $|R| \approx (1-2^{-k})n$ which is rigid in the following sense: if $\chi' : [n] \to \{0,1\}$ is any proper equitable 2-coloring then either: the restriction of $\chi'$ to $R$ is the same as the restriction of $\chi$ to $R$ or $d_n(\chi', \chi)$ is at least $cn/k^t$ for some constants $c, t > 0$. This rigid set is constructed explicitly in terms of local combinatorial data of the coloring $\chi$ on $H_k^\chi(n,m)$.

In summary, these papers study two random models $H_k(n,m)$ and $H_k^\chi(n,m)$. When $r = m/n$ is in the interval $(r_{\text{second}}, r_{\text{second}} + \frac{1-\log(2)}{2})$, the typical number of proper colorings of $H_k(n,m)$ grows exponentially in $n$ but is smaller (by an exponential factor) than the expected number of proper colorings of $H_k^\chi(n,m)$. It is these facts that we will generalize, by replacing $H_k(n,m), H^\chi(n,m)$ with random sofic approximations to a group $\Gamma$ so that the exponential growth rate of the number of proper colorings roughly corresponds with sofic entropy.

Although the models that we study in this paper are similar to the models in [AM06, COZ11, COZ12], they are different enough that we develop all results from scratch. Moreover, although the strategies we employ are roughly same, the proof details differ substantially. The reader need not be familiar with these papers to read this paper.

### 2.1.3 The action

In the rest of this introduction, we introduce the action $\Gamma \curvearrowright X$ in Theorem 2.1.1 and outline the first steps of its proof. So fix positive integers $k, d$. Let

$$\Gamma = \langle s_1, \ldots, s_d : \; s_1^k = s_2^k = \cdots = s_d^k = 1 \rangle$$

be the free product of $d$ copies of $\mathbb{Z}/k\mathbb{Z}$.

The **Cayley hyper-tree** of $\Gamma$, denoted $G = (V, E)$, has vertex set $V = \Gamma$. The edges are the left-cosets of the generator subgroups. That is, each edge $e \in E$ has the form $e = \{gs_i^j : \; 0 \leq j \leq k-1\}$ for some $g \in \Gamma$ and $1 \leq i \leq d$.

*Remark* 8. It can be shown by considering each element of $\Gamma$ as a reduced word in the generators $s_1, ..., s_d$ that $G$ is a hyper-tree in the sense that there exists a unique "hyper-path" between any two vertices. More precisely, for any $v, w \in V$, there exists a unique sequence of edges $e_1, .., e_l$ such that $v \in e_1$, $w \in e_l$, $|e_i \cap e_{i+1}| = 1$, $e_i \neq e_j$ for any $i \neq j$, and $v \notin e_2$, $w \notin e_{l-1}$. More intuitively, there are no "hyper-loops" in $G$.

The group $\Gamma$ acts on $\{0,1\}^\Gamma$ by $(gx)_f = x_{g^{-1}f}$ for $g, f \in \Gamma, x \in \{0,1\}^\Gamma$. Let $X \subset \{0,1\}^\Gamma$ be the subset of proper 2-colorings. It is a closed $\Gamma$-invariant subspace. Furthermore, $\Gamma \curvearrowright X$ is topologically mixing:

**Claim 1.** *For any nonempty open sets $A$, $B$ in $X$, there exists $N$ such that for any $g \in \Gamma$ with $|g| > N$, $gA \cap B \neq \emptyset$. Here $|g|$ denotes the shortest word length of representations of $g$ by generators $s_1, ..., s_d$.*

*Proof.* It suffices to show the claim for $A$,$B$ being cylinder sets. We make a further simplification by assuming each $A$, $B$ is a cylinder set on a union of hyperedges, and a yet further simplification that each $A$, $B$ is a cylinder set on a connected union hyperedges. Informally, by shifting the "coordinates" on which $A$ depends so that they are far enough separated from the coordinates on which $B$ depends, we can always fill in the rest of the graph to get a proper coloring.

More precisely, suppose $A = \{x \in X : x \upharpoonright F_A = \chi_A\}$, where $\mathcal{F}_A \subset E$ such that for any $e \in \mathcal{F}_A$ there exists $f \in \mathcal{F}_A$ such that $e \cap f \neq \emptyset$, $F_A = \cup_{e \in \mathcal{F}_A} e$, and $\chi_A : F_A \to \{0, 1\}$ and similarly $B = \{x \in X : x \upharpoonright F_B = \chi_B\}$. $\chi_A$ and $\chi_B$ must be bichromatic on each edge in their respective domains since $A$ and $B$ are nonempty.

Let $N = \max\{|h| : h \in F_A\} + \max\{|h| : h \in F_B\} + k$. Then it can be shown for any $g$ with $|g| > N$, that $g^{-1}F_A \cap F_B = \emptyset$. It follows from our earlier remark that there exists a unique hyper-path connecting $g^{-1}F_A$ to $F_B$ (otherwise there would be a hyperloop in $G$). Thus for example one can recursively fill in a coloring on the rest of $\Gamma$ by levels of hyperedges - first the hyperedges adjacent to $g^{-1}F_A$ and $F_B$, then the next layer of adjacent hyperedges, and so on. At each step, most hyperedges only have one vertex whose color is determined, so it is always possible to color another vertex of an edge to make it bichromatic. Only along the hyper-path connecting $g^{-1}F_A$ to $F_B$ at some step there will be a hyperedge with two vertices whose colors are already determined, but since $k$ is large there is still another vertex to color to make the edge bichromatic.

□

We will show that for certain values of $k, d$, the action $\Gamma \curvearrowright X$ satisfies the conclusion of Theorem 2.1.1.

### 2.1.4 Sofic entropy of the shift action on proper colorings

Given a homomorphism $\sigma : \Gamma \to \mathrm{sym}(V)$, let $G_\sigma = (V, E_\sigma)$ be the hyper-graph with vertices $V$ and edges equal to the orbits of the generator subgroups. That is, a subset $e \subset V$ is an edge if and only if $e = \{\sigma(s_i^j)v\}_{j=0}^{k-1}$ for some $1 \leq i \leq d$ and $v \in V$.

A hyper-graph is **$k$-uniform** if every edge has cardinality $k$. We will say that a homomorphism $\sigma : \Gamma \to \mathrm{sym}(V)$ is **uniform** if $G_\sigma$ is $k$-uniform. Equivalently, this occurs if for all $1 \leq i \leq d$, $\sigma(s_i)$ decomposes into a disjoint union of $k$-cycles.

A 2-coloring $\chi : V \to \{0, 1\}$ of a hyper-graph $G$ is **$\epsilon$-proper** if the number of monochromatic edges is $\leq \epsilon|V|$. Using the formulation of sofic entropy in [Bow17] (which was inspired by [Aus16]), we show in §2.2 that if $\Sigma = \{\sigma_n\}_{n\ge 1}$ is a sofic approximation to $\Gamma$ by uniform homomorphisms then the $\Sigma$-entropy of $\Gamma \curvearrowright X$ is:

$$h_\Sigma(\Gamma \curvearrowright X) := \inf_{\epsilon>0} \limsup_{i\to\infty} |V_i|^{-1} \log \#\{\epsilon\text{-proper 2-colorings of } G_{\sigma_i}\}.$$

### 2.1.5 Random hyper-graph models

**Definition 1.** Let $\mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ denote the set of all uniform homomorphisms from $\Gamma$ to $\mathrm{sym}(n)$. Let $\mathbb{P}_n^u$ be the uniform probability measure on $\mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ and let $\mathbb{E}_n^u$ be its expectation operator. The measure $\mathbb{P}_n^u$ is called the **uniform model**. We will always assume $n \in k\mathbb{Z}$ so that $\mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ is non-empty. In §2.3 we show that $\{\mathbb{P}_n^u\}_{n\ge1}$ is a random sofic approximation. We will use the uniform model to obtain the sofic approximation $\Sigma_1$ which appears in Theorem 2.1.1.

If $V$ is a finite set, then a 2-coloring $\chi : V \to \{0,1\}$ is **equitable** if $|\chi^{-1}(0)| = |\chi^{-1}(1)|$. We will assume from now on that $n$ is even so that equitable colorings of $[n]$ exist.

**Definition 2.** Fix an equitable coloring $\chi : [n] \to \{0,1\}$. Let $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ be the set of all uniform homomorphisms $\sigma : \Gamma \to \mathrm{sym}(n)$ such that $\chi$ is proper as a coloring on $G_\sigma$. Let $\mathbb{P}_n^\chi$ be the uniform probability measure on $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ and let $\mathbb{E}_n^\chi$ be its expectation operator. The measure $\mathbb{P}_n^\chi$ is called the **planted model** and $\chi$ is the **planted coloring**. When $\chi$ is understood, we will write $\mathbb{P}_n^p$ and $\mathbb{E}_n^p$ instead of $\mathbb{P}_n^\chi$ and $\mathbb{E}_n^\chi$. In §2.3 we show that $\{\mathbb{P}_n^p\}_{n\ge1}$ is a random sofic approximation. We will use the planted model to obtain the sofic approximation $\Sigma_2$ which appears in Theorem 2.1.1.

*Remark* 9. If $\chi$ and $\chi'$ are both equitable 2-colorings then there are natural bijections from $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ to $\mathrm{Hom}_{\chi'}(\Gamma, \mathrm{sym}(n))$ as follows. Given a permutation $\pi \in \mathrm{sym}(n)$ and $\sigma : \Gamma \to \mathrm{sym}(n)$, define $\sigma^\pi : \Gamma \to \mathrm{sym}(n)$ by $\sigma^\pi(g) = \pi\sigma(g)\pi^{-1}$. Because $\chi$ and $\chi'$ are equitable, there exists $\pi \in \mathrm{sym}(n)$ such that $\chi = \chi' \circ \pi$. The map $\sigma \mapsto \sigma^\pi$ defines a bijection from $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ to $\mathrm{Hom}_{\chi'}(\Gamma, \mathrm{sym}(n))$. Moreover $\pi$ defines an hyper-graph-isomorphism from $G_\sigma$ to $G_{\sigma^\pi}$. Therefore, any random variable on $\mathrm{Hom}(\Gamma, \mathrm{sym}(n))$ that depends only on the hyper-graph $G_\sigma$ up to hyper-graph-isomorphism has the same distribution under $\mathbb{P}_n^\chi$ as under $\mathbb{P}_n^{\chi'}$. This justifies calling $\mathbb{P}_n^\chi$ *the* planted model.

### 2.1.6 The strategy and a key lemma

The idea behind the proof of Theorem 2.1.1 is to show that for some choices of $(k, d)$, the uniform model admits an exponential number of proper 2-colorings, but it has exponentially fewer proper 2-colorings than the expected number of proper colorings of the planted model (with probability that decays at most sub-exponentially in $n$).

To make this strategy more precise, we introduce the following notation. Let $Z(\epsilon; \sigma)$ denote the number of $\epsilon$-proper 2-colorings of $G_\sigma$. A coloring is **$\sigma$-proper** if it is $(0, \sigma)$-proper. Let $Z(\sigma) = Z(0; \sigma)$ be the number of $\sigma$-proper 2-colorings.

In §2.3, the proof of Theorem 2.1.1 is reduced to the Key Lemma:

**Lemma 2.1.2** (Key Lemma)**.** *Let $f(d, k) := \log(2) + \frac{d}{k}\log(1 - 2^{1-k})$. Also let $r = d/k$. Then*

$$f(d,k) \;=\; \lim_{n\to\infty} n^{-1}\log \mathbb{E}_n^u[Z(\sigma)] = \inf_{\epsilon>0}\limsup_{n\to\infty} n^{-1}\log \mathbb{E}_n^u[Z(\epsilon;\sigma)]. \tag{2.1}$$

*Moreover, for any*

$$0 < \eta_0 < \eta_1 < (1 - \log 2)/2$$

*there exists $k_0$ (depending on $\eta_0, \eta_1$) such that for all $k \geq k_0$ if*

$$r = d/k = \frac{\log(2)}{2} \cdot 2^k - (1 + \log(2))/2 + \eta$$

*for some $\eta \in [\eta_0, \eta_1]$ then*

$$f(d,k) \;<\; \liminf_{n\to\infty} n^{-1}\log \mathbb{E}_n^p[Z(\sigma)]. \tag{2.2}$$

*Also,*

$$0 \;=\; \inf_{\epsilon>0}\liminf_{n\to\infty} n^{-1}\log\left(\mathbb{P}_n^u\left(\left|n^{-1}\log Z(\sigma) - f(d,k)\right| < \epsilon\right)\right). \tag{2.3}$$

*In all cases above, the limits are over $n \in 2\mathbb{Z} \cap k\mathbb{Z}$.*

Equations (2.1) and (2.2) are proven in §2.4 and §2.5 using first and second moment arguments respectively. This part of the paper is similar to the arguments used in [AM06].

Given $\sigma : \Gamma \to \mathrm{sym}(n)$ and $\chi : [n] \to \{0,1\}$, let $\mathcal{C}_\sigma(\chi)$ be the set of all proper equitable colorings $\chi' : [n] \to \{0,1\}$ with $d_n(\chi,\chi') \le 2^{-k/2}$. In section §2.5.2, second moment arguments are used to reduce equation (2.3) to the following:

**Proposition 2.5.9**. *Let $0 < \eta_0 < (1-\log 2)/2$. Then for all sufficiently large $k$ (depending on $\eta_0$), if*

$$r := d/k = \frac{\log(2)}{2} \cdot 2^k - (1+\log(2))/2 + \eta$$

*for some $\eta$ with $\eta_0 \le \eta < (1-\log 2)/2$ then with high probability in the planted model, $|\mathcal{C}_\sigma(\chi)| \le \mathbb{E}^u_n(Z_e)$. In symbols,*

$$\lim_{n\to\infty} \mathbb{P}^\chi_n\big(|\mathcal{C}_\sigma(\chi)| \le \mathbb{E}^u_n(Z_e)\big) = 1.$$

In §2.6, Proposition 2.5.9 is reduced as follows. First, certain subsets of vertices are defined through local combinatorial constraints. There are two main lemmas concerning these subsets; one of which bounds their density and the other proves they are 'rigid'. Proposition 2.5.9 is proven in §2.6 assuming these lemmas.

The density lemma is proven in §2.7 using a natural Markov model on the space of proper colorings that is the local-on-average limit of the planted model. Rigidity is proven in §2.8 using an expansitivity argument similar to the way random regular graphs are proven to be good expanders. This completes the last step of the proof of Theorem 2.1.1.

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## 2.2 Topological sofic entropy

This section defines topological sofic entropy for subshifts using the formulation from [Aus16]. The main result is:

**Lemma 2.2.1.** *For any sofic approximation $\Sigma = \{\sigma_n\}$ with $\sigma_n \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$,*

$$h_\Sigma(\Gamma \curvearrowright X) = \inf_{\epsilon>0} \limsup_{n\to\infty} n^{-1} \log Z(\epsilon; \sigma_n).$$

Let $\Gamma$ denote a countable group, $\mathcal{A}$ a finite set (called the **alphabet**). Let $T = (T^g)_{g\in\Gamma}$ be the shift action on $\mathcal{A}^\Gamma$ defined by $T^g x(f) = x(g^{-1}f)$ for $x \in A^\Gamma$. Let $X \subset \mathcal{A}^\Gamma$ be a closed $\Gamma$-invariant subspace. We denote the restriction of the action to X by $\Gamma \curvearrowright X$. Also let $\Sigma = \{\sigma_i : \Gamma \to \mathrm{Sym}(V_i)\}_{i\in\mathbb{N}}$ be a sofic approximation to $\Gamma$.

Given $\sigma : \Gamma \to \mathrm{Sym}(V)$, $v \in V$ and $x : V \to \mathcal{A}$ the **pullback name of $x$ at $v$** is defined by

$$\Pi_v^\sigma(x) \in \mathcal{A}^\Gamma, \quad \Pi_v^\sigma(x)(g) = x_{\sigma(g^{-1})v} \quad \forall g \in \Gamma.$$

For the sake of building some intuition, note that when $\sigma$ is a homomorphism, the map $v \mapsto \Pi_v^\sigma(x)$ is $\Gamma$-equivariant (in the sense that $\Pi_{\sigma(g)v}^\sigma(x) = g\Pi_v^\sigma(x)$). In particular $\Pi_v^\sigma(x) \in \mathcal{A}^\Gamma$ is periodic. In general, we think of $\Pi_v^\sigma(x)$ as an approximate periodic point.

Given an open set $\mathcal{O} \subset \mathcal{A}^\Gamma$ containing $X$ and an $\epsilon > 0$, a map $x : V \to \mathcal{A}$ is called an $(\mathcal{O}, \epsilon, \sigma)$**-microstate** if

$$\#\{v \in V :\ \Pi_v^\sigma(x) \in \mathcal{O}\} \geq (1-\epsilon)|V|.$$

Let $\Omega(\mathcal{O}, \epsilon, \sigma) \subset \mathcal{A}^V$ denote the set of all $(\mathcal{O}, \epsilon, \sigma)$-microstates. Finally, the $\Sigma$**-entropy** of the action is defined by

$$h_\Sigma(\Gamma \curvearrowright X) := \inf_{\mathcal{O}} \inf_{\epsilon>0} \limsup_{i\to\infty} |V_i|^{-1} \log \#\Omega(\mathcal{O}, \epsilon, \sigma_i)$$

where the infimum is over all open neighborhoods of $X$ in $\mathcal{A}^\Gamma$. This number depends on the action $\Gamma \curvearrowright X$ only up to topological conjugacy. It is an exercise in [Bow17] to show that this definition agrees with the definition in [KL13]. We include a proof in Appendix 2.9 for completeness.

*Proof of Lemma 2.2.1.* Let $\epsilon > 0$ be given. Let $S(\epsilon; \sigma_n) \subset 2^{V_n}$ be the set of $(\epsilon, \sigma_n)$-proper 2-colorings. Let $\mathcal{O}_0 \subset 2^\Gamma$ be the set of all 2-colorings $\chi : \Gamma \to \{0, 1\}$ such that for each generator hyper-edge $e \subset \Gamma$, $\chi(e) = \{0, 1\}$. A **generator hyper-edge** is a subgroup of the form $\{s_i^j :\ 0 \leq j < k\}$ for some $i$. Note $\mathcal{O}_0$ is an open superset of $X$.

We claim that $\Omega(\mathcal{O}_0, k\epsilon/d, \sigma_n) \subset S(\epsilon; \sigma_n)$. To see this, let $\chi \in \Omega(\mathcal{O}_0, k\epsilon/d, \sigma_n)$. Then $\Pi_v^{\sigma_n}(\chi) \in \mathcal{O}_0$ if and only if all hyper-edges of $G_\sigma$ containing $v$ are bi-chromatic (with respect to $\chi$). So if $\Pi_v^{\sigma_n} \notin \mathcal{O}_0$, then $v$ is contained in up to $d$ monochromatic hyperedges. On the other hand, each monochromatic hyperedge contains exactly $k$ vertices whose pullback name is not in $\mathcal{O}_0$. It follows that $\chi \in S(\epsilon; \sigma_n)$. This implies $h_\Sigma(\Gamma \curvearrowright X) \leq \inf_{\epsilon>0} \limsup_{n\to\infty} n^{-1} \log Z(\epsilon; \sigma_n)$.

Given a finite subset $\mathcal{F}$ of hyper-edges of the Cayley hyper-tree, let $\mathcal{O}_\mathcal{F}$ be the set of all $\chi \in 2^\Gamma$ with the property that $\chi(e) = \{0, 1\}$ for all $e \in \mathcal{F}$. If $\mathcal{O}'$ is any open neighborhood of $X$ in $2^\Gamma$ then $\mathcal{O}'$ contains $\mathcal{O}_\mathcal{F}$ for some $\mathcal{F}$. To see this, suppose that there exist elements $\chi_\mathcal{F} \in \mathcal{O}_\mathcal{F} \setminus \mathcal{O}'$ for every finite $\mathcal{F}$. Let $\chi$ be a cluster point of $\{\chi_\mathcal{F}\}$ as $\mathcal{F}$ increases to the set $E$ of all hyper-edges. Then $\chi \in X \setminus \mathcal{O}'$, a contradiction. It follows that

$$h_\Sigma(\Gamma \curvearrowright X) = \inf_{\mathcal{F}} \inf_{\epsilon>0} \limsup_{i\to\infty} |V_i|^{-1} \log \#\Omega(\mathcal{O}_\mathcal{F}, \epsilon, \sigma_i).$$

Next, fix a finite subset $\mathcal{F}$ of hyper-edges of the Cayley hyper-tree. We claim that $S\left(\frac{\epsilon}{k|\mathcal{F}|}; \sigma_n\right) \subset \Omega(\mathcal{O}_\mathcal{F}, \epsilon, \sigma_n)$. To see this, let $\chi \in S\left(\frac{\epsilon}{k|\mathcal{F}|}; \sigma_n\right)$ and $B(\chi, \sigma_n) \subset V_n$ be the set of vertices contained in a monochromatic edge of $\chi$. Now for $v \in V_n$, $\Pi_v^{\sigma_n}(\chi) \notin \mathcal{O}_\mathcal{F}$ if and only if $\Pi_v^{\sigma_n}(\chi)$ is monochromatic on some edge in $\mathcal{F}$. This occurs if and only if there is an element $f \in \Gamma$ in the union of $\mathcal{F}$ such that $\sigma_n(f^{-1})v \in B(\chi, \sigma_n)$. There are at most $|\mathcal{F}|B(\chi, \sigma_n)$ such vertices. But $|B(\chi, \sigma_n)| \leq (\frac{\epsilon}{|F|})n$, so there are at most $\epsilon n$ such vertices. It follows that $\chi \in \Omega(\mathcal{O}_\mathcal{F}, \epsilon, \sigma_n)$. Therefore,

$$\inf_{\epsilon>0} \limsup_{n\to\infty} n^{-1} \log Z(\epsilon; \sigma_n) \leq \inf_{\mathcal{F}} \inf_{\epsilon>0} \limsup_{i\to\infty} |V_i|^{-1} \log \#\Omega(\mathcal{O}_\mathcal{F}, \epsilon, \sigma_i) = h_\Sigma(\Gamma \curvearrowright X).$$

□

## 2.3 Reduction to the key lemma

The purpose of this section is to show how Lemma 2.1.2 implies Theorem 2.1.1. This requires replacing the (random) uniform and planted models with (deterministic) sofic approximations. The next lemma facilitates this replacement.

**Lemma 2.3.1.** *Let $D \subset \Gamma$ be finite and $\delta > 0$. Then there are constants $\epsilon, N_0 > 0$ such that* *for all $n > N_0$ with $n \in 2\mathbb{Z} \cap k\mathbb{Z}$,*

$$\mathbb{P}_n^u\{\sigma : \ \sigma \text{ is not } (\mathrm{D}, \delta)\text{−sofic}\} \le \mathrm{n}^{-\epsilon \mathrm{n}},$$

$$\mathbb{P}_n^p\{\sigma : \ \sigma \text{ is not } (\mathrm{D}, \delta)\text{−sofic}\} \le \mathrm{n}^{-\epsilon \mathrm{n}}.$$

*Proof.* The proof given here is for the uniform model. The planted model is similar.

The proof begins with a series of four reductions. By taking a union bound, it suffices to prove the special case in which $D = \{w\}$ for $w \in \Gamma$ nontrivial. (This is the first reduction).

Let $w = s_{i_l}^{r_l} \cdots s_{i_1}^{r_1}$ be the **reduced form** of $w$. This means that $i_j \in \{1, \dots, d\}$, $i_j \neq i_{j+1}$ for all $j$ with indices mod $l$ and $1 \le r_j < k$ for all $j$. Let $|w| = r_1 + \cdots + r_l$ be the **length** of $w$.

For any $g \in \Gamma$, the fixed point sets of $\sigma(gwg^{-1})$ and $\sigma(w)$ have the same size. So after conjugating if necessary, we may assume that either $l = 1$ or $i_1 \neq i_l$.

For $1 \le j \le l$, the **$j$-th beginning subword** of $w$ is the element $w_j = s_{i_j}^{r_j} \cdots s_{i_1}^{r_1}$. Given a vertex $v \in V_n$ and $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$, let $p(v, \sigma) = (e_1, \dots, e_l)$ be the path defined by: for each $j$, $e_j$ is the unique hyper-edge of $G_\sigma$ labeled $i_j$ containing $\sigma(w_j)v$. A vertex $v \in V_n$ **represents a $(\sigma, w)$-simple cycle** if $\sigma(w)v = v$ and for every $1 \le a < b \le l$, either

- $e_a \cap e_b = \emptyset$,
- $b = a + 1$ and $|e_a \cap e_b| = 1$,
- or $(a, b) = (1, l)$ and $|e_a \cap e_b| = 1$.

We say that $v$ **represents a $(\sigma, w)$-simple degenerate cycle** if $\sigma(w)v = v$ and $l = 2$ and $|e_1 \cap e_2| \ge 2$.

If $\sigma(w)v = v$ then either

- $v$ represents a $(\sigma, w)$-simple cycle,
- there exists nontrivial $w' \in \Gamma$ with $|w'| \le |w| + k$ such that some vertex $v_0 \in \cup_j e_j$ represents a $(\sigma, w')$-simple cycle,

- or there exists nontrivial $w' \in \Gamma$ with $|w'| \leq |w| + k$ such that some vertex $v_0 \in \cup_j e_j$ represents a $(\sigma, w')$-simple degenerate cycle.

So it suffices to prove there are constants $\epsilon, N_0 > 0$ such that for all $n > N_0$,

$$\mathbb{P}_n^u \big\{ \sigma : \ \#\{v \in [n] : \ v \text{ represents a } (\sigma, w)\text{-simple cycle}\} \geq \delta n \big\} \leq n^{-\epsilon n}.$$

and

$$\mathbb{P}_n^u \big\{ \sigma : \ \#\{v \in [n] : \ v \text{ represents a } (\sigma, w)\text{-simple degenerate cycle}\} \geq \delta n \big\} \leq n^{-\epsilon n}.$$

(This is the second reduction).

Two vertices $v, v' \in V_n$ **represent vertex-disjoint** $(\sigma, w)$**-cycles** if $p(v, \sigma) = (e_1, \dots, e_l), p(v', \sigma) = (e_1', \dots, e_l')$ and $e_i \cap e_j' = \emptyset$ for all $i, j$.

Let $G_n(\delta, w)$ be the set of all $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ such that there exists a subset $S \subset [n]$ satisfying

1. $|S| \geq \delta n$,
2. every $v \in S$ represents a $(\sigma, w)$-simple cycle,
3. the cycles $p(v, \sigma)$ for $v \in S$ are pairwise vertex-disjoint.

If $v$ represents a simple $(\sigma, w)$-cycle then there are at most $(kl)^2$ vertices $v'$ such that $v'$ also represents a simple $(\sigma, w)$-cycle but the two cycles are not vertex-disjoint. Since this bound does not depend on $n$, it suffices to prove there exist $\epsilon > 0$ and $N_0$ such that

$$\mathbb{P}_n^u(G_n(\delta, w)) \leq n^{-\epsilon n}$$

for all $n \geq N_0$. (This is the third reduction. The argument is similar for simple degenerate cycles).

Let $m = \lceil \delta n \rceil$ and $v_1, \dots, v_m$ be distinct vertices in $[n] = V_n$. For $1 \leq i \leq m$, let $F_i$ be the set of all $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ such that for all $1 \leq j \leq i$

1. $v_j$ represents a $(\sigma, w)$-simple cycle,

2. the cycles $p(v_1, \sigma), \ldots, p(v_i, \sigma)$ are pairwise vertex-disjoint.

By summing over all subsets of size $m$, we obtain

$$\mathbb{P}_n^u(G_n(\delta, w)) \leq \binom{n}{m} \mathbb{P}_n^u(F_m).$$

Since $\binom{n}{m} \approx e^{H(\delta, 1-\delta)n}$ grows at most exponentially, it suffices to show there exist $\epsilon > 0$ and $N_0$ such that $\mathbb{P}_n^u(F_m) \leq n^{-\epsilon n}$ for all $n \geq N_0$. (This is the fourth reduction. The argument is similar for simple degenerate cycles).

Set $F_0 = \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$. By the chain rule

$$\mathbb{P}_n^u(F_m) = \prod_{i=0}^{m-1} \mathbb{P}_n^u(F_{i+1}|F_i).$$

In order to estimate $\mathbb{P}_n^u(F_{i+1}|F_i)$, $F_i$ can be expressed a disjoint union over the cycles involved in its definition. To be precise, define an equivalence relation $\mathcal{R}_i$ on $F_i$ by: $\sigma, \sigma'$ are $\mathcal{R}_i$-equivalent if for every $1 \leq j \leq i$, $1 \leq q \leq l$ and $r > 0$

$$\sigma(s_{i_q}^r w_q)v_j = \sigma'(s_{i_q}^r w_q)v_j.$$

In other words, $\sigma, \sigma'$ are $\mathcal{R}_i$-equivalent if they define the same paths according to all vertices up to $v_i$ (so $p(v_j, \sigma) = p(v_j, \sigma')$) and their restrictions to every edge in these paths agree. Of course, $F_i$ is the disjoint union of the $\mathcal{R}_i$-classes. Note that $\mathcal{R}_0$ is trivial (everything is equivalent).

In general, if $A, B_1, \ldots, B_m$ are measurable sets and the $B_i$'s are pairwise disjoint then $\mathbb{P}(A| \cup_i B_i)$ is a convex combination of $\mathbb{P}(A|B_i)$ (for any probability measure $\mathbb{P}$). Therefore, $\mathbb{P}_n^u(F_{i+1}|F_i)$ is a convex combination of probabilities of the form $\mathbb{P}_n^u(F_{i+1}|B_i)$ where $B_i$ is an $\mathcal{R}_i$-class.

Now fix a $\mathcal{R}_i$-class $B_i$ (for some $i$ with $0 \leq i < m$). Let $K$ be the set of all vertices covered by the cycles defining $B_i$. To be precise, this means $K$ is the set of all $u \in [n] = V_n$ such that there exists an edge $e$ with $u \in e$ such that $e$ is contained in a path $p(v_j, \sigma)$ with $1 \leq j \leq i$ and $\sigma \in B_i$. Since each path covers at most $kl$ vertices, $|K| \leq ikl$.

If $l > 1$ (the case $l = 1$ is similar), fix subsets $e_1, \ldots, e_{l-1} \subset [n]$ of size $k$. Conditioned on $B_i$ and the event that the first $(l-1)$ edges of $p(v_{i+1},\sigma)$ are $e_1,\dots,e_{l-1}$, the $\mathbb{P}^u_n$-probability that $v_{i+1}$ represents a simple $(\sigma,w)$-cycle vertex-disjoint from $K$ is bounded by the probability that a uniformly random $k$-element subset of

$$[n] \setminus \left( \bigcup_{2\le j\le l-1} e_j \cup K \right)$$

conditioned to intersect $e_{l-1}$ nontrivially contains $v_{i+1}$. Since

$$\left| \bigcup_{2\le j\le l-1} e_j \cup K \right| \le (i+1)kl \le mkl = kl\lceil \delta n\rceil,$$

this probability is bounded by $C/n$ where $C = C(w,d,k,\delta)$ is a constant not depending on $n$ or the choice of $B_i$. It follows that $\mathbb{P}^u_n(F_{i+1}|F_i) \le C/n$ for all $0 \le i \le m-1$ and therefore

$$\mathbb{P}^u_n(F_m) \le (C/n)^m \le (C/n)^{\delta n}.$$

This implies the lemma (the argument is similar for simple degenerate cycles).

□

*Proof of Theorem 2.1.1 from Lemma 2.1.2.* Choose $d,k$ according to the hypotheses of Lemma 2.1.2 and construct $\Gamma \curvearrowright X$ according to Section 1.3.

Let $\epsilon,\delta > 0$ and $D \subset \Gamma$ be finite. Then there exists $\epsilon', N_1$ such that if $n > N_1$, $n \in 2\mathbb{Z}\cap k\mathbb{Z}$ and $\sigma_n$ is chosen at random with law $\mathbb{P}^u_n$, then with positive probability,

1. $\sigma_n$ is $(D,\delta)$-sofic,

2. $|n^{-1}\log(Z(\epsilon';\sigma_n)) - f(d,k)| < \epsilon$.

This is implied by Lemma 2.3.1 and Lemma 2.1.2 equations (1) and (3) respectively.

Now consider decreasing sequences $\epsilon_m \to 0, \delta_m \to 0$ and $D_m \subset \Gamma$ finite subsets increasing to $\Gamma$. We can repeated apply the above to get a decreasing $\epsilon'_m$ and increasing $N_m$ such that if $n > N_m, n \in 2\mathbb{Z}\cap k\mathbb{Z}$ and $\sigma_n$ is chosen at random with law $\mathbb{P}^u_n$ then with positive probability

1. $\sigma_n$ is $(D_m,\delta_m)$-sofic,

2. $|n^{-1}\log(Z(\epsilon'_m;\sigma_n)) - f(d,k)| < \epsilon_m$.

So there exists a sofic approximation $\Sigma_1 = \{\sigma_n\}$ to $\Gamma$ such that

$$\inf_{\epsilon>0} \limsup_{n\to\infty} n^{-1}\log Z(\epsilon;\sigma_n) = f(d,k).$$

By Lemma 2.2.1, this implies $h_{\Sigma_1}(\Gamma \curvearrowright X) = f(d,k) > 0$.

Equation (2) of Lemma 2.1.2 implies the existence of a number $f_p$ with

$$f(d,k) < f_p < \liminf_{n\to\infty} n^{-1}\log \mathbb{E}_n^p[Z(\sigma)].$$

Since $Z(\sigma) \le 2^n$ for every $\sigma$, there exist constants $c, N_0 > 0$ such that

$$\mathbb{P}_n^p\{\sigma : \ Z(\sigma) \ge \exp(nf_p)\} \ge \exp(-cn) \tag{2.4}$$

for all $n \ge N_0$.

Now let $\delta > 0$ and $D \subset \Gamma$ be finite. Then there exists $N_2$ such that if $n > N_2$, $n \in 2\mathbb{Z} \cap k\mathbb{Z}$ and $\sigma_n$ is chosen at random with law $\mathbb{P}_n^p$, then with positive probability,

1. $\sigma_n$ is $(D,\delta)$-sofic,

2. $n^{-1}\log Z(\sigma_n) \ge f_p$.

This is implied by Lemma 2.3.1 and equation (2.4). So there exists a sofic approximation $\Sigma_2 = \{\sigma'_n\}$ to $\Gamma$ such that

$$\limsup_{n\to\infty} n^{-1}\log Z(\sigma'_n) \ge f_p.$$

Since $Z(\sigma'_n) \le Z(\epsilon;\sigma'_n)$, Lemma 2.2.1 implies $h_{\Sigma_2}(\Gamma \curvearrowright X) \ge f_p > f(d,k) = h_{\Sigma_1}(\Gamma \curvearrowright X)$. $\square$

## 2.4 The first moment

To simplify notation, we assume throughout the paper that $n \in 2\mathbb{Z} \cap k\mathbb{Z}$ without further mention. This section proves (2.1) of Lemma 2.1.2. The proof is in two parts. Part 1, in §2.4.1, establishes:

**Theorem 2.4.1.**

$$\lim_{\epsilon\searrow 0}\limsup_{n\to\infty}(1/n)\log\mathbb{E}_n^u[Z(\epsilon;\sigma)] = \limsup_{n\to\infty}(1/n)\log\mathbb{E}_n^u[Z(\sigma)].$$

Part 2 has to do with equitable colorings, where a 2-coloring $\chi : [n] \to \{0,1\}$ is **equitable** if

$$|\chi^{-1}(0)| = |\chi^{-1}(1)| = n/2.$$

Let $Z_e(\sigma)$ be the number of proper equitable colorings of $G_\sigma$. §2.4.2 establishes

**Theorem 2.4.2.**

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_n^u[Z(\sigma)] = \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_n^u[Z_e(\sigma)].$$

*Moreover,*

$$\frac{1}{n}\log\mathbb{E}_n^u[Z_e(\sigma)] = f(d,k) + O(n^{-1}\log(n))$$

*where* $f(d,k) = \log(2) + \frac{d}{k}\log(1-2^{1-k})$.

Combined, Theorems 2.4.1 and 2.4.2 imply (2.1) of Lemma 2.1.2.

*Remark* 10. If $r := (d/k)$ then the formula for $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_n^u[Z(\sigma)]$ above is the same as the formula found in [AM06, COZ11, COZ12] for the exponential growth rate of the number of proper 2-colorings of $H_k(n,m)$.

*Remark* 11. When we write an error term, such as $O(n^{-1}\log(n))$, we always assume that $n \geq 2$ and the implicit constant is allowed to depend on $k$ or $d$.

### 2.4.1 Almost proper 2-colorings

For $0 < x \leq 1$, let $\eta(x) = -x\log(x)$. Also let $\eta(0) = 0$. If $\vec{T} = (T_i)_{i\in I}$ is a collection of numbers with $0 \leq T_i \leq 1$, then let

$$H(\vec{T}) := \sum_{i\in I}\eta(T_i)$$

be the **Shannon entropy** of $\vec{T}$.

**Definition 3.** A **$k$-partition** of $[n]$ is an unordered partition of $[n]$ into sets of size $k$. Of course, such a partition exists if and only if $n/k \in \mathbb{N}$ in which case there are

$$\frac{n!}{k!^{n/k}(n/k)!} \tag{2.5}$$

such partitions. By Stirling's formula,

$$\frac{1}{n}\log(\#\{k\text{-partitions}\}) = (1-1/k)(\log(n)-1) - (1/k)\log(k-1)! + O(n^{-1}\log(n)). \tag{2.6}$$

**Definition 4.** The **orbit-partition** of a permutation $\rho \in \mathrm{sym}(n)$ is the partition of $[n]$ into orbits of $\rho$. Fix a $k$-partition $\pi$. Then the number of permutations $\rho$ whose orbit partition is $\pi$ equals $(k-1)!^{n/k}$.

Given $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$, define the $d$-tuple $(\pi_1^\sigma, \ldots, \pi_d^\sigma)$ of $k$-partitions by: $\pi_i^\sigma$ is the orbit-partition of $\sigma(s_i)$. Fix a $d$-tuple of $k$-partitions $(\pi_1, \ldots, \pi_d)$. Then the number of uniform homomorphisms $\sigma$ such that $\pi_i^\sigma = \pi_i$ for all $i$ is $[(k-1)!^{n/k}]^d$. Combined with (2.5), this shows the number of uniform homomorphisms into $\mathrm{sym}(n)$ is

$$\left[\frac{n!(k-1)!^{n/k}}{k!^{n/k}(n/k)!}\right]^d.$$

By Stirling's formula,

$$\frac{1}{n}\log\#\mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n)) = d(1-1/k)(\log n - 1) + O(n^{-1}\log(n)). \tag{2.7}$$

**Definition 5.** Let $\pi$ be a $k$-partition, $\chi : [n] \to \{0,1\}$ a 2-coloring and $\vec{t} = (t_j)_{j=0}^k \in [0,1]^{k+1}$ a vector with $\sum_j t_j = 1/k$. The pair $(\pi, \chi)$ has **type** $\vec{t}$ if for all $j$,

$$\#\left\{e \in \pi :\ |e \cap \chi^{-1}(1)| = j\right\} = nt_j.$$

**Lemma 2.4.3.** *Let $\vec{t} = (t_0, t_1, \ldots, t_k) \in [0,1]^{k+1}$ be such that $\sum_j t_j = 1/k$ and $nt_j \in \mathbb{Z}$. Let $p = \sum_j jt_j$. Let $\chi : [n] \to \{0,1\}$ be a map such that $|\chi^{-1}(1)| = pn$. Let $f(\vec{t})$ be the number* *of $k$-partitions $\pi$ of $[n]$ such that $(\pi, \chi)$ has type $\vec{t}$. Then*

$$(1/n)\log f(\vec{t}) = (1-1/k)(\log(n)-1)-H(p,1-p)+H(\vec{t})-\sum_{j=0}^{k} t_j \log(j!(k-j)!)+O(n^{-1}\log(n)).$$

*Proof.* The following algorithm constructs all such partitions with no duplications:

Step 1. Choose an unordered partition of the set $\chi^{-1}(1)$ into $t_j n$ sets of size $j$ $(j = 0, \ldots, k)$.

Step 2. Choose an unordered partition of the set $\chi^{-1}(0)$ into $t_j n$ sets of size $k-j$ $(j = 0, \ldots, k)$.

Step 3. Choose a bijection between the collection of subsets of size $j$ constructed in part 1 with the collection of subsets of size $k-j$ constructed in part 2.

Step 4. The partition consists of all sets of the form $\alpha \cup \beta$ where $\alpha \subset \chi^{-1}(1)$ is a set of size $j$ constructed in Step 1 and $\beta \subset \chi^{-1}(0)$ is a set of size $(k-j)$ constructed in Step 2 that it is paired with under Step 3.

The number of choices in Step 1 is $\frac{(pn)!}{\prod_{j=1}^{k}(j)!^{t_j n}(t_j n)!}$. The number of choices in Step 2 is $\frac{((1-p)n)!}{\prod_{j=0}^{k-1}(k-j)!^{t_j n}(t_j n)!}$. The number of choices in Step 3 is $\prod_{j=1}^{k-1}(t_j n)!$. So

$$f(\vec{t}) = \frac{(pn)!((1-p)n)!}{\prod_{j=0}^{k} j!^{t_j n}(k-j)!^{t_j n}(t_j n)!}.$$

The lemma follows from this and Stirling's formula. $\square$

Let $\mathscr{M}$ be the set of all matrices $\vec{T} = (T_{ij})_{1 \le i \le d, 0 \le j \le k}$ such that

1. $T_{ij} \geq 0$ for all $i, j$,
2. $\sum_{j=0}^{k} T_{ij} = 1/k$ for all $i$,
3. there exists a number, denoted $p(\vec{T})$, such that $p(\vec{T}) = \sum_{j=0}^{k} jT_{ij}$ for all $i$,
4. $n\vec{T}$ is integer-valued.

**Lemma 2.4.4.** *Given a matrix $\vec{T} \in \mathscr{M}$ define*

$$F(\vec{T}) \quad := \quad H(\vec{T}) + (1-d)H(p, 1-p) - (d/k)\log k + \sum_{i=1}^{d}\sum_{j=0}^{k} T_{ij} \log \binom{k}{j}$$

*where $p = p(\vec{T})$. Then for any $\epsilon \geq 0$,*

$$(1/n)\log \mathbb{E}_n^u[Z(\epsilon;\sigma)] = \sup \left\{ F(\vec{T}) : \ \vec{T} \in \mathscr{M} \text{ and } \sum_{i=1}^{d}\sum_{j=0,k} T_{ij} \leq \epsilon \right\} + O(n^{-1}\log(n))$$

*where the constant implicit in the error term does not depend on $\epsilon$.*

*Proof.* Given $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ and $1 \leq i \leq d$, let $\pi_i^\sigma$ be the orbit-partition of $\sigma(s_i)$. For $\vec{T}$ as above, let $Z_\sigma(\vec{T})$ be the number of $\epsilon$-proper colorings $\chi : [n] \to \{0,1\}$ such that $(\pi_i^\sigma, \chi)$ has type $\vec{T}_i = (T_{i,0}, \ldots, T_{i,k})$. It suffices to show that

$$(1/n)\log \mathbb{E}_n^u[Z_\sigma(\vec{T})] = F(\vec{T}) + O(n^{-1}\log(n))$$

for all $n \geq 2$ such that $n\vec{T}$ is integer-valued. This is because the size of $\mathscr{M}$ is a polynomial (depending on $k, \epsilon, d$) in $n$ so the supremum above determines the exponential growth rate of $\mathbb{E}_n^u[Z(\epsilon;\sigma)]$.

To prove this, fix a $\vec{T}$ as above and let $n$ be such that $n\vec{T}$ is integer-valued. Fix a coloring $\chi : [n] \to \{0,1\}$ such that $|\chi^{-1}(1)| = pn$. By symmetry,

$$\mathbb{E}_n^u[Z_\sigma(\vec{T})] = \binom{n}{pn} \mathbb{P}_n^u[(\pi_i^\sigma, \chi) \text{ has type } \vec{T}_i \ \forall i].$$

The events $\{(\pi_i^\sigma, \chi) \text{ has type } \vec{T}_i\}_{i=1}^d$ are jointly independent. So

$$\mathbb{E}_n^u[Z_\sigma(\vec{T})] = \binom{n}{pn} \prod_{i=1}^{d} \mathbb{P}_n^u[(\pi_i^\sigma, \chi) \text{ has type } \vec{T}_i]. \tag{2.8}$$

By symmetry, $\mathbb{P}_n^u[(\pi_i^\sigma, \chi)$ has type $\vec{T}_i]$ is the number of $k$-partitions $\pi$ such that $(\pi, \chi)$ has type $\vec{T}_i$ divided by the number of $k$-partitions. By Lemma 2.4.3 and (2.6),

$$
\begin{aligned}
&\frac{1}{n}\log \mathbb{P}^u_n[(\pi^\sigma_i,\chi) \text{ has type } \vec{T}_i] \\
= \quad & -H(p,1-p) + H(\vec{T}_i) - \sum_{j=0}^{k} T_{ij}\log(j!(k-j)!) + (1/k)\log(k-1)! + O(n^{-1}\log(n)).
\end{aligned}
$$

Combine this with (2.8) to obtain

$$
\begin{aligned}
&(1/n)\log \mathbb{E}^u_n[Z_\sigma(\vec{T})] \\
= \quad & (1-d)H(p,1-p) + H(\vec{T}) - \sum_{i=1}^{d}\sum_{j=0}^{k} T_{ij}\log(j!(k-j)!) + (d/k)\log(k-1)! + O(n^{-1}\log(n)).
\end{aligned}
$$

This simplifies to the formula for $F(\vec{T})$ using the assumption that $\sum_{j=0}^{k} T_{ij} = 1/k$ for all $i$.

□

Theorem 2.4.1 follows from Lemma 2.4.4 because $F$ is continuous and the space of vectors $\vec{T}$ satisfying the constraints of the Lemma is compact.

### 2.4.2 Equitable colorings

*Proof of Theorem 2.4.2.* Let $\mathscr{M}_0$ be the set of all $\vec{T} \in \mathscr{M}$ such that $T_{ij} = 0$ whenever $j \in \{0,k\}$. By Lemma 2.4.4, it suffices to show that $F$ admits a unique global maximum on $\mathscr{M}_0$ and moreover if $\vec{T} \in \mathscr{M}_0$ is the global maximum then $p(\vec{T}) = 1/2$ and $F(\vec{T}) = f(d,k)$.

The function $F$ is symmetric in the index $i$. To exploit this, let $\mathscr{M}'$ be the set of all vectors $\vec{t} = (t_j)_{j=1}^{k-1}$ such that $t_j \ge 0$ for all $j$ and $\sum_{j=1}^{k-1} t_j = 1/k$. Let

$$
\begin{aligned}
p(\vec{t}) &= \sum_{j=1}^{k-1} j t_j \\
F(\vec{t}) &= dH(\vec{t}) + (1-d)H(p,1-p) - (d/k)\log k + d\sum_{j=1}^{k-1} t_j \log\binom{k}{j}.
\end{aligned}
$$

Note that $F(\vec{t}) = F(\vec{T})$ if $\vec{T}$ is defined by $\vec{T}_{ij} = \vec{t}_j$ for all $i,j$. Moreover, since Shannon entropy is strictly concave, for any $\vec{T} \in \mathscr{M}_0$, if $\vec{t}$ is defined to be the average: $\vec{t}_j = d^{-1}\sum_{i=1}^{d} \vec{T}_{ij}$ then

$F(\vec{t}) \geq F(\vec{T})$ with equality if and only if $\vec{t_j} = \vec{T}_{ij}$ for all $i, j$. So it suffices to show that $F$ admits a unique global maximum on $\mathscr{M}'$ and moreover if $\vec{t} \in \mathscr{M}'$ is the global maximum then $p(\vec{t}) = 1/2$ and $F(\vec{t}) = f(d,k)$.

Because $\frac{\partial H(\vec{t})}{\partial t_j} = -[\log(t_j)+1]$, $\frac{\partial p}{\partial t_j} = j$, and $\frac{\partial H(p,1-p)}{\partial t_j} = j \log\left(\frac{1-p}{p}\right)$,

$$\frac{\partial F}{\partial t_j} = -d[\log(t_j)+1] + (1-d)j\log\left(\frac{1-p}{p}\right) + d\log\binom{k}{j}.$$

Since this is positive infinity whenever $t_j = 0$, it follows that every maximum of $F$ occurs in the interior of $\mathscr{M}'$. The method of Lagrange multipliers implies that, at a critical point, there exists $\lambda \in \mathbb{R}$ such that

$$\nabla F = \lambda \nabla \left(\vec{t} \mapsto \sum_j t_j\right) = (\lambda, \lambda, \ldots, \lambda).$$

So at a critical point,

$$\frac{\partial F}{\partial t_j} = -d[\log(t_j)+1] + (1-d)j\log\left(\frac{1-p}{p}\right) + d\log\binom{k}{j} = \lambda.$$

Solve for $t_j$ to obtain

$$t_j = \exp(-\lambda/d - 1)\binom{k}{j}\left(\frac{1-p}{p}\right)^{j(1-d)/d}.$$

Note

$$1 = k\sum_{j=1}^{k-1} t_j$$

$$1 = 1/p\sum_{j=1}^{k-1} jt_j$$

implies

$$0 = \sum_{j=1}^{k-1}(k - j/p)t_j = \sum_{j=1}^{k-1}(pk - j)\binom{k}{j}\left(\frac{1-p}{p}\right)^{j(1-d)/d}.$$

So define

$$g(x) := \sum_{j=1}^{k-1} (kx - j) \binom{k}{j} \left( \frac{1-x}{x} \right)^{j(1-d)/d}.$$

It follows from the above that $g(p(\vec{t})) = 0$ whenever $\vec{t}$ is a critical point.

We claim that $g(x) = 0$ if and only if $x = 1/2$ (for $x \in (0,1)$). The change of variables $j \mapsto k - j$ in the formula for $g$ shows that $g(1-x) = -\left(\frac{x}{1-x}\right)^{k(1-d)/d} g(x)$. So it is enough to prove that $g(x) < 0$ for $x \in (0, 1/2)$.

To obtain a simpler formula for $g$, set $y(x) = \left(\frac{1-x}{x}\right)^{(1-d)/d}$. The binomial formula implies

$$\begin{aligned} g(x) &= \sum_{j=1}^{k-1} (kx - j) \binom{k}{j} y^j \\ &= kx[(1+y)^k - 1 - y^k] - ky[(1+y)^{k-1} - y^{k-1}] \\ &= k[(x(1+y) - y)(1+y)^{k-1} - x + (-x+1)y^k]. \end{aligned}$$

Because $0 < x < 1/2$, $y > \left(\frac{x}{1-x}\right)$ which implies that the middle coefficient $(x(1+y) - y) = x - y(1-x) < 0$. So

$$g(x)/k < (1-x)y^k - x < 0$$

where the last inequality holds because

$$y^k = \left(\frac{x}{1-x}\right)^{k(d-1)/d} < \frac{x}{1-x}$$

assuming $k(d-1)/d > 1$. This proves the claim.

So if $\vec{t}$ is a critical point then $p(\vec{t}) = 1/2$. Put this into the equation above for $t_j$ to obtain

$$t_j = C \binom{k}{j}$$

where $C = \exp(-\lambda/d - 1)$. Because

$$1/k = \sum_{j=1}^{k-1} t_j = C \sum_{j=1}^{k-1} \binom{k}{j} = C(2^k - 2)$$

it must be that

$$t_j = \frac{1}{k(2^k - 2)} \binom{k}{j}. \tag{2.9}$$

The formula $F(\vec{t}) = f(d, k)$ now follows from a straightforward computation. □

## 2.5 The second moment

This section gives an estimate on the expected number of proper colorings at a given Hamming distance from the planted coloring. This computation yields (2.2) of Lemma 2.1.2 as a corollary. It also reduces the proof of (2.3) to obtaining an estimate on the typical number of proper colorings near the planted coloring.

Before stating the main result, it seems worthwhile to review notation. Fix $n > 0$ with $n \in 2\mathbb{Z} \cap k\mathbb{Z}$. Fix an equitable 2-coloring $\chi : [n] \to \{0, 1\}$. This is the **planted coloring**. The planted model $\mathbb{P}_n^p$ is the uniform probability measure on the set $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ of all uniform homomorphisms $\sigma$ such that $\chi$ is $\sigma$-proper. Also let $Z_e : \mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n)) \to \mathbb{N}$ be the number of equitable proper 2-colorings. For $\delta \in [0, 1]$, let $Z_\chi(\delta; \cdot) : \mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n)) \to \mathbb{N}$ be the number of equitable proper 2-colorings $\widetilde{\chi}$ such that $|d_n(\chi, \widetilde{\chi}) - \delta| < 1/2n$ where $d_n$ is the normalized Hamming distance defined by

$$d_n(\chi, \widetilde{\chi}) = n^{-1} \#\{v \in [n] : \ \chi(v) \neq \widetilde{\chi}(v)\}.$$

We will also write $Z_\chi(\delta; \sigma) = Z_\chi(\delta) = Z(\delta)$ when $\chi$ and/or $\sigma$ are understood.

The main result of this section is:

**Theorem 2.5.1.** *With notation as above, for any $0 \leq \delta \leq 1$ such that $\delta n/2$ is an integer,*

$$\frac{1}{n} \log \mathbb{E}_n^p[Z(\delta)] = \psi_0(\delta) + O(n^{-1} \log(n))$$

*(for $n \geq 2$) where*

$$\psi_0(\delta) = (1 - d) H(\delta, 1 - \delta) + d H_0(\delta, 1 - \delta) + \frac{d}{k} \log \left( 1 - \frac{1 - \delta_0^k - (1 - \delta_0)^k}{2^{k-1} - 1} \right),$$

$\delta_0$ *is defined to be the unique solution to*

$$\delta_0 \frac{1 - 2^{2-k} + (\delta_0/2)^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} = \delta$$

*and*

$$\begin{aligned} H(\delta, 1-\delta) &:= -\delta \log \delta - (1-\delta)\log(1-\delta), \\ H_0(\delta, 1-\delta) &:= -\delta \log \delta_0 - (1-\delta)\log(1-\delta_0). \end{aligned}$$

*Moreover, the constant implicit in the error term $O(n^{-1}\log(n))$ may depend on $k$ but not on $\delta$.*

*Remark* 12. If $\delta_0 = \delta$ then $\delta = 1/2$. In the general case, $\delta_0 = \delta + O(2^{-k})$. Theorem 2.5.1 parallels similar results in [AM06, COZ11] for the random hyper-graph $H_k(n,m)$. This is explained in more detail in the next subsection.

The strategy behind the proof of Theorem 2.5.1 is as follows. We need to estimate the expected number of equitable colorings at distance $\delta$ from the planted coloring. By symmetry, it suffices to fix another coloring $\widetilde{\chi}$ that is at distance $\delta$ from the planted coloring and count the number of uniform homomorphisms $\sigma$ such that both $\chi$ and $\widetilde{\chi}$ are proper with respect to $G_\sigma$. This can be handled one generator at a time. Moreover, only the orbit-partition induced by a generator is used in this computation. So, for fixed $\chi, \widetilde{\chi}$, we need to estimate the number of $k$-partitions of $[n]$ that are bi-chromatic under both $\chi$ and $\widetilde{\chi}$. To make this strategy precise, we need the next definitions.

**Definition 6.** Let $\widetilde{\chi}$ be an equitable 2-coloring of $[n]$. An edge $P \subset [n]$ is $(\chi, \widetilde{\chi})$**-bichromatic** if $\chi(P) = \widetilde{\chi}(P) = \{0, 1\}$. Recall that a $k$-partition is a partition $\pi = \{P_1, \ldots, P_{n/k}\}$ of $[n]$ such that every part $P \in \pi$ has cardinality $k$. A $k$-partition $\pi$ is $(\chi, \widetilde{\chi})$**-bichromatic** if every part $P \in \pi$ is $(\chi, \widetilde{\chi})$-bichromatic.

Given a $(\chi, \widetilde{\chi})$-bichromatic edge $P \subset [n]$ of size $k$, there is a $2 \times 2$ matrix $\vec{e}(\widetilde{\chi}, P)$ defined by

$$\vec{e}_{i,j}(\widetilde{\chi}, P) = |P \cap \chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)|.$$

Let $\mathcal{E}$ denote the set of all such matrices (over all $P, \widetilde{\chi}$). This is a finite set. To be precise,

$\mathcal{E}$ is the set of all $2 \times 2$ matrices $\vec{e} = (e_{ij})_{i,j=0,1}$ such that

- $e_{ij} \in \{0, 1, \ldots, k\}$ for all $i, j$
- $0 < e_{10} + e_{11} < k$
- $0 < e_{01} + e_{11} < k$
- $\sum_{i,j} e_{ij} = k$.

If $\pi$ is a $(\chi, \widetilde{\chi})$-bichromatic $k$-partition then it induces a function $t_{\widetilde{\chi},\pi} : \mathcal{E} \to [0, 1]$ by

$$t_{\widetilde{\chi},\pi}(\vec{e}) = n^{-1} \# \left\{ P \in \pi : \ \vec{e} = \vec{e}(\widetilde{\chi}, P) \right\}.$$

Let $\mathcal{T}$ denote the set of all such functions. To be precise, $\mathcal{T}$ is the set of all functions $t : \mathcal{E} \to [0, 1]$ satisfying

- $\sum_{\vec{e} \in \mathcal{E}} t(\vec{e}) = 1/k$,
- $\sum_{\vec{e} \in \mathcal{E}} (e_{10} + e_{11}) t(\vec{e}) = 1/2$,
- $\sum_{\vec{e} \in \mathcal{E}} (e_{01} + e_{11}) t(\vec{e}) = 1/2$,
- $nt(\vec{e})$ is integer-valued

A $k$-partition $\pi$ has **type** $(\chi, \widetilde{\chi}, t)$ if $t = t_{\widetilde{\chi},\pi}$.

**Lemma 2.5.2.** *Given an equitable 2-coloring $\widetilde{\chi} : [n] \to \{0, 1\}$, let $p^{\widetilde{\chi}} = (p^{\widetilde{\chi}}_{ij})$ be the $2 \times 2$ matrix*

$$p^{\widetilde{\chi}}_{ij} = (1/n) |\chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)|.$$

*Then*

$$p^{\widetilde{\chi}} = \begin{bmatrix} 1/2 - d_n(\chi, \widetilde{\chi})/2 & d_n(\chi, \widetilde{\chi})/2 \\ d_n(\chi, \widetilde{\chi})/2 & 1/2 - d_n(\chi, \widetilde{\chi})/2 \end{bmatrix}$$

*In particular, $p^{\widetilde{\chi}}$ is determined by the Hamming distance $d_n(\chi, \widetilde{\chi})$.*

*Proof.* Let $p = p^{\widetilde{\chi}}$. The lemma follows from this system of linear equations:

$$\begin{aligned} 1/2 &= p_{01} + p_{11} \\ 1/2 &= p_{10} + p_{11} \\ d_n(\chi, \widetilde{\chi}) &= p_{01} + p_{10} \\ 1 &= p_{00} + p_{01} + p_{10} + p_{11}. \end{aligned}$$

The first two occur because both $\chi$ and $\widetilde{\chi}$ are equitable. The third follows from the definition of normalized Hamming distance and the last holds because $\{\chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)\}_{i,j\in\{0,1\}}$ partitions $[n]$. □

For $t \in \mathcal{T}$, define the $2 \times 2$ matrix $p^t = (p^t_{ij})$ by

$$p^t_{ij} := \sum_{\vec{e}\in\mathcal{E}} e_{ij} t(\vec{e}).$$

If $\pi$ is a $k$-partition that has type $(\chi, \widetilde{\chi}, t)$ (for some equitable $\widetilde{\chi}$) then $p^{\widetilde{\chi}} = p^t$. This motivates the definition.

The main combinatorial estimate we will need is:

**Lemma 2.5.3.** *Let $t \in \mathcal{T}$ and $\widetilde{\chi} : [n] \to \{0, 1\}$ be equitable. Suppose $nt$ is integer-valued and $p^t = p^{\widetilde{\chi}}$. Let $g(\widetilde{\chi}, t)$ be the number of $k$-partitions of type $(\chi, \widetilde{\chi}, t)$. Also let*

$$G(t) := (1 - 1/k)(\log(n) - 1) - H(p^t) - (1/k)\log(k!) + H(t) + \sum_{\vec{e}} t(\vec{e}) \log \binom{k}{\vec{e}}$$

*where $\binom{k}{\vec{e}}$ is the multinomial $\frac{k!}{e_{00}!e_{01}!e_{10}!e_{11}!}$. Then*

$$(1/n) \log g(\widetilde{\chi}, t) = G(t) + O(n^{-1} \log(n))$$

*(for $n \geq 2$) where the constant implicit in the error term depends on $k$ but not on $\widetilde{\chi}$ or $t$.*

*Proof.* The following algorithm constructs all such partitions with no duplications:

Step 1. Choose a partition $\{Q^{\vec{e}}_{ij} :\ i,j \in \{0,1\}, \vec{e} \in \mathcal{E}\}$ of $\chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)$ such that

$$|Q^{\vec{e}}_{ij}| = e_{ij} t(\vec{e}) n.$$

Step 2. For $i,j \in \{0,1\}$ and $\vec{e} \in \mathcal{E}$, choose an unordered partition $\pi^{\vec{e}}_{ij}$ of $Q^{\vec{e}}_{ij}$ into $t(\vec{e})n$ sets of size $e_{ij}$.

Step 3. For $i,j \in \{0,1\}$ with $(i,j) \neq (0,0)$ and $\vec{e} \in \mathcal{E}$, choose a bijection $\alpha^{\vec{e}}_{ij} : \pi^{\vec{e}}_{00} \to \pi^{\vec{e}}_{ij}$.

Step 4. The $k$-partition consists of all sets of the form $P \bigcup_{i,j\in\{0,1\},(i,j)\neq(0,0)} \alpha^{\vec{e}}_{ij}(P)$ over all $P \in \pi^{\vec{e}}_{00}$ and $\vec{e} \in \mathcal{E}$.

The number of choices in Step 1 is

$$\prod_{i,j\in\{0,1\}} |\chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)|! \prod_{\vec{e}\in\mathcal{E}} (e_{ij} t(\vec{e}) n)!^{-1}.$$

The combined number of choices in Steps 1 and 2 is

$$\prod_{i,j\in\{0,1\}} |\chi^{-1}(i)\cap\widetilde{\chi}^{-1}(j)|! \prod_{\vec{e}\in\mathcal{E}} e_{ij}!^{-t(\vec{e})n} (t(\vec{e})n)!^{-1} = \left(\prod_{\vec{e}\in\mathcal{E}} (t(\vec{e})n)!\right)^{-4} \prod_{i,j\in\{0,1\}} |\chi^{-1}(i)\cap\widetilde{\chi}^{-1}(j)|! \prod_{\vec{e}\in\mathcal{E}} e_{ij}!^{-t(\vec{e})n}.$$

The number of choices in Step 3 is $\prod_{\vec{e}\in\mathcal{E}} (t(\vec{e})n)!^3$. So

$$g(\widetilde{\chi}, t) = \left(\prod_{i,j\in\{0,1\}} |\chi^{-1}(i) \cap \widetilde{\chi}^{-1}(j)|!\right) \left(\prod_{\vec{e}\in\mathcal{E}} (t(\vec{e})n)!\right)^{-1} \left(\prod_{i,j\in\{0,1\}} \prod_{\vec{e}\in\mathcal{E}} e_{ij}!^{-t(\vec{e})n}\right).$$

An application of Stirling's formula gives

$$(1/n)\log g(\widetilde{\chi}, t) = (1 - 1/k)(\log(n) - 1) - H(p^t) + H(t) - \sum_{\vec{e},i,j} t(\vec{e}) \log(e_{ij}!) + O(n^{-1}\log(n))$$

(for $n \geq 2$) where the constant implicit in the error term depends on $k$ but not on $\widetilde{\chi}$ or $t$.

Since $\sum_{\vec{e}} t(\vec{e}) = 1/k$,

$$\sum_{\vec{e}} t(\vec{e}) \log \binom{k}{\vec{e}} = (1/k)\log(k!) - \sum_{\vec{e},i,j} t(\vec{e}) \log(e_{ij}!).$$

Substitute this into the formula above to finish the lemma.

$\square$

Next we use Lagrange multipliers to maximize $G(t)$. To be precise, for $\delta \in [0,1]$, let $\mathcal{T}(\delta)$ be the set of all $t \in \mathcal{T}$ such that $p^t_{01} = \delta/2$. To motivate this definition, observe that if $\widetilde{\chi}$ is an equitable 2-coloring and $\delta = d_n(\chi, \widetilde{\chi})$ then $p^{\widetilde{\chi}}_{01} = \delta/2$. So if $\pi$ is a $k$-partition with type $(\chi, \widetilde{\chi}, t)$ then $p^t_{01} = \delta/2$.

**Lemma 2.5.4.** *Let $\delta \in [0,1]$. Then there exists a unique $s_\delta \in \mathcal{T}(\delta)$ such that*

$$\max_{t \in \mathcal{T}(\delta)} G(t) = G(s_\delta).$$

*Moreover, if $\delta_0$, $C > 0$ and $t_\delta \in \mathcal{T}(\delta)$ are defined by*

$$\begin{aligned}
\frac{\delta}{2} &= \frac{\delta_0}{2} \frac{1 - 2^{2-k} + (\delta_0/2)^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} \\
C &= \frac{1}{k[1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k]} \\
t_\delta(\vec{e}) &= C \left(\frac{1-\delta_0}{2}\right)^{e_{00}+e_{11}} \left(\frac{\delta_0}{2}\right)^{e_{01}+e_{10}} \binom{k}{\vec{e}}
\end{aligned}$$

*then $s_\delta = t_\delta$.*

*Proof.* Define $F : \mathcal{T} \to \mathbb{R}$ by

$$F(t) = H(t) + \sum_{\vec{e}} t(\vec{e}) \log \binom{k}{\vec{e}}.$$

For all $t \in \mathcal{T}(\delta)$, $G(t) - F(t)$ is constant in $t$. Therefore, it suffices to prove the lemma with $F$ in place of $G$.

The function $F$ is concave over $t \in \mathcal{T}(\delta)$. This implies the existence of a unique

$s_\delta \in \mathcal{T}(\delta)$ such that

$$\max_{t \in \mathcal{T}(\delta)} F(t) = F(s_\delta).$$

By definition, $\mathcal{T}(\delta)$ is the set of all functions $t : \mathcal{E} \to [0,1]$ satisfying

$$\begin{aligned} 1/k &= \sum_{\vec{e} \in \mathcal{E}} t(\vec{e}) \\ p_{ij} &= \sum_{\vec{e} \in \mathcal{E}} e_{ij} t(\vec{e}), \end{aligned}$$

where $p = (p_{ij})$ is the matrix

$$p = \begin{bmatrix} 1/2 - \delta/2 & \delta/2 \\ \delta/2 & 1/2 - \delta/2 \end{bmatrix}.$$

For any $\vec{e} \in \mathcal{E}$,

$$\frac{\partial F}{\partial t(\vec{e})} = -\log t(\vec{e}) - 1 + \log \binom{k}{\vec{e}}. \tag{2.10}$$

Since this is positive infinity when $t(\vec{e}) = 0$, $s_\delta$ must lie in the interior of $\mathcal{T}(\delta)$. By the method of Lagrange multipliers there exists $\lambda \in \mathbb{R}$ and a $2 \times 2$ matrix $\vec{\mu}$ such that

$$\frac{\partial F}{\partial t(\vec{e})}(s_\delta) = \lambda + \vec{\mu} \cdot \vec{e}. \tag{2.11}$$

Evaluate (2.10) at $s_\delta$, use (2.11) and solve for $s_\delta(\vec{e})$ to obtain

$$s_\delta(\vec{e}) = C_0 \binom{k}{\vec{e}} x_{00}^{e_{00}} x_{01}^{e_{01}} x_{10}^{e_{10}} x_{11}^{e_{11}}$$

for some constants $C_0, x_{ij}$. In fact, since $F$ is concave, $s_\delta$ is the unique critical point and so it is the only element of $\mathcal{T}(\delta)$ of this form. So it suffices to check that the purported $t_\delta$ given in the statement of the lemma has this form and that it is in $\mathcal{T}(\delta)$ as claimed. The former is immediate while the latter is a tedious but straightforward computation. For example, to check that $\sum_{\vec{e}} t_\delta(\vec{e}) = 1/k$, observe that, by the multinomial formula for any $(x_{ij})_{i,j\in\{0,1\}}$,

$$\begin{aligned}
&\sum_{\vec{e}\in\mathcal{E}} \binom{k}{\vec{e}} x_{00}^{e_{00}} x_{01}^{e_{01}} x_{10}^{e_{10}} x_{11}^{e_{11}} \\
= \;& \Big[(x_{00}+x_{01}+x_{10}+x_{11})^k \\
&-(x_{00}+x_{01})^k-(x_{00}+x_{10})^k-(x_{11}+x_{01})^k-(x_{11}+x_{10})^k+x_{00}^k+x_{01}^k+x_{10}^k+x_{11}^k\Big].
\end{aligned}$$

Substitute $x_{00} = x_{11} = \frac{1-\delta_0}{2}$ and $x_{01} = x_{10} = \delta_0/2$ to obtain

$$\sum_{\vec{e}\in\mathcal{E}} t_\delta(\vec{e}) \;=\; C\left[1-4(1/2)^k+2\left(\frac{1-\delta_0}{2}\right)^k+2\left(\frac{\delta_0}{2}\right)^k\right] = 1/k.$$

The rest of the verification that $t_\delta \in \mathcal{T}(\delta)$ is left to the reader. $\square$

*Proof of Theorem 2.5.1.* Let $\mathscr{E}(\delta)$ be the set of all equitable 2-colorings $\widetilde{\chi} : [n] \to \{0,1\}$ such that $d_n(\widetilde{\chi},\chi) = \delta$. Also let $F_{\widetilde{\chi}} \subset \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{sym}(n))$ be the set of all $\sigma$ such that $\widetilde{\chi}$ is a proper 2-coloring of the hyper-graph $G_\sigma$. By linearity of expectation,

$$\mathbb{E}_n^p[Z_\chi(\delta)] = \sum_{\widetilde{\chi}\in\mathscr{E}(\delta)} \mathbb{P}_n^u(F_{\widetilde{\chi}}|F_\chi).$$

The cardinality of $\mathscr{E}(\delta)$ is $\binom{n/2}{\delta n/2}^2$. By Stirling's formula

$$n^{-1}\log\binom{n/2}{\delta n/2}^2 = H(\delta, 1-\delta) + O(n^{-1}\log(n)). \tag{2.12}$$

We have $\mathbb{P}_n^u(F_{\widetilde{\chi}}|F_\chi)$ is the same for all $\widetilde{\chi} \in \mathscr{E}(\delta)$. This follows by noting that the distribution of hyper-graphs in the planted model is invariant under any permutation which fixes $\chi$. If $\eta, \eta'$ are two configurations with $d_n(\eta,\chi) = d_n(\eta',\chi) = \delta$ then there is a permutation $\pi \in \mathrm{sym}(n)$ which fixes $\chi$ and such that $\eta \circ \pi = \eta'$. To see this note that we simply need to find a $\pi \in \mathrm{sym}(n)$ which maps the sets $\chi^{-1}(i)\cap\eta^{-1}(j)$ to $\chi^{-1}(i)\cap\eta'^{-1}(j)$ for each $i,j \in \{0,1\}$. Such a map exists since for each $i,j$ the two sets have the same size. It follows that

$$n^{-1}\log\mathbb{E}_n^p[Z_\chi(\delta)] = H(\delta,1-\delta) + n^{-1}\log\mathbb{P}_n^u(F_{\widetilde{\chi}}|F_\chi) + O(n^{-1}\log(n)) \tag{2.13}$$

for any fixed $\widetilde{\chi} \in \mathscr{E}(\delta)$.

For $1 \le i \le d$, let $F_{\chi,i}$ be the set of uniform homomorphisms $\sigma$ such that the orbit-partition of $\sigma(s_i)$ is $\chi$-bichromatic in the sense that $\chi(P) = \{0,1\}$ for every $P$ in the orbit-partition of $\sigma(s_i)$. Then the events $\{F_{\chi,i} \cap F_{\widetilde{\chi},i}\}_{i=1}^d$ are i.i.d. and

$$F_\chi \cap F_{\widetilde{\chi}} = \bigcap_{i=1}^d F_{\chi,i} \cap F_{\widetilde{\chi},i}.$$

Therefore,

$$\mathbb{P}_n^u(F_{\widetilde{\chi}} | F_\chi) = \frac{\mathbb{P}_n^u(F_{\widetilde{\chi},1} \cap F_{\chi,1})^d}{\mathbb{P}_n^u(F_\chi)}. \tag{2.14}$$

Note $\mathbb{P}_n^u(F_{\chi,1} \cap F_{\widetilde{\chi},1})$ is, up to sub-exponential factors, equal to the maximum of $g(\widetilde{\chi}, t)$ over $t \in \mathfrak{T}(\delta)$ divided by the number of $k$-partitions of $[n]$. So Lemmas 2.5.3, 2.5.4 and equation (2.6) imply

$$\frac{1}{n} \log \mathbb{P}_n^u(F_{\widetilde{\chi},1} \cap F_{\chi,1}) = -H(\vec{p}) + H(t_\delta) - \sum_{i,j,\vec{e}} t_\delta(\vec{e}) \log(e_{ij}!) + (1/k)\log(k-1)! + O(n^{-1}\log(n)).$$

Since $\vec{p} = (\delta/2, \delta/2, (1-\delta)/2, (1-\delta)/2)$, $H(\vec{p}) = H(\delta, 1-\delta) + \log(2)$. So

$$\begin{aligned} \frac{1}{n} \log \mathbb{P}_n^u(F_{\widetilde{\chi},1} \cap F_{\chi,1}) &= -H(\delta, 1-\delta) - \log(2) + H(t_\delta) \\ &\quad - \sum_{i,j,\vec{e}} t_\delta(\vec{e}) \log(e_{ij}!) + (1/k)\log(k-1)! + O(n^{-1}\log(n)) \end{aligned} \tag{2.15}$$

On the other hand, Theorem 2.4.2 implies

$$\begin{aligned} \frac{1}{n} \log \mathbb{P}_n^u(F_\chi) &= \frac{1}{n} \log \left( \binom{n}{n/2}^{-1} \mathbb{E}_n^u[Z_e(\sigma)] \right) \\ &= (d/k)\log(1 - 2^{1-k}) + O(n^{-1}\log(n)). \end{aligned}$$

Combine this result with (2.13), (2.14) and (2.15) to obtain

$$\begin{aligned} n^{-1}\log \mathbb{E}_n^p[Z_\chi(\delta)] &= (1-d)H(\delta, 1-\delta) - d\log(2) + dH(t_\delta) - d\sum_{i,j,\vec{e}} t_\delta(\vec{e})\log(e_{ij}!) \\ &\quad +(d/k)\log(k-1)! - (d/k)\log(1-2^{1-k}) + O(n^{-1}\log(n)). \end{aligned}$$

Since $\sum_{\vec{e}} t_\delta(\vec{e}) = 1/k$,

$$\sum_{\vec{e}} t_\delta(\vec{e}) \log\binom{k}{\vec{e}} = (1/k)\log k! - \sum_{i,j,\vec{e}} t_\delta(\vec{e})\log(e_{ij}!).$$

Substitute this into the previous equation to obtain

$$n^{-1}\log \mathbb{E}_n^p[Z_\chi(\delta)] = \psi_0(\delta) + O(n^{-1}\log(n))$$

where

$$\begin{aligned} \psi_0(\delta) &= (1-d)H(\delta, 1-\delta) - d\log(2) + dH(t_\delta) + d\sum_{\vec{e}\in\mathcal{E}} t_\delta(\vec{e})\log\binom{k}{\vec{e}} \\ &\quad -(d/k)\log k - (d/k)\log(1-2^{1-k}). \end{aligned}$$

Observe that in every estimate above, the constant implicit in the error term does not depend on $\delta$. To finish the lemma, we need only simplify the expression for $\psi_0$.

By Lemma 2.5.4,

$$\begin{aligned} H(t_\delta) &= -\sum_{\vec{e}} t_\delta(\vec{e})\log t_\delta(\vec{e}) \\ &= -\sum_{\vec{e}} t_\delta(\vec{e})\left(\log C + (e_{00}+e_{11})\log\left(\frac{1-\delta_0}{2}\right) + (e_{01}+e_{10})\log\left(\frac{\delta_0}{2}\right) + \log\binom{k}{\vec{e}}\right) \\ &= -(1/k)(\log C) - (1-\delta)\log(1-\delta_0) - \delta\log(\delta_0) + \log 2 - \sum_{\vec{e}} t_\delta(\vec{e})\log\binom{k}{\vec{e}} \\ &= -(1/k)(\log C) + H_0(\delta, 1-\delta) + \log 2 - \sum_{\vec{e}} t_\delta(\vec{e})\log\binom{k}{\vec{e}}. \end{aligned}$$

Combined with the previous formula for $\psi_0$, this implies

$$\psi_0(\delta) \quad = \quad (1-d)H(\delta,1-\delta)-(d/k)\log C+dH_0(\delta,1-\delta)-(d/k)\log k-(d/k)\log(1-2^{1-k}).$$

To simplify further, use the formula for $C$ in Lemma 2.5.4 to obtain

$$\begin{aligned} -(d/k)\left(\log C+\log k+\log(1-2^{1-k})\right) &= (d/k)\log\frac{1-2^{2-k}+2(\delta_0/2)^k+2((1-\delta_0)/2)^k}{1-2^{1-k}} \\ &= (d/k)\log\left(1-\frac{1-\delta_0^k-(1-\delta_0)^k}{2^{k-1}-1}\right). \end{aligned}$$

Thus $\psi_0(\delta)=(1-d)H(\delta,1-\delta)+dH_0(\delta,1-\delta)+\frac{d}{k}\log(1-\frac{1-\delta_0^k-(1-\delta_0)^k}{2^{k-1}-1})$.

□

### 2.5.1 Analysis of $\psi_0$ and the proof of Lemma 2.1.2 inequality (2.2)

Theorem 2.5.1 reduces inequality (2.2) to analyzing the function $\psi_0$. A related function $\psi$, defined by

$$\psi(x):=H(x,1-x)+\frac{d}{k}\log\left(1-\frac{1-x^k-(1-x)^k}{2^{k-1}-1}\right)$$

has been analyzed in [AM06, COZ11]. It is shown there $\psi(x)$ is the exponential rate of growth of the number of proper colorings at normalized distance $x$ from the planted coloring in the model $H_k(n,m)$. Moreover, if $r=d/k$ is close to $\frac{\log(2)}{2}\cdot 2^k-(1+\log(2))/2$ then the global maximum of $\psi(x)$ is attained at some $x\in(0,2^{-k/2})$. Moreover, $\psi$ has a local maximum at $x=1/2$ and is symmetric around $x=1/2$. It is negative in the region $(2^{-k/2},1/2-2^{-k/2})$. We will not need these facts directly, and mention them only for context, especially because we will obtain similar results for $\psi_0$.

The relevance of $\psi$ to $\psi_0$ lies in the fact that

$$\psi_0(\delta)=\psi(\delta_0)-(H(\delta_0,1-\delta_0)-H_0(\delta,1-\delta))+(d-1)\left[H_0(\delta,1-\delta)-H(\delta,1-\delta)\right].\qquad(2.16)$$

As an aside, note that $H_0(\delta,1-\delta)-H(\delta,1-\delta)$ is the Kullback-Leibler divergence of the distribution $(\delta,1-\delta)$ with respect to $(\delta_0,1-\delta_0)$.

To prove inequality (2.2), we first estimate the difference $\psi_0(\delta)-\psi(\delta_0)$ and then estimate $\psi(\delta_0)$. Because the estimates we obtain are useful in the next subsection, we prove more than what is required for just inequality (2.2).

**Lemma 2.5.5.** *Suppose* $0 \le \delta_0 \le 1/2$. *Define* $\varepsilon \ge 0$ *by* $\delta = \delta_0(1-\varepsilon)$. *Then*

$$\begin{aligned} H(\delta_0, 1-\delta_0) - H_0(\delta, 1-\delta) &= \delta_0 \varepsilon \log\left(\frac{1-\delta_0}{\delta_0}\right) \ge 0, \\ H_0(\delta, 1-\delta) - H(\delta, 1-\delta) &= O(\delta_0\varepsilon^2), \\ \varepsilon &= O(2^{-k}), \\ (1-\delta)_0 &= 1-\delta_0. \end{aligned}$$

*The last equation implies* $\psi_0(1-\delta) = \psi_0(\delta)$.

*Proof.* The first equality follows from:

$$\begin{aligned} H(\delta_0, 1-\delta_0) - H_0(\delta, 1-\delta) &= -\delta_0 \log \delta_0 - (1-\delta_0)\log(1-\delta_0) + \delta \log \delta_0 + (1-\delta)\log(1-\delta_0) \\ &= (\delta_0 - \delta)\log(1/\delta_0) + (\delta_0-\delta)\log(1-\delta_0) \\ &= \delta_0\varepsilon \log\left(\frac{1-\delta_0}{\delta_0}\right). \end{aligned}$$

The second estimate follows from:

$$\begin{aligned} H_0(\delta, 1-\delta) - H(\delta, 1-\delta) &= \delta\left(\log \delta - \log \delta_0\right) + (1-\delta)\left(\log(1-\delta) - \log(1-\delta_0)\right) \\ &= \delta \log(1-\varepsilon) + (1-\delta)\log\left(\frac{1-\delta}{1-\delta_0}\right) \\ &= -\delta\varepsilon + (1-\delta)\log\left(1 + \frac{\delta_0\varepsilon}{1-\delta_0}\right) + O(\delta_0\varepsilon^2) \\ &= -\delta\varepsilon + \delta_0\varepsilon + O(\delta_0\varepsilon^2) = O(\delta_0\varepsilon^2). \end{aligned}$$

The third estimate follows from:

$$\begin{aligned}
\varepsilon &= 1 - \frac{\delta}{\delta_0} && (2.17)\\
&= 1 - \frac{1 - 2^{2-k} + (\delta_0/2)^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} && (2.18)\\
&= \frac{2(\delta_0/2)^k - (\delta_0/2)^{k-1} + 2((1-\delta_0)/2)^k}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} && (2.19)\\
&= \frac{(\delta_0/2)^{k-1}(\delta_0 - 1) + 2((1-\delta_0)/2)^k}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} && (2.20)\\
&= 2^{1-k} \cdot (1-\delta_0) \cdot \frac{(1-\delta_0)^{k-1} - \delta_0^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k}. && (2.21)
\end{aligned}$$

The denominator is $1 + O(2^{-k})$ and the numerator is $O(2^{-k})$. The result follows.

The last equation follows from:

$$\begin{aligned}
1 - \delta &= 1 - \delta_0 \left( \frac{1 - 2^{2-k} + (\delta_0/2)^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} \right)\\
&= \frac{1 - 2^{2-k} + 2(\delta_0/2)^k + 2\left((1-\delta_0)/2\right)^k - \delta_0(1 - 2^{2-k} + (\delta_0/2)^{k-1})}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k}\\
&= \frac{(1-\delta_0)\left(1 - 2^{2-k} + ((1-\delta_0)/2)^{k-1}\right)}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k}.
\end{aligned}$$

The last expression shows that $(1-\delta)_0 = 1 - \delta_0$.

□

**Lemma 2.5.6.** *Let $0 \leq \eta$ be constant with respect to $k$. If*

$$r = d/k = \frac{\log(2)}{2} \cdot 2^k - (1 + \log(2))/2 + \eta$$

*then*

$$\begin{aligned}
f(d,k) &= \psi(1/2) = \psi_0(1/2) = (1 - 2\eta)2^{-k} + O(2^{-2k})\\
\psi(2^{-k}) &= 2^{-k} + O(2^{-2k}).
\end{aligned}$$

*In particular, if $k$ is sufficiently large then $\psi(2^{-k}) > f(d,k)$.*

*Proof.* By direct inspection $f(d,k) = \psi(1/2) = \psi_0(1/2)$. By Taylor series expansion, $\log(1 - 2^{1-k}) = -2^{1-k} - 2^{1-2k} + O(2^{-3k})$. So

$$\begin{aligned} f(d,k) &= \log(2) + r\log(1-2^{1-k}) \\ &= \log(2) + \left(\frac{\log(2)}{2}\cdot 2^k - (1+\log(2))/2 + \eta\right)\left(-2^{1-k} - 2^{1-2k}\right) + O(r2^{-3k}) \\ &= (1-2\eta)2^{-k} + O(2^{-2k}). \end{aligned}$$

Next we estimate $\psi(2^{-k})$. For convenience, let $x = 2^{-k}$. Then

$$1 - x^k - (1-x)^k = k\cdot 2^{-k} + O(k^2 2^{-2k}).$$

Since $\log(1-x) = -x - x^2/2 + O(x^3)$,

$$\log\left(1 - \frac{1-x^k-(1-x)^k}{2^{k-1}-1}\right) = -2k\cdot 2^{-2k} + O(k^2 2^{-3k}).$$

So

$$r\log\left(1 - \frac{1-x^k-(1-x)^k}{2^{k-1}-1}\right) = -k\log(2)\cdot 2^{-k} + O(k^2 2^{-2k}).$$

Also,

$$H(x, 1-x) = (k\log(2)+1)\cdot 2^{-k} + O(2^{-2k}).$$

Add these together to obtain

$$\psi(2^{-k}) = 2^{-k} + O(k^2 2^{-2k}).$$

$\square$

**Corollary 2.5.7.** *Inequality (2.2) of Lemma 2.1.2 is true. To be precise, let $0 < \eta_0$ be constant with respect to $k$. Then for all sufficiently large $k$ (depending on $\eta_0$), if*

$$r = d/k = \frac{\log(2)}{2}\cdot 2^k - (1+\log(2))/2 + \eta$$

*for some $\eta \geq \eta_0$ constant with respect to $k$ then*

$$f(d,k) < \liminf_{n\to\infty} n^{-1} \log \mathbb{E}_n^p[Z(\sigma)].$$

*Proof.* By definition,

$$n^{-1} \log \mathbb{E}_n^p[Z(\sigma)] \geq \max_{\delta\in[0,1/2]} n^{-1} \log \mathbb{E}_n^p[Z(\delta)].$$

By Theorem 2.5.1,

$$\liminf_{n\to\infty} n^{-1} \log \mathbb{E}_n^p[Z(\sigma)] \geq \max_{\delta\in[0,1/2]} \psi_0(\delta). \tag{2.22}$$

Because $H_0(\delta, 1-\delta) - H(\delta, 1-\delta) \geq 0$ (since it is a Kullback-Liebler divergence), the first equality of Lemma 2.5.5 implies

$$\begin{aligned} \psi_0(\delta) &= \psi(\delta_0) - (H(\delta_0, 1-\delta_0) - H_0(\delta, 1-\delta)) + (d-1)\left[H_0(\delta, 1-\delta) - H(\delta, 1-\delta)\right] \\ &\geq \psi(\delta_0) - \delta_0 \varepsilon \log\left(\frac{1-\delta_0}{\delta_0}\right). \end{aligned}$$

By Lemma 2.5.6, $\psi(2^{-k}) = f(d,k) + 2\eta 2^{-k} + O(2^{-2k})$. By Lemma 2.5.5, $\varepsilon = O(2^{-k})$. As $\delta$ varies over $[0,1/2]$, $\delta_0$ also varies over $[0,1/2]$, so there exists $\delta$ such that $\delta_0 = 2^{-k}$. For this value of $\delta$,

$$\psi_0(\delta) \geq \psi(2^{-k}) - 2^{-k}\varepsilon \log\left(\frac{1-2^{-k}}{2^{-k}}\right) \geq f(d,k) + 2\eta 2^{-k} + O(k2^{-2k}).$$

Combined with (2.22) this implies the Corollary. $\square$

In the next subsection, we will need the following result.

**Proposition 2.5.8.** *Let*

$$0 < \eta_0 < \eta_1 < (1-\log 2)/2.$$

*Then there exists $k_0$ (depending on $\eta_0, \eta_1$) such that for all $k \geq k_0$ if*

$$r = d/k = \frac{\log(2)}{2} \cdot 2^k - (1+\log(2))/2 + \eta$$

*for some $\eta \in [\eta_0, \eta_1]$ then in the interval $[2^{-k/2}, 1-2^{-k/2}]$, $\psi_0$ attains its unique maximum at 1/2. That is,*

$$\max\{\psi_0(\delta) : \ 2^{-k/2} \leq \delta \leq 1-2^{-k/2}\} = \psi_0(1/2) = f(d,k) = \log(2) + r\log(1-2^{1-k})$$

*and if $\delta \in [2^{-k/2}, 1-2^{-k/2}]$ and $\delta \neq 1/2$ then $\psi_0(\delta) < \psi_0(1/2)$.*

*Proof.* By Lemma 2.5.5, it suffices to restrict $\delta$ to the interval $[2^{-k/2}, 1/2]$ (because $\psi_0(\delta) = \psi_0(1-\delta)$). So we will assume $\delta \in [2^{-k/2}, 1/2]$ without further mention.

Define $\psi_1$ by

$$\psi_1(\delta_0) = \frac{d}{k}\log\left(1 - \frac{1-\delta_0^k-(1-\delta_0)^k}{2^{k-1}-1}\right).$$

Observe

$$\begin{aligned}
\psi_1(\delta_0) &= r\left(-\frac{1-\delta_0^k-(1-\delta_0)^k}{2^{k-1}-1} + O(4^{-k})\right) && (2.23)\\
&= -\log(2)[1-(1-\delta_0)^k] + O(2^{-k}). && (2.24)
\end{aligned}$$

By (2.16) and the first inequality of Lemma 2.5.5,

$$\begin{aligned}
\psi_0(\delta) &\leq \psi(\delta_0) + (d-1)[H_0(\delta, 1-\delta) - H(\delta, 1-\delta)] && (2.25)\\
&= H(\delta_0, 1-\delta_0) + \psi_1(\delta_0) + (d-1)[H_0(\delta, 1-\delta) - H(\delta, 1-\delta)]. && (2.26)
\end{aligned}$$

Moreover, $(d-1) = O(k2^k)$ and, by Lemma 2.5.5, $H_0(\delta, 1-\delta) - H(\delta, 1-\delta) = O(\delta_0 4^{-k})$. Therefore,

$$\begin{aligned}
\psi_0(\delta) &\leq H(\delta_0, 1-\delta_0) + \psi_1(\delta_0) + O(\delta_0 k 2^{-k}) && (2.27)\\
&\leq H(\delta_0, 1-\delta_0) - \log(2)[1-(1-\delta_0)^k] + O((\delta_0 k+1)2^{-k}). && (2.28)
\end{aligned}$$

Observe that $\delta_0 \geq \delta$. We divide the rest of the proof into five cases depending on where $\delta_0$ lies in the interval $[2^{-k/2}, 1/2]$.

**Case 1**. Suppose $2^{-k/2} \leq \delta_0 \leq \frac{1}{2k}$. We claim that $\psi_0(\delta) < 0$. Note $-\log(\delta_0) \leq$

$(k/2)\log(2)$ and $-(1-\delta_0)\log(1-\delta_0) = \delta_0 + O(\delta_0^2)$. So

$$\begin{aligned} H(\delta_0, 1-\delta_0) &= -\delta_0 \log \delta_0 - (1-\delta_0)\log(1-\delta_0) \\ &\leq \delta_0(k/2)\log(2) + \delta_0 + O(\delta_0^2). \end{aligned}$$

By Taylor series expansion,

$$1-(1-\delta_0)^k \geq k\delta_0 - \binom{k}{2}\delta_0^2 \geq 3k\delta_0/4.$$

So by (2.28)

$$\begin{aligned} \psi_0(\delta) &\leq \delta_0(k/2)\log(2) + \delta_0 - 3k\delta_0\log(2)/4 + O(\delta_0^2) \\ &= \delta_0[1 - k\log(2)/4] + O(\delta_0^2). \end{aligned}$$

Thus $\psi_0(\delta) < 0$ if $k$ is sufficiently large.

**Case 2**. Let $0 < \xi_0 < 1/2$ be a constant such that $H(\xi_0, 1-\xi_0) < \log(2)(1-e^{-1/2})$. Suppose $\frac{1}{2k} \leq \delta_0 \leq \xi_0$. We claim that $\psi_0(\delta) < 0$ if $k$ is sufficiently large.

By monotonicity, $H(\delta_0, 1-\delta_0) \leq H(\xi_0, 1-\xi_0)$. Since $1-x \leq e^{-x}$ (for $x > 0$),

$$[1-(1-\delta_0)^k] \geq 1 - e^{-k\delta_0} \geq 1 - e^{-1/2}.$$

By (2.28),

$$\psi_0(\delta) \leq H(\xi_0, 1-\xi_0) - \log(2)(1-e^{-1/2}) + O(k2^{-k}).$$

This implies the claim.

**Case 3**. Let $\xi_1$ be a constant such that $\max(\xi_0, 1/3) < \xi_1 < 1/2$. Suppose $\xi_0 \leq \delta_0 \leq \xi_1$. We claim that $\psi_0(\delta) < 0$ for all sufficiently large $k$ (depending on $\xi_1$).

By (2.28),

$$\psi_0(\delta) \leq H(\xi_1, 1-\xi_1) - \log(2)[1-(1-\delta_0)^k] + O(k2^{-k}) \leq H(\xi_1, 1-\eta_1) - \log(2) + O((1-\xi_0)^k).$$

This proves the claim.

**Case 4**. We claim that if $\xi_1 \le \delta_0 \le 0.5 - 2^{-k}$ then $\psi_0(\delta) < f(d,k)$ for all sufficiently large $k$ (independent of the choice of $\xi_1$).

Recall that we define $\varepsilon$ by $\delta = \delta_0(1-\varepsilon)$. By (2.21),

$$\begin{aligned} \varepsilon &= 2^{1-k} \cdot (1-\delta_0) \cdot \frac{(1-\delta_0)^{k-1} - \delta_0^{k-1}}{1 - 2^{2-k} + 2(\delta_0/2)^k + 2((1-\delta_0)/2)^k} \\ &\le 2^{1-k}(1-\xi_1)^k + O\left(4^{-k}\right) \le 2 \cdot 3^{-k} \end{aligned}$$

since $\xi_1 > 1/3$, assuming $k$ is sufficiently large.

The assumption on $r$ implies $d = O\left(k2^k\right)$. So the second equality of Lemma 2.5.5 implies

$$(d-1)[H_0(\delta, 1-\delta) - H(\delta, 1-\delta)] = O\left(4.5^{-k}\right).$$

By the discussion before equation (32) on page 19 of [COZ11] and Taylor's theorem,

$$\psi(\delta_0) = \psi(1/2) - (2 + o_k(1))(0.5-\delta_0)^2 + o_k\left((0.5-\delta_0)^3\right). \tag{2.29}$$

Since $\psi(1/2) = f(d,k)$, (2.25) implies

$$\psi_0(\delta) \le f(d,k) - (2+o_k(1))(0.5-\delta_0)^2 + o_k\left((0.5-\delta_0)^3\right) + O\left(4.5^{-k}\right)$$

is strictly less than $f(d,k)$ if $k$ is sufficiently large. This implies the claim.

**Case 5**. Suppose $0.5 - 2^{-k} \le \delta_0 < 0.5$. Let $\gamma = 0.5 - \delta_0$. By (2.21),

$$\varepsilon = O\left(\left[(1/2+\gamma)^{k-1} - (1/2-\gamma)^{k-1}\right] 2^{-k}\right).$$

Define $L(x) := (1/2+x)^{k-1} - (1/2-x)^{k-1}$. We claim that $L(\gamma) \le \gamma$. Since $L(0) = 0$, it suffices to show that $L'(x) \le 1$ for all $x$ with $|x| \le 0.01$. An elementary calculation shows

$$L'(x) = (k-1)\left[(1/2+x)^{k-2} + (1/2-x)^{k-2}\right].$$

So $L'(x) \le 1$ if $|x| \le 0.01$ and $k$ is sufficiently large. Altogether this proves $\varepsilon = O\left(\gamma 2^{-k}\right)$.

So the second equality of Lemma 2.5.5 implies

$$(d-1)\left[H_0(\delta,1-\delta)-H(\delta,1-\delta)\right]=O\left(k2^{-k}\gamma^2\right).$$

By (2.29) and (2.25),

$$\psi_0(\delta)\leq f(d,k)-(4+o_k(1))\gamma^2+O\left(k2^{-k}\gamma^2\right).$$

This is strictly less than $f(d,k)$ if $k$ is sufficiently large.

□

### 2.5.2 Reducing Lemma 2.1.2 inequality (2.3) to estimating the local cluster

As in the previous section, fix an equitable coloring $\chi : V_n \to \{0,1\}$. Given a uniform homomorphism $\sigma \in \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{Sym}(n))$, the **cluster around** $\chi$ is the set

$$\mathcal{C}_\sigma(\chi):=\left\{\widetilde{\chi}\in Z_e(\sigma): d_n(\chi,\widetilde{\chi})\leq 2^{-k/2}\right\}.$$

We also call this the **local cluster** if $\chi$ is understood.

In §2.6 we prove:

**Proposition 2.5.9.** *Let $0<\eta_0<(1-\log 2)/2$. Then for all sufficiently large $k$ (depending on $\eta_0$), if*

$$r:=d/k=\frac{\log(2)}{2}\cdot 2^k-(1+\log(2))/2+\eta$$

*for some $\eta$ with $\eta_0\leq\eta<(1-\log 2)/2$ then with high probability in the planted model, $|\mathcal{C}_\sigma(\chi)|\leq \mathbb{E}^u_n(Z_e)$. In symbols,*

$$\lim_{n\to\infty}\mathbb{P}^\chi_n\big(|\mathcal{C}_\sigma(\chi)|\leq \mathbb{E}^u_n(Z_e)\big)=1.$$

The rest of this section proves Lemma 2.1.2 inequality (2.3) from Proposition 2.5.9 and the second moment estimates from earlier in this section. So we assume the hypotheses of Proposition 2.5.9 without further mention.

We say that a coloring $\chi$ is **$\sigma$-good** if it is equitable and $|\mathcal{C}_\sigma(\chi)| \leq \mathbb{E}_n^u(Z_e(\sigma))$. Let $S_g(\sigma)$ be the set of all $\sigma$-good proper colorings and let $Z_g(\sigma) = |S_g(\sigma)|$ be the number of $\sigma$-good proper colorings.

We will say a positive function $G(n)$ is **sub-exponential in** $n$ if $\lim_{n\to\infty} n^{-1}\log G(n) = 0$. Also we say functions $G$ and $H$ are **asymptotic**, denoted by $G(n) \sim H(n)$, if $\lim_{n\to\infty} G(n)/H(n) = 1$. Similarly, $G(n) \lesssim H(n)$ if $\limsup_{n\to\infty} G(n)/H(n) \leq 1$.

**Lemma 2.5.10.** $\mathbb{E}_n^u(Z_g) \sim \mathbb{E}_n^u(Z_e) = F(n)\mathbb{E}_n^u(Z)$ *where* $F(n)$ *is sub-exponential in* $n$.

*Proof.* For brevity, let $\mathcal{H} = \mathrm{Hom}_{\mathrm{unif}}(\Gamma, \mathrm{Sym}(n))$. Let $\mathbb{P}_n^\chi$ be the probability operator in the planted model of $\chi$. By definition,

$$\begin{aligned}
\mathbb{E}_n^u(Z_g) &= |\mathcal{H}|^{-1}\sum_{\sigma\in\mathcal{H}} Z_g(\sigma) = |\mathcal{H}|^{-1}\sum_{\sigma\in\mathcal{H}}\sum_{\chi:V\to\{0,1\}} 1_{S_g(\sigma)}(\chi)\\
&= \sum_{\chi}\mathbb{P}_n^u(\chi\in S_g(\sigma))\\
&= \sum_{\chi \text{ equitable}} \mathbb{P}_n^u\left(|\mathcal{C}_\sigma(\chi)| \leq \mathbb{E}_n^u(Z_e)|\chi \text{ proper}\right)\mathbb{P}_n^u(\chi \text{ proper})\\
&= \sum_{\chi \text{ equitable}} \mathbb{P}_n^\chi\left(|\mathcal{C}_\sigma(\chi)| \leq \mathbb{E}_n^u(Z_e)\right)\mathbb{P}_n^u(\chi \text{ proper})\\
&\sim \sum_{\chi \text{ equitable}} \mathbb{P}_n^u\left(\chi \text{ proper}\right) = \mathbb{E}_n^u(Z_e)
\end{aligned}$$

where the asymptotic equality $\sim$ follows from Proposition 2.5.9. The equality $\mathbb{E}_n^u(Z_e) = F(n)\mathbb{E}_n^u(Z)$ holds by Theorem 2.4.2. $\square$

**Lemma 2.5.11.** $\mathbb{E}_n^u(Z_g^2) \leq C(n)\mathbb{E}_n^u(Z_g)^2$, *where* $C(n) = C(n,k,r)$ *is sub-exponential in* $n$.

*Proof.*

$$\begin{aligned}
\mathbb{E}_n^u(Z_g^2) &= |\mathcal{H}|^{-1}\sum_{\sigma\in\mathcal{H}}\left(\sum_{\chi}1_{S_g(\sigma)}(\chi)\right)^2 && (2.30)\\
&= |\mathcal{H}|^{-1}\sum_{\sigma\in\mathcal{H}}\sum_{\chi,\widetilde{\chi}}1_{S_g(\sigma)}(\chi)1_{S_g(\sigma)}(\widetilde{\chi}) && (2.31)\\
&= \sum_{\chi,\widetilde{\chi}}\mathbb{P}_n^u\left(\chi\in S_g \text{ and } \widetilde{\chi}\in S_g\right) && (2.32)\\
&= \sum_{\chi,\widetilde{\chi}}\mathbb{P}_n^u(\chi\in S_g)\mathbb{P}_n^u(\widetilde{\chi}\in S_g|\chi\in S_g) && (2.33)\\
&= \sum_{\chi}\mathbb{P}_n^u(\chi\in S_g)\mathbb{E}_n^u(Z_g|\chi\in S_g). && (2.34)
\end{aligned}$$

For a fixed $\chi\in S_g(\sigma)$ we analyze $\mathbb{E}_n^u(Z_g|\chi\in S_g)$ by breaking the colorings into those that are close (i.e. in the local cluster) and those that are far. So let $Z_g(\delta):\mathrm{Hom}_\chi(\Gamma,\mathrm{sym}(n))\to\mathbb{N}$ be the number of good proper colorings such that $d_n(\chi,\widetilde{\chi})=\delta$. (We will also use $Z_e(\delta)=Z_\chi(\delta)$ for the analogous number of equitable proper colorings). Then

$$\mathbb{E}_n^u(Z_g|\chi\in S_g)\leq 2\mathbb{E}_n^u\left(\sum_{0\leq\delta\leq 2^{-k/2}}Z_g(\delta)\Big|\chi\in S_g\right)+2\mathbb{E}_n^u\left(\sum_{2^{-k/2}<\delta\leq 1/2}Z_g(\delta)\Big|\chi\in S_g\right). \quad (2.35)$$

The coefficient 2 above accounts for the following symmetry: if $\widetilde{\chi}$ is a good coloring with $d_n(\chi,\widetilde{\chi})=\delta$ then $1-\widetilde{\chi}$ is a good coloring with $d_n(\chi,1-\widetilde{\chi})=1-\delta$. Note that

$$\mathbb{E}_n^u\left(\sum_{0\leq\delta\leq 2^{-k/2}}Z_g(\delta)\Big|\chi\in S_g\right)\leq\mathbb{E}_n^u\big(\#\mathcal{C}_\sigma(\chi)|\chi\in S_g\big)\leq\mathbb{E}_n^u(Z_e) \quad (2.36)$$

where the last inequality holds by definition of $S_g$.

For colorings not in the local cluster,

$$\begin{aligned}
\mathbb{E}_n^u\left(\sum_{2^{-k/2}<\delta\leq 1/2}Z_g(\delta)\Big|\chi\in S_g\right) &\leq \mathbb{E}_n^u\left(\sum_{2^{-k/2}<\delta\leq 1/2}Z_e(\delta)\Big|\chi\in S_g\right)\\
&\leq \mathbb{E}_n^u\left(\sum_{2^{-k/2}<\delta\leq 1/2}Z_e(\delta)\Big|\chi \text{ proper}\right)\frac{\mathbb{P}_n^u(\chi \text{ proper})}{\mathbb{P}_n^u(\chi\in S_g)}
\end{aligned}$$

where the sum is over all $\delta \in \mathbb{Z}[1/n]$ in the given range. By definition and Proposition 2.5.9,

$$\frac{\mathbb{P}_n^u(\chi \text{ proper})}{\mathbb{P}_n^u(\chi \in S_g)} = \frac{1}{\mathbb{P}_n^u(\chi \in S_g|\chi \text{ proper})} = \frac{1}{\mathbb{P}_n^\chi\big(|\mathcal{C}_\sigma(\chi)| \leq \mathbb{E}_n^u(Z_e)\big)} \to 1$$

as $n \to \infty$. Since $\mathbb{E}_n^u(\cdot|\chi \text{ proper}) = \mathbb{E}_n^\chi(\cdot)$, the above inequality now implies

$$\begin{aligned} \mathbb{E}_n^u\left(\sum_{2^{-k/2}<\delta\leq 1/2} Z_g(\delta)\Big|\chi \in S_g\right) &\lesssim \sum_{2^{-k/2}<\delta\leq 1/2} \mathbb{E}_n^\chi(Z_e(\delta)) \leq C_1 \sum_{2^{-k/2}<\delta\leq 1/2} e^{n\psi_0(\delta)} \quad (2.37)\\ &\leq C_1 n e^{nf(d,k)} \leq C_2 \mathbb{E}_n^u(Z_e) \quad (2.38) \end{aligned}$$

where the second inequality holds by Theorem 2.5.1 for some function $C_1 = C_1(n,k,r)$ which is sub-exponential in $n$. The second-to-last inequality holds because the number of summands is bounded by $n$ since $\delta$ is constrained to lie in $\mathbb{Z}[1/n]$ and by Proposition 2.5.8, $\psi_0(\delta) \leq f(d,k)$. The last inequality holds for some function $C_2 = C_2(n,k,r)$ that is sub-exponential in $n$ since by Theorem 2.4.2, $n^{-1}\log \mathbb{E}_n^u(Z_e)$ converges to $f(d,k)$.

Combine (2.35), (2.36) and (2.38) to obtain

$$\mathbb{E}_n^u(Z_g|\chi \in S_g) \leq 2(1+C_2)\mathbb{E}_n^u(Z_e).$$

Plug this into (2.34) to obtain

$$\mathbb{E}_n^u(Z_g^2) \leq 2(1+C_2)\mathbb{E}_n^u(Z_e)^2 \sim 2(1+C_2)\mathbb{E}_n^u(Z_g)^2$$

where the asymptotic $\sim$ holds by Lemma 2.5.10. This proves the lemma.

□

**Corollary 2.5.12.** *Lemma 2.1.2 inequality (2.3) is true. That is:*

$$0 = \inf_{\epsilon>0}\liminf_{n\to\infty} n^{-1}\log\left(\mathbb{P}_n^u\left(\left|n^{-1}\log Z(\sigma) - f(d,k)\right| < \epsilon\right)\right).$$

*Proof.* By Theorem 2.4.2,

$$n^{-1}\log \mathbb{E}_n^u(Z(\sigma)) \to f(d,k)$$

as $n \to \infty$. In particular, for every $\epsilon > 0$, for large enough $n$, $\mathbb{P}_n^u\left(n^{-1}\log Z(\sigma) > f(d,k)+\epsilon\right) < 1/2$. So it suffices to prove

$$0 = \inf_{\epsilon>0} \liminf_{n\to\infty} n^{-1} \log\left(\mathbb{P}_n^u\left(n^{-1}\log Z(\sigma) \ge f(d,k)-\epsilon\right)\right).$$

Since $Z(\sigma) \ge Z_g(\sigma)$, it suffices to prove the same statement with $Z_g(\sigma)$ in place of $Z(\sigma)$. By Lemma 2.5.10 and Theorem 2.4.2, $n^{-1}\log\left(\mathbb{E}_n^u[Z_g(\sigma)]\right)$ converges to $f(d,k)$ as $n \to \infty$. So we may replace $f(d,k)$ in the statement above with $n^{-1}\log\left(\mathbb{E}_n^u[Z_g(\sigma)]\right)$. Then we may multiply by $n$ both sides and exponentiate inside the probability. So it suffices to prove

$$0 = \inf_{\epsilon>0} \liminf_{n\to\infty} n^{-1} \log\left(\mathbb{P}_n^u\left(Z_g(\sigma) \ge \mathbb{E}_n^u[Z_g(\sigma)]e^{-n\epsilon}\right)\right). \tag{2.39}$$

By the Paley-Zygmund inequality and Lemma 2.5.11

$$\mathbb{P}_n^u\left(Z_g(\sigma) > \mathbb{E}_n^u[Z_g(\sigma)]e^{-n\epsilon}\right) \ge (1-e^{-n\epsilon})^2 \frac{\mathbb{E}_n^u[Z_g(\sigma)]^2}{\mathbb{E}_n^u[Z_g(\sigma)^2]} \ge \frac{1}{C}$$

where $C = C(n)$ is sub-exponential in $n$. This implies (2.39). □

## 2.6 The local cluster

To prove Proposition 2.5.9, we show that with high probability in the planted model, there is a 'rigid' set of vertices with density approximately $1-2^{-k}$. Rigidity here means that any proper coloring either mostly agrees with the planted coloring on the rigid set or it must disagree on a large density subset. Before making these notions precise, we introduce the various subsets, state precise lemmas about them and prove Proposition 2.5.9 from these lemmas which are proven in the next two sections.

So suppose $G = (V,E)$ is a $k$-uniform $d$-regular hyper-graph and $\chi : V \to \{0,1\}$ is a proper coloring. An edge $e \in E$ is **$\chi$-critical** if there is a vertex $v \in e$ such that $\chi(v) \notin \chi(e \setminus \{v\})$. If this is the case, then we say $v$ **supports** $e$ **with respect to** $\chi$. If $\chi$ is understood then we will omit mention of it. We will apply these notions both to the case when $G$ is the Cayley hyper-tree of $\Gamma$ and when $G = G_\sigma$ is a finite hyper-graph.

For $l \in \{0,1,2,\ldots\}$, define the **depth $l$-core of** $\chi$ to be the subset $C_l(\chi) \subset V$

satisfying

$$C_0(\chi) = V,$$

$$C_{l+1}(\chi) = \{v \in C_l(\chi) : \ v \text{ supports at least 3 edges which are contained in } C_l\}.$$

Also let $C_\infty(\chi) = \cap_l C_l(\chi)$.

The set $C_l(\chi)$ is defined to consist of vertices $v$ so that if $v$ is re-colored (in some proper coloring) then this re-coloring forces a sequence of re-colorings in the shape of an immersed hyper-tree of degree at least 3 and depth $l$. Re-coloring a vertex of $C_\infty(\chi)$ would force re-coloring an infinite immersed tree of degree at least 3.

Also define the **attached vertices** $A_l(\chi) \subset V$ by: $v \in A_l(\chi)$ if $v \notin C_l(\chi)$ but there exists an edge $e$, supported by $v$ such that $e \setminus \{v\} \subset C_l(\chi)$. Thus if $v \in A_l(\chi)$ is re-colored then it forces a re-coloring of some vertex in $C_l(\chi)$. In this definition, we allow $l = \infty$.

In order to avoid over-counting, we also need to define the subset $A'_l(\chi)$ of vertices $v \in A_l(\chi)$ such that there exists a vertex $w \in A_l(\chi)$, with $w \neq v$, and edges $e_v$, $e_w$ supported by $v, w$ respectively such that

1. $e_v \cup e_w \setminus \{v, w\} \subset C_l(\chi)$,

2. $e_v \cap e_w \neq \emptyset$.

In this definition, we allow $l = \infty$.

We will need the following constants:

$$\lambda_0 = \frac{1}{2^{k-1}-1}, \quad \lambda := d\lambda_0 = \frac{d}{2^{k-1}-1}.$$

The significance of $\lambda_0$ is: if $e$ is an edge and $v \in e$ a vertex then $\lambda_0$ is the probability $v$ supports $e$ in a uniformly random proper coloring of $e$. So $\lambda = d\lambda_0$ is the expected number of edges that $v$ supports. For the values of $d$ and $k$ used in the Key Lemma 2.1.2, $\lambda = \log(2)k + O(k2^{-k})$.

For the next two lemmas, we assume the hypotheses of Proposition 2.5.9.

**Lemma 2.6.1.** *For any $\delta > 0$ there exists $k_0$ such that $k \geq k_0$ implies*

$$\lim_{l\to\infty} \liminf_{n\to\infty} \mathbb{P}_n^\chi\left(\frac{|C_l(\chi) \cup A_l(\chi) \setminus A'_l(\chi)|}{n} > 1 - e^{-\lambda}(1+\delta)\right) = 1.$$

Lemma 2.6.1 is proven in §2.7.

**Definition 7.** Fix a proper 2-coloring $\chi : V \to \{0,1\}$. Let $\rho > 0$. A subset $R \subset V$ is **$\rho$-rigid** (with respect to $\chi$) if for every proper coloring $\chi' : V \to \{0,1\}$, $|\{v \in R : \chi(v) \neq \chi'(v)\}|$ is either less than $\rho|V|$ or greater than $2^{-k/2}|V|$.

**Lemma 2.6.2.** *For any $\rho > 0$,*

$$\lim_{l\to\infty} \liminf_{n\to\infty} \mathbb{P}_n^\chi \left(C_l(\chi) \cup A_l(\chi) \setminus A_l'(\chi) \text{ is } \rho\text{-rigid}\right) = 1.$$

Lemma 2.6.2 is proven in §2.8. We can now prove Proposition 2.5.9:

**Proposition 2.5.9**. *Let $0 < \eta_0 < \eta_1 < (1 - \log 2)/2$. Then for all sufficiently large $k$ (depending on $\eta_0$), if*

$$r := d/k = \frac{\log(2)}{2} \cdot 2^k - (1 + \log(2))/2 + \eta$$

*for some $\eta$ with $\eta_0 \le \eta \le \eta_1$ then with high probability in the planted model, $|\mathcal{C}_\sigma(\chi)| \le \mathbb{E}_n^u(Z_e)$. In symbols,*

$$\lim_{n\to\infty} \mathbb{P}_n^\chi\big(|\mathcal{C}_\sigma(\chi)| \le \mathbb{E}_n^u(Z_e)\big) = 1.$$

*Proof.* Let $0 < \rho, \delta$ be small constants satisfying

$$\log(2)\delta + H(\rho, 1-\rho) + \log(2)\rho < (1 - 2\eta - \log(2))2^{-k}. \tag{2.40}$$

Let $l$ be a natural number. Also let $\sigma : \Gamma \to \mathrm{sym}(n)$ be a uniform homomorphism and $\chi : [n] \to \{0,1\}$ a proper coloring. To simplify notation, let

$$R = C_l(\chi) \cup A_l(\chi) \setminus A_l'(\chi).$$

By Lemmas 2.6.1 and 2.6.2 it suffices to show that if $|R|/n > 1 - e^{-\lambda} - \delta$ and $R$ is $\rho$-rigid then $|\mathcal{C}_\sigma(\chi)| \le \mathbb{E}_n^u(Z_e)$ (for all sufficiently large $n$). So assume $|R|/n > 1 - e^{-\lambda} - \delta$ and $R$ is $\rho$-rigid.

Let $\chi' \in \mathcal{C}_\sigma(\chi)$. By definition, this means $d_n(\chi', \chi) \leq 2^{-k/2}$. Since $R$ is $\rho$-rigid, this implies

$$|\{v \in R : \chi(v) \neq \chi'(v)\}| \leq \rho n.$$

Since this holds for all $\chi' \in \mathcal{C}_\sigma(\chi)$, it follows that

$$|\mathcal{C}_\sigma(\chi)| \leq \binom{|R|}{\rho n} 2^{\rho n} 2^{n-|R|}.$$

By Stirling's formula

$$n^{-1} \log \binom{|R|}{\rho n} \leq n^{-1} \log \binom{n}{\rho n} \leq H(\rho, 1-\rho) + O(n^{-1} \log(n)).$$

Since $|R|/n > 1 - e^{-\lambda} - \delta = 1 - 2^{-k} - \delta + O(k2^{-2k})$,

$$n^{-1} \log(2^{n-|R|}) \leq \log(2)[2^{-k} + \delta] + O(k2^{-2k}).$$

Thus,

$$n^{-1} \log |\mathcal{C}_\sigma(\chi)| \leq \log(2)2^{-k} + \log(2)\delta + H(\rho, 1-\rho) + \log(2)\rho + O(k2^{-2k} + n^{-1} \log(n)).$$

On the other hand,

$$n^{-1} \log \mathbb{E}^u_n(Z_e) = f(d,k) + O(n^{-1} \log(n)) = (1-2\eta)2^{-k} + O(2^{-2k}) + O(n^{-1} \log(n))$$

by Lemma 2.5.6 and Theorem 2.4.2.

Therefore, the choice of $\rho, \delta$ in (2.40) implies $|\mathcal{C}_\sigma(\chi)| \leq \mathbb{E}^u_n(Z_e)$ for all sufficiently large $n$. This also depends on $k$ being sufficiently large, but the lower bound on $k$ is uniform in $n$.

□

## 2.7 A Markov process on the Cayley hyper-tree

Let $(x_g)_{g \in \Gamma}$ be a family of random variables satisfying the following conditions:

- For each $g \in \Gamma$, $x_g$ is uniformly distributed on $\{0, 1\}$,

- Let $v \in \Gamma$ and let $e \subset \Gamma$ be a hyperedge containing $v$. Let $\mathrm{Past}(e, v)$ be the set of all $g \in \Gamma$ such that every path in the Cayley hyper-tree from $g$ to an element of $e$ passes through $v$. In particular, $e \cap \mathrm{Past}(e, v) = \{v\}$. Then the distribution of $(x_g)_{g \in e \setminus \{v\}}$ conditioned on $\{x_g : g \in \mathrm{Past}(e, v)\}$ is uniformly distributed on the set of all colorings $y : (e \setminus \{v\}) \to \{0, 1\}$ such that there exists some $h \in e \setminus \{v\}$ with $y(h) \neq x(v)$.

By definition, the latter condition means that $(x_g)_{g \in \Gamma}$ is a Markov random field on the Cayley hyper-tree. Let $\mu$ be the law of $(x_g)_{g \in \Gamma}$. So $\mu$ is a $\Gamma$-invariant Borel probability measure on $X$.

### 2.7.1 Local convergence

We will prove the following lemma.

**Lemma 2.7.1.** *Let $\chi : V \to \{0, 1\}$ be an equitable coloring with $|V| = n$. If $B \subseteq X$ is clopen, then for every $\epsilon > 0$*

$$\lim_{n \to \infty} \mathbb{P}_n^{\chi} \left( \left| \frac{1}{n} \sum_{v \in V} \mathbb{1}_B(\Pi_v^{\sigma_n}(\chi)) - \mu(B) \right| > \epsilon \right) = 0.$$

To prove this lemma we will first show that if $f : \mathrm{Hom}_{\chi}(\Gamma, \mathrm{sym}(n)) \to \mathbb{R}$ is the function

$$f(\sigma_n) := \frac{1}{n} \sum_{v \in V} \mathbb{1}_B(\Pi_v^{\sigma_n}(\chi))$$

then $f$ concentrates about its expectation using Theorem 2.10.1, and then we will show that this expectation is given by $\mu(B)$.

**Proposition 2.7.2.** *We have*

$$\lim_{n \to \infty} \mathbb{P}_n^{\chi} \left( |f - \mathbb{E}_n^{\chi}[f]| > \epsilon \right) = 0.$$

*Proof.* For $g \in \Gamma$, let $\mathrm{pr}_g : X \to \{0, 1\}$ be the projection map $\mathrm{pr}_g(x) = x_g$. For $D \subset \Gamma$, let $\mathcal{F}_D$ be the smallest Borel sigma-algebra such that $\mathrm{pr}_g$ is $\mathcal{F}_D$-measurable for every $g \in D$.

Note that every clopen subset $B$ of $X$ is a finite union of cylinder sets. Thus the function $\mathbb{1}_B$ is $\mathcal{F}_D$-measurable for some finite set $D \subset \Gamma$.

We will use the normalized Hamming metrics $d_{\mathrm{Sym}(n)}$ and $d_{\mathrm{Hom}}$ on $\mathrm{Sym}(n)$ and $\mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$ respectively. These are defined in the beginning of Appendix 2.10. We claim $f$ is $L$-Lipschitz for some $L < \infty$. Let $\sigma, \sigma' \in \mathrm{Hom}_\chi(\Gamma, \mathrm{sym}(n))$. Because $\mathbb{1}_B$ is $\mathcal{F}_D$-measurable,

$$\begin{aligned} |f(\sigma) - f(\sigma')| &\leq n^{-1}\#\{v \in [n] : \chi(\sigma(\gamma^{-1})(v)) \neq \chi(\sigma'(\gamma^{-1})(v)) \text{ for some } \gamma \in D\} \\ &\leq n^{-1}\#\{v \in [n] : \sigma(\gamma^{-1})(v) \neq \sigma'(\gamma^{-1})(v) \text{ for some } \gamma \in D\} \\ &\leq \sum_{\gamma \in D} d_{\mathrm{Sym}(n)}(\sigma(\gamma^{-1}), \sigma'(\gamma^{-1})). \end{aligned}$$

Now $d_{\mathrm{Sym}(n)}$ is both left and right invariant. So

$$d_{\mathrm{Sym}(n)}(gh, g'h') \leq d_{\mathrm{Sym}(n)}(gh, gh') + d_{\mathrm{Sym}(n)}(gh', g'h') = d_{\mathrm{Sym}(n)}(h, h') + d_{\mathrm{Sym}(n)}(g, g')$$

for any $g, g', h, h' \in \mathrm{Sym}(n)$. By induction, this implies $d_{\mathrm{Sym}(n)}(\sigma(\gamma), \sigma'(\gamma)) \leq |\gamma| d_{\mathrm{Hom}}(\sigma, \sigma')$ for any $\gamma \in \Gamma$ where $|\gamma|$ is the distance from $\gamma$ to the identity in the word metric on $\Gamma$. Thus if we take $L = \sum_{\gamma \in D} |\gamma| < \infty$ we see that $|f(\sigma) - f(\sigma')| \leq L d_{\mathrm{Hom}}(\sigma, \sigma')$ as desired.

The Proposition now follows from Theorem 2.10.1. $\square$

To finish the proof of Lemma 2.7.1, it now suffices to show the expectation of $f$ with respect to the planted model converges to $\mu(B)$ as $n \to \infty$. We will prove this by an inductive argument, the inductive step of which is covered in the next lemma. In general, if $F$ is a function and $D$ is a subset of the domain of $F$ then we denote the restriction of $F$ to $D$ by $F \upharpoonright D$.

**Lemma 2.7.3.** *Let $h \in \Gamma$ and $e$ be a hyperedge containing $h$. Let $D \subset \mathrm{Past}(e, h)$ be either the singleton $\{h\}$ or a connected finite union of hyperedges containing $h$. Let $\tilde{e} = e \setminus h$ and $\xi \in \{0,1\}^{D \cup \tilde{e}}$ be a proper coloring. Let $F_{D,\xi}$ be the event that $\Pi^{\sigma_n}_{v_n}(\chi) \upharpoonright D = \xi \upharpoonright D$ and define $F_{\tilde{e},\xi}$ similarly. Then for every fixed $v_n \in V$,*

$$\mathbb{P}^\chi_n \left( F_{\tilde{e},\xi} \middle| F_{D,\xi} \right) = \frac{1 + o_n(1)}{2^{k-1} - 1}$$

*where for fixed $n$, $o_n(1)$ does not depend on the choice of $v_n \in V$.*

*Proof.* Let $E_{v_n}$ be the event that $\sigma_n(g^{-1})(v_n) \neq \sigma_n(g'^{-1})(v_n)$ for any $g, g' \in D \cup \tilde{e}$, $g \neq g'$.

Since the planted model is a random sofic approximation, we have $\mathbb{P}_n^\chi(E_{v_n}) = 1 - o_n(1)$. Now

$$\liminf_{n\to\infty} \mathbb{P}_n^\chi\left(F_{D,\xi}\right)$$

rther conditioning on $E_{v_n}$ changes the left hand side by at most a factor of $1 + o_n(1)$. Let $m = |(\xi \restriction \tilde{e})^{-1}(1)|$. We claim that

$$\frac{\binom{n/2-|D|}{m}\binom{n/2-|D|}{k-1-m}m!(k-1-m)!}{(\binom{n-|D|}{k-1}-\binom{n/2}{k-1})(k-1)!} \leq \mathbb{P}_n^\chi\left(F_{\tilde{e},\xi}\middle|F_{D,\xi}, E_{v_n}\right) \leq \frac{\binom{n/2}{m}\binom{n/2}{k-1-m}m!(k-1-m)!}{(\binom{n-|D|}{k-1}-\binom{n/2-|D|}{k-1})(k-1)!} \tag{2.41}$$

Since $|D|$, $m$, and $k$ are constants we have for example $\binom{n-|D|}{k} = \frac{(n-|D|)^k}{k!} + O(n^{k-1}) = (1 + o_n(1))\frac{n^k}{k!} + O(n^{k-1})$. From this and similar computations the desired equality follows from (2.41).

We justify (2.41) as follows:

Since we are conditioning on $E_{v_n}$ and $F_{D,\xi}$, after $\sigma_n(D^{-1})(v_n)$ has been chosen, the total number of choices for $\sigma_n(\tilde{e}^{-1})(v_n)$ to form any bichromatic edge is between $\left(\binom{n-|D|}{k-1} - \binom{n/2-|D|}{k-1}\right)$ $(k-1)!$ and $\left(\binom{n-|D|}{k-1} - \binom{n/2}{k-1}\right)(k-1)!$, depending on the color distribution of $\xi$.

On the other hand, the number of choices for $\sigma_n(\tilde{e}^{-1})(v_n)$ to form an edge with colors matching $\xi$ on $\tilde{e}$, so that $F_{\tilde{e},\xi}$ occurs, is calculated by counting the number of ways to choose $m$ vertices from $\chi^{-1}(1)$ and $k-1-m$ vertices from $\chi^{-1}(0)$ that have not already been fixed by conditioning on $F_{D,\xi}$. There are at least $\binom{n/2-|D|}{m}\binom{n/2-|D|}{k-1-m}m!(k-1-m)!$ and at most $\binom{n/2}{m}\binom{n/2}{k-1-m}m!(k-1-m)!$ ways to choose our vertices with these restrictions, depending again on the color distribution of $\xi$.

Since the hyperedges in $V$ are chosen uniformly at random, the result follows. $\square$

*Proof of Lemma 2.7.1.* Let $\mu_n^\chi$ be the Borel probability measure on $X$ defined by

$$\mu_n^\chi(B) = \mathbb{E}_n^\chi\left(\frac{1}{\#V}\sum_{v\in V}\mathbb{1}_B\left(\Pi_v^{\sigma_n}(\chi)\right)\right)$$

for any Borel set $B \subset X$. By Proposition 2.7.2, it suffices to show that $\mu_n^\chi(B) \to \mu(B)$ as $n \to \infty$ for any clopen set $B \subset X$. Because clopen sets are finite unions of cylinder sets, it suffices to show that if $D \subset \Gamma$ is a finite subset and $\xi \in \{0,1\}^D$ then $\lim_{n\to\infty}\mu_n^\chi([\xi]) = \mu([\xi])$

where $[\xi]$ is the cylinder set $\{x \in X : x \restriction D = \xi\}$. We can further assume $D$ to be a connected finite union of hyperedges, since such objects contain all finite subsets of $\Gamma$.

Let $D$ be a connected finite union of $L$ hyperedges. By properties of $\Gamma$ as a Cayley hyper-tree, $D$ is also a "finite hyper-tree": there exists a sequence $D_1, ..., D_L$ of hyperedges satisfying the following.

1. $\cup_{i=1}^{L} D_i = D$.

2. Let $F_i = \cup_{j=1}^{i} D_j$ for $1 \le i \le L-1$. Let $F_0$ be a fixed singleton in $D_1$. Then there exists a unique $v_i \in D_{i+1} \cap F_i$.

3. $F_i \subset \mathrm{Past}(D_{i+1}, v_i)$.

It follows that if $\xi \in \{0,1\}^D$ is a proper coloring, then

$$\mu([\xi]) = \frac{1}{2} \prod_{i=1}^{L} \mu\left([\xi \restriction D_i] \middle| [\xi \restriction F_{i-1}]\right) = \frac{1}{2}(2^{k-1} - 1)^{-L}.$$

Similarly, along with linearity of expectation, Lemma 2.7.3 implies that

$$\mu_n^{\chi}([\xi]) = (1 + o_n(1))\frac{1}{2}(2^{k-1} - 1)^{-L}.$$

□

### 2.7.2 The density of the rigid set

This subsection proves Lemma 2.6.1. So we assume the hypotheses of Proposition 2.5.9. An element $x \in X$ is a 2-coloring of the Cayley hyper-tree of $\Gamma$. Interpreted as such, $C_l(x), A_l(x), A'_l(x)$ are well-defined subsets of $\Gamma$ (see §2.6 to recall the definitions).

For $l \in \mathbb{N} \cup \{\infty\}$, let

$$\begin{aligned} \tilde{C}_l &= \{x \in X : 1_\Gamma \in C_l(x)\}, \\ \tilde{A}_l &= \{x \in X : 1_\Gamma \in A_l(x)\}, \\ \tilde{A}'_l &= \{x \in X : 1_\Gamma \in A'_l(x)\}. \end{aligned}$$

Recall that $\lambda_0 = \frac{1}{2^{k-1}-1}$ and $\lambda = d\lambda_0$. Since we assume the hypothesis of Prop. 2.5.9, $\lambda$ is asymptotic to $\log(2)k$ as $k \to \infty$.

**Proposition 2.7.4.**

$$\begin{aligned}\mu(\tilde{C}_\infty) &\geq 1 - \lambda^2 e^{-\lambda} + O(k^6 2^{-2k}),\\ \mu(\tilde{C}_\infty \cup \tilde{A}_\infty) &\geq 1 - e^{-\lambda} + O(k^4 2^{-2k}).\end{aligned}$$

*Proof.* For brevity, let $e_i \subset \Gamma$ be the subgroup generated by $s_i$. So $e_i$ is a hyper-edge of the Cayley hyper-tree. Let $F_l^i \subset X$ be the set of all $x$ such that

1. $1_\Gamma$ supports the edge $e_i$ with respect to $x$ and

2. $e_i \setminus \{1_\Gamma\} \subset C_l(x)$.

Since $C_{l+1}(x) \subset C_l(x)$, it follows that $F_{l+1}^i \subset F_l^i$. The events $F_l^i$ for $i = 1, \ldots, d$ are i.i.d. Let $p_l = \mu(F_l^i)$ be their common probability.

We write $\mathrm{Prob}(\mathrm{Bin}(n,p) = m) = \binom{n}{m} p^m (1-p)^{n-m}$ for the probability that a binomial random variable with $n$ trials and success probability $p$ equals $m$. Since the events $F_{l-1}^1, \ldots, F_{l-1}^d$ are i.i.d., $\tilde{A}_{l-1}$ is the event that either 1 or 2 of these events occur and $\tilde{C}_l$ is the event that at least 3 of these events occur, it follows that

$$\mu(\tilde{A}_{l-1}) = \mathrm{Prob}(\mathrm{Bin}(d, p_{l-1}) \in \{1,2\}).$$

$$\mu(\tilde{C}_l) = \mathrm{Prob}(\mathrm{Bin}(d, p_{l-1}) \geq 3).$$

Because the sets $\tilde{A}_{l-1}, \tilde{C}_{l-1}$ are disjoint, so too are the sets $\tilde{A}_{l-1}$ and $\tilde{C}_l$ which implies

$$\mu(\tilde{C}_l \cup \tilde{A}_{l-1}) = \mathrm{Prob}(\mathrm{Bin}(d, p_{l-1}) > 0).$$

**Claim 2.** *$p_0 = \lambda_0$ and for $l \geq 0$, $p_{l+1} = f(p_l)$ where*

$$f(t) = \lambda_0 \, \mathrm{Prob}(\mathrm{Bin}(d-1, t) \geq 3)^{k-1}.$$

*Proof.* To reduce notational clutter, let $F_l = F_l^1$. Note that $p_0 = \mu(F_0) = \lambda_0$ is the probability that the edge $e_1$ is critical. So

$$p_{l+1} = \mu(F_0)\mu(F_{l+1} \mid F_0) = \lambda_0 \mu(F_{l+1} \mid F_0).$$

Conditioned on $F_0$, $F_{l+1}$ is the event that $e_1 \setminus \{1_\Gamma\} \in C_{l+1}(x)$. By symmetry and the Markov property $\mu(F_{l+1} \mid F_0)$ is the $(k-1)$-st power of the probability that $s_1 \in C_{l+1}(x)$ given that $1_\Gamma$ supports $e_1$. By translation invariance, that probability is the same as the probability that $1_\Gamma \in C_{l+1}(x)$ given that $1_\Gamma$ does not support the edge $e_1$. By definition of $C_{l+1}(x)$ and the Markov property, this is the same as the probability that a binomial random variable with $(d-1)$ trials and success probability $p_l$ is at least 3. This implies the claim. □

The next step is to bound $\mathrm{Prob}(\mathrm{Bin}(d-1,t) \geq 3)$ from below:

$$\begin{aligned}
\mathrm{Prob}(\mathrm{Bin}(d-1,t) \geq 3) &= 1 - (1-t)^{d-1} - (d-1)t(1-t)^{d-2} - \binom{d-1}{2} t^2 (1-t)^{d-3} \\
&\geq 1 - e^{-(d-1)t}\left(1 + \frac{(d-1)t}{1-t} + \frac{(d-1)^2 t^2}{2(1-t)^2}\right). \qquad (2.42)
\end{aligned}$$

The last inequality follows from the fact that $(1-t)^{d-1} \leq e^{-(d-1)t}$. This motivates the next claim:

**Claim 3.** *Suppose $t$ is a number satisfying $\lambda_0 \left(1 - \lambda^2 e^{1-\lambda}\right)^{k-1} \leq t \leq \lambda_0$. Then for all sufficiently large $k$,*

$$\begin{aligned}
0 \leq \lambda - (d-1)t &\leq 1 \\
1 + \frac{(d-1)t}{1-t} + \frac{(d-1)^2 t^2}{2(1-t)^2} &\leq (d-1)^2 t^2.
\end{aligned}$$

*Proof.* The first inequality follows from:

$$\lambda - (d-1)t \geq \lambda - (d-1)\lambda_0 = \lambda_0 > 0.$$

The second inequality follows from:

$$\begin{aligned}\lambda-(d-1)t &\le \lambda-(d-1)\lambda_0\left(1-\lambda^2e^{1-\lambda}\right)^{k-1}\\ &= d\lambda_0-(d-1)\lambda_0\left(1-\lambda^2e^{1-\lambda}\right)^{k-1}\\ &\le d\lambda_0-(d-1)\lambda_0(1-(k-1)\lambda^2e^{1-\lambda})\\ &\le \lambda_0+(d-1)\lambda_0(k-1)\lambda^2e^{1-\lambda}\le\lambda_0+k\lambda^3e^{1-\lambda}\to_{k\to\infty}0.\end{aligned}$$

The third line follows from the general inequality $(1-x)^{k-1}\ge 1-(k-1)x$ valid for all $x\in[0,1]$. To see the limit, observe that under the hypotheses of Proposition 2.5.9, $d\sim(\log(2)/2)k2^k$. So $\lambda\sim\log(2)k$. In particular, $k\lambda^3e^{1-\lambda}\to 0$ and $\lambda_0\to 0$ as $k\to\infty$. The implies the limit. Thus if $k$ is large enough then the second inequality holds.

To see the last inequality, observe that since $t\le\lambda_0$, $t\to 0$ as $k\to\infty$. On the other hand, $(d-1)t\sim\lambda\sim\log(2)k$. Thus $\frac{(d-1)t}{1-t}$ and $(d-1)t$ are asymptotic to $\log(2)k$. Since $1+\log(2)k+\frac{\log(2)^2k^2}{2}\le\log(2)^2k^2$ for all sufficiently large $k$, this proves the last inequality assuming $k$ is sufficiently large. $\square$

Now suppose that $t$ is as in Claim 3. Then

$$\begin{aligned}f(t) &\ge \lambda_0\left(1-e^{-(d-1)t}\left(1+\frac{(d-1)t}{1-t}+\frac{(d-1)^2t^2}{2(1-t)^2}\right)\right)^{k-1}\\ &\ge \lambda_0\left(1-e^{1-\lambda}(d-1)^2t^2\right)^{k-1}\ge\lambda_0\left(1-\lambda^2e^{1-\lambda}\right)^{k-1}.\end{aligned}$$

The first inequality is implied by (2.42). The second and third inequalities follow from Claim 3. For example, since $\lambda-(d-1)t\le 1$, $e^{-(d-1)t}\le e^{1-\lambda}$.

Therefore, if $p_l$ satisfies the bounds $\lambda_0\left(1-\lambda^2e^{1-\lambda}\right)^{k-1}\le p_l\le\lambda_0$ then $f(p_l)=p_{l+1}$ satisfies the same bounds. Since $p_\infty=\lim_{l\to\infty}f^l(\lambda_0)$, it follows that

$$\lambda_0\ge p_\infty\ge\lambda_0\left(1-\lambda^2e^{1-\lambda}\right)^{k-1}=\lambda_0+O(k^32^{-2k}).$$

Because $(1-\frac{t}{n})^n \le e^{-t}$ for any $t, n > 0$,

$$\begin{aligned} \mu(\tilde{C}_\infty \cup \tilde{A}_\infty) &= \lim_{l\to\infty} \mu(\tilde{C}_l \cup \tilde{A}_{l-1}) = \lim_{l\to\infty} \mathrm{Prob}(\mathrm{Bin}(d, p_l) > 0) = \mathrm{Prob}(\mathrm{Bin}(d, p_\infty) > 0) \\ &= 1-(1-p_\infty)^d \ge 1-\exp(-p_\infty d) = 1-e^{-\lambda} + O(k^4 2^{-2k}). \end{aligned}$$

By (2.42) and Claim 3 (with $d$ in place of $d-1$),

$$\begin{aligned} \mu(\tilde{C}_\infty) &= \mathrm{Prob}(\mathrm{Bin}(d, p_\infty) \ge 3) \ge \mathrm{Prob}(\mathrm{Bin}(d, \lambda_0 + O(k^3 2^{-2k})) \ge 3) \\ &\ge 1-\exp(-\lambda_0 d)\left(1 + \frac{d\lambda_0}{1-\lambda_0} + \frac{d^2\lambda_0^2}{2(1-\lambda_0)^2}\right) + O(k^6 2^{-2k}) \\ &\ge 1-\lambda^2 e^{-\lambda} + O(k^6 2^{-2k}). \end{aligned}$$

□

**Lemma 2.7.5.** *$\mu(\tilde{A}'_\infty) = o(e^{-\lambda})$ where the implied limit is as $k \to \infty$ and $\eta$ is bounded.*

*Proof.* As in the previous proof, let $e_i \subset \Gamma$ be the subgroup generated by $s_i$. So $e_i$ is a hyper-edge of the Cayley hyper-tree.

Let $x \in X$. We say that an edge $e$ is **attaching** (for $x$) if it is supported by a vertex $v \in A_\infty(x)$ and $e \setminus \{v\} \subset C_\infty(x)$. Let $F(x) = 0$ if $1_\Gamma \notin C_\infty(x)$. Otherwise, let $F(x)$ be the number of attaching edges containing $1_\Gamma$. Then by translation invariance,

$$\mu(\tilde{A}'_\infty) \le \sum_{m=2}^{d} m\mu(F(x) = m). \tag{2.43}$$

Let $G \subset X$ be the set of all $x$ such that

1. $e_1$ is a critical edge supported by some vertex $v \ne 1_\Gamma$,
2. $e_1 \setminus \{v, 1_\Gamma\} \subset C_\infty(x)$,
3. $v \in A_\infty(x)$.

By the Markov property and symmetry,

$$\mu(F(x) = m) \le \binom{d}{m} \mu(G)^m (1-\mu(G))^{d-m}. \tag{2.44}$$

Let

- $G_1 \subset X$ be the set of all $x$ such that $e_1$ is supported by $s_1$,
- $G_2 \subset X$ be the set of all $x$ such that $e_1 \setminus \{s_1, 1_\Gamma\} \subset C_\infty(x)$,
- $G_3 \subset X$ be the set of all $x$ such that $s_1 \in A_\infty(x)$.

By symmetry
$$\mu(G) = (k-1)\mu(G_3 | G_2 \cap G_1)\mu(G_2|G_1)\mu(G_1).$$

Conditioned on $G_1 \cap G_2$, if $G_3$ occurs then there are no more than 2 attaching edges $e$ supported by $s_1$ with $e \neq e_1$. By the Markov property and symmetry,

$$\mu(G_3 | G_2 \cap G_1) \leq \mathrm{Prob}(\mathrm{Bin}(d-1, p_\infty) \leq 2) = O(\lambda^2 e^{-\lambda}).$$

Also $\mu(G_1) = \lambda_0$. Thus $\mu(G) \leq O(k^3 e^{-2\lambda})$. So (2.43) and (2.44) along with straightforward estimates imply $\mu(\tilde{A}'_\infty) = o(e^{-\lambda})$.

□

**Lemma 2.7.6.** $\limsup_{l\to\infty} \mu(\tilde{A}'_l) \leq \mu(\tilde{A}'_\infty)$.

*Proof.* Given a coloring $\chi : \Gamma \to \{0,1\}$ of the Cayley hyper-tree and $l \in \mathbb{N}$, define $A''_l(\chi) = \cup_{m\geq l} A'_m(\chi)$. Also define $\tilde{A}''_l = \{x \in X : \ 1_\Gamma \in A''_l(x)\}$. Since $\tilde{A}''_l \supset \tilde{A}'_l$ and the sets $\tilde{A}''_l$ are decreasing in $l$, it suffices to prove that $\cap_{l\geq 0} \tilde{A}''_l \subset \tilde{A}'_\infty$.

Suppose $x \in \cap_{l\geq 0}\tilde{A}''_l$. Then there exists an infinite set $S \subset \mathbb{N}$ such that $x \in \tilde{A}'_l$ $(\forall l \in S)$. So $1_\Gamma \in A'_l(x)$ $(\forall l \in S)$. So for each $l \in S$, there exist $g_l \in \Gamma \setminus \{1_\Gamma\}$ and hyper-edges $e_l, f_l \subset \Gamma$ such that

1. $1_\Gamma$ supports $e_l$ (with respect to $x$),
2. $g_l$ supports $f_l$ (with respect to $x$),
3. $e_l \cup f_l \setminus \{1_\Gamma, g_l\} \subset C_l(x)$,
4. $e_l \cap f_l \neq \emptyset$.

Because $e_l \cap f_l \neq \emptyset$, $g_l$ is necessarily contained in the finite set $\{s_i^{p_i} s_j^{p_j} : 1 \le i, j \le d, 0 \le p_i \le k\}$. So after passing to an infinite subset of $S$ if necessary, we may assume there is a fixed element $g \in \Gamma$ such that $g = g_l$ $(\forall l \in S)$. Similarly, we may assume there are edges $e, f \subset \Gamma$ such that $e_l = e$ and $f_l = f$ $(\forall l \in S)$.

Observe that $1_\Gamma \notin C_\infty(x)$ because $1_\Gamma \in A_l(x)$ implies $1_\Gamma \notin C_l(x)$ $(\forall l \in S)$. Similarly, $g \notin C_\infty(x)$. Because $e_l \cup f_l \setminus \{1_\Gamma, g_l\} \subset C_l(x)$ $(\forall l \in S)$ and the sets $C_l(x)$ are decreasing in $l$, it follows that $e \cup f \setminus \{1_\Gamma, g\} \subset C_\infty(x)$. Therefore $\{1_\Gamma, g\} \subset A_\infty(x)$. This verifies all of the conditions showing that $1_\Gamma \in A'_\infty(x)$ and therefore $x \in \tilde{A}'_\infty$ as required.

□

We can now prove Lemma 2.6.1:

*Proof of Lemma 2.6.1.* Observe that the sets $\tilde{C}_l, \tilde{A}_l, \tilde{A}'_l$ are clopen for finite $l$. By Lemma 2.7.1,

$$\lim_{\delta \searrow 0} \liminf_{n \to \infty} \mathbb{P}_n^\chi \left( \left| \frac{|C_l(\chi) \cup A_l(\chi) \setminus A'_l(\chi)|}{n} - \mu\left(\tilde{C}_l \cup \tilde{A}_l \setminus \tilde{A}'_l\right) \right| < \delta \right) = 1. \tag{2.45}$$

for any finite $l$. Since $\tilde{A}_\infty \cup \tilde{C}_\infty$ is the decreasing limit of $\tilde{A}_l \cup \tilde{C}_l$, Lemma 2.7.6 implies

$$\liminf_{l \to \infty} \mu(\tilde{C}_l \cup \tilde{A}_l \setminus \tilde{A}'_l) \ge \mu(\tilde{C}_\infty \cup \tilde{A}_\infty \setminus \tilde{A}'_\infty).$$

By Proposition 2.7.4 and Lemma 2.7.5,

$$\mu(\tilde{C}_\infty \cup \tilde{A}_\infty \setminus \tilde{A}'_\infty) \ge 1 - e^{-\lambda} + O(k^2 e^{-2\lambda}) = 1 - e^{-\lambda} + o(e^{-\lambda}).$$

Together with (2.45), this implies the lemma. □

## 2.8 Rigid vertices

This section proves Lemma 2.6.2. So we assume the hypotheses of Proposition 2.5.9.

As in the previous section, fix an equitable coloring $\chi : V \to \{0, 1\}$. We assume $|V| = n$ and let $\sigma : \Gamma \to \mathrm{Sym}(V)$ be a uniformly random uniform homomorphism conditioned on the event that $\chi$ is proper with respect to $\sigma$.

**Lemma 2.8.1** (Expansivity Lemma). *There is a constant $k_0 > 0$ such that the following holds. If $k \geq k_0$ then with high probability (with respect to the planted model), as $n \to \infty$, for any $T \subset V$ with $|T| \leq 2^{-k/2}n$ the following is true. For a vertex $v$ let $E_v$ denote the set of hyperedges supported by $v$. Let $E_T$ be the set of all edges $e \in \cup_{v \in T} E_v$ such that $|e \cap T| \geq 2$. Then*

$$\#E_T \leq 2\#T.$$

*Proof.*

**Claim 4.** *There exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies*

- $k/2 \leq \lambda \leq k$,
- $2^{-k/2} \leq 1/(8k)$,
- *and for any $0 < t \leq 2^{-k/2}$ and $k/2 \leq \lambda' \leq k$*

$$H(t, 1-t) + \lambda' H(2t/\lambda', 1 - 2t/\lambda') + 2t\log(4k) + 4t\log(t) \leq t\log(t)/2.$$

*Proof.* Recall that $\lambda = \log(2)k + O(k2^{-k})$. So the first two requirements are immediate for $k_0$ large enough.

We estimate each of the first three terms on the left as follows. Because $1 = \lim_{t \searrow 0} \frac{H(t,1-t)}{-t\log(t)}$, there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $\frac{H(t,1-t)}{-t\log(t)} \leq 1.1$.

Note,

$$\begin{aligned}
\lambda' H(2t/\lambda', 1 - 2t/\lambda') &= -2t\log(2t/\lambda') - (\lambda' - 2t)\log(1 - 2t/\lambda') \\
&= -2t\log(2t/\lambda') + O(t) \leq -2t\log(t) + 2t\log(\lambda') + O(t) \\
&\leq -2t\log(t) + 2t\log(k) + O(t).
\end{aligned}$$

So by making $k_0$ larger if necessary, we may assume

$$\frac{\lambda' H(2t/\lambda', 1 - 2t/\lambda')}{-t\log(t)} \leq 2.1.$$

Since

$$\frac{2t\log(4k)}{-t\log(t)} \leq \frac{2\log(4k)}{(k/2)\log(2)}$$

we may also assume $\frac{2t\log(4k)}{-t\log(t)} \leq 0.1$. Combining these inequalities, we obtain

$$H(t,1-t)+\lambda' H(2t/\lambda',1-2t/\lambda')+2t\log(4k)+4t\log(t) \leq (1.1+2.1+0.1-4)(-t\log(t)) \leq t\log(t)/2.$$

$\square$

From now on, we assume $k \geq k_0$ with $k_0$ as above. To simplify notation, let $\zeta = 2^{-k/2}$.

For $1 \leq l \leq n$, let $\mathcal{T}_l$ be the collection of all subsets $T \subset V = [n]$ such that $|T| = l$ and $|E_T| > 2|T|$. To prove the lemma, by a first moment argument, it suffices to show that the expected value of $|\mathcal{T}_l|$ tends to zero exponentially in $n$ (with respect to the planted model).

Given a $d$-tuple $c = (c_1, \ldots, c_d)$ of natural numbers, let $E_c$ be the event that there are exactly $c_i$ critical edges of the form $\{\sigma(s_i)^j(v) : 0 \leq j \leq k-1\}$. We denote $|c| = \max_{1\leq i\leq d} c_i$. Let $\mathbb{P}^{\chi}_{c,n}$ be the planted model conditioned on $E_c$.

**Claim 5.** *If $n$ is sufficiently large (depending only on $k,d$), $l \leq \zeta n$ and $kn/2 \leq |c| \leq kn$ then*

$$\mathbb{E}^{\chi}_{c,n}[\#\mathcal{T}_l] \leq \zeta^{\zeta n/2}.$$

Before proving this claim, we show how it implies the lemma. By Lemma 2.7.1, with high probability, the total number of critical edges is asymptotic to $\lambda n$ as $n \to \infty$. Let $E'_n$ be the event that the number of critical edges is between $(k/2)n$ and $kn$. So $\mathbb{P}^{\chi}_n(E'_n) \to 1$ as $n \to \infty$.

By Claim 5,

$$\sum_{1\leq l\leq \zeta n} \mathbb{E}^{\chi}_n[\#\mathcal{T}_l | E'_n] \leq \sum_{1\leq l\leq \zeta n} \sum_{kn/2\leq |c|\leq kn} \mathbb{E}^{\chi}_{c,n}[\#\mathcal{T}_l] \leq (kn)^d \zeta^{\zeta n/2}.$$

Since this decays exponentially in $n$, it implies the lemma.

To prove the claim, we first need to introduce the planted model conditioned on $E_c$ which we denote by $\mathbb{P}^{\chi}_{c,n}$. This measure can be constructed as follows. Let $I_c$ be the set of pairs $(i,j)$ with $1 \leq i \leq d$ and $1 \leq j \leq c_i$. First choose edges $\{e_{i,j}\}_{(i,j)\in I_c}$ uniformly at random subject to the conditions:

1. each $e_{i,j} \subset [n]$ has cardinality $k$ and $e_{i,j} \cap e_{i,j'} = \emptyset$ for $j \neq j'$,

2. each $e_{i,j}$ is critical with respect to $\chi$.

Next choose a uniformly random uniform homomorphism $\sigma$ subject to:

1. $\chi$ is a proper coloring with respect to $\sigma$,

2. each $e_{i,j}$ is of the form $\{\sigma(s_i)^j(v) : 0 \leq j \leq k-1\}$ with respect to $\sigma$,

3. the edges $\{e_{i,j}\}_{(i,j)\in I_c}$ are precisely the critical edges of $\chi$ with respect to $\sigma$.

Then $\sigma$ is distributed according to $\mathbb{P}^{\chi}_{c,n}$.

For $s \in I_c$ and $T \subset V$, let $F_{T,s}$ be the event that $e_s$ is supported by a vertex in $T$ and $|e_s \cap T| \geq 2$. For $S \subset I_c$, let $F_{T,S} = \cap_{s\in S} F_{T,S}$.

Before proving the claim above, we need to prove:

**Claim 6.** *For any $T \subset V$ with cardinality $|T| = l$, $S \subset I_c$ with $|S| \leq 2l-1$ and $s_0 \in I_c \setminus S$, one has $\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S}) \leq \frac{4kl^2}{n^2}$.*

*Proof of Claim 6.* For $s \in S$, let $e_s$ be a random edge in $[n]$ with cardinality $k$ satisfying the conditions above. Let $s_0 = (i_0, j_0)$. Let $V_0$ be the union of all edges of the form $\{\sigma(s_{i_0})^j(v) : 0 \leq j \leq k-1\}$ in $\{e_s\}_{s\in S}$. That is, $v \in V_0$ if and only if there is $s \in S$ with $e_s \ni v$ and $s = (i_0, j)$ for some $j$.

We will condition on $F_{T,S}$ and $\{e_s\}_{s\in S}$. For $i = 0, 1$, let

- $n_i$ be the number of vertices $v \in [n] \setminus V_0$ such that $\chi(v) = i$;

- $l_i$ be the number of vertices $v \in T \setminus V_0$ such that $\chi(v) = i$.

The probability that $e_{s_0}$ is supported by a vertex in $T$ given $\{e_s\}_{s\in S}$ and $F_{T,S}$ is the same as the probability that a randomly chosen vertex in $[n] \setminus V_0$ lies in $T$:

$$\frac{l_0 + l_1}{n_0 + n_1}.$$

Given that $e_{s_0}$ is supported by a vertex in $T$ (and given $\{e_s\}_{s\in S}$ and $F_{T,S}$) the probability that $|e_{s_0}\cap T| = 1$ is

$$\frac{l_0}{l_0+l_1}\frac{\binom{n_1-l_1}{k-1}}{\binom{n_1}{k-1}} + \frac{l_1}{l_0+l_1}\frac{\binom{n_0-l_0}{k-1}}{\binom{n_0}{k-1}}.$$

The reason for this expression is that $\frac{l_0}{l_0+l_1}$ is the probability that the vertex $v$ that supports $e_{s_0}$ has $\chi(v) = 0$. Conditioned on this event, there are $k-1$ non-supporting vertices of $e_{s_0}$ that must be chosen amongst the $n_1$ vertices in $[n]\setminus V_0$. The probability that these vertices are all chosen in the complement of $T$ is $\frac{\binom{n_1-l_1}{k-1}}{\binom{n_1}{k-1}}$. This explains the first summand; the second is justified similarly.

It follows that

$$\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S},\{e_s\}_{s\in S}) = \frac{l_0+l_1}{n_0+n_1}\left[1-\frac{l_0}{l_0+l_1}\frac{\binom{n_1-l_1}{k-1}}{\binom{n_1}{k-1}} - \frac{l_1}{l_0+l_1}\frac{\binom{n_0-l_0}{k-1}}{\binom{n_0}{k-1}}\right].$$

In order to bound this expression, consider

$$\begin{aligned}\frac{\binom{n_1-l_1}{k-1}}{\binom{n_1}{k-1}} &= \left(\frac{n_1-l_1}{n_1}\right)\cdots\left(\frac{n_1-l_1-k+2}{n_1-k+2}\right)\\ &\geq \left(\frac{n_1-l_1-k+2}{n_1-k+2}\right)^{k-1} = \left(1-\frac{l_1}{n_1-k+2}\right)^{k-1}\\ &\geq 1-\frac{(k-1)l_1}{n_1-k+2} \geq 1-\frac{kl_1}{n_1}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S},\{e_s\}_{s\in S}) &\leq \frac{l_0+l_1}{n_0+n_1}\left[1-\frac{l_0}{l_0+l_1}\left(1-\frac{kl_1}{n_1}\right)-\frac{l_1}{l_0+l_1}\left(1-\frac{kl_0}{n_0}\right)\right]\\ &= \frac{kl_0l_1}{n_0n_1}.\end{aligned}$$

Because $l_0+l_1 \leq |T| = l$, the product satisfies $l_0l_1 \leq l^2/4$. Note $n_0 \geq n/2-2kl \geq n/4$. Similarly, $n_1 \geq n/4$. Substitute these inequalities above to obtain,

$$\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S},\{e_s\}_{s\in S}) \leq \frac{4kl^2}{n^2}.$$

Since $\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S})$ is a convex sum of such expressions, it follows that $\mathbb{P}^{\chi}_{c,n}(F_{T,s_0}|F_{T,S}) \leq \frac{4kl^2}{n^2}$. This proves the claim. $\square$

Apply the chain rule and Claim 6 to obtain: if $S \subset I_c$ has $|S| = 2l$ then

$$\mathbb{P}^{\chi}_{c,n}(F_{T,S}) \leq \left(\frac{4kl^2}{n^2}\right)^{2l}.$$

Note

$$\mathbb{E}^{\chi}_{c,n}[\#\mathfrak{T}_l] \leq \sum_{S,T} \mathbb{P}^{\chi}_{c,n}(F_{T,S}). \tag{2.46}$$

where the sum is over all $T \subset [n]$ and $S \subset I_c$ with $|T| = l$ and $|S| = 2l$. Therefore,

$$\mathbb{E}^{\chi}_{c,n}[\#\mathfrak{T}_l] \quad \leq \quad \binom{n}{l}\binom{|c|}{2l}\left(\frac{4kl^2}{n^2}\right)^{2l}.$$

Define $t, \lambda'$ by $tn = l$ and $|c| = \lambda' n$. By hypothesis $k/2 \leq \lambda' \leq k$. We make the following estimates:

$$\begin{aligned}\binom{n}{l} &= \exp(nH(t, 1-t) + o(n)),\\ \binom{|c|}{2l} &= \exp(\lambda' n H(2t/\lambda', 1 - 2t/\lambda') + o(n))\end{aligned}$$

so that

$$\mathbb{E}^{\chi}_{c,n}[\#\mathfrak{T}_l] \leq \exp(n(H(t, 1-t) + \lambda' H(2t/\lambda', 1-2t/\lambda') + 2t\log(4k) + 4t\log(t)) + o(n)).$$

For $n$ sufficiently large and $t \leq 2^{-k/2}$, this is bounded above by $\exp(nt\log(t)/2) = t^{tn/2}$ by the choice of $k_0$.

This proves Claim 5 and finishes the lemma.

$\square$

**Lemma 2.8.2.** *Let $\rho > 0$. Then there exists $L$ such that $l > L$ implies $C_l(\chi) \subset [n]$ is $\rho$-rigid (with high probability in the planted model as $n \to \infty$).*

*Proof.* Without loss of generality, we may assume that $0 < \rho < \mu(\tilde{C}_\infty)$.

Observe that the sets $\tilde{C}_l$ are clopen for finite $l$. By Lemma 2.7.1,

$$\lim_{\eta \searrow 0} \liminf_{n\to\infty} \mathbb{P}_n^\chi \left( \left| \frac{|C_l(\chi)|}{n} - \mu\left(\tilde{C}_l\right) \right| < \eta \right) = 1.$$

Since the sets $C_l(\chi)$ are decreasing with $l$, this implies the existence of $L$ such that $l > L$ implies

$$\liminf_{n\to\infty} \mathbb{P}_n^\chi \left( \left| \frac{|C_l(\chi)|}{n} - \frac{|C_{l+1}(\chi)|}{n} \right| < \rho/3 \right) = 1.$$

Choose $l > L$. Let $\psi : V \to \{0, 1\}$ be a $\sigma$-proper coloring. Let

$$T_l = \{v \in C_l(\chi) : \ \chi(v) \neq \psi(v)\}.$$

Define $T_{l+1}$ similarly. Since $|C_l(\chi) \setminus C_{l+1}(\chi)| < \rho n/3$ (with high probability) and $T_l \setminus T_{l+1} \subset C_l(\chi) \setminus C_{l+1}(\chi)$, it follows that $|T_l \setminus T_{l+1}| < \rho n/3$ (with high probability).

For every $v \in T_{l+1}$, let $F_v \subset E_v$ be the subset of $\chi$-critical edges $e$ such that $e \subset C_l(\chi)$.

We claim that if $v \in T_{l+1}$ then $F_v \subset E_{T_l}$ where

$$E_{T_l} = \{e \in \cup_{v\in T_l} E_v : \ |e \cap T_l| \geq 2\}.$$

Because $\psi : [n] \to \{0, 1\}$ is a proper coloring and $v \in T_{l+1}$, $\psi(v) \neq \chi(v)$. So if $e \in F_v$ then $v$ supports $e$ with respect to $\chi$. Therefore, there must exist a vertex $w \in e \setminus \{v\}$ such that $\chi(w) \neq \chi(v)$. Since $\{v, w\} \subset e \subset C_l(\chi)$, this means that $|e \cap T_l| \geq 2$ and therefore $e \in E_{T_l}$, which proves the claim.

For every $v \in T_{l+1}$, $|F_v| \geq 3$. Since edges can only be supported by one vertex, the sets $F_v$ are pairwise disjoint. So

$$|E_{T_l}| \geq \left| \bigcup_{v\in T_{l+1}} F_v \right| \geq 3|T_{l+1}| \geq 3|T_l| - \rho n.$$

If $|T_l| > \rho n$ then $|E_{T_l}| \geq 3|T_l| - \rho n > 2|T_l|$. So it follows from Lemma 2.8.1 that (with high probability), $|T_l| > 2^{-k/2} n$. Thus $C_l$ is $\rho$-rigid. $\square$

We can now prove Lemma 2.6.2.

*Proof of Lemma 2.6.2.* Let $\rho > 0$. By Lemma 2.8.2, there exists $L$ such that $l > L$ implies $C_l(\chi)$ is $(\rho/2)$-rigid with high probability in the planted model as $n \to \infty$. So without loss of generality we condition on the event that $C_l(\chi)$ is $(\rho/2)$-rigid.

Now let $l > L$. Let $\psi : V \to \{0, 1\}$ be a $\sigma$-proper coloring. Let

$$T = \{v \in C_l(\chi) : \ \chi(v) \neq \psi(v)\}.$$

$$T' = \{v \in A_l(\chi) \setminus A'_l(\chi) : \ \chi(v) \neq \psi(v)\}.$$

We claim that $|T| \geq |T'|$. To see this, let $v \in T'$. Then there exists an edge $e$ supported by $v$ (with respect to $\chi$) with $e \setminus \{v\} \subset C_l(\chi)$. Since $\psi$ is proper and $\psi(v) \neq \chi(v)$, there must exist a vertex $w \in e \setminus \{v\}$ with $\psi(w) \neq \psi(v)$. Necessarily, $w \in T$. So there exists a function $f : T' \to T$ such that $f(v)$ is contained in an edge $e$ supported by $v$ with $e \setminus \{v\} \subset C_l(\chi)$. Because $v \notin A'_l(\chi)$, $f$ is injective. This proves the claim.

Now suppose that $|T \cup T'| > \rho n$. Since $T$ and $T'$ are disjoint, either $|T| > (\rho/2)n$ or $|T'| > (\rho/2)n$. So the claim implies $|T| > (\rho/2)n$. Since $C_l(\chi)$ is $(\rho/2)$-rigid,

$$|T \cup T'| > 2^{-k/2} n.$$

This proves the lemma. $\square$

## 2.9 Appendix A: Topological sofic entropy notions

In this appendix, we recall the notion of topological sofic entropy from [KL13] and prove that it coincides with the definition given in §2.2.

Let $T$ be an action of $\Gamma$ on a compact metrizable space $X$. So for $g \in \Gamma$, $T^g : X \to X$ is a homeomorphism and $T^{gh} = T^g T^h$. We will also denote this action by $\Gamma \curvearrowright X$. Let $\sigma : \Gamma \to \mathrm{Sym}(n)$ be a map, $\rho$ be a pseudo-metric on $X$, $F \Subset \Gamma$ be finite and $\delta > 0$. For

$x, y \in X^n$, let

$$\rho_\infty(x,y) = \max_i \rho(x_i, y_i), \quad \rho_2(x,y) = \left(\frac{1}{n}\sum_i \rho(x_i,y_i)^2\right)^{1/2}$$

be pseudo-metrics on $X^n$. Also let

$$\mathrm{Map}(T,\rho,F,\delta,\sigma) = \{x \in X^n : \forall f \in F,\ \rho_2(T^f x, x \circ \sigma(f)) < \delta\}.$$

Informally, elements of $\mathrm{Map}(T,\rho,F,\delta,\sigma)$ are "good models" that approximate partial periodic orbits with respect to the chosen sofic approximation.

For a pseudo-metric space $(Y,\rho)$, a subset $S \subset Y$ is $(\rho,\epsilon)$**-separated** if for all $s_1 \neq s_2 \in S$, $\rho(s_1,s_2) \geq \epsilon$. Let $N_\epsilon(Y,\rho) = \max\{|S| : S \subset Y, S \text{ is } (\rho,\epsilon)\text{-separated}\}$ be the maximum cardinality over all $(\rho,\epsilon)$-separated subsets of $Y$.

Given a sofic approximation $\Sigma$ to $\Gamma$, we define

$$\tilde{h}_\Sigma(\Gamma \curvearrowright X, \rho) = \sup_{\epsilon>0} \inf_{F \Subset \Gamma} \inf_{\delta>0} \limsup_{i\to\infty} |V_i|^{-1} \log(N_\epsilon(\mathrm{Map}(T,\rho,F,\delta,\sigma_i),\rho_\infty))$$

where the symbol $F \Subset \Gamma$ means that $F$ varies over all finite subsets of $\Gamma$.

We say that a pseudo-metric $\rho$ on $X$ is **generating** if for every $x \neq y$ there exists $g \in \Gamma$ such that $\rho(gx, gy) > 0$. By [KL13, Proposition 2.4], if $\rho$ is continuous and generating, $\tilde{h}_\Sigma(T,\rho)$ is invariant under topological conjugacy and does not depend on the choice of $\rho$. So we define $\tilde{h}_\Sigma(T) = \tilde{h}_\Sigma(T,\rho)$ where $\rho$ is any continuous generating pseudo-metric. The authors of [KL13] define the topological sofic entropy of $\Gamma \curvearrowright X$ to be $\tilde{h}_\Sigma(T)$. The main result of this appendix is:

**Proposition 2.9.1.** *Let $\mathcal{A}$ be a finite set and $X \subset \mathcal{A}^\Gamma$ a closed shift-invariant subspace. Let $T$ be the shift action of $\Gamma$ on $X$. Then $h_\Sigma(\Gamma \curvearrowright X) = \tilde{h}_\Sigma(T)$ where $h_\Sigma(\Gamma \curvearrowright X)$ is as defined in §2.2.*

*Proof.* To begin, we choose a pseudo-metric on $\mathcal{A}^\Gamma$ as follows. For $x, y \in \mathcal{A}^\Gamma$, let $\rho(x,y) = 1_{x_e \neq y_e}$. Then $\rho$ is continuous and generating. So $\tilde{h}_\Sigma(\Gamma \curvearrowright X) = \tilde{h}_\Sigma(\Gamma \curvearrowright X, \rho)$.

Let $\epsilon > 0$, $\mathcal{O} \subset \mathcal{A}^\Gamma$ be an open set. We first analyze $\Omega(\mathcal{O},\epsilon,\sigma)$ from the definition of $h_\Sigma(\Gamma \curvearrowright X)$. Note that the topology for $\mathcal{A}^\Gamma$ is generated by the base $\mathscr{B} = \{[a] : a \in$

$\mathcal{A}^F, F \Subset \Gamma\}$ where if $a \in \mathcal{A}^F$ then $[a] = \{x \in \mathcal{A}^\Gamma : x|_F = a\}$. In other words, open sets of $\mathscr{B}$ are those that specify a configuration on a finite subset of coordinates. For $F \Subset \Gamma$ let $\mathcal{O}(F) = \{y \in \mathcal{A}^\Gamma : \exists x \in X, y|_F = x|_F\} = \cup_{\substack{a \in \mathcal{A}^F \\ [a] \cap X \neq \emptyset}} [a]$ be the open set containing all elements containing some configuration that appears in $X$ in the finite window $F$.

**Claim 7.** *Every open superset $\mathcal{O} \supset X$ contains some open set of the form $\mathcal{O}(F)$.*

Because $\Omega(\mathcal{O}, \epsilon, \sigma)$ decreases as $\mathcal{O}$ decreases, it suffices to only consider open sets of the form $\mathcal{O}(F)$ in the definition of $h_\Sigma(\Gamma \curvearrowright X)$.

*Proof.* $\mathcal{O}$ is a union of elements in $\mathscr{B}$ and $X$ is compact, so that there exists $X \subset \mathcal{O}' \subset \mathcal{O}$ with $\mathcal{O}'$ containing only finitely many base elements. Let $F$ be the union of all coordinates specified by base elements in $\mathcal{O}'$. It follows that $\mathcal{O}'$ contains $\mathcal{O}(F)$. □

Without loss of generality and for convenience we can assume that $F$ is symmetric, i.e. $F = F^{-1}$, and contains the identity. This is because we can replace any $F$ with the larger set $F \cup F^{-1} \cup \{e\}$, and both $\mathrm{Map}(T, \rho, F, \delta, \sigma_i)$ and $\Omega(\mathcal{O}(F), \epsilon, \sigma_i)$ are monotone decreasing in $F$.

Let $n = |V_i|$. We assume $\lim_{i\to\infty} |V_i| = \infty$. Now for each $x \in \Omega(\mathcal{O}(F), \epsilon, \sigma_i)$ we obtain an element $\tilde{x} \in X^n$ and then show that these partial orbits form a good estimate for $\tilde{h}_\Sigma$. Let $G(x) = \{v \in V_i : \Pi_v^{\sigma_i}(x) \in \mathcal{O}(F)\}$. For every $v \in G(x)$, choose some $\tilde{x}_v \in X$ that agrees with $\Pi_v^{\sigma_i}(x)$ on $F$. For $v \notin G(x)$ choose an arbitrary element $\tilde{x}_v \in X$. Thus $\tilde{x} \in X^n$ .

Now for $v \in G(x), f \in F$, $T^f \tilde{x}_v(e) = \tilde{x}_v(f^{-1}) = x_{\sigma_i(f)v}$. On the other hand we also want $\tilde{x}_{\sigma_i(f)v}(e) = x_{\sigma_i(f)v}$, which is true if $v \in \sigma_i(f)^{-1}G(x)$ and $\sigma_i(e)\sigma_i(f)v = \sigma_i(f)v$. It follows that $\rho_2(T^f \tilde{x}, \tilde{x} \circ \sigma_i(f)) < \sqrt{2\epsilon}$.

Now consider separation of $\{\tilde{x} : x \in \Omega(\mathcal{O}(F), \epsilon, \sigma_i)\}$. We will show that a slightly smaller subset is $(\rho_\infty, 1)$-separated. By the pigeonhole principle there exists a subset $\bar{V}_i$ of size at least $(1-\epsilon)n$ such that $\Omega(\mathcal{O}(F), \epsilon, \sigma_i, \bar{V}_i) := \{x \in \Omega(\mathcal{O}(F), \epsilon, \sigma_i) : G(x) = \bar{V}_i\}$ has cardinality at least $e^{-n(H(\epsilon, 1-\epsilon)+o(1))} \# \Omega(\mathcal{O}(F), \epsilon, \sigma_i)$. Furthermore, if $x, y \in \Omega(\mathcal{O}(F), \epsilon, \sigma_i, \bar{V}_i)$ then $\rho_\infty(\tilde{x}, \tilde{y}) = 1$ if $x(v) \neq y(v)$ for some $v \in \bar{V}_i \cap \mathrm{Fix}(1_\Gamma)$, where $\mathrm{Fix}(1_\Gamma) = \{v \in V_i : \sigma_i(1_\Gamma)v = v\}$. Since there are at most $|\mathcal{A}|^{(\epsilon + o(1))n}$ configurations in $\mathcal{A}^{V_i}$ with some fixed configuration on $\bar{V}_i \cap \mathrm{Fix}(1_\Gamma)$, there exists a $(\rho_\infty, 1)$-separated subset of $\Omega(\mathcal{O}(F), \epsilon, \sigma_i, \bar{V}_i)$ of size at least $|\mathcal{A}|^{-(\epsilon+o(1))n}\#\Omega(\mathcal{O}(F)), \epsilon, \sigma_i, \bar{V}_i)$. It follows that

$$N_1(\mathrm{Map}(T, \rho, F, \sqrt{2\epsilon}, \sigma_i), \rho_\infty) \geq |\mathcal{A}|^{-(\epsilon+o(1))n} e^{-n(H(\epsilon, 1-\epsilon)+o(1))}\#\Omega(\mathcal{O}(F), \epsilon, \sigma_i).$$

On the other hand, suppose we have some $\tilde{x} \in \mathrm{Map}(T, \rho, F, \delta, \sigma_i)$. This means that for every $f \in F$, there exists a set $\tilde{V}_i(f)$ of size $> (1-\delta^2)n$ such that for $v \in \tilde{V}_i(f)$, $\tilde{x}_{\sigma_i(f)v}(1_\Gamma) = T^f\tilde{x}_v(1_\Gamma) = \tilde{x}_v(f^{-1})$. Let $\tilde{V}_i = \cap_{f\in F}\tilde{V}_i(f)$. Then $|\tilde{V}_i| > (1-|F|\delta^2)n$ and for $v \in \tilde{V}_i$, for every $f \in F$, $\tilde{x}_{\sigma_i(f)v}(1_\Gamma) = T^f\tilde{x}_v(1_\Gamma) = \tilde{x}_v(f^{-1})$.

Define $x \in \mathcal{A}^{V_i}$ by $x_v = \tilde{x}_v(1_\Gamma)$. Then for any fixed $v \in \tilde{V}_i$, for every $f \in F$, $\Pi_v^{\sigma_i}(x)(f) = x_{\sigma_i(f^{-1})v} = \tilde{x}_{\sigma_i(f^{-1})v}(1_\Gamma) = T^{f^{-1}}\tilde{x}_v(1_\Gamma) = \tilde{x}_v(f)$. Since $\tilde{x}_v \in X$, it follows that $x \in \Omega(\mathcal{O}(F), \delta^2|F|, \sigma_i)$.

Also note that $\tilde{x}, \tilde{y} \in \mathrm{Map}(T, \rho, F, \delta, \sigma_i)$ are $(\rho, \epsilon)$-separated for any $\epsilon \leq 1$ if and only if $\tilde{x}_v(1_\Gamma) \neq \tilde{y}_v(1_\Gamma)$ for some $v \in V_i$, so that $x \neq y$. It follows that

$$N_\epsilon(\mathrm{Map}(T, \rho, F, \delta, \sigma_i), \rho_\infty) \leq \#\Omega(\mathcal{O}(F), \delta^2|F|, \sigma_i).$$

Note that in the definitions of $h_\Sigma$ and $\tilde{h}_\Sigma$, $F$ is fixed with respect to $\delta$.

□

## 2.10 Appendix B: Concentration for the planted model

**Definition 8** (Hamming metrics). Define the **normalized Hamming metric** $d_{\mathrm{Sym}(n)}$ on $\mathrm{Sym}(n)$ by

$$d_{\mathrm{Sym}(n)}(\sigma_1, \sigma_2) = n^{-1}\#\{i \in [n] : \ \sigma_1(i) \neq \sigma_2(i)\}.$$

Define the **normalized Hamming metric** $d_{\mathrm{Hom}}$ on $\mathrm{Hom}(\Gamma, \mathrm{Sym}(n))$ by

$$d_{\mathrm{Hom}}(\sigma_1, \sigma_2) = \sum_{i=1}^{d} d_{\mathrm{Sym}(n)}(\sigma_1(s_i), \sigma_2(s_i)).$$

The purpose of this section is to prove:

**Theorem 2.10.1.** *There exist constants $c, \lambda > 0$ (depending only on $k, d$) such that for every $\delta > 0$ there exists $N_\delta$ such that for all $n > N_\delta$, for every 1-Lipschitz $f : \mathrm{Hom}_\chi(\Gamma, \mathrm{Sym}(n)) \to$*

$\mathbb{R}$,

$$\mathbb{P}_n^{\mathcal{X}}\left(|f - \mathbb{E}_n^{\mathcal{X}}[f]| > \delta\right) \leq c \exp(-\lambda\delta^2 n).$$

### 2.10.1 General considerations

To begin the proof we first introduce some general-purpose tools.

**Definition 9.** A **metric measure space** is a triple $(X, d_X, \mu)$ where $(X, d_X)$ is a metric space and $\mu$ is a Borel probability measure on $X$. We will say $(X, d_X, \mu)$ is **$(c, \lambda)$-concentrated** if for any 1-Lipschitz function $f : X \to \mathbb{R}$,

$$\mu\left(\left|f - \int f \, d\mu\right| > \epsilon\right) < ce^{-\lambda\epsilon^2}.$$

If $(X, d_X, \mu)$ is $(c, \lambda)$-concentrated and $f : X \to \mathbb{R}$ is $L$-Lipschitz, then since $f/L$ is 1-Lipschitz

$$\mu\left(\left|f - \int f \, d\mu\right| > \epsilon\right) = \mu\left(\left|f/L - \int f/L \, d\mu\right| > \epsilon/L\right) < c \exp(-\lambda\epsilon^2/L^2). \tag{2.47}$$

**Lemma 2.10.2.** *Let $(X, d_X, \mu)$ be $(c, \lambda)$-concentrated. If $\phi : X \to Y$ is an $L$-Lipschitz map onto a measure metric space $(Y, d_Y, \nu)$ and $\nu = \phi_*\mu$ is the push-forward measure, then $(Y, d_Y, \nu)$ is $(c, \lambda/L^2)$-concentrated.*

*Proof.* This follows from the observation that if $f : Y \to \mathbb{R}$ is 1-Lipschitz, then the pullback $f \circ \phi : X \to \mathbb{R}$ is $L$-Lipschitz. So equation (2.47) implies

$$\nu\left(\left|f - \int f \, d\nu\right| > \epsilon\right) = \mu\left(\left|f \circ \phi - \int f \circ \phi \, d\mu\right| > \epsilon\right) < c \exp(-\lambda\epsilon^2/L^2).$$

$\square$

The next lemma is concerned with the following situation. Suppose $X = \sqcup_{i \in I} X_i$ is a finite disjoint union of spaces $X_i$. Even if we have good concentration bounds on the spaces $X_i$, this does not imply concentration on $X$ because it is possible that a 1-Lipschitz function $f$ will have different means when restricted to the $X_i$'s. However, if most of the mass of $X$

is concentrated on a sub-union $\cup_{j\in J}X_j$ (for some $J \subset I$) and the sets $X_i$ are all very close to each other, then there is a weak concentration inequality on $X$.

**Lemma 2.10.3.** *Let $(X, d_X, \mu)$ be a measure metric space with diameter $\leq 1$. Suppose $X = \sqcup_{i\in I}X_i$ is a finite disjoint union of spaces $X_i$, each with positive measure ($\mu(X_i) > 0$). Let $\mu_i$ be the induced probability measure on $X_i$. Suppose there exist $J \subset I$ and constants $\eta, \delta, \lambda, c > 0$ satisfying:*

1. *$\mu(\cup_{j\in J}X_j) \geq 1 - \eta \geq 1/2$.*
2. *For every $j, k \in J$, there exists a measure $\mu_{j,k}$ on $X_j \times X_k$ with marginals $\mu_j, \mu_k$ respectively such that*

$$\mu_{j,k}(\{(x_j, x_k) : \ d_X(x_j, x_k) \leq \delta\}) = 1.$$

3. *For each $j \in J$, $(X_j, d_X, \mu_j)$ is $(c, \lambda)$-concentrated.*

*Then for every 1-Lipschitz function $f : X \to \mathbb{R}$ and every $\epsilon > \delta + 2\eta$,*

$$\mu\left(\left|f - \int f \ d\mu\right| > \epsilon\right) \leq \eta + c\exp\left(-\lambda\left(\epsilon - \delta - 2\eta\right)^2\right).$$

*Proof.* Let $f : X \to \mathbb{R}$ be a 1-Lipschitz function. After adding a constant to $f$ if necessary, we may assume $\int f \ d\mu = 0$. Note that the mean of $f$ is a convex combination of its restrictions to the $X_i$'s:

$$\begin{aligned} 0 = \int f(x) \ d\mu(x) &= \sum_{i\in I}\mu(X_i)\int f(x_i) \ d\mu_i(x_i) \\ &= \sum_{i\in I\setminus J}\mu(X_i)\int f(x_i) \ d\mu_i(x_i) + \sum_{j\in J}\mu(X_j)\int f(x_j)d\mu_j(x_j). \end{aligned}$$

Since $f$ is 1-Lipschitz with zero mean, $|f| \leq \mathrm{diam}(X) \leq 1$. So

$$\begin{aligned} \left|\mu(\cup_{j\in J}X_j)^{-1}\sum_{j\in J}\mu(X_j)\int f(x_j) \ d\mu_j(x_j)\right| &= \left|\mu(\cup_{j\in J}X_j)^{-1}\sum_{i\in I\setminus J}\mu(X_i)\int f(x_i) \ d\mu_i(x_i)\right| \\ &\leq \frac{\eta}{1-\eta} \leq 2\eta \end{aligned}$$

where the last inequality uses that $\mu(\cup_{j\in J}X_j) \geq 1-\eta$ and $\eta \leq 1/2$.

For any $j,k \in J$, the $\mu_j$ and $\mu_k$-means of $f$ are $\delta$-close:

$$\begin{aligned}\left|\int f(x_j)\ d\mu_j(x_j) - \int f(x_k)\ d\mu_k(x_k)\right| &= \left|\int f(x_j) - f(x_k)\ d\mu_{j,k}(x_j,x_k)\right| \\ &\leq \int |f(x_j) - f(x_k)|\ d\mu_{j,k}(x_j,x_k) \leq \delta.\end{aligned}$$

So for any $j_0 \in J$,

$$\left|\int f(x_{j_0})\ d\mu_{j_0}(x_{j_0}) - \mu(\cup_{j\in J}X_j)^{-1}\sum_{j\in J}\mu(X_j)\int f(x_j)\ d\mu_j(x_j)\right| \leq \delta.$$

Combined with the previous estimate, this gives

$$\left|\int f(x_{j_0})\ d\mu_{j_0}(x_{j_0})\right| \leq \delta + 2\eta.$$

Now we estimate the $\mu$-probability that $f$ is $> \epsilon$ (assuming $\epsilon > \delta + 2\eta$):

$$\begin{aligned}\mu\left(\left|f - \int f\ d\mu\right| > \epsilon\right) &= \mu(|f| > \epsilon) \\ &\leq \eta + \sum_{j\in J}\mu_j(|f| > \epsilon)\mu(X_j) \\ &\leq \eta + \sum_{j\in J}\mu_j\left(\left|f - \int f(x_j)\ d\mu_j(x_j)\right| > \epsilon - \left|\int f(x_j)\ d\mu_j(x_j)\right|\right)\mu(X_j) \\ &\leq \eta + \sum_{j\in J}\mu_j\left(\left|f - \int f(x_j)\ d\mu_j(x_j)\right| > \epsilon - \delta - 2\eta\right)\mu(X_j) \\ &\leq \eta + c\exp\left(-\lambda\left(\epsilon - \delta - 2\eta\right)^2\right).\end{aligned}$$

□

The next lemma is essentially the same as [Led01, Proposition 1.11]. We include a proof for convenience.

**Lemma 2.10.4.** *[Led01] Suppose $(X, d_X, \mu)$ is $(c_1, \lambda_1)$-concentrated and $(Y, d_Y, \nu)$ is $(c_2, \lambda_2)$-*

*concentrated. Define a metric on $X \times Y$ by $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Then $(X \times Y, d_{X\times Y}, \mu \times \nu)$ is $(c_1 + c_2, \min(\lambda_1, \lambda_2)/4)$-concentrated.*

*Proof.* Let $F : X\times Y \to \mathbb{R}$ be 1-Lipschitz. For $y \in Y$, define $F^y : X \to \mathbb{R}$ by $F^y(x) = F(x, y)$. Define $G : Y \to \mathbb{R}$ by $G(y) = \int F^y(x)\ d\mu(x)$. Then $F^y$ and $G$ are 1-Lipschitz.

If $|F(x, y) - \int F\ d\mu\times\nu| > \epsilon$ then either $|F^y(x) - \int F^y\ d\mu| > \epsilon/2$ or $|G(y) - \int G\ d\nu| > \epsilon/2$. Thus

$$\begin{aligned} & \mu \times \nu \left( \left\{ (x, y) : \ \left| F(x, y) - \int F\ d\mu \times \nu \right| > \epsilon \right\} \right) \\ \leq\ & \mu \times \nu \left( \left\{ (x, y) : \ \left| F^y(x) - \int F^y\ d\mu \right| > \epsilon/2 \right\} \right) + \nu \left( \left\{ y : \ \left| G(y) - \int G\ d\nu \right| > \epsilon/2 \right\} \right) \\ \leq\ & c_1 e^{-\lambda_1 \epsilon^2/4} + c_2 e^{-\lambda_2 \epsilon^2/4} \leq (c_1 + c_2) \exp(-\min(\lambda_1, \lambda_2)\epsilon^2/4). \end{aligned}$$

□

**Lemma 2.10.5.** *Let $(X, d_X, \mu)$ and $(Y, d_Y, \nu)$ be metric-measure spaces. Suppose*

1. *$X, Y$ are finite sets, $\mu$ and $\nu$ are uniform probability measures,*
2. *there is a surjective map $\Phi : X \to Y$ and a constant $C > 0$ such that $|\Phi^{-1}(y)| = C$ for all $y \in Y$,*
3. *$(Y, d_Y, \nu)$ is $(c_1, \lambda_1)$-concentrated,*
4. *for each $y \in Y$, the fiber $\Phi^{-1}(y)$ is $(c_2, \lambda_2)$-concentrated (with respect to the uniform measure on $\Phi^{-1}(y)$ and the restricted metric),*
5. *for each $y_1, y_2 \in Y$ there is a probability measure $\mu_{y_1,y_2}$ on $\Phi^{-1}(y_1) \times \Phi^{-1}(y_2)$ with marginals equal to the uniform measures on $\Phi^{-1}(y_1)$ and $\Phi^{-1}(y_2)$ such that*

$$\mu_{y_1,y_2}(\{(x_1, x_2) : \ d_X(x_1, x_2) \leq d_Y(y_1, y_2)\}) = 1.$$

*Then $(X, d_X, \mu)$ is $(c_1 + c_2, \min(\lambda_1, \lambda_2)/4)$-concentrated.*

*Proof.* Let $f : X \to \mathbb{R}$ be 1-Lipschitz. Let $\mathbb{E}[f|Y] : Y \to \mathbb{R}$ be its conditional expectation defined by

$$\mathbb{E}[f|Y](y) = |\Phi^{-1}(y)|^{-1} \sum_{x \in \Phi^{-1}(y)} f(x).$$

Also let $\mathbb{E}[f] = |X|^{-1} \sum_{x \in X} f(x)$ be its expectation.

We claim that $\mathbb{E}[f|Y]$ is 1-Lipschitz. So let $y_1, y_2 \in Y$. By hypothesis (5)

$$\begin{aligned}
\mathbb{E}[f|Y](y_1) - \mathbb{E}[f|Y](y_2) &= \int f(x_1) - f(x_2)\ d\mu_{y_1,y_2}(x_1, x_2) \\
&\leq \int d(x_1, x_2)\ d\mu_{y_1,y_2}(x_1, x_2) \\
&\leq d(y_1, y_2).
\end{aligned}$$

The first inequality holds because $f$ is 1-Lipschitz and the second by hypothesis (5). This proves $\mathbb{E}[f|Y]$ is 1-Lipschitz.

Let $\epsilon > 0$. Because $\Phi$ is $C$-to-1, it makes the measure $\mu$ to $\nu$. Because $(Y, d_Y, \nu)$ is $(c_1, \lambda_1)$-concentrated,

$$\mu\left(\left|\mathbb{E}[f|Y] \circ \Phi - \int f\ d\mu\right| > \epsilon/2\right) = \nu\left(|\mathbb{E}[f|Y] - \mathbb{E}[f]| > \epsilon/2\right) < c_1 e^{-\lambda_1 \epsilon^2/4}. \tag{2.48}$$

Because each fiber $\Phi^{-1}(y)$ is $(c_2, \lambda_2)$-concentrated, for any $y \in Y$,

$$|\Phi^{-1}(y)|^{-1} \# \left\{x \in \Phi^{-1}(y) : \ |f(x) - \mathbb{E}[f|Y](y)| > \epsilon/2\right\} < c_2 e^{-\lambda_2 \epsilon^2/4}.$$

Average this over $y \in Y$ to obtain

$$\mu\left(\{x \in X : \ |f(x) - \mathbb{E}[f|Y](\Phi(x))| > \epsilon\right) < c_2 e^{-\lambda_2 \epsilon^2}.$$

Combine this with (2.48) to obtain

$$\begin{aligned}
\mu\left(\left|f - \int f\ d\mu\right| > \epsilon\right) &\leq \mu\left(|f - \mathbb{E}[f|Y](\Phi(x))\ d\mu| > \epsilon/2\right) + \mu\left(\left|\mathbb{E}[f|Y](\Phi(x)) - \int f\ d\mu\right| > \epsilon/2\right) \\
&\leq c_2 e^{-\lambda_2 \epsilon^2/4} + c_1 e^{-\lambda_1 \epsilon^2/4}
\end{aligned}$$

which implies the lemma.

$\square$

### 2.10.2 Specific considerations

Given an equitable coloring $\chi : [n] \to \{0,1\}$, let $H_\chi$ be the stabilizer of $\chi$:

$$H_\chi = \{g \in \mathrm{Sym}(n) : \ \chi(gv) = \chi(v) \ \forall v \in [n]\}.$$

**Lemma 2.10.6.** *The group $H_\chi$ is $(4, n/16)$-concentrated (when equipped with the uniform probability measure and the restriction of the normalized Hamming metric $d_{\mathrm{Sym}(n)}$).*

*Proof.* The group $H_\chi$ is isomorphic to the direct product $\mathrm{Sym}(\chi^{-1}(0)) \times \mathrm{Sym}(\chi^{-1}(1))$ which is isomorphic to $\mathrm{Sym}(n/2)^2$. By [Led01, Corollary 4.3], $\mathrm{Sym}(n/2)$ is $(2, n/16)$-concentrated. So the result follows from Lemmas 2.10.4 and 2.10.2. This uses that the inclusion map from $\mathrm{Sym}(n/2)^2$ to itself is $(1/2)$-Lipschitz when the source is equipped with the sum of the $d_{\mathrm{Sym}(n/2)}$-metrics and the target equipped with the $d_{\mathrm{Sym}(n)}$ metric. $\square$

We need to show that certain subsets of the group $\mathrm{Sym}(n)$ are concentrated. To define these subsets, we need the following terminology.

Recall that a **$k$-partition** of $[n]$ is an unordered partition $\pi = \{P_1, \ldots, P_{n/k}\}$ of $[n]$ such that each $P_i$ has cardinality $k$. Let $\mathrm{Part}(n,k)$ be the set of all $k$-partitions of $[n]$. The group $\mathrm{Sym}(n)$ acts on $\mathrm{Part}(n,k)$ by $g\pi = \{gP_1, \ldots, gP_{n/k}\}$.

Let $\sigma \in \mathrm{Sym}(n)$. The **orbit-partition** of $\sigma$ is the partition $\mathrm{Orb}(\sigma)$ of $[n]$ into orbits of $\sigma$. For example, for any $v \in [n]$ the element of $\mathrm{Orb}(\sigma)$ containing $v$ is $\{\sigma^i v : \ i \in \mathbb{Z}\} \subset [n]$. Let $\mathrm{Sym}(n,k) \subset \mathrm{Sym}(n)$ be the set of all permutations $\sigma \in \mathrm{Sym}(n)$ such that the orbit-partition of $\sigma$ is a $k$-partition.

Recall from §2.4.1 that a $k$-partition $\pi$ **has type $\vec{t} = (t_j)_{j=0}^k \in [0,1]^{k+1}$ with respect to a coloring** $\chi$ if the number of partition elements $P$ of $\pi$ with $|P \cap \chi^{-1}(1)| = j$ is $t_j n$. We will also say that a permutation $\sigma \in \mathrm{Sym}(n,k)$ **has type $\vec{t} = (t_j)_{j=0}^k \in [0,1]^{k+1}$ with respect to a coloring** $\chi$ if its orbit-partition $\mathrm{Orb}(\sigma)$ has type $\vec{t}$ with respect to $\chi$.

Let $\mathrm{Sym}(n,k;\chi,\vec{t})$ be the set of all permutations $\sigma \in \mathrm{Sym}(n,k)$ such that $\sigma$ has type $\vec{t}$ with respect to $\chi$.

**Lemma 2.10.7.** *The subset* $\mathrm{Sym}(n,k;\chi,\vec{t})$ *is either empty or* $(6,\lambda n)$*-concentrated (when equipped with the normalized Hamming metric* $d_{\mathrm{Sym}(n)}$ *and the uniform probability measure) where* $\lambda > 0$ *is a constant depending only on* $k$*.*

*Proof.* Let $\mathrm{Part}(n,k;\chi,\vec{t})$ be the set of all (unordered) $k$-partitions of $[n]$ with type $\vec{t}$ (with respect to $\chi$). We will consider this set as a metric space in which the distance between partitions $\pi,\pi' \in \mathrm{Part}(n,k;\chi,\vec{t})$ is $d(\pi,\pi') = \frac{k|\pi \triangle \pi'|}{2n}$ where $\triangle$ denotes symmetric difference.

Let $\mathrm{Orb} : \mathrm{Sym}(n,k;\chi,\vec{t}) \to \mathrm{Part}(n,k;\chi,\vec{t})$ be the map which sends a permutation to its orbit-partition. We will verify the conditions of Lemma 2.10.5 with $X = \mathrm{Sym}(n,k;\chi,\vec{t})$, $Y = \mathrm{Part}(n,k;\chi,\vec{t})$ and $\Phi = \mathrm{Orb}$. Condition (1) is immediate.

Observe that Orb is surjective and constant-to-1. In fact for any partition $\pi \in \mathrm{Part}(n,k;\chi,\vec{t})$, $|\mathrm{Orb}^{-1}(\pi)| = (k-1)!^{n/k}$ since an element $\sigma \in |\mathrm{Orb}^{-1}(\pi)|$ is obtained by choosing a $k$-cycle for every part of $\pi$. To be precise, if $\pi = \{P_1,\ldots,P_{n/k}\}$ then $\mathrm{Orb}^{-1}(\pi)$ is the set of all permutations $\sigma$ of the form $\sigma = \prod_{i=1}^{n/k} \sigma_i$ where $\sigma_i$ is a $k$-cycle with support in $P_i$. This verifies condition (2) of Lemma 2.10.5.

Observe that $H_\chi$ acts transitively on $\mathrm{Part}(n,k;\chi,\vec{t})$. Fix $\pi \in \mathrm{Part}(n,k;\chi,\vec{t})$ and define a map $\phi : H_\chi \to \mathrm{Part}(n,k;\chi,\vec{t})$ by $\phi(h) = h\pi$. We claim that $\phi$ is $k^2/2$-Lipschitz. Indeed, if $h_1,h_2 \in H_\chi$ then

$$\begin{aligned} d(h_1\pi,h_2\pi) &= \frac{k|h_1\pi \triangle h_2\pi|}{2n} \\ &\le \frac{k^2\#\{p\in[n]:\ h_1(p)\neq h_2(p)\}}{2n} \\ &= \frac{k^2}{2} d_{\mathrm{Sym}(n)}(h_1,h_2). \end{aligned}$$

Because $H_\chi$ is $(4,n/16)$-concentrated by Lemma 2.10.6, Lemma 2.10.2 implies $\mathrm{Part}(n,k;\chi,\vec{t})$ is $(4,n/4k^4)$-concentrated. This verifies condition (3) of Lemma 2.10.5.

We claim that $\mathrm{Orb}^{-1}(\pi)$ is $(2,n/2k)$-concentrated. To see this, let $\pi = \{P_1,\ldots,P_{n/k}\}$ and let $\mathrm{Sym}_k(P_i) \subset \mathrm{Sym}(n)$ be the set of all $k$-cycles with support in $P_i$. Then $\mathrm{Orb}^{-1}(\pi)$ is isometric to $\mathrm{Sym}_k(P_1)\times\cdots\times\mathrm{Sym}_k(P_{n/k})$. The diameter of $\mathrm{Sym}_k(P_i)$, viewed as a subset of $\mathrm{Sym}(n)$ with the normalized Hamming metric on $\mathrm{Sym}(n)$, is $k/n$. So the claim follows from [Led01, Corollary 1.17]. This verifies condition (4) of Lemma 2.10.5.

For $\pi_1,\pi_2 \in \mathrm{Part}(n,k;\chi,\vec{t})$, let $X_{\pi_1,\pi_2}$ be the set of all pairs $(\sigma_1,\sigma_2) \in \mathrm{Orb}^{-1}(\pi_1)\times$

$\mathrm{Orb}^{-1}(\pi_2)$ such that if $P \in \pi_1 \cap \pi_2$ then the restriction of $\sigma_1$ to $P$ equals the restriction of $\sigma_2$ to $P$. Observe that $X_{\pi_1,\pi_2}$ is non-empty and the projection maps $X_{\pi_1,\pi_2} \to \mathrm{Orb}^{-1}(\pi_i)$ $(i = 1, 2)$ are constant-to-1. In fact, for any $\sigma_1 \in \mathrm{Orb}^{-1}(\pi_1)$, the set of $\sigma_2$ with $(\sigma_1, \sigma_2) \in X_{\pi_1,\pi_2}$ is bijective with the set of assignments of $k$-cycles to parts in $\pi_2 \setminus \pi_1$.

Let $\mu_{\pi_1,\pi_2}$ be the uniform probability measure on $X_{\pi_1,\pi_2}$. Since the projection maps are constant-to-1, the marginals of $\mu_{\pi_1,\pi_2}$ are uniform. Moreover, if $(\sigma_1, \sigma_2) \in X_{\pi_1,\pi_2}$ then

$$\{i \in [n] : \ \sigma_1(i) \neq \sigma_2(i)\} \subset \cup_{P \in \pi_1 \setminus \pi_2} P.$$

Thus

$$d_{\mathrm{Sym}}(\sigma_1, \sigma_2) \leq n^{-1} | \cup_{P \in \pi_1 \setminus \pi_2} P| = n^{-1} k |\pi_1 \vartriangle \pi_2|/2 = d(\pi_1, \pi_2).$$

This verifies condition (5) of Lemma 2.10.5.

We have now verified all of the conditions of Lemma 2.10.5. The lemma follows.

□

Let $\mathrm{Sym}(n, k; \chi)$ be the set of all $\sigma \in \mathrm{Sym}(n, k)$ such that if $\vec{t} = (t_j)_{j=0}^k$ is the type of $\sigma$ with respect to $\chi$ then $t_0 = t_k = 0$. In other words, $\sigma \in \mathrm{Sym}(n, k; \chi)$ if and only if the orbit-partition $\pi$ of $\sigma$ is proper with respect to $\chi$ (where we think of $\pi$ as a collection of hyper-edges).

Let $\vec{s} = (s_j)$ with $s_0 = s_k = 0$ and $s_j = \frac{1}{k(2^k-2)}\binom{k}{j}$ for $0 < j < k$. For $\delta > 0$ let $\mathrm{Sym}_\delta(n, k; \chi)$ be the set of all $\sigma \in \mathrm{Sym}(n, k; \chi)$ such that if $\vec{t} = (t_i)_{i=0}^k$ is the type of $\sigma$ (with respect to $\chi$) then

$$\sum_{i=0}^{k} |s_i - t_i|^2 < \delta^2.$$

**Lemma 2.10.8.** *With notation as above, for sufficiently large $n$*

$$\frac{|\mathrm{Sym}_\delta(n, k; \chi)|}{|\mathrm{Sym}(n, k; \chi)|} \geq 1 - e^{-\lambda_1 \delta^2 n}$$

*where $\lambda_1 > 0$ is a constant depending only on $k$.*

*Proof.* Let

$$\begin{aligned}
\mathrm{Part}(n,k;\chi,\vec{t}) &= \{\pi \in \mathrm{Part}(n,k) : \ \pi \text{ has type } \vec{t} \text{ with respect to } \chi\},\\
\mathrm{Part}(n,k;\chi) &= \{\pi \in \mathrm{Part}(n,k) : \ \chi \text{ is proper with respect to } \pi\},\\
\mathrm{Part}_\delta(n,k;\chi) &= \left\{\pi \in \mathrm{Part}(n,k;\chi) : \ \text{ if } \vec{t} \text{ is the type of } \pi \text{ with respect to } \chi \text{ then } \sum_{i=0}^{k} |t_i - s_i|^2 < \delta^2\right\}.
\end{aligned}$$

The orbit-partition map from $\mathrm{Sym}(n,k) \to \mathrm{Part}(n,k)$ is constant-to-1 and maps $\mathrm{Sym}(n,k;\chi)$ onto $\mathrm{Part}(n,k;\chi)$ and $\mathrm{Sym}_\delta(n,k;\chi)$ onto $\mathrm{Part}_\delta(n,k;\chi)$. Therefore, it suffices to prove

$$\frac{|\,\mathrm{Part}_\delta(n,k;\chi)|}{|\,\mathrm{Part}(n,k;\chi)|} \geq 1 - e^{-\lambda\delta^2 n}$$

where $\lambda > 0$ is a constant depending only on $k$.

Let $\widetilde{\mathscr{M}}$ be the set of all vectors $\vec{t} = (t_i)_{i=0}^k \in [0,1]^{k+1}$ such that $t_0 = t_k = 0$, $\sum_i t_i = 1/k$ and $\sum_j jt_j = 1/2$. By the proof of Theorem 2.4.2 (specifically equation (2.9)), $J$ is uniquely maximized in $\widetilde{\mathscr{M}}$ by the vector $\vec{s}$.

Recall from Lemma 2.4.3 that if $\vec{t} \in \widetilde{\mathscr{M}}$ and $n\vec{t}$ is $\mathbb{Z}$-valued then

$$(1/n)\log|\,\mathrm{Part}(n,k;\chi,\vec{t})| = (1-1/k)(\log(n)-1) - \log(2) + J(\vec{t}) + O(n^{-1}\log(n))$$

where $J(\vec{t}) = H(\vec{t}) - \sum_{j=0}^k t_j \log(j!(k-j)!)$.

In order to get a lower bound on $|\,\mathrm{Part}(n,k;\chi)|$, observe that there exists $\vec{r} \in \widetilde{\mathscr{M}}$ such that $n\vec{r}$ is $\mathbb{Z}$-valued and $|s_i - r_i| \leq k/n$ for all $i$. Thus $J(\vec{r}) - J(\vec{s}) = O(1/n)$. It follows that

$$|\,\mathrm{Part}(n,k;\chi)| \ \geq \ |\,\mathrm{Part}(n,k;\chi,\vec{r})| = (1-1/k)(\log(n)-1) - \log(2) + J(\vec{s}) + O(n^{-1}\log(n)).$$

We claim that the Hessian of $J$ is negative definite. To see this, one can consider $J$ to be a function of $[0,1]^{k+1}$. The linear terms in $J$ do not contribute to its Hessian. Since the second derivative of $x \mapsto -x\log x$ is $-1/x$,

$$\frac{\partial^2 J}{\partial t_i \partial t_j} = \begin{cases} 0 & i \neq j \\ -1/t_i & i = j \end{cases}$$

Thus the Hessian is diagonal and every eigenvalue is negative; so it is negative definite.

Thus if $\vec{t} \in \widetilde{\mathscr{M}}$ is such that $\sum_i |t_i - s_i|^2 \geq \delta^2$ then

$$(1/n)\log|\operatorname{Part}(n,k;\chi,\vec{t})| \leq (1-1/k)(\log(n)-1) - \log(2) + J(\vec{s}) - \delta^2\lambda_1' + O(n^{-1}\log(n))$$

where $\lambda_1' = \frac{1}{2}\min_{\vec{t}\in\widetilde{\mathscr{M}}}\min_{1\leq i\leq k-1} 1/s_i$ is half the smallest absolute value of an eigenvalue of the Hessian of $J$ on $\widetilde{\mathscr{M}}$.

If $\vec{t}$ is the type of a $k$-partition $\pi$ of $n$ then $t_i \in \{0, 1/n, 2/n, \ldots, 1\}$. Thus the number of different types of $k$-partitions of $[n]$ is bounded by a polynomial in $n$ (namely $(n+1)^{k+1}$). Thus

$$\begin{aligned}\frac{|\operatorname{Part}_\delta(n,k;\chi)|}{|\operatorname{Part}(n,k;\chi)|} &\geq 1-(n+1)^{k+1}\frac{\exp(n[(1-1/k)(\log(n)-1)-\log(2)+J(\vec{s})-\delta^2\lambda_1'+O(n^{-1}\log(n))])}{\exp(n[(1-1/k)(\log(n)-1)-\log(2)+J(\vec{s})+O(n^{-1}\log(n))])}\\ &= 1-n^c\exp(-\delta^2\lambda_1' n)\end{aligned}$$

where $c > 0$ is a constant. This implies the lemma.

$\square$

Recall that a **$k$-cycle** is a permutation $\pi \in \operatorname{Sym}(n)$ of the form $\pi = (v_1, \ldots, v_k)$ for some $v_1, \ldots, v_k \in [n]$. In other words, $\pi$ has $n-k$ fixed points and one orbit of size $k$. The **support** of $\pi \in \operatorname{Sym}(n)$ is the complement of the set of $\pi$-fixed points. It is denoted by $\operatorname{supp}(\pi)$. Two permutations are **disjoint** if their supports are disjoint. A permutation $\pi \in \operatorname{Sym}(n)$ is a **disjoint product of $k$-cycles** if there exist pairwise disjoint $k$-cycles $\pi_1, \ldots, \pi_m$ such that $\pi = \pi_1 \cdots \pi_m$. In this case we say that each $\pi_i$ is **contained in** $\pi$.

**Lemma 2.10.9.** *Let $\vec{t}, \vec{u} \in [0,1]^{k+1}$. Suppose*

$$\sum_{i=0}^{k} |t_i - u_i| < \delta.$$

*Suppose* $\operatorname{Sym}(n,k;\chi,\vec{t})$ *and* $\operatorname{Sym}(n,k;\chi,\vec{u})$ *are non-empty (for some integer $n$ and equitable coloring $\chi$).*

*For $\sigma, \sigma' \in \operatorname{Sym}(n,k)$, let $|\sigma \vartriangle \sigma'|$ be the number of $k$-cycles $\tau$ that are either in $\sigma$ or* *in $\sigma'$ but not in both. Let*

$$Z = \{(\sigma, \sigma') \in \mathrm{Sym}(n, k; \chi, \vec{t}) \times \mathrm{Sym}(n, k; \chi, \vec{u}) : \ |\sigma \vartriangle \sigma'| \leq \delta n\}.$$

*Then $Z$ is non-empty and there exists a probability measure $\mu$ on $Z$ with marginals equal to the uniform probability measures on $\mathrm{Sym}(n, k; \chi, \vec{t})$ and $\mathrm{Sym}(n, k; \chi, \vec{u})$ respectively.*

*Proof.* Let $\rho \in \mathrm{Sym}(n)$ be a disjoint product of $k$-cycles. The **type of $\rho$ with respect to $\chi$** is the vector $\vec{r} = (r_i)_{i=0}^k$ defined by: $r_i$ is $1/n$ times the number of $k$-cycles $\rho'$ contained in $\rho$ such that $|\operatorname{supp}(\rho') \cap \chi^{-1}(1)| = i$.

Let $\sigma \in \mathrm{Sym}(n, k; \chi, \vec{t})$. Then there exist disjoint $k$-cycles $\sigma'_1, \ldots, \sigma'_m$ in $\sigma$ such that if $\rho = \sigma'_1 \cdots \sigma'_m$ and $\vec{r} = (r_i)_{i=0}^k$ is the type of $\rho$ then $r_i = \min(t_i, u_i)$. Note $m \geq n(1/k - \delta/2)$ by assumption on $\vec{t}$ and $\vec{u}$. Moreover, there exist $k$-cycles $\sigma'_{m+1}, \cdots, \sigma'_{n/k}$ such that the collection $\sigma'_1, \ldots, \sigma'_{n/k}$ is pairwise disjoint and the type of $\sigma' = \sigma'_1 \cdots \sigma'_{n/k}$ is $\vec{u}$. Then $|\sigma \vartriangle \sigma'| = 2(n/k - m) \leq \delta n$. So $(\sigma, \sigma') \in Z$ which proves $Z$ is non-empty.

We claim that there is a constant $C_1 > 0$ such that for every $\sigma \in \mathrm{Sym}(n, k; \chi, \vec{t})$ the number of $\sigma' \in \mathrm{Sym}(n, k; \chi, \vec{u})$ with $(\sigma, \sigma') \in Z$ is $C_1$. Indeed the following algorithm constructs all such $\sigma'$ with no duplications:

**Step 1**. Let $\sigma = \sigma_1 \cdots \sigma_{n/k}$ be a representation of $\sigma$ as a disjoint product of $k$-cycles. Choose a vector $\vec{r} = (r_i)_{i=0}^k$ such that

1. there exists a subset $S \subset [n/k]$ with cardinality $|S| \geq n(1/k - \delta/2)$ such that if $\rho = \prod_{i \in S} \sigma_i$ then $\vec{r}$ is the type of $\rho$;

2. $r_i \leq u_i$ for all $i$.

**Step 2**. Choose a subset $S \subset [n/k]$ satisfying the condition in Step 1.

**Step 3**. Choose pairwise disjoint $k$-cycles $\sigma'_1, \ldots, \sigma'_{n/k-|S|}$ such that

1. $\operatorname{supp}(\sigma_i) \cap \operatorname{supp}(\sigma'_j) = \emptyset$ $(\forall i \in S)$ $(\forall j)$;

2. $\sigma'_j$ is not contained in $\sigma$ $(\forall j)$;

3. if $\sigma' = \prod_{i \in S} \sigma_i \prod_j \sigma'_j$ then $\sigma'$ has type $\vec{u}$;

The range of possible vectors $\vec{r}$ in Step 1 depends only on $k, n, \vec{t}, \vec{u}$. The number of choices in Steps 2 and 3 depends only on the choice of $\vec{r}$ in Step 1 and on $k, n, \vec{t}, \vec{u}$. This proves the claim.

Similarly, there is a constant $C_2 > 0$ such that for every $\sigma' \in \mathrm{Sym}(n, k; \chi, \vec{u})$ the number of $\sigma \in \mathrm{Sym}(n, k; \chi, \vec{t})$ with $(\sigma, \sigma') \in Z$ is $C_2$. It follows that the uniform probability measure on $Z$ has marginals equal to the uniform probability measures on $\mathrm{Sym}(n, k; \chi, \vec{t})$ and $\mathrm{Sym}(n, k; \chi, \vec{u})$ respectively.

$\square$

**Corollary 2.10.10.** *Let $U_{\mathrm{Sym}(n,k;\chi)}$ denote the uniform probability measure on $\mathrm{Sym}(n, k; \chi)$ and let $\mathbb{E}_{\mathrm{Sym}(n,k;\chi)}$ be the associated expectation operator. There are constants $c, \lambda > 0$ (depending only on $k$) such that for every $\delta > 0$, there exists $N_\delta$ such that for all $n > N_\delta$, for every 1-Lipschitz $f : \mathrm{Sym}(n, k; \chi) \to \mathbb{R}$,*

$$U_{\mathrm{Sym}(n,k;\chi)} \left( \left| f - \mathbb{E}_{\mathrm{Sym}(n,k;\chi)}[f] \right| > \delta \right) \leq c \exp(-\lambda \delta^2 n).$$

*Moreover $\delta \mapsto N_\delta$ is monotone decreasing.*

*Proof.* The set $\mathrm{Sym}(n, k; \chi)$ is the disjoint union of $\mathrm{Sym}(n, k; \chi, \vec{t})$ over $\vec{t} \in [0, 1]^{k+1}$. Let $\delta > 0$. Lemmas 2.10.7, 2.10.8 and 2.10.9 imply that for all sufficiently large $n$, this decomposition of $\mathrm{Sym}(n, k; \chi)$ satisfies the criterion in Lemma 2.10.3 where we set $c = 3$, $\eta = \exp(-\lambda_1 \delta^2 n)$ and $\lambda = \lambda_0 n$ where $\lambda_0, \lambda_1 > 0$ depend only on $k$. So for every 1-Lipschitz function $f : \mathrm{Sym}(n, k; \chi) \to \mathbb{R}$, every $\epsilon > \delta + 2\eta$ and all sufficiently large $n$,

$$U_{\mathrm{Sym}(n,k;\chi)} \left( \left| f - \mathbb{E}_{\mathrm{Sym}(n,k;\chi)}[f] \right| > \epsilon \right) \leq \exp(-\lambda_1 \delta^2 n) + c \exp\left( -\lambda_0 n \left( \epsilon - \delta - 2\eta \right)^2 \right).$$

In particular, there exist $N_\delta$ such that if $n > N_\delta$ the inequality above holds and $2\eta < \delta$. By choosing $N_\delta$ to satisfy the equation $2 \exp(-\lambda_1 \delta^2 N_\delta) = \delta$, we can ensure that $\delta \mapsto N_\delta$ is decreasing.

Set $\epsilon = 3\delta$ to obtain

$$U_{\mathrm{Sym}(n,k;\chi)} \left( \left| f - \mathbb{E}_{\mathrm{Sym}(n,k;\chi)}[f] \right| > 3\delta \right) \leq \exp(-\lambda_1 \delta^2 n) + c \exp\left( -\lambda_0 n \left( \delta \right)^2 \right) \leq (1 + c) \exp(-\lambda \delta^2 n)$$

where $\lambda = \min(\lambda_0, \lambda_1)$. The corollary is now finished by changing variables.

$\square$

*Proof of Theorem 2.10.1.* The space of homomorphisms $\mathrm{Hom}_\chi(\Gamma, \mathrm{Sym}(n))$ is the $d$-fold direct power of the spaces $\mathrm{Sym}(n, k; \chi)$. So the Theorem follows from Corollary 2.10.10 and the proof of Lemma 2.10.4.

$\square$

# Chapter 3

# A multiplicative ergodic theorem for von Neumann algebra valued cocycles

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## 3.1 Introduction

### 3.1.1 The finite dimensional MET

Here is a version of the classical Multiplicative Ergodic Theorem (MET). Let $(X, \mu)$ be a standard probability space, $f : X \to X$ a measure-preserving transformation, and $c : \mathbb{N} \times X \to \mathrm{GL}(n, \mathbb{R})$ a measurable cocycle:

$$c(n+m, x) = c(n, f^m x)c(m, x) \quad \forall n, m \in \mathbb{N},\ \mu - a.e.\ x \in X.$$

Assume the first moment condition:

$$\int \log^+ \|c(1, x)\| \, d\mu(x) < \infty,$$

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Then there is a **limit operator**

$$\Lambda(x) := \lim_{n\to\infty} [c(n,x)^* c(n,x)]^{1/2n}$$

for a.e. $x$. Let $e^{\lambda_1(x)} > \cdots > e^{\lambda_k(x)}$ be the distinct eigenvalues of $\Lambda(x)$. Then $\lambda_1, \ldots, \lambda_k$ are the **Lyapunov exponents**. They are invariant in the sense that $\lambda_i(fx) = \lambda_i(x)$ for a.e. $x$. If $m_i \in \mathbb{N}$ is the multiplicity of $\lambda_i$ then the **Lyapunov distribution** is the discrete measure $\sum_{i=1}^k m_i \delta_{\lambda_i}$.

Let $W_i$ be the $e^{\lambda_i(x)}$-eigenspace of $\Lambda(x)$ and define

$$V_i = \sum_{j\geq i} W_j$$

so that $V_k(x) \subset \cdots \subset V_1(x) = \mathbb{R}^n$ is a flag. The $V_i(x)$ are the **Oseledets subspaces**. They are cocycle-invariant in the sense that $V_i(fx) = c(1,x)V_i(x)$ (for a.e. $x$).

Finally, for a.e. $x \in X$ and every vector $v \in V_i(x) \setminus V_{i+1}(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|c(n,x)v\| = \lambda_i(x).$$

This last condition can be expressed without reference to Lyapunov exponents by:

$$\lim_{n\to\infty} \|c(n,x)v\|^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n v\|^{1/n}.$$

### 3.1.2 Previous literature

Infinite-dimensional generalizations of the MET have appeared in [Rue82, Mn83, Blu16, LL10, Thi87, GTQ15, Sch91]. Each of these assumes the operators $c(n,x)$ satisfy a quasi-compactness condition and consequently the limit operators $\Lambda(x)$ have discrete spectrum.

On the other hand, one does not expect there to be an unconditional generalization to infinite dimensions. For example, Voiculescu's example in [HS09, Example 8.4] shows there is a bounded operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ such that $|T^n|^{1/n}$ does not converge in the strong operator topology. We could define the cocycle $c$ above by $c(n,x) = T^n$ to see that convergence cannot be guaranteed in the general setting of bounded operators on Hilbert spaces.

### 3.1.3 von Neumann algebras

The purpose of this paper is to establish a new MET in which the cocycle takes values in the group of invertible elements of a tracial von Neumann algebra. To explain in more detail, let $\mathcal{H}$ be a separable Hilbert space, $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. A **von Neumann algebra** is a sub-algebra $M \subset B(\mathcal{H})$ containing the identity ($I \in M$) that is closed under taking adjoints and closed in the weak operator topology. Let $M_+ \subset M$ be the positive operators on $M$. A **trace** on $M$ is a map $\tau : M_+ \to [0, \infty]$ satisfying

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in M_+$;
2. $\tau(\lambda x) = \lambda\tau(x)$ for all $\lambda \in [0, \infty)$, $x \in M_+$ (agreeing that $0(+\infty) = 0$);
3. $\tau(x^*x) = \tau(xx^*)$ for all $x \in M$.

We will always assume $\tau$ is

- faithful, which means $\tau(x^*x) = 0 \Rightarrow x = 0$;
- normal, which means $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ for every increasing net $(x_i)_i$ in $M_+$;
- semifinite, which means for every $x \in M_+$ there exists $y \in M_+$ such that $0 < y < x$ and $0 < \tau(y) < \infty$.

The pair $(M, \tau)$ is a **finite tracial von Neumann algebra** if $\tau(I) < \infty$.

The trace $\tau$ on $M$ is unique (up to scale) if and only if $M$ has trivial center. Many constructions considered here depend on the choice of trace but we will suppress this dependence from the notation.

### 3.1.4 Example: the abelian case

Fix a standard probability space $(Y, \nu)$ and let $M = \mathrm{L}^\infty(Y, \nu)$. For every $\phi \in M$, define the multiplication operator

$$m_\phi : \mathrm{L}^2(Y, \nu) \to \mathrm{L}^2(Y, \nu), \quad (m_\phi f)(y) = \phi(y)f(y).$$

The map $\phi \mapsto m_\phi$ embeds $M$ into the algebra of bounded operators on $\mathrm{L}^2(Y,\nu)$. We will identify $\phi$ with $m_\phi$. Define the trace $\tau : M \to \mathbb{C}$ by

$$\tau(\phi) = \int \phi \, \mathrm{d}\nu.$$

With this trace, $(M,\tau)$ is a finite von Neumann algebra.

### 3.1.5 Main results

#### The limit operator

Our first main result shows the existence of a limit operator. We state the result here and afterwards explain the notions of convergence and the notation used, such as $\mathrm{GL}^2(M,\tau)$

**Theorem 3.1.1.** *Let $(X,\mu)$ be a standard probability space, $f : X \to X$ an ergodic measure-preserving transformation, $(M,\tau)$ a von Neumann algebra with semi-finite faithful normal trace $\tau$. Let $M^\times \subset M$ be the subgroup of elements of $M$ with bounded inverse. Let $c : \mathbb{N} \times X \to M^\times \cap \mathrm{GL}^2(M,\tau)$ be a cocycle in the sense that*

$$c(n+m,x) = c(n, f^m x) c(m,x)$$

*for all $n,m \in \mathbb{N}$ and a.e. $x \in X$. We assume $c$ is measurable with respect to the strong operator topology on $M^\times$.*

*Assume the first moment condition:*

$$\int_X \| \log(|c(1,x)|)\|_2 \; d\mu(x) < \infty.$$

*Then for almost every $x \in X$, the following limit exists:*

$$\lim_{n\to\infty} \frac{\| \log(c(n,x)^* c(n,x))\|_2}{n} = D.$$

*Moreover, if $D > 0$ then for a.e. $x$, there exists a limit operator $\Lambda(x) \in L^0(M,\tau)$ satisfying*

- $\lim_{n\to\infty} \frac{1}{n} d_{\mathcal{P}}(|c(n,x)|, \Lambda(x)^n) = 0$ *in $\mathcal{P}$;*

- $\lim_{n\to\infty} |c(n,x)|^{1/n} \to \Lambda(x)$ *in* $\mathcal{P}$ *and in measure;*

- $\lim_{n\to\infty} n^{-1}\log|c(n,x)| \to \log\Lambda(x)$ *in* $L^2(M,\tau)$.

**The regular representation**

Let $\mathcal{N} = \{x \in M : \ \tau(x^*x) < \infty\}$ The trace induces an inner-product on $\mathcal{N}$ by

$$\langle x, y\rangle := \tau(x^*y).$$

Let $\mathrm{L}^2(M,\tau)$ denote the Hilbert space completion of $\mathcal{N}$ with respect to this inner product. For $x \in M$, the left-multiplication operator $L_x : M \to M$ defined by $L_x(y) = xy$ extends to a bounded linear operator on $\mathrm{L}^2(M,\tau)$. Therefore, we may view $M$ as a sub-algebra of the algebra $B(\mathrm{L}^2(M,\tau))$ of bounded linear operators on $\mathrm{L}^2(M,\tau)$. This is the **regular representation of** $M$ (this is explained in more detail in §3.3).

An operator $x$ on $\mathrm{L}^2(M,\tau)$ is affiliated with $(M,\tau)$ if it is closed, dense defined and commutes with every element in the commutant $M' = \{x \in B(\mathrm{L}^2(M,\tau)) : \ xy = yx \,\forall y \in M\}$. Let $\mathrm{L}^0(M,\tau)$ denote the algebra of operators affiliated with $(M,\tau)$. This is a $*$-algebra in the measure topology. Moreover, the trace $\tau$ extends to $\tau : \mathrm{L}^0(M,\tau)_+ \to [0,\infty]$ where $\mathrm{L}^0(M,\tau)_+ \subset \mathrm{L}^0(M,\tau)$ is the cone of positive affiliated operators. Also if $x \in \mathrm{L}^0(M,\tau)_+$ then $x^{-1/2}$ and $\log x$ are well-defined via the spectral calculus. See §3.3.4 and §3.5.2 for details.

Let $\mathrm{GL}^2(M,\tau)$ consist of those elements $x \in \mathrm{L}^0(M,\tau)$ such that $\log|x| \in \mathrm{L}^2(M,\tau)$. We prove in §3.4 that $\mathrm{GL}^2(M,\tau)$ is a group. Let $\mathcal{P} = \mathrm{GL}^2(M,\tau) \cap \mathrm{L}^0(M,\tau)_+$. For $x, y \in \mathcal{P}$, define $d_{\mathcal{P}}(x,y) = \|\log(x^{-1/2}yx^{-1/2})\|_2$. We prove in §3.5.3-3.5.4 that $(\mathcal{P}, d_{\mathcal{P}})$ is a complete CAT(0) metric space on which $\mathrm{GL}^2(M,\tau)$ acts transitively by isometries. This extends work of Andruchow-Larotonda who previously studied the geometry of $\mathcal{P} \cap M$ [AL06].

*Example* 1. Continuing with our running example, if $M = \mathrm{L}^\infty(Y,\nu)$ then the above-mentioned inner product on $M$ is the restriction of the inner product on $\mathrm{L}^2(Y,\nu)$ to $M$. Therefore, $\mathrm{L}^2(M,\tau)$ is naturally isomorphic to $\mathrm{L}^2(Y,\nu)$. The algebra of affiliated operators $\mathrm{L}^0(M,\tau)$ is identified with the algebra of all complex-valued measurable functions on $(Y,\nu)$ (mod null sets). The exponential map $\exp : \mathrm{L}^2(Y,\nu) \to \mathrm{GL}^2(M,\tau)$ is a surjective homomorphism of groups (where we consider $\mathrm{L}^2(Y,\nu)$ as an abelian group under addition). The kernel consists of all maps $\phi \in \mathrm{L}^2(Y,\nu)$ with essential range in $2\pi i\mathbb{Z}$. The restriction of exp to the real Hilbert space $\mathrm{L}^2(Y,\nu;\mathbb{R})$ is an isometry onto $(\mathcal{P}, d_{\mathcal{P}})$.

### Remarks on the limit operator

*Remark* 13. Let $\|\cdot\|_\infty$ denote the operator norm. If the cocycle is uniformly bounded in operator norm (this means there is a constant $K$ such that $\|c(1,x)\|_\infty \le K$ for a.e. $x$) then $\|\Lambda(x)\|_\infty \le K$ as well. Therefore, $\Lambda(x) \in M$ for a.e. $x$.

*Remark* 14. This theorem is a special case of a more general result (Theorem 3.6.2) which allows the cocycle to take values in $\mathrm{GL}^2(M,\tau)$.

*Remark* 15. The reader might wonder whether a stronger form of convergence holds in the theorem above. Namely, whether convergence $\log\Lambda(x) = \lim_{n\to\infty}\log\left([c(n,x)^*c(n,x)]^{1/2n}\right)$ occurs in operator norm. The answer is 'no'. We provide an explicit example of this in §3.2 below with $M = \mathrm{L}^\infty(Y,\nu)$.

**Conjecture 1.** *For a.e. $x$, $\log\left([c(n,x)^*c(n,x)]^{1/2n}\right)$ converges to $\log\Lambda(x)$ almost uniformly in the sense of [Pad67]. This means that for every $\epsilon > 0$ there exists a closed subspace $S \subset L^2(M,\tau)$ such that the projection operator $p_S \in M$, $\tau(p_S) > 1-\epsilon$ and*

$$\lim_{n\to\infty}\log\left([c(n,x)^*c(n,x)]^{1/2n}\right)p_S = \log(\Lambda(x))p_S$$

*in operator norm.*

### Oseledets subspaces and Lyapunov distribution

One of the main advantages of working with a tracial von Neumann algebra $(M,\tau)$ is that if $x \in M$ is normal (this means $xx^* = x^*x$) then $x$ has a spectral measure. If $M = \mathrm{M}_n(\mathbb{C})$ is the algebra of $n\times n$ complex matrices, then the spectral measure is the uniform probability measure on the eigenvalues of $x$ (with multiplicity). To define it more generally, recall that there is a projection-valued measure $E_x$ on the complex plane such that $x = \int \lambda \; dE_x(\lambda)$. The **spectral measure of** $x$ is the composition $\mu_x = \tau\circ E_x$. It is a positive measure with total mass equal to $\tau(\mathrm{id})$. Moreover, if $p$ is any polynomial then $\tau(p(x)) = \int p \; d\mu_x$ where $p(x)$ is defined via spectral calculus.

*Example* 2. If $M = \mathrm{L}^\infty(Y, \nu)$ then every operator $\phi \in M$ is normal. The spectral measure of $\phi$ is its distribution $\mu_\phi$ defined by

$$\mu_\phi(R) = \nu(\{y \in Y : \ \phi(y) \in R\})$$

for all measurable regions $R \subset \mathbb{C}$.

This definition of spectral measure extends to $x \in \mathrm{L}^0(M, \tau)$. In the context of Theorem 3.1.1, we define the **Lyapunov distribution** to be the spectral measure $\mu_{\log \Lambda(x)}$ of the log limit operator $\log \Lambda(x)$. If $M = \mathrm{M}_n(\mathbb{C})$ is the algebra of $n \times n$ complex matrices and $\tau$ is the usual trace then this definition agrees with the previous definition.

To further justify this definition, we recall the notion of von Neumann dimension. If $S \subset \mathrm{L}^2(M, \tau)$ is a closed subspace and the orthogonal projection operator $p_S$ lies in $M$ then the **von Neumann dimension of** $S$ is $\dim_M(S) = \tau(p_S)$. For example, the vN-dimension of $\mathrm{L}^2(M, \tau)$ itself is $\tau(\mathrm{id})$. This notion of dimension satisfies many desirable properties such as being additive under direct sums and continuous under increasing and decreasing limits [Lüc02].

*Example* 3. If $M = \mathrm{L}^\infty(Y, \nu)$ and if $p \in M$ is a projection operator then there is a measurable subset $Z \subset Y$ such that $p$ is the characteristic function $p = 1_Z$ and the range of $p$ is the space of all $\mathrm{L}^2$-functions with support in $Z$. The vN-dimension of this space is the measure $\nu(Z)$.

Let

$$\mathcal{H}_t(x) = 1_{(-\infty,t]}(\log \Lambda(x))(\mathrm{L}^2(M, \tau)) \subset \mathrm{L}^2(M, \tau)$$

where $1_{(-\infty,t]}(\log \Lambda(x))$ is defined via functional calculus. Alternatively, $\mathcal{H}_t(x)$ is the range of the projection $E_{\log \Lambda(x)}(-\infty, t]$. This is analogous to the Oseledets subspaces defined previously.

**Theorem 3.1.2.** *[Invariance principle] With notation as above, for a.e. $x \in X$ and every $t \in [0, \infty)$,*

$$c(1, x)\mathcal{H}_t(x) = \mathcal{H}_t(fx), \quad \mu_{\log \Lambda(x)} = \mu_{\log \Lambda(fx)}.$$

**Fuglede-Kadison determinants**

The **Fuglede-Kadison determinant** of an arbitrary $x \in M$ is defined by

$$\Delta(x) = \exp\left(\int_0^\infty \log(\lambda)\ d\mu_{|x|}(\lambda)\right)$$

where $|x| = (x^*x)^{1/2}$ is a positive operator defined via the spectral calculus. The FK-determinant is multiplicative in the sense that $\Delta(ab) = \Delta(a)\Delta(b)$ [FK52]. From [HS07] it follows the definition of FK-determinant extends to operators in $\mathrm{GL}^2(M,\tau)$ and therefore can be applied to the limit operator $\Lambda(x)$.

*Example* 4. If $M = \mathrm{L}^\infty(Y,\nu)$ then the FK-determinant of a function $\phi \in M$ is $\exp \int \log|\phi(y)|\,\mathrm{d}\nu(y)$.

**Theorem 3.1.3.** *With notation as above, for a.e. $x \in X$,*

$$\lim_{n\to\infty} (\Delta|c(n,x)|)^{1/n} = \Delta\Lambda(x).$$

**Growth rates**

Assume the notation of Theorem 3.1.1.

**Definition 10.** Given $\xi \in \mathrm{L}^2(M,\tau)$, let $\Sigma(\xi)$ be the set of all sequences $(\xi_n)_n \subset \mathrm{L}^2(M,\tau)$ such $\lim_{n\to\infty} \|\xi - \xi_n\|_2 = 0$. Define the **upper and lower smooth growth rates of the system** $(X,\mu,f,c)$ **with respect to** $\xi$ **at** $x \in X$ by

$$\begin{aligned}
\underline{\mathrm{Gr}}(x|\xi) &= \inf\big\{ \liminf_{n\to\infty} \|c(n,x)\xi_n\|_2^{1/n} :\ (\xi_n)_n \in \Sigma(\xi)\big\}\\
\overline{\mathrm{Gr}}(x|\xi) &= \inf\big\{ \limsup_{n\to\infty} \|c(n,x)\xi_n\|_2^{1/n} :\ (\xi_n)_n \in \Sigma(\xi)\big\}.
\end{aligned}$$

**Theorem 3.1.4.** *Assume the hypotheses of Theorem 3.1.1. Then for a.e. $x \in X$ and every $\xi \in L^2(M,\tau)$,*

$$\underline{\mathrm{Gr}}(x|\xi) = \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n} = \overline{\mathrm{Gr}}(x|\xi).$$

*Remark* 16. In §3.2 we give an explicit example in which a strict inequality

$$\liminf_{n\to\infty} \|c(n,x)\xi\|_2^{1/n} > \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n}$$

occurs.

**Conjecture 2.** *Theorem 3.1.4 can be strengthened to: for a.e. $x \in X$ there exists an essentially dense subspace $\mathcal{H}_x \subset L^2(M,\tau)$ such that for every $\xi \in \mathcal{H}_x$,*

$$\lim_{n\to\infty} \|c(n,x)\xi\|_2^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n}.$$

*Essentially dense subspaces are reviewed in §3.6.4.*

*Remark* 17. In §3.6.5, we prove the conjecture with lim inf in place of lim. To be precise: for a.e. $x \in X$ there exists an essentially dense subspace $\mathcal{H}_x \subset \mathrm{L}^2(M,\tau)$ such that for every $\xi \in \mathcal{H}_x$,

$$\liminf_{n\to\infty} \|c(n,x)\xi\|_2^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n}.$$

### 3.1.6 The abelian case

As in previous examples, suppose $M = \mathrm{L}^\infty(Y,\nu)$. In §3.2, we show that with this choice of $(M,\tau)$, Theorem 3.1.1 follows readily from von Neumann's mean ergodic theorem. Moreover, Conjectures 1 and 2 follow from Birkhoff's pointwise ergodic theorem. We also provide explicit examples where the limit operator $\Lambda(x)$ has continuous spectrum, where convergence to the limit operator does not occur in operator norm, and where there exist vectors $\xi$ satisfying the strict inequality

$$\liminf_{n\to\infty} \|c(n,x)\xi\|_2^{1/n} > \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n}.$$

The section §3.2 can be read independently of the rest of the paper.

### 3.1.7 Powers of a single operator

As above, let $(M,\tau)$ be a finite von Neumann algebra and let $T \in M$. It is a famous open problem to determine whether $T$ admits a proper invariant subspace. The main results of [HS09] show that the limit $\lim_{n\to\infty} |T^n|^{1/n} = \Lambda$ exists in the strong operator topology (SOT) and moreover, if $\mathcal{H}_t = 1_{[0,t]}(\Lambda)(\mathrm{L}^2(M,\tau))$ then $\mathcal{H}_t$ is an invariant subspace. The spectral measure of $\Lambda$ is the same as the Brown measure of $T$ radially projected to the positive real axis. Moreover, if the Brown measure of $T$ is not a Dirac mass then there exists a proper invariant subspace.

Now suppose that $T$ has a bounded inverse $T^{-1} \in M$. Regardless of the dynamics, we may choose to define the cocycle $c$ by $c(n,x) = T^n$. Theorems 3.1.1 and 3.1.4 then recover the main results of [HS09] with the exception that our results say nothing of the Brown measure and they only apply to the invertible case. Our methods are completely different. In particular, we do not use [HS09].

### 3.1.8 Proof overview

We will make use of a general Multiplicative Ergodic Theorem due to Karlsson-Margulis based on non-positive curvature (see also [Kau87] which seems to be the first paper that develops this geometric approach). To accommodate their cocycle convention (which is different than ours), let us say that a measurable map $\check{c} : \mathbb{N} \times X \to G$ is a **reverse cocycle** if

$$\check{c}(n+m,x) = \check{c}(n,x)\check{c}(m,f^n x)$$

for any $n,m \in \mathbb{N}$ (where $G$ is a group).

The following is a special case of the Karlsson-Margulis Theorem.

**Theorem 3.1.5** ([KM99])**.** *Let $(X,\mu)$ be a standard probability space, $f : X \to X$ an ergodic measure-preserving invertible transformation, $(Y,d)$ a complete CAT(0) space, $y_0 \in Y$ and $\check{c} : \mathbb{N} \times X \to \mathrm{Isom}(Y,d)$ a measurable reverse cocycle taking values in the isometry group of $(Y,d)$, where measurable means with respect to the compact-open topology on* $\mathrm{Isom}(Y,d)$*. Assume that*

$$\int_X d(y_0, \check{c}(1,x)y_0)\ d\mu(x) < \infty.$$

*Then for almost every $x \in X$, the following limit exists:*

$$\lim_{n\to\infty} \frac{d(y_0, \check{c}(n,x)y_0)}{n} = D.$$

*Moreover, if $D > 0$ then for almost every $x$ there exists a unique unit-speed geodesic ray*

*$\gamma(\cdot, x)$ in $Y$ starting at $y_0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} d(\gamma(Dn, x), \check{c}(n, x)y_0) = 0.$$

As remarked in [KM99], this result implies the classical MET as follows. Let $P(n, \mathbb{R})$ be the space of positive definite $n \times n$ matrices. Then $\mathrm{GL}(n, \mathbb{R})$ acts on $P(n, \mathbb{R})$ by $g.p := gpg^*$. The tangent space to $p \in P(n, \mathbb{R})$, denoted $T_p(P(n, \mathbb{R}))$, is naturally identified with $S(n, \mathbb{R})$, the space of $n \times n$ real symmetric matrices. Define an inner product on $T_p(P(n, \mathbb{R}))$ by

$$\langle x, y \rangle_p := \mathrm{trace}(p^{-1}xp^{-1}y).$$

This gives a complete Riemannian metric on $P(n, \mathbb{R})$. All sectional curvatures are non-positive and so $P(n, \mathbb{R})$ is CAT(0). Moreover the $\mathrm{GL}(n, \mathbb{R})$ action is isometric and transitive. Every geodesic ray from $I$ (the identity matrix) has the form $t \mapsto \exp(tx)$ for $x \in S(n, \mathbb{R})$.

Substitute $Y = P(n, \mathbb{R})$ and $y_0 = I$ (the identity matrix) in the Karlsson-Margulis Theorem to obtain the classical multiplicative ergodic theorem.

Our proof of Theorem 3.1.1 follows in a similar way from the Karlsson-Margulis Theorem. In [AL06], Andruchow and Larotonda construct a Riemannian metric on the positive cone $\mathcal{P}^\infty(M)$ of a finite von Neumann algebra. They prove that it is non-positively curved. We go over the needed facts from their construction in §3.5.1.

However, $\mathcal{P}^\infty(M)$ is not metrically complete. We prove that its metric completion can naturally be identified with the log-square-integrable positive operators in the affiliated algebra of the von Neumann algebra and that a natural subgroup of its isometry group acts transitively.

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## 3.2 The abelian case

As in §3.1.4, let $M = \mathrm{L}^\infty(Y,\nu)$ and define the trace $\tau$ on $M$ by $\tau(\phi) = \int \phi \, \mathrm{d}\nu$. This section studies the MET under the hypothesis that the cocycle $c$ takes values in $M$. It serves as motivation and can be read independently of the rest of the paper.

This special case might seem trivial and indeed, we will see that the conclusions of Theorem 3.1.1 are implied by the pointwise ergodic theorem. However, there are curious features not present in previous versions of the MET. Below we will give examples in which $\Lambda(x)$ has continuous spectrum and examples where $|c(n,x)|^{1/n}$ converges in $\mathrm{L}^2$-norm to $\Lambda(x)$ but not in operator norm. We will also show that growth rates do not necessarily exist for every vector, but do exist for an essentially dense subspace of vectors.

### 3.2.1 Theorem 3.1.1 from the pointwise ergodic

Let us assume hypotheses as in Theorem 3.1.1 with $M = \mathrm{L}^\infty(Y,\nu)$. In particular, we have $\log|c(1,x)| \in \mathrm{L}^\infty(Y,\nu) \cap \mathrm{L}^2(Y,\nu)$, and $\|\log|c(1,x)|\|_2^{Y,\nu} \in \mathrm{L}^1(X,\mu)$. For a fixed $(x,y) \in X \times Y$, consider

$$A_n(x,y) = \frac{1}{n}\log|c(n,x)(y)| = \frac{1}{n}\sum_{i=0}^{n-1}\log|c(1,f^i x)(y)|$$

.

$$\|A_n(x)\|_2^{Y,\nu} = \|\frac{1}{n}\log|c(n,x)|\|_2^{Y,\nu}$$

We apply the pointwise ergodic theorem in $X$ to both $A_n(x,y)$ and $\|A_n(x)\|_2^{Y,\nu}$. It then follows by what is known as Scheffe's lemma (which has also been shown by Riesz) that for a.e. $x \in X$, $A_n(x)$ converges in $\mathrm{L}^2(Y,\nu)$. This implies one of the conclusions of Theorem 3.1.1. The other conclusions follow easily because we are in the abelian case.

That we can apply the pointwise ergodic theorem to the second equation above follows directly from the condition $\|\log|c(1,x)|\|_2^{Y,\nu} \in \mathrm{L}^1(X,\mu)$. That we can apply the pointwise ergodic theorem to the first equation follows from the following arguments to show that $\log|c(1,x)(y)| \in L^1(X,\mu)$ for $\nu$-a.e. $y$. First notice that if $(Y,\nu)$ is a finite measure space, then $\log|c(1,x)| \in \mathrm{L}^1(Y,\nu)$ and $\|\log|c(1,x)|\|_1^{Y,\nu} \in L^1(X,\mu)$, so by Fubini's theorem $\log|c(1,x)(y)| \in L^1(X,\mu)$. Now if $(Y,\nu)$ is an infinite measure space, let $\epsilon > 0$ and let (note that $A_1(x,y) = \log|c(1,x)(y)|$) $\tilde{A}_1(x,y) = A_1(x,y)$ if $|A_1(x,y)| > \epsilon$, otherwise let $\tilde{A}_1(x,y) = 0$. Then $\tilde{A}_1(x)$ is supported on a finite measure subspace of $Y$ for every fixed $x$. It follows that for a fixed $y$, $\tilde{A}_1(x,y) \in L^1(X,\mu)$. But $\|A_1(x,y)\|_1^{X,\mu} \le \|\tilde{A}_1(x,y)\|_1^{X,\mu} + \epsilon$, so $A_1(x,y) \in L^1(X,\mu)$.

Similarly, if $M = \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}^\infty(Y,\nu)$ where $\mathrm{M}_n(\mathbb{C})$ denotes the algebra of $n \times n$ complex matrices, then the non-ergodic version of the classical multiplicative ergodic theorem implies the conclusions of Theorem 3.1.1.

### 3.2.2 Examples with continuous spectrum

This example is almost trivial. Let $\psi \in \mathrm{L}^\infty(Y,\nu)$ be such that $\log|\psi| \in \mathrm{L}^2(Y,\nu)$. Define $c(n,x) = \psi^n$. Then the limit operator satisfies $\Lambda(x) = |\psi|$ for a.e. $x$ and the spectral measure of $\Lambda$ is the distribution of $|\psi|$. In particular, if $|\psi|$ has continuous distribution then $\Lambda(x)$ has continuous spectrum.

### 3.2.3 Almost uniform convergence and growth rates

In this subsection, we prove Conjectures 1 and 2 in the special case $M = \mathrm{L}^\infty(Y,\nu)$.

**Theorem 3.2.1.** *Assume hypotheses as in Theorem 3.1.1 with $M = L^\infty(Y,\nu)$ and $\mathcal{H} = L^2(Y,\nu)$. For $Z \subset X$ and $\phi \in L^\infty(Y,\nu)$, let $\phi \upharpoonright Z \in L^\infty(Z, \nu \upharpoonright Z)$ denote the restriction of $\phi$ to $Z$.*

*Then for every $\epsilon > 0$ and a.e. $x \in X$, there exists a measurable subset $Z(x) \subset Y$ with $\nu(Z(x)) > 1 - \epsilon$ such that*

$$\lim_{n\to\infty} n^{-1} \log|c(n,x) \upharpoonright Z(x)| = \log(\Lambda(x) \upharpoonright Z(x))$$

*where convergence is in $L^\infty(Z(x))$. In particular, Conjecture 1 is true if $M = L^\infty(Y,\nu)$.*

*Proof.* Define $F, \phi$ and $A_n$ as in §3.2.1. By Birkhoff's Pointwise Ergodic Theorem, $A_n$ converges pointwise a.e. to $\mathbb{E}[\phi|\mathcal{I}]$ where $\mathcal{I}$ denotes the sigma-algebra of $F$-invariant measurable subsets. Since $\exp A_n(x,y) = |c(n,x)(y)|^{1/n}$, it follows that, for a.e. $(x,y)$, $A_n(x,y)$ converges to $\log \Lambda(x)(y)$.

Define $A_n(x) \in L^1(Y, \nu)$ by $A_n(x)(y) = A_n(x, y)$. By Fubini's Theorem, there exists a subset $X' \subset X$ with full measure such that for a.e. $x \in X'$, $A_n(x)$ converges pointwise a.e. (as $n \to \infty$) to $\log \Lambda(x)$.

By Egorov's Theorem, for every $x \in X'$ there exists a measurable subset $Z(x) \subset Y$ with $\nu(Z(x)) \geq 1 - \epsilon$ such that $A_n(x)$ converges uniformly to $\log \Lambda(x)$ on $Z(x)$. $\square$

**Proposition 3.2.2.** *With notation as above, suppose that for some $x \in X$ and $Z(x) \subset Y$ that,*

$$\lim_{n\to\infty} n^{-1} \log |c(n, x) \restriction Z(x)| = \log(\Lambda(x) \restriction Z(x))$$

*where convergence is in $L^\infty(Z(x))$. Then for every $\xi \in L^2(Y, \nu)$ with support in $Z(x)$*

$$\lim_{n\to\infty} \|c(n, x)\xi\|_2^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n \xi\|_2^{1/n} = \|\Lambda(x) 1_{support(\xi)}\|_\infty. \tag{3.1}$$

*In particular, Conjecture 2 is true if $M = L^\infty(Y, \nu)$. Also, if $n^{-1} \log |c(n, x)| \to \log \Lambda(x)$ in operator norm, then (3.1) holds for all $\xi \in L^2(Y, \nu)$.*

*Proof.* Let $\xi \in \mathrm{L}^2(Y, \nu)$. We first prove $\lim_{n\to\infty} \|\Lambda(x)^n \xi\|_2^{1/n} = \|\Lambda(x) 1_{support(\xi)}\|_\infty$.

Without loss of generality, we may assume $\|\xi\|_2 = 1$. Recall that $\lim_{n\to\infty} \|\phi\|_n = \|\phi\|_\infty$ for $\phi \in \mathrm{L}^\infty$. So tends to $\|\Lambda(x)\|_{\mathrm{L}^\infty(Y, |\xi|^2 \mathrm{d}\nu)}$ as $n \to \infty$. The latter is the same as $\|\Lambda(x) 1_{support(\xi)}\|_{\mathrm{L}^\infty(Y,\nu)}$.

Now suppose $\xi$ has support in $Z(x)$. Then

$$\|c(n, x)\xi\|_2 \;\; \leq \;\; \|\Lambda(x)^n \xi\|_2 \left\| \left( y \in Z(x) \mapsto \frac{|c(n, x)(y)|}{\Lambda(x)^n(y)} \right) \right\|_{\mathrm{L}^\infty(Z(x),\nu)}.$$

Since $n^{-1} \log |c(n, x) \restriction Z(x)|$ converges to $\log \Lambda(x)$ uniformly on $Z(x)$, this implies

$$\limsup_{n\to\infty} \|c(n, x)\xi\|_2^{1/n} \leq \lim_{n\to\infty} \|\Lambda(x)^n \xi\|_2^{1/n}.$$

Similarly,

$$\|\Lambda(x)^n \xi\|_2 \;\; \leq \;\; \|c(n, x)\xi\|_2 \left\| \left( y \in Z(x) \mapsto \frac{\Lambda(x)^n(y)}{|c(n, x)(y)|} \right) \right\|_{\mathrm{L}^\infty(Z(x),\nu)}.$$

So $\liminf_{n\to\infty} \|c(n,x)\xi\|_2^{1/n} \geq \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_2^{1/n}$. This proves (3.1).

Next we prove Conjecture 2. In general, a subspace $S \subset \mathrm{L}^2(M,\tau)$ is **essentially dense** if for every $\epsilon > 0$ there exists a projection $p \in M$ with range contained in $S$ such that $\tau(1-p) < \epsilon$. In the special case $M = \mathrm{L}^\infty(Y,\nu)$ this means a subspace $S \subset \mathrm{L}^2(Y,\nu)$ is essentially dense if and only if for every $\epsilon > 0$ there is a subset $Z \subset Y$ with $\nu(Z) > 1-\epsilon$ such that $S$ contains all functions with support in $Z$. So the Conjecture 2 follows from Theorem 3.2.1 and (3.1).

□

*Remark* 18. The same result holds if $M = \mathrm{M}_n(\mathbb{C})\otimes\mathrm{L}^\infty(Y,\nu)$ with essentially the same proof. One needs only use the non-ergodic version of Oseledet's Multiplicative Ergodic Theorem instead of Birkhoff's Pointwise Ergodic Theorem.

### 3.2.4 A counterexample

**Theorem 3.2.3.** *There exist standard probability spaces $(X,\mu),(Y,\nu)$, an ergodic pmp invertible transformation $f : X \to X$, a measurable cocycle $c : \mathbb{Z}\times X \to M = L^\infty(Y,\nu)$ satisfying the hypotheses of Theorem 3.1.1 and a vector $\xi \in L^2(Y,\nu)$ such that*

$$\limsup_{n\to\infty} \|c(n,x)\xi\|_{L^2(Y,\nu)}^{1/n} = \lim_{n\to\infty} \|c(n,x)\|_{L^2(Y,\nu)}^{1/n} > \lim_{n\to\infty} \|\Lambda(x)^n\xi\|_{L^2(Y,\nu)}^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n\|_{L^2(Y,\nu)}^{1/n}.$$

*Moreover, we can choose the cocycle so that $\|c(1,x)\|_\infty \leq C$ for some constant $C$ and a.e. $x$. Moreover, $n^{-1}\log|c(n,x)|$ does not converge to $\log\Lambda(x)$ in operator norm (for a.e. $x$).*

*Proof.* Let $X = \mathbb{Z}_2$ be the compact group of 2-adic integers. An element of $\mathbb{Z}_2$ is written as a formal sum $x = \sum_{i=0}^\infty x_i 2^i$ with $x_i \in \{0,1\}$ and the usual multiplication and addition rules. Let $\mu$ be the Haar probability measure on $X$. There is a bijection between $X$ and $\{0,1\}^{\mathbb{N}\cup\{0\}}$ given by $x \mapsto (x_0,x_1,\ldots)$. This bijection maps the measure $\mu$ to the $(\mathbb{N}\cup\{0\})$-th power of the uniform measure on $\{0,1\}$.

Define $f : X \to X$ by $f(x) = x+1$. It is well-known that a translation on a compact abelian group is ergodic if and only if every orbit is dense. Thus $f$ is an ergodic measure-preserving transformation. Alternatively, $f$ is the standard odometer which is well-known to be ergodic.

Let $(Y, \nu)$ be a probability space that is isomorphic to the unit interval with Lebesgue measure. Let $Y = \sqcup_{n=1}^{\infty} Y_n$ be a partition of $Y$ into positive measure subsets. We will choose the partition more carefully later. Define the cocycle $c : \mathbb{Z} \times X \to \mathrm{L}^{\infty}(Y, \nu)$ by

$$c(1, x)(y) = \begin{cases} 1 & \text{if } y \in Y_m \text{ for some } m \text{ and } x_m = 0 \\ 2 & \text{otherwise} \end{cases}$$

This extends to a cocycle via $c(n, x) = c(1, f^{n-1}x) \cdots c(1, fx)c(1, x)$.

For every $y \in Y$,

$$\int \log c(1, x)(y)\ d\mu(x) = (1/2) \log(2).$$

Since $f$ is ergodic, it follows that the limit operator $\Lambda(x)$ defined by $\log \Lambda(x) = \lim_{n\to\infty} n^{-1} \log c(n, x)$ (where convergence is in $\mathrm{L}^2$ and pointwise a.e.) is the constant function $\Lambda(x) = \sqrt{2}$ for a.e. $x$.

For $n, m \in \mathbb{N}$, let

$$S_{n,m} = \{x \in X :\ x_m = 1, x_n = 0\}.$$

We claim that if $x \in S_{n,m}$ and $n < m$ then $c(n, x)(y) = 2^n\ \forall y \in Y_m$. Indeed,

$$2^m \leq \sum_{i=0}^{m} x_i 2^i \leq 2^{m+1} - 2^n - 1.$$

Therefore, $c(1, f^k x)(y) = c(1, x + k)(y) = 2$ for all $0 \leq k \leq 2^n - 1$ and

$$c(n, x) = c(1, f^{n-1}x) \cdots c(1, x) = 2^n.$$

Note that $\mu(S_{n,m}) = 1/4$. Moreover, if $n_1 \neq n_2$ and $m_1 \neq m_2$ then $S_{n_1,m_1}$ and $S_{n_2,m_2}$ are independent events. It follows that if $T_n = S_{n,n+10}$ then the events $\{T_n\}_{n=1}^{\infty}$ are jointly independent and, by Borel-Cantelli, a.e. $x$ is contained in infinitely many of the sets $T_n$.

If $x \in T_n$ then

$$\|c(n, x)1_Y\|^2_{\mathrm{L}^2(Y,\nu)} \geq \|c(n, x)1_{Y_{n+10}}\|^2_{\mathrm{L}^2(Y,\nu)} \geq \nu(Y_{n+10})2^{2n}.$$

We could choose the subsets $\{Y_m\}$ so that $\nu(Y_m) \geq Cm^{-2}$ for some constant $C$. With this choice and $x \in T_n$,

$$\|c(n,x)1_Y\|^2_{\mathrm{L}^2(Y,\nu)} \geq C2^{2n}/(n+10)^{-2}.$$

Since a.e. $x$ is contained in infinitely many $T_n$'s it follows that

$$\limsup_{n\to\infty} \|c(n,x)1_Y\|^{1/n}_{\mathrm{L}^2(Y,\nu)} = 2.$$

On the other hand,

$$\lim_{n\to\infty} \|\Lambda(x)^n 1_Y\|^{1/n}_{\mathrm{L}^2(Y,\nu)} = \sqrt{2}.$$

This proves the theorem with $\xi = 1_Y$. By Theorem 3.2.1, $|c(n,x)|^{1/n}$ does not converge to $\Lambda(x)$ in operator norm (for a.e. $x$). $\square$

*Remark* 19. The essential phenomena behind this counterexample is that there is no uniform rate of convergence in the pointwise ergodic theorem. Precisely, while $\frac{1}{n}\sum_{k=0}^{n-1}\log c(1,f^k x)(y)$ converges to $\log(2)/2$ for every $y$ and a.e. $x$, the convergence is not uniform in $y$.

## 3.3 Preliminaries

Throughout these notes, by a *tracial von Neumann algebra* we mean a pair $(M,\tau)$ where $M$ is a von Neumann algebra, and $\tau$ is a faithful, tracial state which is ultraweakly continuous. By a *semifinite von Neumann algebra* we mean a pair $(M,\tau)$ where $M$ is a von Neumann algebra and $\tau$ is a faithful, ultraweakly continuous (or normal?), and semifinite. For concreteness, we consider $M$ to be a sub-algebra of the algebra $B(\mathcal{H})$ of all bounded operators on a separable Hilbert space $\mathcal{H}$. We will consider many constructions that depend on the choice of trace $\tau$ but we suppress this dependence from the notation.

### 3.3.1 Spectral measures

Suppose $x$ is a (bounded or unbounded) self-adjoint operator on $\mathcal{H}$. By the Spectral Theorem ([RS80, Theorem VIII.6]), there exists a projection valued measure $E_x$ on the real line such that

$$x = \int \lambda \, \mathrm{d}E_x(\lambda).$$

The support of $E_x$ is contained in the spectrum of $x$. The projections of the form $E_x(R)$ (for Borel sets $R \subset \mathbb{R}$) are the **spectral projections of** $x$. If $f : \mathbb{R} \to \mathbb{R}$ is Borel then $f(x)$ is a self-adjoint operator on $\mathcal{H}$ defined by

$$f(x) := \int f(\lambda) \, \mathrm{d}E_x(\lambda).$$

In the case of unbounded $x$, $f(x)$ has the same domain as $x$. The **absolute value** of $x$ is defined by $|x| = (x^*x)^{1/2} = \int \sqrt{\lambda} \, \mathrm{d}E_{x^*x}(\lambda)$ and is equal to $\int |\lambda| \, \mathrm{d}E_x(\lambda)$.

If $x$ is such that all of its spectral projections lie in the von Neumann algebra $M$, then the composition $\tau \circ E_x$ is a Borel probability measure on $\mathbb{C}$ called the **spectral measure of** $x$ and denoted by $\mu_x$. In particular, if $x \in M$ then $\mu_x$ is well-defined.

*Example* 5 (The abelian case). If $M = \mathrm{L}^\infty(Y, \nu)$ (as in §3.1.4), then every operator $\phi \in M$ is self-adjoint if and only if it is real-valued. Then projection-valued measure $E_\phi$ satisfies $E_\phi(R)$ is the projection onto the subspace of $\mathrm{L}^2$-functions with support in $\phi^{-1}(R)$ (for Borel $R \subset \mathbb{R}$) and $\mu_\phi = \phi_* \nu$ is the distribution of $\phi$.

### 3.3.2 Polar decomposition

We will frequently have to use the polar decomposition, see [RS80, Theorem VIII.32]. We restate it here.

**Proposition 3.3.1.** *Let $x$ be a closed densely defined operator on $\mathcal{H}$. Then there is a positive self-adjoint operator $|x|$ with* $\mathrm{dom}(|x|) = \mathrm{dom}(x)$ *and a partial isometry $u$ with initial space* $\ker(x)^\perp$ *and final space* $\overline{\mathrm{Im}(x)}$ *so that $x = u|x|$ (where* $\mathrm{Im}(x)$ *denotes the image of $x$). Moreover $|x|$ and $u$ are uniquely determined by these properties together with the additional condition* $\ker(|x|) = \ker(x)$.

The expression $x = u|x|$ is called the **polar decomposition** of $x$.

### 3.3.3 The regular representation

Recall from the introduction that $\mathrm{L}^2(M,\tau)$ is the Hilbert space completion of $\mathcal{N}$ with respect to the inner product defined on $M$ by

$$\langle x, y\rangle = \tau(x^* y).$$

Let $\|x\|_2 = \langle x, x\rangle^{1/2}$ and $\|x\|_\infty$ be the operator norm of $x$ (as an operator on $\mathcal{H}$).

For any $x, y \in M$,

$$\|xy\|_2 \leq \|x\|_\infty \|y\|_2 \text{ and } \|xy\|_2 \leq \|x\|_2 \|y\|_\infty.$$

(e.g., [Dix81, Part I, Chapter 6, Theorem 8]). Therefore, the operator $L_x : \mathcal{N} \to \mathcal{N}$ defined by $L_x(y) = xy$ admits a unique continuous extension from $\mathrm{L}^2(M,\tau)$ to itself. Moreover, the operator norm of $L_x$ is bounded by $\|x\|_\infty$. In fact, they are equal. This follows, for example, from [AP16, Proposition 8.2.2]. Similarly, the map $R_x : \mathcal{N} \to \mathcal{N}$ defined by $R_x(y) = yx$ admits a unique continuous extension to $\mathrm{L}^2(M,\tau)$ and the operator norm of $R_x$ is $\|x\|_\infty$.

We will identify $M$ with its image $\{L_x : \ x \in M\}$ (viewed as a sub-algebra of the algebra of bounded operators on $\mathrm{L}^2(M,\tau)$).

### 3.3.4 The algebra of affiliated operators

**Definition 11** ($\mathrm{L}^0(M,\tau)$)**.** The **commutant** of $M$, denoted $M'$, is the algebra of bounded operators $y$ on $\mathrm{L}^2(M,\tau)$ such that $xy = yx$ for all $x \in M$. An unbounded operator $x$ on $\mathrm{L}^2(M,\tau)$ is **affiliated** with $M$ if for every unitary $u \in M'$, $xu = ux$. Let $\mathrm{L}^0(M,\tau)$ denote the set of closed densely defined operators affiliated with $M$. By [AP16, Proposition 7.2.3], if $x$ is a closed densely defined operator and $x = u|x|$ is its polar decomposition, then $x \in \mathrm{L}^0(M,\tau)$ if and only if $u$ and the spectral projections of $|x|$ are in $M$. By [AP16, Theorem 7.2.8], $\mathrm{L}^0(M,\tau)$ is closed under adjoint, addition and multiplication and is a $*$-algebra under these operations.

**Domains**

For $x \in \mathrm{L}^0(M,\tau)$ we write $\mathrm{dom}(x) \subset \mathrm{L}^2(M,\tau)$ for its domain. We remark now that for $a, b \in \mathrm{L}^0(M,\tau)$ the sum $a+b$ is defined as the *closure* of the operator $T$ with $\mathrm{dom}(T) = \mathrm{dom}(a) \cap \mathrm{dom}(b)$ and with $T\xi = a\xi + b\xi$ for $\xi \in \mathrm{dom}(T)$. Similarly $ab$ is defined as the *closure* of the operator $T$ with domain $b^{-1}(\mathrm{dom}(a)) \cap \mathrm{dom}(a)$ and $T\xi = a(b\xi)$ for $\xi \in \mathrm{dom}(T)$. Thus, for example, the domain of $ab$ is often larger than $b^{-1}(\mathrm{dom}(a)) \cap \mathrm{dom}(b)$. This will occasionally cause us some headaches, and we will try to remark when it actually presents an issue. Regardless, this paragraph should be taken as a blanket warning that $ab$ is not literally defined to be the composition, and $a+b$ is not the literal sum.

$\mathbf{L}^2(M,\tau) \subset \mathbf{L}^0(M,\tau)$

We can include $\mathrm{L}^2(M,\tau)$ in $\mathrm{L}^0(M,\tau)$ as follows. For $x \in \mathrm{L}^2(M,\tau)$ and $y \in M$, define $L_x^0(y) = R_y(x) = xy$. Then $L_y^0$ is closable but not bounded in general. Let $L_y$ denote the closure of $L_y^0$. The map $y \mapsto L_y$ defines a linear bijection from $\mathrm{L}^2(M,\tau)$ into $\mathrm{L}^0(M,\tau)$. By abuse of notation, we will identify $\mathrm{L}^2(M,\tau)$ with its image in $\mathrm{L}^0(M,\tau)$. While $\mathrm{L}^2(M,\tau)$ is a subspace of $\mathrm{L}^0(M,\tau)$, it is not a sub-algebra in general.

For $x \in \mathrm{L}^0(M,\tau)$ we set $|x| = (x^*x)^{1/2}$ and

$$\|x\|_2 = \left(\int t^2 \, d\mu_{|x|}(t)\right)^{1/2} \in [0,\infty].$$

Then $\mathrm{L}^2(M,\tau)$ is identified with the set of all $x \in \mathrm{L}^0(M,\tau)$ which have $\|x\|_2 < \infty$.

**Extending the adjoint**

The anti-linear map $x \mapsto x^*$ on $M$ uniquely extends to an anti-linear isometry $J : \mathrm{L}^2(M,\tau) \to \mathrm{L}^2(M,\tau)$. By [AP16, Proposition 7.3.3], if $x \in \mathrm{L}^2(M,\tau)$ then the following are equivalent: (1) $x$ is self-adjoint, (2) $Jx = x$, (3) $x$ is in the $\mathrm{L}^2$-closure of $M_{sa} = \{y \in M : \ y^* = y\}$. Let $\mathrm{L}^2(M,\tau)_{sa} = \{x \in \mathrm{L}^2(M,\tau) : \ Jx = x\}$.

**Invertible affiliated operators**

We say an operator $x \in \mathrm{L}^0(M,\tau)$ is invertible if there exists an operator $y \in \mathrm{L}^0(M,\tau)$ such that $xy = yx = 1$ where, following our abuse of notation, $xy$ and $yx$ denote the closures of the compositions of the operators $x$ and $y$. In this case we write $y = x^{-1}$. Let $\mathrm{L}^0(M,\tau)^\times \subset \mathrm{L}^0(M,\tau)$ be the set of invertible affiliated operators $x$.

**Lemma 3.3.2.** *If $(M,\tau)$ is semi-finite and $x \in L^0(M,\tau)^\times$ has polar decomposition $x = u|x|$ then $u$ is unitary, $|x| \in L^0(M,\tau)^\times$ and $x^* \in L^0(M,\tau)^\times$ with $(x^*)^{-1} = (x^{-1})^*$. If $(M,\tau)$ is finite then $x \in L^0(M,\tau)$ is invertible if and only if it is injective.*

*Proof.* Because $u$ is a partial isometry, $u^*u$ is the orthogonal projection onto $\ker(u)^\perp$. If $x$ is invertible, then $u$ is injective, so $u^*u = 1$. Similarly, $uu^*$ is projection onto the closure of the image of $u$. So if $x$ is invertible then $uu^* = 1$. This proves $u$ is unitary.

Because $x$ is injective, the equality $\|x\xi\| = \||x|\xi\|$ for $\xi \in \mathrm{dom}(x)$ implies that $|x|$ is injective. Thus $1_{\{0\}}(|x|) = p_{\ker(|x|)} = 0$, and so $|x|^{-1}$ may be defined as a closeable operator in $\mathrm{L}^0(M,\tau)$. The computation $(x^*)^{-1} = (x^{-1})^*$ is straightforward.

Now suppose $(M,\tau)$ is finite. Without loss of generality, $\tau(1) = 1$. Suppose $x$ is injective. Because $u$ also injective, $u^*u = 1$. Because $\tau(1) = 1$, $1 = \tau(u^*u) = \tau(u^*u)$. So $u^*u = 1$ and $u$ is unitary. As above, $|x|$ is invertible. So $|x|^{-1}u^*$ is an inverse to $x$.

□

## 3.4 The log-square integrable general linear group

Given a semifinite von Neumann algebra $(M,\tau)$, let

$$\mathrm{GL}^2(M,\tau) = \{a \in \mathrm{L}^0(M,\tau)^\times : \log(|a|) \in \mathrm{L}^2(M,\tau)\}$$

be the **log-square integrable general linear group** of $(M,\tau)$. For brevity we will write $G = \mathrm{GL}^2(M,\tau)$. Although we call this set a group, it is not at all obvious that $G$ is closed under multiplication. The main result of this section is:

**Theorem 3.4.1.** *$G$ is a subgroup of $L^0(M,\tau)^\times$. Moreover, for every $a \in G$ we have that $a^* \in G$ and additionally:*

$$\|\log(|a|)\|_2 = \|\log(|a^*|)\|_2 = \|\log(|a^{-1}|)\|_2.$$

We start with some basic facts about spectral measures.

**Proposition 3.4.2.** *Let $(M,\tau)$ be a semifinite von Neumann algebra and $a \in L^0(M,\tau)^\times$. Then:*

1. $\mu_{|a|} = \mu_{|a^*|}$,
2. $\mu_{|a^{-1}|} = \mu_{|a|^{-1}} = r_*(\mu_{|a|})$, *where $r\colon (0,\infty) \to (0,\infty)$ is the map $r(t) = t^{-1}$.*

*Proof.* (1): Let $a = u|a|$ be the polar decomposition. Since $a \in \mathrm{L}^0(M,\tau)^\times$ by Lemma 3.3.2 we have that $u \in \mathcal{U}(M)$ (which is the unitary group of $M$). Then $a^* = |a|u^*$, and $|a^*|^2 = aa^* = u|a|^2u^*$. From this, it is easy to see that $|a^*| = u|a|u^*$, because $(u|a|u^*)^2 = u|a|^2u^*$ Thus, for every Borel $E \subseteq [0,\infty)$

$$\mu_{|a^*|}(E) = \tau(1_E(|a^*|)) = \tau(1_E(u|a|u^*)) = \tau(u1_E(|a|)u^*) = \tau(1_E(|a|)) = \mu_{|a|}(E).$$

(2): Again, let $a = u|a|$ be the polar decomposition. So $a^{-1} = |a|^{-1}u^*$. As in (1), it is direct to show that $|a^{-1}| = u|a|^{-1}u^*$. The proof then proceeds exactly as in (1), using that $r_*(\mu_{|a|}) = \mu_{|a|^{-1}}$ (which follows from functional calculus).

□

Because expressions like $1_{(\lambda,\infty)}(|a|)(\mathrm{L}^2(M,\tau))$ will show up frequently, it will be helpful to introduce the following notation. Given $a \in \mathrm{L}^0(M,\tau)$ and $E \subseteq [0,\infty)$ Borel, we let $\mathcal{H}^a_E = 1_E(|a|)(\mathrm{L}^2(M,\tau))$. It will be helpful for us to derive an alternate expression for $\|\log(|a|)\|_2$. We have that

$$\begin{aligned} \|\log(|a|)\|_2^2 &= \int_0^\infty t^2 \,\mathrm{d}\mu_{|\log(|a|)|}(t) \\ &= \int_0^\infty \left(\int_0^t 2\lambda \,\mathrm{d}\lambda\right) \mathrm{d}\mu_{|\log(|a|)|}(t) \\ &= 2\int_0^\infty \lambda \int_0^\infty 1_{(\lambda,\infty)}(t)\,\mathrm{d}\mu_{|\log(|a|)|}(t)\,\mathrm{d}\lambda = 2\int_0^\infty \lambda\mu_{|\log(|a|)|}(\lambda,\infty)\,\mathrm{d}\lambda. \quad (3.2) \end{aligned}$$

Note that we have used Fubini's theorem, which is valid since semifiniteness of $\tau$ implies that the associated spectral measure is $\sigma$-finite.

By functional calculus, $\mu_{|\log(|a|)|}$ is the pushforward of $\mu_{|a|}$ under the map $t \mapsto |\log(t)|$. So

$$\|\log(|a|)\|_2^2 = 2\int_0^\infty \lambda\left[\mu_{|a|}(e^\lambda,\infty) + \mu_{|a|}(0,e^{-\lambda})\right]\,\mathrm{d}\lambda.$$

By Proposition 3.4.2,

$$\|\log(|a|)\|_2^2 = 2\int_0^\infty \lambda[\mu_{|a|}(e^\lambda,\infty) + \mu_{|a^{-1}|}(e^\lambda,\infty)]\,\mathrm{d}\lambda. \tag{3.3}$$

**Proposition 3.4.3.** *Let $(M,\tau)$ be a semifinite von Neumann algebra and $a, b \in L^0(M,\tau)$. Given $\lambda_1, \lambda_2, \lambda \in (0,\infty)$ with $\lambda_1\lambda_2 = \lambda$, we have that*

$$\tau(1_{(\lambda,\infty)}(|ab|)) \leq \tau(1_{(\lambda_1,\infty)}(|a|)) + \tau(1_{(\lambda_2,\infty)}(|b|)).$$

*Proof.* The proof is almost immediate from [FK86, Lemma 2.5 (vii) and Proposition 2.2]. To explain, for $a \in \mathrm{L}^0(M,\tau)$, let $\widetilde{\mu}_t(a)$ be the infimum of $\|ap\|_\infty$ over all projections $p \in M$ with $\tau(1-p) \leq t$. This is the **$t$-th generalized $s$-number of** $a$. Also let $\widetilde{\lambda}_t(a) = \tau(1_{(t,\infty)}(|a|))$. So $t \mapsto \widetilde{\lambda}_t(a)$ is the **distribution function of** $a$. These invariants are related by [FK86, Proposition 2.2] which states

$$\widetilde{\mu}_t(a) = \inf\{s \geq 0 :\ \widetilde{\lambda}_s(a) \leq t\}.$$

It follows that $\widetilde{\mu}_t(a) \leq s$ if and only if $\widetilde{\lambda}_s(a) \leq t$. In particular, $\widetilde{\mu}_{\widetilde{\lambda}_t(a)}(a) \leq t$ always holds.

[FK86, Lemma 2.5 (vii)] states

$$\widetilde{\mu}_{t+s}(ab) \leq \widetilde{\mu}_t(a)\widetilde{\mu}_s(b)$$

for any $t, s > 0$ and any $a, b \in \mathrm{L}^0(M,\tau)$.

Now that the tools above are ready, we return to the proposition we want to prove. The inequality $\tau(1_{(\lambda,\infty)}(|ab|)) \leq \tau(1_{(\lambda_1,\infty)}(|a|)) + \tau(1_{(\lambda_2,\infty)}(|b|))$ is equivalent to the statement

$$\widetilde{\lambda}_{ts}(ab) \leq \widetilde{\lambda}_t(a) + \widetilde{\lambda}_s(b).$$

By [FK86, Proposition 2.2], this is true if and only if

$$\widetilde{\mu}_{\widetilde{\lambda}_t(a)+\widetilde{\lambda}_s(b)}(ab) \leq ts.$$

By [FK86, Lemma 2.5 (vii)],

$$\widetilde{\mu}_{\widetilde{\lambda}_t(a)+\widetilde{\lambda}_s(b)}(ab) \leq \widetilde{\mu}_{\widetilde{\lambda}_t(a)}(a)\widetilde{\mu}_{\widetilde{\lambda}_t(b)}(b).$$

By [FK86, Proposition 2.2] again, $\widetilde{\mu}_{\widetilde{\lambda}_t(a)}(a) \leq t$ and $\widetilde{\mu}_{\widetilde{\lambda}_s(b)}(b) \leq s$. Combining these inequalities proves the proposition.

□

*Proof of Theorem 3.4.1.* The fact that $G$ is closed under inverses and the $*$ operation is obvious from Proposition 3.4.2. Let $a, b \in G$. By Proposition 3.4.3:

$$\begin{aligned}
\|\log(|ab|)\|_2^2 &= 2\int_0^\infty \lambda\left[\mu_{|ab|}(e^\lambda, \infty) + \mu_{|b^{-1}a^{-1}|}(e^\lambda, \infty)\right] \mathrm{d}\lambda \\
&\leq 2\int_0^\infty \lambda\left[\mu_{|a|}(e^{\lambda/2}, \infty) + \mu_{|a^{-1}|}(e^{\lambda/2}, \infty)\right] \mathrm{d}\lambda + 2\int_0^\infty \lambda\left[\mu_{|b|}(e^{\lambda/2}, \infty) + \mu_{|b^{-1}|}(e^{\lambda/2}, \infty)\right] \mathrm{d}\lambda \\
&= 4\int_0^\infty \lambda\left[\mu_{|a|}(e^t, \infty) + \mu_{|a^{-1}|}(e^t, \infty)\right] \mathrm{d}t + 4\int_0^\infty \lambda\left[\mu_{|b|}(e^t, \infty) + \mu_{|b^{-1}|}(e^t, \infty)\right] \mathrm{d}\mu \\
&= 2(\|\log(|a|)\|_2^2 + \|\log(|b|)\|_2^2).
\end{aligned}$$

□

We also need the following fact analogous to Proposition 3.4.3, but whose proof is easier.

**Proposition 3.4.4.** *Let $(M, \tau)$ be a semifinite von Neumann algebra, and $a, b \in L^0(M, \tau)$. Then for all $\lambda_1, \lambda_2 > 0$ we have that*

$$\mu_{|a+b|}(\lambda_1 + \lambda_2, \infty) \leq \mu_{|a|}(\lambda_1, \infty) + \mu_{|b|}(\lambda_2, \infty).$$

*Proof.* We use [FK86, Lemma 2.5, (v)], which shows that

$$\widetilde{\mu}_{\mu_{|a|}(\lambda_1,\infty)+\mu_{|b|}(\lambda_2,\infty)}(a+b) \leq \widetilde{\mu}_{\mu_{|a|}(\lambda_1,\infty)}(a) + \widetilde{\mu}_{\mu_{|b|}(\lambda_2,\infty)}(b) \leq \lambda_1 + \lambda_2.$$

Apply [FK86, Proposition 2.2] to the inequality above to obtain

$$\widetilde{\lambda}_{\lambda_1+\lambda_2}(|a+b|) \leq \mu_{|a|}(\lambda_1, \infty) + \mu_{|b|}(\lambda_2, \infty).$$

By definition of $\widetilde{\lambda}$, $\widetilde{\lambda}_{\lambda_1+\lambda_2}(|a+b|) = \mu_{|a+b|}(\lambda_1+\lambda_2, \infty)$ so this finishes the proof.

$\square$

## 3.5 The geometry of positive definite operators

Let $\mathcal{P} = \{x \in \mathrm{GL}^2(M,\tau) : \ x > 0\}$ be the positive definite elements of $\mathrm{GL}^2(M,\tau)$. Most of this section, except for §3.5.5, will deal with a tracial von Neumann algebra. In §3.5.1, we review work of Andruchow-Larontonda [AL06] on the geometry of $\mathcal{P} \cap M$. In §3.5.2, we review the measure topology on $\mathrm{L}^0(M,\tau)$. By approximating $\mathcal{P}$ by $\mathcal{P} \cap M$ (in the measure topology), we show in §3.5.3 that $d_{\mathcal{P}}$ (as defined in the introduction) is a metric on $\mathcal{P}$. Moreover, $\mathrm{GL}^2(M,\tau)$ acts transitively and by isometries on $(\mathcal{P}, d_{\mathcal{P}})$. In §3.5.4, we show that the exponential map $\exp : \mathrm{L}^2(M,\tau)_{sa} \to \mathcal{P}$ is a homeomorphism. From this, we conclude that $(\mathcal{P}, d_{\mathcal{P}})$ is a complete CAT(0) metric space and characterize its geodesics. In §3.5.5 we generalize results to semifinite von Neumann algebras.

### 3.5.1 The space $\mathcal{P}^\infty(M,\tau)$ of bounded positive operators

In this section $(M,\tau)$ is restricted to a tracial von Neumann algebra.

Let $M_{sa} \subset M$ be the subspace of self-adjoint elements and

$$\mathcal{P}^\infty = \mathcal{P}^\infty(M,\tau) := \{\exp(x) : \ x \in M_{sa}\} \subset M_{sa}$$

be the positive definite elements with bounded inverse. This section studies $\mathcal{P}^\infty$ equipped with a natural metric, as introduced in [AL06]. The results in this section are obtained directly from [AL06].

Let $\mathrm{GL}^\infty(M,\tau)$ be the group of elements $x \in M$ such that $x$ has a bounded inverse

$x^{-1}$ in $M$. This group acts on $M_{sa}$ by

$$g.w := gwg^*.$$

For $w \in \mathcal{P}^\infty$, the **tangent space to $\mathcal{P}^\infty$ at** $w$, denoted $T_w(\mathcal{P}^\infty)$, is a copy of $M_{sa}$ with the inner product $\langle \cdot, \cdot \rangle_w$ defined by

$$\langle x, y \rangle_w := \langle w^{-1/2}.x, w^{-1/2}.y \rangle = \tau(w^{-1/2}x^*w^{-1}yw^{-1/2}) = \tau(w^{-1}xw^{-1}y).$$

Let $\| \cdot \|_{w,2}$ denote the L$^2$-norm with respect to this inner product. In the special case that $w = I$ is the identity, this is just the restriction of the standard inner product to $M_{sa}$.

These inner products induce a Riemannian metric on $\mathcal{P}^\infty(M, \tau)$. The reader might be concerned that the tangent spaces $T_w(\mathcal{P}^\infty)$ are not complete with respect to their inner products. This causes no difficulty in defining the metric on $\mathcal{P}^\infty$ but it does mean that Theorem 3.1.5 cannot be directly applied to $\mathcal{P}^\infty$.

Here is a more detailed explanation of the metric. Let $\gamma : [a, b] \to \mathcal{P}^\infty$ be a path. The L$^2$-derivative of $\gamma$ at $t$ is defined by

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

where the limit is taken with respect to the L$^2$-metric on $T_{\gamma(t)}(\mathcal{P}^\infty)$. Then the length of $\gamma$ is defined as in the finite-dimensional case:

$$\text{length}_{\mathcal{P}}(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t),2} \, dt.$$

Define distance on $\mathcal{P}^\infty(M, \tau)$ by $d_{\mathcal{P}}(x, y) = \inf_\gamma \text{length}_{\mathcal{P}}(\gamma)$ where the infimum is taken over all piece-wise smooth curves $\gamma$ with derivatives in $M$. For this to be well-defined, it needs to be shown that there exists a piecewise smooth curve between any two points of $\mathcal{P}^\infty$. For any $\exp(x) \in \mathcal{P}^\infty$, the map $t \mapsto \exp(tx)$ defines a smooth curve from $I$ to $\exp(x)$. A piecewise smooth curve between any two points can be obtained by concatenating two of these special curves.

**Lemma 3.5.1.** *The action of* $\mathrm{GL}^\infty(M, \tau)$ *on* $\mathcal{P}^\infty$ *is transitive and by isometries.*

*Proof.* The action of $\mathrm{GL}^\infty(M,\tau)$ on $\mathcal{P}^\infty$ is by isometries since the Frechet derivative of $g$ at $w$ is the map

$$x \in T_w(\mathcal{P}^\infty) \mapsto g.x = gxg^* \in T_{g.w}(\mathcal{P}^\infty)$$

and

$$\begin{aligned}
\langle g.x, g.y\rangle_{g.w} &= \tau((g.w)^{-1}(g.x)(g.w)^{-1}(g.y)) \\
&= \tau((g^*)^{-1}w^{-1}g^{-1}(gxg^*)(g^*)^{-1}w^{-1}g^{-1}(gyg^*)) \\
&= \tau(w^{-1}xw^{-1}y) = \langle x, y\rangle_w.
\end{aligned}$$

The action $\mathrm{GL}^\infty(M,\tau) \curvearrowright \mathcal{P}^\infty$ is transitive since for any $w \in \mathcal{P}^\infty$, $w^{1/2} \in \mathrm{GL}(M,\tau)$ and

$$w^{1/2}.I = w.$$

□

**Lemma 3.5.2.** *[AL06, Lemma 3.5] For any $a, b \in \mathcal{P}^\infty$,*

$$d_{\mathcal{P}}(a,b) = \|\log(b^{-1/2}ab^{-1/2})\|_2 \geq \|\log(a) - \log(b)\|_2.$$

**Theorem 3.5.3.** *$\mathcal{P}^\infty(M,\tau)$ is a CAT(0) space.*

*Proof.* This follows from [AL06, Lemma 3.6] and [BH99, Chapter II.1, Proposition 1.7 (3)]. □

**Corollary 3.5.4.** *Let $x, y \in M_{sa}$ and $\sigma \geq 1$ be a scalar. Then*

$$d_{\mathcal{P}}(e^{\sigma x}, e^{\sigma y}) \geq \sigma d_{\mathcal{P}}(e^x, e^y).$$

*Proof.* Let $x', y' \in M_{sa}$ and let $f(t) = d_{\mathcal{P}}(e^{tx'}, e^{ty'})$. By [AL06, Corollary 3.4], $f$ is convex. Therefore,

$$f(t) \leq tf(1) + (1-t)f(0) = tf(1)$$

for any $0 \leq t \leq 1$. Set $t = 1/\sigma$, $x' = \sigma x$, $y' = \sigma y$ to obtain

$$f(t) = d_{\mathcal{P}}(e^x, e^y) \leq \sigma^{-1} d_{\mathcal{P}}(e^{\sigma x}, e^{\sigma y}).$$

□

### 3.5.2 The measure topology

This section reviews the measure topology on $\mathrm{L}^0(M, \tau)$. The results here are probably well-known but being unable to find them explicitly stated in the literature, we give proofs for completeness. We will need this material in the next two sections.

Let $(M, \tau)$ be a semifinite von Neumann algebra. By [Tak03, Theorem IX.2.2], the sets

$$\mathcal{O}_{\varepsilon,\delta}(a) = \left\{ b \in \mathrm{L}^0(M, \tau) : \tau(1_{(\varepsilon,\infty)}(|a - b|)) < \delta \right\}$$

ranging over $a \in \mathrm{L}^0(M, \tau)$ and $\varepsilon, \delta > 0$ form a basis for a metrizable vector space topology on $\mathrm{L}^0(M, \tau)$, and this topology turns $\mathrm{L}^0(M, \tau)$ into a topological $*$-algebra (i.e. the product and sum operations are continuous as a function of two variables, as is the adjoint). We shall call this topology the **measure topology**. This motivates the following definition.

**Definition 12.** Let $(M, \tau)$ be a semifinite von Neumann algebra. Given a sequence $(a_n)_n$ in $\mathrm{L}^0(M, \tau)$, and an $a \in \mathrm{L}^0(M, \tau)$ we say that $a_n \to a$ **in measure** if for every $\varepsilon > 0$ we have that

$$\tau(1_{(\varepsilon,\infty)}(|a - a_n|)) \to_{n\to\infty} 0.$$

**Proposition 3.5.5.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Let $C > 0$ and let $M_C \subset M$ be the set of all elements $x$ with $\|x\|_\infty \leq C$. Then the measure, strong operator and $L^2$ topologies all coincide on $M_C$. By strong operator topology we mean with respect to either of the inclusions $M \subset B(\mathcal{H})$ or $M \subset B(L^2(M, \tau))$.*

*Proof.* By [AP16, Corollary 2.5.9 and Proposition 2.5.8], the topology induced on $M_C$ from the SOT on $B(\mathcal{H})$ is the same as the topology it inherits from the SOT on $B(\mathrm{L}^2(M, \tau))$.

Let $x \in M_C$ and let $(x_n)_n \subset M_C$ be a sequence. We will show that if $x_n \to x$ in one of the three topologies then $x_n \to x$ in the other topologies. After replacing $x_n$ with $x_n - x$

and $C$ with $2C$ if necessary, we may assume $x = 0$.

Suppose that $x_n \to 0$ in measure. We will show that $x_n \to 0$ in $\mathrm{L}^2$. For any $\varepsilon > 0$,

$$\|x_n\|_2^2 = \tau(x_n^* x_n) \leq C^2 \tau(1_{(\varepsilon,\infty)}(|x_n|)) + \varepsilon^2.$$

Since $x_n \to x$ in measure, $\limsup_{n\to\infty} \|x_n\|_2^2 \leq \varepsilon^2$. Since $\varepsilon$ is arbitrary, this shows $x_n \to 0$ in $\mathrm{L}^2$.

Now suppose that $x_n \to 0$ in $\mathrm{L}^2$. We will show that $x_n \to 0$ in the SOT. So let $\xi \in L^2(M, \tau)$. If $\xi \in M$ then

$$\limsup_{n\to\infty} \|x_n \xi\|_2 \leq \limsup_{n\to\infty} \|x_n\|_2 \|\xi\|_\infty = 0.$$

In general, for any $\xi \in L^2(M, \tau)$ and $\epsilon > 0$, there exists $\xi' \in M$ with $\|\xi - \xi'\|_2 < \epsilon$. Then

$$\limsup_{n\to\infty} \|x_n \xi\|_2 \leq \limsup_{n\to\infty} \|x_n \xi'\|_2 + \|x_n(\xi - \xi')\|_2 \leq \limsup_{n\to\infty} \|x_n(\xi - \xi')\|_2 \leq C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows $x_n \to 0$ in SOT.

Now suppose that $x_n \to 0$ in SOT. Since $1 \in L^2(M, \tau)$ and $x_n 1 = x_n$, $\|x_n\|_2 \to 0$. This shows $x_n \to 0$ in $\mathrm{L}^2$.

Now suppose $x_n \to 0$ in $\mathrm{L}^2$. Let $\varepsilon > 0$. Then $\tau(1_{(\varepsilon,\infty)}(|x_n|)) \leq \varepsilon^{-2} \|x_n\|_2^2$. Since $\|x_n\|_2^2 \to 0$ this implies

$$\limsup_{n\to\infty} \tau(1_{(\varepsilon,\infty)}(|x_n|)) = 0.$$

So $x_n \to 0$ in measure.

□

*Remark* 20. It is possible that the topology on $M$ inherited from the SOT on $B(\mathcal{H})$ is not the same as topology it inherits from the SOT on $B(\mathrm{L}^2(M, \tau))$ [AP16, Exercise 1.3].

**Definition 13.** If $(\mu_n)_n$ is a sequence of Borel probability measures on a topological space $X$ and $\mu$ is another Borel probability measure on $X$ then we write $\mu_n \to \mu$ **weakly** if for every bounded continuous function $f : \mathbb{C} \to \mathbb{C}$, $\int f \, \mathrm{d}\mu_n$ converges to $\int f \, \mathrm{d}\mu$ as $n \to \infty$.

Recall that $C_0(\mathbb{R})$ denotes continuous functions on $\mathbb{R}$ that vanish at infinity while

$C_c(\mathbb{R}) \subset C_0(\mathbb{R})$ denotes those functions with compact support.

**Proposition 3.5.6.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Suppose that $(a_n)_n, (b_n)_n \in L^0(M, \tau)$, $a_n \to a$ in measure, $b_n \to b$ in measure and $b_n$ is self-adjoint for all $n$. Then:*

1. *$\mu_{b_n} \to \mu_b$ weakly.*
2. *For all but countably many $\lambda \in \mathbb{R}$ we have that $\mu_{b_n}(\lambda, \infty) \to \mu_b(\lambda, \infty)$.*
3. *For every bounded, continuous $f : \mathbb{R} \to \mathbb{R}$ we have that $\|f(b_n) - f(b)\|_2 \to_{n\to\infty} 0$.*
4. *For every continuous $f : \mathbb{R} \to \mathbb{R}$ we have that $f(b_n) \to f(b)$ in measure.*
5. *$|a_n| \to |a|$ in measure.*

*Proof.* (1): Let $f \in C_0(\mathbb{R})$ (where $C_0(\mathbb{R})$ is the space of continuous functions that vanish at infinity). By [Sti59, Corollary 5.4], we know that

$$\lim_{n\to\infty} \|f(b_n) - f(b)\|_2 = 0.$$

Since $|\tau(x) - \tau(y)| \leq \|x - y\|_2$ for all $x, y \in M$, the above convergence shows that

$$\lim_{n\to\infty} \int f \, d\mu_{b_n} = \lim_{n\to\infty} \tau(f(b_n)) = \tau(f(b)) = \int f \, d\mu_b.$$

Once we know convergence when integrated against $C_0$ functions, convergence when integrated against bounded continuous functions is a consequence of the fact that $\mu_{b_n}, \mu_b$ are all probability measures (see, e.g. [Fol99, Exercise 20 of Chapter 7]).

(2): This follows from (1) and the Portmanteau theorem.

(3): Let $M$ be such that $|\phi(t)| \leq M$ for all $t \in \mathbb{R}$. Let $\varepsilon > 0$, and choose a $T > 0$ so that $\mu_b(\{t : |t| \geq T\}) < \varepsilon$. Choose a function $\psi \in C_c(\mathbb{R})$ with $\psi(t) = 1$ for $|t| \leq T$ and so that $0 \leq \psi \leq 1$. Then,

$$\begin{aligned}\|\phi(b_n) - \phi(b)\|_2 &\leq \|\phi\psi(b_n) - \phi\psi(b)\|_2 + \|\phi(1-\psi)(b_n)\|_2 + \|\phi(1-\psi)(b)\|_2 \\ &= \|\phi\psi(b_n) - \phi\psi(b)\|_2 + \left(\int |\phi(t)|^2(1-\psi(t))^2 \, d\mu_{b_n}(t)\right)^{1/2} \\ &+ \left(\int |\phi(t)|^2(1-\psi(t))^2 \, d\mu_b(t)\right)^{1/2}.\end{aligned}$$

By [Sti59, Corollary 5.4] and (1) we thus have that

$$\limsup_{n\to\infty} \|\phi(b_n) - \phi(b)\|_2 \leq 2\left(\int |\phi(t)|^2(1-\psi(t))^2 \, d\mu_b(t)\right)^{1/2} < 2M\varepsilon.$$

Letting $\varepsilon \to 0$ completes the proof.

(4): Let $\phi \in C_c(\mathbb{R})$. By [Sti59, Theorem 5.5], it suffices to show that

$$\|\lim_{n\to\infty} \phi(f(b_n)) - \phi(f(b))\|_2 = 0.$$

Since $\phi \circ f$ is bounded and continuous, this follows from (3).

(5): Since $\mathrm{L}^0(M,\tau)$ is a topological $*$-algebra in the measure topology, $a_n^* a_n \to a^* a$ in the measure topology. Let $g\colon [0,\infty) \to [0,\infty)$ be the function $g(t) = \sqrt{t}$. Then $|a_n| = g(a_n^* a_n)$. So this follows from (4) with $b_n = a_n^* a_n$.

□

We can give a more refined improvement of Proposition 3.5.6.4 which we will need later. We first note the following.

**Corollary 3.5.7.** *Let $(M,\tau)$ be a tracial von Neumann algebra, and let $K \subseteq L^0(M,\tau)$ have compact closure in the measure topology. Then for every $\varepsilon > 0$, there is an $M > 0$ so that*

$$\tau(1_{(M,\infty)}(|a|)) < \varepsilon$$

*for all $a \in K$.*

*Proof.* Replacing $K$ with its closure, we may as well assume $K$ is compact. By Proposition 3.5.6 (5) and (1), the map $\mathrm{L}^0(M,\tau) \to \mathrm{Prob}(\mathbb{R})$ sending $x \mapsto \mu_{|x|}$ is continuous if we give

$\mathrm{L}^0(M,\tau)$ the measure topology, and $\mathrm{Prob}(\mathbb{R})$ the weak topology. So $\{\mu_{|a|} : a \in K\}$ is compact in the weak topology, and thus tight. Tightness means there exists an $M > 0$ so that $\mu_{|a|}(M,\infty) < \varepsilon$ for all $a \in K$. As $\tau(1_{(M,\infty)}(|a|)) = \mu_{|a|}(M,\infty)$, we are done.

□

**Corollary 3.5.8.** *Let $(M,\tau)$ be a tracial von Neumann algebra. Then the map*

$$\mathcal{E}\colon L^0(M,\tau)_{sa} \times C(\mathbb{R},\mathbb{R}) \to L^0(M,\tau)_{sa}$$

*given by $\mathcal{E}(a,f) = f(a)$ is continuous if we give $L^0(M,\tau)_{sa}$ the measure topology and $C(\mathbb{R},\mathbb{R})$ the topology of uniform convergence on compact sets.*

*Proof.* Suppose we are given sequences $(f_n)_n \subset C(\mathbb{R},\mathbb{R})$, $(a_n)_n \subset \mathrm{L}^0(M,\tau)_{sa}$ and $f \in C(\mathbb{R})$, $a \in \mathrm{L}^0(M,\tau)_{sa}$ with $f_n \to f$ uniformly on compact sets and $a_n \to a$ in measure.

To prove $f_n(a_n) \to f(a)$ in measure, fix $\lambda > 0$. Let $\varepsilon > 0$ be given. By Corollary 3.5.7, we may choose an $M > 0$ so that

$$\sup_{n\in\mathbb{N}} \tau(1_{(M,\infty)}(|a_n|)) < \varepsilon, \quad \tau(1_{(M,\infty)}(|a|)) < \varepsilon.$$

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function with $g(t) = t$ for $|t| \leq M$, and define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(t) = f(t) - f(g(t))$ and $h_n\colon \mathbb{R} \to \mathbb{R}$ by $h_n(t) = f_n(t) - f_n(g(t))$. Then:

$$f_n(a_n) - f(a) = f_n(g(a_n)) - f(g(a)) + h_n(a_n) - h(a).$$

Then, by Proposition 3.4.4 we have that:

$$\begin{aligned}
&\tau(1_{(\lambda,\infty)}(|f_n(a_n) - f(a_n)|)\\
&\leq \tau(1_{(\lambda/4,\infty)}(|h_n(a_n)|)) + \tau(1_{(\lambda/4,\infty)}(|h(a)|)) + \tau(1_{(\lambda/2,\infty)}(|f_n(g(a_n)) - f(g(a))|))\\
&\leq \tau(1_{(\lambda/4,\infty)}(|h_n(a_n)|)) + \tau(1_{(\lambda/4,\infty)}(|h(a)|)) + \frac{4}{\lambda^2}\|f_n(g(a_n)) - f(g(a))\|_2.
\end{aligned}$$

Since $\lambda > 0$, and $h = 0$ in $[-M,M]$ it follows that for all $n \in \mathbb{N}$ :

$$\tau(1_{(\lambda/4,\infty)}(|h_n(a_n)|)) \leq \tau(1_{(M,\infty)}(|a_n|)) < \varepsilon.$$

Similarly,

$$\tau(1_{(\lambda/4,\infty)}(|h(a)|)) < \varepsilon.$$

For the last term: let $T > 0$ be such that $\|g\|_\infty \leq T$. Then:

$$\begin{aligned}\|f_n(g(a_n)) - f(g(a))\|_2 &\leq \|f_n(g(a_n)) - f(g(a_n))\|_2 + \|f(g(a_n)) - f(g(a))\|_2 \\ &\leq \|f_n(g(a_n)) - f(g(a_n))\|_\infty + \|f(g(a_n)) - f(g(a))\|_2 \\ &\leq \sup_{t\in\mathbb{R}:|t|\leq T} |f_n(t) - f(t)| + \|f(g(a_n)) - f(g(a))\|_2.\end{aligned}$$

We have that $\sup_{t\in\mathbb{R}:|t|\leq T} |f_n(t) - f(t)| \to_{n\to\infty} 0$ as $n \to \infty$ since $f_n \to f$ uniformly on compact sets. We also have that $\|f(g(a_n)) - f(g(a))\|_2 \to 0$ by Proposition 3.5.6 (3). Hence $\|f_n(g(a_n)) - f(g(a))\|_2 \to_{n\to\infty} 0$. Altogether, we have shown that

$$\limsup_{n\to\infty} \tau(1_{(\lambda,\infty)}(|f_n(a_n) - f(a_n)|)) \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can let $\varepsilon \to 0$ to show that

$$\tau(1_{(\lambda,\infty)}(|f_n(a_n) - f(a)|)) \to_{n\to\infty} 0.$$

Since this is true for every $\lambda > 0$, we have that $f_n(a_n) \to_{n\to\infty} f(a)$ in measure.

$\square$

### 3.5.3 The space $\mathcal{P}(M,\tau)$ of positive log-square integrable operators

**Definition 14.** Let $(M,\tau)$ be a tracial von Neumann algebra, and let

$$G = \mathrm{GL}^2(M,\tau) = \{a \in \mathrm{L}^0(M,\tau) : \log(|a|) \in \mathrm{L}^2(M,\tau)\}.$$

We then set $\mathcal{P} = \mathcal{P}(M,\tau) = \{a \in G : a \geq 0\}$. For $a, b \in \mathcal{P}$, we set

$$d_{\mathcal{P}}(a,b) = \|\log(b^{-1/2}ab^{-1/2})\|_2.$$

This is well-defined by Theorem 3.4.1. Note that $\mathcal{P}^\infty = \mathcal{P} \cap M$ and $d_\mathcal{P}$ restricted to $\mathcal{P}^\infty$ agrees with the formula above by Corollary 3.5.2.

The main result of this section is:

**Theorem 3.5.9.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Then,*

1. *$d_\mathcal{P}$ is a metric.*
2. *The group $G$ acts on $\mathcal{P}$ by isometries by $g \cdot a = gag^*$.*
3. *The action $G \curvearrowright \mathcal{P}$ is transitive.*
4. *$\mathcal{P}^\infty$ is dense in $\mathcal{P}$.*

To prove this theorem, we will approximate elements of $\mathcal{P}$ by elements of $\mathcal{P}^\infty$ in the measure topology and then apply results from the previous section on $\mathcal{P}^\infty$. Because we will use the Dominated Convergence Theroem, we need some basic facts about operator monotonicity. Recall that if $a, b$ are operators then by definition, $a \leq b$ if and only if $b - a$ is a positive operator.

**Proposition 3.5.10.** *Let $(M, \tau)$ be a tracial von Neumann algebra.*

1. *Suppose $a, b \in L^0(M, \tau)$ and $|a| \leq |b|$. Then for every $\lambda > 0$ we have that*

$$\mu_{|a|}(\lambda, \infty) \leq \mu_{|b|}(\lambda, \infty).$$

2. *If $a, b \in L^0(M, \tau)$ are self-adjoint and $a \leq b$, then $cac^* \leq cbc^*$ for every $c \in L^0(M, \tau)$.*

*Proof.* (1): This is implied by [BK90, Lemma 3.(i)]

(2): We may write $b - a = d^*d$ for some $d \in \mathrm{L}^0(M, \tau)$. Then $cac^* - cbc^* = (dc^*)^* dc^*$. □

The next proposition contains the approximations results we will need.

**Proposition 3.5.11.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Suppose that $(a_n)_n$, $(b_n)_n$ are sequences in $G$, that $a \in L^0(M, \tau)$ and that $a_n^{\pm 1} \to a^{\pm 1}$, $b_n^{\pm 1} \to b^{\pm 1}$ in measure. Further assume that there are $A_1, A_2, B_1, B_2 \in \mathcal{P}$ with $A_1 \leq |a_n| \leq A_2$, $B_1 \leq |b_n| \leq B_2$ for all $n \in \mathbb{N}$.*

1. *Then* $a \in G$ *and* $\|\log(|a_n|)\|_2 \to \|\log(|a|)\|_2$.

2. *If* $a_n$ *and* $b_n \in \mathcal{P}$ *for all* $n$*, then* $d_{\mathcal{P}}(a_n, b_n) \to_{n\to\infty} d_{\mathcal{P}}(a, b)$.

*Proof.* (1): As in (3.3),

$$\|\log(a_n)\|_2^2 = 2\int_0^\infty \lambda[\mu_{|a_n|}(e^\lambda, \infty) + \mu_{|a_n^{-1}|}(e^\lambda, \infty)]\, d\lambda.$$

Moreover, since $|a_n| \leq A_2$, we have that $\mu_{|a_n|}(\lambda, \infty) \leq \mu_{A_2}(\lambda, \infty)$. Let $a_n = u_n|a_n|$ be the polar decomposition. Since $a_n^{-1} = u_n^{-1}(u_n|a_n|^{-1}u_n^{-1})$, it follows that $|a_n^{-1}| = u_n|a_n|^{-1}u_n^*$. So by operator monotonicity of inverses, $|a_n^{-1}| \leq u_n A_1^{-1} u_n^*$ and thus by Proposition 3.5.10 (1),

$$\mu_{|a_n^{-1}|}(e^\lambda, \infty) \leq \mu_{u_n A_1^{-1} u_n^*}(e^\lambda, \infty).$$

Since $a_n \in L^0(M, \tau)^\times$ we know that each $u_n$ is a unitary, so $\mu_{|a_n^{-1}|}(e^\lambda, \infty) \leq \mu_{A_1^{-1}}(e^\lambda, \infty)$. Thus

$$\lambda[\mu_{|a_n|}(e^\lambda, \infty) + \mu_{|a_n^{-1}|}(e^\lambda, \infty)] \leq \lambda[\mu_{A_2}(e^\lambda, \infty) + \mu_{A_1^{-1}}(e^\lambda, \infty)].$$

Since $A_1, A_2 \in \mathcal{P}$, the right hand side of this expression is in $L^1(\mathbb{R})$.

Since $a_n^{\pm 1} \to a^{\pm 1}$ in measure, Proposition 3.5.6 implies that $\mu_{|a_n^{\pm 1}|}(\lambda, \infty) \to \mu_{|a^{\pm 1}|}(\lambda, \infty)$ for all but countably many $\lambda$. So by the dominated convergence theorem,

$$\begin{aligned}
\|\log(|a|)\|_2^2 &= 2\int_0^\infty \lambda[\mu_{|a|}((e^\lambda, \infty)) + \mu_{|a^{-1}|}((e^\lambda, \infty))]\, d\lambda \\
&= \lim_{n\to\infty} 2\int_0^\infty \lambda[\mu_{|a_n|}((e^\lambda, \infty)) + \mu_{|a_n^{-1}|}((e^\lambda, \infty))]\, d\lambda \\
&= \lim_{n\to\infty} \|\log(|a_n|)\|_2^2
\end{aligned}$$

Moreover, we already saw that

$$2\int_0^\infty \lambda\left[\mu_{|a_n|}(e^\lambda, \infty) + \mu_{|a_n^{-1}|}(e^\lambda, \infty)\right] \mathrm{d}\lambda \leq 2\int \lambda\left[\mu_{A_2}(e^\lambda, \infty) + \mu_{A_1^{-1}}(e^\lambda, \infty)\right] \mathrm{d}\lambda < \infty.$$

Thus $\log(|a|) \in \mathrm{L}^2(M, \tau)$ and we have established that $\|\log(|a_n|)\|_2 \to \|\log(|a|)\|_2$.

(2): We have that

$$d_{\mathcal{P}}(a_n, b_n) = \| \log(b_n^{-1/2} a_n b_n^{-1/2}) \|_2,$$

and as in (1) we have that

$$d_{\mathcal{P}}(a_n, b_n) = 2 \int_0^\infty \lambda \left[ \mu_{b_n^{-1/2} a_n b_n^{-1/2}}(e^\lambda, \infty) + \mu_{b_n^{1/2} a_n^{-1} b_n^{1/2}}(e^\lambda, \infty) \right] \, \mathrm{d}\lambda. \tag{3.4}$$

By Proposition 3.5.6 (4) we know that $b_n^{-1/2} a_n b_n^{-1/2} \to b^{-1/2} a b^{-1/2}$ in measure, and similarly $b_n^{1/2} a_n^{-1} b_n^{1/2} \to b a^{-1} b$ measure. Hence by Proposition 3.5.6 (5) we know that

$$\lim_{n\to\infty} \mu_{b_n^{-1/2} a_n b_n^{-1/2}}(e^\lambda, \infty) + \mu_{b_n^{1/2} a_n^{-1} b_n^{1/2}}(e^\lambda, \infty) = \mu_{b^{-1/2} a b^{-1/2}}(e^\lambda, \infty) + \mu_{b^{1/2} a^{-1} b^{1/2}}(e^\lambda, \infty), \tag{3.5}$$

for all but countably many $\lambda$. Moreover, by Proposition 3.4.3 we have that

$$\mu_{b_n^{-1/2} a_n b_n^{-1/2}}(e^\lambda, \infty) \leq 2\mu_{b_n^{-1/2}}(e^{\lambda/4}, \infty) + \mu_{a_n}(e^{\lambda/2}, \infty) = 2\mu_{b_n^{-1}}(e^{\lambda/2}, \infty) + \mu_{a_n}(e^{\lambda/2}, \infty).$$

By operator monotonicity of inverses, we have that $b_n^{-1} \leq B_1^{-1}$ and so by Proposition 3.5.10 (1) we have

$$\mu_{b_n^{-1/2} a_n b_n^{-1/2}}(e^\lambda, \infty) \leq 2\mu_{B_1^{-1}}(e^{\lambda/2}, \infty) + \mu_{A_2}(e^{\lambda/2}, \infty). \tag{3.6}$$

Similarly,

$$\mu_{b_n^{1/2} a_n^{-1} b_n^{1/2}}(e^\lambda, \infty) \leq 2\mu_{B_2}(e^{-\lambda/2}, \infty) + \mu_{A_1^{-1}}(e^{-\lambda/2}, \infty). \tag{3.7}$$

As in the proof of (1),

$$\lambda \mapsto \lambda \left[ \mu_{B_1^{-1}}(e^{\lambda/2}, \infty) + \mu_{B_2}(e^{-\lambda/2}, \infty) + \mu_{A_1^{-1}}(e^{-\lambda/2}, \infty) + \mu_{A_2}(e^{\lambda/2}, \infty) \right]$$

is in $L^1(\mathbb{R})$. So by (3.6),(3.7), and (3.5) we may apply the dominated convergence theorem to (3.4) to see that

$$\lim_{n\to\infty} d_{\mathcal{P}}(a_n, b_n) = 2 \int_0^\infty \lambda \left[ \mu_{b^{-1/2} a b^{-1/2}}(e^\lambda, \infty) + \mu_{b^{1/2} a^{-1} b^{1/2}}(e^\lambda, \infty) \right] \, \mathrm{d}\lambda = d_{\mathcal{P}}(a, b).$$

□

*Proof of Theorem 3.5.9.* (1):

We first prove non-degeneracy. So suppose that $a, b \in \mathcal{P}$ and $d_{\mathcal{P}}(a,b) = 0$. Then $\log(a^{-1/2}ba^{-1/2}) = 0$, and so $a^{-1/2}ba^{-1/2} = 1$. Multiplying this equation on the left and right by $a^{1/2}$ proves that $b = a$.

For the triangle inequality, we already know by Corollary 3.5.2 that $d_{\mathcal{P}}$ is a metric when restricted to $\mathrm{GL}^{\infty}(M,\tau) = M^{\times}$. Here $M^{\times}$ is the set of elements of $M$ with a *bounded* inverse. Define $f_n \colon [0,\infty) \to [0,\infty)$ by

$$f_n(t) = \begin{cases} n, & \text{if } t > n \\ t, & \text{if } \frac{1}{n} \leq t \leq n \\ \frac{1}{n}, & \text{if } 0 \leq t < \frac{1}{n}. \end{cases}$$

Given $a, b \in \mathcal{P}$, set $a_n = f_n(a), b_n = f_n(b)$, $A = |\log(a)|, B = |\log(b)|$ and observe that:

- $a_n^{\pm 1} \to a^{\pm 1}$, $b_n^{\pm 1} \to b^{\pm 1}$ in measure,
- $\exp(-A) \leq a_n \leq \exp(A)$, $\exp(-B) \leq b_n \leq \exp(B)$ for all $n \in \mathbb{N}$.

By Proposition 3.5.11 (2),

$$\lim_{n\to\infty} d_{\mathcal{P}}(a_n, b_n) = d_{\mathcal{P}}(a,b).$$

Since $d_{\mathcal{P}}$ is a metric when restricted to $\mathcal{P} \cap M^{\times}$, and $f_n(\mathcal{P}) \subseteq \mathcal{P} \cap M^{\times}$, the above equation implies the triangle inequality for $d_{\mathcal{P}}$. It also implies $d_{\mathcal{P}}$ is symmetric. So it is a metric.

(2): It is easy to see that (2) is true if $g \in \mathcal{U}(M)$ (where $\mathcal{U}(M) \leq M^{\times}$ is the group of unitaries in $M$). Every $g \in G$ can be written as $g = u|g|$ where $u \in \mathcal{U}(M)$. Since $|g| \in \mathcal{P}$ for every $g \in G$, and (2) is true when $u \in \mathcal{U}(M)$, it suffices to show (2) for $g \in \mathcal{P}$. So we will assume throughout the rest of the proof that $g \in \mathcal{P}$.

We first show that $d_{\mathcal{P}}(gag^*, gbg^*) = d_{\mathcal{P}}(a,b)$ for $a, b \in \mathcal{P} \cap M^{\times}$. Since $g \in \mathcal{P}$, as in (1) we may find a sequence $g_n \in \mathcal{P} \cap M^{\times}$ so that

- $g_n^{\pm 1} \to g^{\pm 1}$ in measure,
- $g_n = f_n(g)$ for some $f_n \colon [0,\infty) \to [0,\infty)$
- $\exp(-H) \leq g_n \leq \exp(H)$ for some self-adjoint $H \in \mathrm{L}^2(M,\tau)$.

Since $\mathrm{L}^0(M,\tau)$ is a topological $*$-algebra in the measure topology, we have that $g_nag_n \to_{n\to\infty} gag$ in measure. Moreover by Proposition 3.5.10 (2)

$$\|a^{-1}\|_\infty^{-1}\exp(-2H) \leq \|a^{-1}\|_\infty^{-1} g_ng_n \leq g_nag_n \leq \|a\|_\infty g_n^2 \leq \|a\|_\infty \exp(2H),$$

and similarly

$$\|b^{-1}\|_\infty^{-1}\exp(-2H) \leq g_nbg_n \leq \|b\|_\infty \exp(2H).$$

So as in (1) we may apply Proposition 3.5.11 (2) to see that

$$d_{\mathcal{P}}(gag, gbg) = \lim_{n\to\infty} d_{\mathcal{P}}(g_nag_n, g_nbg_n). \tag{3.8}$$

By Lemma 3.5.1, $d_{\mathcal{P}}(g_nag_n, g_nbg_n) = d_{\mathcal{P}}(a,b)$. We thus have that

$$d_{\mathcal{P}}(gag, gbg) = d_{\mathcal{P}}(a,b).$$

We now handle the case of general $a, b \in \mathcal{P}$. As in (1), we find may sequences $a_n, b_n \in \mathcal{P} \cap M^\times$ so that:

- $a_n^{\pm 1} \to a^{\pm 1}$, $b_n \to b^{\pm 1}$ in measure
- $\exp(-A) \leq a_n \leq \exp(A)$, $\exp(-B) \leq b_n \leq \exp(B)$ for some $A, B \in \mathrm{L}^2(M,\tau)$.

As in (1), we have that

$$d_{\mathcal{P}}(a,b) = \lim_{n\to\infty} d_{\mathcal{P}}(a_n, b_n). \tag{3.9}$$

$$d_{\mathcal{P}}(gag^*, gbg^*) = \lim_{n\to\infty} d_{\mathcal{P}}(ga_ng^*, gb_ng^*).$$

So combining (3.9) with the first case shows that

$$d_{\mathcal{P}}(gag^*, gbg^*) = d_{\mathcal{P}}(a,b).$$

(3) Let $p, q \in \mathcal{P}$. Then $p^{-1/2}, q^{1/2} \in G = \mathrm{GL}^2(M,\tau)$. Moreover,

$$(q^{1/2}p^{-1/2}) \cdot p = q.$$

(4) Let $a \in \mathcal{P}$ and define $a_n = f_n(a)$ as in (1). Then $a_n \in \mathcal{P}^\infty$ and $a_n \to a$ in measure. Apply Proposition 3.5.11 with $b_n = a = B_1 = B_2$ to obtain $d_{\mathcal{P}}(a_n, a) \to d_{\mathcal{P}}(a, a) = 0$ as $n \to \infty$. Since $a \in \mathcal{P}$ is arbitrary, this proves $\mathcal{P}^\infty$ is dense in $\mathcal{P}$.

□

### 3.5.4 Continuity of the exponential map

This section proves that the exponential map $\exp : \mathrm{L}^2(M, \tau)_{sa} \to \mathcal{P}$ is a homeomorphism and obtains as a corollary that $\mathcal{P}$ is a complete CAT(0) metric space. We also obtain a formula for the geodesics in $\mathcal{P}$. First we need the following estimate which extends the $\mathcal{P}^\infty$ case proven earlier.

**Proposition 3.5.12.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Then for $a, b \in L^2(M, \tau)_{sa}$ we have that*

$$\|a - b\|_2 \leq d_{\mathcal{P}}(e^a, e^b).$$

*If $a$ and $b$ commute then $\|a - b\|_2 = d_{\mathcal{P}}(e^a, e^b)$.*

*Proof.* Define a function $f_n \colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} n, & \text{if } t > n \\ t, & \text{if } |t| \leq n \\ -n, & \text{if } t < -n. \end{cases}$$

Set $a_n = f_n(a), b_m = f_n(b)$. Then:

- $e^{a_n} \to e^a$, $e^{b_n} \to e^b$ in measure,
- $\exp(-A) \leq e^{a_n} \leq \exp(A)$, $\exp(-B) \leq e^{b_n} \leq \exp(B)$ for all $n \in \mathbb{N}$.

So as in Theorem 3.5.9 (1) we have that

$$d_{\mathcal{P}}(e^a, e^b) = \lim_{n\to\infty} d_{\mathcal{P}}(e^{a_n}, e^{b_n}).$$

Additionally, it is direct to see from the spectral theorem that

$$\lim_{n\to\infty} \|a - a_n\|_2 = \lim_{n\to\infty} \|b - b_n\|_2 = 0.$$

So, by Corollary 3.5.2,

$$d_{\mathcal{P}}(e^a, e^b) = \lim_{n\to\infty} d_{\mathcal{P}}(e^{a_n}, e^{b_n}) \geq \lim_{n\to\infty} \|a_n - b_n\|_2 = \|a - b\|_2.$$

Suppose $a$ and $b$ commute. By definition

$$d_{\mathcal{P}}(e^a, e^b) = \|\log(e^{-b/2} e^a e^{-b/2})\|_2.$$

Since $a$ and $b$ commute, $e^{-b/2} e^a e^{-b/2} = e^{a-b}$. So $\|\log(e^{-b/2} e^a e^{-b/2})\|_2 = \|a - b\|_2$.

□

**Theorem 3.5.13.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Then the exponential map $\exp\colon L^2(M, \tau)_{sa} \to \mathcal{P}$ is a homeomorphism.*

*Proof.* By Proposition 3.5.12, we know that $\log\colon \mathcal{P} \to \mathrm{L}^2(M, \tau)_{sa}$ is continuous. So it just remains to show that $\exp\colon \mathrm{L}^2(M, \tau)_{sa} \to \mathcal{P}$ is continuous.

Suppose that $(a_n)_n$ is a sequence in $\mathrm{L}^2(M, \tau)$ and $a \in \mathrm{L}^2(M, \tau)$ with $\|a - a_n\|_2 \to 0$. Let $\varepsilon > 0$, and for $\lambda > 0$, define $f_\lambda\colon \mathbb{R} \to \mathbb{R}$ by

$$f_\lambda(t) = \begin{cases} \lambda, & \text{if } t > \lambda \\ t, & \text{if } |t| \leq \lambda \\ -\lambda, & \text{if } t < -\lambda. \end{cases}$$

If $\lambda > 0$ is large enough, then $\|a - f_\lambda(a)\|_2 < \varepsilon$. Fix such a choice of $\lambda$.

Since $a$ and $f_\lambda(a)$ commute,

$$\begin{aligned} d_{\mathcal{P}}(e^{a_n}, e^a) &\leq d_{\mathcal{P}}(e^a, e^{f_\lambda(a)}) + d_{\mathcal{P}}(e^{f_\lambda(a_n)}, e^{f_\lambda(a)}) + d_{\mathcal{P}}(e^{a_n}, e^{f_\lambda(a_n)}) \qquad (3.10) \\ &= \|a - f_\lambda(a)\|_2 + \|a_n - f_\lambda(a_n)\|_2 + d_{\mathcal{P}}(e^{f_\lambda(a_n)}, e^{f_\lambda(a)}). \end{aligned}$$

Since $a_n \to a$ in $\mathrm{L}^2(M, \tau)$, $a_n \to a$ in measure. By Proposition 3.5.6 (3), $\lim_{n\to\infty} \|f_\lambda(a_n) - f_\lambda(a)\|_2 = 0$. Furthermore, $\max(\|f_\lambda(a_n)\|_\infty, \|f_\lambda(a)\|_\infty) \leq \lambda$ for all $n \in \mathbb{N}$.

By Proposition 3.5.6 (4), $e^{-f_\lambda(a_n)/2} \to e^{-f_\lambda(a)/2}$ in measure. Since $\mathrm{L}^0(M, \tau)$ is a topological $*$-algebra in the measure topology, $e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)} e^{-f_\lambda(a_n)/2} \to 1$ in measure. We claim that

$$\log(e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)} e^{-f_\lambda(a_n)/2}) \to 0$$

in measure. To see this, observe that

$$e^{-2\lambda} \leq e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)} e^{-f_\lambda(a_n)/2} \leq e^{2\lambda}.$$

Choose a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) = \log(x)$ for all $e^{-2\lambda} \leq x \leq e^{2\lambda}$. Then $\phi(e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)} e^{-f_\lambda(a_n)/2}) = \log(e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)} e^{-f_\lambda(a_n)/2})$. So the claim follows from Proposition 3.5.6 (4).

By Proposition 3.5.5, the claim above implies $\log(e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)/2} e^{-f_\lambda(a_n)/2}) \to 0$ in $\mathrm{L}^2(M, \tau)$. Since

$$d_{\mathcal{P}}(e^{f_\lambda(a_n)}, e^{f_\lambda(a)}) = \|\log(e^{-f_\lambda(a_n)/2} e^{f_\lambda(a)/2} e^{-f_\lambda(a_n)/2})\|_2$$

this shows

$$d_{\mathcal{P}}(e^{f_\lambda(a_n)}, e^{f_\lambda(a)}) \to_{n\to\infty} 0. \tag{3.11}$$

Since $a_n \to a$ and $f_\lambda(a_n) \to f_\lambda(a)$ in $\mathrm{L}^2(M, \tau)$, $\|a_n - f_\lambda(a_n)\|_2 \to_{n\to\infty} \|a - f_\lambda(a)\|_2$. Combining with (3.11), (3.10) we have shown that

$$\limsup_{n\to\infty} d_{\mathcal{P}}(e^{a_n}, e^a) \leq 2\|a - f_\lambda(a)\|_2 < 2\varepsilon.$$

Letting $\varepsilon \to 0$ proves that $d_{\mathcal{P}}(e^{a_n}, e^a) \to 0$.

□

**Corollary 3.5.14.** *Let $(M, \tau)$ be a tracial von Neumann algebra. Then $(\mathcal{P}, d_{\mathcal{P}})$ is a complete metric space.*

*Proof.* Let $(a_n)$ be a Cauchy sequence in $\mathcal{P}$. Set $b_n = \log(a_n)$. By Proposition 3.5.12, we know that $(b_n)$ is Cauchy in $\mathrm{L}^2(M,\tau)$. By completeness of $\mathrm{L}^2(M,\tau)$, there is a $b \in \mathrm{L}^2(M,\tau)$ with $\|b_n - b\|_2 \to_{n\to\infty} 0$. Then $a = e^b \in \mathcal{P}$, and by Theorem 3.5.13 we know that $a_n = e^{b_n} \to e^b = a$. $\square$

**Corollary 3.5.15.** *$\mathcal{P}$ is CAT(0).*

*Proof.* Recall that $\mathcal{P}^\infty$ is CAT(0) by Theorem 3.5.3. By Theorem 3.5.9 $\mathcal{P}^\infty$ is dense in $\mathcal{P}$. Because metric completions of CAT(0) spaces are CAT(0) by [BH99, II.3, Corollary 3.11], this implies $\mathcal{P}$ is CAT(0). $\square$

**Corollary 3.5.16.** *Let $(M,\tau)$ be a tracial von Neumann algebra. Then the measure topology on $\mathcal{P}(M,\tau)$ is weaker than the $d_{\mathcal{P}}$-topology.*

*Proof.* Let $(b_n)_n$ be a sequence in $\mathcal{P}(M,\tau)$ and $b \in \mathcal{P}(M,\tau)$ with $\lim_{n\to\infty} d_{\mathcal{P}}(b_n, b) = 0$. Let $a_n = \log b_n$, $a = \log(b)$. Then $\|a_n - a\|_2 \to_{n\to\infty} 0$, since the logarithm map is continuous. So $a_n \to a$ in measure. But then by applying the exponential map in Proposition 3.5.6 (4) we have that $b_n \to b$ in measure. $\square$

**Corollary 3.5.17.** *For $\xi \in L^2(M,\tau)_{sa}$, the map $\gamma_\xi : \mathbb{R} \to \mathcal{P}$ defined by*

$$\gamma_\xi(t) = \exp(t\xi)$$

*is a minimal geodesic with speed $\|\xi\|_2$. Moreover every geodesic $\gamma$ with $\gamma(0) = I$ is equal to $\gamma_\xi$ for some $\xi$. Moreover, for any $a, b \in \mathcal{P}$, the unique unit-speed geodesic from $a$ to $b$ is the map $\gamma : [0, d_{\mathcal{P}}(a,b)] \to \mathcal{P}$ defined by*

$$\gamma(t) = a^{1/2}\gamma_\xi(t)a^{1/2}$$

*where*

$$\xi = \frac{\log(a^{-1/2}ba^{-1/2})}{\|\log(a^{-1/2}ba^{-1/2})\|_2} = \frac{\log(a^{-1/2}ba^{-1/2})}{d_{\mathcal{P}}(a,b)}. \tag{3.12}$$

*Proof.* For any $t > 0$,

$$d_{\mathcal{P}}(I, \gamma_\xi(t)) = \|\log \gamma_\xi(t)\|_2 = t\|\xi\|_2.$$

This proves $\gamma_\xi$ is a minimal geodesic with speed $\|\xi\|_2$. Because $\mathcal{P}$ is CAT(0), there is a unique unit-speed geodesic between any two points. By uniqueness of geodesics, every geodesic $\gamma$ with $\gamma(0) = I$ has the above form.

In particular, if $a, b \in \mathcal{P}$ and $\xi$ is defined by (3.12) then $\gamma_\xi : [0, d_\mathcal{P}(a,b)] \to \mathcal{P}$ is a unit-speed geodesic from $I$ to $a^{-1/2}ba^{-1/2}$. Because the action of $\mathrm{GL}^2(M,\tau)$ on $\mathcal{P}$ is by isometries, $\gamma(t) = a^{1/2}.\gamma_\xi(t)$ is a unit-speed geodesic from $a = a^{1/2}.I$ to $b = a^{1/2}.a^{-1/2}ba^{-1/2}$.

□

### 3.5.5 Semi-finite case

Let $(M,\tau)$ be a semi-finite tracial von Neumann algebra. Let $G = \mathrm{GL}^2(M,\tau)$ and $\mathcal{P} = \mathcal{P}(M,\tau) = \exp(L^2_{sa}(M,\tau))$ as before. We want to show that $d_\mathcal{P}(a,b) := ||\log(b^{-1/2}ab^{-1/2})||_2$ is a distance function which makes $\mathcal{P}$ into a complete CAT(0) space. Since we have shown this fact for the case of a finite tracial state, our approach will often involve reducing to the finite case. To this end we first need to identify the following objects.

For a finite projection $p$, let $\mathcal{P}_p = \exp(L^2_{sa}(pMp, \tau \circ p)) \subset \mathrm{L}^0(pMp, \tau \circ p)$ and $\tilde{\mathcal{P}}_p = \exp(pL^2_{sa}(M,\tau)p) \subset \mathrm{L}^0(M,\tau)$. For $a, b \in \mathcal{P}_p$ define $d_{\mathcal{P}_p}(a,b) = ||\log(b^{-1/2}ab^{-1/2})||_{L^2(pMp,\tau\circ p)}$ and define $d_{\tilde{\mathcal{P}}_p} = d_\mathcal{P}|_{\tilde{\mathcal{P}}_p}$. Since $(pMp, \tau \circ p_n)$ is a finite von Neumann algebra, Theorem 3.5.9 implies $d_{\mathcal{P}_p}$ is a metric and Corollar 3.5.15 implies $\mathcal{P}_p$ is complete CAT(0).

**Proposition 3.5.18.** *The following are true:*

1. *For every finite projection p there exists a bijective map $f : \mathcal{P}_p \to \tilde{\mathcal{P}}_p$ such that for all $a, b$ in $\mathcal{P}_p$, $d_{\mathcal{P}_p}(a,b) = d_{\tilde{\mathcal{P}}_p}(f(a), f(b))$.*

2. *$d_\mathcal{P}$ is a metric.*

3. *$G$ acts on $\mathcal{P}$ by isometries.*

4. *$G$ acts on $\mathcal{P}$ transitively.*

5. *$\mathcal{P}$ is complete.*

6. *$\mathcal{P}$ is CAT(0).*

7. *Let $\mathcal{P}^\infty = \exp(M_{sa})$. Then $\mathcal{P}^\infty$ is dense in $\mathcal{P}$ and is equal to $\mathcal{P} \cap M^\times$*

*Proof.* (1)

Let $p \in M$ be any finite projection. Let $L = L^2_{sa}(pMp, \tau \circ p)$, $\tilde{L} = L^2_{sa}(M, \tau)$. First let $\iota : pMp \to M$ be the inclusion map. We will show that $\iota$ extends to an isometry from $L$ to $p\tilde{L}p$.

Observe that $\iota$ is an isometric embedding in the sense that

$$\|\iota(x)\|_{\tilde{L}} = \|x\|_L \quad (\forall x \in pMp).$$

Because $pMp$ is dense in $L$, $\iota$ extends to an isometric embedding, which we also denote by $\iota$, from $L$ to $\tilde{L}$.

**Claim 8.** *The map $\tilde{L} \to p\tilde{L}p$, $x \to pxp$ is continuous*

*Proof.* Suppose $x_n \to x$ in $L^2$. Then $\|pxp - px_np\|_2 = \|p(x_n - x)p\|_2 \leq \|p\|_\infty \|x_n - x\|_2 \|p\|_\infty = \|x_n - x\|_2 \to 0$ □

We claim that $\iota(L) = p\tilde{L}p$. To see this, note that $p\tilde{L}p$ is a closed subspace of $\tilde{L}$: suppose $px_ip \xrightarrow{\tilde{L}} y$. Then $p(px_ip)p \xrightarrow{\tilde{L}} pyp$ by the above claim. But $p(px_ip)p = px_ip$, so $pyp = y$. It follows that $y \in p\tilde{L}p$. Next, note that $pMp \subset p\tilde{L}p$ and $pMp$ is dense in $L$. These facts together imply $\iota(L) \subset p\tilde{L}p$. On the other hand, $pMp$ is also dense in $p\tilde{L}p$ because $M$ is dense in $\tilde{L}$ and because of the above claim. It follows that $\iota(L) = p\tilde{L}p$.

Now we define $f : \mathcal{P}_p \to \tilde{\mathcal{P}}_p$ by $f(e^x) = e^{\iota(x)}$. First note that for $y \in p\tilde{L}p$, for any $v \in (p\mathcal{H})^\perp$ $yv = 0$, so $e^y v = v$ On the other hand $w \in p\mathcal{H}$, $yw \in p\mathcal{H}$, and by functional calculus $e^y w \in p\mathcal{H}$ also. Also note that if $y \in p\tilde{L}p$, so is $-y$ and $y/2$. Now suppose $e^x, e^y \in \mathcal{P}_p$. Then $d_{\mathcal{P}_p}(e^x, e^y) = \|\log(e^{-y/2}e^x e^{-y/2})\|_L$ while $d_{\tilde{\mathcal{P}}_p}(f(e^x), f(e^y)) = \|\log(e^{-\iota(y)/2}e^{\iota(x)}e^{-\iota(y)/2})\|_{\tilde{L}}$. By the above discussion, the spectral measures of $e^{-y/2}e^x e^{-y/2}$ and $e^{-\iota(y)/2}e^{\iota(x)}e^{-\iota(y)/2}$ are identical on $\mathbb{R} \setminus \{1\}$ and similarly for their inverses. It follows by equation 3.3 that $d_{\mathcal{P}_p}(e^x, e^y) = d_{\tilde{\mathcal{P}}_p}(f(e^x), f(e^y))$. Thus $L$ and $p\tilde{L}p$ are naturally identified and so are $\mathcal{P}_p$ and $\tilde{\mathcal{P}}_p$ and we do not distinguish between them from now on.

(2)

**Claim 9.** *Let $(M, \tau)$ be a semifinite von Neumann algebra. Suppose $a, b \in L^0(M, \tau)$ and $|a| \leq |b|$. Then for every $\lambda > 0$*

$$\mu_{|a|}(\lambda,\infty) \le \mu_{|b|}(\lambda,\infty)$$

*Proof.* Use [FK86, Lemma 2.5(iii)] . $\square$

**Lemma 3.5.19.** *Suppose $x_k, x \in L^0_{sa}(M,\tau)$ and $x_k \to x$ in measure and $f : \mathbb{R} \to \mathbb{R}$ is a Borel function continuous on the spectrum of $x$ and bounded on bounded subsets of $\mathbb{R}$. Then $f(x_k) \to f(x)$ in measure.*

*Proof.* [Tik87, Theorem 2.4]. $\square$

**Lemma 3.5.20.** *Suppose $x_k, x \in L^0(M,\tau)$ and $x_k \to x$ in measure. Suppose $\lambda \mapsto \mu_{|x|}(\lambda,\infty)$ is continuous at $\lambda_0$. Then $\mu_{|x_k|}(\lambda_0,\infty) \to \mu_{|x|}(\lambda_0,\infty)$.*

*Proof.* Let $\lambda \mapsto \mu_{|x|}(\lambda,\infty)$ be continuous at $\lambda_0$. By Proposition 3.4.4, for any $\delta$ and $k$,

$$\mu_{|x|}(\lambda+\delta,\infty) \le \mu_{|x-x_k|}(\delta,\infty) + \mu_{|x_k|}(\lambda,\infty);$$

$$\mu_{|x_k|}(\lambda) \le \mu_{|x_k-x|}(\delta,\infty) + \mu_{|x|}(\lambda-\delta,\infty).$$

It follows by first choosing $\delta$ small and then $k$ large, and the assumptions of continuity and of $x_k \to x$ in measure that $\mu_{|x|}(\lambda,\infty) \le \mu_{|x|}(\lambda+\delta,\infty) + \epsilon_1 \le \mu_{|x_k|}(\lambda,\infty) + \epsilon_1 + \epsilon_2$ and similarly $\mu|x|(\lambda,\infty) \ge \mu_{|x_k|}(\lambda,\infty) - \epsilon_3 - \epsilon_4$ $\square$

We now prove that $d_{\mathcal{P}}$ satisfies the triangle inequality and symmetry properties; the identity property is similar to the finite case. We do this by approximating via elements from a "reduced" finite von Neumann algebra.

**Lemma 3.5.21.** *For any $x \in L^2_{sa}(M,\tau)$, let $p^x_n = 1_{(-\infty,-1/n)\cup(1/n,\infty)}(x)$ and $x_n := p^x_n x p^x_n = x p^x_n$. Then for any $x, y \in L^2_{sa}(M,\tau)$, $d_{\mathcal{P}}(e^{x_n}, e^{y_n}) \to d_{\mathcal{P}}(e^x, e^y)$*

*Proof.* Then $x_n$ is an increasing sequence converging in measure to $x$ (that $|x_n| \le |x_{n+m}| \le |x|$ for all $n, m \in \mathbb{N}$). Similarly for $y$. Let $z = e^{-y/2}e^x e^{-y/2}$ and $\tilde{z}_n = e^{-y_n/2}e^{x_n}e^{-y_n/2}$. Since $e$ is continuous and bounded on bounded subsets, and $L^0$ is a topological $*$-algebra (see Takesaki and Lemma 3.5.19), $\tilde{z}_n \to z$ in measure. Similarly $\tilde{z}_n^{-1} \to z^{-1}$.

We will next proceed in a similar fashion as in the finite case.

Now by Lemma 3.5.20, and $z, \tilde{z}_n$ being positive, $\mu_{z_n}(\lambda, \infty) \to \mu_z(\lambda, \infty)$ for all but countably many $\lambda > 0$.

Next use Proposition 3.4.3 and the operator monotonicity claim to get

$$\begin{aligned}
\mu_{\tilde{z}_n}(e^\lambda, \infty) &\leq 2\mu_{e^{-y_n/2}}(e^{\lambda/4}, \infty) + \mu_{e^{x_n}}(e^{\lambda/2}, \infty) \\
&= 2\mu_{-y_n}(\lambda/2, \infty) + \mu_{x_n}(\lambda/2, \infty) \\
&\leq 2\mu_{|y_n|}(\lambda/2, \infty) + \mu_{|x_n|}(\lambda/2, \infty) \\
&\leq 2\mu_{|y|}(\lambda/2, \infty) + \mu_{|x|}(\lambda/2, \infty). \\
&= 2\mu_{2|y|}(\lambda, \infty) + \mu_{2|x|}(\lambda, \infty)
\end{aligned}$$

A similar calculation shows that $\mu_{\tilde{z}_n^{-1}}(e^\lambda, \infty) \leq 2\mu_{2|y|}(\lambda, \infty) + \mu_{2|x|}(\lambda, \infty)$.

Now since $x, y \in L^2_{sa}(M, \tau)$, by equation (3.2) we conclude that $\lambda \mapsto \lambda(\mu_{2|y|}(\lambda, \infty) + \mu_{2|x|}(\lambda, \infty))$ is integrable. It follows that $\lambda \mapsto \lambda(\mu_{\tilde{z}_n}(e^\lambda, \infty) + \mu_{\tilde{z}_n^{-1}}(e^\lambda, \infty))$ is dominated by an integrable function, so that by the dominated convergence theorem and using equation (3.3), $d_{\mathcal{P}}(e^{x_n}, e^{y_n}) \to d_{\mathcal{P}}(e^x, e^y)$.

□

Now for each $n$, $x, y, z \in L^2_{sa}(M, \tau)$, $x_n, y_n, z_n$ are in fact contained in $L^2_{sa}(p_n M p_n, \tau \circ p)$ for a finite projection $p_n$, so they are in a finite trace setting and triangle inequality and symmetry property hold (e.g. $d_{\mathcal{P}}(e^{x_n}, e^{z_n}) \leq d_{\mathcal{P}}(e^{x_n}, e^{y_n}) + d_{\mathcal{P}}(e^{y_n}, e^{z_n})$) , so we conclude that triangle inequality and symmetry hold in general on $\mathcal{P}$.

**Corollary 3.5.22.** $\cup_{n\in\mathbb{N}, x \in L^2_{sa}(M,\tau)} \mathcal{P}_{p_n^x}$ *is dense in* $\mathcal{P}$

*Proof.* This follows from $x_n \to x$ in measure, $|x_n - x| \leq |x|$, and arguments using the dominated convergence theorem similar to the above. □

(3)

**Lemma 3.5.23.** *Let* $x, y \in L^2_{sa}(M, \tau)$ *and* $u \in M$ *is unitary. Then*

1. $(ue^x u^*)^{-1} = ue^{-x}u^*$;

2. $(ue^x u^*)^{1/2} = ue^{x/2}u^*$;

3. $e^{uxu^*} = ue^x u^*$;

4. $d_{\mathcal{P}}(ue^x u^*, ue^y u^*) = d_{\mathcal{P}}(e^x, e^y)$.

*Proof.* The first claim is obvious. The second follows from observing that $ue^{x/2}u^*$ is positive and its square is $ue^x u^*$.

Note: for the third claim it seems that we can only use the series definition of exp when we are in a unital Banach algebra - in particular I'm not sure we can use it on $L^2(M,\tau)$ since it isn't an algebra with respect to the $L^2$ norm.

For the third claim, first consider a sequence $x_k \in M$ converging to $x$ in $L^2$. For each $x_k$, because $M$ is a unital Banach algebra,

$$e^{ux_k u^*} = \sum_{n=0}^{\infty} \frac{(ux_k u^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{ux_k^n u^*}{n!} = ue^{x_k} u^*.$$

Now since $x_k \to x$ in $L^2$, by an argument similar to that in Proposition 3.5.5, $x_k \to x$ in measure, and so by Lemma 3.5.19, $ue^{x_k}u^* \to ue^x u^*$ in measure, and also $e^{ux_k u^*} \to e^{uxu^*}$. But since $ue^{x_k}u^* = e^{ux_k u^*}$, by uniqueness of limits $ue^x u^* = e^{uxu^*}$.

The last claim now follows using the previous three claims:

$$\begin{aligned}
d_{\mathcal{P}}(ue^x u^*, ue^y u^*) &= \|\log[(ue^x u^*)^{-1/2} ue^y u^* (ue^x u^*)^{-1/2}]\|_2 \\
&= \|\log[(ue^{-x/2}u^*) ue^y u^* (ue^{-x/2}u^*)]\|_2 \\
&= \|\log[ue^{-x/2}e^y e^{-x/2}u^*]\|_2 \\
&= \|u \log[e^{-x/2}e^y e^{-x/2}]u^*\|_2 \\
&= \tau(u \log[e^{-x/2}e^y e^{-x/2}]^2 \log[e^{-x/2}e^y e^{-x/2}]u^*) \\
&= \tau(\log[e^{-x/2}e^y e^{-x/2}]^2 \log[e^{-x/2}e^y e^{-x/2}]) \\
&= \|\log[e^{-x/2}e^y e^{-x/2}]\|_2 \\
&= d_{\mathcal{P}}(e^x, e^y).
\end{aligned}$$

The first equality is by definition of $d_{\mathcal{P}}$. The second follows from the first two claims above. The third equality uses $uu^* = 1$. The fourth follows from the third item of this lemma. The fifth is by definition of $\|\cdot\|_2$. The sixth holds because $\tau$ is a trace. $\square$

Now by polar decomposition it suffices to consider the action of $\mathcal{P}$ on $\mathcal{P}$. Let $g, a, b \in \mathcal{P}$, where $g = e^h, a = e^x, b = e^y$, $h, x, y \in L^2_{sa}(M, \tau)$. We want to show that $d_{\mathcal{P}}(gag^*, gbg^*) = d_{\mathcal{P}}(e^x, e^y)$. As before consider reduced versions $h_n, x_n, y_n$ of $h, x, y$. Let $g_n = e^{h_n}$, $a_n = e^{x_n}$, $b_n = e^{y_n}$. We claim that $d_{\mathcal{P}}(g_n a_n g_n^*, g_n b_n g_n^*) \to d_{\mathcal{P}}(gag^*, gbg^*)$. This follows from a dominated convergence argument similar to that used in proving that $d_{\mathcal{P}}$ is a metric.

Let $z_n = (g_n b_n g_n^*)^{-1/2} g_n a_n g_n^* (g_n b_n g_n^*)^{-1/2}$, $z = (gbg^*)^{-1/2} gag^* (gbg^*)^{-1/2}$.

As before, using the fact that $L^0(M, \tau)$ is a topological *-algebra and Lemma 3.5.19, $z_n \to z$ in measure. Then Lemma 3.5.20 shows that $\mu_{z_n}(e^\lambda, \infty) \to \mu_z(e^\lambda, \infty)$ for all but countably many $\lambda$. Then Proposition 3.4.3 can be used to get a dominating function for $\lambda \mapsto \lambda(\mu_{z_n}(e^\lambda, \infty)$, and then the dominated convergence theorem can be applied.

Now because we are again in a finite von Neumann algebra, $d_{\mathcal{P}}(g_n a_n g_n, g_n b_n g_n) = d_{\mathcal{P}}(a_n, b_n) \to d_{\mathcal{P}}(a, b)$ (convergence by Lemma 3.5.21, so it must be that $d_{\mathcal{P}}(gag, gbg) = d_{\mathcal{P}}(a, b)$.

(4 Same as in the finite case

(5) As in the finite case we first show that $exp$ is a homeomorphism between $L^2(M, \tau)_{sa}$ and $\mathcal{P}$.

**Claim 10.** *Let $x, y \in L^2(M, \tau)_{sa}$. Then $||x - y||_2 \leq d_{\mathcal{P}}(e^x, e^y)$*

*Proof.* As before consider $x_n = p_n^x x p_n^x$ and similarly for $y$. Then $||x_n - y_n||_2 \leq d_{\mathcal{P}}(e^{x_n}, e^{y_n})$ by Proposition 3.5.12 since this is the finite von Neumann algebra case. We also know from item 2 that $d_{\mathcal{P}}(e^{x_n}, e^{y_n}) \to d_{\mathcal{P}}(e^x, e^y)$. It remains to show that $||x_n - y_n||_2 \to ||x - y||_2$. Now we know that $x_n - y_n \to x - y$ in measure. Furthermore we can write $||x_n - y_n||_2^2$, in a similar fashion as equation 3.3, as $2\int_0^\infty \lambda \mu_{|(x_n - y_n)|}(\lambda, \infty) \mathrm{d}\lambda$, and by Lemma 3.5.20 $\mu_{|x_n - y_n)}(\lambda, \infty) \to \mu_{|x-y|}(\lambda, \infty)$, while by Proposition 3.4.4 and Claim 9 $\mu_{|x_n - y_n|}(\lambda, \infty) \leq \mu_{|x|}(\lambda, \infty) + \mu_{|y|}(\lambda, \infty)$. So by the dominated convergence theorem $\|x_n - y_n\|_2 \to \|x - y\|_2$.

□

**Claim 11.** $\exp : L^2(M, \tau)_{sa} \to \mathcal{P}$ *is continuous*

*Proof.* Suppose $(a_k)$ is a sequence in $L^2(M, \tau)_{sa}$ converging to $a$ in $L^2$. By similar arguments as in Proposition 3.5.5 $(a_k)$ converges to $a$ in measure. Let $\lambda > 0$ be large. Let $f_\lambda : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that for $f_\lambda(x) = 0$ on $[-1/2\lambda, 1/2\lambda]$, $f_\lambda(x) = \lambda$

for $x > \lambda$, $f_\lambda(x) = -\lambda$ for $x < -\lambda$, and $f_\lambda(x) = x$ on $[-\lambda, -1/\lambda] \cup [1/\lambda, \lambda]$. It follows by Lemma 3.5.19 that $f_\lambda(a_k)$ converges to $f_\lambda(a)$ in measure, and also $e^{f_\lambda(a_k)}$ converges to $e^{f_\lambda(a)}$ in measure. Since multiplication is jointly continuous with respect to the convergence in measure topology, $z_k := e^{-f_\lambda(a_k)/2}e^{f_\lambda(a)}e^{-f_\lambda(a_k)/2} \to 1$ in measure. Now since $e^{-2\lambda} \leq e^{-f_\lambda(a_k)/2}e^{f_\lambda(a)}e^{-f_\lambda(a_k)/2} \leq e^{2\lambda}$, log is a continuous function on the spectrum of $z_k$, so that by Lemma 3.5.19 $\log z_k \to 0$ in measure.

We now show that $\log z_k$ converges to 0 in $L^2$. Now $-2\lambda \leq \log z_k \leq 2\lambda$ is uniformly bounded by $2\lambda$, $\log z_k$ is also in $L^2(M)_{sa}$.

**Claim 12.** $\sup_k \tau(1_{(0,\infty)}(|\log z_k|)) < \infty$

*Proof.* Note that $\ker(f_\lambda(a_k)) \cap \ker(f_\lambda(a)) \leq \ker(\log z_k)$ (where $\leq$ means "is a subspace of"), so $\ker(\log z_k)^\perp \leq (\ker(f_\lambda(a_k)) \cap \ker(f_\lambda(a)))^\perp = (span(\ker(f_\lambda(a_k))^\perp \cup \ker(f_\lambda(a_k))^\perp))^{cl}$. Equivalently, $1_{(0,\infty)}(|\log z_k|) \leq 1_{(0,\infty)}(|f_\lambda(a_k)|) \vee 1_{(0,\infty)}(|f_\lambda(a)|) \leq 1_{(0,\infty)}(|f_\lambda(a_k)|) + 1_{(0,\infty)}(|f_\lambda(a)|)$. Now $\tau(1_{(0,\infty)}|f_\lambda(a_k)|) = \tau(1_{(1/2\lambda,\infty)}|a_k|) \leq 4\lambda^2\|a_k\|_2^2$. Since $a_k$ is converging to $a$ in $L^2$, the right hand side is bounded for large enough $k$. The claim follows. ☐

Now $\|\log z_k\|_2^2 \leq \epsilon^2 \mu_{|\log z_k|}(0, \epsilon] + \lambda^2 \mu_{|\log z_k|}(\epsilon, \infty)$. By first letting $\epsilon \to 0$ and then $k \to \infty$ we see that $\log z_k \to 0$ in $L^2$, showing that $d_{\mathcal{P}}(e^{f_\lambda(a_k)}, e^{f_\lambda(a)}) \to 0$. Then we also have

$$\begin{aligned} d_{\mathcal{P}}(e^{a_k}, e^a) &\leq d_{\mathcal{P}}(e^{a_k}, e^{f_\lambda(a_k)}) + d_{\mathcal{P}}(e^{f_\lambda(a_k)}, e^{f_\lambda(a)}) + d_{\mathcal{P}}(e^{f_\lambda(a)}, e^a) \\ &= \|a - f_\lambda(a)\|_2 + \|a_k - f_\lambda(a_k)\|_2 + d_{\mathcal{P}}(e^{f_\lambda(a_k)}, e^{f_\lambda(a)}). \end{aligned} \tag{3.13}$$

Where e.g. $d_{\mathcal{P}}(e^{a_k}, e^{f_\lambda(a_k)}) = \|a - f_\lambda(a)\|_2$ come from $a$ and $f_\lambda$ commuting. Now $\|a - f_\lambda(a)\|_2 \to 0$ as $\lambda \to \infty$ by standard arguments. Furthermore, $\|a_k - f_\lambda(a_k)\|_2 \leq \|a - f_\lambda(a)\|_2 + \|a - a_k\|_2 + \|f_\lambda(a) - f_\lambda(a_k)\|_2$. A similar argument to that found in Claim 12 shows that $\|f_\lambda(a) - f_\lambda(a_k)\|_2 \to 0$. Thus for all large enough $\lambda$, $\|a_k - f_\lambda(a_k)\|_2 \to \|a - f_\lambda(a)\|_2$ as $k \to 0$.

We conclude that $d_{\mathcal{P}}(e^{a_k}, e^a) \to 0$. By similar arguments as in the finite case 6) follows.

☐

(6) We use arguments similar to those found in [BH99, Theorem II.3.9] in order to apply [BH99, Proposition II.1.11]. In particular, given $a_i \in \mathcal{P}$, $1 \leq i \leq 4$ consider the reduced

versions $a_{i,n} \in \mathcal{P}_{p_n}$, where $p_n = \vee_{i=1}^4 p_n^{a_i}$. Then we have already shown in 2. that for each $i, j$ $d_{\mathcal{P}_{p_n}}(a_{i,n}, a_{j,n}) \to d_{\mathcal{P}}(a_i, a_j)$. For each $n$, since $\mathcal{P}_{p_n}$ is CAT(0), $\mathcal{P}_{p_n}$ satisfies the 4-point condition. In particular, each 4-tuple $(a_{i,n})_i$ has a subembedding in Euclidean space $\mathbb{E}^2$: a 4-tuple of points $(\tilde{a}_{i,n})_i$ such that $d_{\mathcal{P}_n}(a_{1,n}, a_{2,n}) \leq d(\tilde{a}_{1,n}, \tilde{a}_{2,n})$, $d_{\mathcal{P}_n}(a_{3,n}, a_{4,n}) \leq d(\tilde{a}_{3,n}, \tilde{a}_{4,n})$, $d_{\mathcal{P}_n}(a_{i,n}, a_{j,n}) = d(\tilde{a}_{i,n}, \tilde{a}_{j,n})$ for all other $i, j$.

By translation invariance of the standard metric on $\mathbb{E}^2$, we can assume $\tilde{a}_{i,n} = \tilde{a}_i$ is the same for all $n$. Then the subembedding condition and triangle inequality show that $(\tilde{a}_{i,n})$ is contained in a compact set as $i$ and $n$ vary. In particular, by passing to a subsequence if necessary, $\tilde{a}_{i,n}$ converges to $\tilde{a}_i$ for each $i$. It follows that $\tilde{a}_i$ is a subembedding of $a_i$ for each $i$.

Now for the approximate midpoint condition, let $x, y \in \mathcal{P}$. Then the reduced version $x_n, y_n$ has approximate midpoints $z_\delta$ for every $\delta$ in the definition given in the paragraph before [BH99, Proposition II.1.11]. By considering $n$ large enough and Lemma 3.5.21 it follows that $(z_\delta)_n$ is an approximate midpoint for $x, y$ for a slightly different $\delta$. Thus $\mathcal{P}$ is CAT(0) by [BH99, Proposition II.1.11].

(7) That $\mathcal{P}^\infty$ is dense in $\mathcal{P}$ follows from $M_{sa}$ being dense in $L^2(M, \tau)$ and Claim 11 (exp: $L^2(M, \tau) \to \mathcal{P}$ is continuous). Now if $x \in \mathcal{P}^\infty$, then $x = e^y$ for $y \in M_{sa}$. Then $\|e^y\|_\infty \leq exp(\|y\|_\infty)$ and similarly for $x^{-1} = e^{-y}$, so $x \in M^\times$. Thus $\mathcal{P}^\infty \subset \mathcal{P} \cap M^\times$. Conversely if $x \in \mathcal{P} \cap M^\times$ then $\log x \in L^2_{sa}(M, \tau) \cap M = M_{sa}$, so $x \in \mathcal{P}^\infty$ □

**Corollary 3.5.24.** *Corollaries 3.5.16 and 3.5.17 also hold for $(M, \tau)$ semifinite.*

*Proof.* Arguments are similar to the finite case. □

**Lemma 3.5.25.** *Suppose $(M, \tau)$ is semifinite. Suppose $x_n, x \in M_C$, where $M_C = \{x \in M : \|x\|_\infty \leq C\}$ and $x_n \to x$ in measure. Then $x_n \to x$ in SOT*

*Proof.* As in Proposition 3.5.5 we can assume that $x = 0$. Let $\xi \in L^2(M, \tau)$. We want to show that $x_n \xi \to 0$ in $L^2$. Now as in the proof in 3.5.5 of convergence in $L^2$ implies convergence in SOT, let $\xi' \in M \cap L^2(M, \tau)$ with $\|\xi' - \xi\|_2 < \epsilon$. We first show that $x_n \xi' \to 0$ in $L^2$.

Let $p_k^n = 1_{(1/k, \infty)}(|x_n|)$ and $x_{n,k} = x_n p_k^n$. Write $x_n = x_{n,k} + x_{n,k}^\perp$. We show that $\|x_{n,k} \xi'\|_2 \to 0$ and $\|x_{n,k}^\perp \xi'\|_2 \to 0$. Now since $\|x_{n,k} \xi'\|_2 \leq \|x_{n,k}\|_2 \|\xi'\|_\infty$, and $\|x_{n,k}\|_2 \leq$

$C\mu_{x_{n,k}}(0,\infty) \to 0$ as $n \to \infty$, $\|x_{n,k}\xi'\|_2 \to 0$ as $n \to \infty$, independent of $k$. On the other hand, $\|x_{n,k}^{\perp}\xi'\|_2 \leq \|x_{n,k}^{\perp}\|_\infty \|\xi'\|_2 \to 0$ as $k \to \infty$, independent of $n$. It follows that $x_n\xi' \to 0$ in $L^2$. Now $\|x_n\xi\|_2 \leq \|x_n\xi'\|_2 + \|x_n(\xi - \xi')\|_2 \to 0$ by letting $\xi' \to \xi$ in $L^2$. □

## 3.6 Proofs of the main results

### 3.6.1 The limit operator

This subsection proves a generalization of Theorem 3.1.1. We first need a lemma.

**Lemma 3.6.1.** *Let $(M,\tau)$ be a semi-finite tracial von Neumann algebra. For any $a, b \in \mathcal{P}$ and $\sigma \geq 1$,*

$$d_{\mathcal{P}}(a^\sigma, b^\sigma) \geq \sigma d_{\mathcal{P}}(a,b).$$

*Proof.* First, assume $\tau$ is a finite trace. Let $x = \log a$, $y = \log b$. Recall that $M_{sa} \subset M$ is the set of self-adjoint elements in $M$. Because $M_{sa}$ is dense in $\mathrm{L}^2(M,\tau)_{sa}$, there exist $x_n, y_n \in M_{sa}$ with $x_n \to x$ and $y_n \to y$ in $\mathrm{L}^2(M,\tau)_{sa}$ as $n \to \infty$. Thus

$$\begin{aligned} d_{\mathcal{P}}(a^\sigma, b^\sigma) &= d_{\mathcal{P}}(e^{\sigma x}, e^{\sigma y}) = \lim_{n\to\infty} d_{\mathcal{P}}(e^{\sigma x_n}, e^{\sigma y_n}) \\ &\geq \lim_{n\to\infty} \sigma d_{\mathcal{P}}(e^{x_n}, e^{y_n}) = \sigma d_{\mathcal{P}}(e^x, e^y) = \sigma d_{\mathcal{P}}(a,b) \end{aligned}$$

where the second and third equalities follow from continuity of the exponential map (Theorem 3.5.13) and the inequality follows from Corollary 3.5.4.

Next we consider the general semi-finite case. Let $x_n, y_n$ be the reduced versions of $x, y \in L^2_{sa}(M,\tau)$. Then

$$\begin{aligned} d_{\mathcal{P}}(a^\sigma, b^\sigma) &= d_{\mathcal{P}}(e^{\sigma x}, e^{\sigma y}) = \lim_{n\to\infty} d_{\mathcal{P}}(e^{\sigma x_n}, e^{\sigma y_n}) \\ &\geq \lim_{n\to\infty} \sigma d_{\mathcal{P}}(e^{x_n}, e^{y_n}) = \sigma d_{\mathcal{P}}(e^x, e^y) = \sigma d_{\mathcal{P}}(a,b) \end{aligned}$$

Where the first inequality follows from an argument similar to that found in Lemma 3.5.21, the inequality follows from the above finite case, and the second inequality follows form Lemma 3.5.21 □

We can now prove a slight generalization of Theorem 3.1.1 by expanding the range of the cocycle.

**Theorem 3.6.2.** *Let $(X, \mu)$ be a standard probability space, $f : X \to X$ an ergodic measure-preserving transformation, $(M, \tau)$ a semi-finite von Neumann algebra with faithful normal trace $\tau$. Let $c : \mathbb{N} \times X \to \mathrm{GL}^2(M, \tau)$ be a cocycle:*

$$c(n + m, x) = c(n, f^m x)c(m, x) \quad \forall n, m \in \mathbb{N},\ \mu - a.e.\ x \in X.$$

*Let $\pi : \mathrm{GL}^2(M, \tau) \to \mathrm{Isom}(\mathcal{P})$ be the map $\pi(g)x = gxg^*$ where $\mathrm{Isom}(\mathcal{P})$ is the group of isometries of $\mathcal{P}$. Suppose $\pi \circ c$ is measurable with respect to the compact-open topology on $\mathrm{Isom}(\mathcal{P})$ and*

$$\int_X L(c(1, x)^* c(1, x))\ d\mu(x) = \int_X \| \log(|c(1, x)|^2)\|_2\ d\mu(x) = \int_X d_{\mathcal{P}}(1, |c(1, x)|^2)\ d\mu(x) < \infty.$$

*Then for almost every $x \in X$, the following limit exists:*

$$\lim_{n\to\infty} \frac{L(c(n, x)^* c(n, x))}{n} = D.$$

*Moreover, if $D > 0$ then for a.e. $x$, there exists $\Lambda(x) \in L^2(M, \tau)$ with $\Lambda(x) \geq 0$ such that*

$$\log \Lambda(x) := \lim_{n\to\infty} \log \left( [c(n, x)^* c(n, x)]^{1/2n} \right) \in L^2(M, \tau)$$

*exists for a.e. $x$ and*

$$\lim_{n\to\infty} \frac{1}{n} d_{\mathcal{P}}(\Lambda(x)^n, |c(n, x)|) = 0.$$

*Proof.* We will use Theorem 3.1.5. So let $(Y, d) = (\mathcal{P}, d_{\mathcal{P}})$. By Corollaries 3.5.14 and 3.5.15 for the finite case and Proposition 3.5.18 for the semifinite case, $(\mathcal{P}, d_{\mathcal{P}})$ is a complete CAT(0) metric space. Let $y_0 = I \in Y$. Observe that the map

$$\mathbb{N} \times X \to \mathrm{GL}^2(M, \tau), \quad (n, x) \mapsto c(n, x)^*$$

is a reverse cocycle. Also

$$d_{\mathcal{P}}(y_0, c(1,x)^*.y_0) = \| \log(c(1,x)^* c(1,x))\|_2 = L(c(1,x)^* c(1,x)).$$

So

$$\int_X d_{\mathcal{P}}(y_0, c(1,x)^*.y_0)\ d\mu(x) < \infty.$$

Theorem 3.1.5 implies: for almost every $x \in X$, the following limit exists:

$$\lim_{n\to\infty} \frac{d_{\mathcal{P}}(y_0, c(n,x)^*.y_0)}{n} = \lim_{n\to\infty} \frac{L(c(n,x)^* c(n,x))}{n} = D.$$

Moreover, if $D > 0$ then for almost every $x$ there exists a unique unit-speed geodesic ray $\gamma(\cdot, x)$ in $\mathcal{P}$ starting at $I$ such that

$$\lim_{n\to\infty} \frac{1}{n} d_{\mathcal{P}}(\gamma(Dn,x), c(n,x)^*.y_0) = 0.$$

By Corollaries 3.5.17 for the finite case and 3.5.18 for the semifinite case,

$$\gamma(t,x) = \exp(t\xi(x))$$

for some unique unit norm element $\xi(x) \in \mathrm{L}^2(M,\tau)_{sa}$. Let $\Lambda(x) = \exp(D\xi(x)/2)$. Thus we have

$$\lim_{n\to\infty} \frac{1}{n} d_{\mathcal{P}}(\Lambda(x)^{2n}, c(n,x)^* c(n,x)) = 0.$$

Equivalently,

$$\lim_{n\to\infty} \frac{1}{n} \| \log(\Lambda(x)^{-n} c(n,x)^* c(n,x) \Lambda^{-n})\|_2 = \lim_{n\to\infty} \frac{1}{n} L(\Lambda(x)^{-n} c(n,x)^* c(n,x) \Lambda^{-n}) = 0.$$

Observe that

$$\begin{aligned}
\lim_{n\to\infty} \| \log \Lambda(x) - \log[c(n,x)^* c(n,x)]^{1/2n}\|_2 &\leq \lim_{n\to\infty} d_{\mathcal{P}}(\Lambda(x), [c(n,x)^* c(n,x)]^{1/2n}) \\
&\leq \lim_{n\to\infty} \frac{1}{n} d_{\mathcal{P}}(\Lambda(x)^{2n}, c(n,x)^* c(n,x)) = 0
\end{aligned}$$

where the first inequality is by Proposition 3.5.12 for the finite case and Claim 10 for the semifinite case, and the second is by Lemma 3.6.1. This concludes the proof.

□

In order to show that Theorem 3.6.2 implies Theorem 3.1.1, we need to show how SOT-measurability of the cocycle $c$ in Theorem 3.1.1 implies that $\pi \circ c$ is measurable with respect to the compact-open topology.

We will need the next few lemmas to clarify the measurability hypothesis on the cocycle. The next lemma is probably well-known.

**Lemma 3.6.3.** *Let $(Y, d)$ be a metric space. Then the pointwise convergence topology on the isometry group $\mathrm{Isom}(Y, d)$ is the same as the compact-open topology.*

*Proof.* It is immediate that the pointwise convergence topology is contained in the compact-open topology. To show the opposite inclusion, let $K \subset Y$ be compact, $O \subset Y$ be open and suppose $g \in \mathrm{Isom}(Y, d)$ is such that $gK \subset O$. Let $g_n \in \mathrm{Isom}(Y, d)$ and suppose $g_n \to g$ pointwise. It suffices to show that $g_n K \subset O$ for all sufficiently large $n$.

Because $K$ is compact, there are a finite subset $F \subset K$ and for every $x \in F$, a radius $\epsilon_x > 0$ such that if $B(x, \epsilon_x) \subset Y$ is the open ball of radius $\epsilon_x$ centered at $x$ then

$$K \subset \cup_{x \in F} B(x, \epsilon_x) \subset O.$$

By compactness again, there exist $0 < \epsilon'_x < \epsilon_x$ such that

$$K \subset \cup_{x \in F} B(x, \epsilon'_x).$$

Since $g_n \to g$ pointwise, there exists $N$ such that $n > N$ implies $d(g_n x, gx) \leq \epsilon_x - \epsilon'_x$ for all $x \in F$. Therefore,

$$g_n K \subset \cup_{x \in F} B(g_n x, \epsilon'_x) \subset \cup_{x \in F} B(gx, \epsilon_x) \subset O$$

as required. □

**Theorem 3.6.4.** *Suppose $(M, \tau)$ is $\sigma$-finite and semi-finite. Then*

1. *the operator norm $M \to \mathbb{R}$, $T \mapsto \|T\|_\infty$ is SOT-Borel;*

2. *the inverse operator norm* $M^\times \to \mathbb{R}$, $T \mapsto \|T^{-1}\|_\infty$ *is SOT-Borel;*

3. *a subset* $E \subset M$ *is SOT-Borel if and only if it is WOT-Borel;*

4. *the adjoint* $M \to M$, $T \mapsto T^*$ *is SOT-Borel;*

5. *the multiplication map* $M \times M \to M$, $(S,T) \mapsto ST$ *is SOT-Borel;*

6. *the map* $\mathcal{P} \cap M^\times \to M$ *defined* $T \mapsto \log T$ *is SOT-Borel;*

7. *the map* $M \cap L^2(M,\tau)$, $T \mapsto \|T\|_2$ *is SOT-Borel;*

8. *for any* $x, y \in \mathcal{P} \cap M^\times$ *the map* $M^\times \cap \mathrm{GL}^2(M,\tau) \to \mathbb{R}$ *defined by* $T \mapsto d_{\mathcal{P}}(TxT^*, y)$ *is SOT-Borel.*

*Proof.* (1)Let $\mathcal{H}_0 \subset \mathcal{H}$ be a countable dense subset not containing zero. For $h \in \mathcal{H}_0$, let $\phi_h : M \to [0,\infty)$ be the function

$$\phi_h(a) = \|ah\|/\|h\|.$$

By definition, $\phi_h$ is continuous in the strong operator topology. Moreover, $\|a\| = \sup_{h \in \mathcal{H}_0} \phi_h(a)$. So the operator norm is the supremum of a countable set of continuous functions. This implies it is Borel.

(2) If $T$ is invertible with bounded inverse then

$$\|a^{-1}\|_\infty^{-1} = \inf_{h \in \mathcal{H}_0} \phi_h(a).$$

This proves $a \mapsto \|T^{-1}\|_\infty$ is Borel in the strong operator topology.

(3) This follows from

Because every WOT-open set is open in the SOT, it follows that every WOT-Borel set is SOT-Borel.

Given $v, w \in \mathrm{L}^2(M,\tau)$, define $\phi_{v,w} : M \to [0,\infty)$ by $\phi_{v,w}(T) = \|Tv - w\|$. The SOT-topology is the smallest topology that makes all of the functions $\phi_{v,w}$ continuous. By [Johnson Corollary 1], it suffices to show that each $\phi_{v,w}$ is WOT-Borel.

Observe that

$$\|Tv - w\|^2 = \|Tv\|^2 - 2\,\mathrm{Re}(\langle Tv, w\rangle) + \|w\|^2.$$

Since $T \mapsto \langle Tv, w\rangle$ is WOT-Borel, it suffices to show that $T \mapsto \|Tv\|$ is WOT-Borel. Let $S \subset \mathcal{H}$ be a countable dense subset of the unit sphere. Then

$$\|Tv\| = \sup_{u \in S} \langle Tv, u\rangle.$$

Since each of the functions $T \mapsto \langle Tv, u\rangle$ is WOT-Borel, this proves that $T \mapsto \|Tv\|$ is WOT-Borel.

(4) Since $\langle Tv, w\rangle = \overline{\langle T^* w, v\rangle}$, it follows that $T \mapsto T^*$ is WOT-Borel. So item (3) implies the adjoint map is SOT-Borel.

(5) Because the operator norm is SOT-Borel by item (1), it suffices to prove that if $B \subset M$ is a ball in the operator norm then the map $B \times B \to M$ defined by $(S, T) \mapsto ST$ is SOT-continuous. To see this, let $v \in \mathrm{L}^2(M, \tau)$ and suppose $(T_n), (S_n) \subset B$ are sequences with $T_n \to T, S_n \to S$ in SOT (as $n \to \infty$). Then

$$\begin{aligned} \|T_n S_n v - TSv\| &\leq \|T_n S_n v - T_n S v\| + \|T_n S v - TSv\| \\ &\leq \|T_n\|_\infty \|S_n v - Sv\| + \|T_n S v - TSv\|. \end{aligned}$$

By assumption, $\|T_n\|_\infty$ is uniformly bounded in $n$. So the inequality above implies $\lim_{n\to\infty} \|T_n S_n v - TSv\| = 0$. Since $v$ is arbitrary, this implies $T_n S_n \to TS$ in SOT as $n \to \infty$.

(6) For $D > 0$, let

$$M_D = \{T \in \mathcal{P} : \ \|T\|_\infty \leq D \text{ and } \|T^{-1}\|_\infty \leq D\}.$$

By items (1) and (2), $M_D$ is SOT-Borel. So it suffices to show that the map $M_D \to M$ given by $T \mapsto \log T$ is SOT-continuous. This follows from [Corollary, Kaplansky, A theorem on rings of operators] since log is bounded on $[D^{-1}, D]$.

(7) Let $p_1, p_2, \ldots$ be a sequence of pairwise-orthogonal finite projections $p_i \in M$ with $\sum_{i=1}^\infty p_i = I$.

**Claim 13.** *Let $T \in L^2(M,\tau)$. Then $\sum_{i=1}^n p_i T$ converges to $T$ in $L^2$.*

*Proof.* Let $S_n = \sum_{i=1}^n p_i L^2(M,\tau)$. It suffices to show that $\cup_{n=1}^\infty S_n$ is dense in $L^2(M,\tau)$, for then $\sum_{i=1}^n p_i T$, being the orthogonal projection of $T$ onto $S_n$, minimizes the distance from $T$ to $S_n$. Now suppose that there exists $v \in L^2(M,\tau$ such that for all $T$ and $n$, $\langle \sum_{i=1}^n p_i T, v\rangle = 0$. Then since $\sum_{i=1}^n p_i$ converges to $I$ in SOT, the convergence also happens in WOT. Since WOT is intrinsic on norm-bounded sets of $M$, $\langle (\sum_{i=1}^n p_i)T, v\rangle \to \langle T, v\rangle = 0$. Since this happens for all $T \in L^2(M,\tau)$, $v = 0$. The result follows. □

Now $\langle p_i T, p_j T\rangle = 0$ for all $i \neq j$ and $\sum_{i=1}^\infty p_i T = T$. Therefore, along with the above claim

$$\|T\|_2^2 = \sum_{i=1}^\infty \|p_i T\|_2^2 = \sum_{i=1}^\infty \|T^* p_i\|_2^2$$

where the last equality follows from the tracial property of $\tau$. So it suffices to prove that for any fixed finite projection $p \in M$, the map $T \mapsto \|T^* p\|_2^2$ is SOT-Borel. This follows from item (4) which states that the adjoint map is SOT-Borel (since $p \in \mathrm{L}^2(M,\tau)$).

(8) By definition,

$$d_{\mathcal{P}}(TxT^*, y) = \|\log(y^{-1/2} TxT^* y^{-1/2}\|_2.$$

So this item follows from the previous items. □

**Corollary 3.6.5.** *Suppose $(M,\tau)$ is $\sigma$-finite and semi-finite. Then $\pi : M^\times \to \mathrm{Isom}(\mathcal{P})$ is Borel as a map from $M^\times$ with the SOT to $\mathrm{Isom}(\mathcal{P})$ with the pointwise convergence topology.*

*Proof.* By definition of the pointwise convergence topology, it suffices to show that for every $x, y \in \mathcal{P}$, the map $T \mapsto d_{\mathcal{P}}(TxT^*, y)$ is SOT-Borel. By (8) of Proposition 3.5.18, $M^\times \cap \mathcal{P}$ is dense in $\mathcal{P}$. So it suffices to show that for every $x, y \in M^\times \cap \mathcal{P}$, the map $T \mapsto d_{\mathcal{P}}(TxT^*, y)$ is SOT-Borel. This is item (8) of Theorem 3.6.4. □

**Corollary 3.6.6.** *The hypotheses of Theorem 3.1.1 imply the hypotheses of Theorem 3.6.2. In particular, Theorem 3.1.1 is true.*

*Proof of Theorem 3.1.1.* We assume the hypotheses of Theorem 3.1.1. In particular, $c : \mathbb{N} \times X \to M^\times \cap \mathrm{GL}^2(M,\tau)$ is a strongly measurable cocycle (which means measurable with respect to the strong operator topology). By Lemma 3.6.3, it suffices to show that $\pi \circ c$ is measurable with respect to the pointwise convergence topology on $\mathrm{Isom}(\mathcal{P})$. Let $S(x) = c(1,x)$. By Corollary 3.6.5, $\pi \circ S$ is measurable with respect to the pointwise convergence topology

By the cocycle equation, $c(n,x) = S(f^n x) \cdots S(x)$. If $(Y,d)$ is any metric space and $\mathrm{Isom}(Y,d)$ is equipped with the pointwise convergence topology then the multiplication map $(g,h) \mapsto gh$ from $\mathrm{Isom}(Y,d) \times \mathrm{Isom}(Y,d) \to \mathrm{Isom}(Y,d)$ is jointly continuous. It follows that $x \mapsto \pi(c(n,x))$ is measurable with respect to the pointwise convergence topology on $\mathrm{Isom}(\mathcal{P})$ for any $n \in \mathbb{Z}$.

□

### 3.6.2 Determinants

This section proves Theorem 3.1.3. Following [HS07, Definition 2.1], we let $M^\Delta$ be the set of all $x \in \mathrm{L}^0(M,\tau)$ such that

$$\int_0^\infty \log^+(t)\ d\mu_{|x|}(t) < \infty.$$

For $x \in M^\Delta$, the integral $\int_0^\infty \log t\ d\mu_{|x|}(t) \in [-\infty,\infty)$ is well-defined. The **Fuglede-Kadison determinant of** $x$ is

$$\Delta(x) := \exp\left(\int_0^\infty \log t\ d\mu_{|x|}(t)\right) \in [0,\infty).$$

For the sake of context, we mention that by [HS07, Lemma 2.3 and Proposition 2.5], if $x, y \in M^\Delta$ and $\Delta(x) > 0$ then $x^{-1} \in M^\Delta$, $\Delta(x^{-1}) = \Delta(x)^{-1}$, $xy \in M^\Delta$, and $\Delta(xy) = \Delta(x)\Delta(y)$.

**Proposition 3.6.7.** *Suppose $\tau$ is a finite trace. Then $\mathrm{GL}^2(M,\tau) \subset M^\Delta$. Moreover, $\Delta : \mathcal{P} \to (0,\infty)$ is continuous.*

*Proof.* Let $x \in \mathrm{GL}^2(M,\tau)$. By definition, $\log|x| \in \mathrm{L}^2(M,\tau)$. Since $\tau$ is a finite trace, $\mathrm{L}^2(M,\tau) \subset \mathrm{L}^1(M,\tau)$. Thus $\log|x| \in \mathrm{L}^1(M,\tau)$ and therefore $\log^+|x| \in \mathrm{L}^1(M,\tau)$. So $x \in M^\Delta$.

Now let $(x_n)_n \subset \mathcal{P}$ and suppose $\lim_{n\to\infty} x_n = x \in \mathcal{P}$. By Proposition 3.5.12, $\log|x_n|$ converges to $\log|x|$ in $\mathrm{L}^2(M,\tau)$. Therefore, $\log|x_n|$ converges to $\log|x|$ in $\mathrm{L}^1(M,\tau)$. But the trace $\tau : \mathrm{L}^1(M,\tau) \to \mathbb{C}$ is norm-continuous. So $\tau(\log|x_n|) \to \tau(\log|x|)$. Since $\exp : \mathbb{R} \to \mathbb{R}$ is continuous and $\Delta(x) = \exp(\tau(\log|x|))$, this finishes the proof. □

Theorem 3.1.3 follows immediately from Proposition 3.6.7 and Theorem 3.1.1.

### 3.6.3 Growth rates

In this subsection, we prove Theorem 3.1.4. The proof uses Theorem 3.1.1 as a black-box. The extra ingredients needed to prove the theorem are general approximation results for powers of a single operator. These results will also be needed in later subsections to prove Theorem 3.1.2.

**Definition 15.** Let $a \in \mathrm{L}^0(M,\tau)$ be a positive operator and $\xi \in \mathrm{dom}(a)$. By the spectral theorem there exists a unique positive measure $\nu$ on $\mathbb{C}$ such that $\nu([0,\infty)) = \|\xi\|_2^2$ and for every bounded, Borel function $f\colon [0,\infty) \to \mathbb{C}$,

$$\langle f(a)\xi, \xi\rangle = \int f(s)\,\mathrm{d}\nu(s).$$

Moreover, for a Borel function $f\colon [0,\infty) \to \mathbb{C}$ we have that $\xi \in \mathrm{dom}(f(a))$ if and only if $\int |f(s)|^2\,\mathrm{d}\nu(s) < \infty$, and $\langle f(a)\xi,\xi\rangle = \int f\,\mathrm{d}\nu$ if $\xi \in \mathrm{dom}(f(a))$. The measure $\nu$ is called the **spectral measure of $a$ with respect to $\xi$**. Let $\rho(\nu) \in [0,\infty]$ be the smallest number such that $\nu$ is supported on the interval $[0,\rho(\nu)]$.

**Lemma 3.6.8.** *Let $a \in L^0(M,\tau)_{sa}$, $\xi \in \bigcap_{n=1}^\infty \mathrm{dom}(a^n)$ and let $\nu$ be the spectral measure of $a$ with respect to $\xi$. Then*

$$\rho(\nu) = \lim_{n\to\infty} \|a^n\xi\|_2^{1/n} \in [0,\infty].$$

*Moreover, $\xi \in 1_{[0,t]}(a)(L^2(M,\tau))$ if and only if $\lim_{n\to\infty}\|a^n\xi\|_2^{1/n} \le t$ (for any $t \in [0,\infty]$).*

*Proof.* Since $\xi \in \mathrm{dom}(a^n)$, $\int s^{2n}\,\mathrm{d}\nu(s) < \infty$ for every $n \in \mathbb{N}$. Thus

$$\|a^n\xi\|_2^{1/n} = \langle a^{2n}\xi,\xi\rangle^{1/2n} = \left(\int s^{2n}\,\mathrm{d}\nu(s)\right)^{1/2n}. \tag{3.14}$$

It is a standard measure theory exercise that the limit of $\left(\int s^{2n}\,\mathrm{d}\nu(s)\right)^{1/2n}$ as $n \to \infty$ exists and equals $\rho(\nu)$.

Now suppose that $t > 0$ and that $\lim_{n\to\infty} \|a^n\xi\|_2^{1/n} \leq t$. Then, by the above comment we have that $\nu$ is supported on $[0,t]$. Thus:

$$\|\xi - 1_{[0,t]}(a)\xi\|_2^2 = \int |1 - 1_{[0,t]}(s)|^2\,\mathrm{d}\nu(s) = 0.$$

So $\xi \in 1_{[0,t]}(a)(\mathrm{L}^2(M,\tau))$.

For the converse, suppose $\xi \in 1_{[0,t]}(a)(\mathrm{L}^2(M,\tau))$. Then

$$\lim_{n\to\infty} \|a^n\xi\|_2^{1/n} \quad = \quad \lim_{n\to\infty} \|a^n 1_{[0,t]}(a)\xi\|_2^{1/n} = \lim_{n\to\infty} \left(\int_0^t s^{2n}\,\mathrm{d}\nu(s)\right)^{1/2n} \leq t.$$

□

**Lemma 3.6.9.** *For $a \in L^0(M,\tau)_{sa}$ and $\xi \in L^2(M,\tau)$, let $\nu_{a,\xi}$ be the spectral measure of $a$ with respect to $\xi$. Then the map*

$$L^0(M,\tau)_{sa} \times L^2(M,\tau) \to \mathrm{Prob}([0,\infty)), \quad (a,\xi) \mapsto \nu_{a,\xi}$$

*is continuous with respect to the measure topology on $L^0(M,\tau)_{sa}$, the norm topology on $L^2(M,\tau)$ and the weak topology on* $\mathrm{Prob}([0,\infty))$.

*Proof.* For $n \in \mathbb{N}$, let $a, b_n \in \mathrm{L}^0(M,\tau)_{sa}$ and $\xi, \xi_n \in \mathrm{L}^2(M,\tau)$. Assume:

- $b_n \to a$ in measure,
- $\|\xi_n - \xi\|_2 \to 0$ as $n \to \infty$.

Let $\nu$ be the spectral measure of $a$ with respect to $\xi$ and let $\nu_n$ be the spectral measure of $b_n$ with respect to $\xi_n$. It suffices to show $\nu_n \to \nu$ weakly as $n \to \infty$.

Let $f \in C(\mathbb{R})$ be bounded. By Lemma 3.5.19, $f(b_n) \to f(a)$ in measure. Since $\|f(b_n)\|_\infty, \|f(a)\|_\infty \leq \|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$, Lemma 3.5.25 implies that $f(b_n) \to f(a)$

in the strong operator topology. Hence,

$$\begin{aligned}
&\left|\int f(s)\,\mathrm{d}\nu_n(s) - \int f(s)\,\mathrm{d}\nu(s)\right| = |\langle f(b_n)\xi_n,\xi_n\rangle - \langle f(a)\xi,\xi\rangle| \\
\leq\ & |\langle f(b_n)\xi_n,\xi_n\rangle - \langle f(b_n)\xi_n,\xi\rangle| + |\langle f(b_n)\xi_n,\xi\rangle - \langle f(b_n)\xi,\xi\rangle| + |\langle f(b_n)\xi,\xi\rangle - \langle f(a)\xi,\xi\rangle| \\
\leq\ & \|f\|_\infty \|\xi_n - \xi\|_2(\|\xi_n\|_2 + \|\xi\|_2) + |\langle f(b_n)\xi,\xi\rangle - \langle f(a)\xi,\xi\rangle|\,.
\end{aligned}$$

Since $\|\xi_n - \xi\|_2 \to 0$ by assumption and $f(b_n) \to f(a)$ in the SOT, this shows $\int f(s)\,\mathrm{d}\nu_n(s) \to \int f(s)\,\mathrm{d}\nu(s)$ as $n \to \infty$. Since $f$ is arbitrary, $\nu_n \to \nu$ weakly. □

**Definition 16.** If $a \in \mathrm{L}^0(M,\tau)$ and $\xi \in \mathrm{L}^2(M,\tau) \setminus \mathrm{dom}(a)$, then let $\|a\xi\|_2 = +\infty$.

**Definition 17.** Given $\xi \in \mathrm{L}^2(M,\tau)$, let $\Sigma(\xi)$ be the set of all sequences $(\xi_n)_n \subset \mathrm{L}^2(M,\tau)$ such $\lim_{n\to\infty} \|\xi - \xi_n\|_2 = 0$. Given a sequence $\mathbf{c} = (c_n)_n \subset \mathrm{L}^0(M,\tau)$ and $\xi \in \mathrm{L}^2(M,\tau)$, define the **upper and lower smooth growth rates** of $\mathbf{c}$ with respect to $\xi$ by

$$\begin{aligned}
\underline{\mathrm{Gr}}(\mathbf{c}|\xi) &= \inf\big\{\liminf_{n\to\infty} \|c_n\xi_n\|_2^{1/n} :\ (\xi_n)_n \in \Sigma(\xi)\big\} \\
\overline{\mathrm{Gr}}(\mathbf{c}|\xi) &= \inf\big\{\limsup_{n\to\infty} \|c_n\xi_n\|_2^{1/n} :\ (\xi_n)_n \in \Sigma(\xi)\big\}.
\end{aligned}$$

**Lemma 3.6.10.** *For $n \in \mathbb{N}$, let $c_n \in L^0(M,\tau)$, $a \in L^0(M,\tau)$ with $a \geq 0$ and $\xi \in L^2(M,\tau)$. Let $\mathbf{c} = (c_n)_n$. Assume:*

- *$|c_n|^{1/n} \to a$ in measure as $n \to \infty$.*
- *$\xi \in \bigcap_{n=1}^\infty \mathrm{dom}(a^n)$.*

*Then*

$$\underline{\mathrm{Gr}}(\mathbf{c}|\xi) = \lim_{n\to\infty} \|a^n\xi\|_2^{1/n} = \overline{\mathrm{Gr}}(\mathbf{c}|\xi).$$

*Proof.* Let $\nu$ be the spectral measure of $a$ with respect to $\xi$. Let $\rho(\nu) \geq 0$ be the smallest number such that $\nu$ is supported on $[0,\rho(\nu)]$. By Lemma 3.6.8, $\rho(\nu) = \lim_{n\to\infty} \|a^n\xi\|_2^{1/n}$.

It is immediate that $\underline{\mathrm{Gr}}(\mathbf{c}|\xi) \leq \overline{\mathrm{Gr}}(\mathbf{c}|\xi)$. So it suffices to show $\rho(\nu) \leq \underline{\mathrm{Gr}}(\mathbf{c}|\xi)$ and $\overline{\mathrm{Gr}}(\mathbf{c}|\xi) \leq \rho(\nu)$.

We first show $\rho(\nu) \leq \underline{\mathrm{Gr}}(\mathbf{c}|\xi)$. So let $(\xi_n)_n \in \Sigma(\xi)$. Let $b_n = |c_n|^{1/n}$. By hypothesis, $b_n \to a$ in measure. Let $\nu_n$ be the spectral measure of $\xi_n$ with respect to $b_n$. By Lemma

3.6.9, $\nu_n \to \nu$ weakly. By (3.14) and Fatou's Lemma we have for every $m \in \mathbb{N}$ :

$$\begin{aligned}\|a^m\xi\|_2^{1/m} &= \left(\int s^{2m}\,\mathrm{d}\nu(s)\right)^{1/2m} = \left(2m\int \lambda^{2m-1}\nu((\lambda,\infty))\,\mathrm{d}\lambda\right)^{1/2m}\\ &\le \left(2m\liminf_{n\to\infty}\int \lambda^{2m-1}\nu_n((\lambda,\infty))\,\mathrm{d}\lambda\right)^{1/2m} = \left(\liminf_{n\to\infty}\int s^{2m}\,\mathrm{d}\nu_n(s)\right)^{1/2m}\\ &\le \liminf_{n\to\infty}\left(\int s^{2n}\,\mathrm{d}\nu_n(s)\right)^{1/2n} = \liminf_{n\to\infty}\|c_n\xi_n\|_2^{1/n}.\end{aligned}$$

So

$$\sup_m \|a^m\xi\|_2^{1/m} \le \liminf_{n\to\infty}\|c_n\xi_n\|_2^{1/n}.$$

By Lemma 3.6.8,

$$\rho(\nu) = \lim_{n\to\infty}\|a^n\xi\|_2^{1/n} \le \liminf_{n\to\infty}\|c_n\xi_n\|_2^{1/n}.$$

Since $(\xi_n)_n$ is arbitrary, this shows $\rho(\nu) \le \underline{\mathrm{Gr}}(\mathbf{c}|\xi)$.

Next we will show $\overline{\mathrm{Gr}}(\mathbf{c}|\xi) \le \rho(\nu)$. So let $\varepsilon > 0$. Choose a continuous function $f : [0,\infty) \to [0,1]$ such that $f(t) = 1$ for all $t \in [0,\rho(\nu)]$ and $f(t) = 0$ for all $t \ge \rho(\nu)+\varepsilon$. Let $\xi_n = f(b_n)\xi$.

We claim that $\xi_n \to \xi$ in $\mathrm{L}^2(M,\tau)$. First observe that

$$\langle f(a)\xi,\xi\rangle = \int f\ \mathrm{d}\nu = \int 1\ \mathrm{d}\nu = \|\xi\|_2^2.$$

Since $\|f(a)\|_\infty \le 1$, we must have $f(a)\xi = \xi$.

Next, let $\nu_n'$ be the spectral measure of $b_n$ with respect to $\xi$. By Lemma 3.5.19, $f(b_n) \to f(a)$ in measure. Since $\mathrm{L}^0(M,\tau)$ is a topological $*$-algebra in the measure topology, $f(b_n)\xi \to f(a)\xi$ in measure. Note

$$\begin{aligned}\|\xi_n-\xi\|_2^2 &= \|(1-f(b_n))\xi\|_2^2 = \langle(1-f(b_n))\xi,(1-f(b_n))\xi\rangle\\ &= \langle(1-f(b_n))^2\xi,\xi\rangle = \int(1-f)^2\ \mathrm{d}\nu_n'.\end{aligned}$$

By Lemma 3.6.9, $\nu_n' \to \nu$ weakly. So $\int(1-f)^2\ d\nu_n' \to \int(1-f)^2\ d\nu$ as $n\to\infty$. Since $\nu$ is supported on $[0,\rho(\nu)]$ and $1-f = 0$ on $[0,\rho(\nu)]$, it follows that $\int(1-f)^2\ d\nu_n' \to 0$ as $n \to \infty$. This proves that $\|\xi_n-\xi\|_2 \to 0$ as $n\to\infty$. Thus, $(\xi_n)_n \in \Sigma(\xi)$.

Let $\nu_n$ be the spectral measure of $b_n$ with respect to $\xi_n$. We claim that $\mathrm{d}\nu_n = f^2 \mathrm{d}\nu'_n$. To see this, let $g : [0, \infty) \to \mathbb{R}$ be a continuous bounded function. Then

$$\begin{aligned}\int g \, d\nu_n &= \langle g(b_n)\xi_n, \xi_n \rangle = \langle g(b_n) f(b_n)\xi, f(b_n)\xi \rangle \\ &= \langle f(b_n) g(b_n) f(b_n)\xi, \xi \rangle = \int g f^2 \, d\nu'_n.\end{aligned}$$

Since $g$ is arbitrary, this proves $\mathrm{d}\nu_n = f^2 \mathrm{d}\nu'_n$.

By Lemma 3.6.9, $\nu_n \to \nu$ weakly. So

$$\begin{aligned}\overline{\mathrm{Gr}}(\mathbf{c}|\xi) &\leq \limsup_{n\to\infty} \|c_n \xi_n\|_2^{1/n} = \limsup_{n\to\infty} \langle |c_n|^2 \xi_n, \xi_n \rangle^{1/2n} \\ &= \limsup_{n\to\infty} \langle b_n^{2n} \xi_n, \xi_n \rangle^{1/2n} = \limsup_{n\to\infty} \left( \int t^{2n} \, \mathrm{d}\nu_n(t) \right)^{1/2n} \\ &= \limsup_{n\to\infty} \left( \int t^{2n} f(t)^2 \, \mathrm{d}\nu'_n(t) \right)^{1/2n} \leq \rho(\nu) + \varepsilon.\end{aligned}$$

The last inequality occurs because $f(t) = 0$ for all $t > \rho(\nu) + \varepsilon$. Since $\varepsilon$ is arbitrary, $\overline{\mathrm{Gr}}(\mathbf{c}|\xi) \leq \rho(\nu)$.

□

**Corollary 3.6.11.** *Let $X, \mu, f, M, \tau, c, \Lambda$ be as in Theorem 3.6.2. Then for a.e. $x \in X$ and every $\xi \in L^2(M, \tau)$,*

$$\lim_{n\to\infty} \|\Lambda(x)^n \xi\|_2^{1/n} = \overline{\mathrm{Gr}}(\mathbf{c}(x)|\xi) = \underline{\mathrm{Gr}}(\mathbf{c}(x)|\xi)$$

*where $\mathbf{c}(x) = (c(n,x))_n$. In particular, Theorem 3.1.4 is true.*

*Proof.* Apply Lemma 3.6.10 with $a = \Lambda(x)$, $c_n = c(n, x)$. Theorem 3.1.4 now follows from Corollary 3.6.6. □

### 3.6.4 Essentially dense subspaces

In this section, we review the notion of an essentially dense subspace. This is used in the last section to prove Theorem 3.1.2.

**Definition 18.** Let $(M,\tau)$ be a semi-finite tracial von Neumann algebra. A linear subspace $V \subseteq L^2(M,\tau)$ is called **right-invariant** if $Vx \subseteq V$ for all $x \in M$. We say that a right-invariant subspace $\mathcal{D}$ of $L^2(M,\tau)$ is **essentially dense** if for every $\varepsilon > 0$, there is a projection $p \in M$ so that $\tau(1-p) \leq \varepsilon$, and $\mathcal{D} \supseteq pL^2(M,\tau)$. If $\mathcal{H}$ is a closed subspace of $L^2(M,\tau)$ and $W \subseteq \mathcal{H}$ is a right-invariant subspace, we say that $W$ is essentially dense in $\mathcal{H}$ if there exists $\mathcal{D}$ essentially dense in $L^2(M,\tau)$ such that $\mathcal{D} \cap \mathcal{H} = W$.

It is a fact that if $a \in L^0(M,\tau)$, then $\mathrm{dom}(a)$ is essentially dense (see da Silva's lecture notes, Proposition 1.4.17), and that the intersection of countably many essentially dense subspaces is essentially dense.

If $\mathcal{H} \subseteq L^2(M,\tau)$ is closed and right-invariant then the orthogonal projection onto $\mathcal{H}$, denoted $p_{\mathcal{H}}$, is in $M$ as a consequence of the double commutant theorem.

Technically, our definition of *essentially dense in* $\mathcal{H}$ is different from the one in [Lüc02, Definition 8.1]. The next lemma shows that they are in fact equivalent.

**Lemma 3.6.12.** *Let $(M,\tau)$ be a semi-finite tracial von Neumann algebra, let $\mathcal{H} \subseteq L^2(M,\tau)$ be a closed, right-invariant subspace, and let $W \subseteq \mathcal{H}$ be a right-invariant subspace. Then the following are equivalent:*

1. *$W$ is essentially dense in $\mathcal{H}$,*

2. *there is an increasing sequence of projections $p_n \in M$ so that $p_n \to p_{\mathcal{H}}$ in the strong operator topology, $\tau(p_{\mathcal{H}} - p_n) \to 0$, and $W \supseteq p_n L^2(M,\tau)$.*

*Proof.* (2) implies (1): Let $\mathcal{D} = W + (1-p_{\mathcal{H}})L^2(M,\tau)$, then clearly $\mathcal{D} \cap \mathcal{H} = W$. Let $q_n = 1 - p_{\mathcal{H}} + p_n$.

Then $\mathcal{D} \supseteq q_n L^2(M,\tau)$. Since $p_n(1-p_{\mathcal{H}}) = 0$, we also have that $q_n$ is an orthogonal projection. Also $\tau(1-q_n) \to 0$. Thus $\mathcal{D}$ is essentially dense.

(1) implies (2): Write $W = \mathcal{D} \cap \mathcal{H}$, where $\mathcal{D}$ is essentially dense in $L^2(M,\tau)$. By assumption, for every $n \in \mathbb{N}$, we find a projection $f_n$ in $M$ so that $\tau(1-f_n) \leq 2^{-n}$ and $\mathcal{D} \supseteq f_n L^2(M,\tau)$. Set $q_n = \bigwedge_{m=n}^{\infty} f_m$, and $p_n = p_{\mathcal{H}} \wedge q_n$. Observe that

$$\tau(1-q_n) = \tau\left(\bigvee_{m=n}^{\infty} 1 - f_m\right) \leq \sum_{m=n}^{\infty} \tau(1-f_n) \leq 2^{-n+1},$$

where in the first inequality we use that $\tau$ is normal. Then

$$\tau(1-p_n) \leq \tau(1-p_{\mathcal{H}}) + \tau(1-q_n) \leq 2^{-n+1} + \tau(1-p_{\mathcal{H}}),$$

so that $\tau(p_{\mathcal{H}} - p_n) \leq 2^{-n+1}$. Observe that $q_n$ are increasing, and that as a consequence we have that $p_n$ is increasing. Let $p_\infty = \sup_n p_n$. By normality,

$$\lim_{n\to\infty} \tau(p_\infty - p_n) = 0.$$

By definition of least upper bound, $p_\infty \leq p_{\mathcal{H}}$ and since $\tau(p_{\mathcal{H}} - p_\infty) \leq \tau(p_{\mathcal{H}} - p_n) \to 0$, the fact that $\tau$ is faithful implies $p_\infty = p_{\mathcal{H}}$. Now $\langle (p_{\mathcal{H}} - p_n)v, (p_{\mathcal{H}} - p_n)v\rangle = \langle (p_{\mathcal{H}} - p_n)v, v\rangle \to 0$ (otherwise $v$ does not lie in the closure of the subspaces spanned by $p_n$, so that $p_\infty \neq \sup_n p_n$), so $p_n \to p_{\mathcal{H}}$ in the strong operator topology. For each $n \in \mathbb{N}$, we have that

$$p_n L^2(M,\tau) = \mathcal{H} \cap q_n L^2(M,\tau) \subseteq \mathcal{H} \cap f_n L^2(M,\tau) \subseteq \mathcal{H} \cap \mathcal{D} = W.$$

□

**Lemma 3.6.13.** *Let $(M,\tau)$ be a tracial von Neumann algebra, and let $\mathcal{H} \subseteq L^2(M,\tau)$ be a closed and right-invariant subspace, fix an $a \in L^0(M,\tau)$.*

1. *We have that* $\dim_M(\overline{a(\mathcal{H} \cap \mathrm{dom}(a))}) \leq \dim_M(\mathcal{H})$, *with equality if* $ker(a) = \{0\}$
2. *We have that* $(a^{-1}(\mathcal{H}))^\perp = \overline{(a^*)(\mathcal{H}^\perp \cap \mathrm{dom}(a^*))}$.

*Proof.* Throughout, let $p$ be the orthogonal projection onto $\mathcal{H}$.

(1): Let $ap = v|ap|$ be the polar decomposition. Then $v^*v = p_{\ker(ap)^\perp}$, $vv^* = p_{\overline{\mathrm{Im}(ap)}}$. Clearly $\ker(ap) \supseteq (1-p)(L^2(M,\tau))$ so

$$v^*v = p_{\ker(ap)^\perp} \leq p.$$

So:

$$\dim_M(\mathcal{H}) = \tau(p) \geq \tau(v^*v) = \tau(vv^*) = \dim_M(\overline{a(\mathcal{H} \cap \mathrm{dom}(a))}).$$

If $ker(a) = \{0\}$, then in fact $p_{\ker(ap)} = p$.

(2): This is [Sti59, Lemma 3.4]

∎

### 3.6.5 Invariance

In this section, we prove Theorem 3.1.2.

**Lemma 3.6.14.** *For $n \in \mathbb{N}$, let $c_n \in L^0(M,\tau)$ and $a \in L^0(M,\tau)$ with $a > 0$. Let $T_n = a^{-n}|c_n|^2 a^{-n}$, and $S_n = T_n^{1/2n}$. Suppose $S_n \to \mathrm{id}$ in measure and $|c_n|^{1/n} \to a$ in measure. For $0 \leq t$, let*

$$\begin{aligned} \mathcal{V}_t &= \left\{\xi \in L^2(M,\tau) : \ \liminf_{n\to\infty} \|c_n \xi\|_2^{1/n} \leq t\right\} \\ \mathcal{H}_t &= \left\{\xi \in L^2(M,\tau) : \ \liminf_{n\to\infty} \|a^n \xi\|_2^{1/n} \leq t\right\}. \end{aligned}$$

*Then there exists an essentially dense subspace $\mathcal{D}$ of $L^2(M,\tau)$ such that*

$$\mathcal{D} \cap \mathcal{V}_t = \mathcal{D} \cap \mathcal{H}_t.$$

*In particular, we have that $\mathcal{H}_t = \overline{\mathcal{V}_t}$.*

*Proof.* Choose a decreasing sequence $(\varepsilon_k)_k$ of positive real numbers tending to zero. Since $S_n \to \mathrm{id}$ in measure, there exists a strictly increasing sequence $(n_k)_k$ of natural numbers so that

$$\sum_k \mu_{S_{n_k}}((1+\varepsilon_k, \infty)) < \infty.$$

By functional calculus, $\mu_{T_{n_k}}\left((1+\varepsilon_k)^{2n_k}, \infty\right) = \mu_{S_{n_k}}((1+\varepsilon_k,\infty))$. So

$$\sum_k \mu_{T_{n_k}}\left(((1+\varepsilon_k)^{2n_k}, \infty)\right) < \infty.$$

Let

$$\mathcal{D} = \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \Big( \mathrm{dom}(a^{n_k}) \cap a^{-n_k} 1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau)) \cap \mathrm{dom}(c_{n_k}) \Big).$$

We claim that $\mathcal{D}$ is essentially dense. By Lemma 3.6.13 (1), we know that

$$\dim_M \left(\overline{a^{-n_k}1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau))}\right) = \tau\left(1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})\right).$$

Now $\mathrm{dom}(a^{n_k}) \cap \mathrm{dom}(c_{n_k}) \cap \overline{a^{-n_k}1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau))}$ is essentially dense in $\overline{a^{-n_k}1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau))}$ So there exist projections $p_k \in M$ satisfying

- $\mathrm{L}^2(M,\tau)p_k \subseteq \mathrm{dom}(a^{n_k}) \cap a^{-n_k}1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau)) \cap \mathrm{dom}(c_{n_k})$
- $\tau(p_k) \geq 1 - 2\mu_{T_{n_k}}\left(((1+\varepsilon_k)^{2n_k},\infty)\right).$

For $l \in \mathbb{N}$, set $q_l = \bigwedge_{k=l}^{\infty} p_k$. Then for every $l \in \mathbb{N}$, we know that $\mathcal{D} \supseteq \mathrm{L}^2(M,\tau)q_l$ and

$$\tau(1-q_l) \leq 2\sum_{k=l}^{\infty} \mu_{T_{n_k}}\left(((1+\varepsilon_k)^{n_k},\infty)\right) \to_{l\to\infty} 0.$$

So we have shown that $\mathcal{D}$ is essentially dense.

Now suppose that $\xi \in \mathcal{D}$. Without loss of generality, $\|\xi\|_2 = 1$. Then:

$$\begin{aligned}\liminf_{n\to\infty} \|c_n\xi\|_2^{1/n} &\leq \liminf_{k\to\infty} \|c_{n_k}\xi\|_2^{1/n_k} = \liminf_{k\to\infty} \langle |c_{n_k}|^2\xi,\xi\rangle^{1/2n_k} \\ &= \liminf_{k\to\infty} \left\langle a^{-n_k}|c_{n_k}|^2 a^{-n_k}a^{n_k}\xi, a^{n_k}\xi\right\rangle^{\frac{1}{2n_k}} \\ &= \liminf_{k\to\infty} \langle T_{n_k}a^{n_k}\xi, a^{n_k}\xi\rangle^{\frac{1}{2n_k}}.\end{aligned}$$

By choice of $\mathcal{D}$, we have that $a^{n_k}\xi \in 1_{[0,(1+\varepsilon_k)^{2n_k}]}(T_{n_k})(\mathrm{L}^2(M,\tau))$. So

$$\begin{aligned}\liminf_{n\to\infty} \|c_n\xi\|_2^{1/n} &\leq \liminf_{k\to\infty} (1+\varepsilon_k)\langle a^{n_k}\xi, a^{n_k}\xi\rangle^{\frac{1}{2n_k}} \\ &= \liminf_{k\to\infty} (1+\varepsilon_k)\|a^{n_k}\xi\|_2^{1/n_k} = \lim_{n\to\infty} \|a^n\xi\|_2^{1/n},\end{aligned}$$

where the last equality holds by Lemma 3.6.8. So by Lemma 3.6.10,

$$\liminf_{n\to\infty} \|c_n\xi\|_2^{1/n} = \lim_{n\to\infty} \|a^n\xi\|_2^{1/n}.$$

Again by Lemma 3.6.10 we have that $\mathcal{D} \cap \mathcal{V}_t = \mathcal{D} \cap \mathcal{H}_t$. Since Lemma 3.6.10 also shows that $\mathcal{V}_t \subseteq \mathcal{H}_t$, it follows that $\overline{\mathcal{V}_t} = \mathcal{H}_t$ since essentially dense subspaces are always dense. ∎

**Lemma 3.6.15.** *For $n \in \mathbb{N}$, let $c_n \in \mathrm{GL}^2(M,\tau)$ and $a \in \mathcal{P}(M,\tau)$ with $a \geq 0$. Suppose that $A \subseteq \mathbb{N}$ and that*

$$\sum_{n \in A} \left( \frac{1}{2n} d_{\mathcal{P}}(|c_n|^2, a^{2n}) \right)^2 < \infty.$$

*Then there is an essentially dense $\mathcal{D}_A \subseteq L^2(M,\tau)$ so that for every $\xi \in \mathcal{D}_A$ we have*

$$\lim_{\substack{n\to\infty,\\ n\in A}} \|c_n \xi\|_2^{1/n} = \lim_{n\to\infty} \|a^n \xi\|_2^{1/n}.$$

*Proof.* Let $T_n = a^{-n}|c_n|^2 a^{-n}$, so that $\|\log(T_n)\|_2 = d_{\mathcal{P}}(|c_n|^2, a^{2n})$. Fix $\varepsilon > 0$, and observe that

$$\sum_{n\in A} \mu_{T_n}(((1+\varepsilon)^{2n}, \infty)) \leq \frac{1}{\log(1+\varepsilon)^2} \sum_{n\in A} \left( \frac{1}{2n} \|\log(T_n)\|_2 \right)^2 = \frac{1}{\log(1+\varepsilon)^2} \sum_{n\in A} \left( \frac{1}{2n} d_{\mathcal{P}}(|c_n|^2, a_n) \right)^2.$$

By our assumptions,

$$\sum_{n\in A} \mu_{T_n}(((1+\varepsilon)^{2n}, \infty)) < \infty.$$

Now set

$$\mathcal{D}_A^{\varepsilon} = \bigcup_{l\in A} \bigcap_{\substack{k\in A,\\ k\geq l}} \Big( \mathrm{dom}(a^k) \cap a^{-k} 1_{[0,(1+\varepsilon)^{2k}]}(T_k)(\mathrm{L}^2(M,\tau)) \cap \mathrm{dom}(c_k) \Big).$$

As in Lemma 3.6.14, we have that $\mathcal{D}_A^{\varepsilon}$ is essentially dense and for every $\xi \in \mathcal{D}_A^{\varepsilon}$

$$\limsup_{\substack{n\to\infty,\\ n\in A}} \|c_n \xi\|_2^{1/n} \leq (1+\varepsilon) \lim_{n\to\infty} \|a^n \xi\|_2^{1/n}.$$

Now set

$$\mathcal{D}_A = \bigcap_{m=1}^{\infty} \mathcal{D}_A^{\frac{1}{m}}.$$

By Lemma 3.6.10

$$\lim_{n\to\infty} \|a^n \xi\|_2^{1/n} \leq \liminf_{n\to\infty} \|c_n \xi\|_2^{1/n} \leq \liminf_{\substack{n\to\infty,\\ n\in A}} \|c_n \xi\|_2^{1/n}$$

for all $\xi \in \bigcap_{n=1}^{\infty}(\mathrm{dom}(a^n) \cap \mathrm{dom}(c_n))$, so it is not hard to show that $\mathcal{D}_A$ has the desired property.

$\square$

**Corollary 3.6.16.** *Let $X, \mu, f, M, \tau, c, \Lambda$ be as in Theorem 3.6.2. Then the Oseledets subspaces and Lyapunov distributions are invariant in the following sense. For a.e. $x \in X$,*

$$\begin{aligned} c(1,x)\mathcal{H}_t(x) &= \mathcal{H}_t(Tx), \\ \mu_{\Lambda(fx)} &= \mu_{\Lambda(x)}. \end{aligned}$$

*In particular, Theorem 3.1.2 is true.*

*Proof.* By the cocycle equation, $c(n, fx)c(1,x) = c(n+1,x)$. So $c(1,x)\mathcal{V}_t(x) \subseteq \mathcal{V}_t(fx)$. Conversely, using that $c(1,x)^{-1} = c(-1,Tx)$, we can apply the same logic to see that

$$\mathcal{V}_t(x) \supseteq c(1,x)^{-1}\mathcal{V}_t(fx).$$

Hence $c(1,x)\mathcal{V}_t(x) = \mathcal{V}_t(x)$. Applying the “in particular” part of Lemma 3.6.14 shows that $c(1,x)\mathcal{H}_t(x) = \mathcal{H}_t(fx)$. By invertibility of $c(1,x)$ we know that $\dim_M(\mathcal{H}_t(x)) = \dim_M(c(1,x)\mathcal{H}_t(x)) = \dim_M(\mathcal{H}_t(fx))$. Since $\mu_{\Lambda(x)}$ may be regarded as the distributional derivative of $t \mapsto \mathcal{H}_t(x)$, it follows that $\mu_{\Lambda(fx)} = \mu_{\Lambda(x)}$.

Theorem 3.1.2 now follows from Corollary 3.6.6. $\square$

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