NEGATIVE-NORM LEAST-SQUARES METHODS FOR AXISYMMETRIC MAXWELL EQUATIONS

A Dissertation

by

DYLAN MATTHEW COPELAND

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2006

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Joseph E. Pasciak
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ABSTRACT

Negative-norm Least-squares Methods for Axisymmetric
Maxwell Equations. (May 2006)
Dylan Matthew Copeland, B.S., Southeastern Louisiana University;
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Chair of Advisory Committee: Dr. Joseph E. Pasciak

We develop negative-norm least-squares methods to solve the three-dimensional Maxwell equations for static and time-harmonic electromagnetic fields in the case of axial symmetry. The methods compute solutions in a two-dimensional cross section of the domain, thereby reducing the dimension of the problem from three to two. To achieve this dimension reduction, we work with weighted spaces in cylindrical coordinates. In this setting, approximation spaces consisting of low order finite element functions and bubble functions are analyzed.

In contrast to other methods for axisymmetric Maxwell equations, our least-squares methods allow for discontinuous coefficients with large jumps and non-convex, irregular polygonal domains discretized by unstructured meshes. The resulting linear systems are of modest size, are symmetric positive definite, and can be solved very efficiently. Computations demonstrate the robustness of the methods with respect to the coefficients and domain shape.

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CHAPTER I

INTRODUCTION

We are interested in solving three-dimensional Maxwell equations in the case of symmetry with respect to an axis. Because solving three-dimensional vector-valued problems is very expensive computationally, there is a need for numerical solvers which exploit axial symmetry by reducing the problem to a two-dimensional computational domain. The specific problems we shall numerically solve are the axisymmetric Maxwell equations for electrostatics, magnetostatics, and time-harmonic systems. In a simply-connected domain $\Omega \subset \mathbb{R}^3$, the Maxwell system for electrostatics is

$$\begin{cases} \nabla \times \mathbf{e} & = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot (\epsilon \mathbf{e}) & = g \text{ in } \Omega \\ \mathbf{e} \times \mathbf{n} & = \mathbf{0} \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

and for magnetostatics the system is

$$\begin{cases} \nabla \times \mathbf{h} & = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot (\mu \mathbf{h}) & = g \text{ in } \Omega \\ \mathbf{h} \cdot \mathbf{n} & = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Here $\mathbf{e}$ and $\mathbf{h}$ are the electric and magnetic fields, and the coefficients ϵ and μ are the electric permittivity and magnetic permeability, respectively. Both coefficients are assumed to be in $L^\infty(\Omega)$ and uniformly positive. The time-harmonic Maxwell system

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is complex-valued, but can be reduced to solving two real-valued systems of the form

$$\begin{cases} \nabla \times \mathbf{h} &= \omega\epsilon\mathbf{e} + \mathbf{j} \quad \text{in } \Omega \\ \nabla \times \mathbf{e} &= \omega\mu\mathbf{h} + \mathbf{m} \quad \text{in } \Omega \\ \mu\mathbf{h}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega \\ \mathbf{e}\times\mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We assume that the above problems are axisymmetric, meaning that in cylindrical coordinates (r, θ, z) the domain Ω is symmetric with respect to the z-axis and the coefficients and data are independent of the angular variable θ. Under these assumptions, the solution is also axisymmetric, so its derivatives with respect to θ are zero and it suffices to compute the solution on the two-dimensional half section $D = \{(r, z) : (r, 0, z) \in \Omega\}$. Thus a dimension reduction is achieved in the computational domain.

In particular, the electrostatic system (1.1) reduces to the following two decoupled systems:

$$\begin{cases} \nabla \times (e_r, e_z) \equiv \frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r} &= f_\theta \quad \text{in } D \\ \nabla_r \cdot (\epsilon(e_r, e_z)) \equiv \frac{1}{r}\frac{\partial}{\partial r}(r\epsilon e_r) + \frac{\partial}{\partial z}(\epsilon e_z) &= g \quad \text{in } D \\ (e_r, e_z)\cdot(-n_z, n_r) &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{1.4}$$

and

$$\begin{cases} -\frac{\partial e_\theta}{\partial z} &= f_r \quad \text{in } D \\ \frac{1}{r}\frac{\partial}{\partial r}(r e_\theta) &= f_z \quad \text{in } D \\ e_\theta &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{1.5}$$

where $\Gamma_1 = \{(r, z) \in \partial D : r > 0\}$. Similarly, the magnetostatic system (1.2) reduces to the following two decoupled systems:

$$\begin{cases} \nabla_r \times (h_r, h_z) &= f_\theta \quad \text{in } D \\ \nabla_r \cdot (\mu(h_r, h_z)) &= g \quad \text{in } D \\ (h_r, h_z) \cdot (n_r, n_z) &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\frac{\partial h_\theta}{\partial z} &= f_r \quad \text{in } D \\ \frac{1}{r}\frac{\partial}{\partial r}(r h_\theta) &= f_z \quad \text{in } D. \end{cases} \tag{1.7}$$

We define a special curl operator $\nabla_r \times$ on scalar functions ϕ by

$$\nabla_r \times \phi = \left(-\frac{\partial \phi}{\partial z}, \frac{1}{r}\left(\frac{\partial}{\partial r}(r\phi) \right) \right).$$

The time-harmonic system (1.3) reduces to the two decoupled systems

$$\begin{cases} \nabla \times (e_r, e_z) &= \omega\mu h_\theta + m_\theta \quad \text{in } D \\ \nabla_r \times h_\theta &= \omega\epsilon(e_r, e_z) + (j_r, j_z) \quad \text{in } D \\ \nabla_r \cdot \epsilon(e_r, e_z) &= -\omega^{-1}\nabla \cdot \mathbf{j} \quad \text{in } D \\ (e_r, e_z) \cdot \mathbf{t} &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{1.8}$$

$$\begin{cases} \nabla \times (h_r, h_z) &= \omega\epsilon e_\theta + j_\theta \quad \text{in } D \\ \nabla_r \times e_\theta &= \omega\mu(h_r, h_z) + (m_r, m_z) \quad \text{in } D \\ \nabla_r \cdot \mu(h_r, h_z) &= -\omega^{-1}\nabla \cdot \mathbf{m} \quad \text{in } D \\ (h_r, h_z) \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_1. \\ e_\theta &= 0 \quad \text{on } \Gamma_1 \end{cases} \tag{1.9}$$

Here $\mathbf{t} = (-n_z, n_r)$ is the unit tangent vector, oriented counterclockwise. We shall develop negative-norm least-squares methods to solve (1.4) and (1.6) for the meridian components in the electrostatic and magnetostatic equations. For the time-harmonic problem, we shall solve (1.8) and (1.9) for all components of the solution pair $(\mathbf{e}, \mathbf{h})$.

Recently, numerical methods have been developed for related problems. In [4], Börm and Hiptmair present a multigrid method for the $\mathbf{H}(\mathbf{curl};\Omega)$-elliptic variational problem arising from discretization of the time-dependent Maxwell equations. Their analysis requires the rather restrictive assumptions that the coefficients of the bilinear form fit a tensor product structure and have constant ratios. Moreover, their method requires mesh structure, as it performs r-line smoothings and semicoarsening in the z-direction.

Gopalakrishnan and Pasciak [10] demonstrated that line smoothing and semicoarsening are unnecessary for the convergence of geometric multigrid for the axisymmetric Laplace and Maxwell (θ-component only, i.e. (1.5) or (1.7)) equations, in the case of constant coefficients. However, the methods of [10] require structured meshes, and the degradation of convergence with variable coefficients is not addressed. It should also be noted that in [10], only the θ-component of the electric field is solved for (see equation (1.5)). It appears that only [4] treats the (r, z)-components.

For the time-harmonic systems (1.8) and (1.9), Pardo *et al.* [14] have developed a self-adaptive hp finite element method. Although hp methods aim to optimize the computational cost of solving the linear systems of finite element formulations, the methods must contend with the inherent difficulty that variational formulations for the reduced axisymmetric systems have coefficients behaving like r or r^{-1} near the z-axis. Moreover, hp methods are significantly more difficult and expensive to implement than least-squares methods. Therefore, we are interested in least-squares as an efficient alternative method which is robust with respect to the coefficients.

We propose negative-norm least-squares methods which require no special mesh refinement near the z-axis due to the weights r and r^{-1}. Mesh refinement is necessary only in areas where the solution is poorly behaved. Thus we allow for unstructured quasi-uniform meshes discretizing D, which may be any simply-connected polygonal domain in $\mathbb{R}^2$, possibly non-convex. Further, we allow the coefficients ϵ and μ to be piecewise constant with large jump discontinuities. Under these widely applicable conditions, our methods give first order convergence rates with modest problem sizes.

Since the reduced problems in D are derived by restricting the problems in Ω in cylindrical coordinates, the function spaces are weighted with the radial variable r [3]. This complicates the analysis, but the implementation of the resulting discrete systems is simple. For example, the solution space $L^2_1(D)$, defined in Chapter II, is approximated by the space of piecewise constant functions on a given quasi-uniform triangulation of D. The test spaces are approximated by piecewise linear functions, enriched with bubble functions on edges and elements. The discrete analogues of the least-squares problems are then defined by restricting to these discrete subspaces.

The negative-norm least-squares methods involve inner products in dual spaces, which are approximated in computations using preconditioners consisting simply of Gauss-Seidel smoothing on the bubble subspaces and multigrid on the piecewise linear subspaces. The multigrid iteration is computationally inexpensive to perform [10]. The resulting linear system is symmetric and positive definite, and we have shown that it can be preconditioned simply by diagonal scaling. Thus the discrete problems can be solved accurately with few iterations of the conjugate gradient method. Numerical experiments demonstrate quasi-optimal first order convergence rates, as predicted by the theory of [7].

CHAPTER II

PRELIMINARIES

This chapter introduces the function spaces on which our methods are based. We begin by setting the notation for general Hilbert spaces and then proceed to define axisymmetric functions and Sobolev spaces weighted by the radial variable in cylindrical coordinates. We next consider differential operators acting on axisymmetric functions in the weighted Sobolev spaces. Finally, discrete subspaces are introduced, along with inverse estimates and interpolation operators.

A. Hilbert Spaces

In this section, we introduce notation in general Hilbert spaces. Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. We denote by H' the dual of H, i.e. the space of bounded linear functionals on H. Let $< F, x >$ denote the action of a bounded linear functional F in H' on an element x of H.

The operator $T_H : H' \mapsto H$ is defined by

$$(T_H F, x)_H =< F, x > \qquad \text{for all } x \in H. \tag{2.1}$$

The Riesz Representation Theorem guarantees that T_H is a well-defined operator. The dual space H' is a Hilbert space with the inner product

$$(F, G)_{H'} =< F, T_H G > \tag{2.2}$$

and the corresponding norm

$$\|F\|_{H'} = ((F,F)_{H'})^{1/2} = \sup_{x\in H} \frac{< F, x >}{\|x\|_H}. \tag{2.3}$$

If X and Y are Hilbert spaces and $B : X \mapsto Y$ is a linear operator, then we define the *adjoint* operator $B^* : Y \mapsto X$ by

$$(x, B^*y)_X = (Bx, y)_Y \qquad \text{for all } x \in X,\, y \in Y.$$

The kernel of a linear operator B is denoted $\ker B$.

B. Differential Operators and Sobolev Spaces

A subset A of $\mathbb{R}^d$ is said to be a *domain* if it is Lebesgue-measurable and has a non-empty interior. Given an open domain A in $\mathbb{R}^d$, with $d = 2$ or 3, we denote by $C^k(A)$ the space of continuous real-valued functions on A with continuous derivatives up to order k. Let $\mathcal{D}(A)$ denote the space of infinitely smooth real-valued functions with compact support in A and $\mathcal{D}(\overline{A})$ denote the space of functions in $\mathcal{D}(\mathbb{R}^d)$ restricted to the closure of A. To be precise,

$$\mathcal{D}(A) = \{\phi : A \mapsto \mathbb{R} : \operatorname{supp}\phi \text{ is compact in } A, \text{ and } \phi \in C^k(A) \text{ for all } k \geq 0\}, \quad (2.4)$$

$$\mathcal{D}(\overline{A}) = \{\phi|_{\overline{A}} : \phi \in \mathcal{D}(\mathbb{R}^d)\}. \quad (2.5)$$

A linear functional $F : \mathcal{D}(A) \mapsto \mathbb{R}$ is continuous if $F(\phi_n)$ converges to $F(\phi)$ for all sequences $\{\phi_n\}_{n\in\mathbb{N}}$ of functions ϕ_n in $\mathcal{D}(A)$ converging to some ϕ in $\mathcal{D}(A)$. The convergence of such a sequence in $\mathcal{D}(A)$ means that there exists a compact subset K of A such that ϕ_n vanishes on $A \setminus K$ for all $n \in \mathbb{N}$, $\{\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to ϕ on K, and all derivatives of the functions ϕ_n converge uniformly to those of ϕ on K. The space of continuous linear functionals on $\mathcal{D}(A)$, i.e. the space of *distributions*, is denoted by $\mathcal{D}(A)'$.

We now define differential operators on smooth functions in $\mathcal{D}(\mathbb{R}^d)$, for $d = 2$ or 3. Let $\{x_1, \ldots, x_d\}$ be any prescribed coordinate system for $\mathbb{R}^d$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ of length $|\alpha| = \sum_{j=1}^{d} \alpha_j$, where each α_j is a non-negative integer, we define the differential operator $D^\alpha : \mathcal{D}(\mathbb{R}^d) \mapsto \mathcal{D}(\mathbb{R}^d)$ by

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} u.$$

The weak differential operator $D_w^\alpha : \mathcal{D}(\mathbb{R}^d)' \mapsto \mathcal{D}(\mathbb{R}^d)'$ is defined on distributions F by

$$< D_w^\alpha F, \phi >= (-1)^{|\alpha|} < F, D^\alpha \phi > \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d)^3.$$

If $\{x_1, \ldots, x_d\}$ is the Cartesian coordinate system, then the gradient operator $\nabla : \mathcal{D}(\mathbb{R}^d) \mapsto \mathcal{D}(\mathbb{R}^d)^d$ is defined by $(\nabla u)_j = \frac{\partial u}{\partial x_j}$. We define the distributional divergence operator $\nabla \cdot : \left(\mathcal{D}(\mathbb{R}^d)^3\right)' \mapsto \mathcal{D}(\mathbb{R}^d)'$ by

$$< \nabla \cdot \mathbf{F}, \phi >=< \mathbf{F}, \nabla \phi > \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d)^3.$$

In the case $d = 3$, the curl operator $\nabla \times : \mathcal{D}(\mathbb{R}^d)^3 \mapsto \mathcal{D}(\mathbb{R}^d)^3$ is defined (in Cartesian coordinates) by

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ in the standard orthonormal basis of $\mathbb{R}^3$. Then we define the distributional curl operator $\nabla \times : \left(\mathcal{D}(\mathbb{R}^d)^3\right)' \mapsto \left(\mathcal{D}(\mathbb{R}^d)^3\right)'$ by

$$< \nabla \times \mathbf{F}, \phi >=< \mathbf{F}, \nabla \times \phi > \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d)^3.$$

Next we introduce Sobolev spaces of functions with certain differentiability properties. For real $p \geq 1$, $L^p(A)$ is the Banach space of Lebesgue-measurable functions bounded in the norm

$$\|u\|_{L^p(A)} = \left(\int_A |u|^p \, d\mathbf{x}\right)^{1/p}.$$

For integers $k \geq 1$, the Sobolev space $W_p^k(A)$ is defined as the space of distributions in $L^p(A)$ having all weak derivatives up to order k in $L^p(A)$. The norm on $W_p^k(A)$ is

$$\|u\|_{W_p^k(A)} = \left(\sum_{|\alpha| \leq k} \|D_w^\alpha u\|_{L^p(A)}^p \right)^{1/p}.$$

In [1], it is proved that $W_p^k(A)$ is a Banach space. When $p = 2$, $H^k(A) \equiv W_2^k(A)$ is a Hilbert space.

The Sobolev Imbedding Theorem (see [1, 11]) gives conditions under which certain Sobolev spaces are related by continuous imbeddings. Below we state a specific case of the theorem.

Theorem 1 (Sobolev Imbedding Theorem) *Let A be a bounded, open domain in $\mathbb{R}^d$, with a Lipschitz boundary. For all real $t \leq s$ and $p \leq q$ satisfying $s - \frac{d}{p} = t - \frac{d}{q}$, we have the continuous imbedding*

$$W_p^s(A) \subset W_q^t(A).$$

C. Axisymmetric Functions

The methods of this dissertation are designed to compute axisymmetric functions, which we define precisely in this section. It will be convenient to express such functions in terms of cylindrical coordinates (r, θ, z), where r, θ, and z are the *radial*, *azimuthal*, and *axial* variables, respectively. Together, the radial and axial coordinates (r, z) are referred to as the *meridian coordinates*. Recall that the Jacobian of the transformation from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) is

$$J = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = r. \tag{2.6}$$

A domain Ω in $\mathbb{R}^3$ is said to be an *axisymmetric domain* if it is symmetric with respect to rotation about the z-axis.

Assumption 1 *Throughout the dissertation, we assume that* Ω *is a bounded axisymmetric domain in* $\mathbb{R}^3$*, which intersects the* z*-axis. Further, we assume that* Ω *is simply connected, with a polyhedral boundary.*

For the purposes of this dissertation, it is not necessary to assume that Ω intersects the z-axis. We are simply excluding the trivial case of Ω being bounded away from the z-axis, in which the problems can be solved with simpler methods requiring much less analysis. Indeed, the problems involve operators weighted by the radial variable, which have singularities at the z-axis. This necessitates careful analysis in weighted spaces and special bubble functions in the computations. If the domain is bounded away from the z-axis, then there are no such difficulties.

The meridian domain $D = \{(r, z) : (r, 0, z) \in \Omega\}$ shall be the computational domain for our methods. The assumptions on Ω imply that D is simply connected, with a polygonal boundary ∂D. Since Ω intersects the z-axis, the boundary of D has exactly one connected segment on the z-axis, denoted $\Gamma_0 = \{(0, z) \in \partial D\}$. The outer boundary of D is denoted $\Gamma_1 = \partial D \setminus \Gamma_0$. Note that D may be non-convex.

The unit cylindrical coordinate vectors are denoted by $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$. Thus, vector fields are written either as $\mathbf{v} = (v_r, v_\theta, v_z)$ or as $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$. Let $\mathcal{R}_\eta$ be the rotation around the z-axis by the angle $\eta \in [-\pi, \pi)$. A scalar function $u : \Omega \mapsto \mathbb{R}$ is said to be *invariant by rotation* if

$$u \circ \mathcal{R}_\eta = u, \qquad \text{for all } \eta \in [-\pi, \pi), \tag{2.7}$$

and a vector field $\mathbf{v} : \Omega \mapsto \mathbb{R}^3$ is said to be *axisymmetric* if

$$\mathcal{R}_{-\eta}(\mathbf{v} \circ \mathcal{R}_\eta) = \mathbf{v}, \qquad \text{for all } \eta \in [-\pi, \pi). \tag{2.8}$$

The definition (2.7) means that a scalar function u invariant by rotation satisfies $\frac{\partial u}{\partial \theta} = 0$. If $\mathbf{v} = (v_r, v_\theta, v_z)$ is axisymmetric, then (2.8) clearly implies $v_z \circ \mathcal{R}_\eta = v_z$ for all $\eta \in [-\pi, \pi)$, since the rotation $\mathcal{R}_{-\eta}$ has no effect on the z-component of $\mathbf{v} \circ \mathcal{R}_\eta$. On the other hand, $\mathbf{v} \circ \mathcal{R}_\eta(\mathbf{x}) = \mathcal{R}_\eta \mathbf{v}(\mathbf{x})$ implies that $v_r \circ \mathcal{R}_\eta = v_r$ and $v_\theta \circ \mathcal{R}_\eta = v_\theta$, since (v_r, v_θ) are local coordinates in the plane orthogonal the z-axis at the point $\mathbf{x}$. Thus, the component functions of an axisymmetric vector field are invariant by rotation. In particular, derivatives of $\mathbf{v}$ with respect to θ equal zero.

Given a function defined on D, a function invariant by rotation can be defined on Ω by rotating it around the z-axis. The rotation of a scalar function u defined on D is denoted $\breve{u}$, and is given by

$$\breve{u}(r, \theta, z) = u(r, z). \tag{2.9}$$

We say that u is the *trace* of the function $\breve{u}$.

D. Weighted Sobolev Spaces

This section introduces the function spaces on D containing the traces of axisymmetric functions in Sobolev spaces on Ω. For any $\beta \in \mathbb{R}$, let $L^2_\beta(D)$ denote the weighted space of Lebesgue-measurable functions u on D bounded in the norm

$$\|u\|_{L^2_\beta(D)} = \left(\int_D r^\beta u^2 \, dr \, dz \right)^{1/2} .$$

If a function $\breve{u}$ is invariant by rotation and u is its trace, related by (2.9), then (2.6) gives $\|\breve{u}\|^2_{L^2(\Omega)} = 2\pi \|u\|^2_{L^2_1(D)}$. In fact, the trace is an isomorphism (see Proposition 1 below). We denote by $H^k_\beta(D)$ the weighted Sobolev space of functions in $L^2_\beta(D)$ whose weak derivatives up to order k are in $L^2_\beta(D)$. That is, $H^k_\beta(D)$ is the Hilbert space endowed with the seminorm $|\cdot|_{H^k_\beta(D)}$ and norm $\|\cdot\|_{H^k_\beta(D)}$, defined by

$$|u|^2_{H^k_\beta(D)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2_\beta(D)},$$

$$\|u\|^2_{H^k_\beta(D)} = \|u\|^2_{L^2_\beta(D)} + |u|^2_{H^k_\beta(D)}.$$

Further, we define the subspaces

$$H^1_{1,\diamond}(D) = \{w \in H^1_1(D) : w = 0 \text{ on } \Gamma_1\},$$

$$H^1_-(D) = H^1_1(D) \cap L^2_{-1}(D),$$

$$H^1_{-,0}(D) = \{u \in H^1_-(D) : u = 0 \text{ on } \partial D\},$$

$$H^2_-(D) = H^2_1(D) \cap L^2_{-1}(D).$$

The significance of the function spaces introduced above is evident from the following result, proved in [3]. The isomorphisms given below indicate the function spaces for dimension reduced axisymmetric problems. In the sequel, the subspace of axisymmetric functions in a Hilbert space H is denoted $\breve{H}$.

Proposition 1 *The trace mapping, which maps an axisymmetric function $\breve{u}$ to u by* (2.9)*, yields the following isomorphisms:*

$$\breve{L}^2(\Omega) \cong L^2_1(D),$$

$$\breve{H}^k(\Omega) \cong H^k_1(D),$$

$$\breve{H}^1_0(\Omega) \cong H^1_{1,\diamond}(D).$$

In our analysis, we shall often make use of the density of smooth functions in the weighted Sobolev spaces $H^1_1(D)$ and $H^1_-(D)$. The proof of the following density result in $H^k_1(D)$ is given in [12], and the result in $H^1_-(D)$ is proved in [10].

Proposition 2 *The space of smooth functions $\mathcal{D}(\overline{D})$ is dense in $H^k_1(D)$ for all inte-*

gers $k \geq 1$. The set of smooth functions in $\mathcal{D}(\overline{D})$ which vanish in a neighborhood of Γ_0 is dense in $H^1_-(D)$.

The traces of functions in $H^1_1(D)$ comprise the space $H^{1/2}_1(\Gamma)$ (see [3]). However, we shall only use the boundedness of the trace as an operator from $H^1_1(D)$ to $L^2_1(e)$, for straight edges $e \subset \overline{D}$ not contained in Γ_0. The following result can be easily verified for functions in $\mathcal{D}(\overline{D})$, and the proposition then follows by density (see Proposition 2). Note that the norm of the trace operator depends on angles of ∂D. In general, traces of functions in $H^1_1(D)$ do not exist on Γ_0.

Proposition 3 *For any straight edge $e \subset \overline{D}$ with $e \not\subseteq \Gamma_0$, the trace mapping from $H^1_1(D)$ to $L^2_1(e)$ is bounded.*

E. Differential Operators in Cylindrical Coordinates

In cylindrical coordinates with the basis representation $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$, the differential operators curl, divergence, and gradient have the expressions

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z, \quad (2.10)$$

$$\nabla_r \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z},$$

$$\nabla \phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z. \quad (2.11)$$

Assuming axisymmetry, all derivatives with respect to θ equal zero. In particular, we have

$$\nabla \times (0, v_\theta, 0) = -\frac{\partial v_\theta}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\mathbf{e}_z,$$

$$\nabla \times (v_r, 0, v_z) = \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta. \quad (2.12)$$

That is, the curl of an azimuthal vector field is meridian, and the curl of a meridian vector field is azimuthal. Thus the curl decouples the azimuthal and meridian components of an axisymmetric vector field. Consequently, it will be convenient to define the differential operators curl, divergence, and gradient for the azimuthal and meridian components of axisymmetric vector fields. In accordance with (2.11), the gradient is defined by $\nabla\phi = (\frac{\partial\phi}{\partial r}, \frac{\partial\phi}{\partial z})$. The divergence $\nabla_r\cdot$ and the scalar curl $\nabla\times$ of a meridian vector field $\mathbf{v} = (v_r, v_z)$ are defined by

$$\begin{aligned} \nabla\times(v_r, v_z) &= \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\ \nabla_r\cdot(v_r, v_z) &= \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}, \end{aligned} \tag{2.13}$$

respectively. We define the curl of a scalar field ϕ by the r and z components of the curl of $\phi\,\mathbf{e}_\theta$, i.e.

$$\nabla_r\times\phi = \left(-\frac{\partial\phi}{\partial z}, \frac{1}{r}\left(\frac{\partial}{\partial r}(r\phi)\right)\right). \tag{2.14}$$

We use the notation $(\cdot,\cdot)_r = (\cdot,\cdot)_{L^2_1(D)}$, $(\cdot,\cdot)_{r,\tau} = (\cdot,\cdot)_{L^2_1(\tau)}$, and $< \cdot,\cdot >_{r,\Gamma_1} = (\cdot,\cdot)_{L^2_1(\Gamma_1)}$. This notation is also used for the inner products on the vector spaces $L^2_1(D)^2$ and $L^2_1(\tau)^2$, and the meaning will be clear from the context. For vectors $\mathbf{v}\in\mathbb{R}^2$, we denote the Euclidean norm $|\mathbf{v}| = (v_r^2 + v_z^2)^{1/2}$. The Green's formulas stated in the following lemma will be used repeatedly.

Lemma 1 *Let $\mathbf{n} = (n_r, n_z)$ denote the outward unit normal and $\mathbf{t} = (-n_z, n_r)$ denote the unit tangent vector (oriented counterclockwise). Then for all $\mathbf{v}\in H^1_1(D)^2$ and $\phi\in H^1_-(D)$, we have*

$$(\nabla\times\mathbf{v}, \phi)_r = (\mathbf{v}, \nabla_r\times\phi)_r - < \mathbf{v}\cdot\mathbf{t}, \phi >_{r,\Gamma_1}. \tag{2.15}$$

Also, for all $\mathbf{v} \in H^1_-(D) \times H^1_1(D)$ *and* $\phi \in H^1_1(D)$*, we have*

$$(\nabla_r \cdot \mathbf{v}, \phi)_r = -(\mathbf{v}, \nabla\phi)_r + < \mathbf{v} \cdot \mathbf{n}, \phi >_{r,\Gamma_1} . \tag{2.16}$$

Proof. Clearly (2.15) holds for $\mathbf{v} \in \mathcal{D}(\overline{D})^2$ and $\phi \in \mathcal{D}(\overline{D})$, with ϕ vanishing in a neighborhood of Γ_0. By Lemma 3.1 of [10], the subspace of functions in $\mathcal{D}(\overline{D})$ vanishing in a neighborhood of Γ_0 is dense in $H^1_-(D)$. The density of $\mathcal{D}(\overline{D})$ in $H^1_1(D)$ is given by Proposition 2.1(1) of [10]. Thus (2.15) follows by a density argument.

Similarly, (2.16) holds for $\mathbf{v} \in \mathcal{D}(\overline{D})^2$ and $\phi \in \mathcal{D}(\overline{D})$, with v_r vanishing in a neighborhood of Γ_0. By density (see the previous paragraph), (2.16) holds on the spaces stated therein. □

F. Discrete Subspaces

The numerical methods we shall present in this dissertation use finite dimensional subspaces consisting of standard finite element spaces and certain spaces of bubble functions. These spaces are defined on some mesh $\mathcal{T}_h$ which discretizes the computational domain D. Accordingly, their approximation properties depend on the mesh size h, defined as follows. For each element τ of $\mathcal{T}_h$, let h_τ denote the diameter of τ and ρ_τ the diameter of the largest circle inscribed in τ. Then $h = \max_\tau h_\tau$. Let $\mathfrak{T}$ be a given family of meshes discretizing D and containing meshes $\mathcal{T}_h$ for $h > 0$ arbitrarily small. Thus h is considered as a parameter associated with meshes in $\mathfrak{T}$, which tends to zero.

Further, we denote by r_τ the maximum value of the radial variable r at the vertices of τ and by h_e the length of an edge e. The theory and methods of this dissertation apply to both triangular and quadrilateral meshes, but we only consider triangular meshes for simplicity of presentation. We make the following assumptions regarding the family of meshes $\mathfrak{T}$.

Assumption 2 *Each mesh $\mathcal{T}_h$ in $\mathfrak{T}$ consists of non-overlapping triangles τ with boundaries aligned with the jumps of the coefficients ϵ and μ. The family of meshes $\mathfrak{T}$ is locally quasi-uniform, i.e. there exists a constant C independent of h such that*

$$\frac{h_\tau}{\rho_\tau} \leq C \qquad \text{for all } \tau \text{ in } \mathcal{T}_h, \text{ all } \mathcal{T}_h \text{ in } \mathfrak{T}.$$

A consequence of the quasi-uniformity of $\mathfrak{T}$ is the equivalence of r_τ and h_τ for triangles τ intersecting Γ_0. Throughout the dissertation, C represents a generic positive constant which may vary in different instances. The constant C is always independent of the mesh size h and the coefficients ϵ and μ, unless stated otherwise.

The discrete subspaces are defined as piecewise polynomial spaces. For integers $k \geq 0$, we denote by $P_k(A)$ the space of polynomials in r and z of degree less than or equal to k on a domain $A \subseteq D$. We shall use the following spaces of piecewise linear continuous functions:

$$\begin{aligned}
S^h &= \{u \in C^0(D) : u|_\tau \in P_1(\tau) \text{ for all } \tau \in \mathcal{T}_h\} \subset H^1_1(D),\\
S^h_\diamond &= S^h \cap H^1_{1,\diamond}(D),\\
S^h_- &= S^h \cap H^1_-(D),\\
S^h_0 &= \{u \in S^h : u = 0 \text{ on } \partial D\}.
\end{aligned}$$

In addition to the piecewise linear spaces, we shall also use spaces of bubble functions defined on the edges and elements of the mesh $\mathcal{T}_h$. Let τ be a triangle in $\mathcal{T}_h$, and denote by $\lambda_i(r, z)$, $i = 1, 2, 3$, the barycentric coordinate for $(r, z) \in \tau$. Let e be any edge of τ, and assume that λ_3 corresponds to the vertex not in e. Then we define the edge bubble spaces

$$B^{(1)}_e = \operatorname{span}\{\lambda_1\lambda_2\}, \qquad B^{(2)}_e = \operatorname{span}\{\lambda_1\lambda_2, r\lambda_1\lambda_2\},$$

and the element bubble spaces

$$B_\tau^{(1)} = \text{span}\{\lambda_1\lambda_2\lambda_3\}, \qquad B_\tau^{(2)} = \text{span}\{\lambda_1\lambda_2\lambda_3, r\lambda_1\lambda_2\lambda_3\}.$$

The following weighted inverse estimates are proved in [2].

Lemma 2 *For each integer $k \geq 1$, there exists a constant $C_k > 0$ such that for every triangle $\tau \in \mathcal{T}_h$ and all polynomials $f \in P_k(\tau)$ vanishing on Γ_0 if $\tau \cap \Gamma_0 \neq \emptyset$,*

$$\|f\|_{L^2_{-1}(\tau)} \leq C_k r_\tau^{-1} \|f\|_{L^2_1(\tau)}.$$

Lemma 3 *For each integer $k \geq 0$, there exists a constant $C_k > 0$ such that for any triangle τ in $\mathcal{T}_h$,*

$$\|f\|_{H^1_1(\tau)} \leq C_k h_\tau^{-1} \|f\|_{L^2_1(\tau)}, \qquad \textit{for all } f \in P_k(\tau). \tag{2.17}$$

Let $\{\phi_i\}$ denote the standard nodal basis for S^h, so that for each vertex $\mathbf{a}_i$ of $\mathcal{T}_h$, we have $\phi_j(\mathbf{a}_i) = \delta_{ij}$. As a consequence of Lemma 2,

$$S^h_- = \text{span}\{\phi_i : a_i \notin \Gamma_0\}.$$

Now we introduce Clement-like interpolation operators [2], which are bounded in $H^1_1(D)$ or $H^1_-(D)$ and have approximation properties. For each vertex $\mathbf{a}_i$ in $\mathcal{T}_h$, let τ_i be a triangle in $\mathcal{T}_h$ having $\mathbf{a}_i$ as a vertex. Define the $L^2_1(\tau_i)$-orthogonal projector $\pi_i : L^2_1(\tau_i) \mapsto P_1(\tau_i)$ by

$$\int_{\tau_i} r(u - \pi_i u) q \, dr \, dz, \qquad \text{for all } q \in P_1(\tau_i). \tag{2.18}$$

Then we define the Clement operators $\Pi^h : H^1_1(D) \to S^h$, $\Pi^h_\diamond : H^1_{1,\diamond}(D) \to S^h_\diamond$, $\Pi^h_- : H^1_-(D) \to S^h_-$, and $\Pi^h_0 : H^1_{-,0}(D) \to S^h_0$ by

$$\Pi^h u = \textstyle\sum_{\mathbf{a}_i \in \mathcal{T}_h} (\pi_i u)(\mathbf{a}_i)\phi_i, \quad \Pi^h_\diamond u = \textstyle\sum_{\mathbf{a}_i \notin \Gamma_1} (\pi_i u)(\mathbf{a}_i)\phi_i,$$
$$\Pi^h_- u = \textstyle\sum_{\mathbf{a}_i \notin \Gamma_0} (\pi_i u)(\mathbf{a}_i)\phi_i, \quad \Pi^h_0 u = \textstyle\sum_{\mathbf{a}_i \notin \partial D} (\pi_i u)(\mathbf{a}_i)\phi_i.$$

The following approximation results for these projectors are proved in [2]. Here, Δ_τ denotes the union of all triangles in $\mathcal{T}_h$ sharing a common vertex with τ.

Lemma 4 *For all triangles τ in $\mathcal{T}_h$, we have*

$$h_\tau^{-2}\|u-\Pi^h u\|^2_{L^2_1(\tau)} + \|u-\Pi^h u\|^2_{H^1_1(\tau)} \le C\|u\|^2_{H^1_1(\Delta_\tau)} \qquad \text{for all } u \in H^1_1(D), \tag{2.19}$$

$$h_\tau^{-2}\|u-\Pi^h_\diamond u\|^2_{L^2_1(\tau)} + \|u-\Pi^h_\diamond u\|^2_{H^1_1(\tau)} \le C\|u\|^2_{H^1_1(\Delta_\tau)} \qquad \text{for all } u \in H^1_{1,\diamond}(D), \tag{2.20}$$

$$h_\tau^{-2}\|u-\Pi^h_- u\|^2_{L^2_1(\tau)} + \|u-\Pi^h_- u\|^2_{H^1_-(\tau)} \le C\|u\|^2_{H^1_-(\Delta_\tau)} \qquad \text{for all } u \in H^1_-(D), \tag{2.21}$$

$$h_\tau^{-2}\|u-\Pi^h_0 u\|^2_{L^2_1(\tau)} + \|u-\Pi^h_0 u\|^2_{H^1_-(\tau)} \le C\|u\|^2_{H^1_-(\Delta_\tau)} \qquad \text{for all } u \in H^1_{-,0}(D). \tag{2.22}$$

CHAPTER III

NEGATIVE-NORM LEAST-SQUARES METHODS

In this chapter, we introduce negative-norm least-squares methods in an abstract setting. The theory of these methods is surveyed, with theorems on the existence and uniqueness of solutions, as well as convergence rates for discretized least-squares methods. The presentation follows [7].

The abstract setting for this chapter is in Hilbert spaces X and Y. Recall that their duals are denoted by X' and Y'. The weak formulations of problems considered in this dissertation shall be of the following form:

$$\begin{cases} \text{Find } x \in X \text{ satisfying} \\ \qquad b(x,y) = < F, y >, \qquad \text{for all } y \in Y. \end{cases} \tag{3.1}$$

Here, F is in Y' and $b : X \times Y \to \mathbb{R}$ is a continuous bilinear form. That is, $b(x,y) \leq \|b\| \|x\|_X \|y\|_Y$ for all x in X and y in Y, with $\|b\| > 0$. It will be convenient to work with the operator $B : X \to Y'$ associated with b, which is defined by

$$(Bx, y)_Y = b(x,y) \qquad \text{for all } x \in X, y \in Y. \tag{3.2}$$

Note that B is a bounded linear operator and (3.1) may be rewritten as $Bx = F$. The following generalized Lax-Milgram theorem gives sufficient conditions for existence and uniqueness of solutions to (3.1). Clearly, the compatibility condition for F is necessary.

Theorem 2 *Suppose the continuous bilinear form $b : X \times Y \to \mathbb{R}$ satisfies the inf-sup condition*

$$\|x\|_X \leq C_0 \sup_{y \in Y} \frac{b(x,y)}{\|y\|_Y} \qquad \textit{for all } x \in X, \tag{3.3}$$

and $F \in Y'$ *satisfies the compatibility condition*

$$< F, y >= 0 \qquad \textit{for all } y \in \ker B^*. \tag{3.4}$$

Then there exists a unique solution $x \in X$ *to* (3.1)*, satisfying the stability estimate*

$$\|x\|_X \leq C_0 \|F\|_{Y'}. \tag{3.5}$$

Even when the inf-sup condition (3.3) holds, the weak problem (3.1) does not always have a solution due to the compatibility condition. However, there is always a unique solution to the negative-norm least-squares problem: Find $x \in X$ satisfying

$$A(x, z) \equiv (Bx, Bz)_{Y'} = (F, Bz)_{Y'}, \qquad \text{for all } z \in X. \tag{3.6}$$

Indeed, the inf-sup condition (3.3) for b implies that

$$\|x\|_X \leq C \|Bx\|_{Y'} \qquad \text{for all } x \in X. \tag{3.7}$$

Therefore, A is a bounded, coercive bilinear form on $X \times X$. Thus we have the main result for the negative-norm least-squares problem.

Theorem 3 *If the continuous bilinear form* $b : X \times Y \to \mathbb{R}$ *satisfies the inf-sup condition* (3.3) *then there exists a unique solution* $x \in X$ *to* (3.6)*. Moreover, if* (3.1) *has a solution then the solutions of* (3.1) *and* (3.6) *coincide.*

Now we turn our attention to approximation of solutions to (3.6). We consider discrete subspaces $X_h \subset X$ and $Y_h \subset Y$ of finite dimension, where $h > 0$ corresponds to the size of the mesh associated with X_h and Y_h. These discrete subspaces are assumed to have some approximation properties so that for some integers $j, k \geq 1$, we have

$$\inf_{x_h \in X_h} \|x - x_h\|_X \leq C(x)h^j \qquad \text{for all } x \in X, \tag{3.8}$$

$$\inf_{y_h \in Y_h} \|y - y_h\|_Y \leq C(y)h^k \qquad \text{for all } y \in Y. \tag{3.9}$$

In these inequalities, the constant depends on some norm of the function being approximated. In our applications, this will be a higher order weighted Sobolev norm.

The discrete least-squares system is defined by restricting (3.6) to the discrete subspaces. In order to implement the method and perform computations, we must replace the inner product in the dual space Y' by a computable inner product in the discrete space Y_h. Therefore, we introduce the operator $T_{Y_h} : Y_h' \to Y_h$ defined by

$$(T_{Y_h} G, y)_Y = < G, y > \qquad \text{for all } y \in Y_h. \tag{3.10}$$

Now the discrete least-squares problem is to find $x \in X_h$ satisfying

$$A_h(x, z) \equiv < B_h x, T_{Y_h} B_h z > = < F, T_{Y_h} B_h z >, \qquad \text{for all } z \in X_h. \tag{3.11}$$

If X_h and Y_h are such that the discrete inf-sup condition

$$\|x\|_X \leq C_1 \sup_{y \in Y_h} \frac{b(x, y)}{\|y\|_Y} \qquad \text{for all } x \in X_h \tag{3.12}$$

holds then a unique solution to (3.11) exists and provides a quasi-optimal approximation to the solution of the continuous problem (3.6). This is stated precisely in the following theorem, which is proved in [7].

Theorem 4 *Assume the hypotheses of Theorem 2 are satisfied, so that* (3.1) *has a unique solution x. If* (3.12) *also holds, then* (3.11) *has a unique solution x_h in X_h, with*

$$\|x - x_h\|_X \leq (1 + C_1^2 \|b\|^2) \inf_{z \in X_h} \|x - z\|_X. \tag{3.13}$$

The discrete least-squares problem (3.11) requires evaluation of the operator T_{Y_h}, which is essentially the solution of a linear system involving the stiffness matrix for the inner product in Y. In numerical computations, this inversion can be replaced by a more efficient preconditioner $\tilde{T}_{Y_h} : Y_h' \to Y_h$. By this we mean that the operators $\tilde{T}_{Y_h}$ and T_{Y_h} are spectrally equivalent, i.e. there exist constants $0 < a_0 \leq a_1$ satisfying

$$a_0 < G, \tilde{T}_{Y_h} G > \leq < G, T_{Y_h} G > \leq a_1 < G, \tilde{T}_{Y_h} G > \qquad \text{for all } G \in Y_h'. \tag{3.14}$$

Thus the discrete least-squares problem becomes

$$\tilde{A}_h(x, z) \equiv < B_h x, \tilde{T}_{Y_h} B_h z > = < F, \tilde{T}_{Y_h} B_h z >, \qquad \text{for all } z \in X_h. \tag{3.15}$$

The following corollary (cf. [7]) states that the problem (3.15) has a unique solution which is still quasi-optimal, provided the preconditioner $\tilde{T}_{Y_h}$ is spectrally equivalent to T_{Y_h}.

Corollary 1 *Assume the hypotheses of Theorem 4 and the existence of constants* $0 < a_0 \leq a_1$ *satisfying* (3.14). *Then* (3.15) *has a unique solution* x_h *in* X_h, *and*

$$\|x - x_h\|_X \leq \left(1 + \frac{a_1}{a_0} C_1^2 \|b\|^2\right) \inf_{z \in X_h} \|x - z\|_X. \tag{3.16}$$

Remark 1 *Assuming the approximation property* (3.8), *we have the error estimate*

$$\|x - x_h\|_X \leq \left(1 + \frac{a_1}{a_0} C_1^2 \|b\|^2\right) C(x) h^j. \tag{3.17}$$

Thus we have a quasi-optimal method (3.15). The discrete problem involves the solution of a symmetric positive definite linear system. In the applications, this system will be solved iteratively by the conjugate gradient method.

CHAPTER IV

ELECTROSTATICS

In this chapter, we consider as a model problem the Maxwell system for electrostatics, defined as follows. The domain Ω in $\mathbb{R}^3$ satisfies Assumption 1. The static Maxwell system for the electric field $\mathbf{e} \in L^2(\Omega)^3$ is

$$\begin{cases} \nabla \times \mathbf{e} & = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot (\epsilon \mathbf{e}) & = g \text{ in } \Omega \\ \mathbf{e} \times \mathbf{n} & = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that the electric permittivity ϵ is piecewise constant and positive and well behaved enough so that when $f \in L^2(\Omega)$, solutions to the Dirichlet problem

$$-\nabla \cdot \epsilon \nabla u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

are in the Sobolev space $H^{1+s}(\Omega)$ for some $s > 0$. Further, we assume that $\mathbf{f} \in L^2(\Omega)^3$ is axisymmetric and that $g \in L^2(\Omega)$ and ϵ are invariant by rotation. The boundary condition $\mathbf{e} \times \mathbf{n} = \mathbf{0}$ corresponds to a perfect conductor [13]. Inhomogeneous boundary conditions can be handled by simply modifying the data in the weak formulation considered below.

Under the assumption of axisymmetric data, transforming the system (4.1) to cylindrical coordinates (r, θ, z) yields (see [3]) two decoupled systems in the two-dimensional domain $D = \{(r,z) : (r,0,z) \in \Omega\}$:

$$\begin{cases} \nabla \times (e_r, e_z) &= f_\theta \quad \text{in } D \\ \nabla_r \cdot (\epsilon(e_r, e_z)) \equiv \frac{1}{r}\frac{\partial}{\partial r}(r\epsilon e_r) + \frac{\partial}{\partial z}(\epsilon e_z) &= g \quad \text{in } D \\ (e_r, e_z)\cdot(-n_z, n_r) &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{4.2}$$

and

$$\begin{cases} -\frac{\partial e_\theta}{\partial z} &= f_r \quad \text{in } D \\ \frac{1}{r}\frac{\partial}{\partial r}(re_\theta) &= f_z \quad \text{in } D \\ e_\theta &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{4.3}$$

where $\Gamma_1 = \{(r,z) \in \partial D : r > 0\}$. The assumptions on Ω imply that D is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary. Note that D may be non-convex.

The azimuthal component e_θ can be solved for separately in the following scalar equation obtained by taking the curl of equation (4.3):

$$\begin{cases} -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(re_\theta)\right) - \frac{\partial^2 e_\theta}{\partial z^2} &= \nabla \times (f_r, f_z) \quad \text{in } D \\ e_\theta &= 0 \quad \text{on } \Gamma_1. \end{cases} \tag{4.4}$$

Observe that integration by parts yields a variational formulation for (4.4) that is coercive on the space $\{\phi \in H^1_-(D) : \phi = 0 \text{ on } \partial D\}$. Note that this boundary condition is equivalent to the one in (4.4), as all functions in $\phi \in H^1_-(D)$ satisfy $\phi = 0$ on $\Gamma_0 = \partial D \setminus \Gamma_1$.) Thus multigrid methods can be used to numerically solve (4.4), as in [4, 10]. We are interested only in solving (4.2) for the meridian components $\mathbf{e} = (e_r, e_z) \in L^2_1(D)^2$.

A. The Least-Squares Formulation

Applying Green's formulas (Lemma 1) to (4.2) yields the following weak formulation: Find $\mathbf{e} \in L^2_1(D)^2$ satisfying

$$b(\mathbf{e},(\phi,q)) \equiv (\mathbf{e}, \nabla_r \times \phi)_r + (\epsilon\mathbf{e}, \nabla q)_r = (f_\theta, \phi)_r - (g,q)_r, \tag{4.5}$$

for all $(\phi, q) \in H^1_-(D) \times H^1_{1,\diamond}(D)$. (4.5) provides a weak formulation of (4.2), which implicitly enforces the boundary condition $\mathbf{e} \times \mathbf{n} = \mathbf{0}$. Define the operators curl_1 : $L^2_1(D)^2 \to H^1_-(D)'$, $\text{div}_\epsilon : L^2_1(D)^2 \to H^1_{1,\diamond}(D)'$, and $B : L^2_1(D)^2 \to (H^1_-(D) \times H^1_{1,\diamond}(D))'$ by

$$\begin{aligned} < \text{curl}_1\, \mathbf{v}, \psi >&= (\mathbf{v}, \nabla_r \times \psi)_r \quad \text{for all } \mathbf{v} \in L^2_1(D)^2, \psi \in H^1_-(D), \\ < \text{div}_\epsilon\, \mathbf{v}, \psi >&= (\epsilon\mathbf{v}, \nabla\psi)_r \quad \text{for all } \mathbf{v} \in L^2_1(D)^2, \psi \in H^1_{1,\diamond}(D), \\ B\mathbf{v} &= (\text{curl}_1\, \mathbf{v}, \text{div}_\epsilon\, \mathbf{v}) \quad \text{for all } \mathbf{v} \in L^2_1(D)^2. \end{aligned} \tag{4.6}$$

The curl operator curl_1 has the subscript 1 to distinguish it from a different curl operator used later in the magnetostatic and time-harmonic problems. Thus B satisfies $(B\mathbf{e},(\phi,q)) = b(\mathbf{e},(\phi,q))$ for all $\mathbf{e} \in L^2_1(D)^2$ and $(\phi,q) \in H^1_-(D) \times H^1_{1,\diamond}(D)$. Further define the symmetric bilinear form A on $L^2_1(D)^2 \times L^2_1(D)^2$ and the linear functional F by

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) &\equiv (B\mathbf{u}, B\mathbf{v})_{(H^1_-(D)\times H^1_{1,\diamond}(D))'} \\ &= (\text{curl}_1\, \mathbf{u}, \text{curl}_1\, \mathbf{v})_{H^1_-(D)'} + (\text{div}_\epsilon\, \mathbf{u}, \text{div}_\epsilon\, \mathbf{v})_{H^1_{1,\diamond}(D)'}, \\ < F, \mathbf{v} > &\equiv (f_\theta, \text{curl}_1\, \mathbf{v})_{H^1_-(D)'} - (g, \text{div}_\epsilon\, \mathbf{v})_{H^1_{1,\diamond}(D)'}. \end{aligned}$$

Then the dual based least-squares formulation of (4.5) is to find $\mathbf{e} \in L^2_1(D)^2$ satisfying

$$A(\mathbf{e},\mathbf{v}) =< F, \mathbf{v} > \text{ for all } \mathbf{v} \in L^2_1(D)^2. \tag{4.7}$$

To show that the linear operator B and its inverse are bounded, we shall use the following orthogonal decomposition for the space $L^2_1(D)^2$.

Lemma 5 *For any* $\mathbf{v} \in L^2_1(D)^2$*, there exist* $\phi \in H^1_-(D)$ *and* $q \in H^1_{1,\diamond}(D)$ *satisfying* $\mathbf{v} = \nabla_r \times \phi + \epsilon \nabla q$*. This decomposition is orthogonal with respect to the inner-product* $(\epsilon^{-1}\cdot, \cdot)_r$ *on* $L^2_1(D)^2$*.*

Before proving Lemma 5, we first prove a technical lemma.

Lemma 6 *Let* $s > 0$ *and assume that* $A \subset \Omega$ *and* $S \subset \partial A$ *are open sets with Lipschitz continuous boundaries. For any divergence-free vector field* $\mathbf{w}$ *in* $(H^s(\Omega))^3$*, the surface integral*

$$\int_S \mathbf{w} \cdot \mathbf{n} \, ds \tag{4.8}$$

is finite.

Proof. By the Sobolev Imbedding Theorem (Theorem 1), $(H^s(\Omega))^3$ is imbedded in $(L^q(\Omega))^3$ for some $q > 2$. Define $p < 2$ by $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 - \frac{1}{p} < \frac{1}{p}$. Consider the space $\tilde{W}^t_p(S)$ consisting of all functions in $W^t_p(S)$ whose extensions by zero outside S are in $W^t_p(\partial A)$. By Corollary 1.4.4.5 of [11], we have $\tilde{W}^t_p(S) = W^t_p(S)$ when $t < \frac{1}{p}$. In particular, the characteristic function $\chi_S : \partial A \mapsto \mathbb{R}$, defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise}, \end{cases}$$

is in $W_p^{1-\frac{1}{p}}(\partial A)$. Consequently, there exists ϕ in $W^1_p(A)$ such that $\phi|_{\partial A} = \chi_S$. Applying Green's formula and the hypothesis that $\mathbf{w}$ is divergence-free yields

$$\begin{aligned} \int_S \mathbf{w} \cdot \mathbf{n} \, ds = \int_{\partial A} \chi_S \mathbf{w} \cdot \mathbf{n} \, ds &= \int_A \mathbf{w} \cdot \nabla \phi \, dx \\ &\leq \|\mathbf{w}\|_{L^q(A)} \|\nabla \phi\|_{L^p(A)} < \infty. \end{aligned}$$

Thus the surface integral (4.8) is bounded. □

Proof. [Proof of Lemma 5] The Green's formula (2.15) and the density of smooth functions yield $(\nabla_r \times \phi, \nabla q)_r = 0$ for any $\phi \in H^1_-(D)$ and $q \in H^1_{1,\diamond}(D)$. In addition, $\|\nabla_r \times \phi\|_r$ and $\|\epsilon^{1/2}\nabla q\|_r$ provide equivalent norms on $H^1_-(D)$ and $H^1_{1,\diamond}(D)$, respectively. Thus, by density, it suffices to prove the result for $\mathbf{v} \in (C_0^\infty(D))^2$. Associated with $\mathbf{v} = (v_r, v_z)$, we have a rotated function

$$\breve{\mathbf{v}} = v_r \mathbf{e}_r + v_z \mathbf{e}_z$$

defined on Ω. By Theorem II.2.6 of [3], $\breve{\mathbf{v}}$ is smooth. Let $\breve{q} \in H^1_0(\Omega)$ solve

$$(\epsilon \nabla \breve{q}, \nabla \theta)_{L^2(\Omega)} = (\breve{\mathbf{v}}, \nabla \theta)_{L^2(\Omega)} = -(\nabla \cdot \breve{\mathbf{v}}, \theta)_{L^2(\Omega)} \qquad \text{for all } \theta \text{ in } H^1_0(\Omega), \tag{4.9}$$

and set

$$\breve{\mathbf{w}} = \breve{\mathbf{v}} - \epsilon \nabla \breve{q}.$$

Since $\breve{q}$ solves an axisymmetric Laplace equation, $\breve{q}$ is axisymmetric and so is $\breve{\mathbf{w}}$. By restricting to $\theta = 0$, we have the decomposition

$$\mathbf{v} = \mathbf{w} + \epsilon \nabla q.$$

The map $\breve{q} \to q$ is an isomorphism (see [3]) from the space of axisymmetric functions in $H^1_0(\Omega)$ to $H^1_{1,\diamond}(D)$, so we have

$$\|q\|_{H^1_1(D)} \le C \|\mathbf{v}\|_{L^2_1(D)^2}.$$

By regularity assumptions on ϵ, we have that $\breve{q}$ is in $H^{1+s}(\Omega)$ for some $s > 0$. This implies that $\breve{\mathbf{w}}$ is in $(H^s(\Omega))^3$. It now follows from Lemma 6 and the fact that $\breve{\mathbf{w}}$ is divergence-free in Ω that

$$\int_S \breve{\mathbf{w}} \cdot \mathbf{n} \, ds$$

exists on any surface S generated by rotation of a piecewise smooth curve $\gamma \subset D$ which is nowhere tangent to Γ_0. We conclude that

$$\int_S \breve{\mathbf{w}} \cdot \mathbf{n}\, ds = 2\pi \int_\gamma r\mathbf{w} \cdot \mathbf{n}\, ds$$

is finite, where $\mathbf{n}$ denotes the tangent along γ rotated by -90 degrees. If γ is a curve with its endpoints on Γ_0, then

$$2\pi \int_\gamma r\mathbf{w} \cdot \mathbf{n}\, ds = \int_S \breve{\mathbf{w}} \cdot \mathbf{n}\, ds = \int_{\breve{D}_S} \nabla \cdot \breve{\mathbf{w}}\, dx = 0.$$

Here $\breve{D}_S$ is the domain enclosed by S. The same holds true when γ is a closed curve.

For any point $(r_1, z_1) \in D$, let γ be any piecewise smooth path from some point $\mathbf{a}_0$ on Γ_0 to (r_1, z_1). Define

$$\phi(r_1, z_1) = -\frac{1}{r_1} \int_\gamma r\mathbf{w} \cdot \mathbf{n}\, ds. \tag{4.10}$$

The above considerations show that ϕ is well defined and independent of the path γ and starting point $\mathbf{a}_0$ on Γ_0. We claim that this function ϕ satisfies the requirements of the lemma.

First, we verify that $\nabla_r \times \phi = \mathbf{w}$. For $(r_1, z_1) \in D$ and sufficiently small $h > 0$, consider the straight-line path γ_h from $(r_1, z_1 - h)$ to $(r_1, z_1 + h)$. Then $\mathbf{t} = (0, 1)$ and $\mathbf{n} = (1, 0)$, so

$$-\frac{\partial \phi}{\partial z} = \lim_{h \to 0} \frac{1}{2hr_1} \int_{\gamma_h} r w_r\, ds = \frac{1}{r_1} r_1 w_r = w_r.$$

Now let γ_h be the straight-line path from $(r_1 - h, z_1)$ to $(r_1 + h, z_1)$, so that $\mathbf{t} = (1, 0)$ and $\mathbf{n} = (0, -1)$. Then

$$\frac{1}{r_1} \frac{\partial}{\partial r}(r\phi)(r_1, z_1) = \frac{1}{r_1} \lim_{h \to 0} \frac{1}{2h} \int_{\gamma_h} r w_z\, ds = \frac{1}{r_1} r_1 w_z = w_z, \tag{4.11}$$

and we conclude that $\nabla_r \times \phi = \mathbf{w}$.

Next we show that ϕ is in $L^2_{-1}(D)$ by using the Hardy inequality (see Appendix A.4 of [16])

$$\left(\int_0^\infty \left(\int_0^x f(y)\,dy\right)^p x^{-k-1}\,dx\right)^{1/p} \le \frac{p}{k}\left(\int_0^\infty (xf(x))^p x^{-k-1}\,dx\right)^{1/p}. \qquad (4.12)$$

We illustrate the argument when the strip $[0,R]\times\Gamma_0$ is contained in D for some $R>0$. Simple modifications of this argument give the general case. For $z\in\Gamma_0$, applying the Hardy inequality gives

$$\begin{aligned}\int_0^R r_1^{-1}\phi(r_1,z)^2\,dr_1 &= \int_0^R r_1^{-3}\left(\int_0^{r_1} r\mathbf{w}_z(r,z)\,dr\right)^2 dr_1\\ &\le \int_0^R r|\mathbf{w}_z(r,z)|^2\,dr.\end{aligned}$$

Integrating over z gives

$$\|\phi\|^2_{L^2_{-1}([0,R]\times\Gamma_0)} \le \|\mathbf{w}\|^2_{L^2_1([0,R]\times\Gamma_0)^2} \le C\|\mathbf{v}\|^2_{L^2_1([0,R]\times\Gamma_0)^2}.$$

Thus $\|\phi\|_{L^2_{-1}(D)} \le C\|\mathbf{v}\|_{L^2_1(D)^2}$. Using the identity (4.11), we compute

$$\frac{\partial\phi}{\partial r} = -\frac{\phi}{r} + w_z.$$

That $\frac{\partial\phi}{\partial r}$ is in $L^2_1(D)$ follows from the facts that w_z is in $L^2_1(D)$ and ϕ is in $L^2_{-1}(D)$. Finally,

$$\frac{\partial\phi}{\partial z} = -w_r$$

is also in $L^2_1(D)$. This completes the proof of the lemma. □

The main result of this section now follows.

Theorem 5 *For all* $\mathbf{v}\in L^2_1(D)^2$,

$$(\epsilon\mathbf{v},\mathbf{v})_r = \sup_{(\phi,q)\in H^1_-(D)\times H^1_{1,\diamond}(D)} \frac{(\mathbf{v},\nabla_r\times\phi)_r^2}{\|\epsilon^{-1/2}\nabla_r\times\phi\|_r^2} + \frac{(\epsilon\mathbf{v},\nabla q)_r^2}{\|\epsilon^{1/2}\nabla q\|_r^2}.$$

Proof. Applying Lemma 5 to $\epsilon\mathbf{v} \in L^2_1(D)^2$ yields $\phi \in H^1_-(D)$ and $q \in H^1_{1,\diamond}(D)$ satisfying $\mathbf{v} = \epsilon^{-1}\nabla_r \times \phi + \nabla q$. This decomposition is orthogonal in the inner product $(\epsilon\cdot,\cdot)$. Thus,

$$(\epsilon\mathbf{v},\mathbf{v})_r = (\epsilon\mathbf{v}, \epsilon^{-1}\nabla_r \times \phi + \nabla q)_r = \frac{(\mathbf{v}, \nabla_r \times \phi)_r^2}{\|\epsilon^{-1/2}\nabla_r \times \phi\|_r^2} + \frac{(\epsilon\mathbf{v}, \nabla q)_r^2}{\|\epsilon^{1/2}\nabla q\|_r^2}.$$

It follows from the Schwarz inequality and the $(\epsilon\cdot,\cdot)_r$-orthogonality of the decomposition that taking the supremum here preserves the equality. ∎

Remark 2 *The norm $(\epsilon^{-1}\nabla_r \times \phi, \nabla_r \times \phi)_r^{1/2}$ provides an equivalent norm on $H^1_-(D)$ while $(\epsilon\nabla q, \nabla q)_r^{1/2}$ defines an equivalent norm on $H^1_{1,\diamond}(D)$. If we use these norms to define the dual spaces, then the above theorem can be restated*

$$(\epsilon\mathbf{v},\mathbf{v})_r = \|\mathit{curl}_1\,\mathbf{v}\|^2_{(H^1_-(D))'} + \|\mathit{div}_\epsilon\,\mathbf{v}\|^2_{(H^1_{1,\diamond}(D))'}.$$

This immediately implies that the existence and uniqueness of the solution to the least-squares problem (4.7)*.*

Corollary 2 *Using the norm $(\epsilon\cdot,\cdot)^{1/2}$ on $(L^2_1(D))^2$ and the above norms on the dual spaces, the operator $B : (L^2_1(D))^2 \to (H^1_-(D))' \times (H^1_{1,\diamond}(D))'$ is an isometry.*

Proof. By Remark 2 and the generalized Lax-Milgram lemma (see, e.g. [7]), B is an isomorphism onto its image in $(H^1_-(D) \times H^1_{1,\diamond}(D))'$. We need only check that it is onto. It suffices to show that the only pair of functions $\phi \in H^1_-(D)$ and $q \in H^1_{1,\diamond}(D)$ satisfying

$$b(\mathbf{v},(\phi,q)) = 0 \qquad \text{for all } \mathbf{v} \in (L^2_1(D))^2$$

is $(\phi,q) = (0,0)$. This is immediate since setting $\mathbf{v} = \epsilon^{-1}\nabla_r \times \phi + \nabla q$ gives

$$b(\mathbf{v},(\phi,q)) = (\epsilon^{-1}\nabla_r \times \phi, \nabla_r \times \phi)_r + (\epsilon\nabla q, \nabla q)_r.$$

∎

Remark 3 *It is immediate from the above theorem and Remark 2 that* (4.5) *has a unique solution which coincides with that of* (4.7).

B. Stable Approximation

In this section, we describe a stable pair of approximation spaces for the least-squares method (4.7). Let $\mathcal{T}_h$ be a triangulation of D satisfying Assumption 2. For the electrostatic problem, $\mathcal{T}_h$ is assumed to be aligned with the jumps of ϵ. The length of an edge e in $\mathcal{T}_h$ is denoted by h_e. The discrete solution space is $\mathbf{X}_h$, the space of piecewise constant vector fields in $L^2_1(D)^2$. Define the edge and element bubble spaces

$$\begin{aligned} H^h_{e,-} &= \oplus_{e \not\subseteq \Gamma_0} B^{(1)}_e, \\ H^h_{e,\diamond} &= \oplus_{e \not\subseteq \Gamma_1} B^{(1)}_e, \\ H^h_{\tau} &= \oplus_{\tau \in \mathcal{T}_h} B^{(1)}_\tau. \end{aligned} \tag{4.13}$$

The test spaces $H^1_{1,\diamond}(D)$ and $H^1_{-}(D)$ are approximated by

$$\begin{aligned} H^h_\diamond &= S^h_\diamond \oplus H^h_{e,\diamond} \oplus H^h_\tau, \\ H^h_- &= S^h_- \oplus H^h_{e,-}, \end{aligned}$$

respectively. Thus we use edge and element bubble functions in approximating $H^1_{1,\diamond}(D)$, but only edge bubble functions in $H^1_{-}(D)$. The main result of this section is that the spaces $\mathbf{X}_h$, $H^h_\diamond$, and H^h_- satisfy a discrete inf-sup condition. We first prove several lemmas pertaining to these discrete spaces. Recall that the generic constant C may depend on the regularity of the family of meshes $\mathfrak{T}$, but does not depend on the sizes h_τ or h_e.

Lemma 7 *There exists a constant* $C > 0$ *such that if* b_e *is the bubble function in*

$B_e^{(1)}$ *associated with an edge* e *in* $\mathcal{T}_h$ *not contained in* Γ_0*, then*

$$\|b_e\|^2_{L^2_1(\tau)} \le C r_\tau^{-1} \Big(\int_e r b_e \, ds \Big)^2.$$

Here τ *is either triangle having* e *as an edge and* r_τ *is the maximum value of* r *on* τ*.*

Proof. Let $\widehat{B}_e^{(1)}$ denote the edge bubble space on a reference edge of unit length. For edges not intersecting Γ_0, a standard scaling argument and the equivalence of norms on the one-dimensional space $\widehat{B}_e^{(1)}$ gives

$$\|b_e\|^2_{L^2_1(\tau)} \le r_\tau \|b_e\|^2_{L^2(\tau)} \le C r_\tau \Big(\int_e b_e \, ds \Big)^2.$$

Since e does not intersect Γ_0, r_τ can be bounded by a constant times the minimum value of r on e, so

$$\Big(\int_e b_e \, ds \Big)^2 \le C r_\tau^{-2} \Big(\int_e r b_e \, ds \Big)^2.$$

We next consider an edge e which intersects Γ_0 at a point which we denote by $\mathbf{a}_1$. There are two cases. If τ intersects Γ_0 only at $\mathbf{a}_1$ then there are constants C_0, C_1 depending on quasi-uniformity such that

$$C_0 r \le h_\tau(\lambda_2 + \lambda_3) \le C_1 r \qquad \text{for all } (r, z) \in \tau.$$

Here λ_2 and λ_3 are the barycentric coordinates of (r, z) associated with the two vertices of τ not on Γ_0. Scaling and again using equivalence of norms on $\widehat{B}_e$ gives

$$\begin{aligned} \|b_e\|^2_{L^2_1(\tau)} &\le C h_\tau \|(\lambda_1 + \lambda_2)^{1/2} b_e\|^2_{L^2(\tau)} \\ &\le C h_\tau^3 \Big(\int_{\hat{e}} (\widehat{\lambda}_2 + \widehat{\lambda}_3) \hat{b}_e \, ds \Big)^2 \le C h_\tau^{-1} \Big(\int_e r b_e \, ds \Big)^2. \end{aligned} \tag{4.14}$$

The remaining case is when τ intersects Γ_0 along the edge with endpoints, $\mathbf{a}_1$ and $\mathbf{a}_2$. Then on τ, $r = \alpha_\tau h_\tau \lambda_3$ where, because of quasi-uniformity, $0 < C_0 \le \alpha_\tau \le C_1$. Replacing $\lambda_2 + \lambda_3$ by λ_3 in (4.14) completes the proof. □

Lemma 8 *Let τ be a triangle in $\mathcal{T}_h$ and e be an edge of τ not on Γ_0. There exists a constant $C > 0$ such that for all $u \in H^1_1(\tau)$,*

$$\|u\|^2_{L^2_1(e)} \le C(h_e^{-1}\|u\|^2_{L^2_1(\tau)} + h_e\|u\|^2_{H^1_1(\tau)}).$$

Proof. By Propositions 2 and 3, $C^\infty(\overline{\tau})$ is dense in $H^1_1(\tau)$, and the trace operator is continuous from $H^1_1(\tau)$ to $L^2_1(e)$. Therefore, it suffices to prove the result for $u \in C^\infty(\overline{\tau})$. When τ does not intersect Γ_0, the result easily follows from the standard (unweighted) estimate.

We next consider the case when $e \cap \Gamma_0$ is a point. For each point $\mathbf{x} = (x_r, x_z) \in e$, we let $\eta_{\mathbf{x}}$ denote a unit vector pointing from $\mathbf{x}$ to the vertex $\mathbf{a}_3$ of τ not on e. There is a positive number α independent of h such that $\mathbf{x} + \alpha h \eta_{\mathbf{x}}$ is in τ and is of distance greater than Ch from $\mathbf{a}_3$. It follows that the value of r on the line from $\mathbf{x}$ to $\mathbf{x} + \alpha h \eta_{\mathbf{x}}$ is bounded above and below by a constant (independent of h) multiple of x_r. We write

$$u(\mathbf{x})^2 = -\int_0^t \frac{\partial}{\partial y}(u^2)(\mathbf{x} + y\eta_{\mathbf{x}})\,dy + u(\mathbf{x} + t\eta_{\mathbf{x}})^2 \text{ for all } 0 < t < \alpha h.$$

Multiplying the above equation by $r = x_r$, using the above equivalence and integrating over e gives

$$\begin{aligned}\|u\|^2_{L^2_1(e)} \le C \int_e \int_0^t (\mathbf{x}(s) + y\eta_{\mathbf{x}})_r \left| \frac{\partial}{\partial y}(u^2)(\mathbf{x}(s) + y\eta_{\mathbf{x}}) \right| dy\,ds \\ + C \int_e (\mathbf{x}(s) + t\eta_{\mathbf{x}})_r u(\mathbf{x}(s) + t\eta_{\mathbf{x}})^2\,ds.\end{aligned}$$

By quasi-uniformity, the angle between e and $\eta_{\mathbf{x}}$ does not degenerate, so changing the integration variable and applying the Schwarz inequality along with integration over t in $(0, \alpha h)$ gives

$$\alpha h \|u\|^2_{L^2_1(e)} \le C(h\|u\|_{L^2_1(\tau)}\|\nabla u\|_{L^2_1(\tau)} + \|u\|^2_{L^2_1(\tau)}),$$

from which the desired bound immediately follows.

The same argument handles the remaining case as well. When τ intersects Γ_0 and e does not, r behaves like h_τ on e. Fortunately, since $\mathbf{x}+\alpha h_\tau \eta_{\mathbf{x}}$ stays away from $e_3 \in \Gamma_0$, $(\mathbf{x}+y\eta_{\mathbf{x}})_r$ behaves like h_τ for $0 \le y \le \alpha h_\tau$. The above argument gives the desired result. This completes the proof of the lemma. □

Lemma 9 *There exists a constant $C > 0$ independent of h such that*

$$\sup_{\phi \in H^1_-(D)} \frac{(\mathbf{v}, \nabla_r \times \phi)_r}{\|\phi\|_{H^1_-(D)}} \le C \sup_{\phi \in H^h_-} \frac{(\mathbf{v}, \nabla_r \times \phi)_r}{\|\phi\|_{H^1_-(D)}} \text{ for all } \mathbf{v} \in \mathbf{X}_h.$$

Proof. By Lemma 4, there is a Clement-like projector $\Pi^h_- : H^1_-(D) \to S^h_-$ satisfying

$$h_\tau^{-2}\|\phi - \Pi^h_-\phi\|^2_{L^2_1(\tau)} + \|\phi - \Pi^h_-\phi\|^2_{H^1_-(\tau)} \le C\|\phi\|^2_{H^1_-(\Delta_\tau)}, \tag{4.15}$$

where Δ_τ denotes the union of all triangles in $\mathcal{T}_h$ sharing a common vertex with τ.

Let $\mathbf{v} \in \mathbf{X}_h$ and $\phi \in H^1_-(D)$ be given, and set $\psi = \phi - \Pi^h_-\phi$. For each edge e in $\mathcal{T}_h$ not contained in Γ_0, define $w_e \in B_e$ by $\int_e r w_e\, ds = \int_e r\psi\, ds$. Set $q_e = \sum_{e \notin \Gamma_0} w_e$. The above is constructed so that $\phi_h = \Pi^h_-\phi + q_e$ satisfies

$$(\mathbf{v}, \nabla_r \times \phi)_r = (\mathbf{v}, \nabla_r \times \phi_h)_r. \tag{4.16}$$

Indeed, since $\mathbf{v}$ is piecewise constant, $\mathbf{v}\cdot\mathbf{t}$ is constant on each edge and (2.15) gives

$$\begin{aligned}(\mathbf{v}, \nabla_r \times \psi)_{r,\tau} &=< \mathbf{v}\cdot\mathbf{t}, \psi >_{r,\partial\tau\setminus\Gamma_0} \\ &=< \mathbf{v}\cdot\mathbf{t}, q_e >_{r,\partial\tau\setminus\Gamma_0}= (\mathbf{v}, \nabla_r \times q_e)_{r,\tau}.\end{aligned}$$

Summing over all $\tau \in \mathcal{T}_h$ yields (4.16). Thus, the lemma will follow if we show that

$$\|\phi_h\|_{H^1_-(D)} \le C\|\phi\|_{H^1_-(D)}.$$

By (4.15), it suffices to show

$$\|q_e\|_{H^1_-(D)} \le C\|\phi\|_{H^1_-(D)}. \tag{4.17}$$

By Lemma 2, for any polynomial $f \in P_k(\tau)$ vanishing on Γ_0 when $\tau \cap \Gamma_0 \neq \emptyset$,

$$\|f\|_{L^2_{-1}(\tau)} \le C h_\tau^{-1} \|f\|_{L^2_1(\tau)}. \tag{4.18}$$

Thus,

$$\|q_e\|^2_{L^2_{-1}(D)} \le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} \|w_e\|^2_{L^2_{-1}(\tau)} \le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} h_\tau^{-2}\|w_e\|^2_{L^2_1(\tau)}.$$

Applying Lemma 7 gives

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} h_\tau^{-2}\|w_e\|^2_{L^2_1(\tau)} &\le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} r_\tau^{-1} h_\tau^{-2} \left(\int_e r w_e \, ds \right)^2 \\
&= C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} r_\tau^{-1} h_\tau^{-2} \left(\int_e r \psi \, ds \right)^2 \\
&\le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} h_\tau^{-1} \|\psi\|^2_{L^2_1(e)}.
\end{aligned} \tag{4.19}$$

Combining the above inequalities with Lemma 8 and (4.15) gives

$$\sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} h_\tau^{-2}\|w_e\|^2_{L^2_1(\tau)} \le C(h_\tau^{-2}\|\psi\|^2_{L^2_1(D)} + \|\psi\|^2_{H^1_1(D)}) \le C\|\phi\|^2_{H^1_-(D)} \tag{4.20}$$

from which it follows that

$$\|q_e\|_{L^2_{-1}(D)} \le C\|\phi\|_{H^1_-(D)}.$$

By Lemma 3, for any polynomial $f \in P_k(\tau)$ we have

$$\|f\|_{H^1_1(\tau)} \le C h_\tau^{-1} \|f\|_{L^2_1(\tau)}. \tag{4.21}$$

Thus

$$\|q_e\|^2_{H^1_1(D)} \le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} \|w_e\|^2_{H^1_1(\tau)} \le C \sum_{\tau\in\mathcal{T}_h} \sum_{\substack{e\subset\overline{\tau} \\ e\not\subseteq\Gamma_0}} h_\tau^{-2}\|w_e\|^2_{L^2_1(\tau)}. \tag{4.22}$$

Applying (4.20) shows

$$\|q_e\|^2_{H^1_1(D)} \le C\|\phi\|_{H^1_-(D)}.$$

This completes the proof of the Lemma. □

The analogous result for the second term in the discrete inf-sup condition is contained in the following lemma. Its proof is given later.

Lemma 10 *There exists a constant $C > 0$ independent of h such that*

$$\sup_{q\in H^1_{1,\diamond}(D)} \frac{(\epsilon\mathbf{v},\nabla q)_r}{\|q\|_{H^1_1(D)}} \le C \sup_{q\in H^h_\diamond} \frac{(\epsilon\mathbf{v},\nabla q)_r}{\|q\|_{H^1_1(D)}} \text{ for all } \mathbf{v}\in\mathbf{X}_h.$$

In the proof of Lemma 9, we applied the Green's formula (2.15) with $\mathbf{v}$ piecewise constant. In this case, $\mathbf{v}$ is in the appropriate space, namely $H^1_1(\tau)^2$. Similarly, the proof of Lemma 10 will require the Green's formula for the divergence, but (2.16) does not apply since constant vectors $\mathbf{v}$ do not satisfy $v_r \in H^1_-(\tau)$ for triangles τ which intersect Γ_0. However, the Green's formula for the divergence does hold in the specific cases we require for proving Lemma 10. This is stated precisely in the following lemma.

Lemma 11 *If τ is a triangle with no edges contained in Γ_0 (i.e., $\overline{\tau}\cap\Gamma_0$ is empty or a vertex), then*

$$(\nabla_r\cdot\mathbf{v},\phi)_{r,\tau} = -(\mathbf{v},\nabla\phi)_{r,\tau} + <\mathbf{v}\cdot\mathbf{n},\phi>_{r,\partial\tau} \tag{4.23}$$

for all $\phi\in H^1_1(D)$ and constant $\mathbf{v}\in\mathbb{R}^2$.

Proof. The case when τ does not intersect Γ_0 is already contained in Lemma 1. Suppose that τ intersects Γ_0 at a point $\mathbf{a}$. When ϕ is $C^1(\tau)$, the above formula follows immediately from the divergence theorem, i.e.,

$$\int_\tau \nabla \cdot (r\phi \mathbf{v})\, dx = \int_{\partial\tau} r\phi \mathbf{v} \cdot \mathbf{n}\, ds.$$

Thus, by density, it suffices to show that each term in (4.23) is bounded for $\phi \in H_1^1(\tau)$. Applying the Schwarz inequality gives

$$|(\mathbf{v}, \nabla\phi)_{r,\tau}| \le \|\mathbf{v}\|_{L_1^2(\tau)} \|\nabla\phi\|_{L_1^2(\tau)}.$$

It follows from Lemma 8 that

$$\begin{aligned} |<\mathbf{v}\cdot\mathbf{n}, \phi>_{r,\partial\tau}| &\le \|\mathbf{v}\cdot\mathbf{n}\|_{L_1^2(\partial\tau)} \|\phi\|_{L_1^2(\partial\tau)} \\ &\le C\|\mathbf{v}\cdot\mathbf{n}\|_{L_1^2(\partial\tau)} (h_\tau^{-1}\|\phi\|^2_{L_1^2(\tau)} + h_\tau \|\phi\|^2_{H_1^1(\tau)})^{1/2}. \end{aligned}$$

Since $\mathbf{v}$ is a constant and the width of τ in the z-direction is Cr for some constant C depending on an angle, it follows from the Schwarz inequality that

$$|(\nabla_r \cdot \mathbf{v}, \phi)_{r,\tau}| = \left| \int_\tau v_r \phi\, dx \right| \le C(\tau)|v_r| \|\phi\|_{L_1^2(\tau)}. \tag{4.24}$$

Thus all of the terms in (4.23) are bounded, which completes the proof of the lemma.

□

Proof. [Proof of Lemma 10] Given $\phi \in H^1_{1,\diamond}(D)$ and $\mathbf{v} \in \mathbf{X}_h$, we shall construct $\phi_h \in H^h_\diamond$ satisfying

$$\begin{aligned} (\mathbf{v}, \nabla\phi)_r &= (\mathbf{v}, \nabla\phi_h)_r \\ \|\phi_h\|_{H_1^1(D)} &\le C\|\phi\|_{H_1^1(D)}. \end{aligned} \tag{4.25}$$

Again, we use a Clement-like operator $\Pi^h_\diamond : H^1_{1,\diamond}(D) \to S^h_\diamond$ satisfying (see Lemma 4)

$$h_\tau^{-2}\|\phi - \Pi^h_\diamond\phi\|^2_{L_1^2(\tau)} + \|\phi - \Pi^h_\diamond\phi\|^2_{H_1^1(\tau)} \le C\|\phi\|^2_{H_1^1(\Delta_\tau)}. \tag{4.26}$$

Set $\psi = \phi - \Pi^h_\diamond \phi$. As in Lemma 9, for each edge e in $\mathcal{T}_h$ not contained in ∂D, define $w_e \in B_e$ by $\int_e r w_e \, ds = \int_e r\psi \, ds$. Set $q_e = \sum_e w_e$, and note that Lemma 11 gives

$$\begin{aligned}(\mathbf{v}, \nabla\psi)_{r,\tau} &= -v_r \int_\tau \psi \, dx + < \mathbf{v}\cdot\mathbf{n}, \psi >_{r,\partial\tau} \\ &= -v_r \int_\tau \psi \, dx + < \mathbf{v}\cdot\mathbf{n}, q_e >_{r,\partial\tau} .\end{aligned}$$

for all triangles τ without an edge on Γ_0. On each such triangle τ, define w_τ in B_τ by

$$\int_\tau w_\tau \, dx = \int_\tau (\psi - q_e) \, dx.$$

Then

$$(\mathbf{v}, \nabla\psi)_{r,\tau} = -v_r \int_\tau (w_\tau + q_e) \, dx + < \mathbf{v}\cdot\mathbf{n}, w_\tau + q_e >_{r,\partial\tau} = (\mathbf{v}, \nabla(w_\tau + q_e))_{r,\tau}. \quad (4.27)$$

Accordingly, we define $\phi_h = \Pi^h_\diamond \phi + q_e + w_\tau$ on τ when τ does not have an edge on Γ_0.

To deal with the remaining triangles, we have to avoid integration by parts in the r direction with terms involving ψ. Specifically, for a triangle τ with an edge on Γ_0, we choose the bubble function $w_{e_0} \in B_{e_0}$ corresponding to e_0, the edge of τ on Γ_0, so that

$$(\mathbf{v}, \nabla\phi)_{r,\tau} = (\mathbf{v}, \nabla(\Pi^h_\diamond \phi + q_e + w_{e_0}))_{r,\tau}. \quad (4.28)$$

We shall see that this is indeed possible. Let $\tilde{\phi}_h = \Pi^h_\diamond \phi + q_e$ and $\phi_h = \tilde{\phi}_h + w_{e_0}$, with $w_{e_0} \in B_{e_0}$ to be determined. We have

$$\begin{aligned}(1, \frac{\partial\phi}{\partial z})_{r,\tau} &= (1, \frac{\partial\phi_h}{\partial z})_{r,\tau} + \int_{\partial\tau\setminus\Gamma_0} n_z r(\phi - \tilde{\phi}_h) \, ds, \\ (1, \frac{\partial w_{e_0}}{\partial r})_{r,\tau} &= -\int_\tau w_{e_0} \, dx.\end{aligned}$$

Here n_z denotes the z-component of $\mathbf{n}$. Using the first equality above and the definition of q_e gives

$$\left(v_z, \frac{\partial(\phi - \phi_h)}{\partial z}\right)_{r,\tau} = \int_{\partial\tau\setminus\Gamma_0} v_z n_z r(\phi - \tilde{\phi}_h)\, ds$$
$$= \int_{\partial\tau\setminus\Gamma_0} v_z n_z r(\psi - q_e)\, ds = 0.$$

Thus

$$(\mathbf{v}, \nabla\phi)_{r,\tau} = (\mathbf{v}, \nabla\phi_h)_{r,\tau} + v_r\bigg(\int_\tau r\frac{\partial}{\partial r}(\phi - \tilde{\phi}_h)\, dx + \int_\tau w_{e_0}\, dx\bigg). \tag{4.29}$$

Choosing $w_{e_0} \in B_{e_0}$ so that

$$\int_\tau w_{e_0}\, dx = \int_\tau r\frac{\partial}{\partial r}(\tilde{\phi}_h - \phi)\, dx \tag{4.30}$$

gives (4.28) on triangles τ with an edge on Γ_0. Combining (4.27) and (4.28) gives the equality of (4.25).

To complete the proof, we need only show that the inequality of (4.25) holds. The arguments in the proof of Lemma 9, together with (4.26), give

$$\|q_e\|_{H^1_1(\tau)} \le C\|\phi\|_{H^1_1(\Delta_\tau)} \qquad \text{and} \qquad \|q_e\|_{L^2_1(\tau)} \le Ch_\tau\|\phi\|_{H^1_1(\Delta_\tau)}, \tag{4.31}$$

for all triangles $\tau \in \mathcal{T}_h$. For the remaining terms, there are two cases.

First, we consider τ such that τ does not intersect Γ_0 on an edge. Applying a scaling argument and (4.21) gives

$$\begin{aligned}\|w_\tau\|^2_{H^1_1(\tau)} &\le Ch_\tau^{-2}\|w_\tau\|^2_{L^2_1(\tau)} \le Cr_\tau h_\tau^{-4}\bigg(\int_\tau w_\tau\, dx\bigg)^2 \\ &= Cr_\tau h_\tau^{-4}\bigg(\int_\tau (\psi - q_e)\, dx\bigg)^2.\end{aligned} \tag{4.32}$$

If τ intersects Γ_0, we use (4.31), (4.26), and the proof of Lemma 11 to obtain

$$\begin{aligned}\left(\int_\tau (\psi - q_e)\,dx\right)^2 &\leq C r_\tau \|\psi - q_e\|^2_{L^2_1(\tau)} \\ &\leq C r_\tau h_\tau^2 \|\phi\|^2_{H^1_1(\Delta_\tau)},\end{aligned}$$

from which it follows that

$$\|w_\tau\|_{H^1_1(\tau)} \leq C\|\phi\|_{H^1_1(\Delta_\tau)}. \tag{4.33}$$

Otherwise, if $\tau \cap \Gamma_0 = \phi$, then the regularity of the family of meshes $\mathfrak{T}$ implies that r_τ is bounded by a constant multiple of the minimal value of r on τ. Hence

$$\left(\int_\tau (\psi - q_e)\,dx\right)^2 \leq C r_\tau^{-1} h_\tau^2 \|\psi - q_e\|^2_{L^2_1(\tau)} \leq C r_\tau^{-1} h_\tau^4 \|\phi\|^2_{H^1_1(\Delta_\tau)}$$

and (4.33) immediately follows.

We finally consider triangles τ with edges on Γ_0. As in (4.32),

$$\|w_{e_0}\|^2_{H^1_1(\tau)} \leq C h_\tau^{-3}\left(\int_\tau w_{e_0}\,dx\right)^2 \leq C\|\psi - q_e\|^2_{H^1_1(\tau)}, \tag{4.34}$$

where we used (4.30) and the Schwarz inequality to get the second inequality above. The estimate

$$\|w_{e_0}\|_{H^1_1(\tau)} \leq C\|\phi\|_{H^1_1(\Delta_\tau)} \tag{4.35}$$

follows easily from the above inequalities and (4.26). Combining (4.26), (4.31), (4.33), and (4.35) proves the inequality in (4.25). This completes the proof of the lemma. □

As a result of Lemmas 9 and 10 and Theorem 5, we have that the pair of approximation spaces $\mathbf{X}_h$ and $(H^h_-, H^h_\diamond)$ is stable. The inf-sup condition given in the following theorem yields existence and uniqueness of solutions to the discrete least-squares problem.

Theorem 6 *There exists a constant $C > 0$, independent of h and ϵ, such that*

$$\|\epsilon^{1/2}\mathbf{v}\|_{L^2_1(D)^2} \le C\ln(h^{-1})^{1/2}\left(\sup_{(\phi,q)\in H^h_{-}\times H^h_{\diamond}} \frac{(\mathbf{v},\nabla_r\times\phi)_r}{\|\epsilon^{-1/2}\nabla_r\times\phi\|_r} + \frac{(\epsilon\mathbf{v},\nabla q)_r}{\|\epsilon^{1/2}\nabla q\|_r}\right)$$

for all $\mathbf{v}\in\mathbf{X}_h$.

In the case of a constant coefficient ϵ, Theorem 6 holds without the factor of $\ln(h^{-1})^{1/2}$. Although Lemmas 9 and 10 are stated and proved with norms not weighted by ϵ, when ϵ has jumps away from Γ_0 one can construct Clément operators satisfying (4.15) and (4.26) with ϵ-weighted norms and C replaced by $C\ln(h^{-1})$ (see [8]). This gives the appropriate estimates with ϵ-weighted norms, as in Theorem 6. Indeed, the estimates of the bubble functions in the proofs of Lemmas 9 and 10 are made on individual elements where the coefficient ϵ is constant, so the ϵ-weighted estimates follow.

C. Discrete Least-Squares System

Let $\mathbf{X}_h$ and $Y_h = H^h_{-}\times H^h_{\diamond}$ be the stable approximation pair introduced in the previous section and define the operator $B_h : \mathbf{X}_h \to Y_h'$ by

$$< B_h\mathbf{x}, y >= b(\mathbf{x}, y) \text{ for all } \mathbf{x}\in\mathbf{X}_h, y\in Y_h.$$

Further define $T_{Y_h} : Y_h' \to Y_h$ by $(T_{Y_h}f, y)_{Y_h} =< f, y >$ for all $y \in Y_h$. Then the discrete least-squares problem is to find $\mathbf{x}_h \in \mathbf{X}_h$ such that

$$A_h(\mathbf{x}_h,\mathbf{x}) \equiv< B_h\mathbf{x}_h, T_{Y_h}B_h\mathbf{x} >=< F, T_{Y_h}B_h\mathbf{x} > \text{ for all } \mathbf{x}\in\mathbf{X}_h. \tag{4.36}$$

Using Theorem 6, it is easy to see that

$$C_0(h)A_h(\mathbf{x},\mathbf{x}) \le (\epsilon\mathbf{x},\mathbf{x})_r \le C_1(h)A_h(\mathbf{x},\mathbf{x}) \qquad \text{for all } \mathbf{x}\in\mathbf{X}_h \tag{4.37}$$

with constants $C_0(h)$ and $C_1(h)$ satisfying $C_1(h)/C_0(h) \le C\ln(h^{-1})$. Accordingly, the discrete system corresponding to (4.36) can be uniformly preconditioned by the matrix corresponding to $(\epsilon\cdot,\cdot)_r$ on $\mathbf{X}_h$. The following theorem is an immediate consequence of Theorem 6 (see [7]).

Theorem 7 *The problem* (4.36) *has a unique solution which satisfies*

$$\|\mathbf{e}-\mathbf{x}_h\|_r \le C\ln(h^{-1})^{1/2} \inf_{\mathbf{v}\in\mathbf{X}_h} \|\mathbf{e}-\mathbf{v}\|_r.$$

Here $\mathbf{e}$ *is the unique solution of* (3.1).

Remark 4 *When ϵ is constant, the above theorem holds without the factor of* $\ln(h^{-1})^{1/2}$ *and immediately implies that when the solution is in $H_1^s(D)^2$, for $0 < s \le 1$, the convergence rate will be at least of order s.*

In our computations, instead of T_{Y_h} we use a preconditioner $T^h : Y_h' \to Y_h$ defined by $T^h f = (T_-^h f, T_+^h f)$, where the action of T_-^h and T_+^h are computed as follows. We assemble matrices A_- and A_+ in the bases of H_-^h and $H_\diamond^h$ according to the partition

$$A_- = \begin{bmatrix} A_-^{bb} & A_-^{bl} \\ A_-^{lb} & A_-^{ll} \end{bmatrix},\ A_+ = \begin{bmatrix} A_+^{bb} & A_+^{bl} \\ A_+^{lb} & A_+^{ll} \end{bmatrix},$$

where b indicates the space of bubble functions and l the space of piecewise linear functions. The bilinear forms defining A_- and A_+ are, respectively,

$$\begin{aligned} a_-(u,v) &= \int_D \frac{1}{r\epsilon}\partial_r(ru)\partial_r(rv)\,dr\,dz + \int_D \frac{r}{\epsilon}\partial_z u\partial_z v\,dr\,dz, \\ a_+(u,v) &= \int_D r\epsilon(\partial_r u\partial_r v + \partial_z u\partial_z v)\,dr\,dz. \end{aligned}$$

It is shown in [10] that these bilinear forms are continuous and coercive on their respective spaces. Multigrid is used to compute approximate inverses M_-^{ll} and M_+^{ll} to

A_-^{ll} and A_+^{ll}. Point Gauss-Seidel smoothing is used in the multigrid (cf. [10]). The vector $T_-^h f = x = (x_b, x_l)^t \in Y_h$ is computed via the algorithm

1. $x_b \leftarrow$ Forward Gauss-Seidel iteration on A_-^{bb} with data f_b
2. $x_l = M_-^{ll}(f_l - A_-^{lb} x_b)$
3. $x_b \leftarrow x_b +$ Backward Gauss-Seidel iteration on A_-^{bb} with data $f_b - A_-^{bl} x_l$.

The vector $T_+^h f$ is computed similarly. This defines the preconditioner T^h. The weights in $a_-(\cdot,\cdot)$ and $a_+(\cdot,\cdot)$ are chosen in accordance with Theorem 6. Consequently, preconditioning (4.36) with the $(\epsilon\cdot,\cdot)_r$ mass matrix ensures that the iterative convergence rate is independent of ϵ. Note that this mass matrix is diagonal, as there is no interaction between basis functions of the piecewise constant space X_h. According to the theory of [7], the spectral equivalence of T^h and T_{Y_h} ensures that the method (4.36) with T^h will give quasi-optimal convergence. When the solution is in $H_1^1(D)^2$, the convergence rate will be of first order.

CHAPTER V

MAGNETOSTATICS

We now consider the Maxwell system for magnetostatics, given by

$$\begin{cases} \nabla \times \mathbf{h} & = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot (\mu \mathbf{h}) & = g \text{ in } \Omega \\ \mathbf{h} \cdot \mathbf{n} & = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.1}$$

Here, $\mathbf{h}$ is the magnetic field and the coefficient μ is the magnetic permeability. As in the electrostatic case, inhomogeneous boundary conditions can be treated by modifying the data of the weak formulation. We assume that μ is positive and piecewise constant. In contrast to ϵ, we assume that μ is well behaved enough so that for some $s > 0$, we have the regularity $u \in H^{1+s}(\Omega)$ for solutions of the Neumann problem

$$\begin{aligned} -\nabla \cdot \epsilon \nabla u = f \quad & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

where $f \in L^2(\Omega)$ satisfies the condition $\int_\Omega f \, dx = 0$.

The magnetostatic system (5.1) differs from the electrostatic system of the previous chapter only in the boundary conditions. Consequently, the analysis of the least-squares method will be similar, but different spaces will be used. In these spaces, continuous and discrete inf-sup conditions shall be proved, similar to the previous chapter.

Assuming the system (5.1) is axisymmetric (see Chapter IV), the system reduces to the two decoupled systems

$$\begin{cases} \nabla \times (h_r, h_z) &= f_\theta \quad \text{in } D \\ \nabla_r \cdot (\mu(h_r, h_z)) &= g \quad \text{in } D \\ (h_r, h_z) \cdot (n_r, n_z) &= 0 \quad \text{on } \Gamma_1, \end{cases} \tag{5.3}$$

and

$$\begin{cases} -\frac{\partial h_\theta}{\partial z} &= f_r \quad \text{in } D \\ \frac{1}{r}\frac{\partial}{\partial r}(r h_\theta) &= f_z \quad \text{in } D. \end{cases} \tag{5.4}$$

As was done for the electrostatic problem in Chapter IV, taking the curl of (5.4) yields a variational problem with a coercive bilinear form on $H^1_-(D)$. Although system (5.4) for the azimuthal component h_θ has no boundary condition, the bilinear form is coercive on $H^1_-(D)$ because functions in $H^1_-(D)$ have a vanishing trace on Γ_0 (see [10]). In this way, a boundary condition is implied by the function space. Thus multigrid can be applied to solve (5.4) for the azimuthal component h_θ. In this chapter, we shall only solve (5.3) for the meridian components (h_r, h_z).

A. The Least-Squares Formulation

Applying Green's formulas (Lemma 1) yields the following weak formulation of (5.3): Find $\mathbf{h} \in L^2_1(D)^2$ satisfying

$$b(\mathbf{h}, (\phi, q)) \equiv (\mathbf{h}, \nabla_r \times \phi)_r + (\mu \mathbf{h}, \nabla q)_r = (f_\theta, \phi)_r - (g, q)_r, \tag{5.5}$$

for all $(\phi, q) \in H^1_{-,0}(D) \times H^1_1(D)$. Define the operators $\text{curl}_2 : L^2_1(D)^2 \to H^1_{-,0}(D)'$, $\text{div}_\mu : L^2_1(D)^2 \to H^1_1(D)'$, and $B : L^2_1(D)^2 \to (H^1_{-,0}(D) \times H^1_1(D))'$ by

$$\begin{aligned} < \text{curl}_2\, \mathbf{v}, \psi >= (\mathbf{v}, \nabla_r \times \psi)_r \quad & \text{for all } \mathbf{v} \in L^2_1(D)^2, \psi \in H^1_{-,0}(D), \\ < \text{div}_\mu\, \mathbf{v}, \psi >= (\mu \mathbf{v}, \nabla \psi)_r \quad & \text{for all } \mathbf{v} \in L^2_1(D)^2, \psi \in H^1_1(D), \\ B\mathbf{v} = (\text{curl}_2\, \mathbf{v}, \text{div}_\mu\, \mathbf{v}) \quad & \text{for all } \mathbf{v} \in L^2_1(D)^2. \end{aligned} \tag{5.6}$$

Thus B satisfies $(B\mathbf{h},(\phi,q)) = b(\mathbf{h},(\phi,q))$ for all $\mathbf{h} \in L^2_1(D)^2$ and $(\phi,q) \in H^1_{-,0}(D) \times H^1_1(D)$. Further, define the symmetric bilinear form A on $L^2_1(D)^2 \times L^2_1(D)^2$ and the linear functional F by

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) &\equiv (B\mathbf{u}, B\mathbf{v})_{(H^1_{-,0}(D)\times H^1_1(D))'} \\ &= (\operatorname{curl}_2 \mathbf{u}, \operatorname{curl}_2 \mathbf{v})_{H^1_{-,0}(D)'} + (\operatorname{div}_\mu \mathbf{u}, \operatorname{div}_\mu \mathbf{v})_{H^1_1(D)'}, \\ < F, \mathbf{v} > &\equiv (f_\theta, \operatorname{curl}_2 \mathbf{v})_{H^1_{-,0}(D)'} - (g, \operatorname{div}_\mu \mathbf{v})_{H^1_1(D)'}. \end{aligned}$$

Then the dual based least-squares formulation of (5.5) is to find $\mathbf{h} \in L^2_1(D)^2$ satisfying

$$A(\mathbf{h},\mathbf{v}) =< F, \mathbf{v} > \text{ for all } \mathbf{v} \in L^2_1(D)^2. \tag{5.7}$$

To show that the linear operator B and its inverse are bounded, we shall use the following orthogonal decomposition for the space $L^2_1(D)^2$.

Lemma 12 *For any $\mathbf{v} \in L^2_1(D)^2$, there exist $\phi \in H^1_{-,0}(D)$ and $q \in H^1_1(D)$ satisfying $\mathbf{v} = \nabla_r \times \phi + \mu \nabla q$.*

Proof. By density, it suffices to prove the result for $\mathbf{v} \in (C^\infty_0(D))^2$. The proof is entirely similar to the proof of Lemma 5 with only one modification. Instead of defining $\breve{q} \in H^1_0(\Omega)$ as the solution of (4.9), we define $\breve{q} \in H^1(\Omega)/\mathbb{R}$ as the unique solution (see, e.g., [9]) of

$$(\mu\nabla\breve{q}, \nabla\eta)_{L^2(\Omega)} = (\breve{\mathbf{v}}, \nabla\eta)_{L^2(\Omega)} = -(\nabla\cdot\breve{\mathbf{v}}, \eta)_{L^2(\Omega)}, \qquad \text{for all } \eta \in H^1(\Omega)/\mathbb{R}.$$

Since $\breve{\mathbf{v}}$ has compact support, $\breve{\mathbf{v}} \cdot \mathbf{n} = 0$ and $\int_\Omega \nabla \cdot \breve{\mathbf{v}}\, dx = 0$, by the Divergence Theorem. Thus $\breve{q}$ solves a Neumann problem of the form (5.2) and has the regularity $\breve{q} \in H^{1+s}(\Omega)$. Setting

$$\breve{\mathbf{w}} = \breve{\mathbf{v}} - \mu\nabla\breve{q},$$

we have that $\breve{\mathbf{w}}$ is divergence-free in Ω. Hence the trace $\mathbf{w} = \mathbf{v} - \mu\nabla q$ of $\breve{\mathbf{w}}$ on D satisfies $\nabla_r \cdot \mathbf{w} = 0$. Although $\mathbf{w}$ is not necessarily in $H^1_-(D) \times H^1_1(D)$, $\nabla_r \cdot \mathbf{w} = 0$ implies that the Green's formula (2.16) holds for $\mathbf{w}$ (by a simple density argument). Thus we have

$$< \mathbf{w} \cdot \mathbf{n}, \phi >_{r,\Gamma_1} = (\mathbf{w}, \nabla\phi)_r = (\mathbf{v} - \mu\nabla q, \nabla\phi)_r = 0$$

for all ϕ in $H^1_1(D)$. Therefore, $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ_1.

It follows that ϕ, defined in (4.10), is in $H^1_{-,0}(D)$. Indeed, the path γ is independent of the starting point $\mathbf{a}_0$ in Γ_0, so we may take $\mathbf{a}_0$ to be an endpoint of Γ_0. Taking the path γ to be contained in Γ_1 yields $\phi = 0$ at all points in Γ_1. The remainder of the proof proceeds the same as for Lemma 5. □

The analogue of Theorem 5 now follows in the same manner.

Theorem 8 *For all* $\mathbf{v} \in L^2_1(D)^2$,

$$(\mu\mathbf{v}, \mathbf{v})_r = \sup_{(\phi,q) \in H^1_{-,0}(D) \times H^1_1(D)} \frac{(\mathbf{v}, \nabla_r \times \phi)^2_r}{\|\mu^{-1/2}\nabla_r \times \phi\|^2_r} + \frac{(\mu\mathbf{v}, \nabla q)^2_r}{\|\mu^{1/2}\nabla q\|^2_r}.$$

As in Chapter IV, Theorem 8 implies that the weak formulation (5.5) and the least-squares problem (5.7) both have unique solutions, which coincide.

B. Stable Approximation

In this section, discrete subspaces will be introduced and analogues of Lemmas 9 and 10 will be established. Let $\mathcal{T}_h$ be a triangulation of D aligned with the jumps of μ and satisfying Assumption 2. The discrete solution space is again $\mathbf{X}_h$, the space of piecewise constant vector fields in $L^2_1(D)^2$. Define the edge and element bubble spaces

$$H^h_{e,0} = \oplus_{e \not\subseteq \partial D} B^{(1)}_e,$$
$$H^h_e = \oplus_{e \in \mathcal{T}_h} B^{(1)}_e. \tag{5.8}$$

The test spaces $H^1_1(D)$ and $H^1_{-,0}(D)$ are approximated by

$$H^h = S^h \oplus H^h_e \oplus H^h_\tau,$$
$$H^h_{-,0} = S^h_0 \oplus H^h_{e,0},$$

respectively. Thus we use edge and element bubble functions in approximating $H^1_1(D)$, but only edge bubble functions in $H^1_{-,0}(D)$. These discrete test spaces are chosen in such a way that a discrete inf-sup condition can be proved.

Lemma 13 *There exists a constant $C > 0$ independent of h such that*

$$\sup_{\phi \in H^1_{-,0}(D)} \frac{(\mathbf{v}, \nabla_r \times \phi)_r}{\|\phi\|_{H^1_-(D)}} \leq C \sup_{\phi \in H^h_{-,0}} \frac{(\mathbf{v}, \nabla_r \times \phi)_r}{\|\phi\|_{H^1_-(D)}} \text{ for all } \mathbf{v} \in \mathbf{X}_h.$$

Lemma 14 *There exists a constant $C > 0$ independent of h such that*

$$\sup_{q \in H^1_1(D)} \frac{(\mu\mathbf{v}, \nabla q)_r}{\|q\|_{H^1_1(D)}} \leq C \sup_{q \in H^h} \frac{(\mu\mathbf{v}, \nabla q)_r}{\|q\|_{H^1_1(D)}} \text{ for all } \mathbf{v} \in \mathbf{X}_h.$$

The proofs of Lemmas 13 and 14 are nearly identical to the proofs of Lemmas 9 and 10. Indeed, the only differences are in the boundary conditions. The appropriate Clement operators on $H^1_{-,0}(D)$ and $H^1_1(D)$ are given by Lemma 4. The discrete inf-sup condition now follows.

Theorem 9 *There exists a constant $C > 0$, independent of h and μ, such that*

$$\|\mu^{1/2}\mathbf{v}\|_{L^2_1(D)^2} \leq C \ln(h^{-1})^{1/2} \left(\sup_{(\phi,q) \in H^h_{-,0} \times H^h} \frac{(\mathbf{v}, \nabla_r \times \phi)_r}{\|\mu^{-1/2}\nabla_r \times \phi\|_r} + \frac{(\mu\mathbf{v}, \nabla q)_r}{\|\mu^{1/2}\nabla q\|_r} \right)$$

for all $\mathbf{v} \in \mathbf{X}_h$.

Thus the discrete least-squares problem yields a linear system with the same properties as that of the electrostatic case, and can be solved in a similar manner.

CHAPTER VI

TIME-HARMONIC SYSTEMS

As a final application, we consider the time-harmonic Maxwell equations, which are given by

$$\begin{cases} \nabla\times\mathbf{H} &= \lambda\epsilon\mathbf{E}+\mathbf{J} & \text{in } \Omega \\ \nabla\times\mathbf{E} &= -\lambda\mu\mathbf{H}+\mathbf{M} & \text{in } \Omega \\ \mu\mathbf{H}\cdot\mathbf{n} &= 0 & \text{on } \partial\Omega \\ \mathbf{E}\times\mathbf{n} &= 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

with $\lambda = -i\omega$. Here, $i = \sqrt{-1}$ is the imaginary unit and $\omega > 0$ is the temporal frequency of the radiation (cf. [13]). The magnetic and electric fields $\mathbf{H}$ and $\mathbf{E}$, respectively, are complex-valued vector fields. The data $\mathbf{J}$ and $\mathbf{M}$ are complex-valued vector fields representing electric and magnetic current densities. If there exists a solution $(\mathbf{H},\mathbf{E})$ to the homogeneous system

$$\begin{cases} \nabla\times\mathbf{H} &= \lambda\epsilon\mathbf{E} & \text{in } \Omega \\ \nabla\times\mathbf{E} &= -\lambda\mu\mathbf{H} & \text{in } \Omega \\ \mu\mathbf{H}\cdot\mathbf{n} &= 0 & \text{on } \partial\Omega \\ \mathbf{E}\times\mathbf{n} &= 0 & \text{on } \partial\Omega, \end{cases}$$

then λ is said to be a *Maxwell eigenvalue.* In order to solve the inhomogeneous system (6.1), it is necessary to assume that λ is not a Maxwell eigenvalue.

Notice that if $(\mathbf{H},\mathbf{E})$ is a solution to (6.1) then $(\mathbf{h},\mathbf{e},\mathbf{j},\mathbf{m}) = (\Re(\mathbf{H}),\Im(\mathbf{E}),\Re(\mathbf{J}),$

$\Im(\mathbf{M})$) and $(\mathbf{h}, \mathbf{e}, \mathbf{j}, \mathbf{m}) = (-\Im(\mathbf{H}), \Re(\mathbf{E}), -\Im(\mathbf{J}), \Re(\mathbf{M}))$ solve the real-valued system

$$\begin{cases} \nabla \times \mathbf{h} &= \omega\epsilon\mathbf{e} + \mathbf{j} & \text{in } \Omega \\ \nabla \times \mathbf{e} &= \omega\mu\mathbf{h} + \mathbf{m} & \text{in } \Omega \\ \mu\mathbf{h} \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega \\ \mathbf{e} \times \mathbf{n} &= 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Therefore, we develop a solver for (6.2) to avoid complex arithmetic. Observe that taking the divergence of the first two equations of (6.2) yields

$$\begin{cases} \nabla \cdot (\mu\mathbf{h}) &= -\omega^{-1}\nabla \cdot \mathbf{m} & \text{in } \Omega \\ \nabla \cdot (\epsilon\mathbf{e}) &= -\omega^{-1}\nabla \cdot \mathbf{j} & \text{in } \Omega. \end{cases} \tag{6.3}$$

We shall include these redundant equations in the system (6.2) in order to obtain a weak formulation where the operator has a divergence term. This will enable us to prove an inf-sup condition.

An axisymmetric vector field $\mathbf{u}$ is said to be meridian if $u_\theta = 0$, azimuthal if $(u_r, u_z) = \mathbf{0}$. Assuming axisymmetry, the curl of a meridian vector field is azimuthal and the curl of an azimuthal vector field is meridian (see Section E of Chapter II). This property implies that (6.2) and (6.3) reduce to two separate systems for the azimuthal and meridian components of $\mathbf{h}$ and $\mathbf{e}$ in the two dimensional domain D:

$$\begin{cases} \nabla \times (e_r, e_z) &= \omega\mu h_\theta + m_\theta & \text{in } D \\ \nabla_r \times h_\theta &= \omega\epsilon(e_r, e_z) + (j_r, j_z) & \text{in } D \\ \nabla_r \cdot \epsilon(e_r, e_z) &= -\omega^{-1}\nabla \cdot \mathbf{j} & \text{in } D \\ (e_r, e_z) \cdot \mathbf{t} &= 0 & \text{on } \Gamma_1, \end{cases} \tag{6.4}$$

$$
\begin{cases}
\nabla \times (h_r, h_z) & = \omega\epsilon e_\theta + j_\theta & \text{in } D \\
\nabla_r \times e_\theta & = \omega\mu(h_r, h_z) + (m_r, m_z) & \text{in } D \\
\nabla_r \cdot \mu(h_r, h_z) & = -\omega^{-1}\nabla \cdot \mathbf{m} & \text{in } D \\
(h_r, h_z) \cdot \mathbf{n} & = 0 & \text{on } \Gamma_1 \\
e_\theta & = 0 & \text{on } \Gamma_1.
\end{cases}
\tag{6.5}
$$

Here $\mathbf{n} = (n_r, n_z)$ is the unit outward normal and $\mathbf{t} = (-n_z, n_r)$ is the unit tangent vector, oriented counterclockwise. Note that the boundary condition $\mathbf{e} \times \mathbf{n}$ on Γ_1 is equivalent to $e_\theta = 0$ and $(e_r, e_z) \cdot \mathbf{t} = 0$ on Γ_1. The boundary condition $\mathbf{h} \cdot \mathbf{n} = 0$ does not involve h_θ, since axisymmetry ensures that the angular component of the normal vector $\mathbf{n}$ is zero.

A. The Least-squares Formulation

Define the space $\mathbf{H}^1_{1,t}(D) = \{\mathbf{v} \in H^1_1(D)^2 : \mathbf{v} \cdot \mathbf{t} = 0 \text{ on } \Gamma_1\}$ and the bounded linear operators

$$
\begin{aligned}
\mathbf{curl}_1 &: L^2_1(D) \to \mathbf{H}^1_{1,t}(D)', \\
\mathbf{curl}_2 &: L^2_1(D) \to (H^1_1(D)^2)',
\end{aligned}
$$

by

$$
\begin{cases}
< \mathbf{curl}_1\, u, \psi > & = (u, \nabla \times \psi)_r \quad \text{for all } u \in L^2_1(D), \psi \in \mathbf{H}^1_{1,t}(D), \\
< \mathbf{curl}_2\, u, \psi > & = (u, \nabla \times \psi)_r \quad \text{for all } u \in L^2_1(D), \psi \in H^1_1(D)^2,
\end{cases}
$$

Recall that the operators $\mathrm{curl}_1 : L^2_1(D)^2 \to H^1_-(D)'$ and $\mathrm{div}_\epsilon : L^2_1(D)^2 \to H^1_{1,\diamond}(D)'$ were defined in (4.6), and the operators $\mathrm{curl}_2 : L^2_1(D)^2 \to H^1_{-,0}(D)'$ and $\mathrm{div}_\mu : L^2_1(D)^2 \to H^1_1(D)'$ were defined in (5.6). Consider the weak formulation corresponding to (6.4),

$$B^1_\omega(e_r, e_z, h_\theta) \equiv \begin{pmatrix} \text{curl}_1\,(e_r, e_z) - \omega\mu h_\theta \\ \mathbf{curl}_1\, h_\theta - \omega\epsilon(e_r, e_z) \\ \text{div}_\epsilon\,(e_r, e_z) \end{pmatrix} = \begin{pmatrix} m_\theta \\ (j_r, j_z) \\ -\omega^{-1}\nabla_r \cdot (j_r, j_z) \end{pmatrix} \equiv \mathbf{F}^1, \quad (6.6)$$

and the weak formulation corresponding to (6.5),

$$B^2_\omega(h_r, h_z, e_\theta) \equiv \begin{pmatrix} \text{curl}_2\,(h_r, h_z) - \omega\epsilon e_\theta \\ \mathbf{curl}_2\, e_\theta - \omega\mu(h_r, h_z) \\ \text{div}_\mu\,(h_r, h_z) \end{pmatrix} = \begin{pmatrix} j_\theta \\ (m_r, m_z) \\ -\omega^{-1}\nabla_r \cdot (m_r, m_z) \end{pmatrix} \equiv \mathbf{F}^2. \quad (6.7)$$

Here, the operators B^j_ω map $L^2_1(D)^3$ to Y'_j, with $Y_1 = H^1_-(D) \times \mathbf{H}^1_{1,t}(D) \times H^1_{1,\diamond}(D)$ and $Y_2 = H^1_{-,0}(D) \times H^1_1(D)^2 \times H^1_1(D)$. The solutions of (6.4) and (6.6) coincide, as do the solutions of (6.5) and (6.7). In Theorem 10 below, we shall show that the weak formulations (6.6) and (6.7) have unique solutions for any data $\mathbf{F}^j$ in Y'_j satisfying the compatibility conditions

$$< \mathbf{F}^j, y >= 0 \qquad \text{for all } y \in \ker(B^j_\omega)^*,\ j = 1, 2. \qquad (6.8)$$

Theorem 10 *If $\lambda = -i\omega$ is not a Maxwell eigenvalue, then the operator B^j_ω satisfies*

$$C^j_1(\epsilon, \mu, \omega)\|\mathbf{x}\|_{L^2_1(D)^3} \leq \|B^j_\omega(\mathbf{x})\|_{Y'_j} \leq C^j_2(\epsilon, \mu, \omega)\|\mathbf{x}\|_{L^2_1(D)^3} \qquad (6.9)$$

for all $\mathbf{x} \in L^2_1(D)^3$, $j = 1, 2$.

Proof. The upper inequality in (6.9) is trivial, so we only prove the lower inequality. First we prove the result for $j = 1$. The proof is a compactness argument analogous to the one given for Lemma 2.1 of [6]. Suppose, contrary to the result, that there exists a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset L^2_1(D)^3$, $\mathbf{x}_n = (e^n_r, e^n_z, h^n_\theta)$, satisfying $\|\mathbf{x}_n\|^2_{L^2_1(D)^3} = 1$ and $\|B^1_\omega(\mathbf{x}_n)\|^2_{Y'} < 1/n$. We shall arrive at a contradiction by showing that the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ has a subsequence converging to some $\mathbf{x}$ in $L_1^2(D)^3$. To do so, we first show that $\|h_\theta\|_{L_1^2(D)} \leq C\|\mathbf{curl}_1\, h_\theta\|_{\mathbf{H}_{1,t}^1(D)'}$ for all $h_\theta \in L_1^2(D)$. Given $h_\theta \in L_1^2(D)$, consider the variational problem of finding $\phi \in H_-^1(D)$ satisfying

$$a(\phi,\eta) = (\nabla_r \times \phi, \nabla_r \times \eta)_r = (h_\theta, \eta)_r \text{ for all } \eta \in H_-^1(D). \tag{6.10}$$

By Proposition 3.1 of [10], the bilinear form $a(\cdot,\cdot)$ is bounded and coercive. Therefore, the Lax-Milgram lemma gives the unique existence of $\phi \in H_-^1(D)$ solving (6.10). Theorem 3.1 of [10] gives the additional regularity $\phi \in H_-^2(D)$, with $\|\phi\|_{H_-^2(D)} \leq C\|h_\theta\|_{L_1^2(D)}$. In particular, $\nabla_r \times \phi$ is in $H_1^1(D)^2$. By Green's formula, for all η in $C_0^\infty(D) \subset H_-^1(D)$ we have

$$(h_\theta, \eta)_r = (\nabla_r \times \phi, \nabla_r \times \eta)_r = (\nabla \times \nabla_r \times \phi, \eta)_r.$$

Thus $\nabla \times \nabla_r \times \phi = h_\theta$ in $L_1^2(D)$, since $C_0^\infty(D)$ is dense in $L_1^2(D)$. Hence

$$\begin{aligned} < (\nabla_r \times \phi)\cdot \mathbf{t}, \eta >_{r,\Gamma_1} &= (\nabla_r \times \phi, \nabla_r \times \eta)_r - (\nabla \times \nabla_r \times \phi, \eta)_r \\ &= (h_\theta, \eta)_r - (h_\theta, \eta)_r = 0 \end{aligned}$$

for all η in $H_-^1(D)$, so $(\nabla_r \times \phi)\cdot \mathbf{t} = 0$ on Γ_1. Thus $\nabla_r \times \phi$ is in $\mathbf{H}_{1,t}^1(D)$, with

$$\begin{aligned} \|h_\theta\|_{L_1^2(D)} = \frac{(h_\theta, \nabla \times \nabla_r \times \phi)_r}{\|h_\theta\|_{L_1^2(D)}} &\leq C\frac{(h_\theta, \nabla \times \nabla_r \times \phi)_r}{\|\phi\|_{H_-^2(D)}} \\ &\leq C\frac{(h_\theta, \nabla \times \nabla_r \times \phi)_r}{\|\nabla_r \times \phi\|_{H_1^1(D)^2}} \leq C \sup_{\psi \in \mathbf{H}_{1,t}^1(D)} \frac{(h_\theta, \nabla \times \psi)_r}{\|\psi\|_{H_1^1(D)}} \\ &= C\|\mathbf{curl}_1\, h_\theta\|_{\mathbf{H}_{1,t}^1(D)'}. \end{aligned} \tag{6.11}$$

Along with Theorem 5, (6.11) implies

$$C\|(e_r, e_z, h_\theta)\|^2_{L^2_1(D)^3} \le \|\mathrm{curl}_1\,(e_r, e_z)\|^2_{H^1_-(D)'} + \|\mathrm{div}_\epsilon\,(e_r, e_z)\|^2_{H^1_{1,\diamond}(D)'} + \|\mathbf{curl}_1\, h_\theta\|^2_{\mathbf{H}^1_{1,t}(D)'}. \tag{6.12}$$

It follows from (6.12) and

$$\|B^1_\omega(e_r, e_z, h_\theta)\|^2_{Y'} = \|\mathrm{curl}_1\,(e_r, e_z) - \omega\mu h_\theta\|^2_{H^1_-(D)'} + \|\mathrm{div}_\epsilon\,(e_r, e_z)\|^2_{H^1_{1,\diamond}(D)'} + \|\mathbf{curl}_1\, h_\theta - \omega\epsilon(e_r, e_z)\|^2_{\mathbf{H}^1_{1,t}(D)'}$$

that the sequence $\mathbf{x}_n = (e^n_r, e^n_z, h^n_\theta)$ satisfies

$$\|\mathbf{x}_m - \mathbf{x}_n\|^2_{L^2_1(D)^3} \le C\,(\,\|B^1_\omega(\mathbf{x}_m - \mathbf{x}_n)\|^2_{Y'} + \omega^2\|\mu(h^m_\theta - h^n_\theta)\|^2_{H^1_-(D)'} + \omega^2\|\epsilon((e^m_r, e^m_z) - (e^n_r, e^n_z))\|^2_{\mathbf{H}^1_{1,t}(D)'}\,)\,. \tag{6.13}$$

Observe that $L^2_1(D)$ is compactly embedded in $H^{-s}_1(D)$ for all $s > 0$. Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2_1(D)$, then the sequence of rotated functions $\{\breve{f}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is compactly embedded in $H^{-s}(\Omega)$ for all $s > 0$, there is a subsequence $\{\breve{f}_{j_n}\}_{n\in\mathbb{N}}$ which converges in $H^{-s}(\Omega)$. Considering f_{j_n} as an element of $H^{-s}_1(D)$ and $\breve{f}_{j_n}$ as an element of $H^{-s}(\Omega)$, we have

$$< f_{j_n}, \eta >= \int_D r f_{j_n}\eta\, dr\, dz = \frac{1}{2\pi}\int_\Omega \breve{f}_{j_n}\breve{\eta}\, dx\, dy\, dz = \frac{1}{2\pi} < \breve{f}_{j_n}, \breve{\eta} >$$

for all $\eta \in H^s_1(D)$. Note that $\breve{\eta} \in H^s(\Omega)$, since the trace operator is a bijective mapping from $\breve{H}^s(\Omega)$ onto $H^s_1(D)$ (see Theorem II.2.1 of [3]). It immediately follows that $\{f_{j_n}\}_{n\in\mathbb{N}}$ is convergent in $H^{-s}_1(D)$. This shows that $L^2_1(D)$ is compactly embedded in $H^{-s}_1(D)$. Consequently, there is a subsequence of $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, also denoted by $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which converges in $H^{-s}_1(D)^3$.

Clearly $H_1^{-s}(D)$ is continuously embedded in $H_-^1(D)'$. For any ξ in $H_1^{-s}(D)$, we have

$$\|\xi\|_{H_-^1(D)'} = \sup_{\eta \in H_-^1(D)} \frac{<\xi,\eta>}{\|\eta\|_{H_-^1(D)}} \leq \sup_{\eta \in H_1^1(D)} \frac{<\xi,\eta>}{\|\eta\|_{H_1^1(D)}} = \|\xi\|_{H_1^{-1}(D)} \leq \|\xi\|_{H_1^{-s}(D)}.$$

Similarly, $H_1^{-s}(D)^2$ is continuously embedded in $\mathbf{H}_{1,t}^1(D)'$. For $s \in (0, 1/2)$, multiplication by the piecewise smooth coefficient μ is a bounded operator on $H^s(\Omega)$ and hence on $H_1^s(D)$ (see [5]). By (6.13), $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_1^2(D)^3$ and therefore converges to some $\mathbf{x} = (e_r, e_z, h_\theta)$ in $L_1^2(D)^3$. Now $\|\mathbf{x}\|_{L_1^2(D)^3} = 1$ and $B_\omega^1\mathbf{x} = 0$ contradicts the assumption that $\lambda = -i\omega$ is not a Maxwell eigenvalue, as $\mathbf{h} = (0, h_\theta, 0)$ and $\mathbf{e} = (e_r, 0, e_z)$ satisfy the full three-dimensional system (6.2) and (6.3). Therefore, we may conclude that (6.9) holds for $j = 1$.

The proof for $j = 2$ is similar. We only need to verify the analogue of (6.12). By (6.11), we have

$$\|e_\theta\|_{L_1^2(D)} \leq C\|\mathbf{curl}_1\, e_\theta\|_{\mathbf{H}_{1,t}^1(D)'} \leq C\|\mathbf{curl}_2\, e_\theta\|_{(H_1^1(D)^2)'}$$

for all e_θ in $L_1^2(D)$. Together with Theorem 8, this implies

$$\begin{aligned} C\|(h_r, h_z, e_\theta)\|_{L_1^2(D)^3}^2 \leq \|\mathrm{curl}_2\,(h_r, h_z)\|_{H_{-,0}^1(D)'}^2 + \|\mathrm{div}_\mu\,(h_r, h_z)\|_{H_1^1(D)'}^2 \\ + \|\mathbf{curl}_2\, e_\theta\|_{(H_1^1(D)^2)'}^2. \end{aligned}$$

The rest of the proof proceeds in the same manner as for $j = 1$. □

As an immediate consequence of Theorem 10, we have existence and uniqueness of solutions to the least-squares problems of finding $\mathbf{x}$ in $L_1^2(D)^3$ satisfying

$$A_\omega^j(\mathbf{x}, \mathbf{v}) = (B_\omega^j\mathbf{x}, B_\omega^j\mathbf{v})_{Y_j'} = (\mathbf{F}^j, B_\omega^j\mathbf{v})_{Y_j'} \qquad \text{for all } \mathbf{v} \in L_1^2(D)^3, \tag{6.14}$$

for $j = 1, 2$ and any $\mathbf{F}^j$ in Y_j' (the compatibility condition (6.8) is not necessary for the least-squares problem (6.14)). When the compatibility condition (6.8) is satisfied, the solutions of (6.14) coincide with those of the corresponding weak formulations (6.6) and (6.7).

B. Stable Approximation

In this section, we describe finite-dimensional approximation subspaces for which the corresponding discrete least-squares method is stable. Let $\mathcal{T}_h$ be a triangulation of D in the quasi-uniform family of meshes $\mathfrak{T}$ aligned with the discontinuities of the coefficients μ and ϵ. Denote by X_h the space of piecewise constant functions in $L^2_1(D)$, and set $S^h_0 = \{p \in S^h : p = 0 \text{ on } \partial D\}$. Define the edge and element bubble spaces

$$\begin{aligned} H^h_e &= \oplus_{e \in \mathcal{T}_h} B^{(1)}_e, \\ H^{h,j}_{e,\Gamma_0} &= \oplus_{e \subseteq \Gamma_0} B^{(j)}_e, \\ H^{h,j}_\tau &= \oplus_{\tau \in \mathcal{T}_h} B^{(j)}_\tau, \end{aligned}$$

for $j = 1, 2$. Recall that $H^h_{e,-}$ and $H^h_{e,\diamond}$ were defined in (4.13). We approximate $\mathbf{H}^1_{1,t}(D)$ by the finite element subspace $\mathbf{S}^h_t = (S^h)^2 \cap \mathbf{H}^1_{1,t}(D)$. The discrete subspace of Y_1 is defined as $Y^h_1 = H^h_- \times \mathbf{H}^h_t \times H^h_\diamond$, where

$$\begin{aligned} H^h_- &= S^h_- \oplus H^h_{e,-} \oplus H^{h,1}_\tau, \\ \mathbf{H}^h_t &= \mathbf{S}^h_t \oplus (H^h_{e,0})^2 \oplus (H^{h,2}_\tau)^2 \oplus (H^{h,1}_{e,\Gamma_0} \times H^{h,2}_{e,\Gamma_0}), \\ H^h_\diamond &= S^h_\diamond \oplus H^h_{e,\diamond} \oplus H^{h,1}_\tau. \end{aligned}$$

Similarly, the discrete subspace $Y_2^h = H_{-,0}^h \times \mathbf{H}^h \times H^h \subset Y_2$ is defined by

$$\begin{aligned} H_{-,0}^h &= S_-^h \oplus H_{e,0}^h \oplus H_\tau^{h,1}, \\ \mathbf{H}^h &= (S^h)^2 \oplus (H_{e,-}^h)^2 \oplus (H_\tau^{h,2})^2 \oplus (H_{e,\Gamma_0}^{h,1} \times H_{e,\Gamma_0}^{h,2}), \\ H^h &= S^h \oplus H_e^h \oplus H_\tau^{h,1}. \end{aligned}$$

The result of Lemmas 16 and 17 below is an inf-sup condition for the discrete space Y_1^h. The proof of Lemma 16 will utilize the following Green's formula, which is not a consequence of Lemma 1 since constants are not in $H_-^1(\tau)$.

Lemma 15 *If τ is a triangle with no edges contained in Γ_0 (i.e., $\overline{\tau} \cap \Gamma_0$ is empty or a vertex), then for all $\mathbf{v} \in H_1^1(\tau)^2$ we have*

$$(\nabla \times \mathbf{v}, 1)_{r,\tau} = \int_\tau v_r \, dr \, dz - < \mathbf{v} \cdot \mathbf{t}, 1 >_{r,\partial\tau} . \tag{6.15}$$

Proof. We consider the case that $\overline{\tau} \cap \Gamma_0$ is a vertex, as the result follows from Lemma 1 when $\overline{\tau} \cap \Gamma_0 = \emptyset$. Clearly (6.15) holds for the smooth functions in $\mathcal{D}(\overline{\tau})$. The lemma then follows by the density of $\mathcal{D}(\overline{\tau})$ in $H_1^1(D)$ (see Proposition 2), provided that the terms in (6.15) are bounded in $(H_1^1(D))^2$. The desired estimates follow directly from the arguments given in the proof of Lemma 11. □

Lemma 16 *There exists a constant $C > 0$ independent of h satisfying*

$$\sup_{\mathbf{v} \in \mathbf{H}_{1,t}^1(D)} \frac{(h_\theta, \nabla \times \mathbf{v})_r - \omega(\epsilon(e_r, e_z), \mathbf{v})_r}{\|\mathbf{v}\|_{H_1^1(D)^2}} \le C \sup_{\mathbf{v} \in \mathbf{H}_t^h} \frac{(h_\theta, \nabla \times \mathbf{v})_r - \omega(\epsilon(e_r, e_z), \mathbf{v}_h)_r}{\|\mathbf{v}\|_{H_1^1(D)^2}}$$

for all $h_\theta \in X_h$ and $(e_r, e_z) \in (X_h)^2$.

Proof. Consider arbitrary but fixed $h_\theta \in X_h$, $(e_r, e_z) \in (X_h)^2$, and $\mathbf{v} \in \mathbf{H}_{1,t}^1(D)$. By Theorem 12 of the Appendix, there exists a Scott-Zhang type interpolation operator $\Pi_t^h : \mathbf{H}^1_{1,t}(D) \to \mathbf{S}_t^h$ and a constant $C > 0$ such that

$$h_\tau^{-2}\|\mathbf{u} - \Pi_t^h \mathbf{u}\|^2_{L^2_1(\tau)^2} + \|\mathbf{u} - \Pi_t^h \mathbf{u}\|^2_{H^1_1(\tau)^2} \leq C\|\mathbf{u}\|^2_{H^1_1(\Delta_\tau)^2} \tag{6.16}$$

for all $\mathbf{u} \in \mathbf{H}^1_{1,t}(D)$ and all triangles $\tau \in \mathcal{T}_h$. Set $\psi = \mathbf{v} - \Pi_t^h \mathbf{v}$. For each edge e in $\mathcal{T}_h$ not contained in ∂D, let $\mathbf{t}_e = (t_r, t_z)$ be a unit tangent vector on e, with arbitrary orientation. On each such edge e, define $w_e \in B_e$ by $w_e = \alpha\lambda_1\lambda_2$ and $\int_e r w_e \, ds = \int_e r\psi \cdot \mathbf{t}_e \, ds$. Combining estimates similar to (4.22) and (4.19) in the proof of Lemma 9, for the function $q_e = \sum_{e \not\subseteq \partial D} w_e$ one can verify the estimate

$$\|q_e\|^2_{H^1_1(D)} \leq C \sum_{e \not\subseteq \partial D} h_\tau^{-1} \|\psi \cdot \mathbf{t}_e\|^2_{L^2_1(e)}.$$

By Lemma 8 and (6.16), we have

$$\begin{aligned}\|q_e\|^2_{H^1_1(D)} &\leq C \sum_{e \not\subseteq \partial D} h_\tau^{-1} \|\psi\|^2_{L^2_1(e)^2} \leq C \sum_{\tau} (h_\tau^{-2} \|\psi\|^2_{L^2_1(\tau)^2} + \|\psi\|^2_{H^1_1(\tau)^2}) \\ &\leq C\|\mathbf{v}\|^2_{H^1_1(D)^2}. \end{aligned} \tag{6.17}$$

Define $\mathbf{u}_e \in (H^h_{e,0})^2$ on each edge $e \not\subseteq \partial D$ by

$$\mathbf{u}_e = \begin{cases} t_r^{-1}(w_e, 0) & \text{if } t_r \geq t_z, \\ t_z^{-1}(0, w_e) & \text{otherwise}, \end{cases}$$

and put $\mathbf{q}^e = \sum_{e \not\subseteq \partial D} \mathbf{u}_e$. Then (6.17) gives $\|\mathbf{q}^e\|_{H^1_1(D)^2} \leq C\|\mathbf{v}\|^2_{H^1_1(D)^2}$. Moreover, $\mathbf{q}^e$ satisfies $\int_e r\mathbf{q}^e \cdot \mathbf{t}_e \, ds = \int_e r\psi \cdot \mathbf{t}_e \, ds$ for all edges $e \not\subseteq \partial D$. Applying the Green's formula of Lemma 15 yields

$$
\begin{aligned}
(h_\theta, \nabla \times \mathbf{v})_{r,\tau} &= h_\theta \int_\tau v_z\, dr\, dz - h_\theta < \mathbf{v} \cdot \mathbf{t}, 1 >_{r,\partial\tau} \\
&= h_\theta \int_\tau v_z\, dr\, dz - h_\theta < \psi \cdot \mathbf{t}, 1 >_{r,\partial\tau} - h_\theta < \Pi_t^h \mathbf{v} \cdot \mathbf{t}, 1 >_{r,\partial\tau} \\
&= h_\theta \int_\tau v_z\, dr\, dz - h_\theta < \mathbf{q}^e \cdot \mathbf{t}, 1 >_{r,\partial\tau} - h_\theta < \Pi_t^h \mathbf{v} \cdot \mathbf{t}, 1 >_{r,\partial\tau} \\
&= h_\theta \int_\tau (v_z - (\Pi_t^h \mathbf{v})_z - q_z^e)\, dr\, dz + (h_\theta, \nabla \times (\mathbf{q}^e + \Pi_t^h \mathbf{v}))_{r,\tau} \qquad (6.18)
\end{aligned}
$$

for all triangles $\tau \in \mathcal{T}_h$ having no edge contained in Γ_0 (i.e., $\tau \cap \Gamma_0$ is either empty or a vertex). On each such triangle τ, define $\mathbf{w}_\tau = (w_r, w_z)$ in $\left(B_\tau^{(2)}\right)^2$ by

$$
\int_\tau l \mathbf{w}_\tau\, dr\, dz = \int_\tau l(\mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e)\, dr\, dz \text{ for all } l \in \operatorname{span}\{1, r\}. \qquad (6.19)
$$

On all remaining triangles τ, let $\mathbf{w}_\tau = \mathbf{0}$. Observe that (6.19) uniquely defines each component of $\mathbf{w}_\tau$ as the solution of a square system of linear equations. To see that this system is nonsingular, suppose that $\phi = \alpha\lambda_1\lambda_2\lambda_3 + \beta r \lambda_1\lambda_2\lambda_3$ satisfies $\int_\tau l\phi\, dr\, dz = 0$ for all $l \in \operatorname{span}\{1, r\}$. Then $l = \alpha + \beta r \in \operatorname{span}\{1, r\}$, so

$$
\int_\tau l^2 \lambda_1\lambda_2\lambda_3\, dr\, dz = \int_\tau l\phi\, dr\, dz = 0.
$$

Hence $l = 0$ and $\phi = 0$, which shows that the system corresponding to (6.19) is nonsingular. Thus $\mathbf{w}_\tau$ is well-defined. Since $r \in \operatorname{span}\{1, r\}$ and h_θ, ϵ, and (e_r, e_z) are constant on τ, the definition (6.19) and (6.18) imply that $\mathbf{w}_\tau$ satisfies

$$
\begin{aligned}
(h_\theta, \nabla \times \mathbf{w}_\tau)_{r,\tau} - \omega(\epsilon(e_r, e_z), \mathbf{w}_\tau)_{r,\tau} =& (h_\theta, \nabla \times (\mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e))_{r,\tau} \\
& - \omega(\epsilon(e_r, e_z), \mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e)_{r,\tau}.
\end{aligned}
$$

Using the identity (6.19) with $l = 1$ and the same arguments given in the proof of Lemma 10, one can verify that $\mathbf{q}_\tau = \sum_{\tau\in\mathcal{T}_h} \mathbf{w}_\tau$ satisfies the estimate

$$\|\mathbf{q}_\tau\|^2_{H^1_1(D)^2} \le C \sum_{\tau\in\mathcal{T}_h} h_\tau^{-2} \|\mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e\|^2_{L^2_1(\tau)^2}.$$

The assumption that τ has no edge contained in Γ_0 when $\mathbf{q}_\tau \neq \mathbf{0}$ is necessary for the above estimate. It is straightforward to verify that $h_\tau^{-2}\|\mathbf{q}^e\|^2_{L^2_1(\tau)^2} \le C|\mathbf{q}^e|^2_{H^1_1(\tau)^2}$, since $\mathbf{q}^e$ is a sum of bubble functions on interior edges. Hence

$$\|\mathbf{q}_\tau\|^2_{H^1_1(D)^2} \le C \sum_{\tau\in\mathcal{T}_h} h_\tau^{-2} \left(\|\mathbf{v} - \Pi_t^h \mathbf{v}\|^2_{L^2_1(\tau)^2} + \|\mathbf{q}^e\|^2_{L^2_1(\tau)^2} \right) \le \|\mathbf{v}\|^2_{H^1_1(D)^2}.$$

Now we treat the triangles τ having an edge e contained in Γ_0. In this case, define the edge bubble function $\mathbf{w}_e = (w_r, w_z)$ in $H^{h,1}_{e,\Gamma_0} \times H^{h,2}_{e,\Gamma_0}$ by

$$w_r = \alpha\lambda_1\lambda_2, \qquad \int_\tau r w_r \, dr\, dz = \int_\tau r(v_r - (\Pi_t^h \mathbf{v})_r - q_r^e) \, dr\, dz,$$

and

$$\begin{cases} \int_\tau w_z \, dr\, dz = \int_\tau r\nabla \times (\mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e) \, dr\, dz, \\ \int_\tau r w_z \, dr\, dz = \int_\tau r(v_z - (\Pi_t^h \mathbf{v})_z - q_z^e) \, dr\, dz. \end{cases} \tag{6.20}$$

Recall that $\mathbf{q}_\tau = 0$ on τ. By an argument similar to the one given above for element bubble functions, the square system of linear equations defining w_z in (6.20) is nonsingular. Thus $\mathbf{w}_e$ is well-defined. Integration by parts yields the identity

$$\int_\tau r\nabla \times \mathbf{w}_e \, dr\, dz = \int_\tau w_z \, dr\, dz = \int_\tau r\nabla \times (\mathbf{v} - \Pi_t^h \mathbf{v} - \mathbf{q}^e) \, dr\, dz.$$

To show that $\|\mathbf{w}_e\|_{H^1_1(\tau)} \leq C\|\mathbf{v}\|_{H^1_1(\tau)^2}$, we first compute $\int_\tau r\lambda_1\lambda_2\,dr\,dz \approx h_\tau^3$, with constants of equivalence independent of h_τ. Hence

$$\begin{aligned}
\|w_r\|^2_{H^1_1(\tau)} &\leq Ch_\tau^{-2}\|w_r\|^2_{L^2_1(\tau)} \leq Ch_\tau^{-2}|\alpha|^2\|\lambda_1\lambda_2\|^2_{L^2_1(\tau)} \\
&\leq Ch_\tau|\alpha|^2 \leq Ch_\tau^{-5}\left|\int_\tau r(v_r - (\Pi_t^h\mathbf{v})_r - q_r^e)\,dr\,dz\right|^2 \\
&\leq Ch_\tau^{-5}\|1\|^2_{L^2_1(\tau)}\|v_r - (\Pi_t^h\mathbf{v})_r - q_r^e\|^2_{L^2_1(\tau)} \\
&\leq Ch_\tau^{-2}\|v_r - (\Pi_t^h\mathbf{v})_r - q_r^e\|^2_{L^2_1(\tau)}.
\end{aligned}$$

One can easily verify that $\|w_z\|_{L^2(\tau)} \leq Ch_\tau^{-2}|\int_\tau rw_z\,dr\,dz|$ for all $w_z \in B_e$, so

$$\begin{aligned}
\|w_z\|_{L^2_1(\tau)} &\leq h_\tau^{1/2}\|w_z\|_{L^2(\tau)} \leq Ch_\tau^{-3/2}\left|\int_\tau rw_z\,dr\,dz\right| \\
&= Ch_\tau^{-3/2}\left|\int_\tau r(v_z - (\Pi_t^h\mathbf{v})_z - q_z^e)\,dr\,dz\right| \\
&\leq C\|v_z - (\Pi_t^h\mathbf{v})_z - q_z^e\|_{L^2_1(\tau)}.
\end{aligned}$$

It follows that $\|\mathbf{q}_0\|_{H^1_1(D)^2} \leq C\|\mathbf{v}\|_{H^1_1(D)^2}$ for $\mathbf{q}_0 = \sum_{e\subseteq\Gamma_0}\mathbf{w}_e$. In conclusion, the discrete vector field $\mathbf{v}_h = \Pi_t^h\mathbf{v} + \mathbf{q}^e + \mathbf{q}_\tau + \mathbf{q}_0 \in \mathbf{H}_t^h$ satisfies

$$(h_\theta, \nabla\times\mathbf{v}_h)_r - \omega(\epsilon(e_r,e_z),\mathbf{v}_h)_r = (h_\theta,\nabla\times\mathbf{v})_r - \omega(\epsilon(e_r,e_z),\mathbf{v})_r$$

$$\|\mathbf{v}_h\|_{H^1_1(D)^2} \leq \|\mathbf{v}\|_{H^1_1(D)^2}.$$

This completes the proof of the lemma. □

Lemma 17 *There exists a constant $C > 0$ independent of h satisfying*

$$\sup_{\phi\in H^1_-(D)}\frac{((e_r,e_z),\nabla_r\times\phi)_r - \omega(\mu h_\theta,\phi)_r}{\|\phi\|_{H^1_-(D)}} \leq C\sup_{\phi_h\in H^h_-}\frac{((e_r,e_z),\nabla_r\times\phi_h)_r - \omega(\mu h_\theta,\phi_h)_r}{\|\phi_h\|_{H^1_-(D)}}$$

for all $h_\theta \in X_h$ and $(e_r,e_z)\in X_h^2$.

Proof. Consider arbitrary but fixed $h_\theta \in X_h$, $(e_r, e_z) \in X_h^2$, and $\phi \in H^1_-(D)$. As shown in the proof of Lemma 9, there exists $q_h \in S^h_- \oplus H^h_{e,-}$ satisfying

$$\begin{cases} ((e_r, e_z), \nabla_r \times q_h)_r = ((e_r, e_z), \nabla_r \times \phi)_r \\ \|q_h\|_{H^1_-(D)} \leq C\|\phi\|_{H^1_-(D)}. \end{cases}$$

Specifically, $q_h = \Pi^h_- \phi + \phi_e$, where $\phi_e \in H^h_{e,-}$ satisfies $\|\phi_e\|_{H^1_-(D)} \leq C\|\phi\|_{H^1_-(D)}$ and Π^h_- is a Clement interpolation operator satisfying (by Lemma 4)

$$h_\tau^{-2}\|\phi - \Pi^h_-\phi\|^2_{L^2_1(\tau)} + \|\phi - \Pi^h_-\phi\|^2_{H^1_-(\tau)} \leq C\|\phi\|^2_{H^1_-(\Delta_\tau)}. \tag{6.21}$$

On each $\tau \in \mathcal{T}_h$, define w_τ in $B^{(1)}_\tau$ by

$$\int_\tau r w_\tau \, dr \, dz = \int_\tau r(\phi - \Pi^h_-\phi - \phi_e) \, dr \, dz.$$

One can easily verify that $\|w_\tau\|_{L^2(\tau)} \leq C h_\tau^{-1} |\int_\tau w_\tau \, dr \, dz|$ for all $\tau \in \mathcal{T}_h$. If $\tau \cap \Gamma_0 \neq \emptyset$, then $0 < r < Ch_\tau$ on τ. In this case,

$$\begin{aligned} \|w_\tau\|_{L^2_1(\tau)} &\leq C h_\tau^{1/2}\|w_\tau\|_{L^2(\tau)} \leq C h_\tau^{-1/2} \left| \int_\tau w_\tau \, dr \, dz \right| \\ &\leq C h_\tau^{-3/2} \left| \int_\tau r w_\tau \, dr \, dz \right| = C h_\tau^{-3/2} \left| \int_\tau r(\phi - \Pi^h_-\phi - \phi_e) \, dr \, dz \right| \\ &\leq C\|\phi - \Pi^h_-\phi - \phi_e\|_{L^2_1(\tau)}. \end{aligned}$$

On the other hand, if $\tau \cap \Gamma_0 = \emptyset$, then $r_0 < r < 2r_0$ on τ for some $r_0 > 0$. Hence

$$\begin{aligned} \|w_\tau\|_{L^2_1(\tau)} &\leq C r_0^{1/2}\|w_\tau\|_{L^2(\tau)} \leq C r_0^{1/2} h_\tau^{-1} \left| \int_\tau w_\tau \, dr \, dz \right| \\ &\leq C r_0^{-1/2} h_\tau^{-1} \left| \int_\tau r w_\tau \, dr \, dz \right| = C r_0^{-1/2} h_\tau^{-1} \left| \int_\tau r(\phi - \Pi^h_-\phi - \phi_e) \, dr \, dz \right| \\ &\leq C\|\phi - \Pi^h_-\phi - \phi_e\|_{L^2_1(\tau)}. \end{aligned}$$

Thus $\|w_\tau\|_{L^2_1(\tau)} \leq C\|\phi - \Pi^h_- \phi - \phi_e\|_{L^2_1(\tau)}$ for all $\tau \in \mathcal{T}_h$. Since ϕ_e is a sum of edge bubble functions, the estimate $h_\tau^{-2}\|\phi_e\|^2_{L^2_1(\tau)} \leq C\|\phi_e\|^2_{H^1_1(\tau)}$ holds. Now the inverse estimates given by Lemmas 2 and 3, together with (6.21), yield the estimate

$$\|\phi_\tau\|_{H^1_-(D)} \leq C\|\phi\|_{H^1_-(D)}.$$

for the function $\phi_\tau = \sum_{\tau \in \mathcal{T}_h} w_\tau \in H^{h,1}_\tau$. Observe that

$$((e_r, e_z), \nabla_r \times w_\tau)_{r,\tau} = (\nabla \times (e_r, e_z), w_\tau)_{r,\tau} + < (e_r, e_z) \cdot \mathbf{t}, w_\tau >_{r,\partial\tau} = 0$$

for all $\tau \in \mathcal{T}_h$, so $((e_r, e_z), \nabla_r \times \phi_\tau)_r = 0$. Therefore, $\phi_h = \Pi^h_- \phi + \phi_e + \phi_\tau \in H^h_-$ satisfies

$$((e_r, e_z), \nabla_r \times \phi_h)_r - \omega(\mu h_\theta, \phi_h)_r = ((e_r, e_z), \nabla_r \times \phi)_r - \omega(\mu h_\theta, \phi)_r$$

$$\|\phi_h\|_{H^1_-(D)} \leq C\|\phi\|_{H^1_-(D)}.$$

This completes the proof of the lemma. □

Now the discrete inf-sup condition for B^1_ω is an immediate consequence of Theorem 10 and Lemmas 10, 16, and 17. It is clear that the corresponding results for B^2_ω may be obtained by simply modifying the boundary conditions in the proofs of the aforementioned lemmas. Note that the Clement interpolation operators with the appropriate boundary conditions are given by Lemma 4. Thus we have the main result of this section, given in Theorem 11. Unlike the electrostatic problem, the constants in this discrete inf-sup condition depend on the coefficients ϵ, μ, and ω.

Theorem 11 *The discrete inf-sup conditions hold, i.e. there exist positive constants $C_j(\epsilon, \mu, \omega)$ such that*

$$\|\mathbf{x}\|_{L^2_1(D)^3} \leq C_j(\epsilon, \mu, \omega)\|B^j_\omega(\mathbf{x})\|_{(Y^h_j)'} \tag{6.22}$$

for all $\mathbf{x} \in (X_h)^3$ *and* $j = 1, 2.$

CHAPTER VII

NUMERICAL EXPERIMENTS

We now present the results of numerical experiments for the problems studied in this dissertation. First we report the results for the electrostatic problem. Since the magnetostatic system differs from the electrostatic system only in the boundary conditions, the behavior of the numerical methods for these two problems will be essentially identical. Therefore, we neglect to perform numerical experiments for the magnetostatic system. Likewise, for the time-harmonic problem we present results for the components (e_r, h_θ, e_z) in the system (6.4), but not for the other components in the system (6.5).

A. Electrostatics

The linear system representing (4.36) is symmetric and positive definite. We solve the system using the preconditioned conjugate gradient method (PCG), with the $(\epsilon\cdot,\cdot)_r$ mass matrix as the preconditioner. The relative tolerance is 10^{-12}. In the first two experiments, reported in Tables (I) and (II), D is the unit square and the mesh is uniform, with square or triangular elements. Table (I) lists the numerical results for the model problem (4.2) with constant coefficient $\epsilon = 1$ and data

$$f_\theta = \pi(r-1)\cos\pi r\cos\pi z,$$

$$g = 2\cos\pi r\sin\pi z - \pi(r+1)\sin\pi r\sin\pi z.$$

The exact solution is $(e_r, e_z) = (r\cos\pi r\sin\pi z, \sin\pi r\cos\pi z)$. The norm of the error in $L^2_1(D)$ is given only for the square mesh, as the error behaves similarly for the triangular mesh. These results demonstrate the theoretically predicted first order convergence rate.

Table I. Electrostatic problem on square domain, with constant coefficient ($\epsilon = 1$).

	Square mesh				Triangular mesh	
h	L^2_1 error	Ratio	PCG	Unknowns	PCG	Unknowns
1/8	0.0723809	1.96296	7	128	8	256
1/16	0.0363771	1.98974	7	512	9	1024
1/32	0.0182127	1.99735	8	2048	10	4096
1/64	0.00910943	1.99933	9	8192	10	16384
1/128	0.0045551	1.99983	9	32768	11	65536
1/256	0.0022776	1.99996	9	131072	11	262144
1/512	0.00113881	1.99999	10	524288	12	1048576

Table (II) gives the PCG iteration counts when the coefficient has a jump,

$$\epsilon = \begin{cases} 10^4 & \text{if } r, z > 1/2, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the number of iterations does not grow as the mesh is refined, and is not affected by the varying coefficient. The same convergence behavior is also observed when the jump in ϵ occurs along a line intersecting Γ_0.

To be concise, we report three other experiments without tabulating the details of the results. Figure (1) illustrates three non-convex computational domains used in these experiments. For domain (i), the exact solution is unknown, so we cannot confirm the first order convergence rate. The purpose of this experiment is to demonstrate that the number of PCG iterations does not grow when the domain has

Table II. Electrostatic problem on square domain, with large jump in coefficient.

	Square mesh		Triangular mesh	
h	PCG Iterations	Unknowns	PCG Iterations	Unknowns
1/8	7	128	9	256
1/16	7	512	9	1024
1/32	8	2048	10	4096
1/64	9	8192	10	16384
1/128	9	32768	11	65536
1/256	9	131072	11	262144
1/512	10	524288	12	1048576

reentrant corners and the boundary is not rectilinear. Indeed, the number of PCG iterations does not exceed 17 for $1/512 \leq h \leq 1$. For the L-shaped domains (ii) and (iii), the exact solution is known and the observed convergence rate is first order. Furthermore, the PCG iteration count does not exceed 12. In conclusion, the method is efficient and robust with respect to the coefficient and the shape of the domain.

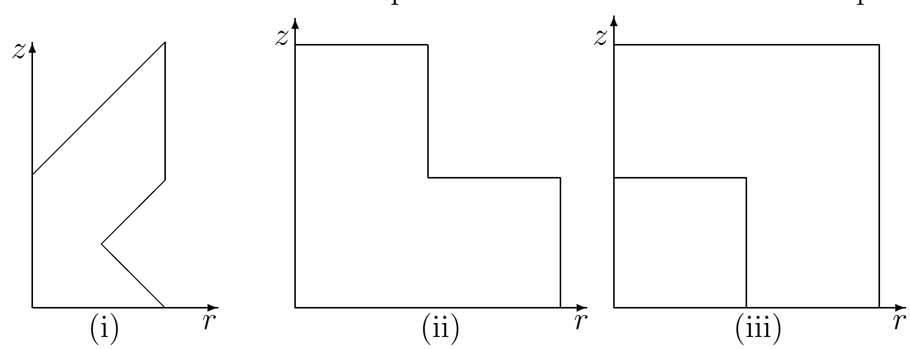

Fig. 1. Computational domains

B. Time-harmonic System

We now present results for the numerical solution of the components (e_r, h_θ, e_z) in the time-harmonic system (6.4). Essentially the same results have been observed for the remaining components, which solve the system (6.5).

With constant coefficients $\epsilon = \mu = 1$ and data

$$\mathbf{j} = (r \sin \pi z\, (\pi \sin \pi r - \cos \pi r)\,, \cos \pi z\, (\sin \pi r + \pi r \cos \pi r))\,,$$

$$m_\theta = \pi(r-1) \cos \pi r \cos \pi z - r \sin \pi r \cos \pi z,$$

$$\nabla \cdot \mathbf{j} = 2 \cos \pi r \sin \pi z - \pi(r+1) \sin \pi r \sin \pi z,$$

the exact solution is $(e_r, h_\theta, e_z) = (r \cos \pi r \sin \pi z, r \sin \pi r \cos \pi z, \sin \pi r \cos \pi z)$. The meshes in the computations for the time-harmonic problem are triangular, which was assumed in Chapter VI. As in the static problem, one would expect similar results for square meshes. The results reported in Table (III) demonstrate the theoretically predicted first order convergence rate.

Table III. Time-harmonic system on square domain, with constant coefficients ($\epsilon = \mu = 1$).

h	L^2_1 error of (e_r, e_z)	Ratio	L^2_1 error of h_θ	Ratio	PCG	Unknowns
1/8	0.0655839	1.99085	0.0309765	1.96871	17	384
1/16	0.0327853	2.00041	0.0155287	1.99479	15	1536
1/32	0.0163932	1.99994	0.0077671	1.9993	16	6144
1/64	0.00819717	1.99986	0.00388316	2.0002	16	24576
1/128	0.00409878	1.9999	0.00194133	2.00025	16	98304
1/256	0.00204944	1.99995	0.000970586	2.00017	17	393216
1/512	0.00102474	1.99997	0.000485271	2.00009	17	1572864

Table (IV) lists the PCG iteration counts for the time-harmonic problem as ω varies between 10^{-3} and 20, with $\epsilon = \mu = 1$. The preconditioner is simply the mass matrix in the inner product of $L^2_1(D)^3$ weighted by μ on the component corresponding to h_θ and by ϵ on the components corresponding to e_r and e_z. The operator T^h is computed in a manner similar to that of the electrostatic problem (see Section (C) of Chapter IV).

We observe that small values of ω do not affect the convergence of the PCG method, but large values have a severely adverse effect. In contrast to the electrostatic and magnetostatic problems, the inf-sup condition for the time-harmonic problem (see Theorem 11) involves constants depending on the coefficients ϵ, μ, and ω. Thus we do not have uniform convergence with respect to the coefficients as in the electrostatic problem.

Table IV. Iteration counts for the time-harmonic problem on square domain, as ω varies with $\epsilon = \mu = 1$.

h	$\omega = 0.001$	$\omega = 0.1$	$\omega = 3$	$\omega = 10$	$\omega = 20$
1/8	16	14	31	54	55
1/16	14	13	31	86	86
1/32	12	12	31	130	129
1/64	11	12	30	117	189
1/128	11	12	30	105	222
1/256	10	12	30	105	223

Although the convergence behavior of the PCG method deteriorates as the magnitude of the coefficients increases, jumps in the coefficients have little effect. Indeed, when $\omega = 1$ and

$$\epsilon = \mu = \begin{cases} 1/2 & \text{if } r, z > 1/2, \\ 1 & \text{otherwise,} \end{cases}$$

the number of PCG iterations is only 16 for h between 1/8 and 1/256.

CHAPTER VIII

CONCLUSION

We have presented negative-norm least-squares methods for dimension-reduced electrostatic, magnetostatic, and time-harmonic Maxwell systems under assumptions of axisymmetry. Theoretical analysis has verified the stability of the methods and has provided quasi-optimal estimates of the error of approximation.

We have shown that the implementation is quite simple, only requiring low order finite element and bubble spaces. Consequently, the resulting linear systems are modest in size. These linear systems can be solved efficiently by iterative techniques. For the electrostatic and magnetostatic systems, we have uniform convergence rates for the iterative solver, independent of the coefficients. It is still an open question how to better precondition the time-harmonic system. The methods for the magnetostatic and time-harmonic systems can be easily implemented by simply modifying the implementation for the electrostatic system, with a minimal amount of additional code.

Compared with other existing methods for numerically solving axisymmetric Maxwell equations, the methods presented in this dissertation appear to be the most widely applicable to real problems in electromagnetics. Indeed, our methods allow for piecewise constant coefficients representing multiple materials. In theory and in numerical experiments, we have shown the methods to be robust with respect to the shape of the domain, which may even be non-convex. Moreover, the domain may be discretized by any unstructured quasi-uniform family of meshes.

Although our methods are quite useful, there are still improvements to be made in order to apply them to more general problems. For instance, it has been assumed throughout the dissertation that the domain has no holes. However, there does not seem to be any prohibitive obstacle to generalizing the theory to treat domains with holes. Another advance that seems feasible is the use of higher order finite element spaces. Unfortunately, this would take much more analysis than that given in this dissertation, which cannot be easily generalized for higher order polynomial spaces. A more interesting and challenging goal would be to extend the methods to handle variable coefficients.

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APPENDIX A

A SCOTT-ZHANG TYPE INTERPOLATION OPERATOR

The result of this appendix is the existence of an interpolation operator $\Pi_t^h : \mathbf{H}^1_{1,t}(D) \mapsto \mathbf{S}_t^h$ satisfying stability and approximation properties. Recall that $\mathbf{S}_t^h = (S^h)^2 \cap \mathbf{H}^1_{1,t}(D)$. This is stated precisely in the following theorem.

Theorem 12 *There exists an interpolation operator* $\Pi_t^h : \mathbf{H}^1_{1,t}(D) \mapsto \mathbf{S}_t^h$ *satisfying*

$$\|\Pi_t^h \mathbf{v}\|_{H^1_1(D)^2} \leq C\|\mathbf{v}\|_{H^1_1(D)^2} \tag{A.1}$$

and

$$h_\tau^{-2}\|\mathbf{v} - \Pi_t^h \mathbf{v}\|^2_{L^2_1(\tau)^2} + \|\mathbf{v} - \Pi_t^h \mathbf{v}\|^2_{H^1_1(\tau)^2} \leq C\|\mathbf{v}\|^2_{H^1_1(\Delta_\tau)^2} \tag{A.2}$$

for all triangles τ *in* $\mathcal{T}_h$ *and all* $\mathbf{v}$ *in* $H^1_1(D)^2$.

We shall construct a weighted Scott-Zhang type interpolation operator $\Pi_t^h : \mathbf{H}^1_{1,t}(D) \mapsto \mathbf{S}_t^h$ based on the unweighted operator defined by Scott and Zhang in [15]. As opposed to the Clement operators which use L^2_1 projection on elements, the operator Π_t^h uses L^2_1 projection on edges. The construction of Π_t^h ensures the property that a function in $H^1_1(D)^2$ with vanishing tangential trace on Γ_1 is mapped to a finite element function in $(S^h)^2$ with vanishing trace on that edge.

In order to enforce the boundary condition $\mathbf{v} \cdot \mathbf{t} = 0$ on Γ_1, Π_t^h must be carefully defined on the corners of Γ_1 where the tangent vector $\mathbf{t}$ changes. Without loss of generality, we assume that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are all of the vertices of the mesh $\mathcal{T}_h$, with $\mathbf{a}_1, \ldots, \mathbf{a}_{n_c}$ being the corners of D on Γ_1. To be clear, the corners $\mathbf{a}_1, \ldots, \mathbf{a}_{n_c}$ are not on Γ_0 and have positive radial coordinates. For each vertex $1 \leq i \leq n$, select an edge

e_i in $\mathcal{T}_h$ having $\mathbf{a}_i$ as a vertex and satisfying the conditions

$$e_i \not\subseteq \Gamma_0 \qquad \text{for all } \mathbf{a}_i \in \mathcal{T}_h, \tag{A.3}$$

$$e_i \subset \Gamma_1 \qquad \text{if } \mathbf{a}_i \in \Gamma_1. \tag{A.4}$$

It is allowed for two adjacent vertices to share an edge, i.e. two vertices $\mathbf{a}_i$ and $\mathbf{a}_j$ may have $e_i = e_j$. The choice of the edges e_i is not unique, and the operator Π_t^h depends on the choice. However, the results we shall prove are independent of the choice of the edges e_i, so the operator Π_t^h has no need for notation indicating the choice of edges. Also for all $1 \le i \le n$, let $\mathbf{a}_{i,1} = \mathbf{a}_i$ and denote by $\mathbf{a}_{i,2}$ the index of the vertex that is the other endpoint of e_i. Thus $\{\phi_{i,1}, \phi_{i,2}\}$ is a basis for the linear finite element space $S^h(e_i)$ on the edge e_i. Let $\{\psi_{i,1}, \psi_{i,2}\}$ be the $L_1^2(e_i)$-dual basis, which satisfies

$$\int_{e_i} r\psi_{i,j}\phi_{i,k}\, ds = \delta_{jk}, \qquad \text{for } j,k = 1,2, \tag{A.5}$$

where δ_{jk} is the Kronecker delta. Here the condition (A.3) is necessary. To simplify notation, we put $\psi_i = \psi_{i,1}$.

The interpolation operator $\Pi_t^h : H_1^1(D)^2 \mapsto (S^h)^2$ is defined by

$$\Pi_t^h \mathbf{v} = \sum_{i=1}^{n_c} \phi_i \mathbf{c}_i(\mathbf{v}) + \sum_{i=n_c+1}^{n} (\phi_i, 0) \int_{e_i} r\psi_i v_r\, ds + (0, \phi_i) \int_{e_i} r\psi_i v_z\, ds, \tag{A.6}$$

where the vector $\mathbf{c}_i(\mathbf{v})$ in $\mathbb{R}^2$ is defined as follows. For a given index $1 \le i \le n_c$, let e_α and e_β denote the two edges of $\mathcal{T}_h$ on Γ_1 having $\mathbf{a}_i$ as an endpoint. The tangent vectors on e_α and e_β are denoted by $\mathbf{t}_\alpha$ and $\mathbf{t}_\beta$, respectively. Then we define $\mathbf{c}_i(\mathbf{v})$ as the solution to the linear system

$$\begin{pmatrix} \mathbf{t}_\alpha^t \\ \mathbf{t}_\beta^t \end{pmatrix} \mathbf{c}_i(\mathbf{v}) = \begin{pmatrix} \int_{e_\alpha} r\psi_\alpha \mathbf{v}\cdot \mathbf{t}_\alpha\, ds \\ \int_{e_\beta} r\psi_\beta \mathbf{v}\cdot \mathbf{t}_\beta\, ds \end{pmatrix}. \tag{A.7}$$

Note that Proposition 3 implies that Π_t^h is well-defined on $H_1^1(D)^2$. The next two lemmas give a stability property for Π_t^h.

Lemma 18 *For all* $1 \leq i \leq n$*, we have*

$$\|\psi_i\|_{L^\infty(e_i)} \leq C h_{e_i}^{-1}. \tag{A.8}$$

Proof. The proof given for Lemma 3.1 of [15] applies here, as there is no weighting by the radial variable in the $L^\infty(e_i)$ norm. ◻

Lemma 19 *For all triangles* τ *in* $\mathcal{T}_h$ *and all* $\mathbf{v}$ *in* $H_1^1(\Delta_\tau)^2$*, we have*

$$\|\Pi_t^h \mathbf{v}\|_{L_1^2(\tau)^2} \leq C \left(\|\mathbf{v}\|_{L_1^2(\Delta_\tau)^2} + h_\tau \|\mathbf{v}\|_{H_1^1(\Delta_\tau)^2} \right), \tag{A.9}$$

$$\|\Pi_t^h \mathbf{v}\|_{H_1^1(\tau)^2} \leq C \left(h_\tau^{-1} \|\mathbf{v}\|_{L_1^2(\Delta_\tau)^2} + \|\mathbf{v}\|_{H_1^1(\Delta_\tau)^2} \right). \tag{A.10}$$

Proof. By Lemma 2 of [2], for all $1 \leq i \leq n$ we have

$$\|\phi_i\|_{L_1^2(\tau)} \leq C r_\tau^{1/2} h_\tau, \qquad \|\phi_i\|_{H_1^1(\tau)} \leq C r_\tau^{1/2}. \tag{A.11}$$

Recall that r_τ denotes the maximum value of the radial variable r on τ. For each $1 \leq i \leq n$, let τ_i be any triangle having e_i as an edge. Since $\phi_i = 0$ on τ for all $1 \leq i \leq n$ such that $\mathbf{a}_i$ is not a vertex of τ, we have by (A.8), (A.11), and Lemma 8 that

$$\begin{aligned}
\left\| \sum_{i=n_c+1}^{n} (\phi_i, 0) \int_{e_i} r\psi_i v_r \, ds \right\|_{L_1^2(D)^2}^2 &\leq \sum_{i=n_c+1}^{n} \|\phi_i\|_{L_1^2(\tau)}^2 \left| \int_{e_i} r\psi_i v_r \, ds \right|^2 \\
&\leq C \sum_{i=n_c+1}^{n} h_\tau^2 h_{e_i}^{-2} \|v_r\|_{L_1^1(e_i)}^2 \leq C \sum_{i=n_c+1}^{n} h_{e_i} \|v_r\|_{L_1^2(e_i)}^2 \\
&\leq C \sum_{i=n_c+1}^{n} h_{e_i} \left(h_{e_i}^{-1} \|v_r\|_{L_1^2(\tau_i)}^2 + h_{e_i} \|v_r\|_{H_1^1(\tau_i)}^2 \right) \\
&\leq C \left(\|v_r\|_{L_1^2(\Delta_\tau)}^2 + h_\tau^2 \|v_r\|_{H_1^1(\Delta_\tau)}^2 \right).
\end{aligned}$$

Similarly,

$$\left\| \sum_{i=1}^{n_c} \phi_i \mathbf{c}_i(\mathbf{v}) \right\|^2_{L^2_1(D)^2} + \left\| \sum_{i=n_c+1}^{n} (0, \phi_i) \int_{e_i} r\psi_i v_z \, ds \right\|^2_{L^2_1(D)^2} \leq C \left(\|v_r\|^2_{L^2_1(\Delta_\tau)} + h^2_\tau \|v_r\|^2_{H^1_1(\Delta_\tau)} \right).$$

The estimate of $\sum_{i=1}^{n_c} \phi_i \mathbf{c}_i(\mathbf{v})$ follows from the facts that $|\mathbf{t}| = 1$ and the angles between tangent vectors on adjacent segments of Γ_1 are bounded away from 0. Thus the inverse of the matrix in (A.7) is bounded by a constant depending only on the domain D. Thus (A.9) holds. Similarly, using the inequality $\|\phi_i\|_{H^1_1(\tau)} \leq C r_\tau^{1/2}$ yields (A.10). □

Proof. [Proof of Theorem 12] Consider Π^h_t, defined above in (A.6), as a map from $H^1_1(D)^2$ to $(S^h)^2$. For all $1 \leq j \leq n_c$, it can be easily verified from (A.7) that $\mathbf{c}_i((\phi_j, 0)) = (\delta_{ij}, 0)$. Therefore,

$$\Pi^h_t(\phi_j, 0) = \sum_{i=1}^{n_c} \phi_i \mathbf{c}_i((\phi_j, 0)) = \sum_{i=1}^{n_c} \phi_i(\delta_{ij}, 0) = (\phi_j, 0)$$

and similarly $\Pi^h_t(0, \phi_j) = (0, \phi_j)$. By (A.5), we have for all $n_c + 1 \leq j \leq n$ that

$$\begin{aligned} \Pi^h_t(\phi_j, 0) &= \sum_{i=n_c+1}^{n} (\phi_i, 0) \int_{e_i} r\psi_i \phi_j \, ds = \sum_{i=n_c+1}^{n} (\phi_i, 0) \int_{e_i} r\psi_{i,1} \phi_{j,1} \, ds \\ &= \sum_{i=1}^{n} (\phi_i, 0) \delta_{ij} = (\phi_j, 0) \end{aligned}$$

and similarly $\Pi^h_t(0, \phi_j) = (0, \phi_j)$. Thus Π^h_t is a projection onto $(S^h)^2$.

By Lemma 19, Π^h_t is bounded on $H^1_1(D)^2$. For all ξ in $(S^h)^2$, we have by (A.9) that

$$\begin{aligned} \|\mathbf{v} - \Pi^h_t \mathbf{v}\|_{L^2_1(\tau)^2} &\leq \|\mathbf{v} - \xi\|_{L^2_1(\tau)^2} + \|\Pi^h_t(\xi - \mathbf{v})\|_{L^2_1(\tau)^2} \\ &\leq C \left(\|\mathbf{v} - \xi\|_{L^2_1(\Delta_\tau)^2} + h_\tau \|\mathbf{v} - \xi\|_{H^1_1(\Delta_\tau)^2} \right). \end{aligned}$$

Now taking the infimum over all ξ in $(S^h)^2$ and using the approximation property

$$\inf_{u_h \in S^h} \|u - u_h\|_{L^2_1(\tau)} + h_\tau \|u - u_h\|_{H^1_1(\tau)} \le h_\tau \|u\|_{H^1_1(\Delta_\tau)} \tag{A.12}$$

for all u in $H^1_1(\Delta_\tau)$, we obtain the inequality

$$\|\mathbf{v} - \Pi^h_t \mathbf{v}\|_{L^2_1(\tau)^2} \le C h_\tau \|\mathbf{v}\|_{H^1_1(\Delta_\tau)^2}. \tag{A.13}$$

Similarly, using (A.10) one can show that

$$\|\mathbf{v} - \Pi^h_t \mathbf{v}\|_{H^1_1(\tau)^2} \le C \|\mathbf{v}\|_{H^1_1(\Delta_\tau)^2}. \tag{A.14}$$

Combining these two inequalities yields the estimate (A.2).

Next we show that Π^h_t maps $\mathbf{H}^1_{1,t}(D)$ onto $\mathbf{S}^h_t$. Let $\mathbf{v}$ be an arbitrary vector field in $\mathbf{H}^1_{1,t}(D)$. It is clear from the definition of the vectors $\mathbf{c}_i(\mathbf{v})$ in (A.7) and the constraint (A.4) that the coefficients of all the nodal basis functions in (A.6) vanish. Indeed, for $1 \le i \le n_c$, the right-hand side of (A.7) is the zero vector, so $\mathbf{c}_i(\mathbf{v}) = \mathbf{0}$.

It remains only to verify that $\Pi^h_t \mathbf{v} \cdot \mathbf{t} = 0$ on the interior of all segments S of Γ_1. Each straight line segment S has a constant tangential vector $\mathbf{t} = (t_r, t_z)$, and the function $u = t_r v_r + t_z v_z$ in $H^1_1(D)$ satisfies $u = 0$ on S. Here we are considering t_r and t_z as constants. Hence

$$\begin{aligned} 0 &= \sum_{i=n_c+1}^{n} \phi_i \int_{e_i} r\psi_i \mathbf{v} \cdot \mathbf{t}\, ds = \sum_{i=n_c+1}^{n} \phi_i \left(t_r \int_{e_i} r\psi_i v_r\, ds + t_z \int_{e_i} r\psi_i v_z\, ds \right) \\ &= \Pi^h_t \mathbf{v} \cdot \mathbf{t} \end{aligned}$$

on S. Since S is an arbitrary segment of Γ_1, we have established that Π^h_t in fact maps $\mathbf{H}^1_{1,t}(D)$ to $\mathbf{S}^h_t$. □

VITA

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