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A study of a nonlocal problem with Robin boundary conditions arising from technology

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In the current work, we study a nonlocal parabolic problem with Robin boundary conditions. The problem arises from the study of an idealized electrically actuated MEMS (micro-electro-mechanical system) device, when the ends of the device are attached or pinned to a cantilever. Initially, the steady-state problem is investigated estimates of the pull-in voltage are derived. In particular, a Pohožaev's type identity is also obtained, which then facilitates the derivation of an estimate of the pull-in voltage for radially symmetric N-dimensional domains. Next a detailed study of the time-dependent problem is delivered and global-in-time as well as quenching results are obtained for generic and radially symmetric domains. The current work closes with a numerical investigation of the presented nonlocal model via an adaptive numerical method. Various numerical experiments are presented, verifying the previously derived analytical results as well as providing new insights on the qualitative behavior of the studied nonlocal model.

KEYWORDS

electrostatic MEMS, non-local parabolic problems, Pohožaev's identity, quenching, touchdown

MSC CLASSIFICATION

35K55; 35J60; 74H35; 74G55; 74K15

1 | INTRODUCTION

In this work we study the following nonlocal parabolic problem:

$$u_t = \Delta u + \frac{\lambda}{(1-u)^2\left[1+\alpha\int_\Omega 1/(1-u)dx\right]^2}, \quad \text{in } Q_T := \Omega\times(0,T),\ T>0, \tag{1.1a}$$

$$\frac{\partial u}{\partial \nu} + \beta u = 0, \ \text{on } \mathrm{fb}_T := \partial\Omega\times(0,T), \tag{1.1b}$$

$$u(x,0) = u_0(x),\ x\in\Omega, \tag{1.1c}$$

Dedicated to Professor Dimitrios E. Tzanetis on the occasion of his retirement where $\lambda > 0$, $\alpha > 0$, and $\beta > 0$ are given positive constants. Especially, λ is proportional to the applied voltage into the system, called pull-in voltage parameter, and it is actually the controlling parameter for the operation of the considered micro-electro-mechanical system (MEMS) device. The initial data $u_0(x)$ is assumed to be a smooth function such that $0 < u_0(x) < 1$ for all $x \in \bar{\Omega}$ and $\frac{\partial u_0}{\partial \nu} + \beta u_0 = 0,$ for $x \in \partial\Omega$; here $\nu = \nu(x)$ stands for the unit outward normal vector on the boundary of the N–dimensional domain Ω. Notably, from the applications point of view only the cases $N = 1, 2$ are viable; however, from the point of view of mathematical analysis cases $N \geq 3$ are also interesting and so they will be investigated. Moreover, here T denotes the maximum existence time of solution u.

When $\alpha = 0$ problem (1.1) reduces to the local parabolic problem

$$u_t = \Delta u + \frac{\lambda}{(1-u)^2}, \text{ in } Q_T, \tag{1.2a}$$

$$\frac{\partial u}{\partial \nu} + \beta u = 0, \text{ on } \mathrm{fb}_T, \tag{1.2b}$$

$$u(x,0) = u_0(x), \ x \in \Omega. \tag{1.2c}$$

It is worth mentioning that for Robin-type boundary conditions, as the ones considered above for $\beta > 0$, there is a limited study for the local problem, cf. Guo[1] while to the best of our knowledge no published works dealing with the nonlocal problem (1.1) can be found in the literature. Our motivation for studying (1.1) comes from the fact that it is actually linked with special applications in MEMS industry, as pointed below. Furthermore, due the imposed Robin-type boundary conditions extra technical difficulties arise compared to the study of the Dirichlet problem, a fact that is indicated through the manuscript.

Problem (1.1) arises as a mathematical model which describes the operation of some electrostatic actuated MEMSs. Those MEMSs are precision devices, which combine mechanical processes with electrical circuits. MEMS devices range in size from millimeters down to microns and involve precision mechanical components that can be constructed using semiconductor manufacturing technologies.

In particular, electrostatic actuation is a popular application of MEMS. Various electrostatic actuated MEMS have been developed and used in a wide variety of devices applied as sensors and have fluid-mechanical, optical, radio frequency (RF), data-storage, and biotechnology applications. Examples of microdevices of this kind include microphones, temperature sensors, RF switches, resonators, accelerometers, micromirrors, micropumps, microvalves, etc., see for example, previous works.[2–4]

In the sequel a derivation for the nonlocal model (1.1), for the one-dimensional case, is presented and also the association of that model with applications in MEMS industry is explained. The main body of the derivation is standard (see, e.g., previous works[3,5–7]); however, in order to justify the inclusion for the Robin boundary conditions in the model and for completeness it is presented here as well. The modifications of this modeling approach are presented in detail in the next section.

1.1 | Derivation of the model

We consider an idealized electrostatiaclly MEMS device, which consists of an elastic membrane and a rigid plate placed parallel to each other as it can be seen in Figure 1. The membrane has two parallel sides usually attached or pinned to a

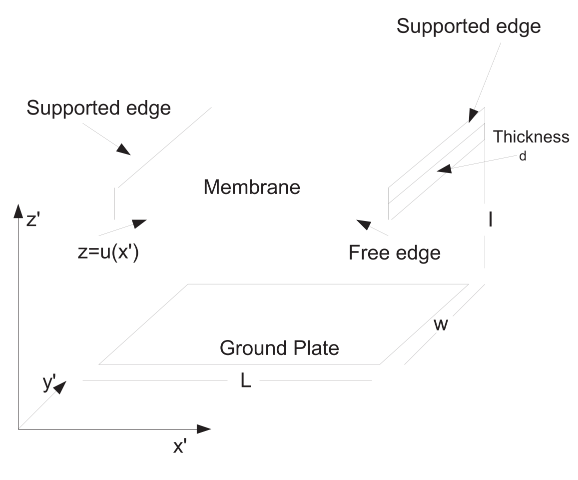

FIGURE 1 Schematic representation of a micro-electro-mechanical system (MEMS) device

cantilever, while the other sides are free. Both membrane and plate have width w and length L, and in the undeformed state (for the membrane) the distance between the membrane and the plate is l. We assume here that the gap between the plate and the membrane is small, that is $l \ll L$ and $l \ll w$. Besides, the area between the elastic membrane and the rigid plate is occupied by some inviscid material with dielectric constant one, so permittivity is that of free space, ϵ_0. A potential difference V is applied between the top surface and the rigid plate and we further assume that the plate is earthed. Besides, the small aspect ratio of the gap gives potential

$$\phi = V(l - z')/(l - u'), \tag{1.3}$$

to leading order, where u' is the displacement of the membrane toward the plate ($u' = l$ corresponds to touch-down, i.e., when the top surface touches the rigid plate) and z' is the distance measured from the undisturbed membrane position toward the plate. The electrostatic force per unit area on the membrane (in the z' direction) is then $\frac{1}{2} \times$ surface charge density $\times$ electric field $= \frac{1}{2}\epsilon_0 \phi_{z'}^2 = \frac{1}{2}\epsilon_0 V^2/(l-u')^2$, recalling that ϵ_0 is the permittivity of the free space.

We take the sides of width w, say at $x' = 0$ and $x' = L$, to be connected with the support of the device, with those of length L, say at $y' = 0$ and $y' = w$, to be free. We also assume there is no variation in the y' direction, so $u' = u'(x', z', t')$ for time t'. The surface density of the membrane is denoted by ρ, while T_m stands for constant surface tension of the membrane. Then its displacement satisfies the forced wave equation with damping (proportional to the membrane speed),

$$\rho u'_{t't'} + a u'_{t'} = T_m u'_{x'x'} + \frac{1}{2}\epsilon_0 V^2/(l-u')^2. \tag{1.4}$$

In many situations is observed that the damping term is dominant compared with the inertia term. According to this ansatz, we get the following parabolic equation:

$$a u'_{t'} = T_m u'_{x'x'} + \frac{1}{2}\epsilon_0 V^2/(l-u')^2. \tag{1.5}$$

In addition to the derived Equation (1.5), appropriate boundary conditions should be imposed. The standard way to do so is to assume that since the edges of the membrane or beam are fixed at the support of the device, Dirichlet boundary conditions, in the case of the flexible membrane or clamped boundary conditions, in the case of a beam should be considered. Although as it is stated in Younis[4, Chapter 6] it is evident that the support or cantilever of MEMS devises might be nonideal and flexible.

More specifically, cantilever microbeams can tilt upward or downward due to the deformation of their support since the anchors or supports of them can have some flexibility making the assumption of perfect clamping inaccurate. This flexibility of the supports of microbeams are accounted for by assuming springs at the beam boundaries and consequently modeling a flexible nonideal support can be done in general by assuming torsional and translational springs at the membrane or beam edge.

As a first step toward this modeling approach, in this work we will assume that we have a device for which its movable upper part is thin enough, so that it can be considered to behave as a membrane while its ends are connected with a flexible nonideal support behaving as a spring moving in the x' direction, see Figure 2A. Torsional or other kind of behavior is assumed to be negligible at this occasion.

Therefore, according to the above assumptions, the appropriate boundary conditions should be those of Robin type, and thus, we set

$$u'_{x'}(-L, t') = k u'(-L, t'), \;\; u'_{x'}(L, t') = -k u'(L, t'),$$

where k is the spring constant.

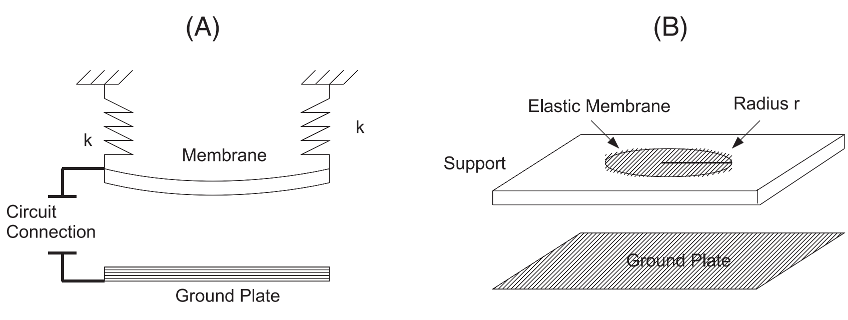

FIGURE 2 (A) Schematic representation of a micro-electro-mechanical system (MEMS) device with nonideal support. (B) Schematic representation of a MEMS device with radial symmetry

Next by introducing the scaling $u' = lu, x' = Lx, t' = \frac{L^2 a}{T_m} t$, we end up with the local equation

$$u_t = u_{xx} + (\epsilon_0 V^2 L^2 / T_m 2l^3)/(1-u)^2, \tag{1.6}$$

associated with the aforementioned boundary conditions and some appropriate initial deformation $0 < u(x, 0) < 1$. Therefore, we end up in the first place with the following local problem:

$$u_t = u_{xx} + \frac{\lambda}{(1-u)^2}, -1 < x < 1, \ t > 0, \tag{1.7a}$$

$$u_x(\mp 1, t) = \pm \beta u(\mp 1, t), \ t > 0, \tag{1.7b}$$

$$u(x, 0) = u_0(x), -1 < x < 1, \tag{1.7c}$$

for $\beta = Lk$ and $\lambda = \frac{\epsilon_0 V^2 L^2}{T_m 2l^3}$.

Since pull-in instability is a ubiquitous feature of electrostatically actuated systems, many researchers have focused on extending the stable operation of electrostatically actuated systems beyond the pull-in regime. In particular, in previous works[8,9] the basic capacitive control scheme was first proposed by Seeger and Crary to elaborate this kind of stabilization, see also.[10] More precisely, this scheme provides control of the voltage by the addition of a series capacitance to the circuit containing the MEMS device, since the added capacitance acts as a voltage divider. So in the event the MEMS device, which has a capacitance C depending on displacement, is connected in series with a capacitor of fixed capacitance C_f and a source of fixed voltage V_s, we have that

$$V_s = \frac{Q}{C_c} = Q\left(\frac{1}{C} + \frac{1}{C_f}\right),$$

where Q is the charge on the device and fixed capacitor, and C_c the series capacitance of the two. Then the potential difference V across the MEMS device, by applying Kirchoff's law is equal to

$$V = \frac{V_s}{1 + C/C_f}. \tag{1.8}$$

In addition, we also have

$$Q = \epsilon_0 \int_0^w \int_0^L \phi_{z'}(x, y, 0)\, dx'\, dy' = V \frac{wL\epsilon_0}{l} \int_0^1 \frac{1}{1-u}\, dx,$$

and by using relation (1.3) we get,

$$C = C_0 \int_0^1 \frac{1}{1-u}\, dx,$$

for $C_0 = \frac{wL\epsilon_0}{l}$ being the capacitance of the undeflected device.

When the latter relation is combined with Equations (1.8) and (1.6), we finally obtain the nonlocal problem

$$u_t = u_{xx} + \frac{\lambda}{(1-u)^2 \left(1 + \alpha \int_{-1}^{1} \frac{1}{1-u}\, dx\right)^2}, -1 < x < 1, \ t > 0, \tag{1.9a}$$

$$u_x(\mp 1, t) = \pm \beta u(\mp 1, t), \ t > 0, \tag{1.9b}$$

$$u(x, 0) = u_0(x), -1 < x < 1, \tag{1.9c}$$

with $\alpha = \frac{wL\epsilon_0}{lC_f} = \frac{C_0}{C_f}$.

Usually, it is supposed that the elastic membrane is initially in its unforced position, so that $u(x, 0) \equiv 0$. However, in this work, we consider more general non-negative initial conditions, reflecting also the situation when the membrane has an initial displacement.

It has been experimentally observed that the applied voltage V_s controls the operation of the MEMS device. Indeed, when V_s exceeds a critical threshold V_{cr}, called the *pull-in voltage*, then the phenomenon of *touch-down* (or *pull-in instability* as it is also known in MEMS literature) occurs when the elastic membrane touches the rigid ground plate. The related mathematical problem has been studied quite extensively in, for example, previous works.[2,6,7,11–16]

Note that the limiting case $\alpha = 0$ corresponds to the configuration where there is no capacitor in the circuit and then we end up with the local problem (1.7), which has been studied in Guo.[1] A stochastic version of problem (1.7) is treated in previous works.[17,18] Besides, the local problem with Dirichlet boundary conditions ($\beta = +\infty$) has been extensively studied among others in previous works.[2,7,11,13] Also, for hyperbolic modifications of the variation of (1.7) an interested reader can check.[6,19]

The quenching behavior of the nonlocal Equation (1.1a) associated with Dirichlet boundary ($\beta = +\infty$) has been treated in Kavallaris et al.[20] and in references therein as well as in previous works.[21–23] Also, non-local alterations of parabolic and hyperbolic problems arising in MEMS technology were tackled in previous works.[5,7,20–22,24,25] However, to the best of our knowledge, there are not similar studies available in the literature for the Robin problem ($0 < \beta < +\infty$,) so in the current work we study problem (1.1) and we extend some of the results given in Guo[1] for the local problem, but we also deliver a further investigation related to the steady-state problem and the quenching behavior of the time-dependent problem. Our mathematical analysis is inspired by ideas developed in previous works;[20,22] however, important modifications are necessary due to the Robin boundary conditions. In particular, a new Pohožaev's type identity for Robin boundary conditions is derived which is then used to derive lower estimates of the pull-in voltage. Moreover, a novel argument, see Theorem 3.15, is developed to derive an upper estimate of the quenching rate; note that such a reasoning is missing from the approach used in Kavallaris et al.[20] Still, the derivation of a key estimate for the nonlocal term, analogous to the one derived in Kavallaris et al.[20, Lemma 3.3] for the Dirichlet problem, needs more work for Robin problem (1.1) and it is finally derived under some extra restriction, cf. Lemma 3.10.

The organization of the paper is as follows. In Section 2 a thorough study of the steady-state problem is delivered, where among other results some estimates of the supremum of its spectrum (pull-in voltage) are derived. Uniqueness and local-in-time existence results for time-dependent problem (1.1) are discussed in the first part of section 3. The second part of section 3 deals with the long-time behavior of the solutions of (1.1). In particular, at first a quenching result is obtained for a genericl domain, while a sharper quenching result is derived for a radially symmetric domain later on. A numerical treatment of (1.1) via an adaptive method is presented in Section 4. We thus numerically verify all the obtained analytical results as well as we determine the quenching profile which cannot be derived via our theoretical approach. We conclude with a discussion of our main results in Section 5.

2 | STEADY-STATE PROBLEM: ESTIMATES OF THE PULL-IN VOLTAGE

The main purpose of the current section is to study the steady-state problem of (1.1). In particular, we are interested in obtaining estimates of the supremum of its spectrum (pull-in voltage) while in the one-dimensional case we are also able to derive the form of its bifurcation diagram.

2.1 | The one-dimensional case

Below we provide a thorough investigation of the steady-state problem in the one-dimensional case. In particular we study the structure of the solution set of

$$w'' + \frac{\lambda}{(1-w)^2\left[1+\alpha\int_{-1}^{1}\frac{dx}{1-w}\right]^2} = 0,\ -1 < x < 1, \tag{2.1a}$$

$$w'(-1) - \beta w(-1) = 0,\ w'(1) + \beta w(1) = 0, \tag{2.1b}$$

where we always have $0 \le w < 1$ in $[-1, 1]$ for a (classical) solution of (2.1).

For convenience we set $W = 1 - w$ and then (2.1) becomes

$$W'' = \frac{\mu}{W^2}, -1 < x < 1, \tag{2.2a}$$

$$W'(-1) + \beta\,(1 - W(-1)) = 0,\ W'(1) - \beta\,(1 - W(1)) = 0, \tag{2.2b}$$

where

$$\mu = \frac{\lambda}{\left[1 + \alpha \int_{-1}^{1} \frac{1}{W} dx\right]^2}. \tag{2.3}$$

Note that W is symmetric and thus $m = \min\{W(x), x \in [-1.1]\} = W(0)$, cf.[25,26] Then multiplying both sides of Equation (2.2a) by W' and integrating from $m = W(0)$ to $W(x) = W$ we derive

$$\int_0^{W'} W' dW' = \int_0^x W'' W' dx = \mu \int_0^x \frac{W'}{W^2} dx = \mu \int_m^W \frac{dW}{W^2},$$

hence,

$$\frac{1}{2}(W')^2 = \mu \left(\frac{1}{m} - \frac{1}{W}\right). \tag{2.4}$$

This gives equivalently

$$\frac{dx}{dW} = \sqrt{\frac{m}{2\mu}} \sqrt{\frac{W}{W - m}}, \tag{2.5}$$

which implies (see previous works[5,25])

$$x = \sqrt{\frac{m}{2\mu}} \left[\sqrt{W(W-m)} - \frac{1}{2} m \ln(m) + m \ln\left(\sqrt{W} + \sqrt{W-m}\right)\right].$$

Additionally, at the point $x = 1$ and for $W(1) = M := \max\{W(x),\ x \in [-1, 1]\}$ we deduce

$$1 = \sqrt{\frac{m}{2\mu}} \left[\sqrt{M(M-m)} - \frac{1}{2} m \ln(m) + m \ln\left(\sqrt{M} + \sqrt{M-m}\right)\right]. \tag{2.6}$$

Moreover combining the boundary condition, $W'(1) = \beta\,(1 - W(1))$, with Equation (2.4) we obtain

$$\frac{\beta^2 (1-M)^2}{2} = \mu \left(\frac{1}{m} - \frac{1}{M}\right). \tag{2.7}$$

At this point, recalling that for $\alpha = 0$ we have $\mu = \lambda$, we can obtain the bifurcation diagram of the local problem. More specifically rearranging (2.7), we have

$$m = \frac{2\lambda M}{2\lambda + M\beta^2 (1-M)^2}, \tag{2.8}$$

which together with (2.6), for $\mu = \lambda$, namely

$$1 = \sqrt{\frac{m}{2\lambda}} \left[\sqrt{M(M-m)} - \frac{1}{2} m \ln(m) + m \ln\left(\sqrt{M} + \sqrt{M-m}\right)\right], \tag{2.9}$$

forms a system of algebraic equations giving an implicit relation of the form $F(\lambda, M) = 0$.

Furthermore, in order to obtain the bifurcation diagram for the nonlocal problem ($\alpha > 0$) we have to express the integral of the nonlocal term in terms of λ, m, M.

That is, on using Equation (2.5)

$$\int_{-1}^{1}\frac{1}{W}dx=\int_{-1}^{1}\frac{dx}{dW}\frac{dW}{W}=2\sqrt{\frac{m}{2\mu}}\int_{m}^{M}\frac{1}{\sqrt{W(W-m)}}dW$$
$$=2\frac{1}{\sqrt{M(M-m)}-\frac{1}{2}m\ln(m)+m\ln\left(\sqrt{M}+\sqrt{M-m}\right)}\int_{m}^{1}\frac{1}{\sqrt{W(W-m)}}dW$$
$$=2\frac{1}{\sqrt{M(M-m)}-\frac{1}{2}m\ln(m)+m\ln\left(\sqrt{M}+\sqrt{M-m}\right)}\ln\left(\frac{2M-m+2\sqrt{M(M-m)}}{m}\right).$$

Therefore, using also (2.3), (2.7) to eliminate μ, we obtain the following system of algebraic equations for λ, M, m:

$$1=\sqrt{\frac{m}{2\mu}}\left[\sqrt{M(M-m)}-\frac{1}{2}m\ln(m)+m\ln\left(\sqrt{M}+\sqrt{M-m}\right)\right],\tag{2.10a}$$

$$\frac{\beta^2(M-1)^2}{2}\frac{mM}{M-m}=\lambda\left[1+\alpha\frac{2\ln\left(\frac{2M-m+2\sqrt{M(M-m)}}{m}\right)}{\sqrt{M(M-m)}-\frac{1}{2}m\ln(m)+m\ln\left(\sqrt{M}+\sqrt{M-m}\right)}\right]^{-2},\tag{2.10b}$$

together with (2.3), which can be solved numerically.

Remark 2.1. Note that by Equation (2.7) for $\beta \gg 1$ we have $(M-1)\sim 0$ or $M\sim 1$ and we retrive the expression for λ and m which gives the bifurcation diagram for the local problem with Dirichlet boundary conditions (see Kavallaris et al.[5]), that is,

$$\lambda=\frac{m}{2}\left[\sqrt{1-m}-\frac{1}{2}m\ln(m)+m\ln\left(1+\sqrt{1-m}\right)\right]^2.$$

In Figure 3A we plot the bifurcation diagram for the stationary local problem (2.1a) for $\alpha=0$. We can observe the existence of a critical value of the parameter λ, say λ^*, usually called the *pull-in voltage* in MEMS literature, above which we have no solution for the steady problem while for values below λ^* we have two solutions. We finally derive that $\lambda^*=0.108711900526435$ and for this value we have that the maximum of the solution $M=W(1)=0.761$.

Regarding the nonlocal stationary problem, Equation (2.1a) with $\alpha=1$ we present a similar plot of the bifurcation diagram in Figure 4A (line indicated with $\alpha=1$). In this case the critical value of the parameter λ is $\lambda^*=2.387086785660011$. In both of the above cases the parameter in the boundary conditions is taken to be $\beta=1$.

In this set of graphs we can see also the variation of the bifurcation diagram of the local problem with respect to the parameter β in Figure 3B.

A similar graph, see Figure 4, investigates the variation of the bifurcation diagram of the nonlocal problem with respect to the parameter α in Figure 4A and with respect to the parameter β in Figure 4B.

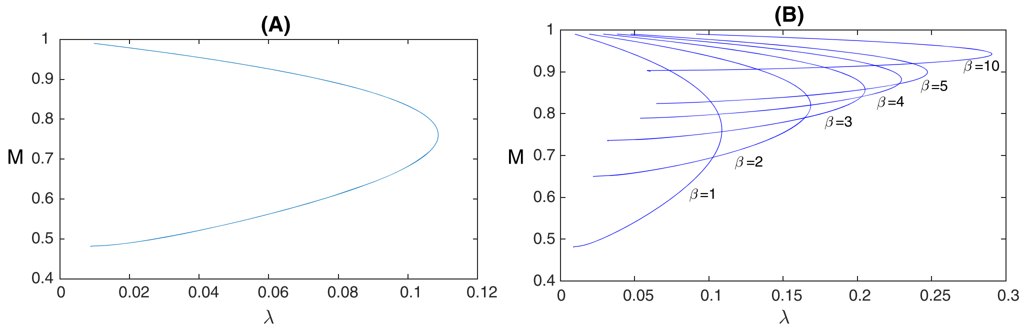

FIGURE 3 (A) Bifurcation diagram for the local problem. (B) Variation of the bifurcation diagram of the local problem with respect to the parameter β [Colour figure can be viewed at wileyonlinelibrary.com]

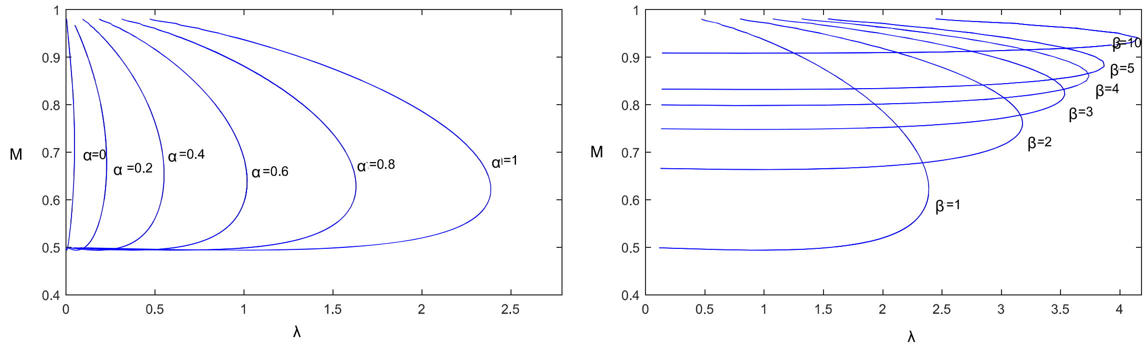

FIGURE 4 (A) Variation of the bifurcation diagramm of the nonlocal problem with respect to the parameter α for $\beta = 1$. (B) Variation of the bifurcation diagramm of the nonlocal problem with respect to the parameter β for $\alpha = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

For a rigorous bifurcation analysis of nonlocal problem (2.1), we kindly advice the reader to check.[25]

2.2 | The higher dimensional case

In this part we study the steady-state problem of the N-dimensional version of (1.1) for $N > 1$. In particular, we perform an investigation of the set of classical solutions $0 \le w = w(x) < 1$ in $\bar{\Omega}$, satisfying the nonlocal problem

$$\Delta w + \frac{\lambda}{(1-w)^2\left(1+\alpha\int_\Omega \frac{1}{1-w}dx\right)^2} = 0, x \in \Omega \subset \mathbb{R}^N,\ N \ge 1, \tag{2.11a}$$

$$\frac{\partial w}{\partial \nu} + \beta w = 0,\ x \in \partial\Omega. \tag{2.11b}$$

In the following, we denote

$$\lambda^* := \sup\{\lambda > 0 :\ \text{problem (2.11) has a classical solution}\}, \tag{2.12}$$

and we recall that λ^* in MEMS terminology is called *pull-in voltage*. By setting

$$\mu := \frac{\lambda}{K} = \frac{\lambda}{\left(1+\alpha\int_\Omega \frac{1}{1-w}dx\right)^2}, \tag{2.13}$$

where

$$K = K(w) := \left(1+\alpha\int_\Omega \frac{1}{1-w}dx\right)^2, \tag{2.14}$$

then (2.11) can be written as a local problem

$$\Delta w + \frac{\mu}{(1-w)^2} = 0,\ x \in \Omega, \tag{2.15a}$$

$$\frac{\partial w}{\partial \nu} + \beta w = 0,\ x \in \partial\Omega, \tag{2.15b}$$

and we also define

$$\mu^* := \sup\{\mu > 0 :\ \text{problem (2.15) has a classical solution}\}. \tag{2.16}$$

It is readily seen that problems (2.11) and (2.15) are equivalent via relation (2.13). More specifically w is a solution of (2.11) corresponding to λ if and only if w satisfies (2.15) for μ given by (2.13).

Next we introduce the notion of weak solution for the problem (2.11) which will be used in an essential way to our approach (cf. Kavallaris et al.[20]) toward the study of the quenching (touching down) phenomenon.

Definition 2.2. A function $w \in H_0^1(\Omega)$ is called weak finite-energy solution of (2.11) if there exists a sequence $\{w_j\}_{j=1}^{\infty} \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying as $j \to \infty$

$$w_j \to w \text{ weakly } in\ H^1(\Omega), \tag{2.17a}$$

$$w_j \to w \text{ a.e.}, \tag{2.17b}$$

$$\frac{1}{(1-w_j)^2} \to \frac{1}{(1-w)^2} \ \text{ in } \ L^1(\Omega), \tag{2.17c}$$

$$\frac{1}{(1-w_j)} \to \frac{1}{(1-w)} \ \text{ in } \ L^1(\Omega), \tag{2.17d}$$

$$\Delta w_j + \frac{\lambda}{(1-w_j)^2\left(1+\alpha\int_\Omega \frac{dx}{1-w_j}\right)^2} \to 0 \ \text{ in } \ L^2(\Omega). \tag{2.17e}$$

A weak finite-energy solution of (2.11) satisfies

$$-\int_\Omega \nabla\phi \cdot \nabla w\, dx + \int_{\partial\Omega} \phi \frac{\partial w}{\partial \nu} ds + \lambda \frac{\int_\Omega \frac{\phi}{(1-w)^2} dx}{\left(1+\alpha\int_\Omega \frac{1}{1-w} dx\right)^2} = 0,$$

for any $\phi \in W^{2,2}(\Omega)$ satisfying $\frac{\partial\phi}{\partial\nu} + \beta\phi = 0$ on $\partial\Omega$.

We also denote

$$\hat{\lambda} := \sup\{\lambda > 0 : \text{ problem (2.11) has a weak finite-energy solution}\}.$$

In addition and in accordance to Kavallaris et al,[20, Proposition 2.2] we have the following:

Proposition 2.3. *For the radial symmetric case, i.e. when $\Omega = B_1(0) := \{x \in \mathbb{R}^N : |x| < 1\}$, the suprema of the spectra for classical and weak energy solutions are identical. In particular, $\lambda^* = \hat{\lambda}$.*

The proof of Proposition 2.3 follows closely the proof of Kavallaris et al[20, Proposition 2.2] and so it is omitted.
Next we show that μ^* defined by (2.16) is well defined and bounded. More precisely,

Lemma 2.4. *There exists a finite μ^* defined by (2.16) such that*

(i) If $\mu < \mu^*$ then problem (2.15) has at least one (classical) solution.
(ii) If $\mu > \mu^*$ then problem (2.15) has no (classical) solution.

Proof. We first establish the existence of μ^* defined by (2.16). Indeed, implicit function theorem implies that problem (2.15) has a solution bifurcating from the trivial solution $w = 0$ at $\mu = 0$. This solution is positive due to the maximum principle, hence μ^* is well-defined and positive.

Next we prove the boundedness of μ^*. Let $(\lambda_1, \phi_1(x))$ be the principal normalized eigenpair of the Laplacian associated with Robin boundary conditions, that is, $\phi_1(x)$ satisfies

$$-\Delta\phi_1 = \lambda_1\phi_1, x \in \Omega, \ \frac{\partial\phi_1}{\partial\nu} + \beta\phi_1 = 0, \ x \in \partial\Omega, \tag{2.18}$$

with

$$\int_\Omega \phi_1\, dx = 1. \tag{2.19}$$

It is known (see, e.g., Amann[27, Theorem 4.3]) that λ_1 is positive and that $\phi_1(x)$ does not change sign in Ω, so by condition (2.19) is positive.

Testing (2.15a) by $\phi_1(x)$ and using second Green's identity in conjunction with (2.19) we obtain for any calssical solution w

$$\lambda_1 \int_\Omega w\phi_1\, dx = \mu \int_\Omega \frac{\phi_1}{(1-w)^2} dx \geq \mu.$$

The latter inequality, by virtue of (2.16), implies

$$\mu^* \leq \lambda_1 \int_\Omega w\phi_1\, dx \leq \lambda_1 < \infty,$$

and so μ^* is finite.

Next we focus on proving statement (i). We pick $\mu \in (0, \mu^*)$, then thanks to the definition of μ^* there exists $\bar{\mu} \in (\mu, \mu^*)$ such that the minimal solution $w_{\bar{\mu}}$ (i.e., the smallest solution corresponding parameter $\bar{\mu}$) of (2.15) satisfies

$$-\Delta w_{\bar{\mu}} = \frac{\bar{\mu}}{(1-w_{\bar{\mu}})^2} \geq \frac{\mu}{(1-w_{\bar{\mu}})^2},\ x \in \Omega,$$
$$\frac{\partial w_{\bar{\mu}}}{\partial \nu} + \beta w_{\bar{\mu}} = 0,\ x \in \partial\Omega,$$

since $\bar{\mu} > \mu$. The latter implies that $w_{\bar{\mu}}$ is an upper solution of (2.15) corresponding to parameter μ. Additionally, it is easily seen that $w \equiv 0$ is a lower solution of (2.15) corresponding to μ. Consequently by using comparison arguments, cf.,[28] we can construct a solution of (2.15) corresponding to parameter μ, and this completes the proof of (i). On the other hand, by the definition of μ^*, we deduce that problem (2.15) has no solution for $\mu > \mu^*$ and statement (ii) is also proven. □

Next we prove the monotonicity of minimal (stable) branch of problem (2.15) with respect to (local) parameter μ.

Lemma 2.5. *Let $\mu_1,\ \mu_2 \in (0,\ \mu^*)$. Assume that w_{μ_1} and w_{μ_2} are the corresponding minimal solutions of problem (2.15), then*

$$0 < w_{\mu_1}(x) < w_{\mu_2}(x) < 1\ for\ x \in \Omega,\ if\ 0 < \mu_1 < \mu_2 < \mu^*. \tag{2.20}$$

Proof. It is known, cf. previous worls,[2,22] that $w(x;\ \mu)$ is differentiable with respect to $\mu \in (0,\ \mu^*)$. Set $z = \frac{\partial w}{\partial \mu}$ then differentiating (2.15a) with respect to μ we derive

$$-\Delta z - \frac{2\mu}{(1-w)^3} z = \frac{1}{(1-w)^2} > 0.$$

In addition due to the boundary conditions, we have similarly $\frac{\partial z}{\partial \nu} + \beta z > 0$. Therefore, by the maximum principle, since $(1 - w)^{-3}$ is bounded for a classical solution, cf. Evans,[29] we obtain that $z > 0$, that is, $\frac{\partial w}{\partial \mu} > 0$.

Using the preceding monotonicity result we can also prove, as in Guo and Kavallaris,[22] the following. □

Theorem 2.6. *There exists a classical solution to problem (2.11) for any $\lambda \in (0,\ (1 + \alpha|\Omega|)^2\mu^*)$, and therefore,*

$$\lambda^* \geq \sup_{(0,\mu^*)} \mu K(w_\mu) \geq (1+\alpha|\Omega|)^2 \mu^*, \tag{2.21}$$

where μ^* is defined by (2.16) and recall that $K(w_\mu) = \left(1 + \alpha \int_\Omega \frac{1}{1-w_\mu} dx\right)^2$.

Proof. By virtue of (2.20) we have

$$(1+\alpha|\Omega|)^2 = K(0) < K(w_\mu),\ if\ 0 < \mu < \mu^*. \tag{2.22}$$

Next, for any $\lambda \in (0,\ (1 + \alpha|\Omega|)^2\mu^*)$ there is a unique $\mu \in (0,\ \mu^*)$ such that

$$\mu = \frac{\lambda}{(1+\alpha|\Omega|)^2}, \tag{2.23}$$

and hence, there is a minimal solution w_μ for local problem (2.15). Since problems (2.11) and (2.15) are equivalent through (2.13), there exists $\lambda_1 \in (0, \lambda^*)$ with

$$\mu = \frac{\lambda_1}{K(w_\mu)}. \tag{2.24}$$

Therefore, (2.23) and (2.24) in conjunction with (2.22) imply that $0 < \lambda < \lambda_1 < \lambda^*$ and thus nonlocal problem (2.11) has at least one (minimal) solution w_λ. This completes the proof. □

Remark 2.7. One can derive lower estimates of μ^* as in the case of Dirichlet boundary conditions, cf. Esposito et al,[2, Proposition 2.2.2] and thus via (2.21) can finally obtain lower estimates of the pull-in voltage λ^*.

Next we provide a more delicate lower estimate of λ^* in the case of the N–dimensional sphere, that is, when

$$\Omega = B_R = B_R(0) =: \{x \in \mathbb{R}^N \,:\, |x| < R\}, \text{ for } R > 0.$$

Such a radial symmetric case is rather of high importance from applications point of view as it is indicated in previous works.[30–32] In order to prove such a lower estimate of λ^* we need to use a Pohožaev's type identity, cf. Pohožaev,[33] for the following problem:

$$\Delta v + \mu f(v) = 0,\ x \in \Omega,\ \mu > 0, \tag{2.25a}$$

$$\frac{\partial v}{\partial \nu} + \beta v = 0,\ for\ x \in \partial\Omega. \tag{2.25b}$$

Since to the best of our knowledge such an identity is not available in the literature for problem (2.25a)-(2.25b), we provide a proof of it below.

Proposition 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with antiderivative $F(v) := \int_0^v f(s)ds$. Assume that $\Omega \subset \mathbb{R}^N$ is open and bounded. If $v \in C^2(\bar{\Omega})$ is a smooth solution of problem (2.25) then the following identity holds*

$$\begin{aligned} \frac{\mu(N-2)}{2}\int_\Omega v f(v)dx - \mu N \int_\Omega F(v)dx = \frac{(N-2)}{2\beta}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu(x)}\right)^2 dS + \frac{1}{2}\int_{\partial\Omega}|\nabla v|^2 \langle \nu(x), x\rangle\ dS \\ - \int_{\partial\Omega}\frac{\partial v}{\partial \nu(x)}\langle \nabla v, x\rangle\, dS - \mu\int_{\partial\Omega}F(v)\frac{\partial}{\partial \nu(x)}\left(\frac{1}{2}|x|^2\right) dS, \end{aligned} \tag{2.26}$$

where $\langle \cdot,\cdot\rangle$ stands for the dot (inner) product in the Euclidean space $\mathbb{R}^N$.

Proof. We first multiply (2.25a) by $\langle x, \nabla v\rangle$ and integrate over Ω to derive

$$-\int_\Omega \Delta v\,\langle x, \nabla v\rangle\, dx = \mu\int_\Omega f(v)\,\langle x, \nabla v\rangle\, dx. \tag{2.27}$$

The LHS of (2.27) via integration by parts gives

$$-\int_\Omega \Delta v\,\langle x, \nabla v\rangle\, dx = \int_\Omega \langle \nabla v, \nabla\langle x, \nabla v\rangle\rangle\, dx - \int_{\partial\Omega}\frac{\partial v}{\partial \nu(x)}\langle x, \nabla v\rangle\, dS. \tag{2.28}$$

Now since

$$\begin{aligned} \langle \nabla v, \nabla\langle x, \nabla v\rangle\rangle &= \frac{1}{2}\left\langle \nabla\left(|\nabla v|^2\right), x\right\rangle + |\nabla v|^2 \\ &= \frac{1}{2}\left\langle \nabla\left(|\nabla v|^2\right), \nabla\left(\frac{1}{2}|x|^2\right)\right\rangle + |\nabla v|^2, \end{aligned}$$

using again integration by parts we obtain

$$\begin{aligned}\int_\Omega \langle \nabla v, \langle x, \nabla v\rangle\rangle\, dx &= \frac{1}{2}\int_\Omega \left\langle \nabla\left(|\nabla v|^2\right), \nabla\left(\frac{1}{2}|x|^2\right)\right\rangle dx + \int_\Omega |\nabla v|^2\, dx \\ &= \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 \frac{\partial}{\partial \nu(x)}\left(\frac{1}{2}|x^2|\right) dS - \frac{N}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega |\nabla v|^2\, dx \\ &= \frac{2-N}{2}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 \langle \nu(x), x\rangle\, dS,\end{aligned} \tag{2.29}$$

taking also into account that $\Delta\left(\frac{1}{2}|x|^2\right) = N$ for any $x \in \Omega$.

Next we estimate the first term on the RHS of (2.29) using (2.25a). Indeed multiplying (2.25a) by v, integrating over Ω and using integration by parts we deduce

$$\begin{aligned}\int_\Omega |\nabla v|^2\, dx &= \mu\int_\Omega v f(v)\, dx + \int_{\partial\Omega} v\frac{\partial v}{\partial \nu(x)} dS \\ &= \mu\int_\Omega v f(v)\, dx - \frac{1}{\beta}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu(x)}\right)^2 dS,\end{aligned} \tag{2.30}$$

where the last equality is a result of boundary condition (2.25b).

Therefore, (2.28) in conjunction with (2.29) and (2.30) implies

$$\begin{aligned}-\int_\Omega \Delta v\, \langle x, \nabla v\rangle\, dx = {} & \frac{(2-N)}{2}\mu\int_\Omega v f(v)\, dx - \frac{(2-N)}{2\beta}\int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu(x)}\right)^2 dS + \frac{1}{2}\int_{\partial\Omega}|\nabla v|^2 \langle \nu(x), x\rangle\, dS \\ & - \int_{\partial\Omega}\frac{\partial v}{\partial \nu(x)}\langle x, \nabla v\rangle\, dS.\end{aligned} \tag{2.31}$$

Furthermore, the RHS of (2.27) by virtue of integration by parts implies

$$\begin{aligned}\mu\int_\Omega f(v)\, \langle x, \nabla v\rangle\, dx &= \mu\int_\Omega \langle x, \nabla F(v)\rangle\, dx = \mu\int_\Omega \left\langle \nabla\left(\frac{1}{2}|x|^2\right), \nabla F(v)\right\rangle dx \\ &= \mu\int_{\partial\Omega} F(v)\frac{\partial}{\partial \nu(x)}\left(\frac{1}{2}|x|^2\right) dS - \mu N\int_\Omega F(v)\, dx,\end{aligned} \tag{2.32}$$

using again the fact that $\Delta\left(\frac{1}{2}|x|^2\right) = N$.

Consequently, identity (2.26) arises immediately by (2.31) and (2.32). □

Now we are ready to provide a rather measurable (computable) lower estimate of λ^* given by the following.

Theorem 2.9. *Consider problem (2.11) defined in* $\Omega := B_R = \{x \in \mathbb{R}^N : |x| < R\}$, *for* $R > 0$. *Then if* $N > 2(1+\beta R)$ *problem (2.11) has a classical solution for any*

$$\lambda \le \lambda_* := \frac{\beta A(\partial B_R)(N-2)}{[N-2(1+\beta R)]}\frac{(1+\alpha\omega_N^R)^2}{\omega_N^R}, \tag{2.33}$$

where $A(\partial B_R)$ *and* ω_N^R *stand for the area of the surface and the volume of the* $N-$*dimensional sphere* B_R *respectively. Consequently, by (2.12) there holds* $\lambda^* \ge \lambda_*$.

Proof. Assume $0 < \lambda < \lambda^*$, in which case problem (2.11) has a classical solution, and we are working toward the derivation of estimate (2.33). Taking $f(w) = \frac{1}{(1-w)^2}$, hence $F(w) = \frac{w}{(1-w)}$, then Pohožaev's type identity (2.26), for $\mu = \frac{\lambda}{K}$ and K given by (2.14), infers

$$
\begin{aligned}
&\frac{\lambda(N-2)}{2K}\int_{B_R}\frac{w}{(1-w)^2}dx-\frac{\lambda N}{K}\int_{B_R}\frac{w}{1-w}dx\\
&=\frac{N-2}{2\beta}\int_{\partial B_R}\left(\frac{\partial w}{\partial\nu(x)}\right)^2 dS+\frac{1}{2}\int_{\partial B_R}|\nabla w|^2\langle\nu(x),x\rangle\,dS-\int_{\partial B_R}\frac{\partial w}{\partial\nu(x)}\langle\nabla w,x\rangle\,dS\\
&\quad-\frac{\lambda}{K}\int_{\partial B_R}\frac{w}{1-w}\frac{\partial}{\partial\nu(x)}\left(\frac{1}{2}|x|^2\right)dS\\
&=\frac{[N-2(1+\beta R)]}{2\beta}\int_{\partial B_R}\left(\frac{\partial w}{\partial\nu(x)}\right)^2 dS+\frac{R}{2}\int_{\partial B_R}|\nabla w|^2 dS-\frac{\lambda}{K}\int_{\partial B_R}\frac{w}{1-w}\langle x,\nu(x)\rangle\,dS,
\end{aligned}
\tag{2.34}
$$

using the fact that $\frac{\partial}{\partial\nu(x)}\left(\frac{1}{2}|x|^2\right)=\langle x,\nu(x)\rangle$ and $\langle\nu(x),x\rangle=R$ when $\Omega=B_R$. Notably for the case of Dirichlet boundary conditions the term

$$\frac{\lambda}{K}\int_{\partial B_R}\frac{w}{1-w}\langle x,\nu(x)\rangle\,dS$$

vanishes and then calculations in that case are simpler, which is not the case for Robin boundary conditions. However, in the sequel we show that even for Robin boundary conditions this term luckily can be estimated in the right direction. Indeed, via the divergence theorem we have

$$\int_{\partial B_R}\frac{w}{1-w}\langle x,\nu(x)\rangle\,dS=\int_{\partial B_R}\left\langle\hat{F},\nu(x)\right\rangle dS=\int_{B_R}div(\hat{F})\,dx$$

where the vector field $\hat{F}$ is defined by $\hat{F}:=\frac{w}{1-w}x$.

Since $div(\hat{F})=\frac{1}{(1-w)^2}\langle\nabla w,x\rangle+N\frac{w}{1-w}$ then

$$\int_{\partial B_R}\frac{w}{1-w}\langle x,\nu(x)\rangle\,dS=\int_{B_R}\frac{1}{(1-w)^2}\langle\nabla w,x\rangle\,dx+N\int_{B_R}\frac{w}{1-w}dx \tag{2.35}$$

and thus by virtue of (2.34) we derive

$$
\begin{aligned}
&\frac{\lambda(N-2)}{2K}\int_{B_R}\frac{w}{(1-w)^2}dx-\frac{\lambda N}{K}\int_{B_R}\frac{w}{1-w}dx\\
&\geq\frac{[N-2(1+\beta R)]}{2\beta}\int_{\partial B_R}\left(\frac{\partial w}{\partial\nu(x)}\right)^2 dS-\frac{\lambda}{K}\int_{B_R}\frac{1}{(1-w)^2}\langle\nabla w,x\rangle\,dx-\frac{\lambda N}{K}\int_{B_R}\frac{w}{1-w}dx,
\end{aligned}
$$

or

$$\frac{\lambda(N-2)}{2K}\int_{B_R}\frac{1}{(1-w)^2}dx\geq\frac{[N-2(1+\beta R)]}{2\beta}\int_{\partial B_R}\left(\frac{\partial w}{\partial\nu(x)}\right)^2 dS-\frac{\lambda}{K}\int_{B_R}\frac{1}{(1-w)^2}\langle\nabla w,x\rangle\,dx, \tag{2.36}$$

since $0\leq w<1$ for any classical solution of (2.11).

Hölder's inequality infers

$$0\leq-\int_{\partial B_R}\frac{\partial w}{\partial\nu(x)}dS\leq\left(\int_{\partial B_R}\left(\frac{\partial w}{\partial\nu(x)}\right)^2 dS\right)^{1/2}\left(\int_{\partial B_R}dS\right)^{1/2},$$

and so (2.15a) and divergence theorem imply

$$\int_{\partial B_R}\left(\frac{\partial w}{\partial \nu(x)}\right)^2 dS \geq \frac{1}{A(\partial B_R)}\left(\int_{\partial B_R} -\frac{\partial w}{\partial \nu(x)}\,dS\right)^2 = \frac{1}{A(\partial B_R)}\left(\int_{B_R} -\Delta w\,dx\right)^2 = \frac{\lambda^2}{K^2 A(\partial B_R)}\left(\int_{B_R}\frac{1}{(1-w)^2}\,dx\right)^2, \tag{2.37}$$

where

$$A(\partial B_R) := \frac{2\pi^{(N+1)/2} R^{N-1}}{\text{fb}\left(\frac{N+1}{2}\right)},$$

and $\Gamma(\cdot)$ is the Eüler's gamma function.

On the other hand,

$$\langle \nabla w, x\rangle = \frac{\partial w}{\partial x} = w_r \frac{\partial r}{\partial x} = w_r |x|, \text{ for } r = |x|, \tag{2.38}$$

where $\frac{\partial w}{\partial x}$ is the directional derivative in the x direction and where $w(r)$ satisfies

$$-w_{rr} - \frac{N-1}{r} w_r = \frac{\lambda}{(1-w(r))^2 K},\ 0 < r < R,$$
$$w_r(0) = 0,\ w_r(R) + \beta w(R) = 0.$$

Let $\psi := w_r$, then θ satisfies

$$-\psi_{rr} - \frac{N-1}{r}\psi_r + \chi(r)\psi = 0,\ 0 < r < R,$$
$$\psi(0) = 0,\ \psi(R) = -\beta w(R) \leq 0,$$

where $\chi(r) := \left(\frac{N-1}{r^2} - \frac{2\lambda}{(1-w(r))^3 K}\right)$ is bounded since $w(r)$ is a classical solution. Thus, maximum principle[29] infers that $\psi(r) \leq 0$ in $[0, R]$; hence, via (2.38) we obtain

$$\frac{\lambda}{K}\int_{B_R}\frac{1}{(1-w)^2}\langle \nabla w, x\rangle\,dx \leq 0. \tag{2.39}$$

Therefore, (2.36) in conjunction with (2.37) and (2.39) implies

$$\frac{\lambda(N-2)}{2}\frac{\int_{B_R}\frac{1}{(1-w)^2}dx}{\left(1+\alpha\int_{B_R}\frac{1}{(1-w)}dx\right)^2} \geq \frac{\lambda^2[N-2(1+\beta R)]}{2\beta A(\partial B_R)}\left(\frac{\int_{B_R}\frac{1}{(1-w)^2}dx}{\left(1+\alpha\int_{B_R}\frac{1}{(1-w)}dx\right)^2}\right)^2,$$

or

$$\frac{(N-2)}{2} \geq \frac{\lambda[N-2(1+\beta R)]}{2\beta A(\partial B_R)}\frac{\int_{B_R}\frac{1}{(1-w)^2}dx}{\left(1+\alpha\int_{B_R}\frac{1}{(1-w)}dx\right)^2},$$

since $N > 2(1+\beta R)$. Then Hölder's inequality suggests that

$$\left(\int_{B_R}\frac{1}{(1-w)}dx\right)^2 \leq \omega_N^R \int_{B_R}\frac{1}{(1-w)^2}dx,$$

and thus,

$$\frac{(N-2)}{2} \geq \frac{\lambda[N-2(1+\beta R)]}{2\beta\omega_N^R A(\partial B_R)}\left[\frac{\left(\int_{B_R}\frac{1}{(1-w)}dx\right)^2}{\left(1+\alpha\int_{B_R}\frac{1}{(1-w)}dx\right)^2}\right] = \frac{\lambda[N-2(1+\beta R)]}{2\beta\omega_N^R A(\partial B_R)}\left[\frac{\int_{B_R}\frac{1}{(1-w)}dx}{1+\alpha\int_{B_R}\frac{1}{(1-w)}dx}\right]^2, \tag{2.40}$$

where

$$\omega_N^R = |B_R| := \frac{\pi^{N/2}R^N}{\mathrm{fb}\left(\frac{N}{2}+1\right)}.$$

Note that for a classical solution w of (2.11) holds

$$\int_{B_R}\frac{1}{(1-w)}dx \geq \omega_N^R,$$

so using that $g(y) = \frac{y}{\alpha y+1}$ is increasing in $(0, +\infty)$, and thus, $g(y) \geq \omega_N^R/(1+\alpha\omega_N^R)$ for any $y \geq \omega_N^R > 0$, then inequality (2.40) yields

$$N-2 \geq \frac{\lambda}{\beta A(\partial B_R)}\frac{\omega_N^R}{(1+\alpha\omega_N^R)^2}.$$

The latter inequality finally gives the desired estimate

$$\lambda \leq \lambda_* := \frac{\beta A(\partial B_R)(N-2)}{[N-2(1+\beta R)]}\frac{(1+\alpha\omega_N^R)^2}{\omega_N^R},$$

and thus,

$$\lambda^* \geq \frac{\beta A(\partial B_R)(N-2)}{[N-2(1+\beta R)]}\frac{(1+\alpha\omega_N^R)^2}{\omega_N^R}, \tag{2.41}$$

by the definition of λ^*. □

Remark 2.10. Estimate (2.33) in the case of the N-dimensional unit sphere $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$, takes the form

$$\lambda \leq \lambda_* := \frac{\beta A(\partial B_1)(N-2)(1+\alpha\omega_N)^2}{[N-2(1+\beta)]\omega_N},$$

provided that $N > 2(1+\beta)$ where

$$A(\partial B_1) = \frac{2\pi^{(N+1)/2}}{\mathrm{fb}\left(\frac{N+1}{2}\right)}$$

and

$$\omega_N = |B_1| = \frac{\pi^{N/2}}{\mathrm{fb}\left(\frac{N}{2}+1\right)}.$$

Remark 2.11. Let Ω be a bounded domain with the same volume as the N–dimensional ball B_R, then we can get a lower estimate of $\lambda^*(\Omega)$ by virtue of (2.41). Indeed, one can adapt the proof of the well known isoperimetric inequality[34, Theorem 4.10] holding for regular inequalities to the case of the singular MEMS nonlinearity $f(u) = \frac{1}{(1-u)^2}$, cf.[2, Proposition 2.2.1] Therefore,

$$\lambda^*(\Omega) \geq \lambda^*(B_R),$$

hence by virtue of Theorem 2.9 we finally derive

$$\lambda^*(\Omega) \geq \lambda_*(B_R) := \frac{\beta A\,(\partial B_R)\,(N-2)}{[N-2(1+\beta R)]}\,\frac{(1+\alpha\omega_N^R)^2}{\omega_N^R}.$$

Next we present an upper estimate of the pull-in voltage λ^* for a general bounded domain Ω. In particular it holds.

Proposition 2.12. *For a general domain Ω the following upper estimate of the pull-in voltage λ^* holds*

$$\lambda^* \leq \frac{2\lambda_1\left(1+\alpha^2|\Omega|^2\right)}{m_1|\Omega|} < \infty, \tag{2.42}$$

where (λ_1, ϕ_1) is the principal eigenpair of the Laplacian associated with Robin boundary conditions, given by (2.18), and $m_1 := \min_{\bar{\Omega}}\phi_1(x) > 0$.

Proof. Testing Equation (2.11a) by ϕ_1 over the domain Ω we obtain

$$\lambda_1\int_\Omega w\phi_1 dx = \frac{\lambda\int_\Omega \frac{\phi_1}{(1-w)^2}dx}{\left(1+\alpha\int_\Omega \frac{1}{(1-w)}dx\right)^2} > \frac{\lambda m_1\int_\Omega \frac{1}{(1-w)^2}dx}{\left(1+\alpha\int_\Omega \frac{1}{(1-w)}dx\right)^2}. \tag{2.43}$$

Next Hölder's and Young's inequality suggest that

$$\left(1+\alpha\int_\Omega \frac{1}{(1-w)}dx\right)^2 \leq 2+2\alpha^2|\Omega|\int_\Omega \frac{1}{(1-w)^2}dx. \tag{2.44}$$

Then inequalities (2.43) and (2.44), and for a classical solution $0 \leq w < 1$, imply

$$\lambda_1 = \lambda_1\int_\Omega \phi_1 dx \geq \int_\Omega w\phi_1 dx \geq \frac{\frac{\lambda m_1}{\alpha^2}\alpha^2\int_\Omega \frac{dx}{(1-w)^2}}{2+2\alpha^2|\Omega|\int_\Omega \frac{dx}{(1-w)^2}} = \frac{\lambda m_1}{\alpha^2}\mathrm{fk}(I_\alpha(w)) \tag{2.45}$$

where $\mathrm{fk}(s) = \frac{s}{2+2\alpha^2|\Omega|s}$, taking also into account (2.19). Note that $\Psi(s)$ is increasing and thus

$$\mathrm{fk}(I_\alpha(w)) > \mathrm{fk}(\alpha^2|\Omega|) = \frac{\alpha^2|\Omega|}{2+2\alpha^2|\Omega|^2}$$

for $I_\alpha(w) := \alpha^2\int_\Omega \frac{dx}{(1-w)^2}$.

The latter by virtue of (2.44) implies

$$\lambda_1 \geq \frac{\lambda m_1|\Omega|}{2(1+\alpha^2|\Omega|^2)}$$

and thus via the definition of λ^* we derive the desired upper bound (2.42). □

3 | THE TIME DEPENDENT PROBLEM: LOCAL, GLOBAL EXISTENCE, AND QUENCHING

3.1 | Local existence and uniqueness

In this subsection we study the local existence and uniqueness of solutions of problem (1.1). Initially we define the notion of lower-upper solution pairs which will be applied for comparison purposes, cf. previous works.[22,35,36]

Definition 3.1. A pair of functions $0 \le v(x,t), z(x,t) < 1$ with $v, z \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is called a lower- upper solution pair of problem (1.1), if $v(x,t) \le z(x,t)$ for $(x,t) \in Q_T$, $0 < v(x,0) \le u_0(x) \le z(x,0) < 1$ in $\bar{\Omega}$, $\frac{\partial v}{\partial \nu}(x,t) + \beta v(x,t) \le 0 \le \frac{\partial z}{\partial \nu}(x,t) + \beta z(x,t)$ for $(x,t) \in \partial\Omega \times [0, T]$, and

$$v_t \le \Delta v + \frac{\lambda}{(1-v)^2\left(1+\alpha\int_\Omega \frac{dx}{1-z}\right)^2}, \text{ in } Q_T,$$

$$z_t \ge \Delta z + \frac{\lambda}{(1-z)^2\left(1+\alpha\int_\Omega \frac{dx}{1-v}\right)^2}, \text{ in } Q_T.$$

Then local-in-time existence and uniqueness of problem (1.1) is then established by the following.

Proposition 3.2. *Let (v, z) is a lower- upper solution pair to problem (1.1) in Q_T for some $T > 0$. There is a unique solution u to problem (1.1) such that $0 < v \le u \le z < 1$ in Q_T.*

Proof. We define $\bar{u}_0 = z, \underline{u}_0 = v$ and we construct a sequence of lower-upper solutions of problem (1.1) in the following way:

$$\begin{aligned}
&\underline{u}_{nt} = \Delta \underline{u}_n + \frac{\lambda}{(1-\underline{u}_{n-1})^2\left(1+\alpha\int_\Omega \frac{dx}{1-\bar{u}_{n-1}}\right)^2}, \text{ in } Q_{T_n} := \Omega \times (0, T_n),\\
&\bar{u}_{nt} = \Delta \bar{u}_n + \frac{\lambda}{(1-\bar{u}_{n-1})^2\left(1+\alpha\int_\Omega \frac{dx}{1-\underline{u}_{n-1}}\right)^2} \text{ in } Q_{T_n},\\
&\frac{\partial \underline{u}_n}{\partial \nu} + \beta \underline{u}_n = 0, \text{ on } \mathrm{fb}_{T_n} := \partial\Omega \times (0, T_n),\\
&\frac{\partial \bar{u}_n}{\partial \nu} + \beta \bar{u}_n = 0, \text{ on } \mathrm{fb}_{T_n},\\
&\underline{u}_n(x,0) = \bar{u}_n(x,0) = u_0(x), \text{ for } x \in \bar{\Omega},
\end{aligned}$$

for $n = 1, 2, \ldots$ where T_n is the maximum existence time for the pair $(\underline{u}_n, \bar{u}_n)$. Note that by the previous definition we have that the pair $(\underline{u}_n, \bar{u}_n)$ exist as long as the pair $(\underline{u}_{n-1}, \bar{u}_{n-1})$ does so, and thus $T_{n-1} \le T_n$. for $n = 2, 3, \ldots$.

The above problems are local and linear and so we can get local-in-time solutions for them via the classical parabolic theory. Furthermore using Definition 3.1 and standard comparison arguments for parabolic problems (see[37]), we deduce that the sequences $\{\underline{u}_n\}_{n=1}^\infty, \{\bar{u}_n\}_{n=1}^\infty \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, for $T =: \min\{T_n | n \in \mathbb{N}\} = T_1$, are positive and satisfy the ordering

$$v \le \underline{u}_{n-1} \le \underline{u}_n \le \ldots \le \bar{u}_n \le \bar{u}_{n-1} \le z.$$

Let $u_1 := \lim_{n\to\infty} \underline{u}_n$ and $u_2 := \lim_{n\to\infty} \bar{u}_n$ then u_1, u_2 satisfy

$$\begin{aligned}
&u_{1t} = \Delta u_1 + \frac{\lambda}{(1-u_1)^2\left(1+\alpha\int_\Omega \frac{dx}{1-u_2}\right)^2}, \text{ in } Q_T,\\
&u_{2t} = \Delta u_2 + \frac{\lambda}{(1-u_2)^2\left(1+\alpha\int_\Omega \frac{dx}{1-u_1}\right)^2}, \text{ in } Q_T,\\
&\frac{\partial u_1}{\partial \nu} + \beta u_1 = 0, \text{ on } \mathrm{fb}_T,\\
&\frac{\partial u_2}{\partial \nu} + \beta u_2 = 0 \text{ on } \mathrm{fb}_T,\\
&u_1(x,0) = u_2(x,0) = u_0(x), \text{ for } x \in \bar{\Omega}.
\end{aligned}$$

Set $\psi(x,t) = u_1(x,t) - u_2(x,t)$ then

$$\psi_t = \Delta\psi + A(x,t)\psi + B(x,t)\int_\Omega\int_0^1 \frac{d\theta}{[1-\theta u_1-(1-\theta)u_2]^2}\psi dx, \text{ in } Q_T,$$
$$\frac{\partial\psi}{\partial\nu} + \beta\psi = 0, \text{ on } \mathrm{fb}_T,$$
$$\psi(x) = 0, \; x\in\bar{\Omega},$$

where

$$A(x,t) := 2\lambda \frac{\int_0^1 \frac{d\theta}{[1-\theta u_1-(1-\theta)u_2]^3}}{\left(1+\alpha\int_\Omega \frac{dx}{1-u_2}\right)^2} > 0, \tag{3.1}$$

and

$$B(x,t) := \frac{\lambda}{(1-u_2)^2}\frac{2+\int_\Omega\frac{dx}{1-u_1}+\int_\Omega\frac{dx}{1-u_2}}{\left(1+\alpha\int_\Omega\frac{dx}{1-u_1}\right)^2\left(1+\alpha\int_\Omega\frac{dx}{1-u_2}\right)^2} > 0, \tag{3.2}$$

cf. Guo and Kavallaris.[22] Applying now[38, Proposition 52.24] we obtain that $\psi(x,t)=0$ and therefore $u_1 = u_2 := u$ in $\bar{Q}_T$.

Now assume there is a second solution U which satisfies $v \le U \le z$. Subsequently by the preceding iteration scheme we have that $\underline{u}_n \le U \le \bar{u}_n$ for every $n = 1, 2, \ldots$. and by taking the limit as $n \to \infty$ we finally deduce that $U = u$ by the uniqueness of the limit. □

Remark 3.3. By the above result we obtain that the solution of (1.1) continues to exist as long as it remains less than or equal to B for some $B < 1$. In this case we say that u ceases to exist only by quenching, if there is a sequence $(x_n, t_n) \to (x^*, t^*)$ as $n \to \infty$ with $t^* \le \infty$ such that $u(x_n, t_n) \to 1$ as $n \to \infty$, cf. Definition 3.6.

Next we provide a local-in-time existence result for (1.1) using comparison arguments. To this end we first note that the following (local) problem

$$z_t = \Delta z + \frac{\lambda}{(1-z)^2(1+\alpha|\Omega|)^2} \text{ on } Q_T, \tag{3.3a}$$

$$\frac{\partial z}{\partial\nu} + \beta z = 0 \text{ on } \mathrm{fb}_T, \tag{3.3b}$$

$$0 \le z(x,0) = z_0(x) < 1 \text{ for } x\in\bar{\Omega}. \tag{3.3c}$$

has a unique solution, see Guo.[1] Therefore, the following holds:

Proposition 3.4. *If $z_0(x) \ge u_0(x)$ for each $x \in \Omega$, then the problem (1.1) has a unique solution u on $\Omega \times [0, T)$, where $[0, T)$ is the maximal existence time interval for the solution $z(x, t)$ of the problem (3.3), and* $0 \le u(x,t) \le z(x,t) < 1$ *on $\Omega \times [0, T)$.*

Proof. Let $v(x,t) = 0$, then it is readily seen that

$$z_t = \Delta z + \frac{\lambda}{(1-z)^2(1+\alpha|\Omega|)^2} \ge \Delta z + \frac{\lambda}{(1-z)^2\left(1+\alpha\int_\Omega\frac{dx}{1-v}\right)^2} \text{ in } Q_T,$$
$$\frac{\partial z}{\partial\nu} + \beta z = 0, \text{ on } \mathrm{fb}_T,$$
$$z(x,0) = z_0(x) \text{ for } x\in\bar{\Omega},$$

while v satisfies

$$v_t - \Delta v = 0 \le \frac{\lambda}{(1-v)^2\left(1+\alpha\int_\Omega \frac{dx}{1-z}\right)^2} \text{ on } Q_T,$$

$$\frac{\partial v}{\partial \nu} + \beta v = 0, \text{ on } \mathrm{fb}_T,$$

$$v(x,0) = 0 \text{ for } x \in \bar{\Omega}.$$

Therefore, according to Definition 3.1 (v, z) is a lower-upper solution pair for the problem (1.1) and thus the result is an immediate consequence of Proposition 3.2. □

3.2 | Global existence and quenching for general domain

In the current subsection we investigate the global existence and quenching of the solutions of problem (1.1).

We first show the following global existence result.

Theorem 3.5. *Assume that $\lambda \in (0, (1+\alpha|\Omega|)^2\mu^*)$, recalling that μ^* defined by (2.16). Then problem (1.1) with initial condition $u_0(x) \le w_\lambda(x)$ has a global-in-time solution converging as $t \to \infty$ to the minimal steady state solution $w_\lambda(x)$ of (2.11), corresponding to λ.*

Proof. By Proposition 3.4 we have that $(0, z)$ is a lower-upper pair for problem (1.1), where z is the unique solution of local problem (3.3) with initial data $0 \le z_0 = u_0 \le w_\lambda < 1$. Then Proposition 3.2 infers that $0 \le u \le z$. Moreover, due to Theorem 2.6 problem (2.11) has a minimal solution w_λ for any $\lambda \in (0, (1+\alpha|\Omega|)^2\mu^*)$ and thus (2.15) has also a minimal solution w_μ for any

$$0 < \mu = \frac{\lambda}{K(w_\mu)} < \mu^*. \tag{3.4}$$

On the other hand, we can find $\mu_1 \in (0, \mu^*)$ such that

$$\mu_1 = \frac{\lambda}{(1+\alpha|\Omega|)^2}. \tag{3.5}$$

Using now (2.22), then by virtue of (3.4) and (3.5) we get that $\mu < \mu_1$ and so Lemma 2.5 finally implies that $w_\mu \le w_{\mu_1}$. Then via comparison, cf. Proposiition 3.2, $0 \le z \le w_{\mu_1}$ since $z_0 = u_0 \le w_\lambda = w_\mu \le w_{\mu_1}$ and thus we finally deduce that

$$0 < u(x,t) \le z(x,t) \le w_{\mu_1}(x) < \infty, \text{ for any } x \in \Omega, \text{ and } t > 0,$$

and therefore, a global-in-time solution for problem (1.1) exists. Using the dissipative property (3.6) of energy $E(t)$, see also,[39] we can prove convergence of $u(x, t)$ toward the steady-state solution $w_\lambda(x)$, since $u_0(x) \le w_\lambda(x)$.

Next we define the notion of finite time quenching, which is closely related to the mechanical phenomenon of touching down. □

Definition 3.6. The solution $u(x, t)$ of problem (1.1) quenches at some point $x^* \in \Omega$ in finite time $0 < T_q < \infty$ if there exist sequences $\{x_n\}_{n=1}^\infty \in \Omega$ and $\{t_n\}_{n=1}^\infty \in (0,\infty)$ with $x_n \to x^*$ and $t_n \to T_q$ as $n \to \infty$ such that $u(x_n, t_n) \to 1$ as $n \to \infty$. When $T_q = \infty$ we say that $u(x, t)$ quenches in infinite time at x^*. Moreover,

$$Q = \{x^* \in \bar{\Omega} | \exists (x_k, t_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_q) : x_k \to x^*, t_k \to T_q \text{ and } u(x_k, t_k) \to 1 \text{ as } k \to \infty\},$$

is called the quenching set of u.

Now we determine the energy of the problem (1.1). Accordingly we multiply (1.1a) by u_t and integrating over Ω to derive

$$\begin{aligned}\int_\Omega u_t^2\,dx &= -\int_\Omega \nabla u_t \nabla u dx - \beta\int_{\partial\Omega} u_t u dS + \frac{\lambda}{\alpha}\frac{d}{dt}\left(-\frac{1}{1+\alpha\int_\Omega \frac{1}{1-u}dx}\right)\\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx - \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} u^2 dS + \frac{\lambda}{\alpha}\frac{d}{dt}\left(-\frac{1}{1+\alpha\int_\Omega \frac{1}{1-u}dx}\right),\end{aligned}$$

taking also into account boundary condition (1.1b).

Therefore, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\Omega}|\nabla u|^2dx+\frac{\beta}{2}\int_{\partial\Omega}u^2dS+\frac{\lambda/\alpha}{1+\alpha\int_{\Omega}\frac{1}{1-u}dx}\right]=-\int_{\Omega}u_t^2dx, \tag{3.6}$$

which implies that the energy functional

$$E(t):=\frac{1}{2}\int_{\Omega}|\nabla u|^2dx+\frac{\beta}{2}\int_{\partial\Omega}u^2dS+\frac{\lambda/\alpha}{1+\alpha\int_{\Omega}\frac{1}{1-u}dx}\le E(0):=E_0<\infty, \tag{3.7}$$

decreases in time along any solution of (1.1).

Below, we present a quenching result for a general domain Ω following an approach introduced in Guo and Kavallaris[22] see also.[40]

Theorem 3.7. *For any fixed $\lambda>0$, there exist initial data such that the solution of problem (1.1) quenches in finite time provided the associated initial energy*

$$E_0:=\frac{1}{2}\int_{\Omega}|\nabla u_0|^2dx+\frac{\beta}{2}\int_{\partial\Omega}u_0^2dS+\frac{\lambda/\alpha}{1+\alpha\int_{\Omega}\frac{1}{1-u_0}dx}$$

is chosen sufficiently small, that is,

$$E_0<\frac{\lambda q_\alpha(|\Omega|)}{2\alpha}, \tag{3.8}$$

where

$$q_\alpha(|\Omega|):=\begin{cases}1, & |\Omega|\le\frac{1}{3\alpha},\\ \frac{1}{3\alpha|\Omega|}, & |\Omega|\ge\frac{1}{3\alpha}.\end{cases} \tag{3.9}$$

Proof. The proof follows closely that of Kavallaris and Suzuki,[7, Theorem 1.2.17] which deals with Dirichlet boundary conditions, however for the sake of completeness a sketch of the proof is provided here.

Assume that problem (1.1) has a global-in-time (classical) solution u, that is, $0<u(x,t)<1$ for any $(x,t)\in\Omega\times(0,\infty)$ and so

$$Z(t):=\int_{\Omega}u^2(x,t)dx<|\Omega|,\ \text{for any } t>0. \tag{3.10}$$

Multiplying Equation (1.1a) by u and integrating by parts over Ω, we deduce

$$\frac{1}{2}\frac{dZ}{dt}=-\int_{\Omega}|\nabla u|^2dx-\beta\int_{\partial\Omega}u^2\,ds+\lambda\frac{\int_{\Omega}\frac{u}{(1-u)^2}dx}{\left(1+\alpha\int_{\Omega}(1-u)^{-1}dx\right)^2}. \tag{3.11}$$

Using (3.6) then (3.11) reads

$$\begin{aligned}\frac{1}{2}\frac{dZ}{dt}&=-2E(t)+\frac{2\lambda}{\alpha}\frac{1}{\left(1+\alpha\int_{\Omega}(1-u)^{-1}dx\right)}+\lambda\frac{\int_{\Omega}\frac{u}{(1-u)^2}dx}{\left(1+\alpha\int_{\Omega}(1-u)^{-1}dx\right)^2}\\&\ge-2E_0+\frac{\lambda}{\alpha}\frac{2+\alpha\int_{\Omega}\frac{2-u}{(1-u)^2}dx}{\left(1+\alpha\int_{\Omega}(1-u)^{-1}dx\right)^2}.\end{aligned} \tag{3.12}$$

Besides, Hölder's and Young's inequalities imply

$$\left(1+\alpha\int_{\Omega}\frac{dx}{1-u}\right)^2\le2+3\alpha^2|\Omega|\int_{\Omega}\frac{dx}{(1-u)^2},$$

and thus, by virtue of (3.12) we obtain

$$\frac{1}{2}\frac{dZ}{dt} \geq -2E_0 + \frac{\lambda}{\alpha} q_\alpha(|\Omega|),$$

or

$$Z(t) \geq 2\left[q_\alpha(|\Omega|)\,\frac{\lambda}{\alpha} - 2E_0\right] t + Z(0),$$

for $q_\alpha(|\Omega|)$ given by (3.9). The latter implies that $Z(t) \to \infty$ as $t \to \infty$ provided that E_0 satisfies (3.8), which contradicts to (3.10). Therefore the theorem follows. □

Remark 3.8. If we fix the initial data u_0, and thus initial energy $E(0)$, then Theorem 3.7 provides a quenching result for big values of the nonlocal parameter λ. In particular, (3.8) provides a threshold for parameter λ above which finite-time quenching occurs. Namely, if

$$\lambda > \tilde{\lambda} := \frac{2\alpha\left(\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u_0^2 dS\right)}{q_\alpha(|\Omega|) - \frac{2\alpha}{1+\alpha\int_\Omega \frac{1}{1-u_0}dx}},$$

then $||u(\cdot, t)||_\infty \to 1^-$ as $t \to T_q < \infty$ provided that $\mathcal{A}_\alpha(|\Omega|) := q_\alpha(|\Omega|) - \frac{2\alpha}{1+\alpha\int_\Omega \frac{1}{1-u_0}dx}$ is positive. Note that

$$\mathcal{A}_\alpha(|\Omega|) \geq q_\alpha(|\Omega|) - \frac{2\alpha}{1+\alpha|\Omega|} = \begin{cases} \frac{1+\alpha(|\Omega|-2)}{1+\alpha|\Omega|}, & |\Omega| \leq \frac{1}{3\alpha}, \\ \frac{1-\alpha(6\alpha-1)|\Omega|}{3\alpha|\Omega|(1+\alpha|\Omega|)}, & |\Omega| \geq \frac{1}{3\alpha}, \end{cases}$$

and so $\mathcal{A}_\alpha(|\Omega|) > 0$ by either choosing $\alpha < \frac{2}{3}$ and $\frac{2\alpha-1}{\alpha} < |\Omega| \leq \frac{1}{3\alpha}$ for the first branch of the inequality, and $\frac{1}{6} < \alpha < \frac{2}{3}$ with $\frac{1}{3\alpha} \leq |\Omega| < \frac{1}{\alpha(6\alpha-1)}$ or just $\alpha < \frac{1}{6}$ for the second branch.

Remarkably, an optimal value of $\tilde{\lambda}$ for the unit sphere $B_1(0)$ is given in Theorem 3.12, where it is actually shown that $\tilde{\lambda} = \lambda^*$.

A first step toward the derivation of sharper quenching results is the following lemma. Henceforth, we use $C_i, i = 1, \dots,$ to denote various positive constants.

Lemma 3.9. *Let u be a global-in-time solution of the problem (1.1). Then there is a sequence* $\{t_j\}_{j=1}^\infty \uparrow \infty$ *as* $j \to \infty$ *such that*

$$\lambda \int_\Omega u_j(1-u_j)^{-2} dx \leq C_1\left(H(u_j)\right)^2, \tag{3.13}$$

for a positive constant C_1*, where* $u_j = u(\cdot, t_j)$ *and*

$$H(u_j) := 1 + \alpha\int_\Omega \frac{1}{1-u_j} dx > 1. \tag{3.14}$$

Proof. The proof follows closely the steps of the proof of Kavallaris et al[20, Lemma 2.1] for the case of Dirichlet boundary conditions and so it is omitted. □

3.3 | Finite time quenching for the radial symmetric case

A wide used situation is a circular MEMS configuration, see Figure 2B, cf. Pelesko and Chen.[31] Especially, in that case the role of the elastic membrane is played by a soap film and such configuration was first suggested by the prolific British scientist, G.I. Taylor, who actually investigated the coalescence of liquid drops held at differing electric potentials.[32] Later, R.C. Ackerberg initiated the mathematically study of Taylor's model in Ackerberg.[30]

Under a circular configuration, that is, when $\Omega = B_1(0)$, then solution of problem (1.1) is radial symmetric, cf. Gidas et al[26] and then we end up with the following

$$u_t - u_{rr} - (N-1)r^{-1}u_r = F(r,t), \;\; (r,t) \in (0,1)\times(0,T), \;\; N \geq 1, \tag{3.15a}$$

$$u_r(0,t)=0,\ u_r(1,t)+\beta u(1,t)=0,\ t\in(0,T), \tag{3.15b}$$

$$0\le u(r,0)=u_0(r)<1,\ 0<r<1, \tag{3.15c}$$

where

$$F(r,t)=\lambda(1-u(r,t))^{-2}k(t), \tag{3.16}$$

and

$$k(t)=\left[1+\alpha N\omega_N\int_0^1 r^{N-1}(1-u(r,t))^{-1}dr\right]^{-2},$$

recalling that ω_N stands for the volume of the N-dimensional unit sphere $B_1(0)$ in $\mathbb{R}^N$. Note that condition $u_r(0,t)=0$ is imposed to guarantee the regularity of the solution u. We also, for simplicity, consider that $u_0'(r)\le 0$ for $0\le r\le 1$, and thus via maximum principle $u_r(r,t)\le 0$ for $(r,t)\in[0,1]\times[0,T)$.

For convenience we define $0<v:=1-u\le 1$ and so v satisfies

$$v_t-v_{rr}-(N-1)r^{-1}v_r=-fv^{-2},\ (r,t)\in(0,1)\times(0,T), \tag{3.17a}$$

$$v_r(0,t)=0,\ v_r(1,t)+\beta v(1,t)=\beta,\ t\in(0,T) \tag{3.17b}$$

$$0<v(r,0)=v_0(r)\le 1,\ 0<r<1, \tag{3.17c}$$

where

$$f=f(t):=\frac{\lambda}{\left[1+\alpha N\omega_N\int_0^1 r^{N-1}v^{-1}dr\right]^2}$$

and

$$v_r(r,t)>0 \text{ for } (r,t)\in(0,1]\times[0,T). \tag{3.18}$$

For the rest of the our analysis we need a lower estimate for v, which infers a uniform in time upper estimate of the nonocal term, and is shown in the following.

Lemma 3.10. *Consider radial symmetric $v_0(r)$ with $v_0'(r)>0$ and assume also that $N>\beta+1$. Then for any $k>2/3$ there is a constant $C=C(k)$ such that*

$$v(r,t)\ge C(k)r^k \text{ for } (r,t)\in(0,1)\times(0,T). \tag{3.19}$$

Moreover, there exists a constant C_2 which is independent of time t and uniform in λ such that

$$H(u)=H(1-v)\le C_2 \text{ for any } 0<t<T. \tag{3.20}$$

Proof. Considering $1<b<2$, there exist some $t_1>0$ and $\epsilon_1>0$ such that

$$v_r>\epsilon_1 r v^{-b} \text{ at } t=t_1 \text{ for } 0<r\le 1, \tag{3.21}$$

since $v>0$ with a bounded spatial derivative is a classical solution of (3.17a)-(3.17c).

Next differentiating Equation (3.17a) with respect to r gives

$$(v_r)_t-(v_{rr})_r-(N-1)(-r^{-2}v_r+r^{-1}(v_r)_r)=2fv^{-3}v_r,$$

which after multiplying with r^{N-1} reads

$$z_t - z_{rr} + (N-1)r^{-1}z_r = 2fv^{-3}z, \tag{3.22}$$

for $z := r^{N-1}v_r$.

As a next step, we define the functional

$$J = z - \epsilon r^N v^{-b} \text{ for } 0 < \epsilon < \epsilon_1, \tag{3.23}$$

and note that

$$J > 0 \text{ for } 0 < r \le 1 \text{ at } t = t_1, \tag{3.24}$$

thanks to (3.21).

Moreover,

$$J_t = z_t + b\epsilon r^N v^{-b-1} v_t,$$
$$J_r = z_r - \epsilon N r^{N-1} v^{-b} + \epsilon b r^N v^{-b-1} v_r,$$

and

$$J_{rr} = z_{rr} + b\epsilon r^N v^{-b-1} v_{rr} + 2Nb\epsilon r^{N-1} v^{-b-1} v_r - b(b+1)\epsilon r^N v^{-b-2} v_r^2 - N(N-1)\epsilon r^{N-2} v^{-b}.$$

Notably, as long as $J > 0$, then $v_r > \epsilon r v^{-b}$ and so

$$v > \left(\frac{b+1}{2}\epsilon\right)^{\frac{1}{b+1}} r^{\frac{2}{b+1}},$$

which retrieves (3.19) for $C = \left(\frac{b+1}{2}\epsilon\right)^{\frac{1}{b+1}}$ and $k = \frac{2}{b+1}$.

The latter inequality infers

$$\begin{aligned} \int_0^1 r^{N-1} v^{-1} dr &< \int_0^1 r^{N-1} \left(\frac{2}{(b+1)\epsilon}\right)^{\frac{1}{b+1}} \frac{1}{r^{\frac{2}{b+1}}} dr \\ &\le \left(\frac{2}{(b+1)\epsilon}\right)^{\frac{1}{b+1}} \left(\frac{b+1}{Nb+N-2}\right) = C_2 \epsilon^{-1/(b+1)}, \end{aligned} \tag{3.25}$$

and thus estimate (3.20) is also retrieved.

We now introduce the function

$$G(\epsilon) := \frac{\epsilon^{\frac{2}{b+1}}}{\left(\epsilon^{\frac{1}{b+1}} + \alpha N \omega_N C_2\right)^2}, \tag{3.26}$$

where parameter ϵ is small enough $0 < \epsilon \ll 1$, and ϵ_2 imposed to fulfill

$$\epsilon_2 < \sup\left\{\epsilon : \epsilon \le \min\left\{\frac{1}{N}, \left(\frac{2-b}{2b}\right)\right\} \lambda G(\epsilon)\right\}. \tag{3.27}$$

Remarkably, such an ϵ_2 satisfying (3.27) exists since $G(\epsilon) = O(\epsilon^{2/(b+1)})) \gg \epsilon$ for ϵ small with $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$, taking also into account that $b > 1$.

By virtue of (3.24) and (3.25) there holds

$$f(t_1) = \frac{\lambda}{\left(1 + \alpha N \omega_N \int_0^1 r^{N-1} v^{-1} dr\right)^2} > \lambda G(\epsilon), \tag{3.28}$$

thus, in a neighborhood of $t = t_1$ we obtain that $f(t) > \lambda G(\epsilon)$.

Now we claim that $f(t) > \lambda G(\epsilon)$ for any $t \in (t_1, T)$. Let us assume to the contrary that:

$$\text{there exists } t_2 \in (t_1, T) \text{ such that } f(t_2) = \lambda G(\epsilon) \text{ with } f(t) > \lambda G(\epsilon) \text{ for } t_1 \le t < t_2. \tag{3.29}$$

By the definition of J and z we immediately get

$$J = 0 \ on \ r = 0,$$

while on the boundary $r = 1$, due to (3.17b), we have

$$\begin{aligned} J &= z(1,t) - \epsilon v^{-b}(1,t) \\ &= \beta(1 - v(1,t)) - \epsilon v^{-b}(1,t) = v_r(1,t) - \epsilon v^{-b}(1,t) > 0, \end{aligned} \tag{3.30}$$

provided that

$$0 < \epsilon \le \epsilon_3 := \inf_{t_1<t<t_2} \frac{v_r(1,t)}{v^{-b}(1,t)},$$

and taking also into account (3.18).

In addition,

$$J_r = z_r - \epsilon N r^{N-1} v^{-b} + \epsilon b r^N v^{-b-1} v_r = (N-1) r^{N-2} v_r + r^{N-1} v_{rr} + \epsilon r^N v^{-b}(-N r^{-1} + b v^{-1} v_r),$$

and for $r = 1$ we obtain

$$J_r = (N-1) v_r(1,t) + v_{rr}(1,t) + \epsilon v^{-b}(1,t)\left[-N + b v^{-1}(1,t) v_r(1,t)\right].$$

Moreover, at $r = 1$

$$\begin{aligned} J_r - b\epsilon J &= (N-1) v_r(1,t) + v_{rr}(1,t) - \epsilon v^{-b}(1,t)\left[N - b\beta v^{-1}(1,t) + b\beta\right] - b\epsilon\left[\beta - (\beta v(1,t) + \epsilon v^{-b}(1,t))\right] \\ &= (N-1) v_r(1,t) + v_{rr}(1,t) \\ &\quad - \epsilon\left[v^{-b}(1,t) N - b\beta v^{-b-1}(1,t) + b\beta v^{-b}(1,t) + b\beta - b\beta v(1,t) - b\epsilon v^{-b}(1,t)\right], \end{aligned}$$

and therefore, after dropping all the positive terms,

$$J_r - b\epsilon J > (N-1) v_r(1,t) + v_{rr}(1,t) - \epsilon\left[v^{-b}(1,t) N + b\beta v^{-b}(1,t) + b\beta\right].$$

Next differentiating the second of the boundary conditions (3.15b) with respect to r we get

$$v_{rr}(1,t) = -\beta v_r(1,t),$$

and thus

$$J_r - b\epsilon J > (N - 1 - \beta)\, v_r(1,t) - \epsilon\left[v^{-b}(1,t) N + b\beta v^{-b}(1,t) + b\beta\right].$$

Therefore, for $J_r(1, t)$ and $J_r(1, t) - b\epsilon J(1, t)$ to be positive we need

$$(N - 1 - \beta)\, v_r(1,t) - \epsilon\left[v^{-b}(1,t) N + b\beta v^{-b}(1,t) + b\beta\right] > 0,$$

or it is sufficient to choose $\epsilon \le \min\{\epsilon_3, \epsilon_4\}$ for

$$\epsilon_4 := \inf_{t_1<t<t_2} \frac{(N - 1 - \beta)\, v_r(1,t)}{(N + b\beta) v^{-b}(1,t) + b\beta} > 0,$$

since $N > \beta + 1$.

Therefore, we have

$$J_t - J_{rr} + (N-1)r^{-1}J_r \geq 2J(fv^{-3} - b\epsilon v^{-b-1}) + \epsilon f r^N v^{-b-3}(2-b) - 2\epsilon^2 r^N v^{-2b-1} b,$$

and hence,

$$J_t - J_{rr} + (N-1)r^{-1}J_r > 2J(fv^{-3} - b\epsilon v^{-b-1}), \tag{3.31}$$

as far as

$$\epsilon f r^N v^{-b-3}(2-b) - 2\epsilon^2 r^N v^{-2b-1} b > 0,$$

or

$$\epsilon f(2-b) > 2\epsilon^2 b,$$

which in turn gives

$$\epsilon < \epsilon_5 := \inf_{t_1<t<t_2} \frac{f(t)(2-b)}{2b}.$$

After all by maximum principle we derive that $J > 0$, for $0 < r \leq 1$, $t_1 \leq t \leq t_2$ and for ϵ small enough satisfying $\epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$. In $0 < r \leq 1$, $t_1 \leq t \leq t_2$, and since $v > 0$ then the coefficient of J in Equation (3.31) is bounded, so we can define a new variable $\tilde{J} = e^{-D_1 t}J$ which then satisfies the boundary condition (3.30), the boundary inequality (3.31) and

$$\tilde{J}_t - \tilde{J}_{rr} + (N-1)r^{-1}\tilde{J}_r > -D_2\tilde{J}, \tag{3.32}$$

where D_1 and D_2 are positive constants. Should $\tilde{J}$ be non-positive, it must take a non-positive minimum at (r_3, t_3) with $0 < r_3 \leq 1$ and $t_1 < t_3 \leq t_2$. At $r_3 = 1$, by the fact that $J_r(1,t) > 0$ we have $\tilde{J}_r(1,t) > 0$ leading to a contradiction. Thus the supposed minimum must have $0 < r_3 < 1$, where $\tilde{J}_t \leq 0, \tilde{J}_r = 0$ and $\tilde{J}_{rr} \geq 0$. If we have $\tilde{J} \leq 0$ then equation (3.32) gives another contradiction. Therefore, $\tilde{J}$ and J remain positive in $0 < r < 1$ for $t_1 \leq t \leq t_2$.

The latter infers that Equation (3.28) holds at $t = t_2$, contradicting to the initial assumption (3.29). So, as long as solution u exists then $f(t) > \lambda G(\epsilon)$ for $t \geq t_1$. It then follows that $J > 0$, and estimate (3.19) holds together with

$$\begin{aligned} \int_0^1 r^{N-1}v^{-1}dr &< \frac{2^{\frac{1}{b+1}}}{((b+1)\epsilon)^{\frac{1}{b+1}}} \frac{b+1}{Nb+N-2} = \frac{1}{Nb+N-2}\left(\frac{2}{\epsilon}\right)^{\frac{1}{b+1}}(b+1)^{1-\frac{1}{b+1}} \\ &= \frac{1}{Nb+N-2}\left(\frac{2}{\epsilon}\right)^{\frac{1}{b+1}}(b+1)^{\frac{b}{b+1}}, \end{aligned} \tag{3.33}$$

for $t \geq t_1$, in case (3.17a)-(3.17c) has a global solution u or up to and including the quenching time T_q when u quenches. Finally by the definition of $H(u)$ and inequality (3.33) we obtain the desired estimate, (3.20), and the lemma follows. □

Remark 3.11. Note that we can alternatively obtain that

$$\epsilon_4 = \inf_{t_1<t<t_2} \frac{fv^{-2}(1,t) + v_t(1,t)}{(N+b\beta)v^{-b}(1,t) + b\beta} > 0$$

without any restrictions on the spatial dimesnion N, by choosing λ large enough, that is, $\lambda > \lambda^{**} \geq \lambda^*$, so that

$$f(t) = \frac{\lambda}{\left(1 + \alpha N\omega_N \int_0^1 r^{N-1}v^{-1}(r,t)dr\right)^2} > -v_t(1,t)v^2(1,t) \text{ for } t \in (t_1, t_2), \tag{3.34}$$

which is always possible for a classical (and thus smooth enough) solution $u(r, t)$. Therefore, we can recover the result of Lemma 3.10 independently of the dimension N, but for $\lambda > \lambda^{**}$ so that (3.34) is satisfied. Consequently, in the sequel all the derived quenching results can alternatively be obtained for λ large enough, in particular for $\lambda > \lambda^{**}$, but without imposing any restrictions on the spatial dimesnion.

Now having in place Lemmata 3.9 and 3.10 we are ready to prove the following quenching result. This result is sharp (optimal) in the sense that predicts quenching in the parameter range for the pull-in voltage λ where no classical steady-states exist.

Theorem 3.12. *Consider radially symmetric initial data $u_0(r)$ with $u_0'(r) < 0$. Assume also that $N > \beta + 1$ then for any $\lambda > \lambda^*$ the solution of the problem (3.15) quenches in finite time $T_q < \infty$.*

Proof. Let assume to the contrary that for some $\lambda > \lambda^*$ problem (3.15) has a global-in-time solution. Then thanks to (3.13) and (3.20), we can get a sequence $\{t_j\}_{j=1}^{\infty}$ with $t_j \to \infty$ as $j \to \infty$ such that

$$\lambda N\omega_N \int_0^1 r^{N-1} u_j (1-u_j)^{-2}\, dr \le C_3, \text{ for any } t > 0, \tag{3.35}$$

where the constant C_3 is independent of j.

Then by (3.20) it is readily seen that

$$\begin{aligned} N\omega_N \int_0^1 \frac{r^{N-1}\, dr}{(1-u_j)^2} &= N\omega_N \int_0^1 \frac{r^{N-1}\, dr}{(1-u_j)} + N\omega_N \int_0^1 \frac{r^{N-1} u_j\, dr}{(1-u_j)^2} \\ &\le (C_2 - 1) + \frac{C_3}{\lambda} := C_4, \end{aligned} \tag{3.36}$$

where C_4 is independent of j.

Additionally, by virtue of (3.7) we have

$$||\nabla u_j||^2_{L^2(B_1)} \le C_5 < \infty, \tag{3.37}$$

where C_5 is again independent of j.

Passing to a sub-sequence, if necessary, relation (3.37) infers the existence of a function w such that

$$u_j \rightharpoonup w \text{ in } H^1(B_1), \tag{3.38}$$

$$u_j \to w \text{ a.e. in } B_1, \tag{3.39}$$

as $j \to \infty$. For $N \ge 2$ and by (3.36) we immediately obtain that $1/(1-u_j)^2$ is uniformly integrable and since

$$\frac{1}{(1-u_j)^2} \to \frac{1}{(1-w)^2},\ j \to \infty \text{ a.e. in } B_1,$$

due to (3.39), we finally deduce

$$\frac{1}{(1-u_j)^2} \to \frac{1}{(1-w)^2} \text{ as } j \to \infty \text{ in } L^1(B_1), \tag{3.40}$$

by virtue of Lebesque dominated convergence theorem. Similarly we also derive

$$H(u_j) \to H(w) \text{ as } j \to \infty \text{ in } L^1(B_1). \tag{3.41}$$

Next note also that by relation (3.6), see also Kavallaris et al,[20] we derive the following estimate

$$\int_\tau^\infty \int_{B_1} u_t^2(x,s)\, dx\, ds \le C < \infty,$$

for a constant C independent of $\tau > 0$, and thus passing to a sub-sequence if it is necessary we obtain

$$||u_t(\cdot, t_j)||_2^2 = \int_{B_1} u_t^2(x, t_j)\, dx \to 0 \text{ as } j \to \infty. \tag{3.42}$$

A weak formulation of (3.15) along the sequence $\{t_j\}_{j=1}^{\infty}$ can be written as

$$\int_{B_1} \frac{\partial u_j}{\partial t} \phi \,\mathrm{d}x = -\int_{B_1} \nabla u_j \cdot \nabla \phi \,\mathrm{d}x + \int_{\partial B_1} \frac{\partial u_j}{\partial \nu} \phi \,ds + \lambda H^{-1}(u_j) \int_{B_1} \phi (1-u_j)^{-2} \,\mathrm{d}x, \tag{3.43}$$

for any $\phi \in H^1(B_1)$.

For any $\phi \in W^{2,2}(B_1)$ with $\frac{\partial \phi}{\partial \nu} + \beta\phi = 0,$ on ∂B_1, then Green's identities imply

$$\begin{aligned}\int_{\partial B_1} \frac{\partial u_j}{\partial \nu} \phi \,ds &= \int_{B_1} \nabla u_j \cdot \nabla \phi \,dx + \int_{B_1} (\Delta u_j)\phi \,dx \\ &= \int_{B_1} \nabla u_j \cdot \nabla \phi \,dx + \int_{B_1} u_j (\Delta \phi) \,dx\end{aligned}$$

and thus by virtue of (3.38), (3.39) and Lebesque dominated convergence theorem we derive

$$\int_{\partial B_1} \frac{\partial u_j}{\partial \nu} \phi \,ds \to \int_{B_1} \nabla w \cdot \nabla \phi \,dx + \int_{B_1} w(\Delta \phi) \,dx = \int_{\partial B_1} \frac{\partial w}{\partial \nu} \phi \,ds, \tag{3.44}$$

since $w \in H^1(B_1)$.

Passing to the limit as $j \to \infty$ in (3.43), and in conjunction with (3.38), (3.40), (3.41),(3.42) and (3.44) we derive

$$-\int_{B_1} \nabla \phi \cdot \nabla w \,dx + \int_{\partial B_1} \phi \frac{\partial w}{\partial \nu} \,ds + \lambda \frac{\int_{B_1} \frac{\phi}{(1-w)^2} dx}{\left(1 + \int_{B_1} \frac{1}{1-w} dx\right)^2} = 0,$$

for any $\phi \in W^{2,2}(B_1)$ satisfying $\frac{\partial \phi}{\partial \nu} + \beta\phi = 0$ on ∂B_1.

The latter, according to Definition 2.2, infers that w is a weak finite-energy solution of problem (3.15) corresponding to $\lambda > \lambda^*$ which contradicts with the result of Proposition 2.3.

For $N = 1$, using a similar approach and trace theorem, see also Kavallaris et al,[20, Theorem 3.5] we obtain that u_j converges to a weak finite-energy solution of problem (3.15) arriving again at a contradiction. This completes the proof of theorem. □

Remark 3.13. Notably the quenching predicted by Theorem 3.12 is single-point quenching. In particular, due to (3.19) we derive that $u(r.t)$ can only quench at the origin $r = 0$.

3.4 | Quenching for large initial data

In the following, we investigate the behavior of the problem (3.15) for large initial data. Namely, the following result holds.

Theorem 3.14. *For any $\lambda > 0$ and for $N > \beta + 1$ we can choose initial data u_0 close enough to 1 such that the solution u of problem (3.15) quenches in finite time $T_q < \infty$.*

Proof. We denote by (λ_1, ϕ_1) be the principal eigenpair of

$$-\Delta\phi_1 = \lambda_1\phi_1, x \in B_1, \; \frac{\partial \phi_1}{\partial \nu} + \beta\phi_1 = 0, \; x \in \partial B_1,$$

where again ϕ is normalized so that

$$\int_{B_1} \phi_1(x) dx = 1.$$

Let us suppose that problem (3.15) has a global-in-time solution $0 < u(x, t) < 1$ for any $(x, t) \in B_1 \times (0, \infty)$.

Testing Equation (3.15a) with ϕ_1 and integrating over B_1 then Green's second identity and Lemma 3.10 infer,

$$\begin{aligned}\frac{d}{dt}\int_{B_1}\phi_1 u\,dx &= \int_{B_1}\phi_1\Delta_r u dx + \lambda\int_{B_1}\phi_1(1-u)^{-2}(H(u))^{-2}dx\\ &= \int_{B_1}\Delta_r\phi_1 u dx + \int_{\partial B_1}\left(u\frac{\partial\phi_1}{\partial\nu}-\frac{\partial u}{\partial\nu}\phi_1\right)ds + \lambda\int_{B_1}\phi_1(1-u)^{-2}(H(u))^{-2}dx\\ &= -\int_{B_1}\lambda_1\phi_1 u dx + \frac{\lambda\int_{B_1}\phi_1(1-u)^{-2}dx}{(H(u))^2}.\end{aligned} \tag{3.45}$$

Set $A(t) := \int_{B_1} u\phi_1 dx$, then applying Jensen's inequality to equation (3.45), we obtain

$$\frac{dA}{dt} \geq -\lambda_1 A(t) + \frac{\lambda}{C_2^2}(1-A(t))^{-2}, \text{ for any } t>0. \tag{3.46}$$

Next we choose suitable $\gamma \in (0,1)$ such that

$$\mathrm{fk}(s) := \frac{\lambda}{C_2^2}(1-s)^{-2} - \lambda_1 s > 0 \text{ for all } s\in[\gamma,1),$$

and then by choosing u_0 such that $A(0) = \int_{B_1} u_0\phi_1 dx \geq \gamma$, then (3.46) infers

$$\frac{dA}{dt} \geq \mathrm{fk}(A(t)) > 0 \text{ for any } t>0,$$

or by integrating

$$t \leq \int_{A(0)}^{A(t)}\frac{ds}{\mathrm{fk}(s)} \leq \int_{A(0)}^{1}\frac{ds}{\mathrm{fk}(s)} < \infty.$$

The latter is in contradiction with our initial assumption that $T=\infty$, and the theorem is proved. □

3.5 | Behavior at quenching

In the current subsection we give more details regarding the behavior of quenching solutions close to quenching time T_q.

We first obtain the quenching rate. Let us recall that a solution $u(r,t)$ of (3.15) with radial decreasing initial data u_0 then u is also radial decreasing and thus,

$$M(t) := \max_{x\in\bar{B}_1} u(x,t) = u(0,t).$$

The next result determines the quenching rate of u for singular solutions of (3.15).

Theorem 3.15. *Let $u(r,t)$ be a quenching solution of (3.15). Then for $N > \beta+1$ there are positive constants $\hat{C},\tilde{C}$ indpendent on time t such that*

$$1-\hat{C}(T_q-t)^{1/3} \leq M(t) \leq 1-\tilde{C}(T_q-t)^{1/3} \textit{ for } 0<t-T_q \ll 1. \tag{3.47}$$

Proof. Since $M(t)$ is Lipschitz continuous then by Rademacher's theorem, is almost everywhere differentiable, cf.[41,42] Furthermore, since u attains a maximum at $r=0$ then $\Delta_r u(0,t) \leq 0$ for all $t \in (0,T_q)$. Therefore, for any t where dM/dt exists, we derive

$$\frac{\mathrm{d}}{\mathrm{d}M}t \leq \lambda\frac{(1-M(t))^{-2}}{\left(1+\int_{B_1}\frac{1}{1-u}\mathrm{d}x\right)^2} \leq \lambda\frac{(1-M(t))^{-2}}{(1+N\omega_N)^2} \text{ for a.e. } t\in(0,T_q),$$

which yields

$$\int_{M(t)}^{1} (1-s)^2 \mathrm{d}s \leq \lambda C(T_q - t),$$

for $C = 1/(1+N\omega_N)^2$. The latter implies

$$M(t) \geq 1 - \hat{C}(T_q - t)^{1/3} \text{ for } 0 < t < T_q, \tag{3.48}$$

where $\hat{C} = (3\lambda C)^{1/3}$.

Note that inequality (3.20) implies that $H(u)$ is uniformly integrable so then via, (3.19) and parabolic regularity estimates in the region $r \in (0, 1)$, cf. Ladyženskaja et al[43] we obtain that

$$\lim_{t\to T_q} u(r,t) = u(r, T_q) \text{ for any } 0 < r < 1. \tag{3.49}$$

Estimate (3.19) also implies that

$$(1-u)^{-1} \leq \bar{C}(k) r^{-k},$$

for $k > \frac{2}{3}$, and $\bar{C}(k) = \frac{1}{C(k)}$ and thus from relation (3.49), and the Lebesque dominated convergence theorem we get that

$$\lim_{t\to T_q} \int_{B(0,1)} \frac{1}{1-u(x,t)} dx = \int_{B(0,1)} \frac{1}{1-u(x,T_q)} dx < \infty$$

and finally,

$$\lim_{t\to T_q} (H(u))^2 = K < \infty.$$

Therefore, for $0 < t - T_q \ll 1$ we have that

$$\begin{aligned} &u_t(x,t) \simeq \Delta u + \frac{\lambda}{K}\frac{1}{(1-u)^2}, \; x \in B(0,1), \\ &\frac{\partial u}{\partial \nu}(x,t) + \beta u(x,t) = 0, \; x \in \partial B(0,1), \\ &u(x,0) = u_0(x). \end{aligned}$$

But for the above local problem it is known, cf. previous works,[44,45] that

$$M(t) = u(0,t) \lesssim 1 - \tilde{C}(T_q - t)^{1/3}, \tag{3.50}$$

for some $\tilde{C} > 0$.

Therefore, combining inequalities (3.48), (3.50), we obtain the required estimation (3.47) □

It is worth noting that due to the uniform bounds of nonlocal term $(H(u))^2$ we can treat nonlocal problem (3.15) as a local one and therefore, the quenching profile is given as follows, cf.[44,45]

$$1 - u(r, T_q) \sim C^* \left[\frac{|r|^2}{|\ln |r||}\right]^{1/3} \text{ as } \quad r \to 0^+, \tag{3.51}$$

for some positive constant C^*. For a more rigorous approach, which is out of the scope of the current work, one should follow similar arguments as in[24,46] to derive (3.51) where it is conjectured that $C^* = \lim_{t\to T_q} H(u(r,t))$.

4 | NUMERICAL APPROACH

In the current section we present a numerical study of problem (1.1) both in the one-dimensional as well as in the two-dimensional radial symmetric case. For that purpose an adaptive method monitoring the behavior of the solution near a singularity, such as the detected quenching behavior of (1.1), is used (e.g., see[6,47]).

4.1 | One-dimensional case

For the one-dimensional case and for the sake of simplicity, taking advantage of the symmetry of the solution, we may consider the problem in the interval $[0,\ 1]$ with Neumann condition at $x = 0$, $u_x(0, t) = 0$ and the original Robin condition at the point $x = 1$.

Initially, we take a partition of $M + 1$ points in the interval [0,1], $\xi_0 = 0, \xi_1 = \xi_0 + \Delta\xi, \dots, \xi_M = 1$. For $u = u(x, t)$, we introduce a computational coordinate ξ in [0,1] and we consider the mesh points X_i to be the images of the points ξ_i under the map $x(\xi, t)$ so that $X_i(t) = x(i\Delta\xi, t)$. By the latter relation we obtain $\frac{du(X(t),t)}{dt} = u_t(X_i, t) + u_x X_i'$ for the approximation of the solution $u_i(t) \simeq u(x_i(t), t)$.

Moreover, the map $x(\xi, t)$ is determined by the function $\mathcal{M}(u)$ which in a sense, follows the evolution of the singularity in case of quenching. This function is determined by the scale invariants of the problem. In particular, for the semilinear parabolic equation

$$v_t = v_{xx} - \frac{\lambda}{v^2\left[1 + \alpha \int_{-1}^{1} 1/v\, dx\right]^2},$$

where $v = 1 - u$, an appropriate monitor function should be of the form $(u) = |1 - u|^{-2}$ or $\mathcal{M}(v) = |v|^{-2}$.

We need also a rescaling of time of the form $\frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt}$ where $\frac{dt}{d\tau} = g(u)$, and $g(u)$ is a function determining the way that the time scale changes as the solution approaches the singularity. In particular, we have $g(u) = \frac{1}{\|\mathcal{M}(u)\|_\infty}$.

In addition the evolution of $X_i(t)$ is given by a moving mesh PDE which is of the form $x_{\tau\xi\xi} = \epsilon^{-1} g(u)(\mathcal{M}(u)x_\xi)_\xi$. Here ϵ is a small parameter accounting for the time scale. Thus finally we obtain a system of ODEs for X_i and u_i. The underlying ODE system takes the form

$$\begin{aligned} \frac{dt}{d\tau} &= g(u), \\ u_\tau - x_\tau u_x &= g(u)\left(u_{xx} + \frac{\lambda}{(1-u)^2\left(1 + \alpha \int_0^1 \frac{1}{1-u} dx\right)^2} \right), \\ -x_{\tau\xi\xi} &= \frac{g(u)}{\epsilon}(\mathcal{M}(u)x_\xi)_\xi. \end{aligned} \tag{4.1}$$

We apply a discretization in space to derive

$$\begin{aligned} u_x(X_i, \tau) &\simeq \Delta_x u_i(\tau) := -\frac{u_{i+1}(\tau) - u_{i-1}(\tau)}{X_{i+1}(\tau) - X_{i-1}(\tau)}, \\ u_{xx}(X_i, \tau) &\simeq \Delta_x^2 u_i(\tau) := \left(\frac{u_{i+1}(\tau) - u_i(\tau)}{X_{i+1}(\tau) - X_i(\tau)} - \frac{u_i(\tau) - u_{i-1}(\tau)}{X_i(\tau) - X_{i-1}(\tau)} \right) \frac{2}{X_{i+1}(\tau) - X_{i-1}(\tau)}, \\ x_{\xi\xi}(\xi_i, \tau) &\simeq \Delta_\xi^2 x_i(\tau) := \frac{X_{i+1}(\tau) - 2X_i(\tau) + X_{i-1}(\tau)}{\delta\xi^2}, \\ (\mathcal{M}(u)x_\xi)_\xi &\simeq \Delta_\xi(\mathcal{M}\Delta_\xi x) := -\left(\frac{\mathcal{M}_{i+1} + \mathcal{M}_i}{2}\frac{x_{i+1} - x_i}{\Delta\xi} - \frac{\mathcal{M}_i + \mathcal{M}_{i-1}}{2}\frac{x_i - x_{i-1}}{\Delta\xi} \right) \frac{1}{\Delta\xi}. \end{aligned}$$

Notably at the boundary point $X_M = 1$ the discretized boundary condition $u_M = u_{M-1} - \beta u_M\,(X_M - X_{M-1})$ has been used. The preceding spatial discretization leads to an ODE system of the form

$$A(\tau, y)\frac{dy}{d\tau} = b(\tau, y), \tag{4.2}$$

with the vector $y \in \mathbb{R}^{2n+1}$ defined as

$$y = (t(\tau), u_1(\tau), u_2(\tau), \ldots, u_M(\tau), X_1(\tau), X_2(\tau), \ldots, X_M(\tau)), = (t(\tau), \mathbf{u}, \mathbf{X}), \mathbf{u}, \mathbf{X} \in \mathbb{R}^M,$$

and $A \in \mathbb{R}^{2n+1,2n+1}$. System (4.2) has the block form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & -\Delta_x u \\ 0 & 0 & -\Delta_\xi^2 \end{bmatrix}, \quad y = \begin{bmatrix} t(\tau) \\ u \\ X \end{bmatrix}, \quad b = g(u) \begin{bmatrix} 1 \\ \Delta_x^2 u + \lambda \frac{1}{(1-u)^2(1+\alpha I(u))^2} \\ \Delta_\xi(\mathcal{M}\Delta_\xi x) \end{bmatrix},$$

where $\boldsymbol{I}(u)$ is an approximation of the integral $\int_0^1 \frac{1}{1-u} dx$, using Simpsons' method. For the solution of (4.2) a standard ODE solver, such as the matlab function "ode15i", can be used.

4.1.1 | The local problem

Initially, we present a simulation for the local problem, (1.2), that is, problem (1.1) for $\alpha = 0$. In Figure 5some numerical experiments presented for the case where a global-in-time solution exists. In the first of these graphs (top left) we plot the solution against space and time. In the second one (top right) we plot the moving mesh $X(i, t)$ against time, while in

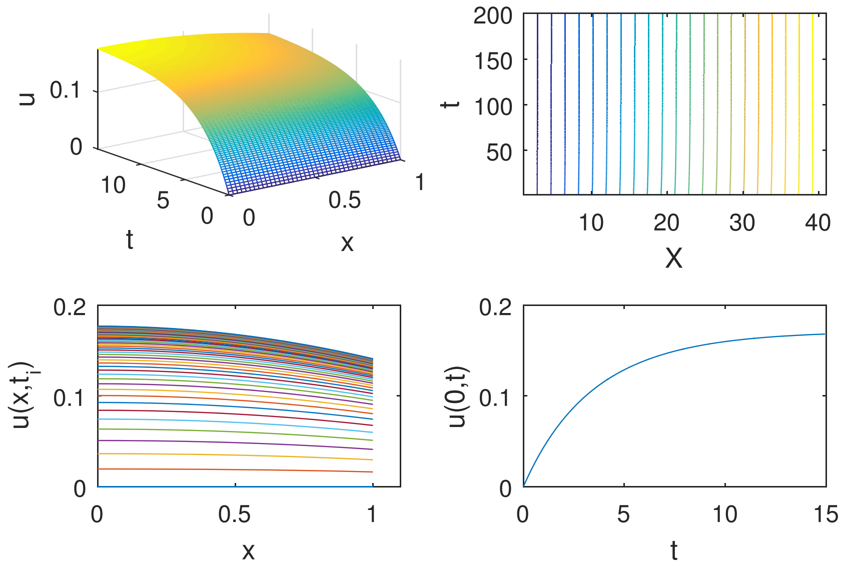

FIGURE 5 Form of the solution and various profiles of the local problem for $\lambda = 0.05$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

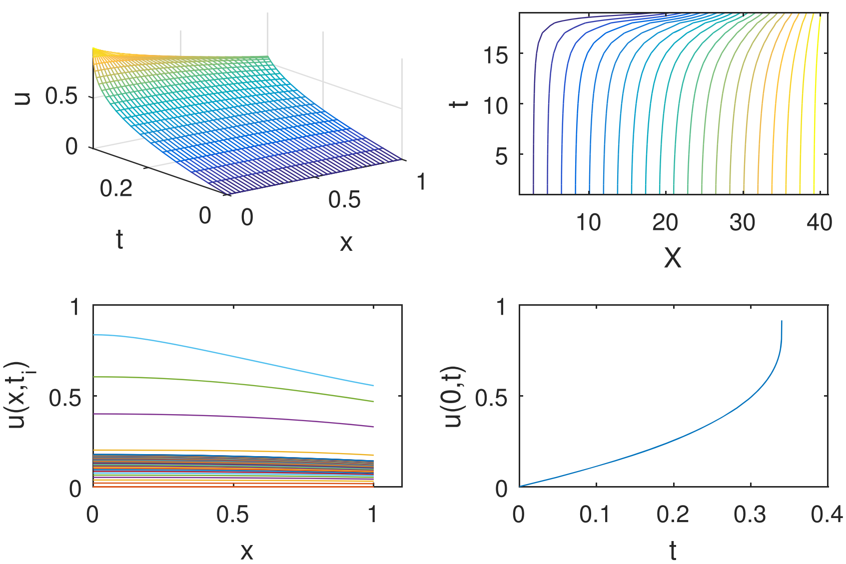

FIGURE 6 Form of the solution and various profiles of the local problem in the case of quenching for $\lambda = 1$, and $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

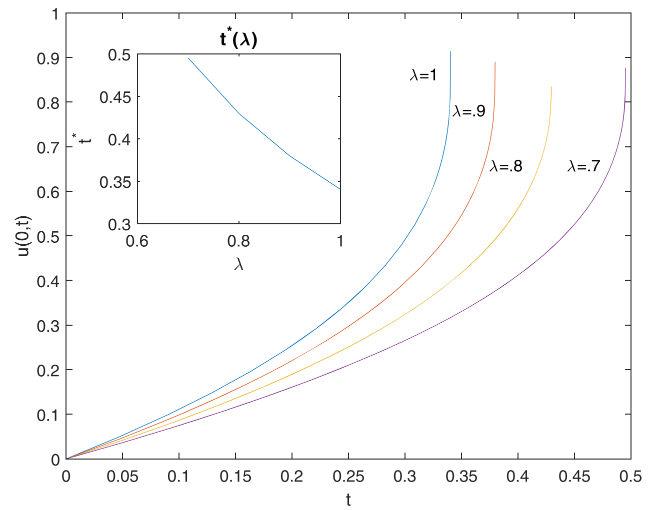

FIGURE 7 Form of the solution maximum against time for various values of the parameter λ for the local problem for $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

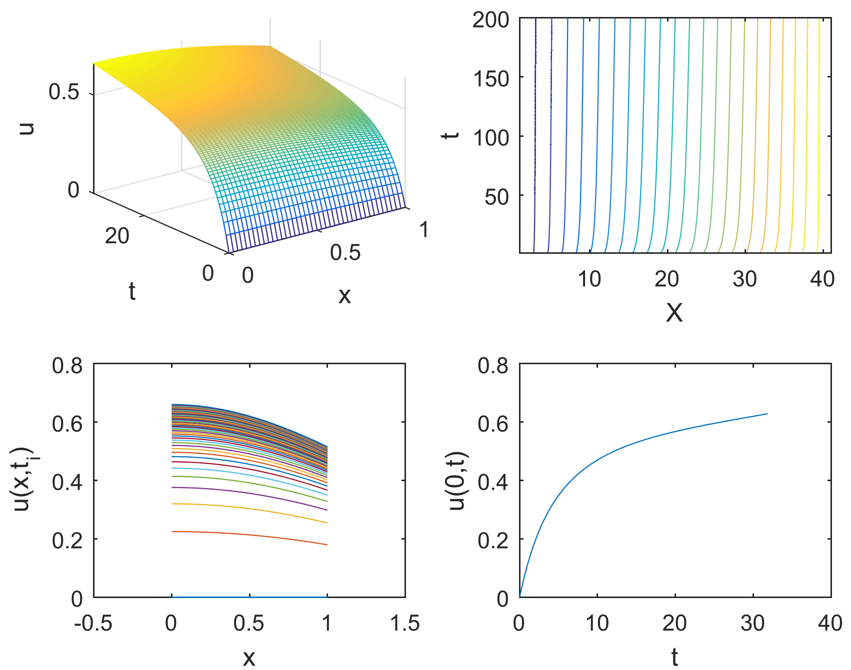

FIGURE 8 Form of the solution and various profiles of the nonlocal problem, for $\lambda = 0.5$, $\alpha = 1$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

the third (bottom left) a sequence of profiles of the solution ($u(x, t_i)$) for various time steps t_i is presented. Finally, in the fourth graph we plot the maximum of the solution $u(0, t)$ against time. The latter plot shows the convergence toward a steady state. The initial condition here, as well as in the rest of the simulations, was taken to be zero, $u_0(x) = 0$. Also, the parameters used here were $\lambda = 0.05$, $\beta = 1$, $t \in [0, T_f]$, $T_f = 40$, $M = 141$.

Figure 6 depicts the situation where the solution quenches in finite time. Again in the first of these graphs (top left) we plot the solution against space and time. In the second one (top right) we plot the moving mesh $X(i, t)$ against time. Here the motion of X_i's captures the observed singularity, that is, the finite-time quenching. In the third (bottom left) a sequence of profiles of the solution ($u(x, t_i)$) for various time steps t_i is presented. We can observe the increasing with time profiles of the solution. Finally in the fourth graph we plot the maximum of the solution $u(0, t)$ against time from which the quenching behavior is revealed. The same parameters as in Figure 5 are used but with $\lambda = 1$. In the next figure, 7 we plot the profiles of the solution maximum, $u(0, t)$ against time, for various λ's and specifically for $\lambda = .7, .8, .9, 1$. We observe that by increasing the value of the parameter λ the quenching time decreases as it is expected.

4.1.2 | The non-local problem

A similar set of simulations is presented for the case that $\alpha = 1$ while the rest of the parameters, unless otherwise stated, are kept the same as in the experiment of Figure 5. In Figure 8and for $\lambda = 0.5$ the convergence of the solution toward a

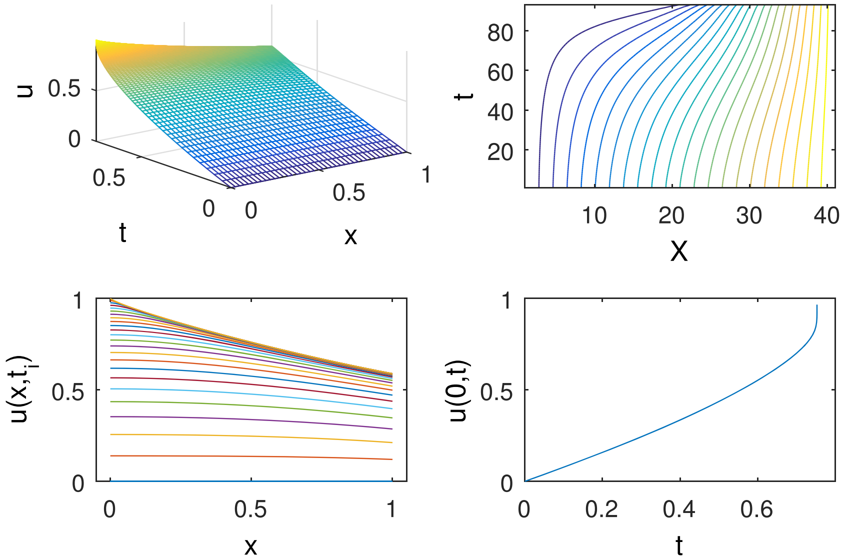

FIGURE 9 Form of the solution and various profiles of the nonlocal problem for $\lambda = 3$, $\alpha = 1$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

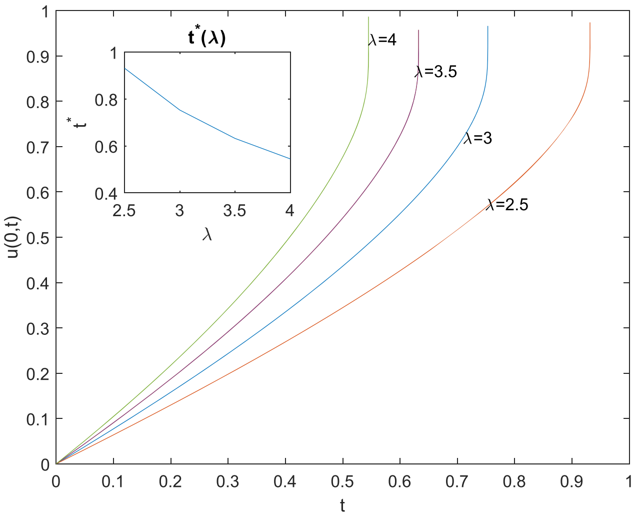

FIGURE 10 Form of the solution maximum against time for various values of the parameter λ and with $\alpha = 1$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

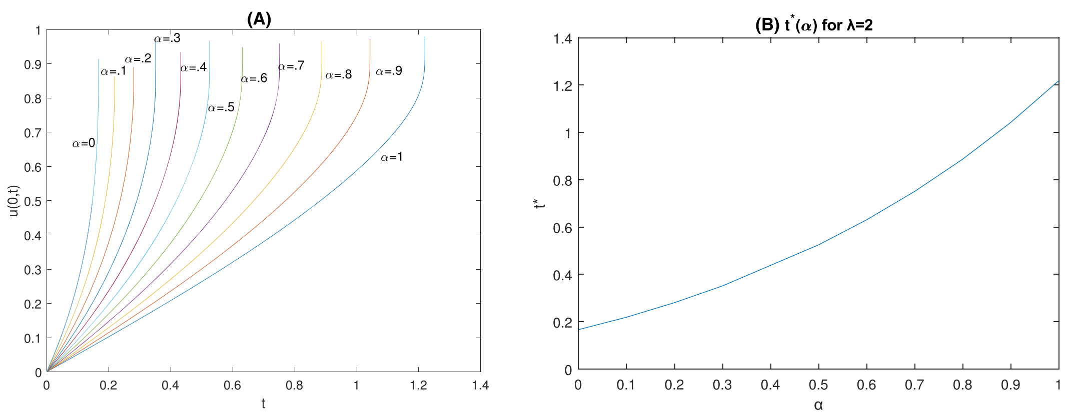

FIGURE 11 (A) Form of the solution maximum against time for various values of the parameter α for $\lambda = 2$ and $\beta = 1$. (B) Variation of the quenching time of the nonlocal problem with respect to the parameter α [Colour figure can be viewed at wileyonlinelibrary.com]

steady state is depicted. In a similar set of graphs, see Figure 9 and for $\lambda = 3$, we present the quenching behavior of the solution. Moreover, in Figure 10 we can observe the evolution of the quenching time as the value of the parameter λ varies, something cannot be seen via our theoretical results. In particular,by increasing the parameter λ results in a decreasing of quenching time. Here $\lambda = 2.5, 3, 3.5, 4$. Next in Figure 11A we plot a series of profiles for the maximum of the solution as the parameter α varies. Again such a behavior cannot be unveiled via our analytical results in Sections 3.2 and 3.3. It is easily seen that by decreasing α the quenching time decreases too. The parameter α decreases from 1 to the value 0 while the parameter λ is kept constant and equal to $\lambda = 2$.

The effect of the boundary parameter β is unveiled by Figure 12A, a fact cannot be easily seen by our theoretical results in Section 3. Indeed, it is seen that by increasing β a long-time behavior resembles the one of the Dirichlet problem is derived. The variation of the quenching time t^* of the nonlocal problem is depicted in a series of plots in Figures 11A and 12B. In the first of them, Figure 11B, we present a plot of $t^*(\alpha)$, while in the second, Figure 12B, a plot of $t^*(\beta)$. In both cases was taken $\lambda = 2$.

4.2 | The radial symmetric case

It has been already pointed out that the two-dimensional problem in the radially symmetric case is very interesting from the point of view of applications and thus we choose to provide a numerical treatment for it in the current subsection. For this purpose the aforementioned adaptive numerical scheme and specifically Equation (4.1) can be modified accordingly with $u_{xx} + (N-1)r^{-1}u_x$ used in place of u_{xx}.

Initially, we solve the local problem, that is, problem (3.15) for $\alpha = 0$ and the results are presented in Figure 13. Here we take $\beta = 1$, $\lambda = 0.05$ and we observe that the solution converges toward a steady state. In Figure 14we present an analogous simulation for the nonlocal problem. In that case we take $\alpha = 1, \beta = 1$, and $\lambda = 0.2$ and we derive that the solution quenches in finite time.

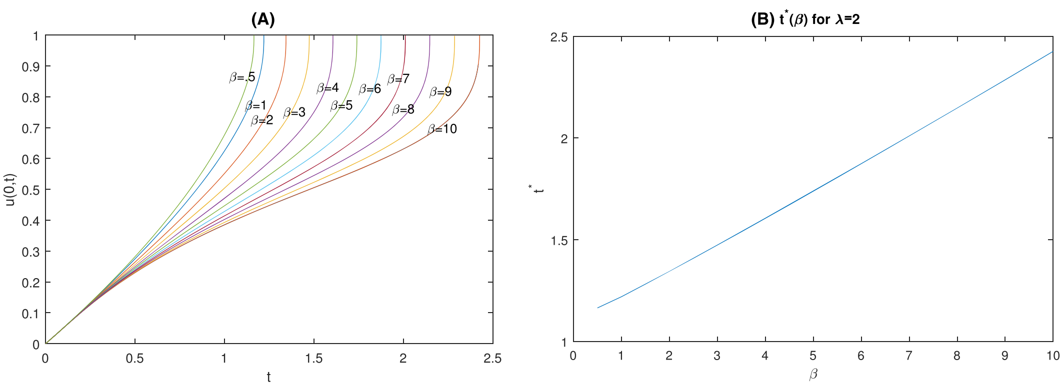

FIGURE 12 (A) Form of the solution maximum against time for various values of the parameter β for $\lambda = 2$ and $\alpha = 1$. (B) Variation of the quenching time of the nonlocal problem with respect to the parameter β [Colour figure can be viewed at wileyonlinelibrary.com]

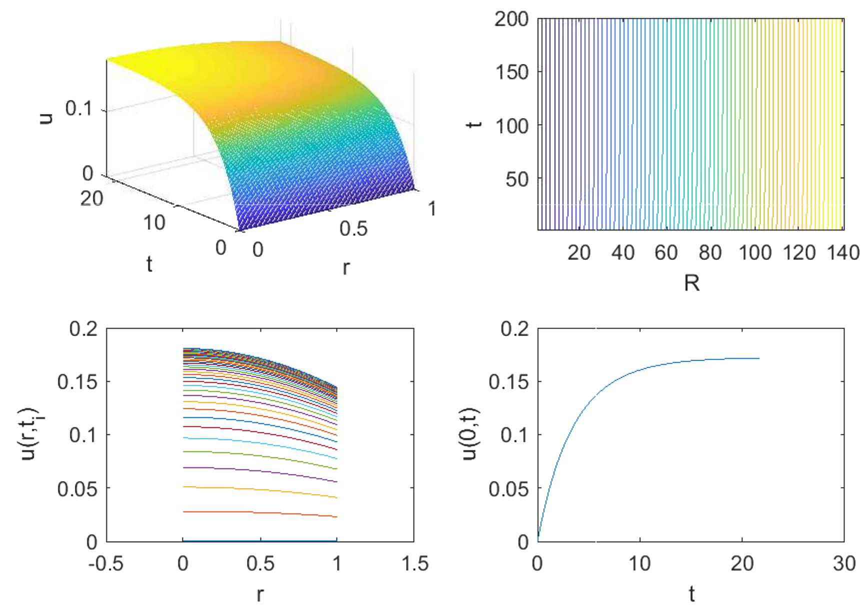

FIGURE 13 Form of the solution and various profiles of the local problem for the radial symmetric case for $\lambda = 0.05$ and $\alpha = 0$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

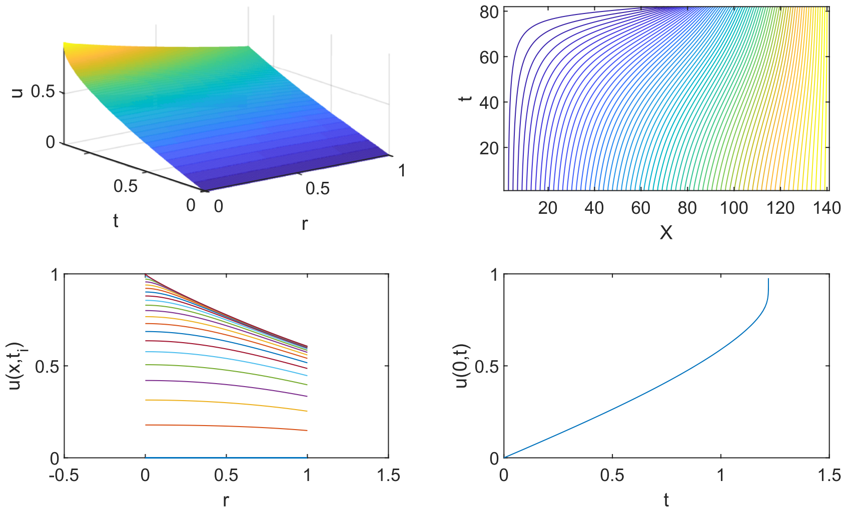

FIGURE 14 Form of the solution and various profiles of the nonlocal problem for the radial symmetric case for $\lambda = 0.2$ and $\alpha = 1$, $\beta = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

5 | DISCUSSION

In the current work we investigate a nonlocal parabolic problem with Robin boundary conditions associated with the operation of some idealized MEMS device. In the first part we deliver a thorough investigation of the associated steady-state problem, and we derive some estimates of the *pull-in voltage*, which is the controlling parameter of the model. In particular, and for the $N-$dimensional case, $N > 1$, in order to derive sharp estimates for the *pull-in voltage* we had to show, as a very interesting by-product, a Pohožaev's type identity for Robin boundary conditions. To the best of our knowledge such a result has not been available in the literature.

In the second part of this work, existence and uniqueness results together with long time behavior of time-dependent problem are discussed. In particular, we focus on the investigation of the phenomenon of quenching (i.e., the so called *touching down* in the context of MEMS literature). We first examine the quenching behavior on a general domain, while later in order to derive an optimal quenching result we restrict ourselves to the radially symmetric case.

Finally, we close our investigation by the implementation of an adaptive numerical method,[47] for the solution of the time-dependent problem. We actually perform a series of numerical experiments verifying the obtained analytical results as well as revealing qualitative features of nonlocal problem (1.1) do not arise from our analytical approach. Additionally, some further numerical experiments are performed to determine the quenching profile of the solution in the radially symmetric case.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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