Comparison of the Stochastic Models
for Double-Differenced GPS Measurements

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Abstract. Double-differenced GPS carrier phase measurements are commonly used in GPS precise positioning applications and processed with algorithms based on the least-squares (LS) principle. In order to apply the LS principle, one needs to define properly both the functional and stochastic models. Whilst the functional model for precise GPS positioning is sufficiently well known, realistic stochastic modeling is still a difficult task to accomplish in practice. Incorrect stochastic models for double-differenced GPS measurements will lead to unreliable estimates for ambiguity resolution and, eventually, it will bias positioning results. The common assumption when we construct the stochastic model is that all raw GPS measurements are independent and have the same variances. In fact, this is not realistic, since due to varying noise levels measurements obtained from different GPS satellites cannot have the same accuracy. A realistic stochastic modeling should be able to capture the ordinary noises in the observables.

In order to specify a realistic stochastic model for precise relative GPS positioning applications, in this paper the performance of three stochastic models namely the commonly used model or the standard model, the outer product of residual data vector model and *Minimum Norm Quadratic Unbiased Estimation* (MINQUE) are examined and effects of each the proposed model on statistic for ambiguity search and positioning accuracy are compared. The results indicate that the MINQUE model tends to perform better than the other models. Using the MINQUE model, the reliability of the ambiguity resolution and the statistics of the baseline components can be improved. It may suggest that the MINQUE model, which is based on modern statistical theory, is capable of capturing the ordinary noises.

1   Introduction

Nowadays, GPS has been playing an increasingly important contribution in high-precision surveying and geodetic applications. In order to ensure high accuracy, double-differenced GPS measurements are favored because they can eliminate some systematic errors such as satellite and receiver clock errors. In

GPS positioning adjustment, as like in traditional geodetic adjustment, the least-squares (LS) principle is applied to estimate baseline components and other nuisance parameters. In order to apply the LS principle, one needs to define properly both the functional and stochastic models. Whilst the functional model for precise GPS positioning is sufficiently well known, realistic stochastic modeling is still a difficult task to accomplish in practice. The stochastic model of the double-differenced measurements is commonly constructed by applying the variance-covariance propagation law. In this case, the variance-covariance (VC) matrix of the zero-differenced measurements, assuming that all raw GPS measurements are independent and have the same variances, is used to construct the VC-matrix of the double-differenced measurements. However, such assumption may be not realistic due to residual or unmodeled systematic errors, which have varying noise level, may be present in the double-differenced observables. A more reliable stochastic model needs to be investigated accordingly.

Kuterer (1999) has studied the sensitivity of some characteristic results of the LS adjustments such as the estimated values of the parameters and their VC-matrix due to imminent uncertainties of the stochastic model. In case of the LS adjustments for GPS measurements, a proper choice of the VC-matrix is of relevance for all the subsequent stages of the data processing. The LS solution for instance, will loose its property of "minimum variance" when mis-specified VC-matrix is used. In addition, the detection power of the statistical test, employed for model validation and quality control (e.g. outliers and cycle-slips), will become smaller when the noise characteristics are not properly taken into account. And finally, the a posteriori quality description of the computed results will also be affected when mis-specified or oversimplified VC-matrices are used (Teunissen et.al, 1998). According to studies carried out by Teunissen *et al* (1998) and also by Kutere (1999), it is obviously shown that incorrect stochastic models will lead to unreliable estimates for ambiguity resolution and, eventually, it will bias positioning results. In order to ensure a realistic stochastic model for double-differenced GPS data, the VC-matrix should be able to capture the ordinary noise in GPS observables.

Recently, some efforts have been made to improve the stochastic models used in GPS relative positioning. Han (1997) has used an exponential formula to approximate the standard deviation of GPS measurements, which are considered to be dependent on the elevation angles of the tracked satellites. The coefficients of the exponential formula are determined using empirical methods with experimental data. El-Rabbany and Kleusberg (2003) considered the temporal correlation of the GPS observables and they assumed that all one-way measurements are independent and have the same variances and temporal correlation. Tiberius and Kanselaar (2000, 2003) considered satellite elevation

dependent, time-correlation and cross-correlation of the GPS observables. Hartinger and Brunner (1999), Wieser and Brunner (2000, 2001) have proposed the family of the stochastic SIGMA model that is SIGMA-ε, SIGMA-Δ and SIGMA-F. These models, which used satellite elevation dependent and signal-to-noise ratio (SNR) as the quality indicators for GPS observations, have successfully eliminated the systematic errors inherent in GPS measurements. Positioning accuracy of the kinematic GPS data test in the presence of multipath and signal diffraction could be well improved. However, Satirapod *et al* (2002) have shown that such quality indicators may not always reflect reality. Therefore, a rigorous statistical method for estimating VC-components should be applied.

A comprehensive study of some rigorous methods for estimating the VC-components has been carried out by Crocetto *et al* (2000), Bera *et al* (2002) and Chang (2003a, 2003b). Regarding these methods, as long as the influence of the gross error can be eliminated, the *Minimum Quadratic Unbiased Estimation* (MINQUE) proposed by Rao (1972) is one of the best methods and most commonly used. This method was successfully introduced by Wang *et al* (1988a) to estimate the VC-components of GPS observations. Furthermore, Wang *et al* (2002) improved their procedure by taking into account the temporal correlation of GPS observables. Due to the computational burden, Satirapod *et al* (2001a) proposed a new computation procedure to simplify the MINQUE method.

In this paper, the performance of three stochastic models are examined and analyzed with various static GPS baseline data set. Those models are: A). The Standard model, which assumes that all raw GPS observables have the same variance and statistically independent. B). The outer product of the residual vector model. C). The MINQUE model. In the following section, the aforementioned models are briefly described. The temporal correlation is assumed absent in our data experiments.

2   The Stochastic Models of GPS Observations

2.1   The Standard Model (Model A)

As it has been described earlier that double-differenced GPS measurements are favored. Geometric configuration of such measurement is depicted in Fig.1. Suppose *m* satellites are tracked at epoch *i* by two receivers 1 and 2, thus total number the double-differenced measurement (*l*) is (*m-1*). The VC-matrix of the double-differenced measurements is constructed with the variance-covariance propagation law.
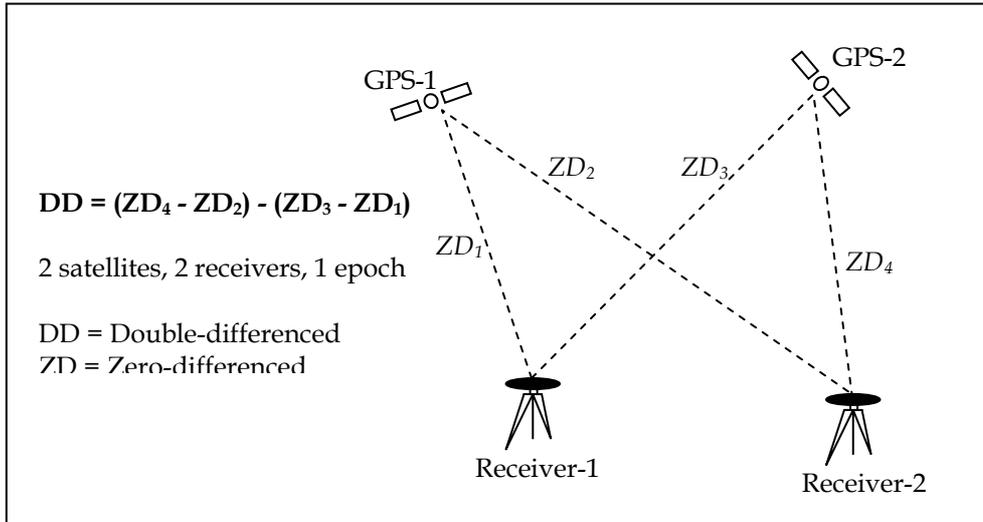
Figure 1 Geometric configuration of double-differenced GPS measurement

By propagating the VC-matrix of the zero-differenced measurements and assuming that all the raw GPS measurements are uncorrelated and have the same a priori variance σ^2, the VC-matrix of the double-differenced measurements at epoch i can be written as:

$$Cov(DD)_{li} = \sigma^2 \begin{bmatrix} 4 & 2 & .. & 2 \\ 2 & 4 & .. & 2 \\ .. & .. & .. & .. \\ 2 & 2 & .. & 4 \end{bmatrix} \tag{1}$$

Note that the VC-matrix of the double-differenced measurements relies heavily on the a priori variance σ^2 and number of satellite m at each epoch. For detailed explanation, the reader is recommended to refer to Hoffman-Wallenhof *et al* (1994) and Leick (1995).

2.2   The Outer Product of the Residual Model (Model B)

The LS residual vector v of the measurements can be used to describe the stochastic properties of the measurements themselves, since it may reflect the characteristic of residual or unmodeled systematic errors. In this model, the outer product of the double-differenced residual is used to construct the VC-matrix.

$$Cov(DD)_{li} = V.V^T \tag{2}$$

Two LS steps are carried out, first the LS is employed by taking into account the VC-matrix constructed from model A. Then, the double-differenced residual vector is computed. In the next step, the LS is re-employed with the VC-matrix constructed from the outer product of the residual vector *v*.

2.3   Minimum Norm Quadratic Unbiased Estimation (Model C)

A rigorous statistical method for estimating VC-components known as the *Minimum Quadratic Unbiased Estimation* (MINQUE) has been proposed by Rao (1972). For a set of double-differenced measurements at epoch *i*, the VC-matrix equation 1 can be also be written as:

$$Cov(DD)_{li} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & ... & \sigma_{1r} \\ \sigma_{21} & \sigma_{22}^2 & ... & \sigma_{2r} \\ ... & ... & ... & ... \\ \sigma_{r1} & \sigma_{r2} & ... & \sigma_{rr}^2 \end{bmatrix} = \sum_{j=1}^{k} \theta_j T_{ji} \tag{3}$$

where θ is the vector of unknown variance components, T are so-called accompanying matrices and $k=r(r+1)/2$ is the number of unknown VC-components. The VC-components can be estimated as:

$$\theta = S^{-1}.q . \tag{4}$$

where the matrix $S=\{S_{ij}\}$ with

$$S_{ij} = Trace(RT_i RT_j) \tag{5}$$

and the vector $q = \{q_i\}$

$$q_i = v^T RT_i Rv \tag{5}$$

and

$$R = PQ_v P \tag{6}$$

Finally Q_v is written as

$$Q_v = \left[Cov(DD)^{-I} - A[A^T Cov(DD)A]^{-I} A^T \right] \tag{7}$$

I is an *n x n* identity matrix. It is noted from *Qv* that estimated VC-components depend on the Cov(DD)-matrix, which includes the VC-components themselves. Therefore, an iterative process must be performed Wang *et al* (1988a, 2002). Flowchart of the iterative MINQUE process is shown in Fig.2.
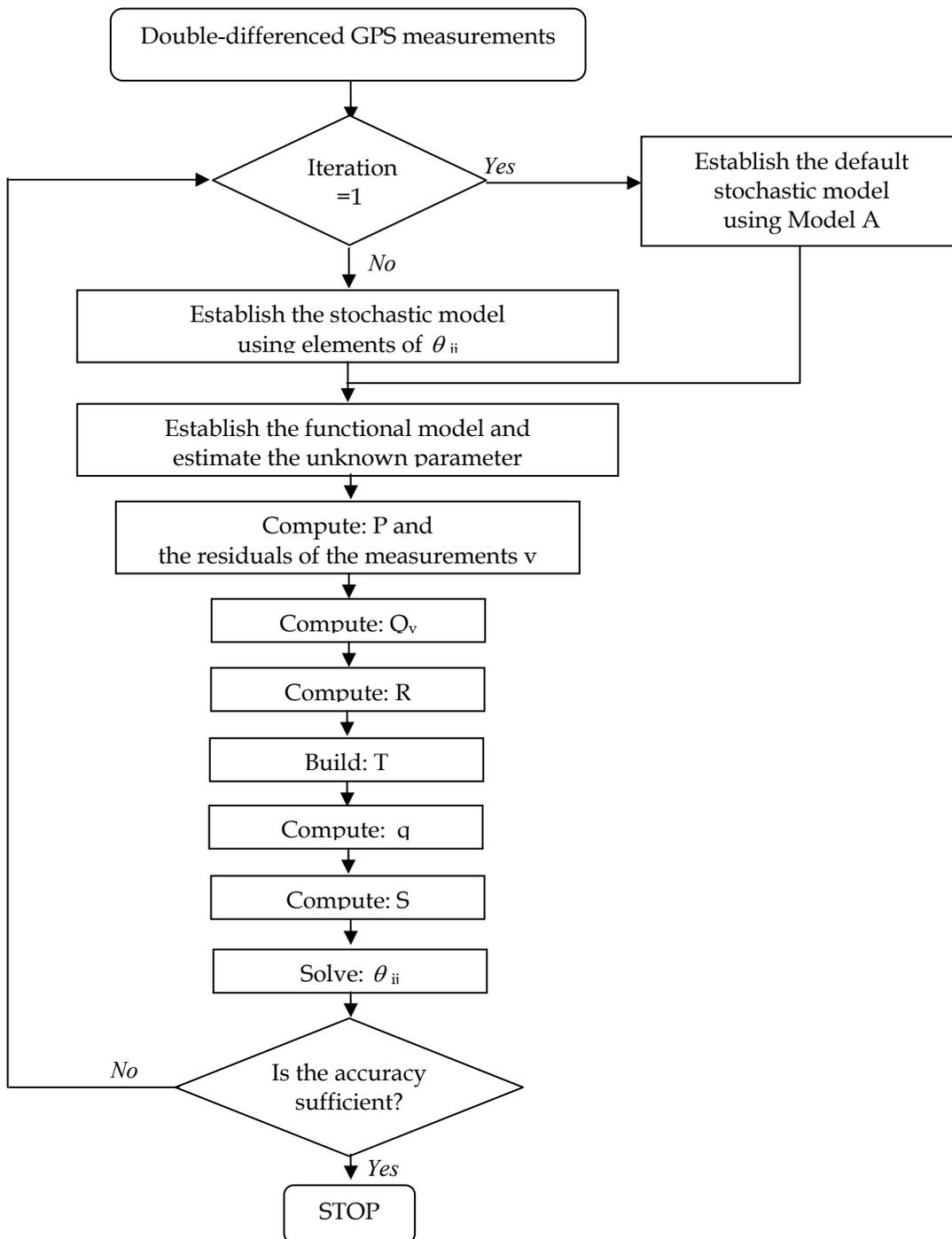
Figure 2 Flowchart of the iterative MINQUE process.

3   Test Result and Analysis

3.1   Experimental data sets

Some GPS static data sets are used to test the performance of the various stochastic models in static data processing. In this test, three GPS static baseline data sets, of which two of them were used in Wang *et al* (2002), are processed using the SNAP baseline processing software (Satirapod et al, 2001b). For all the data sets, the data interval is 3 seconds and the session length is 10 minutes. In the data processing, only L1 frequency data were used.

Baseline name	Receiver	Baseline length (m)	Survey dates
B215M	Ashtech Z-XII	215	7 June 1999
B3KM	Leica CRS-1000	2659	12 October 1999
B13KM	Trimble 4000SSE	13.300	18 December 1996

Table 1   Details of the experimental data sets.

It should be noted that in the case of the Ashtech data set, two receivers were mounted on pillars that are part of the first-order terrestrial survey network. The known baseline length between the two pillars is 215.929 ± 0.001 m (Wang *et al*, 2002), which will be used as ground truth to check the results obtained using the various stochastic modeling procedures.

3.2   Analysis of Results

3.2.1   GPS Measurements Accuracy Due to the Satellite Elevation Angle

The LS residuals of the L1 double-differenced (DD) phase measurements for each data set are presented in Figs. 3 to 5, which show two time series of the DD residuals obtained from the baselines B215M, B3KM and B13KM respectively using different stochastic models. From the figures, it is clearly demonstrated that GPS measurements observed from different satellites do have different noise levels. It is of interest to note that the noise level decreases as the satellite elevation rises. Fig. 3 and 5 clearly show the significant different noise levels between the lowest and highest satellite elevation angles, while in Fig. 4, since the difference angles is not too large, both the DD residuals almost show the same characteristic. The periodic pattern seen in the DD residuals for a low satellite elevation angle, SV02-07 in Fig. 3 and SV15-18 in Fig. 5, indicates that multipath errors seem to have contributed the GPS measurements. In table 2, we compare the GPS measurement accuracy with the satellite elevation angle. The results shown in the table 2 are obtained from the baseline B215M using model

C. It can be seen that two satellite pairs SV02-07 and SV02-19, which have averaged elevation angle lesser than 20 degrees, have the worst accuracy compared with the other satellite pairs.
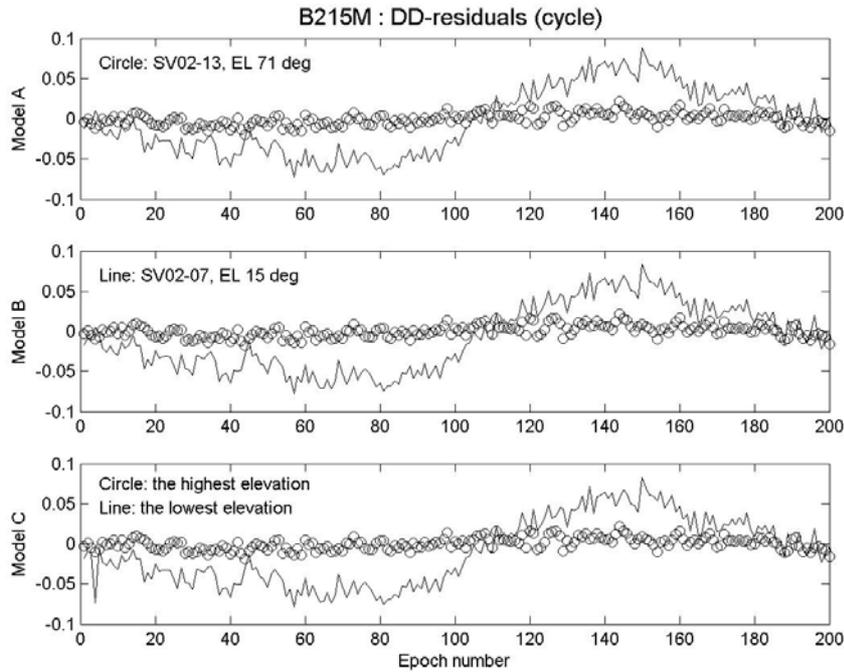

Figure 3 The DD residuals obtained from the baseline B215M for the lowest (SV02-07) and highest (SV02-13) elevation of the satellite pairs. The Line and Circle symbols denote the lowest and highest elevation.

Satellite pairs	L1 DD phase standard deviation (cycle)	Averaged satellite elevation (degrees)
SV 02 - 27	0.0080	52.5
SV 02 - 19	0.0326	19.4
SV 02 - 07	0.0419	15.4
SV 02 - 10	0.0090	51.8
SV 02 - 13	0.0076	71.3

Table 2 Relation between GPS measurement accuracy and satellite elevation angle. Obtained from B215M - Model C.

These results have obviously a good agreement with that reported by Hartinger and Brunner (1999), Wieser and Brunner (2000, 2001). Although, the satellite

elevation may not reflect reality but it still can be used as a quality indicator. This result also suggests that the stochastic models should be chosen properly by considering the various noise levels in the GPS measurements.
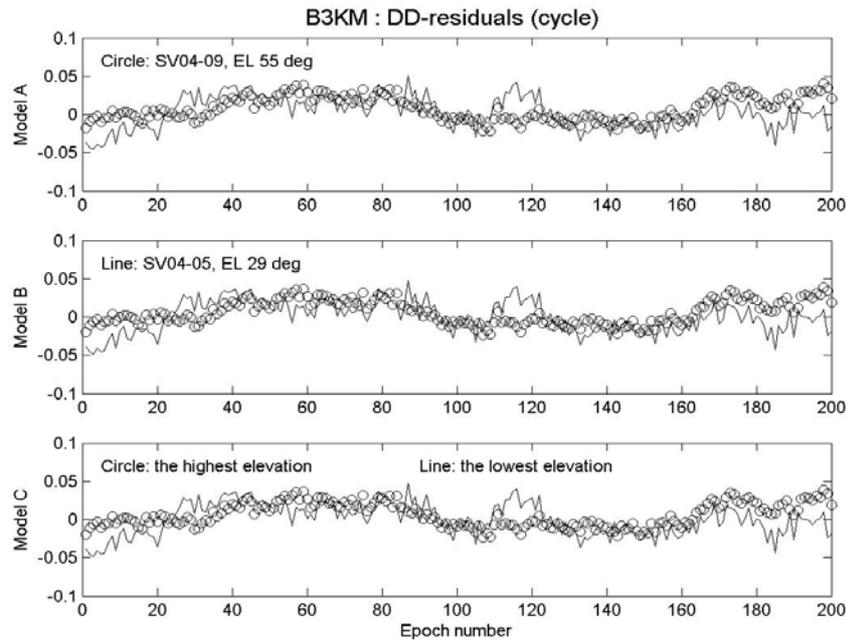

Figure 4 The DD residuals obtained from the baseline B3KM for the lowest (SV04-05) and highest (SV04-09) elevation of the satellite pairs. The Line and Circle symbols denote the lowest and highest elevation.

3.2.2 Ambiguity validation test

It has been shown in the previous section, c.f. Teunissen (1998), that the stochastic models have significance influence on ambiguity resolution. The discrimination test is one of the critical steps. In this study, the F and W-ratio statistics for the ambiguity validation test are used. The larger the values of these statistics, the more reliable the ambiguity resolution can be achieved. The F-ratio statistic is computed by comparing the change of the sum of squares of the best and second best solution of the ambiguity candidates (Leick, 1995):

$$ F < \frac{\left(\Delta V^T P V \right)_{2nd\,smallest}}{\left(\Delta V^T V P \right)_{smallest}} \tag{8} $$
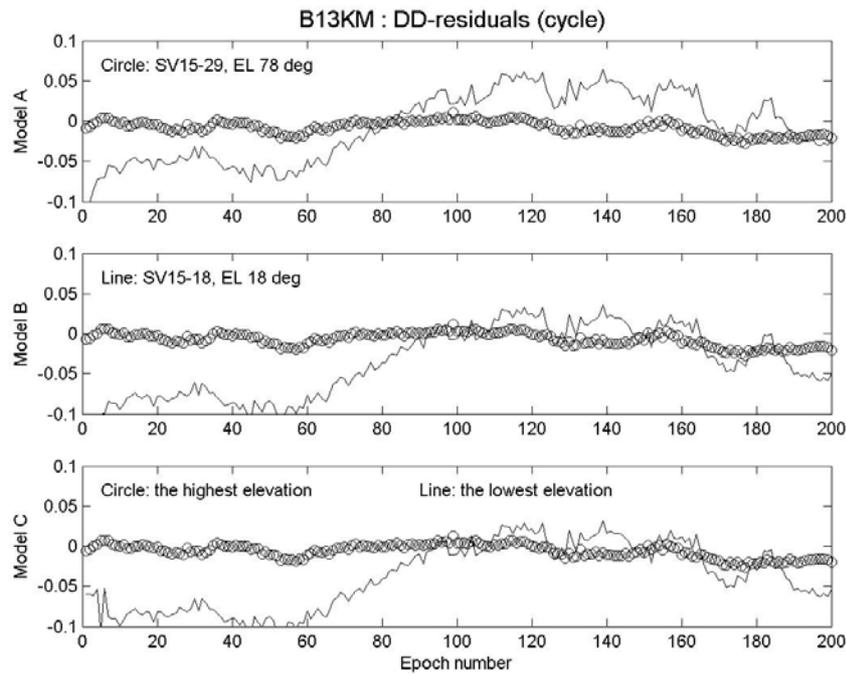
Figure 5 The DD residuals obtained from the baseline B13KM for the lowest (SV15-18) and highest (SV15-29) elevation of the satellite pairs. The Line and Circle symbols denote the lowest and highest elevation.

The critical value of the F-ratio is commonly chosen to be 2.0 or 3.0 (Abidin 1993, Leick 1995, Han 1997). The W-ratio statistic is computed following Wang et al (1998b):

$$W < \frac{d}{2.\hat{\sigma}\sqrt{Q_d}} \qquad (9)$$

where $d = \left(\Delta V^T P V\right)_{\text{2nd smallest}} - \left(\Delta V^T P V\right)_{\text{smallest}}$, while $\hat{\sigma}$ and Q_d are the a posteriori standard deviation and cofactor of d, respectively.

Baseline	F-ratio statistics			W-ratio statistics		
	A	B	C	A	B	C
B215M	2.195	8.437	9.031	13.779	42.241	43.614
B3KM	2.727	3.259	3.255	17.528	20.403	20.382
B13KM	3.936	7.771	14.532	30.942	51.317	67.939

Table 3 F-ratio and W-ratio values for the ambiguity validation test.

From table 3, it can be seen that for model A, all three baselines have small F and W-ratio values. Model C shows a better performance than model A and B since it generally can produce larger F and W-ratio. This may indicate that model C has successfully incorporated the statistical properties of the GPS measurements into the VC-matrix. It is interesting to note, however, in case of the baseline B3KM, the values of the F and W-ratio do not show a large difference among the models. The values of the F- and W-ratio for all the baselines obtained from model B are a little larger than those obtained from model C. These cases might be linked to the systematic errors existing in the measurements. As it has also been mentioned in the previous section, the MINQUE model (model C) is quite sensitive to the systematic errors.

3.2.3   The estimated baseline components

In table 4, the estimated baseline components and their a posteriori standard deviation are presented. In general, although there is no significance difference of the standard deviation among the models, model A produces larger standard deviations than the other models. These results indicate also that there is generally no significance difference both in the horizontal and vertical components. The differences in the estimated horizontal and height components among the models can be as large as 20 mm. However, for high precision applications such a difference becomes significant and then a realistic stochastic model is critical for such applications. For the baseline B215M, the estimated baseline lengths using model C is much closer to the known baseline length than are those using models A and B.

Baseline	Model	Estimated baseline component (m)			Standard deviation (mm)			Baseline length (m)
		North	East	Up	North	East	Up	
B215M	A	-188.5132	105.2934	0.5108	0.4	0.4	0.6	215.9263
	B	-188.5151	105.2932	0.5118	0.2	0.3	0.4	215.9279
	C	-188.5153	105.2931	0.5117	0.2	0.2	0.4	215.9280
B3KM	A	-1832.9362	1926.9867	9.5452	0.2	0.2	0.7	2659.5158
	B	-1832.9368	1926.9871	9.5440	0.1	0.2	0.8	2659.5164
	C	-1832.9368	1926.9871	9.5439	0.1	0.2	0.8	2659.5164
B13KM	A	7209.3642	-11173.7029	-30.0739	1.1	0.4	1.4	13297.6492
	B	7209.3665	-11173.7039	-30.0817	0.6	0.4	0.8	13297.6512
	C	7209.3677	-11173.7046	-30.0829	0.3	0.3	0.4	13297.6525

Table 4   Estimated baseline components and standard deviations.

4   Conclusion

In the commonly used, the stochastic model is constructed by assigning the same variance to all raw GPS measurements and assuming they are statistically independent. This common practice is not realistic in fact, since the measurements do have different noise levels. GPS measurements accuracy relies on the satellite elevation angles. The noise level decreases as the satellite elevation rise. To ensure high accuracy, this varying noise levels should be considered in the stochastic model.

Unrealistic stochastic models will inevitably lead to biased statistic for ambiguity resolution and positioning results. In this paper, three different stochastic models, namely models A, B and C have been examined and analyzed. Various GPS static baseline data sets, namely B215M, B3KM and B13KM are used to examine the performance of the models. Overall, among the three models, model C, which is based on the modern statistical theory, is identified as the best one. Test results indicate that, using model C the reliability of the ambiguity resolution and the estimated baseline component as well as their standard deviation are well improved. For the longest baseline B13KM, the standard deviation for North and Up components are almost 2-3 times better than those obtained using Model B and A. The F and W-ratio statistics are also 3 times larger.

It should be pointed out here that model C is quite sensitive to the systematic errors. It would be quite often that the double-differenced measurements are still contaminated by the systematic errors. Therefore, model C should be applied carefully. A new method that considers the existing of the systematic errors in our measurements needs to be proposed. Simultaneous procedure to remove the systematic errors and to estimate the stochastic parameters is theoretically possible and this would be a topic for further research.

Acknowledgment

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