An Alternative Way to Price Exotic Options

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By

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Abstract

One of the most important aspects of financial options is how they are priced. Although there are a variety of methods for pricing basic financial options, two of the most utilized are the Binomial Option Pricing method and the Black-Scholes Formula. The Binomial Option Pricing method requires the assumption that asset prices only increase or decrease by a certain amount in a time-period. This method also requires the creation of binomial trees to track the asset and option prices. In contrast, the Black-Scholes Formula is a general formula and does not hold the assumption that stocks only go up or down by a certain amount. When looking at more complex exotic options, they are almost always priced via the Black-Scholes Formula. This is partly because the Binomial Option Pricing method is too calculation-heavy; however, through programming languages, such as R, some computations of the Binomial Option Pricing method become more feasible. Due to this, comparisons between these two pricing methods can be made for exotic options.

Contents

1 Introduction

When talking about financial options, one of the most important aspects of them is how they are priced. There are multiple ways to price standard financial options, but the two most popular pricing methods are known as the binomial option pricing model and the Black-Scholes Formula. The binomial option pricing model assumes that a stock price moves either up by a certain amount or down by a certain amount. Using this assumption, the option can be priced through methods we will talk about throughout this paper. The Black-Scholes Formula is more of a standard formula where the main assumption is that the asset price follows a log-normal distribution. This formula will be discussed throughout the paper as well and will be explained much more thoroughly. Regardless, both methods are legitimate ways to price standard financial options.

There are options known as exotic options. Exotic options (also called nonstandard options) are more complex financial options than standard options. They are created by financial engineers and are priced in more unconventional ways than standard options. More often than not, a variation of the Black-Scholes Formula is used to price these options rather than the binomial option pricing model.

Yet, why couldn't we be able to price these select exotic options through an alternative method rather than the established one? In particular, there must be a way to price these exotic options using the binomial option pricing model instead. We will discuss throughout this paper the background of financial options, how to approach pricing options, and other topics that will prepare us to take on our objective. The binomial option pricing model and the Black-Scholes Formula will be discussed heavily and whole chapters with background information will be devoted to these pricing methods.

We will look at multiple examples of exotic options in this paper and decide whether we can or cannot use the binomial pricing model as a replacement method. There are reasons why the binomial pricing method will be more beneficial to use rather than the traditional way the exotic option of choice is priced. These reasons will become apparent throughout the paper and will be discussed later on in the paper. Also, the pricing of any standard or exotic options we go over in this paper will be done using the software program R. While some of these problems could be done by hand, it is much easier and efficient to have code written up so variables can just be inputted, and then the price is found after the code is run.

2 Option Theory Basics

2.1 Basic Options

In order to talk about exotic options and pricing them, it is essential to go over the fundamentals of basic financial options and some common terms to all options.

First, what is exactly an financial option? An option is a kind or example of a derivative. A **derivative** is a financial instrument where the price of some other object determines the value of this instrument (McDonald, 2). For example, a commodity such as corn is not a derivative; however, a derivative could be based around the fluctuating price of corn (McDonald, 2). Let's say we believe the price of corn will move up, then we will enter a derivative where we benefit if the price goes up. For the options in this paper, we will be concerned with stock or asset prices.

Let's talk about some common terms in options. A recurring term that is seen throughout pricing options is the **spot price**, the market price for delivery of an asset immediately (McDonald, 27). This "asset" is also referred to as the **underlying asset** on which the contract is based (McDonald, 25). Every contract also has an expiration date, the date in which the contract is settled. For example, a 1-year contract lasts 1 year from the day the contract is entered into.

Perhaps the simplest derivative contract that could be looked at is a **forward contract**. In this contract, its own terms are settled on in present day; yet, the actual buying or selling of this asset does not happen until later on in the future (McDonald, 25). These terms include the price that will be paid at time of delivery, obligating the buyer to buy and seller to sell, etc. (McDonald, 25). The buyer will receive the monetary amount of the spot price when the contract expires (for example, if the spot price at time of exercise is 40, I will get 40 dollars); however, the buyer will also have to lose money (give out a payment). This monetary amount is known as the **forward**

price, which is agreed upon by both parties in the present day when the terms are settled.

So, let's say we enter into a forward contract as the buyer. Whenever looking in the buyer's perspective of a option, it is generally referred to as a **long** position (McDonald, 29). In contrast, the seller is shown to be in the **short** position (McDonald, 29). So, what exactly is the result, or the incentive, of being in a long position? To answer this question for any type of option, the payoff and the profit of an option should be looked at.

The **payoff** of a contract or option is the value of the position when contract or option expires (McDonald, 29). Recall that if we're in the long position, we will receive the spot price at expiration (S), but need to pay the forward price agreed upon (F). Therefore, the long position's payoff will be:

$$\text{Payoff of Long Forward} = \text{S} - \text{F} \tag{1}$$

If the buyer has a payoff, surely the seller also does. This makes sense since the seller makes a payment to the buyer (the spot price at expiration) and receives a payment from the buyer (the forward price). Therefore, the short position's payoff will be:

$$\text{Payoff of Short Forward } = \text{F} - \text{S} \tag{2}$$

Notice, this is just the opposite of the long forward's payoff. This makes logical sense, as whatever the buyer gains the seller loses and vice versa.

A word of caution is to not confuse payoff with profit. Payoff is **not** often the same thing as profit. Usually, an option requires an initial payment when entering into it, known as a **premium** (McDonald, 25). This premium can be counted into the profit but not the payoff. We will talk more about premiums in the introduction of call and put options; however, forward contracts do not have premiums. The only two transactions that take place during a forward contract are the paying (or receiving) of the spot price and the paying (or receiving) of the forward price. Therefore in this special case, the profit and payoff of forward contracts in either position is the same.

One of the most common types of options is the **call option**. A call option is where the buyer of the option has the right, but not the obligation, to buy an asset (McDonald, 35). Whenever the option will result in the buyer having a negative payoff, the buyer is not going to exercise the option. So how is it known when the payoff will be positive?

When a call option is exercised, the amount the buyer pays is known as the **strike** or **exercise price** (this is usually denoted by the letter K) (McDonald, 35). The other important price to remember is the spot price or the market price. In a call option, the spot price (S) is earned by the buyer when the option is exercised. In summary, when the buyer exercises an option, he receives the monetary amount of the spot price yet has to pay the monetary amount of the strike price. It is important to remember that the buyer does not need to exercise the option and will choose not to if there is a negative payoff. Therefore, it make sense to assume the option will not be exercised when the spot price is less than the strike price. As such, the payoff of a call option can be written as:

$$\text{Purchased Call Payoff} = \max[0, S - K] \tag{3}$$

To clarify on this formula, $\max[A, B]$ indicates the maximum (the largest) of the numbers A and B is used. The payoff of a purchased call is either 0 or $S - K$, if this difference is positive or 0.

Furthermore, notice that the formula states payoff is of a "purchased" payoff. Since one side buys the call, another side sells the call. The seller of the call, logically, has the opposite transaction of the buyer. Whatever the buyer gains, the seller loses.

$$\text{Written Call Payoff} = -\max[0, S - K] \tag{4}$$

Not to be misleading, it should be clarified that the payoff of a call is not how much the buyer makes when purchasing it. There is one last step to make the payoff formula into the profit formula. Whenever someone tries to acquire a call option, there is a cost for the buyer. This initial cost is known as the premium of the option, and it is paid when the option is purchased (McDonald, 36). Therefore, the purchased call profit is:

$$\text{Purchased Call Profit} = \max[0, S - K] \ - \text{F.V. of option premium} \tag{5}$$

Just like payoffs, the seller of a call option has a profit as well. *F.V.* stands for the future value of the premium. The premium is paid at whatever time the option is entered into, which is typically time 0. The payoff and profits are received or given when the contract or option expires at some time t. Therefore, the future value of the premium must be used. Since the buyer pays the future value of the option premium, the seller receives that amount.

$$\text{Written Call Profit } = \text{F.V. of option premium } - \max[0, \text{S} - \text{K}] \tag{6}$$

Another common type of an option is a **put option**. Where a call option gives the buyer the right to buy, a put option gives the seller the right to sell (McDonald, 41). The strike price is the price that is sold, while the spot price is the price that is bought. Since the owner of a put option has the right to sell, he/she would receive the strike price if the option is exercised. But, the seller needs to pay at exercise, and the payment will equal the spot price. This is a reversal of roles of sorts compared to calls. As such, we can follow similar logic of call payoffs and state:

$$\text{Purchased Put Payoff} = \max[0, \text{K} - \text{S}] \tag{7}$$

Similar to call options, puts also have written payoffs.

$$\text{Written Put Payoff} = -\max[0, \text{K} - \text{S}] \tag{8}$$

Put options also have profits just like call options, and they follow the same form as call options (just with the difference of the spot price and the exercise price).

$$\text{Purchased Put Profit} = \max[0, \text{K} - \text{S}] - \text{F.V. of option premium} \tag{9}$$

$$\text{Written Put Profit } = \text{F.V. of option premium } - \max[0, \text{K} - \text{S}] \tag{10}$$

2.2 European and American Exercise Styles

For certain financial options, such as calls and puts, the exercise style must be specified. An exercise style determines when the buyer is allowed to exercise the option (McDonald, 36). The main two exercise styles discussed in this paper are **American** and **European**. In a European-style option, exercise may only occur when the option expires (McDonald, 36). For example, if I buy a one-year European call option, I need to wait one year from the day I entered the option to exercise that option.

In an American-style option, exercise can occur at any point in the life of the option (McDonald, 36). If this time I instead enter in a one-year American call option, I do not have to wait the whole year to exercise the option. I can exercise the option six months later, a week later, etc.

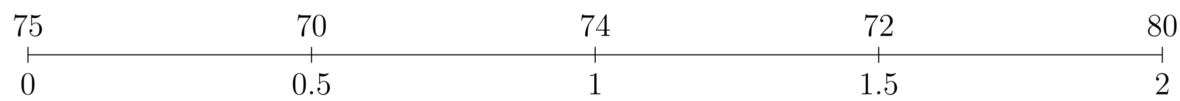
I will demonstrate these exercise styles through an example. Let's say Stock Q follows the following time-line:

75	70	78	84	80
0	0.5	1	1.5	2

If I entered a two-year European call option with the underlying asset Stock Q and a Strike price (K) of 75, my payoff is going to be $S(2)-\text{K}$ = 80 - 75 = 5. Since this is an European option, the spot price is going to be the asset price at expiry. Remember, expiration is the only time European options can be exercised.

Let's now see what would happen if I instead entered into an American call option. Now, I can exercise this option at any point in the whole term. At time 1.5, the stock price is 84, which is higher than the stock price at time 2. If I exercise the option at time 1.5, I get the payoff of $S(1.5)-\text{K}$ = 84 - 75 = 9.

In this example, I can get a higher payoff with an American call than an European call. This isn't always the case. Imagine if the time-line of Stock Q looks like this:



Here, the 2-year European Call Option is the same as before: $S(2)-\text{K}$ = 80 - 75 = 5. The American Call Option will have a different payoff now. Of the time-steps, the one with the largest payoff will be at time 2. Therefore, the American option in this case has the same payoff of the European option. It is always true that an American option will have **at least** the same payoff of an European option. As a result, the prices of American options will be at least the same price of an European option.

3 The Log-normal Distribution

3.1 Introduction

Before getting too far into option theory, we must discuss a significant distribution that will be vital in option pricing: the Log-normal distribution. In most methods of option pricing, one assumes the asset price follows a log-normal distribution. It would be false to say that asset prices are exactly log-normal; however, this assumption is the foundation of popularly-used pricing models (McDonald, 545).

3.2 What is the Log-normal Distribution?

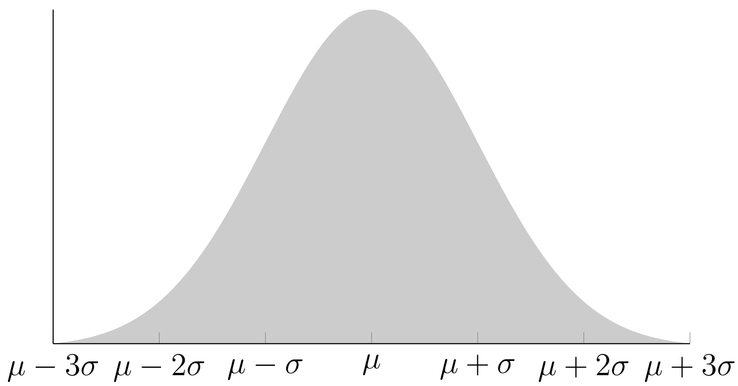
3.2.1 Normal Distribution

Before we can talk about the log-normal distribution, let's introduce what that distribution is built from: the normal distribution.

The normal distribution is a continuous probability distribution that is completely described by two parameters: mean, denoted by μ, and standard deviation, denoted by σ (McDonald, 545). A random variable, let's call it X, is said to be normally distributed if the probability X is in a certain interval can be calculated using the normal density function (McDonald, 545). This function is as follows (Ross, 198):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{given } -\infty < \text{x} < \infty \tag{11}$$

The normal density function is distinguishable with a recognizable bell-shaped curved (Ross, 198).



Additionally, if a random variable x is normally distributed with mean μ and variance σ^2, it is denoted as such (McDonald, 546):

$$x \sim \mathcal{N}(\mu, \sigma^2)$$

It is also commonplace to use the variable z to represent a random variable with the standard normal distribution. The standard normal distribution has a μ of 0 and a σ of 1 (McDonald, 546). (The importance of the standard normal distribution will be discussed shortly).

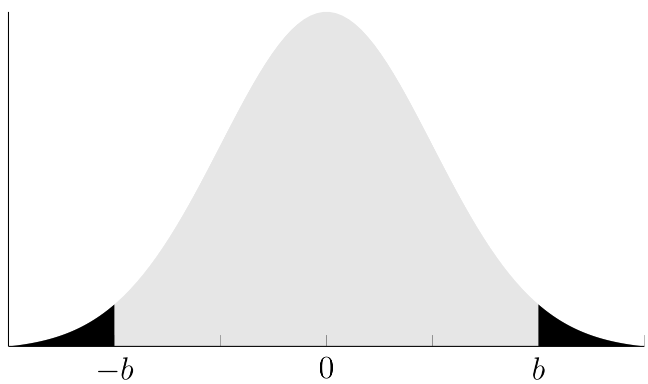
$$z \sim \mathcal{N}(0, 1)$$

Furthermore, this function can be used to find probabilities concerning a normal distribution. In general, probabilities can be found by computing the area under the density function's curve in the specified range (McDonald, 546-547). Let's say I want to find the probability that a standard normal random variable x, with mean 0 and standard deviation 1, is less than a particular value b. This can be found by taking the integral of the standard normal density function from $-\infty$ to

b (McDonald, 547).

$$N(b) = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{12}$$

Just to clarify on notation, $N(b)$ is the probability that a standard normal random variable is less than b. Knowing how to find this probability will be important throughout this paper. There is an important property drawn from this function. Let's say I now want to find the probability that a standard normal random variable x is less than $-b$. I can find this the same way I found the probability that x is less than b but instead replace it with $-b$; however, there is a special relationship to know. Observe the following standard normal curve.



As you can observe from this curve, the area below $-b$ is equal to the area above b. The area I am concerned with is $N(-b)$ (McDonald, 547-548). Recall that the area of the entire density curve is 1. Additionally, we learned earlier that the probability that a standard normal random variable is less than b is $N(b)$. Since the two events are complements of each other, we can say the probability of the standard normal random variable being greater than b is $1 - N(b)$ (McDonald, 548). Finally, since the areas of this event is equal to the probability of a standard normal random variable being less than $-b$, $N(-b) = 1 - N(b)$ (McDonald, 548).

One of the important techniques we will discuss in this paper is converting a Normal Random Variable into a Standard Normal Random Variable. The significance of this technique will be fully realized in later sections.

Let's assume we have a variable x which is normally distributed with mean μ and standard deviation σ ($\mathcal{N}(\mu, \sigma^2)$). From this information, I am able to create a standard normal random variable. By subtracting μ from x and then dividing this result by σ, we create this standard normal random variable, z (McDonald, 548). As such, the formula for z is: (McDonald, 548):

$$z = \frac{x - \mu}{\sigma} \tag{13}$$

Now, let's look back at the probability of the random variable x being less than b. We can now derive this formula (Ross, 201).

$$\begin{aligned} Pr(x < b) &= Pr(\frac{x - \mu}{\sigma} < \frac{b - \mu}{\sigma}) \\ &= Pr(z < \frac{b - \mu}{\sigma}) \\ &= N(\frac{b - \mu}{\sigma}) \end{aligned}$$

$Pr()$ stands for the probability of the event in the parentheses. This result and some other conclusions from it will be important in formulas in future chapters.

The importance of the normal distribution in the context of assets comes from the Central Limit Theorem. What this theorem entails is that if outcomes of events can be described by sums of independent random variables (which have finite variability), these sums will approach a normal distribution as the number of summands increases (McDonald, 550). When thinking about the continuously compounded return in a year, one may stumble upon the conclusion that this return is merely the sum of the daily continuously compounded returns within the year (McDonald, 550). As a result, the Central Limit Theorem can feasibly be used on these returns (McDonald, 550). As it turns out, this is a decent starting point for delving more into asset returns.

3.2.2 An Introduction to The Log-normal Distribution

Recall from the previous section that X stands for the continuously compounded asset return which follows a normal distribution. For our pruposes, we are more concerned with the prices rather than the return.

Since the returns are continuously compounded, we can find the actual prices involved, usually denoted with Y, of the asset by e^X (McDonald, 550).

$$Y = e^X, \text{where } \mathrm{e}^\mathrm{X} \geq 0$$

This can also be written as $\ln(Y) = X$. Using this information, we will be able to derive the Log-Normal density function.

Let's look at the cumulative probability density function for Y, denoted by $F_y(y)$. This is the equivalent of finding the probability of said Y being less than or equal to a value y.

$$F_y(y) = Pr(Y \leq y) = Pr(e^X \leq y)$$

Remember that X follows a Normal Distribution; therefore, we want to have X by itself in order further manipulate this expression.

$$Pr(e^X \leq y) = Pr(X \leq \ln(y))$$

Recall that the Normal Random Variable X has a mean of μ and standard deviation of σ. Now, we convert X to a standard normal random variable as follows.

$$Pr(X \leq \ln(y)) = Pr\left(\frac{X-\mu}{\sigma} \leq \frac{\ln(y)-\mu}{\sigma}\right) = N\left(\frac{\ln(y)-\mu}{\sigma}\right)$$

This result is the cumulative standard normal function. Recall that we can find this probability via an integral with bounds of $-\infty$ and $\dfrac{\ln(y)-\mu}{\sigma}$. This looks as such:

$$N(\frac{\ln(y)-\mu}{\sigma}) = \int_{-\infty}^{\frac{\ln(y)-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}}\, dx$$

The density function for y is the derivative of the cumulative density function. As a result, we will get:

$$\frac{d}{dy}\int_{-\infty}^{\frac{\ln(y)-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(y)-\mu}{\sigma}\right]^2} * \frac{d}{dy}(\frac{\ln(y)-\mu}{\sigma})$$

$$= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(y)-\mu}{\sigma}\right]^2}$$

So finally, this is the Log-Normal density function if $\ln(y) \sim \mathcal{N}(\mu, \sigma)$ (McDonald, 551):

$$g(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(y)-\mu}{\sigma}\right]^2} \tag{14}$$

Lastly, we can find the mean of a random variable that is log-normally distributed, in this case particularly y or e^x. So the mean, or the expected value $E(e^x)$, can be found via this way (Note: we will denote $E(e^x)$ instead as $E(e^y)$ but they are the same thing) (McDonald, 571):

$$E(e^y) = \int_{-\infty}^{\infty} e^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} dx$$

$$E(e^y) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x-\mu)^2 - 2\sigma^2 x]} dx$$

Let's now turn our attention to the exponent term (McDonald, 572):

$$(x-\mu)^2 - 2\sigma^2 x.$$

We can simplify this now into:

$$x^2 - 2x\mu + \mu^2 - 2\sigma^2 x$$

$$x^2 + \mu^2 - 2x(\mu + \sigma^2)$$

$$x^2 + \mu^2 - 2x(\mu + \sigma^2) + (\mu + \sigma^2)^2 - (\mu + \sigma^2)^2$$

$$x^2 - 2x(\mu + \sigma^2) + (\mu + \sigma^2)^2 + \mu^2 - (\mu + \sigma^2)^2$$

$$(x - (\mu + \sigma^2))^2 + \mu^2 - \mu^2 - 2\mu\sigma^2 - \sigma^4$$

$$(x - (\mu + \sigma^2))^2 - \sigma^4 - 2\mu\sigma^2$$

Let's now use this as our new exponent for e.

$$E(e^y) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x-(\mu+\sigma^2))^2 - \sigma^4 - 2\mu\sigma^2]} dx$$

$$E(e^y) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2))^2} e^{-\frac{1}{2\sigma^2}(-\sigma^4 - 2\mu\sigma^2)} dx$$

$$E(e^y) = e^{\mu + \frac{-\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2))^2} dx$$

$$E(e^y) = e^{\mu + \frac{-\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-(\mu+\sigma^2)}{\sigma}\right)^2} dx$$

The integral is now measuring the area under the entire normal density curve with mean $(\mu + \sigma^2)^2$ and variance σ^2. Since it is calculating for entire curve, the area is 1. Therefore, we get our $E(e^y)$ to be (McDonald, 551):

$$E(e^y) = e^{\mu + \frac{\sigma^2}{2}} \tag{15}$$

This calculation is helpful to know in this paper and for future calculations later to come.

3.3 Log-Normal Distribution and Asset Prices

3.3.1 Preliminary Formulas

Now we are introduced to the log-normal distribution, let's see how it is integrated into models for asset prices.

Let's talk about the asset return between time 0 and time t. The rate of return is $\frac{S_t}{S_0}$, where the asset prices we assume to be log-normally distributed (McDonald, 552). This rate of return can also be written by e^x, where x is a normal random variable and is the continuously compounded return for the asset (McDonald, 552). For the next few paragraphs, we will denote the continuously compounded return between two times t_0 and t_t as $R(t_0, t_t)$.

If we wanted to find the asset price at time t_1, we should look at the product of the starting asset price S_{t_0} and the exponentiated rate of return (McDonald, 553):

$$S_{t_1} = S_{t_0} e^{R(t_0,t_1)}$$

Now what if we wanted to find the asset price at time t_2? Following similar logic, we should look at the rate of return between time t_1 and t_2.

$$S_{t_2} = S_{t_1} e^{R(t_1,t_2)}$$

Since we found a formula for S_{t_1}, let's substitute it in here (McDonald, 553).

$$S_{t_2} = S_{t_0} e^{R(t_0,t_1)} e^{R(t_1,t_2)}$$

$$S_{t_2} = S_{t_0} e^{R(t_0,t_1)+R(t_1,t_2)}$$

This implies that $R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$.

We could keep going up to t_3, t_4, etc. The point made here is that the continuously compounded return from time 0 to time T can be found through the sum of returns in smaller periods. So suppose we divide the time period into n periods of time h once. We may form this general formula

(McDonald, 553):

$$R(0,T) = R(0,h) + R(h,2h) + R(2h,3h) + ... + R((n-1)h,T)$$

$$R(0,T) = \sum_{j=1}^{n} R[(j-1)h, jh]$$

Lastly, we will have α_h be the mean for a single period rate of return (R[(j-1)h, jh]), and likewise the variance will be denoted with σ_h^2. So as such, the expected value and the variance will be (McDonald, 553):

$$E[R(0,T)] = n\alpha_h \tag{16}$$

$$Var[R(0,T)] = n\sigma_h^2 \tag{17}$$

Note that this presence of the factor n implies for a fixed h that the expected value and the variance of the compounded returns are proportional to time. Being proportional to time is an important conclusion to meet, since we know the significance of time and will use that fact in our future calculations.

3.3.2 Building a log-normal model

We are trying to find a model for the asset's rate of return. This ultimately will provide us with the groundwork for a pricing model. Remember that $e^x = \frac{S_t}{S_0}$ and x is a normal random variable. If we get x by itself in the equation, we should be able to use the normal distribution. As such,

$$\ln\left(\frac{S_t}{S_0}\right) = x$$

Thus, we justly assume that $\ln\left(\frac{S_t}{S_0}\right)$ is normally distributed. Let's also assume the mean of this normal distribution is $(\alpha - \delta - 0.5\sigma^2)t$, and the variance is $\sigma^2 t$ (McDonald, 553). Notice that both mean and variance have a factor of t, which is the time period. As we discussed in the last section with rate of returns within intervals, time periods have a huge effect. As such, they should play a part in this distribution. Note that α is the expected continuously compounded rate of return, δ is the dividend rate, and σ is the asset's volatility (McDonald, 553).

$$\ln\left(\frac{S_t}{S_0}\right) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t)] \tag{18}$$

In order to find any $\ln\left(\frac{S_t}{S_0}\right)$, we can use this formula (McDonald, 554):

$$\ln\left(\frac{S_t}{S_0}\right) = (\alpha - \delta - 0.5\sigma^2)t + \sigma\sqrt{t}z^*$$

Now, the first part of this formula is the mean of this Normal Distribution. $\sigma\sqrt{t}$ is the standard deviation, and z^* is a standard normal random variable. Recall the standard normal random variable formula from the normal distribution section of this paper.

$$z = \frac{x - \mu}{\sigma} \implies x = \mu + z\sigma$$

This is exactly what we are doing to find $\ln(\frac{S_t}{S_0})$. Furthermore, we are more concerned with the future stock price, so we solve for S_t (McDonald, 554):

$$S_t = S_0 e^{(\alpha-\delta-.5\sigma^2)t+\sigma\sqrt{t}z^*}$$

In order to better understand what this equation is exactly saying, let's try to find the expected asset price at time t. First, we should manipulate the equation so the z^* part is by itself (McDonald, 554).

$$S_t = S_0 e^{(\alpha-\delta-.5\sigma^2)t} e^{\sigma\sqrt{t}z^*}$$

Let's find the expected value of the $e^{\sigma\sqrt{t}z^*}$ (McDonald, 554). Recall that we found in the last section that $E(e^x) = e^{\mu+0.5\sigma^2}$, and since z^* is a Normal Distribution with mean 0 and standard deviation 1, we can say the expectation is (McDonald, 554):

$$E(e^{\sigma\sqrt{t}z^*}) = e^{0.5\sigma^2 t}$$

Using this, we can look at the expected stock price as...

$$E(S_t) = S_0 e^{(\alpha-\delta-0.5\sigma^2)t} e^{0.5\sigma^2 t}$$

$$= S_0 e^{(\alpha-\delta)t-0.5\sigma^2 t+0.5\sigma^2 t} = S_0 e^{(\alpha-\delta)t}$$

Finally, we have that $\ln E(\frac{S_t}{S_0}) = (\alpha - \delta)t$. We get this by dividing both sides of the equation above by S_0 and then taking the natural logarithm of both sides.

$(\alpha - \delta)t$ represents the continuously compounded rate of appreciation of the asset.

Let's discuss more about the S_t formula. We are curious about the stock price moving up or down only by one standard deviation, since we are talking about the normal distribution. If we are only moving by one standard deviation, z^* will equal 1 if we move up one standard deviation or -1 if we move down one standard deviation. Thus if the asset moves up, the stock price can be written as (McDonald, 555):

$$S_u = Se^{\alpha * h + \sigma\sqrt{h}} \tag{19}$$

And similarly, the price if asset moves down, the price will be (McDonald, 555):

$$S_d = Se^{\alpha * h - \sigma\sqrt{h}} \tag{20}$$

α could be different depending on the situations. This will become vital in binomial option pricing model down the line, which talks about up and down asset moves.

3.4 Probabilities associated with Log-normal distributions

3.4.1 Probability

What are the chances that the stock price S_t is less than the strike price K, or $S_t < K$? This is an important question to ask, since not only it is good to know for payoff purposes, but also to help develop pricing models later on. Remember that

$$\ln(S_t/S_0) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t]$$

But if we want just look at $\ln(S_t)$, then...

$$\ln(S_t) \sim N[\ln S_0 + (\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t]$$

Now, if we wanted to find a standard normal random variable out of this information, we will subtract the mean from $\ln S_t$ and then divide by the standard deviation (McDonald, 556).

$$Z = \frac{\ln(S_t) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t)}{\sigma\sqrt{t}}$$

Finally, we will find $Pr(S_t < K)$. We found the standard normal random variable Z in the previous part; therefore, we can find... (McDonald, 556)

$$Pr(S_t < K) = Pr\left[Z < \frac{\ln(K) - \ln S_0 - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right]$$

And since Z is a standard normal random variable, we can find the probability with the Normal distribution (McDonald, 556).

$$Pr(S_t < K) = N\left[\frac{\ln(K) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right]$$

We write the inside part to be as $-\hat{d}_2$ (this will be important later on in the Black-Scholes Formula) (McDonald, 556).

$$Pr(S_t < K) = N(-\hat{d}_2) \tag{21}$$

We can also look at the complement (McDonald, 557):

$$Pr(S_t > K) = N(\hat{d}_2) \tag{22}$$

Both of these formulas will become essential in the Black-Scholes Pricing Section.

3.4.2 Log-normal Prediction Intervals and Expected Prices

Let's say I want to find a prediction interval for our log-normal distribution. The interval will have a lower bound S_t^L and an upper bound S_t^U where $Pr(S_t < S_t^L) = p/2$ and $Pr(S_t > S_t^U) = p/2$ (McDonald, 557). p determines the level of the prediction interval (McDonald, 557). The prediction interval can be notated as (S_t^L, S_t^U) (McDonald, 557).

If we replace K with the lower bound asset price, we can find:

$$Pr(S < S_t^L) = N(-\hat{d}_2) = p/2$$

Thus, using past equations from earlier in this chapter, we can find the lower bound asset price (McDonald, 557).

$$S_t^L = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}N^{-1}(p/2)} \tag{23}$$

Similarly, to find the upper bound, we instead use $1-(p/2)$ (McDonald, 557).

$$S_t^U = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}N^{-1}(1-(p/2))} \tag{24}$$

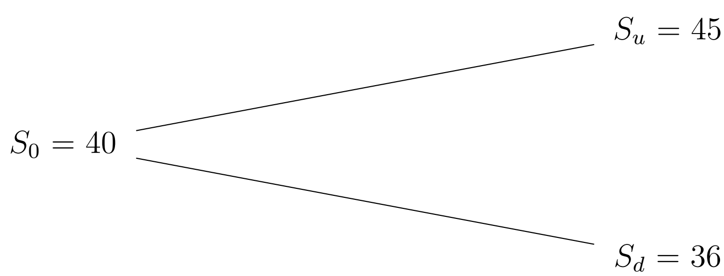
Almost all of the information in this chapter will be referred to in the Black-Scholes Pricing chapter. All of this information will help develop the Black-Scholes Formula for both puts and calls. When reading that chapter, this chapter will be beneficial to look at if any questions arise.

4 Option Pricing: Binomial

4.1 Background and Basics

One of the most important aspects of options is how to price them, especially the price of an option that is relative to the underlying asset's price. One method to price options is the **binomial option pricing model**. The main assumption of this model is the price of the asset follows a binomial distribution, meaning the asset price only goes up by a certain amount or down by a certain amount in a designated time frame (McDonald, 293). This simplifies the process considerably, since there could only be two possible asset prices.

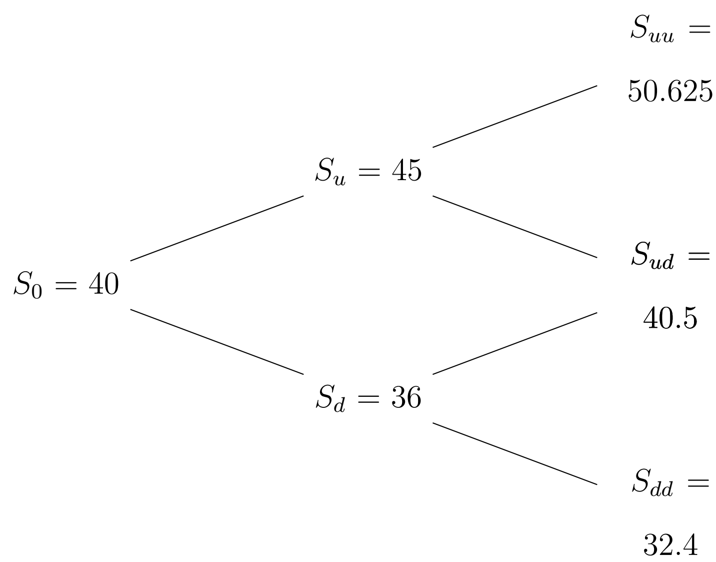
I will demonstrate this concept through an example. Suppose the stock price at time 0 is 40. At time 1, under the binomial option pricing model, there are only two possible stock prices:



So under the binomial assumption, the stock can only be 45 (if it goes up in price) and 36 (if it goes down in price). S_u is the future stock price if there is a stock price increase. S_d is the future stock price if there is a stock price decrease.

In other words, we can say that the initial stock went up by a multiplicative factor of 1.125, which is the same thing as 45 divided by 40. This multiplicative factor will be known as u for all future uses. So in general, $S_u = S_o * u$. In the same manner, the initial stock can go down by a multiplicative factor, in this case it is 0.9 or 36 divided by 40. This factor is also known as d. So, $S_d = S_o * d$.

That tree was just an example of a one period history of a stock price. This one period could be one month, a six month period, one year, etc. Using the same example, what if I wanted to know the possible stock prices at time 2? My new stock tree will look something like this:



The possible stock prices at time 1, 45 and 36, are the same as previous tree. Let's assume the stock went up from time 0 to time 1 and is now currently at the price of 45. From time 1 to time 2, the stock can still only go up by the factor u or down by the factor d. S_{uu} is the stock price at time 2 if the price goes up twice in a row. This can be calculated by taking the initial stock price and multiplying it by the up factor twice. As a formula, $S_{uu} = S_o * u * u = S_o * u^2$. $40 * 1.125 * 1.125$ is equal to 50.625.

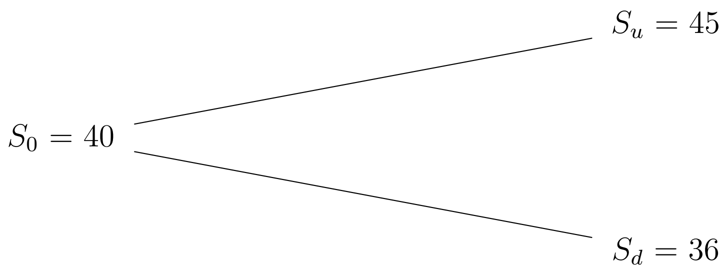
Let's still look at S_u but instead look at if the price now decreases in time 1 to time 2 (S_{ud}). This can also be written as the formula $S_{ud} = S_o * u * d$. The important point to make about this node at time 2 is that it can be reached via two routes. I can go up to S_u and then down to S_{ud} or can go down to S_d and then up to S_{du} (Note: S_{ud} and S_{du} are interchangeable and refer to the same stock price). As a result, both of these paths use the same formula and end in the same result.

Lastly, S_{dd} is the stock price when the stock price decreased from time 0 to time 1 and from time 1 to time 2. This can be written in the formula of $S_{dd} = S_o * d * d = S_o * d^2$.

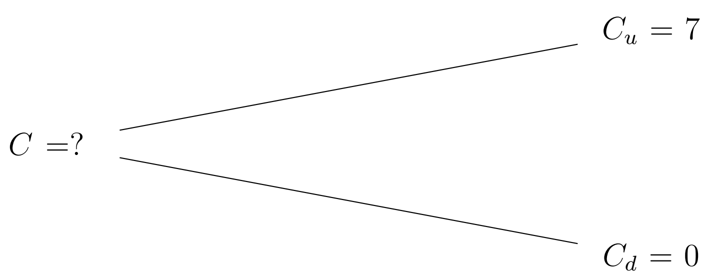
There could be more than 2 time steps; furthermore, there could be a large number of time steps. As such, there can be n many steps, n being an integer greater than 0. In example, a 5-year option may be looked at in a 5-step binomial model (1 year per step) or a 10-step binomial model (6 months per step).

4.2 Option pricing

We now know how stock prices are represented in the binomial model, but how about the price of the option itself? Let's look at the one-step one year example again:



Let's say I enter into an one-year call option on this underlying asset, but I don't know the call price (or the premium). The interest rate is known to be 5%, the strike price is 38, and the asset pays no dividends. Remember, the payoff of a call option is $\max[0, S - K]$ with S being the spot price, and K being the strike price. So let's say the stock price goes up to price 45. Then, the payoff of the call option would be $\max[0, 45 - 38] = 7$. If the stock price goes down to 36, the call option payoff would be $\max[0, 36 - 38] = 0$. This could be demonstrated in a tree diagram as well:



C_u represents the call option price at time 1 when the stock price goes up, while C_d represents the call option price when the stock price goes down at time 1. C is the value we are most interested in, which is the call option price at time 0.

I could get these payoffs via a different method. Imagine I'm in the position of the investor. Let's say I buy $\frac{7}{9}$ shares of the underlying asset and borrow \$26.6344 at the given rate. Since I'm buying $\frac{7}{9}$ shares when the price is \$40, I am paying $\frac{7}{9} * 40 = \$31.1111$. Due to receiving the borrowed amount of \$26.6344, the total cost at time 0 is $31.111 - 26.6344 = \$4.4767$. What is important about this cost is that it is the price of this particular one-year call option. This method of buying stock shares and borrowing money is known as a **synthetic call** (McDonald, 295). We are going to show that this "synthetic call" provides the same payoffs as a typical call option. This will be vital to the fundamentals of option pricing.

If the stock goes down in price, it will go down to \$36. Since I purchased $\frac{7}{9}$ shares, then their worth at time 1 is 36*$\frac{7}{9}$ which is \$28.00. However, I still have the borrowed money. The money gained interest during this one year, so the borrowed money is now 26.6344*$e^{.05}$ = \$28.00. Overall, I earn \$28 from the purchased shares of stock, and I have to repay \$28, which is the future value of the borrowed cash.. Therefore, my payoff is \$28 - \$28 = \$0. Notice, this is the same payoff, C_d I got from the tree above.

Let's check for when the stock price goes up to \$45. Since I purchased 7/9 shares, then the shares are now worth 45*$\frac{7}{9}$ = 35. Just like for the when the asset price decreases, my accumulated borrowed money is \$28.00. So I earn \$35 from the borrowed shares, and I have to repay the borrowed money of \$28. So, the payoff will be \$35 - \$28 = \$7, which is the same as the payoff C_u found in the tree.

Since this method showed the correct payoff for both possible stock prices at time 1, I should be able to find the correct call price for time 0. The stock price started at \$40 and I purchased $\frac{7}{9}$ shares; therefore, I pay $40 * \frac{7}{9} = 31.111$ for the shares. But, I borrowed the \$26.6344 at this time. Finally, the call price is $31.111 - 26.6344$, which equals \$4.4767. This is the synthetic call option that we referred to earlier.

But, how could I know that I needed exactly $\frac{7}{9}$ shares and needed to borrow \$26.6244? There are formulas for these values, of course. Δ is denoted to be the number of shares, and B will represent the amount borrowed. Remember, I found C_u from $35 - 26.6344 * e^{.05}$. Similarly, C_d can be found from $28 - 26.6344 * e^{.05}$ Therefore, I can write the general formulas:

$$\Delta * d * S * e^{\delta * h} + B * e^{r*h} = C_d \tag{25}$$

$$\Delta * u * S * e^{\delta * h} + B * e^{r * h} = C_u \tag{26}$$

Let's clarify some of the variables used in these equations. d is the factor the asset can go down by, and u is the factor the asset can go up by. δ is the continuous dividend rate. In my example, the asset did not pay dividends; however, some assets may pay dividends, and this takes the dividends into account. h is the amount of time passed from one step to the next (in my example, h was 1 for 1 year).

Δ and B are the two variables of interest here. I can use the two equations above to find formulas for both of these variables (McDonald, 296).

$$\Delta = e^{-\delta * h} \frac{C_u - C_d}{S(u - d)} \tag{27}$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u - d} \tag{28}$$

The 7/9 for Δ and 26.6344 for B can be verified through these formulas. These formulas will be of good use when trying to derive future formulas. Remember, I found the call price using $\Delta * S + B$. Therefore, I can derive the following formula (the steps to reach this formula are not shown) (McDonald, 296):

$$\Delta S + B = e^{-rh}[C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d}] \tag{29}$$

There are a couple of points to be made about this formula. $\frac{e^{(r-\delta)h} - d}{u - d}$ is paired with the call price if the asset price moves up, while $\frac{u - e^{(r-\delta)h}}{u - d}$ is paired with the call price if the asset price moves down. These two multipliers resemble each other, so let's see if there is a relationship between them. Suppose I subtract the up-move multiplier from 1.

$$1 - \frac{e^{(r-\delta)h} - d}{u - d} = \frac{u - d}{u - d} - \frac{e^{(r-\delta)h} - d}{u - d} = \frac{u - e^{(r-\delta)h}}{u - d}$$

Notice that the result is the down-move multiplier. So, the sum of these multipliers is 1. Due to this fact and that the multipliers with the call prices resemble a weighted average, these multipliers resemble probabilities. This is why the up-multiplier is known as the risk-neutral probability of an increase in the underlying asset price, denoted by p^*.

This makes logical sense since it's paired with C_u. In the same vein, the probability of of a decrease in the underlying asset price is just one minus p^*. We get this formula for the risk-neutral probability (McDonald, 299):

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \tag{30}$$

An important point that should be made is that p^* **resembles** a probability, but is not a probability in the traditional sense (like the chances of getting a heads on a coin flip, etc.). It is called a probability purely because it resembles a probability by sharing some properties.

Earlier, we made the assumption of the factors u and d being given/known. But, there are two methods to find these factors. First, if we are given future possible asset prices, the factors could be found from them. For example, let's say $S_0 = 50$, $S_u = 55$, and $S_d = 45$. The up multiplicative factor, u, is $\frac{55}{50} = 1.10$, and d is $\frac{45}{50} = 0.90$.

Another way to find u and d is using the following formulas. This is the best way to find the factors when not knowing the future asset prices beforehand (McDonald, 300).

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \tag{31}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} \tag{32}$$

r stands for the continuously compounded interest rate, while δ stands for the continuously compounded dividend rate (McDonald, 300). h stands for the amount of time (in years) each time step occurs during (McDonald, 300). Lastly, σ is the volatility of the asset (McDonald, 300). This is the the standard deviation of the asset's continuously compounded returns.

These formulas for the up-movement factors and the down-movement factors have roots from formulas in the previous chapter concerning the Log-normal distribution. Recall from Chapter 3 that

$$S_u = Se^{\alpha * h + \sigma\sqrt{h}}$$

$$\text{and } S_d = Se^{\alpha * h - \sigma\sqrt{h}}$$

The only difference for u here will be α, but instead we have $r - \delta$.

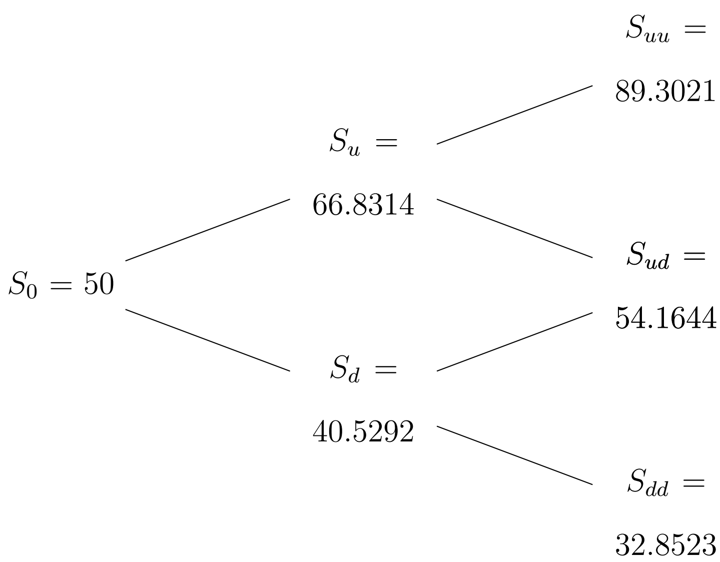
Let's verify these previous formulas and everything we examined in this section by looking into a two-year European call divided into two different time steps. The underlying asset has a current price of \$50.00, a strike price of \$45.00, an interest rate of 5%, a dividend rate of 1%, and volatility of 25%. $h = 1$ since we are looking at a two-year option into two steps, so one year each.

First, let's find the factors u and d.

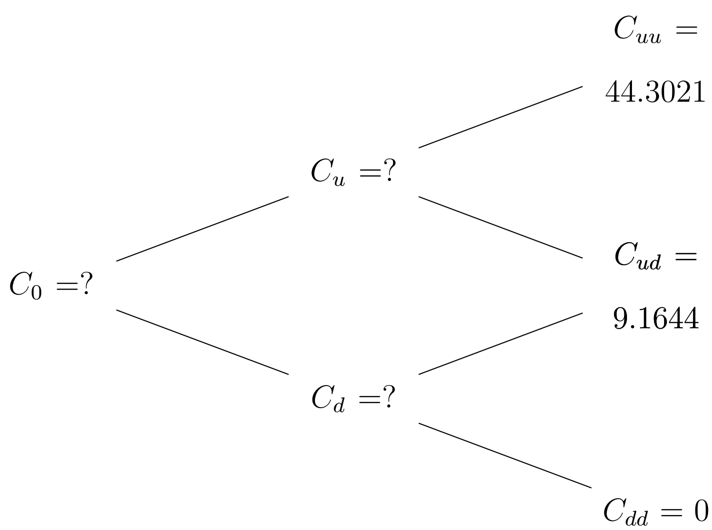
$$u = e^{(.05 - .01)*1 + 0.25*\sqrt{1}} = 1.33643$$

$$d = e^{(.05 - .01)*1 - 0.25*\sqrt{1}} = 0.81058$$

Using these factors, I can find the future stock prices.



Using a call's payoff of $\max[0, S - K]$, we can start off the call pricing tree.



Now, let's find C_u and C_d. I need to find the risk-neutral probability before I can do this, however. So using the formula from earlier for p^*:

$$p^* = \frac{e^{(.05-.01)} - 0.81058}{1.33643 - 0.81058} = 0.43783$$

Essentially, the call price when the stock moves up once (C_u) is a weighted average of the call prices C_{uu} and C_{ud}.

$$C_u = e^{-rh}[C_{uu}p^* + C_{ud}(1 - p^*)]$$

And in this example, C_u is:

$$C_u = e^{-.05}[44.3021(0.43782) + 9.1644(1 - 0.43783)] = \$23.3510616$$

Similarly, C_d can be calculated with:

$$C_d = e^{-.05}[9.1644(0.43782) + 0(1 - 0.43783)] = 3.816673$$

Now that I have the C_u and C_d, I can find the call price at time 0.

$$C = e^{-r*h}[C_u p^* + C_d(1 - p^*)]$$

$$C = e^{-.05}[23.510616(0.43782) + 3.816673(1 - 0.43782)] = 11.832414$$

So, the call price is just about \$11.83. As demonstrated, multi-step trees are just as feasible as single-step trees. Furthermore, since we are speaking strictly about European options, there is the opportunity to make a general formula. Due to the option being unable to be exercised early, the only prices that matter are the end node prices (the ones at the final time-step).

Let's look at the final three call prices. C_{uu} can only be reached by moving up twice. C_{ud} can be reached via two ways: moving up once and then down once, or moving down and then up. Finally, C_{dd} can be reached by moving down twice. Furthermore, each price has an unique probability. C_{uu} has the probability $(p^*)(p^*)$, since p^* is the risk-neutral probability of an up-move, and $(p^*)(p^*)$ is the probability of two consecutive up-moves. The other probabilities can be found using similar logic. Therefore, using this information and their associated probabilities, we can the price using:

$$C = e^{-.10}[44.302(.438)^2 + 2(9.164)(0.438)(1 - 0.438) + 0(1 - .438)^2]$$

$$C = 11.83$$

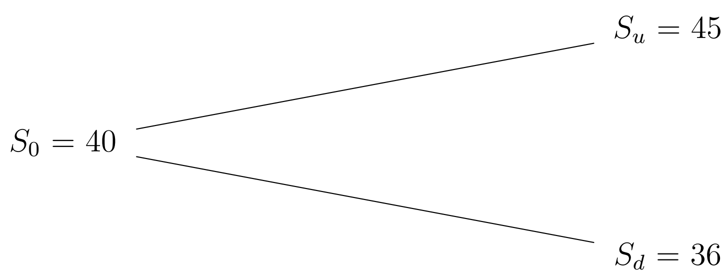
In general, the price for a n-year European call can be found using this formula.

$$C = e^{-rnh} \sum_{k=0}^{n} \binom{n}{k} C_{u^{n-k}d^k} (p^*)^{n-k} (1-p^*)^k \tag{33}$$

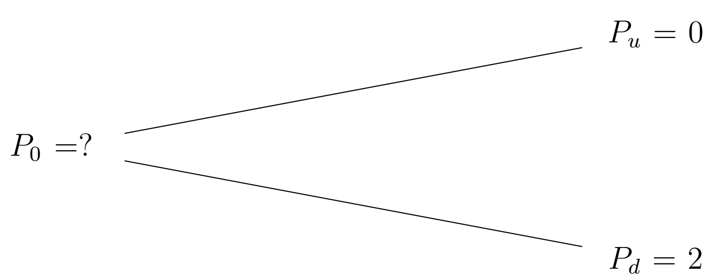
This formula will prove to be useful when looking at options with a large amount of time-steps. This method is less time-consuming than actually calculating each different time-step.

4.3 European Puts

European puts follow a similar pricing procedure to the European calls. The payoff of a Put is $\max[0, K - S]$; therefore, the only difference is going to be using this different payoff (McDonald, 309). Let's look at an earlier example to demonstrate this. Let's say I have own a 1-period European Put with a strike price of 38. Here is the asset's price tree.



Now, let's find the put prices at each of the end nodes. When the asset's price moves up to 45, the option owner will not exercise the put ($\max[0, 38 - 45] = 0$). When the price moves down, the owner will now exercise the option ($\max[0, 38 - 36] = 2$).



In a contrast to the call option, the put option in this example benefits the owner more often when the asset price moves down. However, it is quite possible to benefit from an increase in the asset price for a put.

I could find the put price by then using the same formula as the call option pricing model, but now with put prices.

$$P = e^{-r*h}[P_u(p^*) + P_d(1 - p^*)] \tag{34}$$

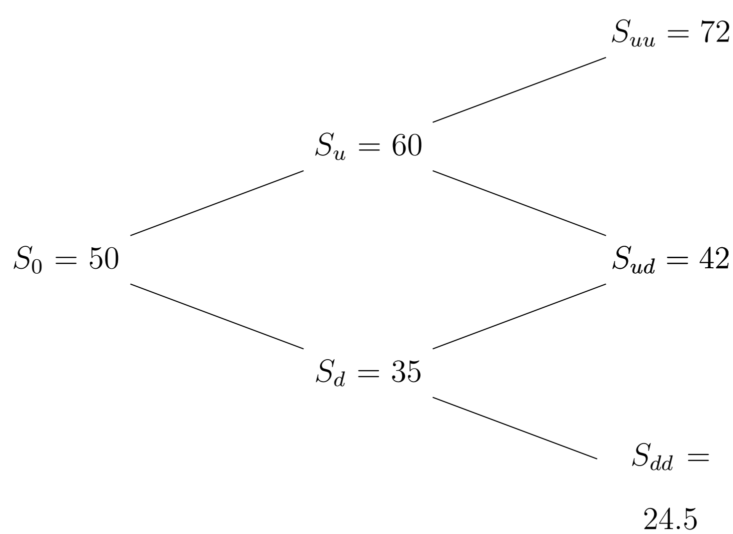
Similarly, the number of time-steps can be expanded to an infinite amount. As such, we could use the formula resembling the binomial function.

$$P = e^{-rnh} \sum_{k=0}^{n} \binom{n}{k} P_{u^{n-k}d^k} (p^*)^{n-k} (1 - p^*)^k \tag{35}$$

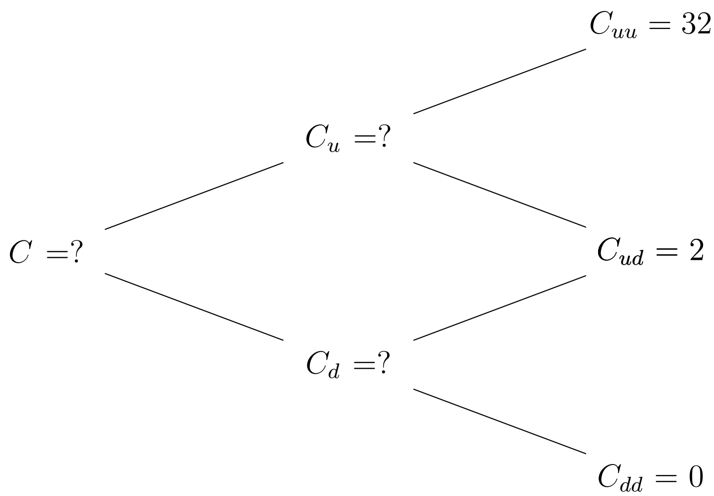
4.4 American calls and American puts

The European exercise style means the option can only be exercised when the option itself expires. Let's now look at American exercise style, which allows the buyer to exercise the option at any point in the life-time. This changes the price compared to European options. Let's start by talking about American call options.

Let's say we have a two-period two-year American call with $K = 40$. The buyer may exercise this option at any of the time-steps, and the stock follows this tree with $u = 1.20$ and $d = 0.70$:

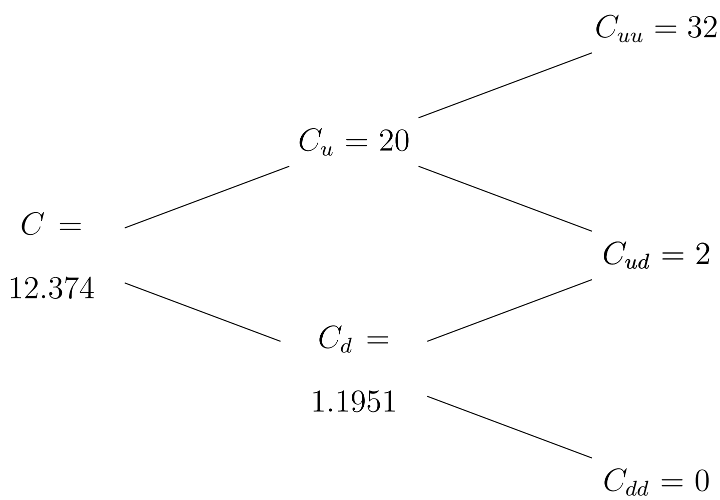


Now, the American call prices at the end nodes are treated the same way as the European end nodes. This is because the American and the European options have the same last opportunity to exercise the option: at expiration. Also, if an American option and and European option were priced using only one time period, the price will be the same. This is due to both options having only one time-step for the buyer to exercise.



So $C_{uu} = \max[0, 72 - 40] = 32$, $C_{ud} = \max[0, 42 - 40] = 2$, and $C_{dd} = \max[0, 24.5 - 40] = 0$. Now we need to move onto the call prices in the first time-steps. Let's assume $r = .05$, $\delta = 0$ and $h = 1$. This would make $p^* = 0.62818$. If this was an European option, C_u would be $e^{-.05}[32(0.62818) + 2(1 - 0.62818)] = \$19.83$; however, there is the option to exercise early. If I exercise the option when the asset moved up once after a year, the payoff will be $\max[0, 60 - 40] = 20$. Notice that if I exercise early, I will receive a higher payoff. As a result, C_u will be 20. So, as a rule for American Call options, the payoff $C_u = \max[S_u - K, e^{-rh}[C_{uu}(p^*) + C_{ud}(1 - p^*)]$. This formula accounts for the ability to exercise early compared to the European-style option. Similarly, $C_d = \max[S_d - K, e^{-rh}[C_{ud}(p^*) + C_{dd}(1 - p^*)]$. As such, $C_d = \max[35 - 40, e^{-.05}[2(0.62818) + 0(1 - .62818)] = 1.1951$. This form is true for any node in the tree besides the end nodes.

Finally, the call price at time 0 follows the same logic. $C = \max[S - K, e^{-rh}[C_u(p^*) + C_d(1 - p^*)]$ (McDonald, 311). Therefore, $C = \max[50 - 40, e^{-.05}[20(0.62818) + 1.1951(1 - .62818)]] = 12.374$.



The same process applies for American puts, just using the put payoff instead of the call payoffs. Early exercise can be applied to exotic options and is a nice introduction to more complicated binomial trees.

4.5 R Programming with Binomial Options

In order to calculate numerous different option prices efficiently, different technological methods can be used. For the purposes of this paper, R is the programming language of choice.

For the binomial pricing model, we constructed four different basic models, each one calculating the price of an unique option. These options are European Calls, European Puts, American Calls, and American Puts. The prices are calculated using the method if we would use if we were to find the prices by hand. In subsequent sections, we will use these binomial models as implemented in R to find the option prices, rather than drawing out the trees and calculating the prices by hand.

The main reason why it is helpful to have this code is the ability to find the prices quickly of options with a high amount of time-steps. For example, if we were trying to calculate the price of a 10-year option with 1-year periods, we probably wouldn't want to try this by hand. The benefit of having these models in R is especially apparent in the American-Style options. For American-style options, it is essential to know the price at each individual node, since there is always the opportunity for early exercise.

Lastly, this code will be helpful for future chapters when we try to price exotic options via the binomial pricing model. These models will serve as a template for the models of the exotic options.

The code for these binomial options can be found in the Appendix.

5 Option Pricing: Black-Scholes Formula

5.1 Background and Basics

Recall the binomial option pricing model. Using the model, there are a discrete number of time-steps, n, in the model. Theoretically, I can keep increasing the number of time-steps I can use into an arbitrarily large number (McDonald, 349). As such, it is possible for $n \to \infty$. It is not possible to have an infinite amount of discrete time-steps in the binomial pricing model (McDonald, 349). Therefore, it is reasonable to think that as $n \to \infty$, another pricing method will need to be used.

This other method of pricing options is the Black-Scholes Formula. This formula is used for multiple different types of options; however, when looking at European Options, this formula is considered a limiting case of the binomial formula (as we discussed in the previous paragraph) (McDonald, 349).

In order to develop the Black-Scholes Formula, we need to start with the assumption that stock prices have a log-normal distribution.

Remember from Chapter 3 in section 3.3.2 of this paper that

$$\ln\left(\frac{S_t}{S_0}\right) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2]$$

And since $\ln\left(\frac{S_t}{S_0}\right) = \ln(S_t) - \ln(S_0)$, we can conclude that

$$\ln(S_t) \sim N[\ln(S_0) + (\alpha - \delta - 0.5\sigma^2)t, \sigma^2]$$

Next, using the log-normal density function $g(y)$ from Section 3.2.2 and filling in the correct mean and standard deviation,

$$g(S_t) = \frac{1}{S_t \sigma \sqrt{t}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(S_t) - [\ln(S_0) + (\alpha - \delta - 0.5\sigma^2)t]}{\sigma\sqrt{t}}\right)^2}$$

This density function for S_t is conditional on S_0, since S_0 is in this function and influences S_t (McDonald, 561).

Let's take a quick aside to talk about conditional expected prices and partial expectations. Let's say we only care about the asset price when the option of interest is in the money and want to know the expected asset price(McDonald, 559). So let's say that the option is a put option; therefore, the option is in-the-money when $S_t < K$. So given that $S_t < K$, we want the expected asset price $E(S_t|S_t < K)$. This is also known as a **partial expectation**, since this is not the full expected value of the stock. This is only the expected value when (in the case of put options) $S < K$.

Now, let's find a standard normal random variable Z for the normal random variable of $\ln(S_t)$. We discussed how to do this in Section 3.4.1.

$$Z = \frac{\ln(S_t) - [\ln(S_0) + (\alpha - \delta - 0.5\sigma^2)t)]}{\sigma\sqrt{t}}$$

If we solve for S_t, we obtain

$$S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z}$$

Let's go back to our function $g(S_t)$ and substitute in our z variable where appropriate. Note that we will not be transforming S_t at all just yet; the reason why will become clear soon.

$$g(S_t) = \frac{1}{S_t \sigma \sqrt{t}\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Let's try to find the partial expectation for the put option now. Put options are in-the-money when the asset price is between 0 and K. And an expected value (partial or full) is found by multiplying each value by its associated probability. Since the probabilities are found by the probability density function, we are looking for

$$\int_0^K S_t g(S_t) dS_t$$

The only term here we do not know is dS_t. Let's find it in terms of z:

$$dS_t = \frac{dS_t}{dz} dz = \frac{d}{dz}(S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z}) dz$$

$$= \sigma\sqrt{t}(S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z}) dz = \sigma\sqrt{t} S_t dz$$

Let's now try to simplify the integral a bit:

$$\int_0^K S_t g(S_t) dS_t = \int_0^K S_t * \frac{1}{S_t\sigma\sqrt{t}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} * \sigma\sqrt{t} S_t dz$$

$$= \int_0^K S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

We now want to put this integral completely in terms of z, so we will substitute in for S_t

$$= \int_a^b S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

The bounds of a and b are chosen here since we do not know the exact bounds for z. 0 and K are the bounds for S_t, not z.

We are getting closer to simplifying this integral, but there are a few more things to do. Let's try to find out what the bounds a and b are exactly.

$$a < z < b \iff 0 < S_t < K \iff 0 < S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z} < K$$

Dividing both sides by S_0,

$$a < z < b \iff 0 < e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}z} < \frac{K}{S_0}$$

Our next step should be to take the natural logarithm of all sides in order to get rid of e. Note that $\ln(0)$ does not exist; however, as a number approaches toward $\ln(0)$, the closer that number gets to $-\infty$ (We are essentially finding the limit of the function $\ln(x)$ as $x \to 0^+$.)

Thus, we get

$$a < z < b \iff -\infty < (\alpha-\delta-0.5\sigma^2)t + \sigma\sqrt{t}z < \ln\left(\frac{K}{S_0}\right)$$

Lastly, we will just solve so z is by itself in the middle of the inequality.

$$a < z < b \iff -\infty < \sigma\sqrt{t}z < \ln\left(\frac{K}{S_0}\right) - (\alpha-\delta-0.5\sigma^2)t$$

$$a < z < b \iff -\infty < z < \frac{\ln\left(\frac{K}{S_0}\right) - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}$$

So the bounds are: a $= -\infty$ and $b = \dfrac{\ln\left(\frac{K}{S_0}\right) - (\alpha-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}$

We must deal once again with the integral. Remember, this integral is terms of z, so this indicates that the variable t is just a constant with respect to z. So we can further simplify the integral to be:

$$S_0 e^{(\alpha-\delta-0.5\sigma^2)t} \int_a^b e^{\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= S_0 e^{(\alpha-\delta-0.5\sigma^2)t} \int_a^b \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{t}z-\frac{1}{2}z^2}\, dz$$

We can simplify the expression in the exponent of e by completing the square and other methods.

$$\sigma\sqrt{t}z - \frac{1}{2}z^2 = -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z) = -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z + \sigma^2 t - \sigma^2 t)$$

$$= -\frac{1}{2}(z - \sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t$$

So now, we can change this integral to

$$S_0 e^{(\alpha-\delta-0.5\sigma^2)t} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t} dz$$

We can simplify this further by pulling out factors constant with respect to z, and we get:

$$S_0 e^{(\alpha-\delta-0.5\sigma^2)t} e^{\frac{1}{2}\sigma^2 t} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz$$

$$S_0 e^{(\alpha-\delta)t} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz$$

The last step is to substitute in for $z - \sigma\sqrt{t}$ with x.

$$S_0 e^{(\alpha-\delta)t} \int_{a-\sigma\sqrt{t}}^{b-\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

The bound $a - \sigma\sqrt{t}$ is still going to remain $-\infty$. This integral is now recognizable as the cumulative standard normal distribution function (which we discussed in Section 3.2.1) with our upper bound being $b - \sigma\sqrt{t}$. However, we found out what b is equal to. So we can simplify this upper bound.

$$b - \sigma\sqrt{t} = \frac{\ln\left(\frac{K}{S_0}\right) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} - \sigma\sqrt{t}$$

$$= \frac{\ln\left(\frac{K}{S_0}\right) - (\alpha - \delta - 0.5\sigma^2)t - \sigma^2 t}{\sigma\sqrt{t}}$$

$$= \frac{\ln\left(\frac{K}{S_0}\right) - (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$$

Therefore, we finally get our partial expectation to equal (McDonald, 561):

$$\int_0^K S_t g(S_t) dS_t = S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln(K) - [\ln(S_0) + (\alpha - \delta + \frac{1}{2}\sigma^2)t]}{\sigma\sqrt{t}}\right)$$

Remember that $N()$ stands for the cumulative normal distribution function. Having the partial expectation in this form makes things easier to compute from now on. The inside of $N()$ can be rewritten as

$$\frac{\ln(K/S_0) - (\alpha - \delta + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

This inside term is denoted as $-d_1$, similar to d_2 we established earlier in this paper (McDonald, 561). So now we have

$$\int_0^K S_t g(S_t) dS_t = S_0 e^{(\alpha-\delta)t} N(-d_1) \quad (36)$$

Lastly, we want the conditional expected value of the stock price given that it is less than the strike price ($E(S_t|S_t < K)$). In order find this, we will divide the partial expectation by the probability that $S_t < K$.

In Section 3.4.1, we found that the probability that $S < K$ is shown to be $N(-d_2)$. Therefore (McDonald, 561),

$$E(S_t|S_t < K) = S_0 e^{(\alpha-\delta)t} \frac{N(-d_1)}{N(-d_2)} \quad (37)$$

Similarly, we can find the partial expectation and conditional expected value for when call options are in the money ($S > K$). We won't go through the whole process to find the partial expectation, since it is very similar to the previous one. The partial expectation of S_t conditional on $S_t > K$ is the following (McDonald, 561).

$$\int_K^\infty S_t g(S_t) dS_t = S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln(S_0) - \ln(K) + (\alpha - \delta + 0.5\sigma^2)t)}{\sigma\sqrt{t}}\right)$$

$$\int_K^\infty S_t g(S_t) dS_t = S_0 e^{(\alpha-\delta)t} N(d_1) \tag{38}$$

Lastly, we can find the conditional expected value of the asset price if $S_t > K$ (McDonald, 561). In Section 3.4.1, we found that $P(S_t > K) = N(d_2)$. Therefore,

$$E(S_t|S_t > K) = S_0 e^{(\alpha-\delta)t} \frac{N(d_1)}{N(d_2)} \tag{39}$$

We can now use some of these past formulas in order to derive the Black-Scholes Pricing Call Formula. The price of a call option is the present value of the expected value of the possible payoffs weighted by the probability the option is in the money. This is very similar to how call and put prices are found using the binomial option pricing model.

Let's start with the call option price being equal to

$$C = e^{-rt} \int_K^\infty (S_t - K) g(S_t) dS_t$$

$$= e^{-rt} E(S - K|S > K) * Pr(S > K)$$

If we split up the expectation, we see more familiar expressions (McDonald, 561).

$$e^{-r*t} E(S|S > K) * Pr(S > K) - e^{-r*t} E(K|S > K) * Pr(S > K)$$

Substituting in the expression derived above for the conditional expectation $E(S|S > K)$ and

$N(\hat{d}_2)$ for $Pr(S > K)$ (McDonald, 561):

$$C = e^{-r*t} S_0 * e^{(r-\delta)t} \frac{N(\hat{d}_1)}{N(\hat{d}_2)} * N(\hat{d}_2) - e^{-r*t} K * N(\hat{d}_2)$$

$$C = e^{-\delta*t} S_0 N(\hat{d}_1) - e^{-r*t} K N(\hat{d}_2)$$

As a result, the Black-Scholes formula for the price of a European call option is:

$$C = S e^{-\delta*T} N(d_1) - K e^{-rT} N(d_2)$$

$$\text{where } d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma * \sqrt{T}}$$

$$\text{and } d_2 = d_1 - \sigma * \sqrt{T}$$

Many of these variables are shared from the binomial option pricing model. S is the spot price of the asset, while K is the exercise price. r is the continuously compounded interest rate, and δ is the continuously compounded dividend rate. σ is the asset's volatility, and lastly, T is the life-time of the option (i.e., a 3-year option will have a $T = 3$).

Recall from earlier sections what $N(d_1)$ and $N(d_2)$ represent. $N(x)$ is the probability that a standard normal random variable is less than x. As such, $N(d_1)$ is the probability that a standard normal random variable is less than d_1, and the same applies for d_2.

Let's look at an example to demonstrate not only pricing European calls, but also compare the prices of this method and the binomial option pricing model. Let's say I enter into an European Call Option with the variables: $S = 40,\ K = 35,\ r = .05,\ \delta = .01,\ \sigma = 0.25,$ and T = 4.

First, we should calculate d_1 and d_2 first so we could use $N(d_1)$ and $N(d_2)$ in the main formula.

$$d_1 = \frac{\ln{(40/35)} + (.05 - .01 + \frac{1}{2}0.25^2)4}{0.25 * \sqrt{4}} = 0.83706$$

$$\text{and } d_2 = 0.83706 - 0.25 * \sqrt{4} = 0.33706$$

We can finally use the call formula in order to find the price of this European Call:

$$C = 40e^{-.01*4}N(0.83706) - 35e^{-.05*4}N(0.33706) = \$12.586798$$

As we can see, the price of this European call will be \$12.58679. Recall earlier when we discussed the binomial model with the number of time-steps approaching infinity and the Black-Scholes Formula. Let's test this claim by finding the option price using the binomial model. In the last section of the chapter on binomial pricing, we discussed methods to find the binomial model price via technology such as R. We will use R to find the prices in this example. The code for binomial pricing models is available in the Appendix.

Let's say we use the same information as the previous example and instead use the binomial method, but with only one time-step, meaning $n = 1$. As a result, the price is \$13.10345 This is not quite the price we got through the Black-Scholes method; however, we need to see what happens when I increase the number of time-steps.

$$n = 10 \rightarrow \text{C} = \$12.51267$$

$$n = 100 \rightarrow \text{C} = \$12.57822$$

$$n = 200 \rightarrow \text{C} = \$12.58592$$

$$n = 400 \rightarrow \text{C} = \$12.58605$$

As you can see, as the number of time-steps increase for the binomial method, the price approaches the Black-Scholes Formula price. This is evidence toward the assumption we discussed earlier.

The past example looked at call options, but let's now look at put options. We will still use d_1 and d_2 for put options; however, we will need a new formula entirely.

Let's derive the European put Black-Scholes Formula. Now remember the put payoff is $K - S$, and this is essential to deriving this formula. So we can start the formula with:

$$P = e^{-rt}E(K - S|K > S) * P(K > S)$$

$P(K > S)$ can be found with $N(-\hat{d}_2)$.

$$P = e^{-rt}E(K - S|K > S)N(-\hat{d}_2))$$

.

We can also expand the expectation further like we did for the call formula.

$$P = e^{-rt}E(K|K > S)N(-\hat{d}_2) - e^{-rt}E(S|K > S)N(-\hat{d}_2)$$

$E(K|K > S)$ will be just K. We will now substitute $E(S|K > S)$ with $S_0e^{(r-\delta)t}\dfrac{N(-\hat{d}_1)}{N(-\hat{d}_2)}$

$$P = Ke^{-rt}N(-\hat{d}_2) - e^{-rt}S_0e^{rt-\delta*t}N(-\hat{d}_2) * \frac{N(-\hat{d}_1)}{N(-\hat{d}_2)}$$

$$P = Ke^{-rt}N(-\hat{d}_2) - S_0e^{-\delta*t}N(-\hat{d}_1)$$

Officially, the Black-Scholes formula for European put options is:

$$P = Ke^{-r*T}N(-d_2) - Se^{-\delta*T}N(-d_1)$$

$$\text{where } d_1 = \frac{\ln{(S/K)} + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma * \sqrt{T}}$$

$$\text{and } d_2 = d_1 - \sigma * \sqrt{T}$$

As you can see, the put option formula resembles the call option formula; however, there are some changes. Since this is a put option, the format for this formula is $K - S$, which is the payoff of a put. The interest rate is still paired with the strike price and the dividend rate is also still paired with the spot price. d_2 is still paired with the strike price; however, now it is $N(-d_2)$ instead of $N(d_2)$. Since $N(d_2)$ and $N(-d_2)$ are probabilities, remember that $N(-d_2) = 1 - N(d_2)$. It makes logical sense to now use $N(-d_2)$. Call options benefit the buyer when $S > K$, while put options benefit the buyer when $S < K$. These two events are complete opposites (or using a more statistical term, they are complements of each other), and the sum of their probabilities will add up to 1. Finally, recall that the probability of $S > K$ is indeed $N(d_2)$.

Let's do a quick example just to make sure everything works.

Let's say I enter into an European Put Option with the variables: $S = 40,\ K = 35,\ r = .05,\ \delta = .01,\ \sigma = 0.25,$ and T = 4.

So d_1 and d_2 will be:

$$d_1 = \frac{\ln{(40/35)} + (.05 - .01 + \frac{1}{2}0.25^2)4}{0.25 * \sqrt{4}} = 0.83706$$

$$\text{and } d_2 = 0.83706 - 0.25 * \sqrt{4} = 0.33706$$

We can finally use the call formula in order to find the price of this European Put:

$$P = 35e^{-.05*4}N(-0.33706) - 40e^{-.01*4}N(-0.83706) = \$2.8107972$$

The price of the put is about \$2.81. Now let's use the binomial option pricing model to verify this price. I will use $n = 10$, $n = 100$, $n = 200$, and $n = 400$.

$$n = 10 \rightarrow \text{P} = \$2.736672$$

$$n = 100 \rightarrow \text{P} = \$2.80222$$

$$n = 200 \rightarrow \text{P} = \$2.80992$$

$$n = 400 \rightarrow \text{P} = \$2.81005$$

Once again, as I increase the number of time-steps the put price fluctuates toward the Black-Scholes Price.

These two formulas will provide a foundation for exotic options when we talk about them in the next chapter. Some exotic options are priced using a variation of the Black-Scholes Formula.

5.2 American Options

While the Black-Scholes Formula is extremely helpful in pricing European options, there are some major drawbacks and limitations.

The Black-Scholes Formula we discussed above concerns only European calls or European puts. As a reminder, the European exercise style only allows for the option to be exercised at its expiration. The American exercise style allows for exercise at any point in the option's lifetime.

Can we use the previous formulas for American exercise style? No, we cannot. In general, American style options are harder to price compared to European options. European options only have one possible exercise price. For American options, the exercise price is changing, since we always want to exercise at the optimal time. As a result of this, it is harder to derive formulas like we have for the European options.

5.3 R Programming

Compared to the code for the binomial formula, the R code for Black-Scholes Formula is relatively straight-forward. This is due to the Black-Scholes Formula just being a standard formula. Unlike the binomial pricing model, we don't have to worry about individual time-steps. It is easy to write a program for this formula in other languages besides R; however for the sake of sameness with the binomial pricing model, we will stick with R.

The code for this formula can be found in the Appendix.

6 Exotic Options

6.1 Introduction

European or American call and put options are known as "standard" options. This is mainly due to how often they are traded and their well-defined properties. They aren't the only options in the market though. Over the years, financial engineers have constructed unique options that can be classified as "nonstandard" options (Hull, 574). More commonly, these options are known as **exotic options**.

The purposes of these options are varied. Some are constructed in order to satisfy a particular hedging need, and others are used to be able to reflect future movements in market variables of interest.

There are multiple different kinds of exotic options, and we will discuss a few of them. The goal of this paper is to delve into the traditional ways these nonstandard options are priced and explore the possibility of alternative pricing methods (Hull, 574). This chapter is central to the whole paper and builds off everything we discussed prior.

6.2 Simple All-or-Nothing Options

6.2.1 Introduction

A simple **All-or-Nothing Option** is an option which gives the buyer an agreed upon amount of money or shares of an asset if a particular event is met (McDonald, 683). This unique method of payoff is where the name of this option comes from. Either you get the money or shares (all) or you get nothing.

There are different kinds of All-or-Nothing options, and their names depend on what the buyer receives if the conditions are met. We will shortly discuss this when we get into specific examples.

6.2.2 Cash-or-Nothing Options

a. Definition and how they are priced

A **Cash-or-Nothing Option** is an option where the buyer receives an agreed-upon amount of money at the time of expiration if certain conditions are met (Hull, 581).

For example, let's say we enter into a cash-or-nothing option where we would receive \$1 if the spot price at expiration (S_T) is greater than the strike price (K). T is the time of expiration. The agreed upon amount of money is 1 dollar, and the conditions that must be met are $S_T > K$.

Notice that the conditions resemble a non-zero payoff of a call option. The payoff of a call option is $\max[0, S - K]$. As such, this kind of option is known as a **cash-or-nothing call** (McDonald, 684).

How is this option typically priced? It appears to be simple. We know how much the buyer will receive if $S_T > K$. In the example above, we said the amount was \$1; however, we can make it a more general amount known as x. Also, if $S < K$, then the payoff in that case will be 0. So we do not really care when the stock is less than the strike rather just when the option is in the money. We just need to know the probability of $S_T > K$ so we can compute an expected value for the option price.

Recall from the log-normal and Black-Scholes Formula chapters the probability of $S > K$. $P(S > K) = N(d_2)$ and $P(S < K) = N(d_2)$ (McDonald, 684). Using this information, we can find the expected value of the option at time $T - t$. Time t is the present time. It would be:

$$xN(d_2) + 0N(-d_2) = xN(d_2)$$

Finally, we just need to take the present value of this value to find the price of the option at time t. We will be using the continuously compounded rate of interest, r. So the price of the cash call is (McDonald, 684):

$$\text{Cash Call} = xe^{-r(T-t)}N(d_2)$$

$$\text{where } d_2 = \frac{\ln(S/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

If we could have a cash-or-nothing call, surely there can be a **cash-or-nothing put**. Remember that the payoff of a ordinary put is $\max[0, K - S]$. So the condition of this all-or-nothing option is that if the strike price is greater than the spot price at time T, then the buyer will receive money (Hull, 582). Otherwise, he or she will get nothing.

The probability corresponding to a positive payoff is the probability of the strike price being greater than the spot price. This can also be seen as the probability of $S < K$, which is also $N(-d_2)$. So the expected value of this cash or nothing put is:

$$0 * N(d_2) + xN(-d_2) = xN(-d_2)$$

Like the cash call, we will need to take the present value of this option. So the price of the cash-or-nothing put will be (McDonald, 685):

$$\text{Cash Put} = xe^{-r(T-t)}N(-d_2)$$

$$\text{where } d_2 = \frac{\ln(S/K) + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

b. Pricing Cash-or-Nothing Options with Binomial pricing model

These options are typically just priced through the method we discussed in the past subsection. It seems completely feasible to price these kinds of options through by using the binomial pricing model. The conditions ($S > K$ for cash calls or $S < K$ for cash puts) are relatively simple and seem to correspond well with the binomial model.

Suppose we enter into a cash-or-nothing call option with $S = 50$, $K = 46$, $r = .05$, $\delta = 0$, $\sigma = 0.15$, $t = 0$, and $T = 1$. If conditions are met, we will get paid \$5 (x). There is no early exercise ability in this option.

Let's just start with the established formula for a cash-or-nothing call. Let's find d_2 first.

$$d_2 = \frac{\ln(50/46) + (.05 - .5 * .15^2)(1-0)}{.15} = 0.814211$$

Now, we can compute the cash call price with the formula

$$\text{Cash Call} = 5 * e^{-.05} N(0.814211) = \$3.768$$

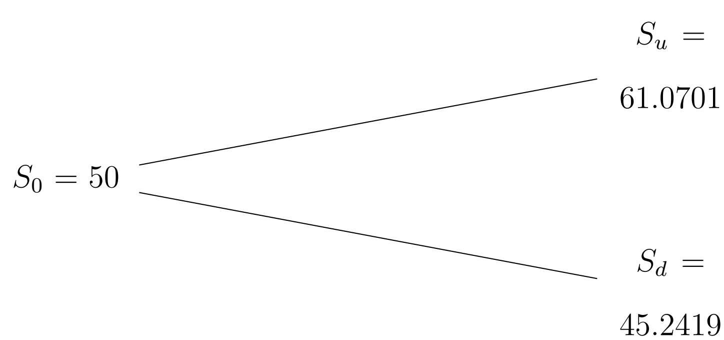
Theoretically, we are going to want our binomial pricing model price to approach the price of \$3.768 as the number of steps increase.

Let's just start with one period for now. Remember we can find the up-move and down-move multiplicative moves.

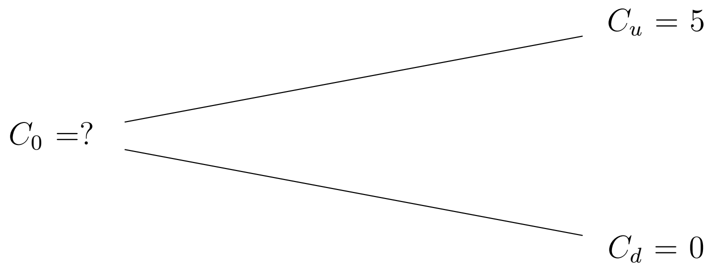
$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{.05+.15} = 1.221403$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{.05-.15} = 0.904837$$

Using these factors, I can find the future stock prices S_u and S_d.



Now when the stock price moves up, we will receive the \$5 since $S_u > K$. When the stock price moves down, we won't receive any money since $S_d < K$.



Finally, we can find the price of the option by using the risk-neutral probability and then setting it to the present value.

$$p^* = \frac{.05 - d}{u - d} = 0.4625701$$

$$C = e^{-r}[C_u * p^* + C_d * (1 - p^*)]$$

$$C = e^{-.05} 5 * 0.4625701 = \$2.20$$

This one period price is way off compared to the regular price. We shouldn't give up now though, let's increase the number of time-steps and see what occurs. The rest of these calculations will be done through code made via R. It follows the same method we just discussed, just using multiple time-steps. Here are a few different prices as I increase the number of steps.

$$n = 2 \rightarrow \text{C} = \$3.43777$$

$$n = 10 \rightarrow \text{C} = \$3.84288$$

$$n = 25 \rightarrow \text{C} = \$3.6397$$

$$n = 100 \rightarrow \text{C} = \$3.78225$$

As you can see, as we increase the number of steps, the price is going to fluctuate toward the price through the established formula. This indicates that the binomial model can be a viable way to price these options.

You may have noticed that I used relatively simple values for my first example. Let's just make the variables a little more complex.

Let's enter a two year Cash-or-Nothing call with $S = 75$, $K = 70$, $r = .06$, $\delta = .02$, $\sigma = 0.35$, $x = 13$, $T = 2$, $t = 0$.

Using the established formula, we find the price to be \$6.01106.

Now, using the binomial pricing model, we will find different option prices.

$$n = 1 \rightarrow \text{C} = \$4.36666$$

$$n = 10 \rightarrow \text{C} = \$6.0418576$$

$$n = 100 \rightarrow \text{C} = \$6.002724$$

Even with the more complex choice of variable values, the price was approached within 100 steps. This is still a working formula and hasn't been proven to be true. More examples and research would need to be done in order to see if this approach is reliable. Yet, it appears to be a good foundation and a decent approximation.

Let's talk about the Cash-or-Nothing Put now. We will only go through one example just for sake of brevity. Suppose the spot price $S = 40$, $K = 45$, $r = .05$, $\delta = 0$, $\sigma = 0.2$, $T = 2$, $t = 0$, and $x = 15$.

The established formula for Cash-or-Nothing Put options is :

$$xe^{-r*(T-t)}N(-d_2)$$

Inputting the information into this formula or the R program created (which can be found in the Appendix), we find the price to be:

$$\text{Cash Put} = 15e^{-.05*2}(.580938) = 7.88482$$

Now, let's look at the binomial pricing model method and see what prices we will obtain.

$$n = 1 \rightarrow \text{P} = \$7.73965$$

$$n = 10 \rightarrow \text{P} = \$9.18412$$

$$n = 100 \rightarrow \text{P} = \$8.0823$$

$$n = 200 \rightarrow \text{P} = \$7.92712$$

As we see, as the number of time-steps increase, it appears the price is heading toward the established formula's price.

6.2.3 Asset-or-Nothing Options

a. Definition and how they are priced

Cash-or-Nothing isn't the only simple type of All-or-Nothing option. There is also something known as the **Asset-or-Nothing Option**. Instead of giving a certain amount of money if conditions are met, this option gives the buyer a share of the asset underlying it (McDonald, 685).

Similar to the Cash-or-Nothing options, Asset-or-Nothing options can be classified as calls or puts. An **Asset-or-Nothing call option** pays the asset price (also called the spot price) if this very price is greater than the Strike Price (Hull, 582). Otherwise, the payoff is 0. An **Asset-or-Nothing put option** pays the asset price instead if this price is less than the strike price. Otherwise, the payoff is 0 (Hull, 582).

In order to find the pricing formula for asset-or-nothing options, let's look at the Black-Scholes Formula for calls (McDonald, 632).

$$C = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

Let's look at the first half of the equation, $Se^{-\delta(T-t)}N(d_1)$. Let's write out what d_1 represents.

$$Se^{-\delta(T-t)}N\left(\frac{\ln{(S/K)}+(r-\delta+0.5*\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Let's look at the limit of this equation when the time approaches T. This would cancel multiple variables and we will get (McDonald, 632):

$$\lim_{t\to\ T} Se^{-\delta(T-t)}N\left(\frac{\ln{(S/K)}+(r-\delta+0.5*\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$\lim_{t\to\ T} S*N(\frac{\ln{(S/K)}}{\sigma\sqrt{T-t}})$$

$$S*N(\infty)=S$$

If $S > K$, the numerator is going to be positive. Since it is being divided by a very small number as it approaches T, then we have $N(\infty)$ which is equal to 1 (McDonald, 632). Finally, we just have S, or the asset price at expiration. Remember, an asset-or-nothing option gives the buyer the asset price if conditions are met, which happens here.

If $S < K$, the numerator is going to be negative. Then, we will have $N(-\infty)$ which is equal to 0. So finally, the expression will equal 0.

As such, we just found the pricing formula for an **asset-or-nothing call option** (McDonald, 633). This is due to the condition of $S > K$ (McDonald, 686).

$$\text{Asset Call} = Se^{-\delta(T-t)}N(d_1)$$

Lastly, the **asset-or-nothing put option** has the condition of $S < K$ in order to receive a payoff. Let's look at $N(-d_1)$ instead of $N(d_1)$.

Then the expression we will look at it is:

$$Se^{-\delta(T-t)}N(-d_1)$$

If $S > K$, d_1 will approach ∞ so $-d_1$ will approach $-\infty$. As such, the expression will become 0. When $S < K$, d_1 will approach $-\infty$ and therefore $-d_1$ will approach ∞. As such, the expression will approach the asset price S.

Therefore, the price of the asset-or-nothing put option is (McDonald, 686):

$$Se^{\delta(T-t)}N(-d_1)$$

b. Pricing Asset-or-Nothing Options with Binomial Pricing Model

Using the binomial pricing model to price asset-or-nothing options is very similar to the way the model was used for the cash-or-nothing options. There is one key difference: the payoff. For the cash-or-nothing option, the payoff, if conditions are met, is an agreed-upon amount of money. While for the asset-or-nothing option, the payoff, if conditions are met, is the asset price at time T. So when writing the code for the binomial pricing model of asset-or-nothing options, minor tweaks were just made to change the payoff appropriately.

Besides that, the binomial pricing models of the options are relatively the same. The code can be found in the Appendix. We will just look at an example of an asset-or-nothing call option and of a put option for completeness.

Let's start with the asset-or-nothing call option. Suppose we are looking at a one-year asset-or-nothing call with $S = 30$, $K = 28$, $r = .04$, $\delta = .01$, $T = 1$, $t = 0$, and $\sigma = 0.25$.

Using the information above, the price of this asset-or-nothing call option using the established formula is \$20.7556. Now, let's look at the price through the binomial-option pricing model with different numbers of steps.

$$n = 1 \rightarrow \text{C} = \$16.6975$$

$$n = 10 \rightarrow \text{C} = \$19.9179$$

$$n = 200 \rightarrow \text{C} = \$20.3122$$

$$n = 1000 \rightarrow \text{C} = \$20.9115$$

So it appears that the option price approaches the formula price as n increases, which is the result we want.

Let's now look at the asset-or-nothing put option to wrap things up. We will use the same parameter values that we used for the call option. Using the established formula, the price of this put is \$8.9459. And with the binomial pricing formula,

$$n = 1 \rightarrow \text{P} = \$13.0040$$

$$n = 10 \rightarrow \text{P} = \$9.7836$$

$$n = 200 \rightarrow \text{P} = \$9.3893$$

$$n = 1000 \rightarrow \text{P} = \$8.79$$

The binomial model price approaches the established formula price. As with the cash-or-nothing options, it appears that the binomial pricing model is a feasible way to price these kinds of options.

6.2.4 Conclusions

There are different kinds of All-or-Nothing options that can be explored; however, we will end our discussion of these options here so we can discuss more options.

As we saw, the binomial option pricing model appears to be a feasible way to price these simple all-or-nothing options. But why would we use the binomial pricing model instead of the established formulas? Remember that one of the advantages of binomial models is the ability to observe opportunities for early exercise. These generalized formulas, however, do not have this luxury. It is much more difficult to find where early exercise is best.

Typically, these options do not give the ability to exercise early; however, remember these exotic options are made by financial engineers and, in a way, are customizable. So if early exercise were possible, these binomial models would appear to be more helpful in finding an accurate price easily.

6.3 Gap Options

6.3.1 Basic Concepts

When talking about simple call option payoffs, we decided that the payoff is $S - K$ when $S > K$ and 0 otherwise. Recall that K is the strike (or exercise) price and is used in both calculating the payoff and also the inequality that determines when there is a positive payoff.

It is possible to have two separate strike values, one for the payoff amount and the other for the inequality. For example, let's say I have an option that pays $S - 50$ only when $S > 55$. $50 is the strike price used in the payoff while $55 is the strike price used as the condition inequality. Stemming from this, if the asset price is less than $55 ($S < 55$), then the payoff is going to be 0. Yet, when the asset price is exactly $55, the payoff is going to be $55 - 50 = 5$. So in this case, the smallest non-zero payoff is going to be $5. Therefore, it is impossible in this case to obtain a payoff between 0 and 5. This creates a "gap" in the possible payoff amounts (McDonald, 422). Hence, these options with two different kinds of strike prices are called **gap options** (McDonald, 422). Specifically in this example, this is a **gap call option** since the payoff is in the form of $S - K$. There also exists **gap put options**, where the payoff resembles $K - S$.

The strike price used for payoff amount will be denoted as K_1 in these gap options. K_2 is the strike price used in the inequality. K_2 is often referred to as the **trigger price** due to the option being "triggered" once the asset price passes this point. So the payoff of a gap call option is $S - K_1$ when $S > K_2$ (McDonald, 422).

Let's now talk about the established pricing formula for a gap call option. It is going to be a modification of the Black-Scholes Call formula. K_1 is going to be the strike price in the pricing portion of the formula, while K_2 is used in the d_1 and d_2 calculations (McDonald, 422).

$$C = Se^{-\delta * T} N(d_1) - K_1 e^{-r * T} N(d_2)$$

$$\text{where } d_1 = \frac{\ln(S/K_2) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T}$$

As mentioned earlier, we also have the gap put option. This option's payoff is $K_1 - S$ where $K_2 < S$. So while the gap call option benefited when the asset price was greater than the trigger price, the gap put option benefits when the asset price is less than the trigger price. The pricing formula for this option is also relatively straight-forward. The Black-Scholes Formula for puts will be modified slightly just to account for the strike and trigger prices.

$$P = K_1 e^{-r*T} N(-d_2) - S e^{-\delta * T} N(-d_1)$$

$$\text{where } d_1 = \frac{\ln{(S/K_2)} + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T}$$

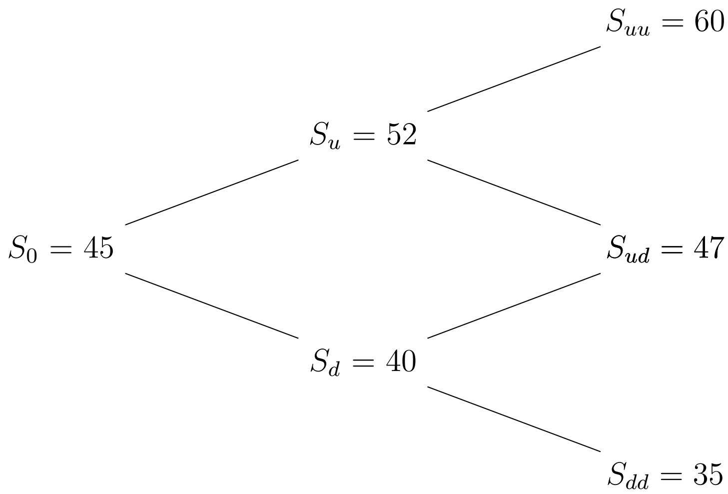
An interesting thing to note about gap options is the payoffs of these options can be negative. Let's say we enter into a gap call option with a strike price (K_1) of \$90 and a trigger price (K_2 of \$70. The option will pay $S - 90$ when $S > 70$; however, when our asset price is in the inequality of $70 < S < 90$, our payoff at expiration is going to be negative (meaning we will have to pay the seller). So if the asset price at expiration is \$80, the payoff is going to be $80 - 90 = -\$10$, and we will pay the seller ten dollars.

This fact makes gap options very unique compared to the previously discussed exotic options. Our minimum payoff was typically 0, but now that is not the case. This is useful to know when conducting calculations so negative payoffs won't be written off as mistakes.

6.3.2 Binomial Pricing Gap Options

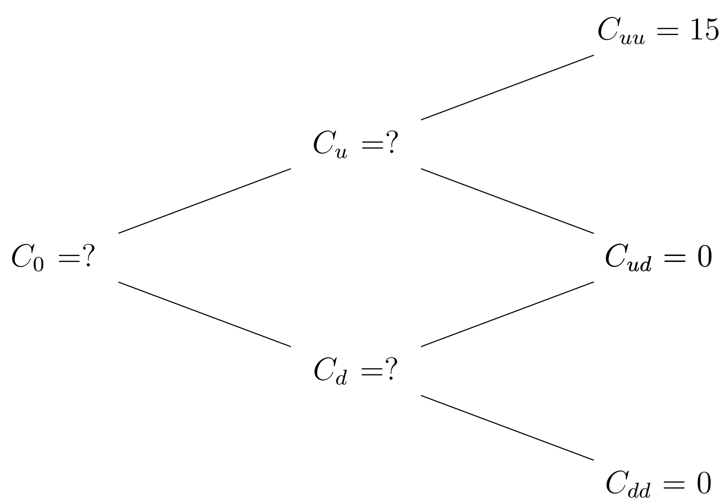
Gap Options appear to be relatively straight forward to price in the binomial pricing model format. The main difference, from let's say European call options, is the presence of this "trigger price."

Let's say we have asset with a price binomial tree as such:



Let's say I enter into a two-step gap call option with a strike price of 45 with a trigger price of 48. This means that the option will have a payoff of $S - K_1$ when $S > K_2$.

The payoffs at the end node will then be:



If this was a typical European call option, the call payoff C_{ud} would be $\max[0, 47 - 45] = 2$. With the trigger price of 48 though, the payoff for a gap call option would be 0 since 47 is less than the trigger price of 48, hence why the payoff is 0. This is the foundation and guidelines of how I made the binomial pricing model for gap options in R. This code can be found in the appendix.

Besides that change, everything else should remain the same. Let's do an example of a gap call and then a gap put just to see if this simple change is all that is needed.

Let's say I want to enter into a one-year gap call option with $S = 40$, $K_1 = 35$, $K_2 = 42$, $r = .05$, $\delta = .01$, $\sigma = 0.30$, and $T = 1$.

Using the standard gap call formula of

$$C = Se^{-\delta * T}N(d_1) - K_1 e^{-r*T}N(d_2)$$

$$\text{where } d_1 = \frac{\ln{(S/K_2)} + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T}$$

we obtain the gap call price of \$7.42555. Let's see if we get a similar result through our binomial pricing model approach. The following are prices with their corresponding number of time-steps.

$n = 1 \rightarrow$ C = \$8.58097

$n = 10 \rightarrow$ C = \$6.62558

$n = 100 \rightarrow$ C = \$7.24121

$n = 1000 \rightarrow$ C = \$7.42055

As we thought, the binomial gap call option price approached the established formula price as the number of time-steps increases. It appears that the the binomial pricing model is a feasible way to price these options.

Lastly, let's quickly do a gap put option example. We have an one-year gap put option with $S = 30$, $K_1 = 40$, $K_2 = 36$, $r = .05$, $\delta = .01$, $\sigma = 0.30$, $cap_T = 1$, and $t = 0$. The established formula gives the price of \$9.29291. With the binomial formula, we will obtain

$n = 1 \rightarrow$ P = \$9.21737

$$n = 10 \rightarrow \text{P} = \$9.129088$$

$$n = 100 \rightarrow \text{P} = \$9.33117$$

$$n = 1000 \rightarrow \text{P} = \$9.293605$$

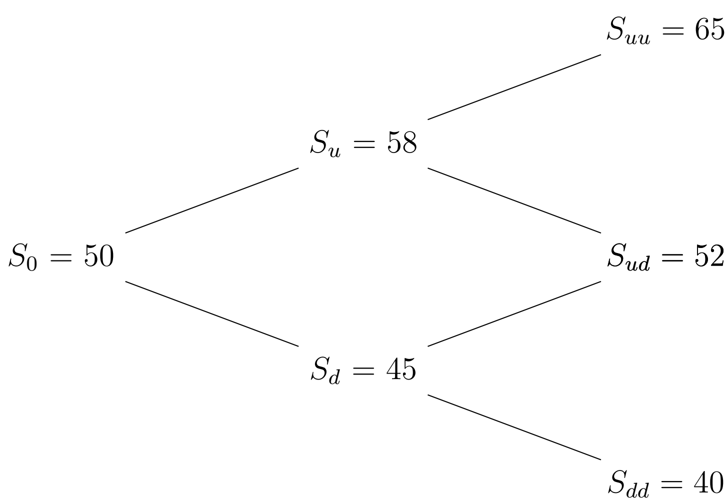
As the number of time-steps increases, the price of the gap put approaches the established formula price. So both binomial pricing formulas seem to be feasible. For more information about these pricing models, look in the appendix of this paper.

7 Further Research in Other Exotic Options

7.1 Barrier Options

7.1.1 Intro to Path-Dependent Options

So far in all of the options we discussed so far, the little details of each individual path of the asset price does not matter until the final asset price is met. Let's say the asset price follows this tree:



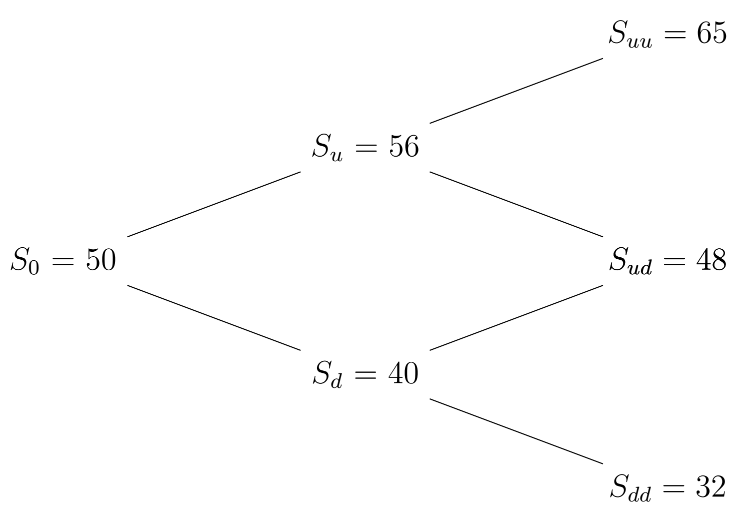
So for past options, it doesn't matter if the asset price moves up to 58 and then down to 52 or the asset price moves down to 45 and then up to 52. All that matters is the asset price is now 52.

But, what if the specific path to the end node does matter? These options are known as **path-dependent options**. The type we will focus on in this section is the barrier option.

A **barrier option** is a path-dependent option where there the payoff is determined on whether the asset price reaches a set level (Hull, 579). This specified level is known as the barrier (McDonald, 414). Once this barrier is reached for the first time, two events can occur: either the option comes into existence or the option goes out of existence (McDonald, 414). If the option comes into existence if the barrier price is met, this barrier option is known as a **knock-in option** (Hull, 579). If the option goes out of existence when the barrier is met, this is the **knock-out option** (Hull, 579).

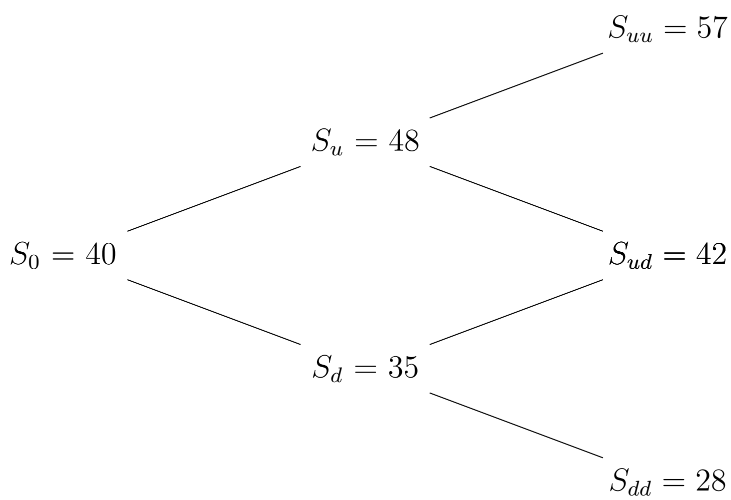
Knock-in and Knock-out options even have their own subset of options as well. A **down-and-in** option is a knock-in option where the asset price must move down in order to be put into existence (McDonald, 415). In contrast, a **up-and-in** option is considered "in" when the asset price moves up to hit the barrier (McDonald, 415). Likewise, there are knock-out options known as **down-and-out** and **up-and-out** that are in a similar vein.

When payoffs are possible in the barrier options, the payoffs are that of either a call or put option. So the main catch about this type of option is the barrier. Let's look at a binomial tree just to see how these barriers work. Let's say we have a down-and-out option with a barrier of 45. The tree is



When the stock moves up two nodes, it never hits the barrier of 45. Therefore, the payoff of the option at Node (uu) is going to be $\max[0, 65 - K]$. If the asset moves down twice in a row to price 32, the barrier has been hit. So automatically, the option is "out" and the payoff is zero no matter what. Where it is interesting is when the stock moves up once and moves down once. If the stock moves up first and then moves down second, the barrier of 45 will never be hit. So the option is considered "in," and the payoff is $\max[0, 48 - K]$. If the stock moves down first to 40 and then up to 58, the barrier of 45 is hit. Therefore, the option is considered "out" and the payoff would automatically be 0. Therefore, we can see that the payoff for node (ud) depends on how the price arrived at this point.

Let's now consider a down-and-in put option to see the subtle differences between these options. This option starts off as "out" and needs to pass the barrier in order to be considered "in." Let's say the option has a barrier of 36. The stock price follows this tree:



If the asset moves down twice in a row, the barrier of 36 is hit. Therefore, the option is "in" since it passed the barrier, and payoff at node (dd) will be $\max[0, K - 28]$. If the asset moves up twice in a row, the barrier is never hit; therefore, the payoff can only be 0. Lastly, node (ud) will have two different scenarios. If the stock moves up to 48 and then down to 42, the barrier is never hit so this option is considered out. But if the stock moves down to price 35 and then up to 42, the barrier is hit so the option is now "in." So the payoff will be $\max[0, K - 42]$. Once again, these barriers are much different from the past options we have explored. This is because of the possibility of end asset prices having two different possible payoffs depending on what path was took to get there.

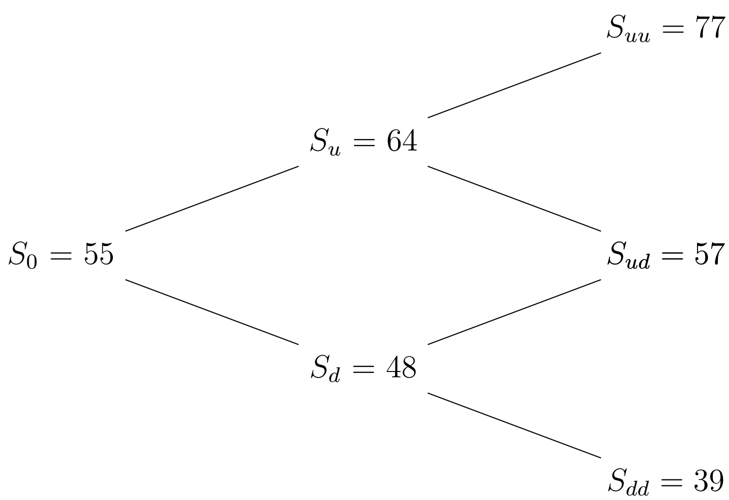
7.1.2 Problems involving Barrier Binomial Trees

Now looking at the past examples we just discussed, it seems pretty simple and straightforward to develop a binomial pricing model for barrier options. If the options are up-and-in or down-and-in, the payoff can be nonzero only if the barrier is hit. The up-and-out or down-and-out options have nonzero payoffs if the barrier is avoided. If we keep track if the paths cross the barriers, we can be able to determine if a nonzero payoff is possible at each node. And this is easily accomplished when the number of steps is 2 or 3.

How do we know how many paths are in each binomial tree? When there is only one time-step in the tree, there are only two paths: the asset price moving up once or moving down once. In the binomial tree on the previous page (with 2 time-steps), there were 4 possible paths: up and up, up and down, down and up, or down and down. Lastly, let's just quickly go over three time-steps. The possible paths are 1) up-up-up, 2) up-up-down, 3) up-down-up, 4) down-up-up 5) up-down-down, 6) down-up-down, 7) down-down-up, and 8) down-down-down; therefore, there are 8 possible paths. With these results, we can infer that the total number of paths (where $n =$ the number of time-steps) will be:

$$\text{Total number of Paths} = 2^n$$

Now that we know how to find how many different paths there could be, let's try to figure out how to find out the option payoffs at the end nodes. Let's look at a down-and-out call option with a Barrier Price (B) of 50 and a Strike Price (K) of 52. The option has the following tree.



We know there are 4 possible paths, these path are

$$\text{Path } 1 = 55 - 64 - 77$$

$$\text{Path } 2 = 55 - 64 - 57$$

$$\text{Path } 3 = 55 - 48 - 57$$

$$\text{Path } 4 = 55 - 48 - 39$$

So paths that cross the barrier of 50 when moving down will be considered "out." As such, it would be safe to say if the minimum price of the path is greater than the barrier, then this path will result in an "in" option. If the minimum price is greater than the barrier price, the barrier will never be hit; therefore, the option will be "in." Otherwise, the option is "out". So, for example,

$$\text{Path } 1 : \min[55, 64, 77] = 55 > 50 \text{ so "IN"}$$

$$\text{Path } 2 : \min[55, 64, 57] = 55 > 50 \text{ so "IN"}$$

$$\text{Path } 3 : \min[55, 48, 57] = 48 < 50 \text{ so "OUT"}$$

$$\text{Path } 4 : \min[55, 48, 39] = 39 < 50 \text{ so "OUT"}$$

This process can be applied for any amount of time-steps. Now regarding this example, we can find the payoffs for each end node. The two "IN" nodes will have a payoff similar to how regular call option payoffs are found. The "OUT" nodes will automatically have a payoff of 0.

$$\text{Path 1 Payoff} = \max\left[77 - 52, 0\right] = 25$$

$$\text{Path 2 Payoff} = \max\left[57 - 52, 0\right] = 5$$

$$\text{Path 3 Payoff} = 0$$

$$\text{Path 4 Payoff} = 0$$

Finally, we can price this as we would previously with the binomial pricing model:

$$C = e^{-rh}[25(p^*)^2 + 5(p^*(1 - p^*) + 0 + 0]$$

This whole process is was coded in R, and the result code can be found in Appendix D.

However, what happens when we try to develop binomial trees with a large number of steps (for example, 100 or 500). The total number of paths for 100 paths alone would be 2^{100}, which is an incredibly massive number. As such, the number of possible paths has now vastly increased to such a degree that doing this by hand is virtually impossible. In the same vein, the number of alternative paths that could be traveled is even too much for most software programs.

As such, looking at all of the possible paths becomes impossible as n increases. Therefore, the method we discussed earlier cannot be used for larger n values. Unfortunately, I was not able to think of another solution to pricing these options without using the alternative paths. Only looking at the end nodes that are above the barrier does limit the number of possible paths; however, this does not solve the bigger issue. Due to time constraints, I was not able to focus more time on solving this problem. Hopefully, future research could be built about the work I have done so far within R, and in this section alone.

7.2 Other Exotic Options and Further Research

There are numerous exotic options that we were not able to go in this paper. Some of these options are very difficult (like the barrier options) to price with the binomial option pricing model. This is due to certain options having path dependency, which requires each path to be taken into account. For anyone interested in learning more about exotic options, *Derivatives Markets* by McDonald and *Options, Futures, and Other Derivatives* by Hull are good textbooks to refer to and look into it.

8 Conclusions

While we still have some unanswered questions for certain exotic options, we have stumbled upon some good conclusions for other exotic options. All-or-Nothing options appear to be the options best suited for transition to the binomial option pricing model. These options are known as binary options, since these options' payoffs could only be two values (for example, Asset-or-Nothing options have payoffs of either of 0 or S_t). There is some perfecting to be done for the code in R for these options, but we are in the right option. The same can be said for Gap Options, as they are pretty straight-forward with the only difference being the presence of the trigger price.

More research needs to be done with barrier options. These options are nice to visualize using the binomial tree. Unfortunately, the computations needed to achieve barrier options with a large number of time-steps is too extreme for us to move on currently. Hopefully, someone else can continue this research on to see if there is a solution available. There are also other exotic options that we were not able to discuss in this paper, and some of these could be worth our attention later on.

The main reason why we were concerned with pricing these exotic options using the binomial pricing model is due to the benefits of this model. If an exotic option has the opportunity for early exercise, it is much easier to calculate this price through the binomial model rather than Black-Scholes Formula. This flexibility with early exercise makes exotic options even more customizable and appealing. Lastly, it is always appealing to have a visual representation of future asset prices and option payoffs via the binomial tree.

9 References

[1] Hull, John C. (2012). *Options, Futures, and Other Derivatives*, Eighth Edition, Prentice Hall.

[2] McDonald, Robert L. (2013). *Derivatives Markets*, Third Edition, Prentice Hall.

[3] Ross, Sheldon (2010). *A First Course in Probability*, Eighth Edition, Prentice Hall. pp198-208.

Appendices

A R Programming: Standard Calls and Puts

European Options

```
###European Options


###Enter your variables here:


So <-40 #Stock (Spot) Price at time 0
K <- 35 #Strike Price
r<- .05 #interest rate
delta <-0.01 #dividend yield rate
n <-400 #number of time-steps
h<- 4/400 #length of each time-step.
v <-0.25 #volatility



u <- exp((r-delta)*h +(v*sqrt(h))) #up-move factor
d <- exp((r-delta)*h -(v*sqrt(h))) #down-move factor
risk_neutral <- (exp((r-delta)*h)-d)/(u-d) #risk-neutral probability

###CALLS


Call_price <- 0 #initial position for the call option.


###This is for call prices at each end node.
```

```
This creates a matrix of these
end node call prices.

for(i in c(0:n+1)) {
  Call_price[i] <-max(0, (So*u^((n+1)-i)*d^(i-1)) - K)
  print(Call_price[i])
}

Weight_Call<- 0

###This is for the Weighted Price from each node.
This goes through each term in the binomial formula.

for(i in c(0:n+1)) {
Weight_Call[i] <-
choose(n, i-1)*Call_price[i]*risk_neutral^(n-(i-1))*
    (1-risk_neutral)^(i-1)
print(Weight_Call[i])
}

###This is for the sum to get the final call price.
The sum of the binomial formula.

Final_Price_Call<- exp(-n*r*h)*sum(Weight_Call)

###PUTS
```

```
Put_price <- 0

###This is for Put prices at each each node.
Creates a matrix for the end node put prices

for(i in c(0:n+1)) {
  Put_price[i] <-max(0, (K - So*u^((n+1)-i)*d^(i-1)))
  print(Put_price[i])
}

Weight_Put<- 0

###This is for the Weighted Price from each node.

for(i in c(0:n+1)) {
Weight_Put[i] <- choose(n, i-1)*Put_price[i]*risk_neutral^(n-(i-1))
  *(1-risk_neutral)^(i-1)
print(Weight_Put[i])
}

###This is for the sum to get the final put price.

Final_Price_Put<- exp(-n*r*h)*sum(Weight_Put)
```

American options

```
###American Options

###Enter your variables here:
```

```
So <-40 #initial stock price
K <- 50 #initial strike price
r<- .05 #interest rate
delta <-0 #dividend rate
n <-3 #number of steps
h<- 1 #amount of time per step (in years)
v <-0.25 #volatility


u <- exp((r-delta)*h +(v*sqrt(h))) #up-factor move
d <- exp((r-delta)*h -(v*sqrt(h))) #down-factor move
risk_neutral <- (exp((r-delta)*h)-d)/(u-d) #risk-neutral prob.

###CALLS

###Setting up the Stock Matrix, Future Stock Prices
Stock_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  for(k in c(0:i)){
    Stock_Matrix[k,i] <- So*u^(i-k)*d^(k-1)
  }
}

###Setting up the Call Matrix
Call_Matrix <-matrix(0L, nrow = n+1, ncol = n+1)
```

```
###Finding the call prices at the end
for(l in c(0:n+1)){
    Call_Matrix[l,(n+1)] <- max(Stock_Matrix[l,(n+1)]-K,0)
                  }

###Finding the rest of the call prices
for(a in c(1:n)){
  for(b in c(1:(n+1)-a)){
    Call_Matrix[b,(n+1)-a] <-max(Stock_Matrix[b,(n+1)-a]-K, exp(-r*h)*
    ((Call_Matrix[b, (n+2)-a]*risk_neutral)+Call_Matrix[b+1, (n+2)-a]*
    (1-risk_neutral)),0)
  }
}

Final_Call_Price <- Call_Matrix[1,1]

###PUTS

###Setting up the Put Matrix
Put_Matrix <-matrix(0L, nrow = n+1, ncol = n+1)

###Finding the put prices at the end
for(x in c(0:n+1)){
  Put_Matrix[x,(n+1)] <- max(K - Stock_Matrix[x,(n+1)],0)
}
```

```
###Finding the rest of the put prices
for(y in c(1:n)){
  for(z in c(1:((n+1)-y))){
    Put_Matrix[z,(n+1)-y] <- max(K-Stock_Matrix[z,(n+1)-y], exp(-r*h)*
    ((Put_Matrix[z, (n+2)-y]*risk_neutral)+Put_Matrix[z+1, (n+2)-y]*
    (1-risk_neutral)),0)
  }
}

Final_Put_Price <- Put_Matrix[1,1]
```

B R Programming: All-or-Nothing Options

Cash-or-Nothing Options

```
###Cash-or-Nothing Options

###CashCalls

###Enter your variables here:

S <-40#initial stock price
K <- 45#strike price
r<- .05#interest rate
delta <- 0#dividend rate
cap_T <-2#Expiration Time
t<- 0#initial time
v <-0.2#volatility
```

```
x <- 15#amount gained if conditions met
n<-200#number of time-steps
h<-2/200#length of each time-step
###Black-Scholes Formula

d2 <- (log(S/K) + (r -delta -.5*v^(2) )*(cap_T-t))/(v*sqrt(cap_T-t))

N_of_d2 <-pnorm(d2)

Cash_Call_BSF <- x*exp(-r*(cap_T-t))*N_of_d2

###Binomial Option Pricing

u <- exp((r-delta)*h +(v*sqrt(h)))
d <- exp((r-delta)*h -(v*sqrt(h)))
risk_neutral <- (exp((r-delta)*h)-d)/(u-d)

Stock_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  for(k in c(0:i)){
    Stock_Matrix[k,i] <- S*u^(i-k)*d^(k-1)
  }
}

Final_Call_Matrix <- matrix(0L, nrow = n+1, ncol = 1)
```

```
for(j in c(1:(n+1))){
Final_Call_Matrix[j,1] <- if(Stock_Matrix[j,n+1] > K){x}
                          else{0}
}

Weight_Call<- 0

###This is for the Weighted Price from each node

for(i in c(0:n+1)) {
  Weight_Call[i] <- exp(-n*r*h)
  *choose(n, i-1)*Final_Call_Matrix[i,1]
  *risk_neutral^((n)-(i-1))*(1-risk_neutral)^(i-1)
}

Final_Price_Call<- sum(Weight_Call)

###Puts

###Black-Scholes Formula

N_of_neg_d2 <-pnorm(-d2)

Cash_Put_BSF <- x*exp(-r*(cap_T-t))*N_of_neg_d2

###Binomial
```

```
Put_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
Put_Matrix[i,n+1] <-if(Stock_Matrix[i,n+1] < K){x}
else{0}
}

for(y in c(1:n)){
  for(z in c(1:((n+1)-y))){
    Put_Matrix[z,(n+1)-y] <- max(exp(-r*h)*
    ((Put_Matrix[z, (n+2)-y]*risk_neutral)+Put_Matrix[z+1, (n+2)-y]
    *(1-risk_neutral)),0)
  }
}

Final_Put_Price <- Put_Matrix[1,1]
```

Asset-or-Nothing Options

```
###Asset-or-Nothing Options

###Enter your variables here:

S <-30 #initial stock price
K <- 28 #strike price
r<- .04 #interest rate
delta <- 0.01 #dividend rate
cap_T <-1 #Expiration time
t<- 0 #starting time
```

```
v <-0.25 #volatility
n<-1000 #total number of time-steps
h<-1/1000 #length of each time-step

##Regular Black Scholes Call formulas

d1 <- (log(S/K) + (r -delta +.5*v^(2) )*(cap_T-t))/(v*sqrt(cap_T-t))

N_of_d1 <-pnorm(d1) #N(d_1)

Asset_Call_BSF <- S*exp(-delta*(cap_T-t))*N_of_d1 #Price

##Call Price using the Binomial Formula

u <- exp((r-delta)*h +(v*sqrt(h))) #up-factor
d <- exp((r-delta)*h -(v*sqrt(h))) #down-factor
risk_neutral <- (exp((r-delta)*h)-d)/(u-d) #risk-neutral prob

Stock_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)
#initializes a stock matrix

#This finds all possible stock prices at each time-step
for(i in c(0:n+1)) {
  for(k in c(0:i)){
    Stock_Matrix[k,i] <- S*u^(i-k)*d^(k-1)
  }
}
```

```
Final_Call_Matrix <- matrix(0L, nrow = n+1, ncol = 1)
#initializes call matrix

#Finds the end-node payoffs
for(j in c(1:(n+1))){
  Final_Call_Matrix[j,1] <- if(Stock_Matrix[j,n+1]
  > K){Stock_Matrix[j, n+1]}
  else{0}
}

Weight_Call<- 0

###This is for the Weighted Price from each node

for(i in c(0:n+1)) {
  Weight_Call[i] <- exp(-n*r*h)*choose(n, i-1)*Final_Call_Matrix[i,1]*
  risk_neutral^((n)-(i-1))*(1-risk_neutral)^(i-1)
}

Final_Price_Call<- sum(Weight_Call)

##Regular Put formulas

neg_d1 <- -d1

N_of_neg_d1 <- pnorm(neg_d1)
```

```
Asset_Put_BSF <- S*exp(-delta*(cap_T-t))*N_of_neg_d1

##Binomial Put Options

Put_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  Put_Matrix[i,n+1] <-if(Stock_Matrix[i,n+1] < K)
  {Stock_Matrix[i, n+1]}
  else{0}
}

for(y in c(1:n)){
  for(z in c(1:((n+1)-y))){
    Put_Matrix[z,(n+1)-y] <- max(exp(-r*h)*
    ((Put_Matrix[z, (n+2)-y]*risk_neutral)+
    Put_Matrix[z+1, (n+2)-y]*(1-risk_neutral)),0)
  }
}

Final_Put_Price <- Put_Matrix[1,1]
```

C R Programming: Gap Options

```
###Enter Variables
```

```
S <- 30 #Spot Price
K_1 <- 40 #Strike Price
K_2 <- 36 #Trigger Price
r <- .05 #interest rate
delta <- .01 #dividend rate
v <- .30 #volatility
t <- 0 #starting time
cap_T <- 1 #total time
h <- 1/1000 #length of time-step
n <- 1000 #number of time-steps

### Gap Calls and Gap Puts Standard Formula

d1 <- (log(S/K_2) + (r -delta +.5*v^(2) )*(cap_T-t))/(v*sqrt(cap_T-t))
d2 <- (log(S/K_2) + (r -delta -.5*v^(2) )*(cap_T-t))/(v*sqrt(cap_T-t))

Gap_Call <- S*exp(-delta*(cap_T - t))*pnorm(d1) - K_1*
exp(-r*(cap_T -t))*pnorm(d2)
Gap_Put <- K_1*exp(-r*(cap_T -t))*pnorm(-d2) - S*exp(-delta*
(cap_T-t))*pnorm(-d1)

### Gap Calls and Gap Puts Binomial Formula

##Stock Matrix

u <- exp((r-delta)*h +(v*sqrt(h)))
d <- exp((r-delta)*h -(v*sqrt(h)))
```

```
risk_neutral <- (exp((r-delta)*h)-d)/(u-d)

Stock_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  for(k in c(0:i)){
    Stock_Matrix[k,i] <- S*u^(i-k)*d^(k-1)
  }
}

#Call Matrix

Call_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  Call_Matrix[i,n+1] <-if(Stock_Matrix[i,n+1]
  > K_2){Stock_Matrix[i, n+1] - K_1}
  else{0}
}

for(y in c(1:n)){
  for(z in c(1:((n+1)-y))){
    Call_Matrix[z,(n+1)-y] <- max(exp(-r*h)*((Call_Matrix[z, (n+2)-y]
    *risk_neutral)+Call_Matrix[z+1, (n+2)-y]
    *(1-risk_neutral)),0)
  }
}
```

```
Final_Call_Price <- Call_Matrix[1,1]

##Put Matrix

Put_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  Put_Matrix[i,n+1] <-if(Stock_Matrix[i,n+1] < K_2)
  {K_1 - Stock_Matrix[i, n+1]}
  else{0}
}

for(y in c(1:n)){
  for(z in c(1:((n+1)-y))){
    Put_Matrix[z,(n+1)-y] <- max(exp(-r*h)
    *((Put_Matrix[z, (n+2)-y]*risk_neutral)+Put_Matrix[z+1, (n+2)-y]
    *(1-risk_neutral)),0)
  }
}

Final_Put_Price <- Put_Matrix[1,1]
```

D R Programming: Barrier Options

```
###This is a starting point for barrier options
#This is in no way complete.
```

```
###Enter your variables here:

S <-55#Stock Price
K <- 45#Strike Price
B <- 48#Barrier Price
r<- .05#interest rate
delta <- 0#dividend rate
cap_T <-3#Expiration Time
t<- 0#initial time
v <-0.2#volatility
n<-3#number of time-steps
h<-1#length of each time-step

u <- exp((r-delta)*h +(v*sqrt(h)))#up-factor
d <- exp((r-delta)*h -(v*sqrt(h)))#down-factor
risk_neutral <- (exp((r-delta)*h)-d)/(u-d)#risk-neutral

Stock_Matrix <-matrix(0L, nrow = n+1, ncol =n+1)

for(i in c(0:n+1)) {
  for(k in c(0:i)){
    Stock_Matrix[k,i] <- S*u^(i-k)*d^(k-1)
  }
}

Path_Matrix <-matrix(0L, nrow = 2^n, ncol =n+1)
#This matrix makes possible paths
```

```
Path_Matrix[,1] <-Stock_Matrix[1,1]
Path_Matrix[1:4,2] <-Stock_Matrix[1,2]
Path_Matrix[5:8,2] <- Stock_Matrix[2,2]
Path_Matrix[1:2,3] <-Stock_Matrix[1,3]
Path_Matrix[3:6,3] <-Stock_Matrix[2,3]
Path_Matrix[7:8,3] <- Stock_Matrix[3,3]
Path_Matrix[1,4] <- Stock_Matrix[1,4]
Path_Matrix[2:3, 4] <- Stock_Matrix[2,4]
Path_Matrix[5, 4] <- Stock_Matrix[2,4]
Path_Matrix[4,4] <- Stock_Matrix[3,4]
Path_Matrix[6:7, 4] <- Stock_Matrix[3,4]
Path_Matrix[8,4] <- Stock_Matrix[4,4]

Condition_Matrix <-matrix(0L, nrow = 2^n, ncol =1)

for(i in c(1:2^n)) #1 stands for IN, 0 for OUT
{ if(min(Path_Matrix[i,] > B)){Condition_Matrix[i,1] = 1}#IN
  else{ Condition_Matrix[i,1] = 0}}#OUT

Complete_Matrix<-cbind(Path_Matrix, Condition_Matrix)

Price_Matrix <-matrix(0L, nrow = 2^n, ncol = 1)

for(i in c(1:2^n))
{ if(Complete_Matrix[i, n+2] > 0.5){Price_Matrix[i,1] =
max(0, Stock_Matrix[i,n+1] - K)}
```

```
else{Price_Matrix[i,1] = 0}}
```

R