SUPERGRAVITIES WITH POSITIVE DEFINITE POTENTIALS AND AdS PP-WAVES

A Dissertation

by

JOHANNES KERIMO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2005

Major Subject: Physics

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Approved as to style and content by:

Christopher N. Pope (Chair of Committee)	
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ABSTRACT

Supergravities with Positive Definite Potentials
and AdS PP-Waves. (May 2005)

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Ten-dimensional superstring theory (or the conjectured nonperturbative M-theory in eleven dimensions) is the most promising candidate for a consistent quantum theory of gravity capable of unifying all known forces of nature. An important question concerning these fundamental theories is how they compactify to lower dimensions and how to obtain a real four dimensional world? In this dissertation we present new avenues for M/string theory to reduce to lower dimensions as well as to four dimensions. For example, we show that by performing a generalized Kaluza-Klein $\mathbb{R}$ reduction on the low-energy field theory of the heterotic string, the resulting lower dimensional theory compactifies spontaneously on S^3 to give rise to (Minkowski)$_6$ spacetime. Furthermore, a generalized reduction of M-theory on $K3 \times \mathbb{R}$ compactifies spontaneously on S^2 to give rise to a (Minkowski)$_4$ spacetime.

The generalized Kaluza-Klein reduction gauges the Cremmer-Julia type global symmetry and the homogeneous rescaling symmetry of the supergravity equations of motion by giving the higher dimensional fields an additional dependence on the circle coordinate. We apply the generalized reduction scheme to half-maximal supergravities which are obtained from the heterotic string (or the NS-NS sector of the type-II string) compactified on a $(10 - D)$-dimensional torus truncated to the pure supergravity multiplet. This gives rise to new gauged supergravities in diverse dimensions with supersymmetric Minkowski $\times$ sphere vacua.

Since two large extra dimensions have received much attention recently, we make a detailed study of the gauged $D = 6$, $\mathcal{N} = (1, 1)$ supergravity. In particular, we show that this theory allows for a consistent sphere reduction on S^2 to give rise to $D = 4$, $\mathcal{N} = 2$ supergravity coupled to a vector multiplet which can further be truncated to $\mathcal{N} = 1$ supergravity with a chiral multiplet.

We also investigate pp-waves in AdS backgrounds, i.e. pp-waves as solutions of gauged supergravities with AdS vacua. These solutions generically preserve $\frac{1}{4}$ of the supersymmetry. We demonstrate supernumerary supersymmetries for both purely gravitational pp-waves and pp-waves supported by fields strengths. These new backgrounds provide interesting novel features of the supersymmetry enhancement for the dual conformal field theory in the infinite-momentum frame.

To my parents

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CHAPTER I

INTRODUCTION

In the 20th century, two successful theories emerged in the realm of fundamental physics. One is Einstein's general relativity which describes the dynamics of our spacetime. The other is quantum mechanics that governs the interactions in the microscopic level. General relativity provides a framework to study large scale physics such as astronomy and cosmology, whilst quantum mechanics has been established in studying the microscopic world. In particular, the standard model, which describes the interaction of three fundamental forces (electromagnetic, weak and strong nuclear interactions) is a quantum field theory.

Thus it is natural to expect that one should be able to incorporate the quantum principle in general relativity. This is essential if one would like to unite all the four fundamental forces in one unified theory. The history of physics suggests the trend of unifications of fundamental forces. The seemingly different electric and magnetic forces which were observed in ancient time turned out to be described by the same set of equations discovered by Maxwell in the second half of the 19th century. The merger of the electromagnetic interaction with the weak interaction gives rise to the electro-weak theory (or the Glashow-Salam-Weinberg theory) which in turn when combined with the strong force is described by the Standard Model with gauge group $SU(3) \times SU(2) \times U(1)$. Although the standard model is in excellent agreement with experiments it is not without drawbacks. For one, it has many arbitrary parameters which are not explained by the theory. In addition, its constituent gauge groups are not truly unified since the theory contains three separate gauge coupling parameters.

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An improved version is the Grand Unified Theory (GUT), based on realistic models with gauge groups $SU(5)$ or $SO(10)$. The coupling parameters of the standard model run toward a common value and (almost) meet at the GUT scale 10^{15} GeV. An important prediction of GUTs is the decay of the proton ($\sim 10^{32}$ years) (which is within the reach of experiments). But GUTs are not free from problems (aside from all their arbitrary parameters and their incompatibility with gravity). The critics of this model point out that it predicts no new interactions from 10^{15} GeV down to the weak scale (an energy range of twelve orders of magnitude). Furthermore, GUTs provide no attractive solution to the hierarchy problem where the two energy scales get mixed at each order in the perturbation series.

Quantum physics has taught us to divide particles into bosons (integer spin) and fermions (half-integer spin). We would like to describe the bosons and fermions by some underlying symmetry principle as in the successful cases of general relativity (general coordinate transformation invariance), the standard model and GUTs. This is achieved by supersymmetry, which exchanges bosons and fermions. In fact, supersymmetry is necessary in order to unify the particle spectrum with gravity. Supersymmetry was introduced in four dimensions in [1, 2] and existed earlier in two dimensions [3]. The paper [2], which used a field theoretic approach, began a major development in theoretical physics.

One attractive feature of supersymmetry is that it is capable of giving us field theories which are perturbatively finite (for example, $D = 4$, $\mathcal{N} = 4$ super Yang-Mills theory). In a supersymmetric GUT theory the situation with the hierarchy problem is almost solved in the sense that there is no longer a mixing of the two energy scales in the perturbation series. However, to achieve this an initial fine tuning is still required and it cannot be explained by the theory. We mentioned earlier that in the (nonsupersymmetric) GUT the coupling parameters of the standard model do not quite meet at a common value. The supersymmetric extension of the model does however rectify this situation and the coupling parameters remarkably unify at energies about 10^{16} GeV.

It was hoped that supersymmetry would be able to solve the difficult problem of the cosmological constant. Observations tells us that its value is less than 10^{-84} GeV^2. But the GUT model based on SU(5) (which is the most realistic one) gives rise from the spontaneous breaking of symmetry to a cosmological constant with a value 100^{100} times the limit set by observations. Taking supersymmetry into account provides no solution to this problem.

Supersymmetry has been around for a fairly long time but there is as yet no experimental evidence for it. Nevertheless, it is generally believed to be a necessary ingredient in any unifications of the fundamental forces. The rigid supersymmetry of Wess and Zumino was gauged in [4, 5]. The remarkable result that came out from this gauging was that consistent local supersymmetry requires the inclusion of a massless spin-2 field and its superpartner of spin-3/2. Hence gauged supersymmetry is nothing but a theory of supergravity. Although much effort was put in the investigations of supergravities it was soon realized that these theories were not problem free and their predictive power had limitations. To give one example, the promising $\mathcal{N} = 8$ theory in four dimensions does not have a large enough symmetry to contain the standard model. Another problem is the lack of chirality. We should however add that it is possible to obtain the gauge group of the standard model by compactifying eleven-dimensional supergravity (which is also the highest dimension a consistent supergravity can exist [6]) on a compact manifold [7] but this does not yield chiral fermions. In addition, supergravities are famously nonrenormalizable. It is clear that a quantum theory of gravity must go beyond the point particle concept.

The reconciliation of gravity with quantum physics is important not just on theoretical grounds; a unified description of the fundamental forces is needed to understand the singularity inside a black hole and the moments after the creation of the universe. Furthermore, a black hole has a temperature and entropy, and a quantum theory of gravity is therefore needed to understand these processes fully as the black hole reaches the final stages of its evolution.

The leading candidate for a quantum theory of gravity is superstring theory, whose vibrational modes represent the elementary particles. On a historical note the string was initially introduced to explain hadronic physics. But since the closed string admits a massless spin-2 particle in its spectrum string theory was recognized instead as providing a quantum theory of gravity.

The superstring avoids many of the problems and inadequacies of theories based on particles. For example, an immediate advantage following the introduction of one-dimensional objects is that the pointlike interaction vertex in a Feynman diagram of a traditional field theory is now smeared out, and hence no UV-divergence arises. Although superstrings were introduced already in the early 1970's, they weren't taken too seriously. One of the reasons was that a consistent superstring requires ten dimensions (the purely bosonic string requires 26 dimensions) while our world is four-dimensional. Around 1973 QCD began emergin as a successful theory of the strong nuclear force and subsequently received much of the attention. In addition, in the early 1980's, it was shown that superstring theory was suffering from anomalies. This all changed completely with the paper [8] where it was shown that the anomalies cancel for the group SO(32) (as well as for the group $E_8 \times E_8$). In short, the superstring theory has all the features to be a consistent quantum theory of gravity, with large enough symmetries to reproduce all known particles and their properties.

As we have already emphasized, supergravities suffer from infinities and have problems producing chiral fermions. But in string theory supergravitiy does still play an important role, as the low energy limit of the theory. In the last ten years supergravity theories have been the focus of much attention. There are many reasons for this. Some of them are presented below and others elsewhere in this chapter.

One motivation is the yet unsolved problem of how to obtain a real four dimensional world? It is very likely that supergravity will play an important role in its solution.

Another motivation is the AdS/CFT correspondence [9] which states that a gravity theory in the AdS-bulk is dual to a conformal field theory on the boundary (for example, type-IIB string theory on $\mathrm{AdS}_5 \times S^5$ is dual to $D = 4$, $\mathcal{N} = 4$ super Yang-Mills theory). As we know, the $SU(3)$ theory of QCD becomes nonperturbative at low energies. Consider instead an $SU(N)$ non-abelian theory. Expanding it in $1/N$, the theory in fact simplifies for large N at low energies. The reason for this simplification is that only planar diagrams survive in the large N expansion. It turns out that in the large N limit, a free string theory emerges from the gauge field theory. Here $1/N$ can be viewed as a string coupling constant. The equivalence between gauge fields and free string theories for large N explains why string theory was able to explain aspects of hadronic physics. By using D3-branes the duality is extended to ten dimensional superstring theory and so becomes a duality which includes gravity. Evidence for the AdS/CFT correspondence can seen by analysing the low-energy limit of the Born-Infeld action for D3-branes and the low-energy limit of the D3-brane solution. In this limit the Born-infeld action for the D3-brane reduces to a free supergravity theory in the bulk, and to a four dimensional gauge field theory on the brane. These two systems do not interact and so are decoupled. Consider next the energy excitations of the classical D3-brane solution at low energies. One needs to analyse two regions: the bulk and the near horizon region of the D3-brane (which is $\mathrm{AdS}_5 \times S^5$). It follows that in the bulk, free gravitons dominate, but close to the near horizon region, string excitations become important. At low energies these two sets of excitations are decoupled, as indicated by absorption cross section calculations. This led Maldacena to conjecture the AdS/CFT correspondence. According to this conjecture, when $g_{YM}^2 N \gg 1$, i.e. when the radius of AdS_5 and S^5 are very large, type-IIB supergravity on $\mathrm{AdS}_5 \times S^5$ is a good approximation to strongly coupled $\mathcal{N} = 4$ Yang-Mills theory. If we instead consider the limit $N \to \infty$ and $g_{YM}^2 N$=finite, then string theory is a good approximation for the gauge field theory. Note that N can be viewed as the radius of AdS_5 and S^5. There is much evidence for the validity of the AdS/CFT conjecture. One example is the fact that the symmetries of type-IIB superstring on $\mathrm{AdS}_5 \times S^5$ are the same as those of $\mathcal{N} = 4$ super Yang-Mills. See [10] for details.

Further interest to supergravities lies in the discovery of duality symmetries in M/superstring theory, which relate the five known consistent string theories to each other. To test the duality conjectures is not always a straightforward task. Dualities which relate two weakly-coupled superstrings can be proven within the theories themselves. But dualities that relate the weak-coupling regime of a string theory to the strong-coupling regime of another are more problematic, since we know how to define superstrings only perturbatively. Fortunately, the strong/weak duality can be investigated by analysing the low energy effective field theories obtained by dimensional reduction of the superstrings on certain internal manifolds. An example where this is done is the duality between type-IIA string theory reduced on $K3$ and the heterotic string reduced on the four dimensional torus T^4. See [11] for an extensive discussion of string dualities.

Going from ten to eleven dimensions, the superstring theories find a common origin in a conjectured nonperturbative theory called M-theory. The low energy limit of M-theory is eleven-dimensional supergravity [12]. The existence of M-theory is revealed by reducing it on a small circle S^1, where it yields the type-IIA superstring. Comparing $D = 11$ supergravity on S^1 with the low energy limit of type-IIA string theory, one obtains a relation between the radius of the circle (R_{11}) and the string coupling parameter ($g_s = e^{<\phi>}$), given by

$$R_{11} = (g_s)^{2/3}. \tag{1.1}$$

We see why the eleventh dimension is not seen in perturabtive string theory since small g_s, i.e. $g_s \to 0$ implies $R_{11} \to 0$, and hence the eleventh dimension appears only in the strong coupling region where $R_{11} \to \infty$.

It is clear that supergravities are central in many developments. In the light of this we shall discuss new aspects of supergravity theories in this dissertation, and supersymmetry and spontaneous compactification to four dimensions. The second part of the dissertation treats pp-waves in an AdS background.

Recent interest in both de Sitter and anti-de Sitter vacua has led to a renewed study of gauged supergravities, where the gauging of some R-symmetry naturally leads to a non-trivial potential. Well-known examples include the gauged supergravities in four, five and seven dimensions that admit maximally supersymmetric anti-de Sitter vacua. In addition, there are also gauged supergravities with run-away potentials. Although such theories do not admit maximally supersymmetric vacua, they typically allow domain-wall solutions where scalar gradient energy is balanced against the scalar potential. What has not been achieved, however, is the construction of conventional gauged supergravities admitting de Sitter vacua. Of course this is not particularly surprising, since de Sitter spacetime is incompatible with conventional supersymmetry.

Supergravities with positive-definite (albeit run-away) potentials do nevertheless exist. A particularly interesting example is the Salam-Sezgin model, which is a gauged $\mathcal{N} = (1, 0)$ supergravity in $D = 6$ coupled to a tensor and an abelian vector multiplet [13]. This model has a supersymmetric $(\text{Minkowski})_4 \times S^2$ vacuum, in which the vector has a non-trivial flux on the 2-sphere. This monopole flux, combined with the single-exponential potential $V \sim \exp(-\varphi/\sqrt{2})$, is responsible for a "self-tuning" of the vacuum, in which the positive energy density is confined to the 2-sphere, thereby ensuring a vanishing 4-dimensional cosmological constant and correspondingly a $(\text{Minkowski})_4$ vacuum. The self-tuning feature of this model has attracted much attention, especially as a means of protecting the cosmological constant from large corrections even after supersymmetry breaking [14, 15]. It was shown in [16] that the Salam-Sezgin chiral theory arises from a consistent reduction of ten-dimensional supergravity on a circle times a hyperbolic 3-space. It was also shown, in [17], that the Salam-Sezgin model can be consistently reduced on S^2 to give rise to $D = 4$, $\mathcal{N} = 1$ supergravity coupled to an $SU(2)$ vector multiplet and a scalar multiplet. There are further aspects of the gauged $\mathcal{N} = (1, 0)$ theory which we shall discuss elsewhere in the dissertation.

We should remark here that there exist other supergravities with a single exponential potential. Such examples can be found in seven and four dimensions but with their potential being negative definite. The theory in $D = 7$ and the $D = 4$, $\mathcal{N} = 4$ Freedman-Schwarz model [18], which are of this kind, have been obtained from the heterotic string by reducing on S^3 and $S^3 \times S^3$ respectively [19, 20].

The attractive features of the Salam-Sezgin model have led us to search for other possible supergravity theories with positive-definite potentials. This search was guided by the realization in [21] that a generalized Kaluza-Klein reduction which gauges a combination of a homogeneous global scaling symmetry together with a Cremmer-Julia type global symmetry yields a consistent reduction with just such a positive-definite potential.

The generalized reduction is introduced by giving the higher dimensional fields an additional dependence on the circle coordinate z. Let us demonstrate this with an example in type-IIB supergravity. The equations of motion are invariant under the shift transformation $\chi \rightarrow \chi + c$ of the axion, since is covered by a derivative everywhere. If we replace c by $m\,z$ and reduce to nine dimensions with the ansatz for the axion given as $\chi(x,z) \rightarrow \chi(x) + m\,z$, the reduction is consistent since no z-dependence will appear in the lower dimension. The resulting massive theory [22] is in fact related by a T-duality to Romans massive theory [23] reduced on a circle.

The generalized reduction scheme was used in [24, 25] to construct a variant $D = 6, \mathcal{N} = (1,1)$ supergravity admitting both $(\text{Minkowski})_4 \times S^2$ and $(\text{Minkowski})_3 \times S^3$ vacua. Consider the bosonic sector of the $D = 7$ (ungauged) minimal theory which is described by the Lagrangian

$$\hat{e}^{-1}\hat{\mathcal{L}} = \hat{R} - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{12} e^{\frac{4}{\sqrt{10}}\hat{\phi}} \hat{H}_{(3)}^2 - \tfrac{1}{4} e^{\frac{2}{\sqrt{10}}\hat{\phi}} (\hat{F}^a_{(2)})^2 \,, \tag{1.2}$$

where $a = 1, 2, 3$. The theory possesses the following rigid symmetry

$$\begin{aligned} &\hat{\phi} \rightarrow \hat{\phi} + \sqrt{10}\,\lambda_1 \,, \qquad d\hat{s}^2 \rightarrow e^{2\lambda_2}\, d\hat{s}^2 \,, \\ &\hat{B}_{(2)} \rightarrow e^{-2\lambda_1 + 2\lambda_2}\, \hat{B}_{(2)} \,, \qquad \hat{A}^a_{(1)} \rightarrow e^{-\lambda_1 + \lambda_2}\, \hat{A}^a_{(1)} \,. \end{aligned} \tag{1.3}$$

The transformations associated with λ_1 leaves the Lagrangian invariant, and therefore describes a symmetry of the Lagrangian. On the other hand, the transformation associated with λ_2 which is applied according to the number of spacetime indices scales the Lagrangian uniformly, and so it is not a symmetry of it or even the action, but a symmetry of the equations of motion. These symmetries are then gauged in the dimensional reduction by replacing the λ_i by $m_i\,z$. Since the scale transformation is not a symmetry of the Lagrangian, the generalized reduction must be performed on the equations of motion rather than the Lagrangian itself. A discussion of global $\mathbb{R}$ symmetries and those of the torus generally are given in chapter II.

The would-be vector multiplet arising from performing the generalized reduction of the $D = 7$ theory may be truncated out by a judicious choice of the gauging parameters. In this manner, the reduction takes one from a pure $(d+1)$-dimensional supergravity without a potential to a pure d-dimensional supergravity with a (positive-definite) single-exponential potential. In fact, a further truncation of the bosonic equations of motion to a subsector is possible, with a Lagrangian description that turns out to be identical to the bosonic sector of the Salam-Sezgin model, albeit with a triplet of gauge fields. Although the work of [24, 25] focused on the reduction from seven to six dimensions, the generalized Kaluza-Klein procedure may be carried out in arbitrary dimensions. This was done in [26], where the generalized reduction scheme was performed on the full class of half maximal supergravities in $D \leq 10$, and in this manner variant supergravities in diverse dimensions were obtained. We present this calculation in chapter III. These new gauged supergravities have supersymmetric Minkowski$\times$sphere vacua. A comment at this stage on the maximal supergravities is in place. We have omitted them because starting from $D = 10$ the 4-form field strength cannot support vacua of the type Minkowski$\times$sphere. For an investigation of the maximal supergravities see [27, 28, 21, 29, 30].

Note that the $D = 7$ ungauged theory we discussed above can be obtained from M-theory compactified on $K3$[1] or, for example, from the heterotic string compactified on T^3 with vector multiplets truncated out.

The variant six dimensional supergravity we obtained is different from the Ro-

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[1]The consistency of the reduction of M-theory on $K3$ is questionable. It is consistent however if one restricts to the pure supergravity multiplet which does not turn on the scalars parameterising the $K3$.

mans $D = 6$, $\mathcal{N} = (1,1)$ gauged supergravity [31], in that the four vectors in our theory are all abelian instead of being $SU(2) \times U(1)$ Yang-Mills fields. The supersymmetric (Minkowski)$_4 \times S^2$ (or (Minkowski)$_3 \times S^3$) vacua of the new theory can be uplifted back to $D = 11$, where it becomes the near-horizon geometry of two intersecting M5-branes wrapping on a supersymmetric two-cycle of $K3$. The solution of the two intersecting M5-branes preserve $\frac{1}{4}$ of the maximal supersymmetry.

In chapter IV we derive the complete supersymmetry transformations of the variant $\mathcal{N} = (1,1)$ supergravity from $D = 7$ dimensions. We investigate some of its spontaneous compactifications. As in the $\mathcal{N} = (1,0)$ model, we find that it can also be consistently reduced on a 2-sphere to give rise to four-dimensional $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet. This can further be truncated to yield $\mathcal{N} = 1$ supergravity coupled to a chiral multiplet. We further demonstrate that, in contrast to the $\mathcal{N} = (1,0)$ theory, this model also admits a supersymmetric (Minkowski)$_3 \times S^3$ vacuum. Using the ansatz we uplift supersymmetric dyonic blackhole solutions of the $\mathcal{N} = 2$ supergravity to six dimensions. In the following chapter we continue our studies of supersymmetry by deriving the complete supersymmetry transformations of the variant supergravity in $D = 9$.

The M-theory origin of the $\mathcal{N} = (1,1)$ theory and the vacua (Minkowski)$_4 \times S^2$ are discussed in detail in chapter VI. In chapter VII we derive the Minkowski$\times$sphere vacua in diverse dimensions of the new gauged supergravities which exist for $D \leq 9$. We demonstrate that these vacua are all supersymmetric by uplifting them to higher dimensions, where they become the near horizon geometries of certain brane solutions. A general discussion of (Minkowski)$_{d-n} \times S^n$ vacua is included.

The generalized Kaluza-Klein reduction can readily be adapted to the string frame. In the case of setting the two cosmological parameters equal the generalized reduction becomes essentially just the standard Kaluza-Klein procedure except for a linear z-dependence in the string frame dilaton. In this construction the variant supergravities take a particularly simple form. The scalar potential of the Einstein frame becomes a pure cosmological constant in the string frame. This is the topic of chapter VIII.

In chapter IX we derive a time dependent supersymmetric solution in the nine dimensional gauged theory with the fluxes turned off. The solution can be viewed as a dilaton driven cosmological solution in both $D = 9$ and $D = 10$ dimensions. In the string frame the solution becomes pure Minkowski spacetime. A further uplift to $D = 11$ yields a solution describing a pp-wave.

Recently the Penrose limit [32] of spacetime solutions which give rise to pp-waves has attracted considerable attention. In particular, superstring theory is exactly solvable [33] on the backgrounds of the maximal supersymmetric pp-waves of the type-IIB string [34, 35] and M-theory [36]. String theory on a pp-wave background reduces to a free massive theory in the light-cone gauge. In this description one can now study the AdS/CFT duality [9, 37, 38] beyond the supergravity approximation by including string states [39]. The pp-waves of the Penrose limit arise when one focuses on the geometry close to a null geodesic. These solutions are plane-fronted gravitational waves with parallell rays, propagating in an asymptotically flat spacetime. On the dual gauge side the Penrose limit corresponds to sending both the rank of the gauge group N and the R-charge to infinity [39].

Let us define a pp-wave more precisely. These are spacetime solutions admitting a covariantly constant null Killing vector. We use the following metric for the pp-wave,

$$ds^2 = -4dx^+dx^- - H(x^+, z_i)\,(dx^+)^2 + dz_i\,dz_i\,. \tag{1.4}$$

Note that the transverse space can be any Ricci flat metric, but we are here taking it to be flat. A sub-class of pp-waves are solutions called plane waves. These solutions have an extra symmetry, i.e. plane symmetry, and are obtained by making the specialization $H(x^+, z_i) = h_{ij}(x^+) z^i z^j$. The pp-waves arising from the Penrose limits of AdS×sphere spacetimes belong to the plane-wave category, but with no x^+-dependence in h_{ij}. The pp-waves themselves belongs to a wider class of null solutions [40]. The null solutions we are going to obtain will always be refered to as pp-waves.

In chapters X and XI, we shall study the pp-waves of gauged AdS supergravities. Taking the limit of vanishing cosmological constant these solutions reduce to pp-waves of the corresponding ungauged theories. Before explaining the motivation behind studying AdS pp-waves, let us first introduce the pp-wave in ungauged supergravity by working out an example in minimal $D = 5$ supergravity. The bosonic sector of this theory is described by the Lagrangian

$$\mathcal{L} = R - \tfrac{1}{4} F_{(2)}^2 + \tfrac{1}{12\sqrt{3}}\, \epsilon^{MNPQR} F_{MN} F_{PQ} A_R \,. \tag{1.5}$$

The pp-wave ansatz is

$$\begin{aligned} ds^2 &= -4dx^+ dx^- - H(z_i)(dx^+)^2 + dz_1^2 + dz_2^2 + dz_3^2 \,, \\ F_{(2)} &= -\mu\, dx^+ \wedge dz_1 \,, \end{aligned} \tag{1.6}$$

where $x^\pm = \frac{1}{2}(t \pm x)$. (In general H has also an arbitrary dependence on x^+). If we set $H = 0$, the metric becomes flat Minkowski spacetime. The function H, which is a harmonic function, is given by

$$\begin{aligned} H &= H_0 + H_1 \,, \\ H_0 &= \frac{m}{(z_1^2 + z_2^2 + z_3^2)^{1/2}} \,, \\ H_1 &= \sum_{i=1}^{3} c_i z_i^2 \,, \end{aligned} \tag{1.7}$$

where $c_1 + c_2 + c_3 = -\mu^2/2$. Here H_0 is a pure gravitational solution and H_1 is supported by the null flux (μ). The above solution generically preserve $\frac{1}{2}$ of the supersymmetry (standard supersymmetry) with the only condition being that H satisfies the second order equations of motion. However, if one sets $H_0 = 0$ an enhancement of the supersymmetry can occur, if the coefficients c_i are chosen appropriately. If we set $c_1 = -\mu^2/3$, $c_2 = c_3 = -\mu^2/12$, the pp-wave supported by the field strength becomes maximally supersymmetric. (A derivation of the maximal supersymmetric pp-wave via the Penrose limit is given in chapter X.)

The metric ansatz for a generic pp-wave in the corresponding gauged supergravity would be given by

$$ds^2 = e^{2g\rho}(-4dx^+dx^- + H(dx^+)^2 + dz_1^2 + dz_2^2) + d\rho^2\,, \tag{1.8}$$

where the cosmological constant is related to the gauge coupling constant g as $\Lambda = -g^2$ and $H(x^+, \rho, z_a)$ is a harmonic function on the space of z_a and ρ. If $H = 0$ the metric describes pure AdS spacetime. The pp-wave with the dependence $H(\rho)$ was constructed in four dimensions by Kaigorodov [41], and its higher dimensional counterparts were obtained in [42]. These solutions have interestingly been shown to be related to boosted p-branes in higher dimensions [42]. To give one example, consider the near horizon geometry of the M2-brane, which is $\mathrm{AdS}_4 \times S^7$. If we now perform a singular boost of an BPS M2-brane, then the near horizon geometry of the boosted brane become $(\mathrm{Kaigorodov})_4 \times S^7$. The Kaigorodov metric and its generalization to higher dimensions are homogeneous spaces admitting $\frac{1}{2}D(D-3)+3$ Killing vectors, where D is the spacetime dimension. The AdS pp-waves can in fact also be viewed as plane-fronted solutions. In chapter X we present a detailed investigation of the supersymmetry of AdS pp-waves. These solutions generically preserve $\frac{1}{4}$ of the supersymmetry for any solution H. We show that purely gravitational solutions can in fact admit supernumerary supersymmetry for appropriately constrained H.

We demonstrate the same phenomenon in the case of solutions supported by a field strength in minimal gauged supergravities in $D = 4$ and $D = 5$.

Some of the reasons that motivate the studies of AdS pp-waves are the following. String theory on the $(\text{Kaigorodov})_5 \times S^5$ background is dual to $D = 4, \mathcal{N} = 4$ Yang-Mills theory on an infinitely-boosted frame, with a constant momentum density background. In our case we turn on a $U(1)$ A_μ field as well, which is related to the R-charge of the Yang-Mills. It is of interest to study the effect of turning on the R-charge.

In chapter XI we investigate AdS pp-waves further by studying the pp-waves of $D = 5$ and $D = 4$ gauged supergravities supported by $U(1)^3$ and $U(1)^4$ gauge fields respectively. We also study the pp-waves of the Freedman-Schwarz model.

CHAPTER II

SYMMETRIES IN THE T^n REDUCTION

The topic of dimensional reduction is a vast and diverse one and of great importance with many applications. In this chapter we shall concentrate just on those Kaluza-Klein reductions where the internal manifold is a circle S^1 or product of circles in the case of n-torus T^n. Our focus here is on the torus symmetries which are needed to introduce the generalized Kaluza-Klein reduction. Since this chapter concerns only with T^n reductions we have omitted the coset sphere reductions, group manifold reductions, brane world reductions and reductions based on Calabai-Yau or $K3$ manifolds. The material in this chapter is based on [43].

A. The standard Kaluza-Klein S^1 reduction

We begin with a discussion of the consistency of the S^1 reduction and show that the symmetries of the reduction ansatz makes sense with the lower dimensional equations of motion. Since in the circle reduction each spacetime point comprises a small circle we can expand all higher dimensional fields and its symmetries into harmonics of S^1. In essence, if we split the $D = d + 1$ dimensional coordiantes as $x^M = (x^\mu, z)$, the Fourier serie of the metric for example would be given by

$$\hat{g}_{MN}(x^\mu, z) = \sum_{n=0}^{\infty} g^{(n)}_{MN}(x^\mu)\, e^{\mathrm{i\,nz/L}} \qquad (2.1)$$

where n is a Fourier mode number and L is the radius of the internal compact manifold. It is clear that there are infinite many fields arising from the harmonic expansions and each characterized by the mode number n. Fields with an $n \neq 0$ are massive with a mass proportional to n/L and those with $n = 0$ are massless. If we now substitute such a serie expansion in the higher dimensional equations of motion (or the Lagrangian) then for the dimensional reduction to make sense it is absolute essential that all massive modes can be subsequently truncated out while keeping only the massless ones. In fact, a detail analysis shows that the equations of motion for the $n = 0$ fields remarkably decouple from the $n \neq 0$ equations, and so the massless fields do not act as sources for the massive fields. This means that one can consistently set to zero all massive fields. This decoupling of fields in the S^1 reduction is in fact nothing but guaranteed by group theoretical arguments. As a definition of a consistent reduction, one can take that when uplifted, all solutions of the lower dimensional theory become solutions of the original theory. As we have inferred in the prelude of this chapter, there are rather many types of dimensional reductions and they are all important, but the internal torus manifold which is extremely utilitarian is clearly special among them. Lets now work out a simple S^1 reduction involving just pure gravity in D dimensions and study the symmetries involved. The ansatz that reduces from $D = d + 1$ to d dimensions is given by

$$d\hat{s}_{_D}^2 = e^{2\alpha\varphi} ds_d^2 + e^{2\beta\varphi}(dz + \mathcal{A}_{(1)})^2 \tag{2.2}$$

where

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}\,, \qquad \beta = -(d-2)\alpha\,. \tag{2.3}$$

The reduction of the Einstein theory

$$\hat{e}^{-1}\hat{\mathcal{L}}_{_D} = \hat{R} \tag{2.4}$$

yields an Einstein-Maxwell-scalar system described by the Lagrangian

$$e^{-1}\mathcal{L}_d = R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{4}e^{-2(d-1)\alpha\varphi}\mathcal{F}_{(2)}^2\,. \tag{2.5}$$

The parameters α and β were determined by the requirement that the Lagrangian in lower dimensions have a gravity term of the form eR plus a canonical normalized kinetic term for the dilaton. Note that it is not allowed to set the dilaton to zero since this would be inconsistent with the (z, z)-component of the higher dimensional Einstein equation. Lets now look at the symmetries of this simple example. The lower dimensional theory we obtained has general coordinate covariance, local $U(1)$ gauge symmetry of the Maxwell field and a constant shift symmetry given by

$$\varphi \to \varphi + c\,, \qquad \mathcal{A}_\mu \to e^{c(d-1)\alpha}\mathcal{A}_\mu\,. \tag{2.6}$$

These residual symmetries constitute an infinitesimal amount symmetries surviving from the higher dimensional general coordinate covariance of the original theory. To see that these symmetries are consistent with the S^1 reduction ansatz we need to analyse the general coordinate reparametrization invariance

$$\delta\hat{x}^M = -\hat{\xi}^M\,, \qquad \delta\hat{g}_{MN} = \hat{\xi}^P\partial_P\hat{g}_{MN} + \hat{g}_{PN}\partial_M\hat{\xi}^P + \hat{g}_{MP}\partial_N\hat{\xi}^P \tag{2.7}$$

of the D-dimensional theory where $\hat{\xi}^M$ depends on all D-dimensional coordinates. It is clear that the S^1 reduction ansatz is not preserved under this transformation. An investigation shows that

$$\hat{\xi}^\mu = \xi^\mu(x)\,, \qquad \hat{\xi}^z = c\,z + \lambda(x) \tag{2.8}$$

is the most general form which leaves the reduction ansatz invariant. Here the $\xi^\mu(x)$ and $\lambda(x)$ now depends on the $(D-1)$-dimensional coordinates and parameterise local transformations while the constant c-parameter is associated with a rigid symmetry. Implementing (2.8) in $\delta\hat{g}_{MN}$ for the S^1 metric ansatz, and if we for now set $c = 0$, we obtain

$$\begin{aligned}\delta\varphi &= \xi^\rho\partial_\rho\varphi\,,\\ \delta\mathcal{A}_\mu &= \xi^\rho\partial_\rho\mathcal{A}_\mu + \mathcal{A}_\rho\partial_\mu\xi^\rho + \partial_\mu\lambda\,,\\ \delta g_{\mu\nu} &= \xi^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu\xi^\rho + g_{\mu\rho}\partial_\nu\xi^\rho.\end{aligned} \tag{2.9}$$

It is clear from these that the various fields transform properly under the $(D-1)$-dimensional general coordinate transformations and that $\mathcal{A}_\mu$ has $U(1)$ symmetry. These results are of course in agreement with the Lagrangian (2.5). As mentioned earlier the Lagrangian also has a rigid symmetry given by (2.6) which we would like to obtain from the general coordinate transformations (2.7). In order to do so we need to make use of a conformal symmetry of the D-dimensional equations of motion. We are here refering to the scaling transformation $\hat{g}_{MN} \longrightarrow k^2\hat{g}_{MN}$. Although this transformation leaves the equations of motion invariant it is not a symmetry of the Lagrangian since it is scaled homogeneously. The scaling transformation in infinitesimal form is $\delta\hat{g}_{MN} = 2a\hat{g}_{MN}$, where a is an infinitesimal constant parameter. Now use this together with $\hat{\xi}^z = c\,z$ in (2.7) we obtain

$$\delta\varphi = -\frac{c}{\alpha(D-2)}\,, \qquad \delta\mathcal{A}_\mu = -c\mathcal{A}_\mu\,, \qquad \delta g_{\mu\nu} = 0\,, \tag{2.10}$$

and $a = -c/(D-2)$. This is precisely the symmetry given in (2.6) after the redefinition $c \to \alpha(D-2)c$. Note that the parameter a was fixed from requiring that the variation of the metric be inert under the constant shift transformation. In the forthcoming chapters we are going to make use of these two global symmetries in the S^1 reduction and in this way be able to obtain new supergravities.

We are going to skip the discussion of supersymmetry in the standard Kaluza-Klein theory since we will treat it instead within the generalized Kaluza-Klein reduction. However, we should say that the S^1 reduction preserve all supersymmetry of the original theory.

B. The torus reduction

In this subsection we shall extend the discussions of the previous section by including some elementary aspects of the T^n reductions, and continue to focus on the symmetries involved. In the T^n reductions the parameter c discussed in the S^1 reduction is now replaced by $\Lambda^i{}_j$ and the metric reduction ansatz is preserved by

$$\hat{\xi}^\mu(x,z) = \xi^\mu(x)\,, \qquad \hat{\xi}^i(x,z) = \Lambda^i{}_j z^j + \lambda^i(x)\,. \tag{2.11}$$

We also have

$$\delta z^i = -\Lambda^i{}_j z^j\,. \tag{2.12}$$

The elements of the matrix Λ are real and satisfy $\det(\Lambda) = 1$. This form the global symmetry group $SL(n, \mathbb{R})$ and acts on all fields of the theory except the metric. Making use of the homogeneous scaling symmetry (if present) of the equations of motion the internal $SL(n, \mathbb{R})$ global symmetry can be expanded to the full $GL(n, \mathbb{R})$. Note that $GL(n, \mathbb{R}) \sim SL(n, \mathbb{R}) \times \mathbb{R}$. Lets consider a T^2 reduction of pure Einstein gravity in D dimensions as an example. The T^2 reduction gives rise to the $(D-2)$-dimensional theory

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\varphi)^2 - \frac{\partial\tau\cdot\partial\bar{\tau}}{2(\mathrm{Im}\,\tau)^2} - \tfrac{1}{4}e^{\phi+q\varphi}(\mathcal{F}^1_{(2)})^2 - \tfrac{1}{4}e^{-\phi+q\varphi}(\mathcal{F}^2_{(2)})^2, \tag{2.13}$$

where $\tau = \chi + \mathrm{i}e^{-\phi}$ and χ is an axion field and $q = \sqrt{(D-2)/(D-4)}$. The field strengths are defined as

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} + \chi d\mathcal{A}^2_{(1)}\,, \qquad \mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)}. \tag{2.14}$$

It is clear that the sector (ϕ, χ) transform under $SL(2,\mathbb{R})$ and the Lagrangian has a global shift symmetry $\varphi \to \varphi + c$. This yields the group $SL(2,\mathbb{R}) \times \mathbb{R}$. However, one can show after some calculations that this global symmetry of the scalar sector of the Lagrangian is remarkably also a symmetry of the full Lagrangian involving the gauge fields. This discussion applies also if the original Lagrangian contained higher rank potentials. In summary, the $SL(n,\mathbb{R})$ global symmetry of the torus can be enlarged to $GL(n,\mathbb{R})$ when combined with a scaling symmetry of the original theory. We should however also add that in certain cases the $SL(n,\mathbb{R})$ group can in fact be enhanced to an even larger group than $GL(n,\mathbb{R})$ due to conspiracies among scalar fields and field strengths. This occurs in the eleven dimensional supergravity, type-IIB supergravity and certain dilatonic supergravities [44].

CHAPTER III

GAUGED SUPERGRAVITIES WITH POSITIVE DEFINITE POTENTIALS

In general, the various (ungauged) supergravities are quite distinct (especially in their fermionic sectors). However, it is noteworthy that the bosonic sector of the half-maximal (16 supercharge) supergravities in $D \leq 10$ is universal, with field content

$$(g_{\mu\nu}, B_{\mu\nu}, \phi, A^a_\mu) \tag{3.1}$$

($a = 1, 2, ..., 10, 10-D$). This is of course the bosonic content of the heterotic string (or the NS-NS sector of the Type-II string) compactified on a $(10-D)$-dimensional torus, with vector multiplets truncated out. Owing to this universality of the field content, we may perform a generalized Kaluza-Klein reduction on the half-maximal supergravities in arbitrary dimensions and in this manner obtain the full class of (16 supercharge) variant supergravities.

A. The generalized reduction ansatz

The Lagrangian describing the bosonic sector of pure supergravity with 16 supercharges can be written as

$$\hat{\mathcal{L}} = \hat{R}\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\hat{\phi} \wedge d\hat{\phi} - \tfrac{1}{2}e^{\hat{a}\hat{\phi}}\,\hat{*}\hat{H}_{(3)} \wedge \hat{H}_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\hat{a}\hat{\phi}}\,\hat{*}\hat{F}^a_{(2)} \wedge \hat{F}^a_{(2)}\,, \tag{3.2}$$

where $\hat{F}^a_{(2)} = d\hat{A}^a_{(1)}$, $\hat{H}_{(3)} = d\hat{B}_{(2)} - \frac{1}{2}\hat{F}^a_{(2)} \wedge \hat{A}^a_{(1)}$, and $a = 1, 2\ldots, (10-D)$. The constant $\hat{a}$ is given by

$$\hat{a}^2 = \frac{8}{D-2}\,. \tag{3.3}$$

The equations of motion are given by

$$
\begin{aligned}
\hat{R}_{MN} &= \tfrac{1}{2}\partial_M\hat{\phi}\,\partial_N\hat{\phi} + \tfrac{1}{4}e^{\hat{a}\hat{\phi}}\Big(\hat{H}_{MPQ}\,H_N{}^{PQ} - \frac{2}{3(D-2)}\hat{H}^2_{(3)}\,\hat{g}_{MN}\Big)\\
&\quad + \tfrac{1}{2}e^{\frac{1}{2}\hat{a}\hat{\phi}}\Big(\hat{F}^a_{MP}\,\hat{F}^{a\ P}_{N} - \frac{1}{2(D-2)}(\hat{F}^a_{(2)})^2\,\hat{g}_{MN}\Big),\\
d(e^{\hat{a}\hat{\phi}}\,\hat{*}\hat{H}_{(3)}) &= 0\,,\\
d(e^{\frac{1}{2}\hat{a}\hat{\phi}}\,\hat{*}\hat{F}^a_{(2)}) &= (-1)^{D+1}e^{\hat{a}\hat{\phi}}\,\hat{*}\hat{H}_{(3)}\wedge\hat{F}^a_{(2)}\,,\\
\widehat{\Box}\hat{\phi} &= \frac{\hat{a}}{12}\,e^{\hat{a}\hat{\phi}}\hat{H}^2_{(3)} + \frac{\hat{a}}{8}\,e^{\frac{1}{2}\hat{a}\hat{\phi}}\,\hat{F}^{a2}_{(2)}\,.
\end{aligned}
\tag{3.4}
$$

The key observation behind the generalized reduction [21] is the realization that the equations of motion are invariant under the symmetry

$$
\begin{aligned}
&\hat{\phi}\to\hat{\phi}+\frac{1}{\hat{a}}\,\lambda_1\,, \qquad d\hat{s}^2\to e^{2\lambda_2}\,d\hat{s}^2\,,\\
&\hat{B}_{(2)}\to e^{-2\lambda_1+2\lambda_2}\,\hat{B}_{(2)}\,, \qquad \hat{A}^a_{(1)}\to e^{-\lambda_1+\lambda_2}\,\hat{A}^a_{(1)}\,.
\end{aligned}
\tag{3.5}
$$

Although the shift of the scalar field by λ_1 is a symmetry of the Lagrangian, the scaling transformation involving λ_2 on the metric is not since the Lagrangian will scale as $\sqrt{-\hat{g}}\,(\hat{R}+\cdots)\longrightarrow e^{(D-2)\,\lambda_2}\sqrt{-\hat{g}}\,(\hat{R}+\cdots)$.

We now reduce from D dimensions to $d=(D-1)$, while simultaneously gauging the above two global symmetries. The D-dimensional pure supergravity multiplet then reduces to d-dimensional supergravity coupled to a single vector multiplet. This is achieved by making the following generalized reduction ansatz

$$
\begin{aligned}
d\hat{s}^2 &= e^{2m_2 z}\Big(e^{2\alpha\varphi}\,ds^2 + e^{2\beta\varphi}\,(dz+\mathcal{A}_{(1)})^2\Big)\,,\\
\hat{B}_{(2)} &= e^{2(m_2-m_1)z}\Big(B_{(2)} + B_{(1)}\wedge dz\Big)\,,\\
\hat{A}^a_{(1)} &= e^{(m_2-m_1)z}\Big(A^a_{(1)} + \chi^a dz\Big)\,,\\
\hat{\phi} &= \phi + \frac{4}{\hat{a}}\,m_1\,z\,,
\end{aligned}
\tag{3.6}
$$

where

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}\,, \qquad \beta = -(d-2)\alpha\,. \tag{3.7}$$

The standard Kaluza-Klein ansatz for an ungauged S^1 reduction would correspond to setting $m_1 = m_2 = 0$.

In general, for unequal mass parameters m_1 and m_2, the lower-dimensional equations of motion are rather complicated. However, a significant simplification occurs if $m_1 = m_2$. In this case, various exponential factors drop out from (3.6), and one can consistently truncate out the vector multiplet, owing to conspiracies between the fields. In this manner, one can obtain variant gauged supergravities with positive-definite scalar potentials and with half-maximal supersymmetry in $d \le 9$ dimensions.

Before writing out the complete reduction of the bosonic equations of motion, we first collect some intermediate results. The reduction of the potentials in (3.6) yields a corresponding reduction on the field strengths:

$$\begin{aligned} \hat{H}_{(3)} &= e^{2(m_2-m_1)z}(H_{(3)} + H_{(2)} \wedge (dz + \mathcal{A}_{(1)}))\,, \\ \hat{F}^a_{(2)} &= e^{(m_2-m_1)z}(F^a_{(2)} + L^a_{(1)} \wedge (dz + \mathcal{A}_{(1)}))\,, \end{aligned} \tag{3.8}$$

where the lower dimensional fields are defined by

$$\begin{aligned} H_{(3)} &= dB_{(2)} - \tfrac{1}{2} F^a_{(2)} \wedge A^a_{(1)} - dB_{(1)} \wedge \mathcal{A}_{(1)} - 2(m_2 - m_1) B_{(2)} \wedge \mathcal{A}_{(1)} + \tfrac{1}{2}\chi^a F^a_{(2)} \wedge \mathcal{A}_{(1)}\,, \\ G_{(2)} &= dB_{(1)} - \tfrac{1}{2}\chi^a\, F^a_{(2)} + \tfrac{1}{2} L^a_{(1)} \wedge A^a_{(1)} - \tfrac{1}{2}\chi^a L^a_{(1)} \wedge \mathcal{A}_{(1)} + 2(m_2 - m_1) B_{(2)}\,, \\ F^a_{(2)} &= dA^a_{(1)} - d\chi^a \wedge \mathcal{A}_{(1)} + (m_2 - m_1) A^a_{(1)} \wedge \mathcal{A}_{(1)}\,, \\ L^a_{(1)} &= d\chi^a - (m_2 - m_1) A^a_{(1)}\,. \end{aligned} \tag{3.9}$$

The Kaluza-Klein potential $\mathcal{A}_{(1)}$ has the standard field strength $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. It is evident at this stage that the vector fields $A^a_{(1)}$ and the tensor field $B_{(2)}$ acquire masses proportional to $|m_2 - m_1|$, in the process eating the axions χ^a and the vector $B_{(1)}$

respectively.

B. Untruncated d-dimensional equations

We are now able to write down the full bosonic equations of motion for the variant d-dimensional gauged supergravity. The bosonic field content is

$$(g_{\mu\nu}, B_{\mu\nu}, \varphi, A^a_\mu, \mathcal{A}_\mu) \quad \text{and} \quad (B_\mu, \chi^a, \phi)\,, \tag{3.10}$$

corresponding to half-maximal supergravity coupled to a single vector multiplet. This representation is schematic in the sense that the scalars ϕ and φ as well as the 1-form potentials $B_{(1)}$ and $\mathcal{A}_{(1)}$ must necessarily be taken as appropriate linear combinations in the actual multiplets.

We find that the equations of motion for the form fields are given by

$$\begin{aligned}
\nabla^\sigma(e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}) &= (2m_1+(d-3)\,m_2)\left(e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}\mathcal{A}^\sigma - e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\right),\\
\nabla^\nu(e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}\mathcal{F}^{\nu\sigma}\\
&\quad +(2m_1+(d-3)\,m_2)e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\,\mathcal{A}^\nu,\\
\nabla^\nu(e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}F^{a\nu\sigma} + e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}L^{a\nu}\\
&\quad +(m_1+(d-2)\,m_2)\left(e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}\mathcal{A}^\nu - e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu\right),\\
\nabla^\mu(e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu) &= -\tfrac12 e^{\hat a\phi+2(d-3)\alpha\varphi}G_{\mu\nu}\,F^{a\mu\nu} + \tfrac12 e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}\mathcal{F}^{\mu\nu}\\
&\quad +(m_1+(d-2)\,m_2)e^{\frac12\hat a\phi+2(d-2)\alpha\varphi}L^a_\mu\mathcal{A}^\mu,\\
\nabla^\nu(e^{-2(d-1)\alpha\varphi}\mathcal{F}_{\mu\nu}) &= \tfrac12 e^{\hat a\phi-4\alpha\varphi}H_{\mu\nu\sigma}G^{\nu\sigma} - e^{\frac12\hat a\phi-2\alpha\varphi}F^a_{\mu\nu}L^{a\nu}\\
&\quad +\frac{4}{\hat a}\,m_1(\partial_\mu\phi - \frac{4}{\hat a}\,m_1\mathcal{A}_\mu) - 2m_2(d-1)\,(\beta\partial_\mu\varphi - m_2\mathcal{A}_\mu)\\
&\quad +m_2(d-1)e^{-2(d-1)\alpha\varphi}\,\mathcal{F}_{\mu\nu}\mathcal{A}^\nu.
\end{aligned} \tag{3.11}$$

The two scalar fields, ϕ and φ satisfy similar m_1 and m_2 dependent equations of motion. The scalar coming from the metric satisfies the equation

$$
\begin{aligned}
-\beta\Box\varphi \;=\;& -\frac{e^{\hat{a}\phi-4\alpha\varphi}}{6(d-1)}H_{(3)}^2 - \frac{e^{\frac12\hat{a}\phi-2\alpha\varphi}}{4(d-1)}(F_{(2)}^a)^2 + \frac{d-3}{4(d-1)}e^{\hat{a}\phi+2(d-3)\alpha\varphi}\,G_{(2)}^2 \\
&+\frac{d-2}{2(d-1)}\,e^{\frac12\hat{a}\phi+2(d-2)\alpha\varphi}(L_{(1)}^a)^2 - \tfrac14 e^{-2(d-1)\alpha\varphi}\mathcal{F}_{(2)}^2 \\
&- m_2\beta(d-1)\mathcal{A}^\mu\partial_\mu\varphi - m_2\nabla_\mu\mathcal{A}^\mu + m_2^2(d-1)\mathcal{A}_{(1)}^2 + \frac{8}{\hat{a}^2}\,m_1^2 e^{2(d-1)\alpha\varphi}\,,
\end{aligned}
\tag{3.12}
$$

while the D-dimensional dilaton equation reduces to

$$
\begin{aligned}
\Box\phi \;=\;& \frac{\hat{a}}{12}\,e^{\hat{a}\phi-4\alpha\varphi}H_{(3)}^2 + \frac{\hat{a}}{4}\,e^{\hat{a}\phi+2(d-3)\alpha\varphi}G_{(2)}^2 + \frac{\hat{a}}{8}\,e^{\frac12\hat{a}\phi-2\alpha\varphi}(F_{(2)}^a)^2 \\
&+\frac{\hat{a}}{4}\,e^{\frac12\hat{a}\phi+2(d-2)\alpha\varphi}(L_{(1)}^a)^2 + m_2(d-1)\mathcal{A}^\mu\partial_\mu\phi + \frac{4}{\hat{a}}\,m_1\nabla_\mu\mathcal{A}^\mu \\
&-\frac{4(d-1)}{\hat{a}}\,m_1 m_2\,(\mathcal{A}_{(1)}^2 + e^{2(d-1)\alpha\varphi})\,.
\end{aligned}
\tag{3.13}
$$

The d-dimensional Einstein equation takes the form

$$
\begin{aligned}
&R_{\mu\nu} - \tfrac12 R g_{\mu\nu} = \tfrac12(\partial_\mu\varphi\,\partial_\nu\varphi - \tfrac12(\partial\varphi)^2\,g_{\mu\nu}) + \tfrac12(\partial_\mu\phi\,\partial_\nu\phi - \tfrac12(\partial\phi)^2\,g_{\mu\nu}) \\
&+\tfrac12 e^{-2(d-1)\alpha\varphi}\,(\mathcal{F}_{\mu\sigma}\mathcal{F}_\nu{}^\sigma - \tfrac14 g_{\mu\nu}\mathcal{F}_{(2)}^2) + \tfrac14 e^{\hat{a}\phi-4\alpha\varphi}(H_{\mu\rho\sigma}\,H_\nu{}^{\rho\sigma} - \tfrac16 g_{\mu\nu}H_{(3)}^2) \\
&+\tfrac12 e^{\frac12\hat{a}\phi-2\alpha\varphi}(F_{\mu\sigma}^a\,F_\nu^{a\,\sigma} - \tfrac14 g_{\mu\nu}(F_{(2)}^a)^2) + \tfrac12 e^{\hat{a}\phi+2(d-3)\alpha\varphi}(G_{\mu\sigma}G_\nu{}^\sigma - \tfrac14 g_{\mu\nu}G_{(2)}^2) \\
&+\tfrac12 e^{\frac12\hat{a}\phi+2(d-2)\alpha\varphi}\,(L_\mu^a L_\nu^a - \tfrac12 g_{\mu\nu}\,(L_{(1)}^a)^2) \\
&-\alpha m_2(d-1)(\mathcal{A}^\sigma\partial_\sigma\varphi\,g_{\mu\nu} - \mathcal{A}_\mu\partial_\nu\varphi - \mathcal{A}_\nu\partial_\mu\varphi) \\
&+\frac{2}{\hat{a}}\,m_1(\mathcal{A}^\sigma\partial_\sigma\phi\,g_{\mu\nu} - \mathcal{A}_\mu\partial_\nu\phi - \mathcal{A}_\nu\partial_\mu\phi) + \Big(\frac{8}{\hat{a}^2}\,m_1^2 - (d-1)m_2^2\Big)\mathcal{A}_\mu\mathcal{A}_\nu \\
&-\tfrac12 m_2(d-1)(\nabla_\mu\mathcal{A}_\nu + \nabla_\nu\mathcal{A}_\mu - 2\nabla_\sigma\mathcal{A}^\sigma\,g_{\mu\nu}) \\
&-\Big(\frac{4m_1^2}{\hat{a}^2} + \tfrac12 m_2^2(d-1)(d-2)\Big)\,(\mathcal{A}_{(1)}^2 + e^{2(d-1)\alpha\varphi})g_{\mu\nu}\,.
\end{aligned}
\tag{3.14}
$$

Note that the last term is associated with a positive-definite scalar potential.

C. Truncated d-dimensional equations

The scalars ϕ and φ may be disentangled between the supergravity and vector multiplets of (3.10) by performing a rotation to ϕ_1 (supergravity) and ϕ_2 (vector) given by

$$\hat{a}\phi - 4\alpha\varphi = a\phi_1\,, \qquad 4\alpha\phi + \hat{a}\varphi = a\phi_2\,, \tag{3.15}$$

where $a = \sqrt{8/(D-3)}$. When $m_1 = m_2$, the vector multiplet may be further truncated away. This is done by setting

$$B_{(1)} = \mathcal{A}_{(1)} \equiv \tfrac{1}{\sqrt{2}} A_{(1)}\,, \qquad \phi_2 = 0\,, \qquad L^a_{(1)} = 0\,. \tag{3.16}$$

The equations of motion for the pure supergravity fields are then given by

$$\begin{aligned}
&\nabla^\rho\Big(e^{a\phi} H_{\mu\nu\rho}\Big) = \frac{d-1}{\sqrt{2}}\, m\Big(e^{a\phi}\, H_{\mu\nu\rho}\, A^\rho - e^{\frac12 a\phi} F_{\mu\nu}\Big)\,,\\
&\nabla^\nu\Big(e^{\frac12 a\phi} F_{\mu\nu}\Big) = \tfrac12 e^{a\phi} H_{\mu\nu\rho} F^{\nu\rho} + \frac{d-1}{\sqrt{2}}\, m e^{\frac12 a\phi} F_{\mu\nu}\, A^\nu\,,\\
&\nabla^\nu\Big(e^{\frac12 a\phi} F^a_{\mu\nu}\Big) = \tfrac12 e^{a\phi} H_{\mu\nu\rho}\, F^{a\,\nu\rho} + \frac{d-1}{\sqrt{2}}\, m e^{\frac12 a\phi} F^a_{\mu\nu}\, A^\nu\,,\\
&\Box\phi = \frac{e^{a\phi}}{3\sqrt{2(d-2)}} H^2_{(3)} + \frac{e^{\frac12 a\phi}}{2\sqrt{2(d-2)}} (F^2_{(2)} + (F^a_{(2)})^2) + \frac{d-1}{\sqrt{2}}\, m A^\mu\, \partial_\mu\phi\\
&\qquad + \frac{d-1}{\sqrt{d-2}}\, m \nabla_\mu\, A^\mu - \frac{\sqrt{2}\,(d-1)^2}{\sqrt{d-2}}\, m^2 (\tfrac12 A^2_{(1)} + e^{-\frac12 a\phi})\,,\\
&R_{\mu\nu} = \tfrac12 \partial_\mu\phi\partial_\nu\phi + \tfrac14 e^{a\phi}(H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - \frac{2}{3(d-2)} H^2_{(3)} g_{\mu\nu})\\
&+\tfrac12 e^{\frac12 a\phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(d-2)} F^2_{(2)} g_{\mu\nu}) + \tfrac12 e^{\frac12 a\phi}(F^a_{\mu\rho}F^{a\,\rho}_\nu - \frac{1}{2(d-2)} (F^a_{(2)})^2 g_{\mu\nu})\\
&-\frac{m(d-1)}{2\sqrt{d-2}}(A_\mu\partial_\nu\phi + A_\nu\partial_\mu\phi) - \frac{m(d-1)}{2\sqrt{2}}(\nabla_\mu A_\nu + \nabla_\nu A_\mu + \frac{2}{d-2}\nabla_\rho A^\rho g_{\mu\nu})\\
&+\frac{m^2(d-1)^2}{2(d-2)}(A^2_{(1)} + 2e^{-\frac12 a\phi}) g_{\mu\nu}\,,
\end{aligned} \tag{3.17}$$

where $H_{(3)} = dB_{(2)} - \frac12 F^a_{(2)} \wedge A^a_{(1)} - \frac12 F_{(2)} \wedge A_{(1)}\,,\ F^a_{(2)} = dA^a_{(1)}\,,\ F_{(2)} = dA_{(1)}$ and we have rewritten ϕ_1 as ϕ. It may be seen that this set of equations cannot be obtained from a

Lagrangian in terms of the physical fields. This is not altogether surprising, since they were derived in a generalized reduction that gauged a symmetry of the equations of motion which was not a symmetry of the Lagrangian. This is demonstrated from the fact that if there were a Lagrangian, it would from the truncated Einstein equation of motion have the term $m^2 A^2$. On the other hand, the equation of motion for the $A_{(1)}$ indicates that such a term should not exist. We furthermore note that all vectors in our theories are abelian but with the gauge symmetry of $A_{(1)}$ broken owing to the higher-order interactions. This is for example different from the Romans $d = 6$ gauged supergravity where the four vectors are the $SU(2) \times U(1)$ Yang-Mills fields.

By examining the linearized equations of motion, it can be seen that $A_{(1)}$ is a massless gauge potential. This gauge field can in fact be consistently set to zero. In this case, the remaining equations of motion can then be obtained from the Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{a\phi}H_{(3)}^2 - \tfrac{1}{4}e^{\frac{1}{2}a\phi}(F_{(2)}^a)^2 - (d-1)^2\, m^2 e^{-\frac{1}{2}a\phi}\,, \tag{3.18}$$

where $e = \sqrt{-g}$. Thus we see once again that the scalar potential is positive definite. The supergravities we have obtained have all vacuum solutions of the type Minkowski$\times$ sphere.

A few remarks are needed at this stage. Owing to the overall z-dependent scaling factor in the ansätze (3.6), the coordinate z cannot be viewed as a circle coordinate. Thus the theory is not compactified. To resolve such a problem, it was proposed in [28, 45] that one can modify the original supergravity by introducing an auxiliary field associated with the gauging of the scaling symmetry, which can be identified with the reduction coordinate in the dimensional reduction. The auxiliary field always appear in the equations through a derivative in the modified theory, and can therefore be defined as a circle coordinate in the reduction. Locally, this approach is the same as our generalized circular reduction, but globally, the internal direction is a circle instead of a real line. In fact, if we consider in our example the string frame, then there is no z-dependence in the metric when $m_1 = m_2$, and so z can be viewed as a circular coordinate at least from the metric point of view. An alternative approach is to introduce a delta function singularity à la Randall-Sundrum. We can then replace the prefactor in the metric $e^{2m\,z}$ by $e^{-2m\,|z|}$. By doing this, the volume of the internal direction will be finite even though z is a non-compact coordinate. Consequently, the gravity will be localized on the brane located at $z = 0$. The exponential nature of the warp factor in the conformal-frame metric implies that the effect of localization is strong with a mass gap. It would be interesting to study further if the delta function singularity in this procedure can be smoothed out.

With the derivation of the bosonic equations of motion completed, we now turn to a consideration of the supersymmetry. Although we have obtained new gauged supergravities in dimensions $d < 10$ we are going to derive the supersymmetry transformations in just $d = 6$ and $d = 9$ dimensions. We begin by investigating the supersymmetry of the $d = 6$, $\mathcal{N} = (1, 1)$ theory, and its spontaneous compactification to $d = 4$.

CHAPTER IV

$\mathcal{N} = (1,1)$ GAUGED SUPERGRAVITY AND (MINKOWSKI)$_4 \times S^2$ VACUA

A. Supersymmetry of the generalized reduction

The bosonic field content of half-maximum supergravity in seven dimensions comprises a metric $\hat{g}_{MN}$, a scalar $\hat{\phi}$, an antisymmetric tensor $\hat{B}_{(2)}$ and three vectors $\hat{A}^a_{(1)}$. The Lagrangian in the bosonic sector is [46, 47]

$$\hat{\mathcal{L}} = \hat{R}\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\hat{\phi} \wedge d\hat{\phi} - \tfrac{1}{2}e^{\frac{4}{\sqrt{10}}\hat{\phi}}\hat{*}\hat{H}_{(3)} \wedge \hat{H}_{(3)} - \tfrac{1}{2}e^{\frac{2}{\sqrt{10}}\hat{\phi}}\hat{*}\hat{F}^a_{(2)} \wedge \hat{F}^a_{(2)}, \tag{4.1}$$

where $\hat{F}^a_{(2)} = d\hat{A}^a_{(1)}$ and $\hat{H}_{(3)} = d\hat{B}_{(2)} - \frac{1}{2}\hat{F}^a_{(2)} \wedge \hat{A}^a_{(1)}$. The generalized Kaluza-Klein reduction of this theory was worked out in the previous chapter.

1. The supersymmetry transformations

The fermionic sector consists of a pair of symplectic-Majorana gravitinos $\hat{\psi}_{Mi}$ as well as a pair of dilatinos $\hat{\lambda}_i$, where $i = 1, 2$ is an Sp(1) index. The three vectors form a triplet under Sp(1), and may equivalently be written as $\hat{A}_{(1)\,i}{}^j = \hat{A}^a_{(1)}(-\tau^a)_i{}^j$ where τ^a are the usual Pauli matrices. In this form, the supersymmetry transformations on the fermions are given by

$$\begin{aligned}
\delta\hat{\psi}_{Mi} &= [\hat{\nabla}_M - \tfrac{1}{60}(\hat{\gamma}_M{}^{NPQ} - \tfrac{9}{2}\delta^N_M\hat{\gamma}^{PQ})\, e^{\frac{1}{2}\hat{a}\hat{\phi}}\hat{H}_{NPQ}]\hat{\epsilon}_i \\
&\quad + \tfrac{i}{20\sqrt{2}}(\hat{\gamma}_M{}^{NP} - 8\delta^N_M\hat{\gamma}^P)e^{\frac{1}{4}\hat{a}\hat{\phi}}\hat{F}_{NP\,i}{}^j\hat{\epsilon}_j, \\
\delta\hat{\lambda}_i &= [-\tfrac{1}{2\sqrt{2}}\hat{\gamma}^M\partial_M\hat{\phi} + \tfrac{1}{12\sqrt{5}}e^{\frac{1}{2}\hat{a}\hat{\phi}}\hat{H}_{MNP}\hat{\gamma}^{MNP}]\hat{\epsilon}_i - \tfrac{i}{4\sqrt{10}}\, e^{\frac{1}{4}\hat{a}\hat{\phi}}\hat{F}_{MNi}{}^j\hat{\gamma}^{MN}\hat{\epsilon}_j,
\end{aligned} \tag{4.2}$$

where $\hat{a} = 4/\sqrt{10}$.

In addition, the transformations on the bosonic fields have the form

$$
\begin{aligned}
\delta\hat{\phi} &= -\tfrac{1}{2\sqrt{2}}\bar{\hat{\epsilon}}^i\hat{\lambda}_i, \\
\delta\hat{g}_{MN} &= \tfrac{1}{2}\bar{\hat{\epsilon}}^i\hat{\gamma}_{(M}\hat{\psi}_{N)\,i}, \\
\delta\hat{A}_{M\,i}{}^j &= \tfrac{i}{\sqrt{2}}e^{-\frac{1}{4}\hat{a}\hat{\phi}}(\bar{\hat{\psi}}{}^j{}_M - \tfrac{1}{\sqrt{5}}\bar{\hat{\lambda}}{}^j\hat{\gamma}_M)\hat{\epsilon}_i, \\
\delta\hat{B}_{MN} &= -\tfrac{1}{2}\hat{A}_{[M\,i}{}^j\delta\hat{A}_{N]\,j}{}^i - \tfrac{1}{2}e^{-\frac{1}{2}\hat{a}\hat{\phi}}(\bar{\hat{\psi}}{}^i{}_{[M}\hat{\gamma}_{N]} - \tfrac{1}{\sqrt{5}}\bar{\hat{\lambda}}{}^i\hat{\gamma}_{MN})\hat{\epsilon}_i,
\end{aligned}
\tag{4.3}
$$

where in the transformation for $\hat{A}_{M\,i}{}^j$, the Sp(1) indices i and j are to be taken in the triplet combination. In particular, this may be enforced by the projection $(\delta_i^{i'}\delta_{j'}^j - \frac{1}{2}\delta_i^j\delta_{j'}^{i'})$ which removes the trace. Note that the transformation for $\delta\hat{B}_{MN}$ is given in a dualized form compared to that of [46].

The above fermionic (4.2) and bosonic (4.3) supersymmetries are normalized according to

$$
[\delta_1,\delta_2]\hat{\Xi} = \tfrac{1}{4}\hat{\xi}^M\partial_M\hat{\Xi} + (\text{local Lorentz}) + (\text{general coordinate}) + (\text{gauge}), \tag{4.4}
$$

where $\hat{\xi}^M = \bar{\hat{\epsilon}}{}_2^i\hat{\gamma}^M\hat{\epsilon}_{1i}$. Furthermore, when working with the fermions, it is often convenient to make use of the Majorana flip conditions

$$
\begin{aligned}
\bar{\hat{\chi}}{}^i\gamma_{M_1M_2\cdots M_n}\hat{\psi}_i &= (-)^n\bar{\hat{\psi}}{}^i\gamma_{M_nM_{n-1}\cdots M_1}\hat{\chi}_i, \\
\bar{\hat{\chi}}{}^j\gamma_{M_1M_2\cdots M_n}\hat{\psi}_i &= (-)^{n+1}\bar{\hat{\psi}}{}^j\gamma_{M_nM_{n-1}\cdots M_1}\hat{\chi}_i,
\end{aligned}
\tag{4.5}
$$

for the singlet and triplet combinations, respectively.

2. The bosonic reduction ansatz

As demonstrated in [24], the generalized S^1 reduction ansatz is given on the bosonic fields by

$$
d\hat{s}_7^2 = e^{2m_2z}\left(e^{2\alpha\varphi}ds_6^2 + e^{2\beta\varphi}(dz + \mathcal{A}_{(1)})^2\right),
$$

$$\begin{aligned}
\hat{B}_{(2)} &= e^{2(m_2-m_1)z}(B_{(2)} + B_{(1)} \wedge dz), \\
\hat{A}^a_{(1)} &= e^{(m_2-m_1)z}(A^a_{(1)} + \Phi^a dz), \\
\hat{\phi} &= \phi + \sqrt{10}\, m_1 z,
\end{aligned} \tag{4.6}$$

where $\alpha^2 = \frac{1}{40}$ and $\beta = -4\alpha$. The resulting reduction yields the six-dimensional fields $(g_{\mu\nu}, \mathcal{A}_{(1)}, A^a_{(1)}, B_{(2)}, \phi_1)$ and $(B_{(1)}, \phi_2, \Phi^a)$ corresponding to the bosonic content of $\mathcal{N} = (1,1)$ supergravity coupled to a vector multiplet. Note that the Sp(1) singlet graviphoton and the matter vector are in actuality given by linear combinations of $\mathcal{A}_{(1)}$ and $B_{(1)}$. However, the scalars ϕ_1 and ϕ_2, given by the rotated combinations

$$\begin{aligned}
\phi_1 &= \tfrac{2}{\sqrt{5}}\phi - \tfrac{1}{\sqrt{5}}\varphi, \\
\phi_2 &= \tfrac{1}{\sqrt{5}}\phi + \tfrac{2}{\sqrt{5}}\varphi,
\end{aligned} \tag{4.7}$$

are diagonal between multiplets.

3. The fermionic reduction

Working out the fermion reduction is straightforward, although somewhat tedious. Since the resulting $D = 6$ theory contains a vector multiplet in addition to the pure supergravity multiplet, the $D = 7$ fermions $\hat{\psi}_{M\,i}$ and $\hat{\lambda}_i$ must reduce to yield a $D = 6$ gravitino and dilatino $(\psi_{\mu\,i}, \lambda_i)$ as well as a gaugino χ_i. The reduction from seven to six dimensions is facilitated by the fact that the $D = 7$ symplectic-Majorana condition $\bar{\hat{\psi}}^i = -\epsilon^{ij}\hat{\psi}^T_j \hat{C}$ continues to apply in $D = 6$, yielding a trivial reduction on the spinors.

Examination of the supersymmetry transformations on the fermions, (4.2), indicates that the proper fermionic reduction is given by

$$\begin{aligned}
\hat{\epsilon}_i &= e^{\frac{1}{2}m_2 z} e^{\frac{1}{2}\alpha\varphi}\epsilon_i, \\
\hat{\lambda}_i &= \tfrac{1}{\sqrt{5}} e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi}(\chi_i + 2\lambda_i),
\end{aligned}$$

$$\begin{aligned}
\hat{\psi}_{z\,i} &= \tfrac{2}{5} e^{\frac{1}{2}m_2 z} e^{(\beta-\frac{1}{2}\alpha)\varphi}\gamma_7(2\chi_i-\lambda_i),\\
\hat{\psi}_{\mu\,i} &= e^{\frac{1}{2}m_2 z} e^{\frac{1}{2}\alpha\varphi}[\psi_{\mu\,i}+(\tfrac{2}{5}e^{(\beta-\alpha)\varphi}\mathcal{A}_\mu\gamma_7-\tfrac{1}{10}\gamma_\mu)(2\chi_i-\lambda_i)]. \qquad (4.8)
\end{aligned}$$

In this case, the resulting $D=6$ fermions have supersymmetry transformations

$$\begin{aligned}
\delta\psi_{\mu\,i} &= [\nabla_\mu - \tfrac{5}{8}m_2\gamma_\mu\gamma^\nu\mathcal{A}_\nu - \tfrac{1}{48}e^{\frac{1}{\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho\sigma}-3\delta_\mu^\nu\gamma^{\rho\sigma})H_{\nu\rho\sigma} + \tfrac{5}{8}m_2 e^{\frac{1}{\sqrt{2}}\phi_2-\frac{1}{2\sqrt{2}}\phi_1}\gamma_\mu\gamma_7\\
&\quad - \tfrac{1}{32}e^{\frac{1}{2\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho}-6\delta_\mu^\nu\gamma^\rho)\gamma_7(e^{\frac{1}{\sqrt{2}}\phi_2}H_{\nu\rho}+e^{-\frac{1}{\sqrt{2}}\phi_2}\mathcal{F}_{\nu\rho})]\epsilon_i\\
&\quad +[-\tfrac{i}{2\sqrt{2}}e^{\frac{1}{\sqrt{2}}\phi_2}\gamma_7 Q_{\mu\,i}{}^j + \tfrac{i}{16\sqrt{2}}e^{\frac{1}{2\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho}-6\delta_\mu^\nu\gamma^\rho)F_{\nu\rho\,i}{}^j]\epsilon_j,\\
\delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}}\gamma^\mu\partial_\mu\phi_1 + \tfrac{1}{4}(4m_1+m_2)\mathcal{A}_\mu\gamma^\mu + \tfrac{1}{24}e^{\frac{1}{\sqrt{2}}\phi_1}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\\
&\quad + \tfrac{1}{16}e^{\frac{1}{2\sqrt{2}}\phi_1}\gamma^{\mu\nu}\gamma_7(e^{\frac{1}{\sqrt{2}}\phi_2}H_{\mu\nu}+e^{-\frac{1}{\sqrt{2}}\phi_2}\mathcal{F}_{\mu\nu}) - \tfrac{1}{4}(4m_1+m_2)e^{\frac{1}{\sqrt{2}}\phi_2-\frac{1}{2\sqrt{2}}\phi_1}\gamma_7]\epsilon_i\\
&\quad - \tfrac{i}{8\sqrt{2}}e^{\frac{1}{2\sqrt{2}}\phi_1}\gamma^{\mu\nu}F_{\mu\nu\,i}{}^j\epsilon_j,\\
\delta\chi_i &= [-\tfrac{1}{2\sqrt{2}}\gamma^\mu\partial_\mu\phi_2 + \tfrac{1}{2}(m_1-m_2)\mathcal{A}_\mu\gamma^\mu - \tfrac{1}{2}(m_1-m_2)e^{\frac{1}{\sqrt{2}}\phi_2-\frac{1}{2\sqrt{2}}\phi_1}\gamma_7\\
&\quad - \tfrac{1}{8}e^{\frac{1}{2\sqrt{2}}\phi_1}\gamma^{\mu\nu}\gamma_7(-e^{\frac{1}{\sqrt{2}}\phi_2}H_{\mu\nu}+e^{-\frac{1}{\sqrt{2}}\phi_2}\mathcal{F}_{\mu\nu})]\epsilon_i - \tfrac{i}{2\sqrt{2}}e^{\frac{1}{\sqrt{2}}\phi_2}\gamma^\mu\gamma_7 Q_{\mu\,i}{}^j\epsilon_j. \qquad (4.9)
\end{aligned}$$

Here we recall that the $D=6$ field strengths are given by

$$\begin{aligned}
H_{(3)} &= dB_{(2)} - \tfrac{1}{2}F^a_{(2)}\wedge A^a_{(1)} - dB_{(1)}\wedge\mathcal{A}_{(1)} - 2(m_2-m_1)B_{(2)}\wedge\mathcal{A}_{(1)} + \tfrac{1}{2}\Phi^a F^a_{(2)}\wedge\mathcal{A}_{(1)},\\
H_{(2)} &= dB_{(1)} - \tfrac{1}{2}\Phi^a F^a_{(2)} + \tfrac{1}{2}Q^a_{(1)}\wedge A^a_{(1)} - \tfrac{1}{2}\Phi^a Q^a_{(1)}\wedge\mathcal{A}_{(1)} + 2(m_2-m_1)B_{(2)},\\
F^a_{(2)} &= dA^a_{(1)} - d\Phi^a\wedge\mathcal{A}_{(1)} + (m_2-m_1)A^a_{(1)}\wedge\mathcal{A}_{(1)},\\
Q^a_{(1)} &= d\Phi^a - (m_2-m_1)A^a_{(1)}, \qquad (4.10)
\end{aligned}$$

with $Q_{(1)\,i}{}^j = Q^a_{(1)}(-\tau^a)_{ij}$, *etc.* The gravitino transformation in (4.9) demonstrates that the Sp(1) singlet graviphoton arises as a linear combination of $H_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$. Note, further, that these transformations reduce to those of ordinary ungauged $\mathcal{N}=(1,1)$ supergravity coupled to a vector multiplet in the limit of vanishing m_1 and m_2.

Given the bosonic (4.6) and fermionic (4.8) reductions, it is now a matter of substituting these expressions into (4.3) to obtain the $D=6$ bosonic transformations. We find

$$\begin{aligned}
\delta\phi_1 &= -\tfrac{1}{2\sqrt{2}}\bar\epsilon^i\lambda_i,\\
\delta\phi_2 &= -\tfrac{1}{2\sqrt{2}}\bar\epsilon^i\chi_i,\\
\delta g_{\mu\nu} &= \tfrac{1}{2}\bar\epsilon^i\gamma_{(\mu}\psi_{\nu)\,i},\\
\delta\mathcal{A}_\mu &= \tfrac{1}{4}e^{-\frac{1}{2\sqrt{2}}\phi_1+\frac{1}{\sqrt{2}}\phi_2}[\bar\epsilon^i\gamma_7(\psi_{\mu\,i}+\tfrac{1}{2}\gamma_\mu\lambda_i)+\bar\epsilon^i\gamma_\mu\gamma_7\chi_i],\\
\delta A_{\mu\,i}{}^j &= -\Phi_i{}^j\delta\mathcal{A}_\mu-\tfrac{i}{\sqrt{2}}e^{-\frac{1}{2\sqrt{2}}\phi_1}\bar\epsilon^j(\psi_{\mu\,i}+\tfrac{1}{2}\gamma_\mu\lambda_i),\\
\delta\Phi_i{}^j &= -\tfrac{i}{\sqrt{2}}e^{-\frac{1}{\sqrt{2}}\phi_2}\bar\chi^j\gamma_7\epsilon_i, \qquad\qquad (4.11)\\
\delta B_\mu &= \tfrac{1}{4}\Phi_i{}^j(\delta A_{\mu\,j}{}^i+\Phi_j{}^i\delta\mathcal{A}_\mu)-\tfrac{1}{4}A_{\mu\,i}{}^j\delta\Phi_j{}^i\\
&\quad+\tfrac{1}{4}e^{-\frac{1}{2\sqrt{2}}\phi_1-\frac{1}{\sqrt{2}}\phi_2}[\bar\epsilon^i\gamma_7(\psi_{\mu\,i}+\tfrac{1}{2}\gamma_\mu\lambda_i)-\bar\epsilon^i\gamma_\mu\gamma_7\chi_i],\\
\delta B_{\mu\nu} &= -\tfrac{1}{2}A_{[\mu\,i}{}^j\Phi_j{}^i\delta\mathcal{A}_{\nu]}-2B_{[\mu}\delta\mathcal{A}_{\nu]}-\tfrac{1}{2}A_{[\mu\,i}{}^j\delta A_{\nu]\,j}{}^i-\tfrac{1}{2}e^{-\frac{1}{\sqrt{2}}\phi_1}\bar\epsilon^i(\gamma_{[\mu}\psi_{\nu]\,i}+\tfrac{1}{2}\gamma_{\mu\nu}\lambda_i).
\end{aligned}$$

This result, combined with (4.9) yield the complete (lowest order) supersymmetry transformations of the variant $\mathcal{N}=(1,1)$ supergravity coupled to a vector multiplet. Note that in obtaining (4.9) and (4.11), it was crucial that the ansatz (4.8) allowed a *consistent* reduction from seven to six dimensions, in which the dependence on the z coordinate cancelled in the seven-dimensional transformation rules. This guarantees that the resulting six-dimensional supersymmetry transformations are symmetries of the six-dimensional variant supergravity.

As noted in [24], the vector multiplet may be truncated away by setting $m_1=m_2$ as well as

$$\phi_2=0,\qquad \Phi_i{}^j=0,\qquad B_\mu=\mathcal{A}_\mu=\tfrac{1}{\sqrt{2}}A_\mu,\qquad \chi_i=0. \qquad (4.12)$$

In this case, the $D=6$ field strengths of (4.10) simplify to

$$
\begin{aligned}
H_{(3)} &= dB_{(2)} - \tfrac{1}{2}F^a_{(2)} \wedge A^a_{(1)} - \tfrac{1}{2}F_{(2)} \wedge A_{(1)}, \\
F_{(2)} &= dA_{(1)}, \qquad F^a_{(2)} = dA^a_{(1)}.
\end{aligned} \tag{4.13}
$$

The resulting six dimensional theory has field content $(g_{\mu\nu}, A_\mu, A^a_\mu, B_{\mu\nu}, \phi_1, \psi_{\mu\, i}, \lambda_i)$ and supersymmetry transformations

$$
\begin{aligned}
\delta\psi_{\mu\, i} &= [\nabla_\mu - \tfrac{5}{8\sqrt{2}} m\gamma_\mu\gamma^\nu A_\nu - \tfrac{1}{48} e^{\frac{1}{\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho\sigma} - 3\delta^\nu_\mu\gamma^{\rho\sigma})H_{\nu\rho\sigma} \\
&\quad - \tfrac{1}{16\sqrt{2}} e^{\frac{1}{2\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho} - 6\delta^\nu_\mu\gamma^\rho)\gamma_7 F_{\nu\rho} + \tfrac{5}{8} m e^{-\frac{1}{2\sqrt{2}}\phi_1}\gamma_\mu\gamma_7]\epsilon_i \\
&\quad + \tfrac{i}{16\sqrt{2}} e^{\frac{1}{2\sqrt{2}}\phi_1}(\gamma_\mu{}^{\nu\rho} - 6\delta^\nu_\mu\gamma^\rho)F_{\nu\rho\, i}{}^j\epsilon_j, \\
\delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}}\gamma^\mu\partial_\mu\phi_1 + \tfrac{5}{4\sqrt{2}} m A_\mu\gamma^\mu + \tfrac{1}{24} e^{\frac{1}{\sqrt{2}}\phi_1} H_{\mu\nu\rho}\gamma^{\mu\nu\rho} \\
&\quad + \tfrac{1}{8\sqrt{2}} e^{\frac{1}{2\sqrt{2}}\phi_1}\gamma^{\mu\nu}\gamma_7 F_{\mu\nu} - \tfrac{5}{4} m e^{-\frac{1}{2\sqrt{2}}\phi_1}\gamma_7]\epsilon_i - \tfrac{i}{8\sqrt{2}} e^{\frac{1}{2\sqrt{2}}\phi_1}\gamma^{\mu\nu}F_{\mu\nu\, i}{}^j\epsilon_j, \\
\delta\phi_1 &= -\tfrac{1}{2\sqrt{2}}\bar{\epsilon}^i\lambda_i, \\
\delta g_{\mu\nu} &= \tfrac{1}{2}\bar{\epsilon}^i\gamma_{(\mu}\psi_{\nu)\, i}, \\
\delta A_\mu &= \tfrac{1}{2\sqrt{2}} e^{-\frac{1}{2\sqrt{2}}\phi_1}\bar{\epsilon}^i\gamma_7(\psi_{\mu\, i} + \tfrac{1}{2}\gamma_\mu\lambda_i), \\
\delta A_{\mu\, i}{}^j &= -\tfrac{i}{\sqrt{2}} e^{-\frac{1}{2\sqrt{2}}\phi_1}\bar{\epsilon}^j(\psi_{\mu\, i} + \tfrac{1}{2}\gamma_\mu\lambda_i), \\
\delta B_{\mu\nu} &= -A_{[\mu}\delta A_{\nu]} - \tfrac{1}{2}A_{[\mu\, i}{}^j\delta A_{\nu]\, j}{}^i - \tfrac{1}{2} e^{-\frac{1}{\sqrt{2}}\phi_1}\bar{\epsilon}^i(\gamma_{[\mu}\psi_{\nu]\, i} + \tfrac{1}{2}\gamma_{\mu\nu}\lambda_i).
\end{aligned} \tag{4.14}
$$

These transformations reduce to those of [31] when $m \to 0$.

On the other hand, for $m \neq 0$, the generalized reduction yields additional terms in $\delta\psi_{\mu\, i}$ and $\delta\lambda_i$. Furthermore, these m-dependent terms do not have the usual structure for a gauged supergravity. In particular, the gauge potential $A_{(1)}$ does not appear in $\delta\psi_{\mu\, i}$ as a minimal coupling term $D_\mu = \nabla_\mu + igA_\mu$ to a charged spinor, yet shows up as a bare potential term in $\delta\lambda_i$. This is consistent with $A_{(1)}$ showing up as well in the bosonic equations of motion [24]. For this reason, it is natural to suspect that the local supersymmetry algebra satisfied by this theory is necessarily modified. To see this, we may examine, *e.g.*, the double variation on ϕ_1. We find

$$[\delta_1, \delta_2]\phi_1 = \tfrac{1}{4}\xi^\mu \partial_\mu \phi_1 - \tfrac{5}{4\sqrt{2}} m \Big(\tfrac{1}{\sqrt{2}} \xi^\mu A_\mu - e^{-\frac{1}{2\sqrt{2}}\phi_1} (\bar\epsilon_2^i \gamma_7 \epsilon_{1i}) \Big), \qquad (4.15)$$

where $\xi^\mu = \bar\epsilon_2^i \gamma^\mu \epsilon_{1i}$. The additional terms vanish when $m = 0$.

B. The (Minkowski)$_4 \times S^2$ reduction

The $D = 6$ theory obtained in [24] does not admit a Lagrangian formulation since the bare potential $A_{(1)}$ appears directly in the equations of motion. This is also apparent from the supersymmetry variations obtained in the previous section. However, for field configurations with vanishing $A_{(1)}$, the resulting bosonic equations of motion may be obtained from the Lagrangian

$$\mathcal{L} = \hat{R} \hat{*} 1\!\!1 - \tfrac{1}{4} \hat{*} d\hat\phi \wedge d\hat\phi - \tfrac{1}{2} e^{\hat\phi} \hat{*} \hat{H}_{(3)} \wedge \hat{H}_{(3)} - \tfrac{1}{2} e^{\frac{1}{2}\hat\phi} \hat{*} \hat{F}^a_{(2)} \wedge \hat{F}^a_{(2)} - 8 g^2 e^{-\frac{1}{2}\hat\phi} \hat{*} 1\!\!1. \qquad (4.16)$$

We have now introduced carets to denote six-dimensional fields, in anticipation of a subsequent reduction to four dimensions. Furthermore, we have defined $\hat\phi = \sqrt{2}\,\phi_1$ to simplify the subsequent expressions and have defined $5m = 2\sqrt{2}\,g$.

Curiously, this bosonic Lagrangian is identical to that of the Salam-Sezgin model, with the exception that there are three vector fields instead of one. As a result, this model clearly admits a bosonic $M_4 \times S^2$ reduction, where M_4 denotes four-dimensional Minkowski spacetime. On the other hand, the supersymmetry of the reduction must still be verified, as the supersymmetry transformations of the variant $\mathcal{N} = (1,1)$ theory differ from that of the gauged $\mathcal{N} = (1,0)$ model. In order to investigate the supersymmetry, it is useful to rewrite the six-dimensional symplectic-Majorana spinors using a Dirac notation. A symplectic-Majorana spinor satisfies the reality condition $(\hat{\psi}_i)^* = -\epsilon^{ij}\hat{C}\hat{\gamma}_0\hat{\psi}_j$, where the charge conjugation matrix $\hat{C}$ satisfies $\hat{C}^T = \hat{C}$ and $\hat{C}^\dagger\hat{C} = 1$. We may now form the Dirac combination $\hat{\psi} = \hat{\psi}_1 + i\hat{\psi}_2$, with complex conjugate $\hat{\psi}^* = -i\hat{C}\hat{\gamma}_0(\hat{\psi}_1 - i\hat{\psi}_2)$. Equivalently, these definitions may be inverted to yield

$$\hat{\psi}_1 = \tfrac{1}{2}(\hat{\psi} - i\hat{\gamma}_0\hat{C}^*\hat{\psi}^*), \qquad \hat{\psi}_2 = \tfrac{1}{2i}(\hat{\psi} + i\hat{\gamma}_0\hat{C}^*\hat{\psi}^*). \tag{4.17}$$

As a result, for $\hat{A}_{(1)} = 0$, the supersymmetry transformations (4.14) may be rewritten as

$$\begin{aligned}
\delta\hat{\psi}_\mu &= \Big[\hat{\nabla}_\mu - \tfrac{1}{48}e^{\frac{1}{2}\hat{\phi}}(\hat{\gamma}_\mu{}^{\nu\rho\sigma} - 3\delta_\mu^\nu\hat{\gamma}^{\rho\sigma})\hat{H}_{\nu\rho\sigma} + \tfrac{1}{2\sqrt{2}}ge^{-\frac{1}{4}\hat{\phi}}\hat{\gamma}_\mu\hat{\gamma}_7\Big]\hat{\epsilon} \\
&\quad + \tfrac{i}{16\sqrt{2}}e^{\frac{1}{4}\hat{\phi}}(\hat{\gamma}_\mu{}^{\nu\rho} - 6\delta_\mu^\nu\hat{\gamma}^\rho)\Big(\hat{F}^2_{\nu\rho}\hat{\epsilon} - (\hat{F}^1_{\nu\rho} - i\hat{F}^3_{\nu\rho})\hat{\gamma}_0\hat{C}^*\hat{\epsilon}^*\Big), \\
\delta\hat{\lambda} &= \Big[-\tfrac{1}{4}\hat{\gamma}^\mu\partial_\mu\hat{\phi} + \tfrac{1}{24}e^{\frac{1}{2}\hat{\phi}}\hat{H}_{\mu\nu\rho}\hat{\gamma}^{\mu\nu\rho} - \tfrac{1}{\sqrt{2}}ge^{-\frac{1}{4}\hat{\phi}}\hat{\gamma}_7\Big]\hat{\epsilon} \\
&\quad - \tfrac{i}{8\sqrt{2}}e^{\frac{1}{4}\hat{\phi}}\hat{\gamma}^{\mu\nu}\Big(\hat{F}^2_{\mu\nu}\hat{\epsilon} - (\hat{F}^1_{\mu\nu} - i\hat{F}^3_{\mu\nu})\hat{\gamma}_0\hat{C}^*\hat{\epsilon}^*\Big),
\end{aligned} \tag{4.18}$$

for the fermions, and

$$\begin{aligned}
\delta\hat{\phi} &= -\tfrac{1}{4}[\bar{\hat{\epsilon}}\hat{\lambda} + \bar{\hat{\lambda}}\hat{\epsilon}], \\
\delta\hat{g}_{\mu\nu} &= \tfrac{1}{2}[\bar{\hat{\epsilon}}\hat{\gamma}_{(\mu}\hat{\psi}_{\nu)} - \bar{\hat{\psi}}_{(\mu}\hat{\gamma}_{\nu)}\hat{\epsilon}], \\
\delta\hat{A}_\mu &= \tfrac{1}{4\sqrt{2}}e^{-\frac{1}{4}\hat{\phi}}\,[\bar{\hat{\epsilon}}\hat{\gamma}_7(\hat{\psi}_\mu + \tfrac{1}{2}\hat{\gamma}_\mu\hat{\lambda}) - (\bar{\hat{\psi}}_\mu - \tfrac{1}{2}\bar{\hat{\lambda}}\hat{\gamma}_\mu)\hat{\gamma}_7\hat{\epsilon}], \\
\delta\hat{A}^1_\mu &= -\tfrac{1}{2\sqrt{2}}e^{-\frac{1}{4}\hat{\phi}}\,\mathrm{Im}[\hat{\epsilon}^{\mathrm{T}}\hat{\mathrm{C}}(\hat{\psi}_\mu + \tfrac{1}{2}\hat{\gamma}_\mu\hat{\lambda})], \\
\delta\hat{A}^2_\mu &= -\tfrac{i}{4\sqrt{2}}e^{-\frac{1}{4}\hat{\phi}}\,[\bar{\hat{\epsilon}}(\hat{\psi}_\mu + \tfrac{1}{2}\hat{\gamma}_\mu\hat{\lambda}) - (\bar{\hat{\psi}}_\mu - \tfrac{1}{2}\bar{\hat{\lambda}}\hat{\gamma}_\mu)\hat{\epsilon}], \\
\delta\hat{A}^3_\mu &= -\tfrac{1}{2\sqrt{2}}e^{-\frac{1}{4}\hat{\phi}}\,\mathrm{Re}[\hat{\epsilon}^{\mathrm{T}}\hat{\mathrm{C}}(\hat{\psi}_\mu + \tfrac{1}{2}\hat{\gamma}_\mu\hat{\lambda})], \\
\delta\hat{B}_{\mu\nu} &= -\hat{A}^a_{[\mu}\delta\hat{A}^a_{\nu]} - \tfrac{1}{4}e^{-\frac{1}{2}\hat{\phi}}[\bar{\hat{\epsilon}}(\hat{\gamma}_{[\mu}\hat{\psi}_{\nu]} + \tfrac{1}{2}\hat{\gamma}_{\mu\nu}\hat{\lambda}) + (\bar{\hat{\psi}}_{[\mu}\hat{\gamma}_{\nu]} - \tfrac{1}{2}\bar{\hat{\lambda}}\hat{\gamma}_{\mu\nu})\hat{\epsilon}],
\end{aligned} \tag{4.19}$$

for the bosons. While we have set $\hat{A}_\mu = 0$, it is important to retain its supersymmetry variation so that it is possible to check later for consistency. These expressions serve as the starting point for the subsequent analysis.

1. Supersymmetry of the $M_4 \times S^2$ vacuum

The bosonic theory, given by (4.16), admits an $M_4 \times S^2$ solution given by

$$d\hat{s}_6^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{8g^2} d\Omega_2^2,$$
$$\hat{F}^2_{(2)} = \frac{1}{2g} \Omega_{(2)}, \tag{4.20}$$

where $\Omega_{(2)} = \sin\theta\, d\theta \wedge d\varphi$ is the volume form on the unit S^2. Note that we have singled out the 2-component of the Sp(1) triplet gauge fields for convenience. While this choice is a natural one corresponding to the Dirac combination in (4.18), any other choice would yield the same result.

To examine the supersymmetry of the vacuum, we insert (4.20) into (4.18) to obtain

$$\begin{aligned}
\delta\hat{\psi}_\alpha &= [\partial_\alpha + \tfrac{1}{\sqrt{2}} g\, \hat{\gamma}_\alpha \hat{\gamma}_7 P_+]\hat{\epsilon}, \\
\delta\hat{\psi}_a &= [\nabla_a - i\sqrt{2} g\, \hat{\gamma}_a \hat{\gamma}_{45}]\hat{\epsilon} + \tfrac{1}{\sqrt{2}} g \hat{\gamma}_a \hat{\gamma}_7 P_+ \hat{\epsilon}, \\
\delta\hat{\lambda} &= -\sqrt{2} g\, \hat{\gamma}_7 P_+ \hat{\epsilon}
\end{aligned} \tag{4.21}$$

where $P_\pm = \frac{1}{2}(1 \pm i\hat{\gamma}^{45}\hat{\gamma}_7)$ is a half-BPS projection. These equations vanish for $\hat{\epsilon} = P_-\hat{\epsilon}_0$ where $\hat{\epsilon}_0$ solves the Killing spinor equation on the round 2-sphere, $[\nabla_a - i\sqrt{2}g\, \hat{\gamma}_a \hat{\gamma}_{45}]\hat{\epsilon}_0 = 0$.

To be more precise, we decompose the six-dimensional Dirac matrices according to

$$\begin{aligned}
&\hat{\gamma}_\alpha = \gamma_\alpha \otimes \sigma_3, \qquad \hat{\gamma}_4 = \mathbb{1} \otimes \sigma_1, \qquad \hat{\gamma}_5 = \mathbb{1} \otimes \sigma_2, \\
&\hat{\gamma}_7 = \hat{\gamma}_0 \hat{\gamma}_1 \cdots \hat{\gamma}_5 = \gamma^5 \otimes \sigma_3, \qquad \hat{C} = C \otimes \sigma_2
\end{aligned} \tag{4.22}$$

where C is now the four-dimensional charge conjugation matrix and $\gamma^5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Six dimensional spinors $\hat{\epsilon}$ may then be written in terms of M_4 and S^2 spinors as

$\hat{\epsilon} = \sum_I \epsilon_I \otimes \eta_I$ where η_I is taken to be commuting. In this case, the Killing spinor equation on S^2 becomes $[\nabla_a + \sqrt{2}g\,\sigma_a\sigma_3]\eta_I = 0$, and yields two independent solutions. Corresponding to the above choice of Dirac matrices, we find that in the basis $e^4 = (2\sqrt{2}g)^{-1}\,d\theta$, $e^5 = (2\sqrt{2}g)^{-1}\sin\theta\,d\varphi$, the two independent Killing spinors can be written as

$$\eta_1 = \begin{pmatrix} \cos\frac{1}{2}\theta \\ -\sin\frac{1}{2}\theta \end{pmatrix} e^{\frac{i}{2}\varphi}, \qquad \eta_2 = \begin{pmatrix} \sin\frac{1}{2}\theta \\ \cos\frac{1}{2}\theta \end{pmatrix} e^{-\frac{i}{2}\varphi}. \tag{4.23}$$

It is easily seen that these satisfy the conditions

$$\bar{\eta}_I\eta_J = \delta_{IJ}, \qquad \eta_I^T\sigma^2\eta_J = -i\epsilon_{IJ}, \qquad \eta_I^* = i\sigma^2\epsilon_{IJ}\eta_J. \tag{4.24}$$

Note that $\bar{\eta}_I \equiv \eta_I^\dagger$. Using the decomposition (4.22), the half-BPS projection operator takes the form $P_\pm = \frac{1}{2}(1 \mp \gamma_5)$. As a result, the Killing spinors in the $M_4 \times S^2$ background are given by

$$\hat{\epsilon} = \epsilon_I \otimes \eta_I \qquad (\epsilon_I = \gamma_5\epsilon_I), \tag{4.25}$$

where the ϵ_I are a pair of constant $D = 4$ Weyl spinors.

2. Reduction to $D = 4$, $\mathcal{N} = 2$ supergravity

The existence of a supersymmetric vacuum suggests that a consistent Kaluza-Klein reduction on S^2 is possible, yielding a Poincaré theory in four dimensions. Since the six-dimensional $\mathcal{N} = (1,1)$ theory has 16 real supersymmetries, and the vacuum breaks exactly half of them, the resulting theory corresponds to $\mathcal{N} = 2$ supersymmetry in four dimensions.

The basic $\mathcal{N} = 2$ supergravity multiplet consists of a graviton $g_{\mu\nu}$, graviphoton $A_{(1)}$ and a pair of Majorana gravitinos $\psi_{\mu\,i}$. In addition, $\mathcal{N} = 2$ vector multiplets are given by a vector $A_{(1)}$, two real scalars ϕ and a, and a pair of Majorana gauginos χ_i.

We find that the six dimensional field content reduces to yield $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet. The reduction ansatz for the bosons is given by

$$\begin{aligned}
&d\hat{s}_6^2 = e^{\frac{1}{2}\phi} ds_4^2 + \frac{1}{8g^2} e^{-\frac{1}{2}\phi} d\Omega_2^2, \\
&\hat{F}^2_{(2)} = 2g e^{\frac{1}{2}\phi} \epsilon_{ab} \hat{e}^a \wedge \hat{e}^b, \qquad \hat{F}^1_{(2)} = F^1_{(2)}, \qquad \hat{F}^3_{(2)} = F^3_{(2)}, \\
&\hat{H}_{(3)} = H_{(3)}, \qquad \hat{\phi} = -\phi.
\end{aligned} \tag{4.26}$$

Note that the graviphoton and matter vector field strengths are given by a combination of $F^1_{(2)}$ and $F^3_{(2)}$ (up to duality) as will be apparent below. The use of the 1- and 3-components of the Sp(1) triplet in the Kaluza-Klein reduction is dictated by the choice of turning on $F^2_{(2)}$ flux on the sphere.

It is straightforward to verify the consistency of the bosonic reduction. The resulting four-dimensional equations of motion may be obtained from the Lagrangian

$$\mathcal{L} = R{*}\mathbb{1} - \tfrac{1}{2}{*}d\phi \wedge d\phi - \tfrac{1}{2} e^{-2\phi} {*}H_{(3)} \wedge H_{(3)} - \tfrac{1}{2} e^{-\phi} ({*}F^1_{(2)} \wedge F^1_{(2)} + {*}F^3_{(2)} \wedge F^3_{(2)}). \tag{4.27}$$

The fermion reduction ansatz may be obtained by substituting the bosonic fields (4.26) into the six-dimensional gravitino and dilatino transformations (4.18). Starting with the latter, we see that

$$\begin{aligned}
\delta\hat{\lambda} \;=\; & \sqrt{2} g e^{\frac{1}{4}\phi} P_+ \otimes \sigma_3 \hat{\epsilon} + e^{-\frac{1}{4}\phi} [\tfrac{1}{4}\gamma^\mu \partial_\mu \phi + \tfrac{1}{24} e^{-\phi} H_{\mu\nu\rho} \gamma^{\mu\nu\rho}] \otimes \sigma_3 \hat{\epsilon} \\
& - \tfrac{i}{8\sqrt{2}} e^{-\frac{1}{4}\phi} [e^{-\frac{1}{2}\phi} (F^1_{\mu\nu} - i F^3_{\mu\nu}) \gamma^{\mu\nu} \gamma_0 C^*] \otimes \sigma_3 \sigma_2 \hat{\epsilon}^*.
\end{aligned} \tag{4.28}$$

The first term vanishes on chiral spinors $P_+ \hat{\epsilon} = 0$, while the remaining terms combine to yield the four-dimensional gaugino transformation.

Turning to the gravitino variation, as usual the $D = 6$ variation splits into a $D = 4$ gravitino term, $\delta\hat{\psi}_\alpha$, as well as two internal variations, $\delta\hat{\psi}_a$. Since the S^2 symmetry is unbroken by the bosonic ansatz, the two internal components of the gravitino variation are related by symmetry. In fact, provided $\hat{\epsilon}$ is decomposed in terms of Killing spinors on the sphere, the $\delta\hat{\psi}_a$ variation has identical content as that of $\delta\hat{\lambda}$. (This is not in general true, but holds in the present case.) As a result, we find the fermionic reduction ansatz to have the form

$$\begin{aligned}\hat{\epsilon} &= e^{\frac{1}{8}\phi}\epsilon_I \otimes \eta_I,\\ \hat{\lambda} &= e^{-\frac{1}{8}\phi}\chi_I \otimes \sigma_3\eta_I,\\ \hat{\psi}_\alpha &= e^{-\frac{1}{8}\phi}[\psi_{\alpha\, I} + \tfrac{1}{2}\gamma_\alpha\chi_I] \otimes \eta_I, \qquad \hat{\psi}_a = e^{-\frac{1}{8}\phi}(-\tfrac{1}{2}\chi_I) \otimes \sigma_a\sigma_3\eta_I. \end{aligned} \tag{4.29}$$

Inserting this ansatz into (4.28) as well as the gravitino variations yields the four-dimensional supersymmetry transformations

$$\begin{aligned}\delta\chi_I &= [\tfrac{1}{4}\gamma^\mu\partial_\mu\phi + \tfrac{1}{24}e^{-\phi}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}]\epsilon_I - \tfrac{1}{4\sqrt{2}}e^{-\frac{1}{2}\phi}(F^1_{\mu\nu} - iF^3_{\mu\nu})\gamma^{\mu\nu}\gamma_0 C^*\epsilon_{IJ}\epsilon^*_J,\\ \delta\psi_{\mu\, I} &= [\nabla_\mu - \tfrac{1}{24}e^{-\phi}\gamma_\mu{}^{\nu\rho\sigma}H_{\nu\rho\sigma}]\epsilon_I - \tfrac{1}{8\sqrt{2}}e^{-\frac{1}{2}\phi}(F^1_{\nu\rho} - iF^3_{\nu\rho})\gamma^{\nu\rho}\gamma_\mu\gamma_0 C^*\epsilon_{IJ}\epsilon^*_J. \end{aligned} \tag{4.30}$$

To obtain this result, we had to make use of the η^*_I relation in (4.24). At this stage, we note that the gauge fields may be dualized in four dimensions, so that $F_{\mu\nu}\gamma^{\mu\nu} = -i{*F}_{\mu\nu}\gamma^{\mu\nu}\gamma_5$. Since the four-dimensional spinors are given in a Weyl basis

$$P_+\,\epsilon_I = 0, \qquad P_+\,\psi_{\alpha\, I} = 0, \qquad P_-\,\chi_I = 0, \tag{4.31}$$

where $P_\pm = \frac{1}{2}(1 \mp \gamma_5)$, the above supersymmetry variations may be rewritten as

$$\begin{aligned}\delta\chi_I &= [\tfrac{1}{4}\gamma^\mu\partial_\mu\phi + \tfrac{1}{24}e^{-\phi}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}]\epsilon_I - \tfrac{1}{4\sqrt{2}}e^{-\frac{1}{2}\phi}(F^1_{\mu\nu} + {*F}^3_{\mu\nu})\gamma^{\mu\nu}\gamma_0 C^*\epsilon_{IJ}\epsilon^*_J,\\ \delta\psi_{\mu\, I} &= [\nabla_\mu - \tfrac{1}{24}e^{-\phi}\gamma_\mu{}^{\nu\rho\sigma}H_{\nu\rho\sigma}]\epsilon_I - \tfrac{1}{8\sqrt{2}}e^{-\frac{1}{2}\phi}(F^1_{\nu\rho} - {*F}^3_{\nu\rho})\gamma^{\nu\rho}\gamma_\mu\gamma_0 C^*\epsilon_{IJ}\epsilon^*_J. \end{aligned} \tag{4.32}$$

This highlights the nature of the $\mathcal{N} = 2$ graviphoton, $F^{(\mathcal{N}=2)}_{(2)} = e^{-\frac{1}{2}\phi}F^1_{(2)} + e^{\frac{1}{2}\phi}\widetilde{F}^3_{(2)}$, where $\widetilde{F}^3_{(2)} = e^{-\phi}{*F}^3_{(2)}$.

Having completed the fermion reduction and supersymmetry variations, we now turn to the reduction of the bosonic variations, (4.19). The six-dimensional dilaton variation $\delta\hat{\phi}$ readily yields $\delta\phi = \frac{1}{2}\bar{\epsilon}_I\chi_I$. Similarly, the four-dimensional components of $\delta\hat{g}_{\mu\nu}$ yield $\delta g_{\mu\nu} = \frac{1}{2}\bar{\epsilon}_I\gamma_{(\mu}\psi_{\nu)\,I}$, while the internal components reduce to give the identical $\delta\phi$ transformation. This is a result of setting the internal components of the six-dimensional gravitino equal to the dilatino in the reduction.

In general, one obtains non-trivial vector field variations from the mixed components of the metric, $\delta\hat{g}_{\mu i}$, as well as directly from $\delta\hat{A}_\mu$. However, these terms vanish identically based on the $P_\pm$ chiralities of the four-dimensional spinors. Likewise, $\delta\hat{A}^2_\mu$ vanishes for the same reason. On the other hand, the additional complex conjugation appearing in $\delta\hat{A}^1_\mu$ and $\delta\hat{A}^3_\mu$ prevents these transformations from vanishing. The resulting four-dimensional variations then have the form

$$
\begin{aligned}
\delta g_{\mu\nu} &= \tfrac{1}{4}[\bar{\epsilon}_I\gamma_{(\mu}\psi_{\nu)\,I} - \bar{\psi}_{(\mu\,I}\gamma_{\nu)}\epsilon_I], \\
\delta\phi &= \tfrac{1}{4}[\bar{\epsilon}_I\chi_I + \bar{\chi}_I\epsilon_I], \\
\delta B_{\mu\nu} &= -\tfrac{1}{4}e^{\phi}\,[\bar{\epsilon}_I\gamma_{[\mu}\psi_{\nu]\,I} + \bar{\psi}_{[\mu\,I}\gamma_{\nu]}\epsilon_I + \bar{\epsilon}_I\gamma_{\mu\nu}\chi_I - \bar{\chi}_I\gamma_{\mu\nu}\epsilon_I], \\
\delta A^1_\mu &= \tfrac{1}{2\sqrt{2}}e^{\frac{1}{2}\phi}\epsilon_{IJ}\mathrm{Re}[\epsilon^{\mathrm{T}}_{\mathrm{I}}\mathrm{C}(\psi_{\mu\,\mathrm{J}} + \tfrac{1}{2}\gamma_\mu\chi_{\mathrm{J}})], \\
\delta A^3_\mu &= -\tfrac{1}{2\sqrt{2}}e^{\frac{1}{2}\phi}\epsilon_{IJ}\mathrm{Im}[\epsilon^{\mathrm{T}}_{\mathrm{I}}\mathrm{C}(\psi_{\mu\,\mathrm{J}} + \tfrac{1}{2}\gamma_\mu\chi_{\mathrm{J}})].
\end{aligned}
\qquad (4.33)
$$

We have verified that all variations of fields initially set to zero vanish, either identically or through four-dimensional chirality. This verifies the consistency of the supersymmetric reduction to $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet.

3. Truncation to $D = 4$, $\mathcal{N} = 1$ supergravity

While we have retained $\mathcal{N} = 2$ supersymmetry in the above reduction, there is a natural truncation to $\mathcal{N} = 1$. This may be accomplished by removing one of the two supersymmetry parameters by setting $\epsilon_I = \hat{n}_I \epsilon$ where $\hat{n}_I$ is any constant unit vector. At the same time, it is necessary to truncate the $\mathcal{N}=1$ gravitino and vector multiplets, leaving $\mathcal{N}=1$ supergravity coupled to a chiral multiplet. In the bosonic sector, this corresponds to setting $A^1_\mu = A^3_\mu = 0$. The resulting bosonic Lagrangian is given by

$$\mathcal{L} = R{*}\mathbb{1} - \tfrac{1}{2}{*}d\phi \wedge d\phi - \tfrac{1}{2}e^{-2\phi}{*}H_{(3)} \wedge H_{(3)}, \tag{4.34}$$

while the relevant supersymmetry transformations are

$$\begin{aligned}
\delta\chi &= [\tfrac{1}{4}\gamma^\mu \partial_\mu \phi + \tfrac{1}{24}e^{-\phi}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}]\epsilon, \\
\delta\psi_\mu &= [\nabla_\mu - \tfrac{1}{24}e^{-\phi}\gamma_\mu{}^{\nu\rho\sigma}H_{\nu\rho\sigma}]\epsilon, \\
\delta g_{\mu\nu} &= \tfrac{1}{4}[\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} - \bar{\psi}_{(\mu}\gamma_{\nu)}\epsilon], \\
\delta\phi &= \tfrac{1}{4}[\bar{\epsilon}\chi + \bar{\chi}\epsilon], \\
\delta B_{\mu\nu} &= -\tfrac{1}{4}e^{\phi}[\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \bar{\psi}_{[\mu}\gamma_{\nu]}\epsilon + \bar{\epsilon}\gamma_{\mu\nu}\chi - \bar{\chi}\gamma_{\mu\nu}\epsilon].
\end{aligned} \tag{4.35}$$

C. BPS solutions

The bosonic Lagrangian (4.27) admits a dyonic black hole solution where $F^1_{(2)}$ is electric and $F^3_{(2)}$ is magnetic (or vice versa). The solution is given by

$$\begin{aligned}
ds_4^2 &= -(\mathcal{H}_1\,\mathcal{H}_3)^{-1}\,dt^2 + \mathcal{H}_1\,\mathcal{H}_3(dr^2 + r^2\,d\widetilde{\Omega}_2^2)\,, \\
F^1_{(2)} &= dt \wedge d\mathcal{H}_1^{-1}\,, \qquad F^3_{(2)} = q_3\,\widetilde{\Omega}_{(2)}\,, \\
\phi &= -\log(\mathcal{H}_1/\mathcal{H}_3)\,,
\end{aligned} \tag{4.36}$$

where $\mathcal{H}_1 = 1 + q_1/r$ and $\mathcal{H}_3 = 1 + q_3/r$ are two harmonic functions in the Euclidean three-dimensional transverse space. It becomes the standard Reissner-Nordström black hole when $\mathcal{H}_1 = \mathcal{H}_3$. We can easily lift the solution back to $D = 6$ dimensions, and it becomes

$$
\begin{aligned}
ds_6^2 &= (\mathcal{H}_1/\mathcal{H}_3)^{\frac{1}{2}} \Big[-\mathcal{H}_1^{-2}\, dt^2 + \mathcal{H}_3^2\,(dr^2 + r^2\, d\widetilde{\Omega}_2^2) + \frac{1}{8g^2}\, d\Omega_2^2\Big]\,,\\
\hat{F}^2_{(2)} &= \tfrac{1}{2} g^{-1}\,\Omega_{(2)}\,, \qquad \hat{F}^1_{(2)} = dt\wedge d\mathcal{H}_1^{-1}\,, \qquad \hat{F}^3_{(2)} = q_3\,\widetilde{\Omega}_{(2)}\,,\\
\hat{\phi} &= \log(\mathcal{H}_1/\mathcal{H}_3)\,.
\end{aligned} \tag{4.37}
$$

In the near horizon limit, the geometry becomes $\mathrm{AdS}_2\times S^2\times S^2$. For $\mathcal{H}_1=\mathcal{H}_3$, the metric is the direct product of an S^2 and the Reissner-Nordström black hole. In the string frame, the metric is given by

$$
ds^2_{\rm str} = -\mathcal{H}_1^{-2}\, dt^2 + \mathcal{H}_3^2\,(dr^2 + r^2\, d\widetilde{\Omega}_2^2) + \frac{1}{8g^2}\, d\Omega_2^2 \tag{4.38}
$$

D. (Minkowski)$_3\times S^3$ vacuum

The variant $\mathcal{N}=(1,1)$ six-dimensional supergravity has the unusual feature that it admits not only a supersymmetric (Minkowski)$_4\times S^2$ vacuum, but also a supersymmetric (Minkowski)$_3\times S^3$ vacuum. This is quite different from the situation in the Salam-Sezgin theory; although the Salam-Sezgin model admits a (Minkowski)$_3\times S^3$ solution as well as a supersymmetric (Minkowski)$_4\times S^2$ solution, the former is non-supersymmetric.

To construct the supersymmetric (Minkowski)$_3\times S^3$ solution in the variant $\mathcal{N}=(1,1)$ supergravity, we make a standard Freund-Rubin type ansatz in which

$$
d\hat{s}_6^2 = dx^\mu\, dx_\mu + ds_3^2\,, \qquad \hat{H}_{(3)} = q\,\epsilon_{(3)}\,, \qquad \hat{\phi}=0\,, \tag{4.39}
$$

where ds_3^2 is the metric on a round S^3, with volume form $\epsilon_{(3)}$, and all other fields are set to zero. We find that this solves the six-dimensional equations of motion if

$$
q = 2\sqrt{2}\, g\,. \tag{4.40}
$$

The S^3 metric has Ricci tensor given by $R_{ij} = 4g^2\, g_{ij}$.

To establish the supersymmetry of the solution, we decompose the six dimensional Dirac matrices as

$$\hat{\gamma}_\mu = \gamma_\mu \otimes 1\!\!1 \otimes \sigma_2\,, \quad \hat{\gamma}_i = 1\!\!1 \otimes \gamma_i \otimes \sigma_1\,, \quad \hat{\gamma}_7 = 1\!\!1 \otimes 1\!\!1 \otimes \sigma_3\,. \tag{4.41}$$

Writing $\hat{\epsilon} = \epsilon \otimes \eta \otimes \nu$, we find from the transformation rules (4.18) that supersymmetry is preserved if ϵ is a constant spinor in the (Minkowski)$_3$ spacetime, $\sigma_2\, \nu = \nu$ and if η is a Killing spinor on S^3, satisfying

$$\nabla_i\, \eta = \frac{i\, g}{\sqrt{2}}\, \gamma_i\, \eta\,. \tag{4.42}$$

Thus the solution has three-dimensional $\mathcal{N} = 4$ supersymmetry.

E. Discussion

In this chapter we have presented the complete supersymmetry of the new gauged $\mathcal{N} = (1,1)$ theory. This theory differs from the conventional supergravities with gauged R-symmetry in the sense that the bare vector potential terms in (4.14) do not correspond to the usual minimal coupling to charged fermions. For a vanishing Sp(1) singlet, the $\mathcal{N} = (1,1)$ theory reduces in its bosonic sector to the Salam-Sezgin $\mathcal{N} = (1,0)$ model, albeit with a triplet of gauge fields. In this truncation the supersymmetry transformation rules of the $\mathcal{N} = (1,1)$ theory do not give rise to the supersymmetry of the gauged $\mathcal{N} = (1,0)$ model. The reason for this is because the singlet and triplet gauge fields of the $\mathcal{N} = (1,1)$ supergravity reside in the gravitino multiplet, and not a vector multiplet, as would be necessary for obtaining a Salam-Sezgin truncation. This implies that, although our theory admits similar solutions to the Salam-Sezgin model, their supersymmetry can be drastically different.

We have shown that the variant $\mathcal{N} = (1, 1)$ supergravity admits a consistent S^2 reduction giving rise to $D = 4$, $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet which can be truncated further to $\mathcal{N} = 1$ supergravity with a chiral mulitplet. Although we have used a Weyl notation for the four-dimensional fermions, and there is a presence of 'left-handed' gravitinos in (4.31), this itself is not an indication of actual chirality. We should emphasize that the $M_4 \times S^2$ reduction of the Salam-Sezgin model likewise is non-chiral. This is understod by that a smooth Kaluza-Klein reduction in the gravitational sector cannot lead to a chiral theory in four dimensions [48]. However, [48] goes on to indicate that chirality may be obtained by starting with chiral fermions coupled to gauge fields in the higher dimensional theory, provided the gauge reduction is non-trivial. In particular, reductions with a monopole flux such as [49] could in principle give rise to four dimensional chirality. This would suggest that the Salam-Sezgin model is chiral, since it precisely involves turning on such a $U(1)$ monopole flux, with all fermions charged under this $U(1)$. However, as emphasized in [14, 17], the $U(1)$ does not survive the reduction to four dimensions. The resulting theory contains only $SU(2)$ gauge fields and uncharged fermions, and is hence non-chiral.

CHAPTER V

GAUGED SUPERGRAVITY IN NINE DIMENSIONS

In this chapter we continue our study of the supersymmetry in the generalized Kaluza-Klein reduction. We shall present the results for two cases. The first is the variant ten-dimensional massive gauged supergravity obtained in [21] by performing a generalized reduction of eleven-dimensional supergravity.[1] The reduction in this case involves just the global scaling symmetry of the $D = 11$ equations of motion. Then, we shall consider the nine-dimensional massive gauged theory obtained from massless $D = 10, \mathcal{N} = 1$ supergravity, using the generalized reduction involving the two global symmetries that we discussed in chapter III.

A. Massive type-IIA supergravity from $D = 11$

The supersymmetry transformations in $D = 11$ are

$$\begin{aligned} &\delta\hat{e}_M{}^A = \hat{\bar{\epsilon}}\hat{\gamma}^A\hat{\psi}_M\,, \qquad \delta\hat{A}_{MNP} = 3\hat{\bar{\epsilon}}\hat{\gamma}_{[MN}\hat{\psi}_{P]}\,, \\ &\delta\hat{\psi}_M = \widehat{\nabla}_M\hat{\epsilon} - \tfrac{1}{288}\hat{F}_{NPQR}(\hat{\gamma}_M{}^{NPQR} - 8\hat{\gamma}^{PQR}\delta^N_M)\hat{\epsilon}\,, \end{aligned} \tag{5.1}$$

where in our conventions

$$\{\hat{\gamma}_A, \hat{\gamma}_B\} = 2\hat{\eta}_{AB} \tag{5.2}$$

and the metric signature is $(-++\cdots+)$. The equations of motion of the eleven-dimensional theory are invariant under a scaling symmetry, which was used in [21] in a generalized reduction to obtain the bosonic sector of a massive ten-dimensional supergravity. Here, we extend that discussion to include the fermionic sector. This

[1] Note that this massive type-IIA supergravity [28, 21] is not the same as the massive IIA theory obtained by Romans [23].

variant maximal supersymmetric $D = 10$ massive theory [28, 21] has also been considered in [30]. The corresponding ansatz for the generalized reduction of the fermions is

$$\begin{aligned} \hat{\epsilon} &= e^{\frac{1}{2}m_2 z} e^{\frac{1}{24}\varphi}\,\epsilon\,, \\ \hat{\psi}_{11} &= \tfrac{2\sqrt{2}}{3} e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{24}\varphi}\hat{\gamma}_{11}\lambda\,, \\ \hat{\psi}_a &= e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{24}\varphi}(\psi_a - \tfrac{\sqrt{2}}{12}\gamma_a\lambda)\,. \end{aligned} \tag{5.3}$$

Performing the reduction of the fermionic transformation rules, we obtain

$$\begin{aligned} \delta\lambda &= -\tfrac{1}{2\sqrt{2}}\gamma^\mu\epsilon\,\partial_\mu\varphi - \tfrac{1}{192\sqrt{2}} e^{-\frac{1}{4}\varphi} F_{\mu\nu\sigma\rho}\gamma^{\mu\nu\sigma\rho}\epsilon + \tfrac{1}{24\sqrt{2}} e^{\frac{1}{2}\varphi} F_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\hat{\gamma}_{11}\epsilon \\ &\quad - \tfrac{3}{16\sqrt{2}} e^{-\frac{3}{4}\varphi}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\hat{\gamma}_{11}\epsilon - \tfrac{3}{4\sqrt{2}} m_2(\mathcal{A}_\mu\gamma^\mu - e^{\frac{3}{4}\varphi}\hat{\gamma}_{11})\epsilon\,, \\ \delta\psi_\mu &= \nabla_\mu\epsilon - \tfrac{1}{256} e^{-\frac{1}{4}\varphi} F_{\nu\alpha\sigma\rho}\Big(\gamma_\mu{}^{\nu\alpha\sigma\rho} - \tfrac{20}{3}\delta^\nu_\mu\gamma^{\alpha\sigma\rho}\Big)\epsilon - \tfrac{1}{96} e^{\frac{1}{2}\varphi} F_{\nu\sigma\rho}\Big(\gamma_\mu{}^{\nu\sigma\rho} - 9\delta^\nu_\mu\gamma^{\sigma\rho}\Big)\hat{\gamma}_{11}\epsilon \\ &\quad - \tfrac{1}{64} e^{-\frac{3}{4}\varphi}\mathcal{F}_{\nu\sigma}\Big(\gamma_\mu{}^{\nu\sigma} - 14\delta^\nu_\mu\gamma^\sigma\Big)\hat{\gamma}_{11}\epsilon - \tfrac{9}{16} m_2(\mathcal{A}_\nu\gamma_\mu\gamma^\nu - e^{\frac{3}{4}\varphi}\gamma_\mu\hat{\gamma}_{11})\epsilon\,. \end{aligned} \tag{5.4}$$

The supersymmetry transformation rules for the bosons are

$$\begin{aligned} &\delta e_\mu{}^a = \bar{\epsilon}\gamma^a\psi_\mu\,, \qquad \delta\phi = -\sqrt{2}\,\bar{\epsilon}\,\lambda\,, \\ &\delta\mathcal{A}_\mu = e^{\frac{3}{4}\phi}\bar{\epsilon}\hat{\gamma}_{11}(\psi_\mu - \tfrac{3\sqrt{2}}{4}\gamma_\mu\lambda)\,, \\ &\delta A_{\mu\nu} = e^{-\frac{1}{2}\phi}\bar{\epsilon}\hat{\gamma}_{11}(2\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{\sqrt{2}}\gamma_{\mu\nu}\lambda)\,, \\ &\delta A_{\mu\nu\rho} = 3e^{\frac{1}{4}\phi}\bar{\epsilon}(\gamma_{[\mu\nu}\psi_{\rho]} - \tfrac{\sqrt{2}}{12}\gamma_{\mu\nu\rho}\lambda) + 3\mathcal{A}_{[\mu}\delta A_{\nu\rho]}\,. \end{aligned} \tag{5.5}$$

As was shown in [21] this theory admits a de Sitter vacuum solution, which necessarily breaks all supersymmetry. Note that the ten dimensional field strengths are those defined in [21]. References for massless type-IIA supergravity are [50, 51, 52].

B. Reduction of $D=10$, $\mathcal{N}=1$ supersymmetry

Since we have obtained the transformation rules for the type-IIA massive gauged supergravity in section A, it is convenient to make use of these here in order to establish our conventions and notation for the transformation rules of the standard massless $\mathcal{N}=1$ supergravity in ten dimensions. These are obtained by setting the mass parameter $m_2=0$ in (5.4), and in addition making the chiral projection that reduces the $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$:

$$\hat\gamma_{11}\epsilon=\epsilon\,,\qquad \hat\gamma_{11}\psi_a=\psi_a\qquad \text{and}\qquad \hat\gamma_{11}\lambda=-\lambda\,. \tag{5.6}$$

The chirality condition is consistent with setting to zero both the 3-form potential and the Kaluza-Klein vector. This yields the ten-dimensional $\mathcal{N}=1$ supersymmetry transformation rules [53, 54]

$$\begin{aligned}
\delta\hat\lambda &= -\tfrac{1}{2\sqrt2}\,\hat\gamma^M\hat\epsilon\,\partial_M\hat\phi+\tfrac{1}{24\sqrt2}\,e^{\frac12\hat\phi}\,\hat H_{MNP}\hat\gamma^{MNP}\,\hat\epsilon\,,\\
\delta\hat\psi_M &= \widehat\nabla_M\hat\epsilon-\tfrac{1}{96}\,e^{\frac12\hat\phi}\hat H_{NPQ}\Big(\hat\gamma_M{}^{NPQ}-9\,\hat\gamma^{PQ}\delta^N_M\Big)\hat\epsilon\,,\\
\delta\hat e_M{}^A &= \bar{\hat\epsilon}\hat\gamma^A\hat\psi_M\,,\qquad \delta\hat\phi=-\sqrt2\,\bar{\hat\epsilon}\,\hat\lambda\,,\\
\delta\hat B_{MN} &= -e^{-\frac12\hat\phi}\bar{\hat\epsilon}\,(2\hat\gamma_{[M}\hat\psi_{N]}+\tfrac{1}{\sqrt2}\hat\gamma_{MN}\hat\lambda)\,.
\end{aligned} \tag{5.7}$$

We can now use these standard $\mathcal{N}=1$ results in a generalized circle reduction to $d=9$. We shall focus just on the pure supergravity multiplet in $d=9$, by performing a (consistent) truncation of the matter multiplet. The required reduction ansatz is obtained from the arbitrary-dimension ansatz of appendix B by setting $m_1=m_2=m$ and $\phi_2=0=\chi$. This gives

$$\begin{aligned}
\hat\epsilon &= e^{\frac12 mz}e^{-\frac{1}{16\sqrt{14}}\phi_1}\,\tilde\epsilon\,,\\
\hat\lambda &= \sqrt{\tfrac78}\,e^{-\frac12 mz}e^{\frac{1}{16\sqrt{14}}\phi_1}\tilde\lambda\,,
\end{aligned}$$

$$\begin{aligned}
\hat{\psi}_{10} &= -\tfrac{\sqrt{7}}{8}\, e^{-\frac{1}{2}mz} e^{\frac{1}{16\sqrt{14}}\phi_1} \tilde{\gamma}_{10}\tilde{\lambda}\,, \\
\hat{\psi}_{a} &= e^{-\frac{1}{2}mz} e^{\frac{1}{16\sqrt{14}}\phi_1} \Big(\tilde{\psi}_a + \tfrac{1}{8\sqrt{7}}\, \tilde{\gamma}_a \tilde{\lambda}\Big)\,, \\
\hat{\phi} &= \tfrac{\sqrt{14}}{4}\, \phi_1 + 4mz\,.
\end{aligned} \tag{5.8}$$

The tildes signify that the fermions and the Dirac matrices are still ten-dimensional. These can be related to the nine-dimensional quantities as follows:

$$\begin{aligned}
&\tilde{\gamma}_a = \gamma_a \times \sigma_1\,, \qquad \tilde{\gamma}_{10} = \mathbb{1} \times \sigma_2 \qquad \text{and} \qquad \hat{\gamma}_{11} = \mathbb{1} \times \sigma_3\,, \\
&\tilde{\epsilon} = \epsilon \times \eta\,, \qquad \tilde{\lambda} = \lambda \times \sigma_1 \eta \qquad \text{and} \qquad \tilde{\psi}_a = \psi_a \times \eta\,,
\end{aligned} \tag{5.9}$$

where η is a 2-component constant spinor. The chiral projections (5.6) imply that we must have $\sigma_3 \eta = \eta$. We present in the subsection below the supersymmetry obtained from the above reduction ansatz applied to the ten dimensional chiral supergravity.

1. $D = 9$ supersymmetry

Reducing the $D = 10$, $\mathcal{N} = 1$ transformation rules, and setting $G_{(2)} = \mathcal{F}_{(2)} = \frac{1}{\sqrt{2}}\, F_{(2)}$, we obtain the following nine-dimensional supersymmetry transformation rules:

$$\begin{aligned}
\delta\lambda &= -\tfrac{1}{2\sqrt{2}}\gamma^{\mu}\epsilon\,\partial_\mu\phi + \tfrac{1}{12\sqrt{7}} e^{\sqrt{\frac{2}{7}}\phi} H_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\epsilon + \tfrac{\mathrm{i}}{4\sqrt{14}} e^{\frac{1}{\sqrt{14}}\phi} F_{\mu\nu}\gamma^{\mu\nu}\epsilon \\
&\quad + \tfrac{4}{\sqrt{7}} m \Big(\tfrac{1}{\sqrt{2}}\gamma^{\mu} A_\mu - \mathrm{i} e^{-\frac{1}{\sqrt{14}}\phi}\Big)\epsilon\,, \\
\delta\psi_\mu &= \nabla_\mu \epsilon - \tfrac{1}{84} e^{\sqrt{\frac{2}{7}}\phi} H_{\nu\sigma\rho}({\gamma_\mu}^{\nu\sigma\rho} - \tfrac{15}{2}\delta^{\nu}_{\mu}\gamma^{\sigma\rho})\epsilon - \tfrac{\mathrm{i}}{28\sqrt{2}} e^{\frac{1}{\sqrt{14}}\phi} F_{\nu\sigma}({\gamma_\mu}^{\nu\sigma} - 12\delta^{\nu}_{\mu}\gamma^{\sigma})\epsilon \\
&\quad - \tfrac{4}{7\sqrt{2}} m A_\nu \gamma_\mu \gamma^{\nu}\epsilon + \tfrac{4\mathrm{i}}{7} m e^{-\frac{1}{\sqrt{14}}\phi}\gamma_\mu \epsilon\,, \\
\delta {e_\mu}^{a} &= \bar{\epsilon}\gamma^{a}\psi_\mu\,, \qquad \delta\phi = -\sqrt{2}\,\bar{\epsilon}\,\lambda\,, \\
\delta A_\mu &= \mathrm{i}\sqrt{2} e^{-\frac{1}{\sqrt{14}}\phi}\bar{\epsilon}(\psi_\mu + \tfrac{1}{\sqrt{7}}\gamma_\mu\lambda)\,, \\
\delta B_{\mu\nu} &= -e^{-\sqrt{\frac{2}{7}}\phi}\bar{\epsilon}(2\gamma_{[\mu}\psi_{\nu]} + \tfrac{2}{\sqrt{7}}\gamma_{\mu\nu}\lambda) - A_{[\mu}\delta A_{\nu]}\,,
\end{aligned} \tag{5.10}$$

where we have dropped the "1" subscript on the scalar field. The field strengths are $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \frac{3}{2}A_{[\mu}F_{\nu\rho]}$ and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. This theory is an Abelian gauged version of $D = 9$, $\mathcal{N} = 1$ supergravity. We shall show that it admits a supersymmetric (Minkowski)$_6 \times S^3$ vacuum solution. We shall also obtain a time-dependent supersymmetric cosmological solution in this theory.

CHAPTER VI

M-THEORY INTERPRETATION OF THE GAUGED $\mathcal{N} = (1,1)$ SUPERGRAVITY AND THE VACUA (MINKOWSKI)$_4 \times S^2$

In chapter IV we showed that the new $D = 6$, $\mathcal{N} = (1,1)$ supergravity admits a consistent sphere reduction to $D = 4$ giving rise to $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet. In this chapter we shall discuss the higher dimensionsl origin of the $\mathcal{N} = (1,1)$ theory and the vacua (Minkowski)$_4 \times S^2$. The vacuum solution is given by

$$\begin{aligned} ds^2 &= dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{1}{25m^2}\, d\Omega_2^2\,, \\ F_{(2)} &= \frac{\sqrt{2}}{5m}\,\Omega_{(2)}\,, \qquad \phi = 0\,, \end{aligned} \tag{6.1}$$

where we have turned on one of the three vector field strengths $F^a_{(2)}$. Lifting this solution back to $D = 7$, it becomes the near-horizon limit of a 3-brane supported by one of the vector field strengths $\hat{F}^a_{(2)}$. To see this, let us start with the 3-brane, given by

$$\begin{aligned} d\hat{s}_7^2 &= H^{-\frac{2}{5}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{8}{5}}\,(dr^2 + r^2\, d\Omega_2^2)\,, \\ \hat{F}_2 &= \sqrt{2}\, Q\, \Omega_{(2)}\,, \qquad e^\phi = H^{-\frac{2}{\sqrt{10}}}\,, \end{aligned} \tag{6.2}$$

where $H = 1 + Q/r$. In the decoupling (or near-horizon) limit, we have $H = Q/r$. Taking the charge parameter Q to be $Q = (5m)^{-1}$, and making a coordinate transformation $Q/r = e^{-5m\,z}$, the solution (6.2) becomes

$$\begin{aligned} d\hat{s}_7^2 &= e^{2m\,z}(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{1}{25m^2}\, d\Omega_2^2 + dz^2)\,, \\ \hat{F}_{(2)} &= \frac{\sqrt{2}}{5m}\,\Omega_{(2)}\,, \qquad \hat{\phi} = \sqrt{10}\, m\, z\,. \end{aligned} \tag{6.3}$$

This fits exactly the reduction ansatz (4.6), giving rise to precisely the lower dimensional solution (6.1). It is worth mentioning that the solution (6.3) can also be viewed as a domain wall with a (Minkowski)$_4 \times S^2$ world-volume.

We can further lift the solution back to $D = 11$, where it becomes the near-horizon structure of two intersecting M5-branes. As in the above, we start with the two intersecting M5-branes in $D = 11$:

$$\begin{aligned} ds_{11}^2 &= (H_1\, H_2)^{-1/3} \Big(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H_2\,(dz_1^2 + dz_2^2) + H_1\,(dz_3^2 + dz_4^2) \\ &\quad + H_1\, H_2\,(dr^2 + r^2\, d\Omega_2^2)\Big)\,, \\ F_{(4)} &= (Q_1\, dz_3 \wedge dz_4 + Q_2\, dz_1 \wedge dz_2) \wedge \Omega_{(2)}\,, \end{aligned} \tag{6.4}$$

with $H_i = 1 + Q_i/r$. Setting $Q_1 = Q_2 = Q$, the solution in the near-horizon limit becomes

$$\begin{aligned} ds_{11}^2 &= \rho^{2/3}\,(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + Q^2\, \frac{d\rho^2}{\rho^2} + Q^2\, d\Omega^2) + \rho^{-\frac{1}{3}}\, ds_4^2\,, \\ F_{(4)} &= Q\, J_{(2)} \wedge \Omega_{(2)}\,. \end{aligned} \tag{6.5}$$

Here we can replace the 4-torus ds_4^2 by a Ricci-flat $K3$ manifold, and $J_{(2)}$ is a self-dual harmonic 2-form in the $K3$. It is straightforward to see that the $D = 11$ solution (6.5) becomes (6.1) in $D = 6$ by first reducing on the K3 manifold followed by the generalized Kaluza-Klein reduction.

It is interesting to note that only by taking the decoupling or near-horizon limit does the brane solution fit the reduction ansatz. This is different from the usual Kaluza-Klein circle reduction where the whole solution can be reduced instead of just the near-horizon limit. Thus the standard S^1 reduction can be viewed as a special case of a DeWitt group-manifold reduction, whose consistency is guaranteed, whilst the generalized Kaluza-Klein reduction can be viewed as a special case of a Pauli sphere reduction, where the consistency requires conspiracies. (A discussion of the terminology is contained in [55].)

Of course, the $\mathcal{N}=1$ supergravity in $D=7$ can also be obtained from a T^3 reduction of the heterotic string theory, which is S-dual to M-theory on $K3$. The vector field strengths $F^a_{(2)}$ in the minimal $D=7$ supergravity come from setting equal the three Kaluza-Klein and the three winding vectors. It follows that the 3-brane in $D=7$ can be lifted to the $D=10$ heterotic theory as an intersection of the heterotic 5-brane and Taub-NUT.

We conclude this chapter by adding that in [56] a proof was constructed demonstrating that the Salam-Sezgin vacuum solution is unique among all nonsingular solutions with a four-dimensional Poincare, de Sitter or anti-de Sitter invariance. The proof of uniqueness applies of course also to the $\mathcal{N}=(1,1)$ supergravity.

We should also mention that in [57], a general class of dyonic strings were obtained in the $D=6$, $\mathcal{N}=(1,0)$ gauged supergravity preserving $\frac{1}{4}$ of the supersymmetry. The near-horizon limit of the dyonic strings, gives rise to $\mathrm{AdS}_3\times S^3$. Here S^3 is a homogeneously squashed 3-sphere. The $\mathrm{AdS}_3\times S^3$ solution which is supported by both 2-form and 3-form charges contains a nontrivial free adjustable parameter associated with the squashing of the sphere. In the limit when this parameter goes to zero (or for a vanishing 3-form charge) one recover the $(\mathrm{Minkowski})_4\times S^2$ vacua.

CHAPTER VII

SUPERSYMMETRIC $M_{d-3}\times S^3$ AND $M_{d-2}\times S^2$ VACUA

The generalized Kaluza-Klein reduction gives rise to gauged supergravities that admit supersymmetric vacuum solutions of the form Minkowski×sphere [26]. The nine-dimensional theory admits just a (Minkowski)$_6\times S^3$ vacuum of this kind, supported by the $H_{(3)}$ flux. The theories in lower dimensions admit (Minkowski)$_{d-3}\times S^3$ vacua supported by $H_{(3)}$, and (Minkowski)$_{d-2}\times S^2$ vacua supported by a 2-form $F_{(2)}$. In this chapter, we shall show that these vacua are all supersymmetric.

A. $M_{d-3}\times S^3$ vacua

Consider first the (Minkowski)$_{d-3}\times S^3$ solution supported by the $H_{(3)}$ field. This is given by

$$\begin{aligned} ds_d^2 &= dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{4}{m^2\,(d-1)^2}\, d\Omega_3^2\,,\\ H_{(3)} &= \frac{8}{m^2\,(d-1)^2}\,\Omega_{(3)}\,, \qquad \phi=0\,. \end{aligned} \tag{7.1}$$

If we lift the solution back to D dimensions using the generalized reduction ansatz, it becomes the near-horizon geometry of a $(D-5)$-brane supported by the field $\hat H_{(3)}$. To see this, we start with the $(D-5)$-brane in D dimensions, given by

$$\begin{aligned} d\hat s_D^2 &= H^{-\frac{2}{D-2}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{D-4}{D-2}}\,(dr^2 + r^2\, d\Omega_3^2)\,,\\ \hat H_{(3)} &= 2Q\,\Omega_{(3)}\,, \qquad \hat\phi = -\tfrac12 \hat a\, \log H\,, \qquad H = 1 + Q/r^2\,. \end{aligned} \tag{7.2}$$

In the near-horizon limit, the additive constant 1 in H is dropped. Making the coordinate transformation $r^2/Q = e^{(D-2)\,m\,z}$, and letting $Q = 4/((D-2)^2\,m^2)$, we obtain

$$
\begin{aligned}
d\hat{s}_D^2 &= e^{2mz}\Big(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + dz^2 + \frac{4}{m^2\,(D-2)^2}\, d\Omega_3^2\Big)\,,\\
\hat{H}_{(3)} &= \frac{8}{m^2\,(D-2)^2}\,\Omega_{(3)}\,,\qquad \hat{\phi} = \frac{4}{\hat{a}}\, mz\,,
\end{aligned}
\tag{7.3}
$$

which fits the reduction ansatz precisely, giving rise to the lower-dimensional solution (7.1).

The supersymmetry of the (Minkowski)$_{d-3} \times S^3$ solution is easily established. Firstly, since its lift to $D = d+1$ dimensions gives the near-horizon limit of the $(D-5)$-brane, as discussed above, it is manifest that *qua* D-dimensional solution, it will preserve one half of the D-dimensional supersymmetry. This halving of supersymmetry comes about from the usual projection condition for supersymmetry of the $(D-5)$-brane, $\hat{\epsilon} = \hat{\Gamma}_*\,\hat{\epsilon}$, where $\hat{\Gamma}_*$ is built from the product of Dirac matrices in the world-volume of the $(D-5)$-brane. As is well known, for any of the BPS brane solutions with metric given by

$$
d\hat{s}^2 = e^{2A}\, dx^\mu\, dx_\mu + e^{2B}\, dy^m\, dy^m\,,
\tag{7.4}
$$

the Killing spinors are given by

$$
\hat{\epsilon} = e^{\frac{1}{2}A}\,\hat{\epsilon}_0\,,\qquad \hat{\Gamma}_*\,\hat{\epsilon}_0 = \hat{\epsilon}_0\,,
\tag{7.5}
$$

where $\hat{\epsilon}_0$ is a constant spinor. We see from (7.3) that $A = mz$, and hence the Killing spinors in D dimensions take the form

$$
\hat{\epsilon} = e^{\frac{1}{2}mz}\,\hat{\epsilon}_0\,.
\tag{7.6}
$$

Since this z dependence matches precisely the z dependence for $\hat{\epsilon}$ in the generalized reduction ansatz (5.8), it immediately follows that the (Minkowski)$_{d-3} \times S^3$ solution will be supersymmetric *qua* solution of the d-dimensional gauged supergravity.

B. $M_{d-2} \times S^2$ vacua

Another class of supersymmetric vacuum is of the form (Minkowski)$_{d-2} \times S^2$, supported by one of the two-form field strengths $F^a_{(2)}$. It is given by

$$
\begin{aligned}
ds_d^2 &= dx^\mu\, dx^\nu\, \eta_{\mu\nu} + \frac{1}{m^2\,(d-1)^2}\, d\Omega_2^2\,,\\
F_{(2)} &= \frac{\sqrt{2}}{m\,(d-1)}\, \Omega_{(2)}\,, \qquad \phi = 0\,.
\end{aligned}
\tag{7.7}
$$

Lifting this solution back to D dimensions, it becomes the near-horizon limit of the $(D-4)$-brane supported by one of the field strengths $\hat F^a_{(2)}$. The $(D-4)$-brane solution is given by

$$
\begin{aligned}
d\hat s_D^2 &= H^{-\frac{2}{D-2}}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{\frac{2(D-3)}{D-2}}\, (dr^2 + r^2\, d\Omega_2^2)\,,\\
\hat F_{(2)} &= \sqrt{2}\, Q\, \Omega_{(2)}, \qquad \hat\phi = -\tfrac{1}{2}\hat a\, \log H\,, \qquad H = 1 + Q/r\,.
\end{aligned}
\tag{7.8}
$$

In the near-horizon limit, the constant 1 in H is dropped. Making the coordinate transformation $r/Q = e^{(D-2)\, mz}$ and setting $Q = 1/(m\,(D-2))$ we have

$$
\begin{aligned}
d\hat s_D^2 &= e^{2mz}\Big(dx^\mu\, dx^\nu\, \eta_{\mu\nu} + dz^2 + \frac{1}{m^2\,(D-2)^2}\, d\Omega_2^2\Big)\,,\\
\hat F_{(2)} &= \frac{\sqrt{2}}{m\,(D-2)}\, \Omega_{(2)}\,, \qquad \hat\phi = \frac{4}{\hat a}\, mz\,.
\end{aligned}
\tag{7.9}
$$

This clearly fits the reduction ansatz exactly to give rise to (7.7).

Again, the supersymmetry of the solution as a lifted D-dimensional configuration is manifest, since it is just the near-horizon limit of a BPS $(D-4)$-brane. Its supersymmetry as a solution in the $d = D-1$ dimensional gauged supergravity itself is again easily seen, from the general form (7.5) of the Killing spinors in the lifted

$(D-4)$-brane. Thus we again find that the D-dimensional Killing spinors are of the form (7.6), and so comparison with the generalized reduction ansatz (5.8) for $\hat{\epsilon}$ shows that the $(\text{Minkowski})_{d-2} \times S^2$ solution will be supersymmetric in the d-dimensional gauged supergravity.

C. A general discussion of $M_{d-n} \times S^n$ vacua

In this section we show that the brane world interpretation of the generalized Kaluza-Klein reduction presented above is unique to half-maximal supergravities and it cannot be applied for example to type-IIA supergravity. For this discussion we need the p-brane solutions in D dimensions of supergravities. These solutions involves beside the metric, a dilaton and an n-index antisymmetric tensor $F_{M_1 \cdots M_n}$ where $n \leq D/2$. The Lagrangian describing this set of fields is given by

$$e^{-1}\mathcal{L}_D = R - \tfrac{1}{2}(\partial\phi)^2 - \frac{1}{2n!} e^{-a\phi} F^2_{(n)} \,. \tag{7.10}$$

The p-branes were obtained in [58] and are given by

$$\begin{aligned} ds^2_D &= \Big(1+\frac{k}{r^{\tilde{d}}}\Big)^{-\frac{4\tilde{d}}{(D-2)\Delta}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(1+\frac{k}{r^{\tilde{d}}}\Big)^{\frac{4d}{(D-2)\Delta}} (dr^2 + r^2 d\Omega^2_{D-d-1}) \,, \\ e^{\phi} &= \Big(1+\frac{k}{r^{\tilde{d}}}\Big)^{\frac{2a}{\Delta}} , \end{aligned} \tag{7.11}$$

where $x^\mu (\mu = 0, ..., d-1)$ are the brane volume coordinates and

$$d + \tilde{d} = D - 2. \tag{7.12}$$

The constant k is defined as $k = \frac{1}{2}\sqrt{\Delta}\lambda/\tilde{d}$ and the dilaton coupling a is given by

$$a^2 = \Delta - \frac{2(n-1)(D-n-1)}{D-2} \,. \tag{7.13}$$

Examples of values on Δ that arise in supergravity theories are $\Delta = 4$ for $n \neq 2$, and $\Delta = 4$ and 2 for $n = 2$. (Note that eq.(7.13) is valid also for cases with no scalar field but with a now set to zero.) Consider the near horizon limit of the p-brane metric given by

$$ds_D^2 = k^{-\frac{4\tilde{d}}{(D-2)\Delta}}\, r^{\frac{4\tilde{d}^2}{(D-2)\Delta}} \Big[\eta_{\mu\nu}dx^\mu dx^\nu + k^{\frac{4}{\Delta}} r^{2-\frac{4\tilde{d}}{\Delta}} \Big(\frac{dr^2}{r^2} + d\Omega_{D-d-1}^2\Big)\Big]\,, \qquad (7.14)$$

where we have made use of the relation (7.12). Now to make contact with the generalized reduction ansatz the r-dependence inside the bracked must drop out and this gives the condition

$$2\tilde{d} = \Delta\,. \qquad (7.15)$$

Making use of the relation $\tilde{d} = n - 1$ we obtain

$$\begin{aligned} \Delta = 2 \qquad &\Longrightarrow \qquad n = 2\,, \\ \Delta = 4 \qquad &\Longrightarrow \qquad n = 3\,. \end{aligned} \qquad (7.16)$$

This result which is independent of the spacetime dimension D is clearly the field content of the NS-NS sector of Type-II string and its torus reduction with vector multiplets truncated out. At the same time the R-R sector has been ruled out by (7.15). Note that the hodge dual field strengths $F_{(D-2)}$ and $F_{(D-3)}$ of course also satisfies the condition (7.15).

CHAPTER VIII

GENERALIZED KALUZA-KLEIN REDUCTION IN THE STRING FRAME AND σ-MODEL ACTION

A. σ-model action

For many purposes it is advantageous to perform the Weyl rescaling of the metric that transforms from the Einstein frame that we used in the previous section to the string frame. One reason is because the half-maximal supergravities that we are considering have a direct relation to the heterotic string, or the NS-NS sector of the Type-II string. Another reason is that many of the formulae become considerably simpler when expressed in the string frame. We shall consider only the case $m_1 = m_2 = m$.

Consistent string propagation demands world-sheet conformal invariance, and hence the vanishing of the beta functions for the background spacetime fields. In this manner one obtains supergravity equations of motion which arise naturally in the string frame. The corresponding equations may be derived from the string-frame Lagrangian

$$\hat{e}^{-1}\hat{\mathcal{L}} = e^{-2\hat{\Phi}}(\hat{R} + 4(\partial\hat{\Phi})^2 - \tfrac{1}{12}\hat{H}_{(3)}^2 - \tfrac{1}{4}(\hat{F}^a_{(2)})^2)\,, \tag{8.1}$$

taken here to have been compactified on a $(10 - D)$-dimensional torus (with the additional truncation of $(10 - D)$ vector multiplets). It is to be understood that all fields in this section are labelled with a suppressed tilde ($\tilde{g}_{\mu\nu}$, $\widetilde{H}_{(3)}$, *etc.*) unless otherwise indicated, to distinguish them from the Einstein frame fields. The complete transformation between the two frames in dimensions $D \leq 10$ is given in appendix C.

The equations of motion following from the Lagrangian (8.1) are

$$\hat{R}_{MN} \;\; = \;\; -2\widehat{\nabla}_M\widehat{\nabla}_N\hat{\Phi} + \tfrac{1}{4}\hat{H}_{MPQ}\hat{H}_N{}^{PQ} + \tfrac{1}{2}\hat{F}^a_{MP}\hat{F}^a_N{}^P,$$

$$
\begin{aligned}
d(e^{-2\hat\Phi}\hat{*}\hat H_{(3)}) &= 0\,,\\
d(e^{-2\hat\Phi}\hat{*}\hat F^a_{(2)}) &= (-1)^{D+1}\, e^{-2\hat\Phi}\hat{*}\hat H_{(3)}\wedge \hat F^a_{(2)}\,,\\
\widehat{\Box}\hat\Phi &= 2(\partial\hat\Phi)^2 - \tfrac{1}{12}\hat H^2_{(3)} - \tfrac{1}{8}(\hat F^a_{(2)})^2. \qquad (8.2)
\end{aligned}
$$

By tracing the Einstein equation and substituting in the dilaton equation, we may obtain an expression for the Ricci scalar:

$$\hat R = -4(\partial\,\hat\Phi)^2 + \tfrac{5}{12}\hat H^2_{(3)} + \tfrac{3}{4}(\hat F^a_{(2)})^2. \qquad (8.3)$$

In D dimensions, the Einstein-frame and the string-frame metrics are related by

$$d\hat s^2_{\rm Ein} = e^{\frac{1}{2}\hat a\hat\phi}\, d\hat s^2_{\rm str} = e^{-\frac{1}{2}\hat a^2\hat\Phi}\, d\hat s^2_{\rm str}\,, \qquad (8.4)$$

where we have defined $\hat\Phi = -\hat\phi/\hat a$ and $\hat\phi$ is the Einstein-frame dilaton field. For the case where $m_1 = m_2$, the reduction ansatz (3.6) converted to the string frame is rather simple, namely

$$
\begin{aligned}
d\hat s^2_{\rm str} &= ds^2_{\rm str} + e^{-\sqrt{2}\varphi}(dz + \mathcal{A}_{(1)})^2,\\
\hat B_{(2)} &= B_{(2)} + B_{(1)}\wedge dz\,,\\
\hat\Phi &= \Phi - \tfrac{1}{\sqrt{8}}\,\varphi - \tfrac{1}{2}(d-1)\,mz\,. \qquad (8.5)
\end{aligned}
$$

In other words, the reduction is exactly the same as a standard Kaluza-Klein reduction, except for a linear z-dependence in the dilaton $\hat\Phi$. The string frame reduction ansatz can be obtained by using in D-dimensions the Ricci tensor for a Weyl transformed metric $\hat g_{MN} = e^{2\sigma}\tilde g_{MN}$ which yields

$$\hat R_{MN} = \widetilde R_{MN} + (D-2)(\partial_M\sigma\partial_N\sigma - \widetilde\nabla_M\partial_N\sigma - \tilde g_{MN}\tilde g^{PQ}\partial_P\sigma\partial_Q\sigma) - \tilde g_{MN}\widetilde\Box\sigma. \qquad (8.6)$$

It follows that the σ-model action for this generalized circle reduction is given by

$$I = \frac{1}{4\pi\,\alpha'}\int d\sigma\, d\tau\Big[\sqrt{\gamma}\,\gamma^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,\hat g_{\mu\nu} + \epsilon^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,\hat B_{\mu\nu} + \alpha'\hat R\,(\Phi - \tfrac12(D-2)\,mz)\Big]\,,$$

where Φ, $\hat g_{\mu\nu}$ and $\hat B_{\mu\nu}$ are independent of z, and X^0 (the circle coordinate) is given by $X^0 = z$. However, the z dependence of the string action implies that T-duality is now broken. This can also be seen from the low-energy effective action obtained in chapter III, where the Kaluza-Klein vector $\mathcal{A}_{(1)}$ and the winding vector $B_{(1)}$ are clearly not on a parallel footing.

B. Untruncated d-dimensional string-frame equations

We give here the complete set of bosonic equations of motion for the untruncated system, expressed in the string frame. It will be seen that these are considerably simpler than the previous expressions that were obtained in the Einsten frame.

For the form fields in the string frame we find

$$\begin{aligned}
\nabla^\rho(e^{-2\Phi}H_{\mu\nu\rho}) &= m(d-1)\Big(e^{-2\Phi}H_{\mu\nu\sigma}\mathcal{A}^\sigma - e^{-2\Phi+\sqrt2\varphi}G_{\mu\nu}\Big)\,,\\
\nabla^\nu(e^{-2\Phi+\sqrt2\varphi}G_{\mu\nu}) &= \tfrac12 e^{-2\Phi}H_{\mu\nu\sigma}\mathcal{F}^{\nu\sigma} + m(d-1)e^{-2\Phi+\sqrt2\varphi}G_{\mu\nu}\mathcal{A}^\nu\,,\\
\nabla^\nu(e^{-2\Phi}F^a_{\mu\nu}) &= \tfrac12 e^{-2\Phi}H_{\mu\nu\sigma}F^{a\,\nu\sigma} + e^{-2\Phi+\sqrt2\varphi}G_{\mu\nu}L^{a\,\nu}\\
&\quad + m(d-1)\Big(e^{-2\Phi}F^a_{\mu\nu}\mathcal{A}^\nu - e^{-2\Phi+\sqrt2\varphi}L^a_\mu\Big)\,,\\
\nabla^\mu(e^{-2\Phi+\sqrt2\varphi}L^a_\mu) &= \tfrac12 e^{-2\Phi}F^a_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac12 e^{-2\Phi+\sqrt2\varphi}G_{\mu\nu}F^{a\,\mu\nu}\\
&\quad + m(d-1)e^{-2\Phi+\sqrt2\varphi}L^a_\mu\mathcal{A}^\mu\,,\\
\nabla^\nu(e^{-\frac{3}{\sqrt2}\varphi}\mathcal{F}_{\mu\nu}) &= e^{-\frac{1}{\sqrt2}\varphi}(\tfrac12 H_{\mu\nu\sigma}G^{\nu\sigma} - F^a_{\mu\nu}L^{a\nu}) + 2e^{-\frac{3}{\sqrt2}\varphi}(\partial_\nu\Phi - \tfrac{1}{\sqrt8}\partial_\nu\varphi)\mathcal{F}_\mu{}^\nu\\
&\quad + m(d-1)(\sqrt2\,e^{-\frac{1}{\sqrt2}\varphi}\partial_\mu\varphi + e^{-\frac{3}{\sqrt2}\varphi}\mathcal{A}^\nu\mathcal{F}_{\mu\nu})\,.
\end{aligned}\tag{8.7}$$

For the scalar fields, we find

$$\Box\varphi = \tfrac{1}{2\sqrt{2}}(e^{\sqrt{2}\varphi}G_{(2)}^2 - e^{-\sqrt{2}\varphi}\mathcal{F}_{(2)}^2) + \tfrac{1}{\sqrt{2}}e^{\sqrt{2}\varphi}(L_{(1)}^a)^2 + 2\partial_\mu\varphi\partial^\mu\Phi + m(d-1)\mathcal{A}^\mu\partial_\mu\varphi\,,$$

$$\Box\Phi = -\tfrac{1}{12}H_{(3)}^2 - \tfrac{1}{8}(F_{(2)}^a)^2 - \tfrac{1}{8}(e^{\sqrt{2}\varphi}G_{(2)}^2 + e^{-\sqrt{2}\varphi}\mathcal{F}_{(2)}^2) + 2(\partial\Phi)^2 + 2m(d-1)\mathcal{A}^\mu\,\partial_\mu\Phi - \tfrac{1}{2}m(d-1)\nabla_\mu\mathcal{A}^\mu + \tfrac{1}{2}m^2(d-1)^2(\mathcal{A}_{(1)}^2 + e^{\sqrt{2}\varphi})\,. \tag{8.8}$$

The Einstein equation in the string frame is given by

$$R_{\mu\nu} = \tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi - 2\nabla_\mu\partial_\nu\Phi + \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \tfrac{1}{2}e^{\sqrt{2}\varphi}G_{\mu\rho}\,G_\nu{}^\rho + \tfrac{1}{2}e^{-\sqrt{2}\varphi}\mathcal{F}_{\mu\rho}\,\mathcal{F}_\nu{}^\rho + \tfrac{1}{2}F_{\mu\rho}^a\,F_\nu^{a\,\rho} + \tfrac{1}{2}e^{\sqrt{2}\varphi}L_\mu^aL_\nu^a - \tfrac{1}{2}m(d-1)(\nabla_\mu\mathcal{A}_\nu + \nabla_\nu\mathcal{A}_\mu)\,. \tag{8.9}$$

C. Truncated d-dimensional string-frame equations

In the string frame, we may again truncate out the vector multiplet by setting $\varphi = 0$, $L_{(1)}^a = 0$ and $\mathcal{A}_{(1)} = B_{(1)} \equiv A_{(1)}/\sqrt{2}$. The equations of motion for the bosonic fields of the pure supergravity multiplet now become

$$\begin{aligned}
\nabla^\sigma H_{\mu\nu\sigma} &= 2H_{\mu\nu\sigma}M^\sigma - \tfrac{1}{\sqrt{2}}\,m(d-1)F_{\mu\nu}\,,\\
\nabla^\nu F_{\mu\nu} &= \tfrac{1}{2}H_{\mu\nu\sigma}F^{\nu\sigma} + 2F_{\mu\nu}M^\nu\,,\\
\nabla^\nu F_{\mu\nu}^a &= \tfrac{1}{2}H_{\mu\nu\sigma}F^{a\,\nu\sigma} + 2F_{\mu\nu}^aM^\nu\,,\\
\nabla^\mu M_\mu &= 2M_{(1)}^2 - \tfrac{1}{12}H_{(3)}^2 - \tfrac{1}{8}(F_{(2)}^2 + (F_{(2)}^a)^2) + \tfrac{1}{2}m^2(d-1)^2\,,\\
R_{\mu\nu} &= -\nabla_\mu M_\nu - \nabla_\nu M_\mu + \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \tfrac{1}{2}(F_{\mu\rho}\,F_\nu{}^\rho + F_{\mu\rho}^a\,F_\nu^{a\,\rho})\,,
\end{aligned} \tag{8.10}$$

where we have introduced the field

$$M_{(1)} = d\Phi + \frac{m(d-1)}{2\sqrt{2}}A_{(1)}\,. \tag{8.11}$$

It is evident that the massive field $M_{(1)}$ arises because the dilaton Φ is eaten by the gauge field $A_{(1)}$.

As in the Einstein frame, these equations cannot be obtained from a Lagrangian. However, if we set $A_{(1)}$ to zero, the equations of motion for the remaining fields can be obtained from a Lagrangian, given by

$$e^{-1}\mathcal{L} = e^{-2\Phi}\Big(R + 4(\partial\Phi)^2 - \tfrac{1}{12}H_{(3)}^2 - \tfrac{1}{4}(F_{(2)}^a)^2 - (d-1)^2 m^2\Big)\,. \qquad (8.12)$$

Although this truncation is consistent within the bosonic theory, it cannot be consistent with the full supergravity, as it would be incompatible with the structure of the supermultiplets. Nevertheless, we see from (8.12) that in the string frame the scalar potential becomes a pure positive cosmological constant.

D. Supersymmetry in the string frame

The supersymmetry transformation rules for the fermions are readily expressed in terms of the fields of the string frame, using the formulae given in appendix C.

1. $D = 6$

The transformation formulae in six dimensions is

$$\begin{aligned}
&g_{\mu\nu} = e^{-\phi}\,\tilde g_{\mu\nu}\,, \quad F_{(2)}^a = \widetilde F_{(2)}^a\,, \quad B_{(2)} = \widetilde B_{(2)}\,, \quad d\phi + \tfrac{5}{2\sqrt2} m A_{(1)} = \widetilde M_{(1)}\,, \qquad (8.13)\\
&F_{(2)} = \widetilde F_{(2)}\,, \quad \phi_1 = -\sqrt2\,\phi\,, \quad \epsilon = e^{-\frac14\phi}\,\tilde\epsilon\,, \quad \lambda = e^{\frac14\phi}\,\tilde\lambda\,, \quad \psi_\mu = e^{-\frac14\phi}\,\tilde\psi_\mu\,,
\end{aligned}$$

The supersymmetric variations for the fermions take the form

$$\begin{aligned}
\delta\tilde\lambda_i &= [\tfrac12\widetilde M_\mu\,\tilde\gamma^\mu + \tfrac{1}{24}\widetilde H_{\mu\nu\rho}\tilde\gamma^{\mu\nu\rho} + \tfrac{1}{8\sqrt2}\tilde\gamma^{\mu\nu}\gamma_7\widetilde F_{\mu\nu} - \tfrac54 m\gamma_7]\tilde\epsilon_i - \tfrac{i}{8\sqrt2}\tilde\gamma^{\mu\nu}\widetilde F_{\mu\nu\,i}{}^j\tilde\epsilon_j,\\
\delta\tilde\psi_{\mu\,i} &= [\widetilde\nabla_\mu - \tfrac14\widetilde M_\nu\,\tilde\gamma_\mu\tilde\gamma^\nu - \tfrac{1}{48}(\tilde\gamma_\mu{}^{\nu\rho\sigma} - 3\delta_\mu^\nu\tilde\gamma^{\rho\sigma})\widetilde H_{\nu\rho\sigma} + \tfrac58 m\tilde\gamma_\mu\gamma_7\\
&\quad - \tfrac{1}{16\sqrt2}(\tilde\gamma_\mu{}^{\nu\rho} - 6\delta_\mu^\nu\tilde\gamma^\rho)\gamma_7\widetilde F_{\nu\rho}]\tilde\epsilon_i + \tfrac{i}{16\sqrt2}(\tilde\gamma_\mu{}^{\nu\rho} - 6\delta_\mu^\nu\tilde\gamma^\rho)\widetilde F_{\nu\rho\,i}{}^j\tilde\epsilon_j\,. \qquad (8.14)
\end{aligned}$$

It is of interest to note that the gravitino variation for the shifted gravitino, $\tilde{\tilde{\psi}}_\mu = \widetilde{\psi}_\mu + \frac{1}{2}\tilde{\gamma}_\mu\,\tilde{\lambda}$, given by

$$\delta\tilde{\tilde{\psi}}_{\mu\,i} = \Big[\widetilde{\nabla}_\mu + \tfrac{1}{8}\widetilde{H}_{\mu\nu\rho}\tilde{\gamma}^{\nu\rho} + \tfrac{1}{2\sqrt{2}}\tilde{\gamma}^\nu\gamma_7\widetilde{F}_{\mu\nu}\Big]\tilde{\epsilon}_i - \tfrac{i}{2\sqrt{2}}\tilde{\gamma}^\nu\widetilde{F}_{\mu\nu\,i}{}^j\tilde{\epsilon}_j\,, \qquad (8.15)$$

does not depend on m.

2. $D = 9$

In nine dimensions the transformation formulae is

$$\begin{aligned} &g_{\mu\nu} = e^{\sqrt{\frac{2}{7}}\phi_1}\tilde{g}_{\mu\nu}\,, \qquad F_{(2)} = \widetilde{F}_{(2)}\,, \qquad H_{(3)} = \widetilde{H}_{(3)}\,, \qquad d\Phi + \sqrt{8}mA_{(1)} = \widetilde{M}_{(1)}\,, \\ &\phi_1 = -\sqrt{\tfrac{8}{7}}\,\Phi\,, \quad \epsilon = e^{\frac{1}{2\sqrt{14}}\phi_1}\tilde{\epsilon}\,, \quad \lambda = e^{-\frac{1}{2\sqrt{14}}\phi_1}\tilde{\lambda}\,, \quad \psi_\mu = e^{\frac{1}{2\sqrt{14}}\phi_1}\tilde{\psi}_\mu\,, \end{aligned} \qquad (8.16)$$

The supersymmetric fermionic transformation in the string frame then take the form

$$\begin{aligned} \delta\tilde{\lambda} &= \Big(\tfrac{1}{\sqrt{7}}\widetilde{M}_\mu\tilde{\gamma}^\mu + \tfrac{1}{12\sqrt{7}}\widetilde{H}_{\mu\nu\sigma}\tilde{\gamma}^{\mu\nu\sigma} + \tfrac{\mathrm{i}}{4\sqrt{14}}\widetilde{F}_{\mu\nu}\tilde{\gamma}^{\mu\nu} - \tfrac{4\mathrm{i}}{\sqrt{7}}m\Big)\tilde{\epsilon}\,, \\ \delta\tilde{\psi}_\mu &= \Big(\widetilde{\nabla}_\mu - \tfrac{1}{7}\widetilde{M}_\nu\tilde{\gamma}_\mu\tilde{\gamma}^\nu - \tfrac{1}{84}\widetilde{H}_{\nu\sigma\rho}(\tilde{\gamma}_\mu{}^{\nu\sigma\rho} - \tfrac{15}{2}\delta_\mu^\nu\tilde{\gamma}^{\sigma\rho}) \\ &\quad - \tfrac{\mathrm{i}}{28\sqrt{2}}\widetilde{F}_{\nu\sigma}(\tilde{\gamma}_\mu{}^{\nu\sigma} - 12\delta_\mu^\nu\tilde{\gamma}^\sigma) + \tfrac{4\mathrm{i}}{7}m\tilde{\gamma}_\mu\Big)\tilde{\epsilon}\,. \end{aligned} \qquad (8.17)$$

CHAPTER IX

SUPERSYMMETRIC TIME-DEPENDENT SOLUTIONS AND PP-WAVES

In this chapter we construct a time-dependent solution of the new gauged nine-dimensional supergravity, and we show that it is supersymmetric. It can be thought of as a cosmological solution in the gauged supergravity. The solution is of a form analogous to a standard domain wall, except that here the "transverse space coordinate" is timelike rather than spatial.

A. Cosmological solutions and pp-waves

It is easily seen that the configuration

$$\begin{aligned} ds_9^2 &= -dt^2 + (\tfrac{8}{7}m\,t)^2 dx^i dx^i\,, \\ e^{\frac{1}{\sqrt{14}}\phi} &= \tfrac{8}{7}m\,t\,. \end{aligned} \tag{9.1}$$

solves the nine-dimensional equations of motion that follow from (3.18). Note that the form-fields are all zero in this solution.

The fermionic transformation rules (5.10) in this background reduce to

$$\begin{aligned} \delta\lambda &= -\tfrac{1}{2\sqrt{2}}\Gamma^M(\partial_M\phi)\,\epsilon - \tfrac{4\mathrm{i}}{\sqrt{7}}me^{-\frac{1}{\sqrt{14}}\phi}\epsilon\,, \\ \delta\psi_M &= \nabla_M\,\epsilon + \tfrac{4\mathrm{i}}{7}me^{-\frac{1}{\sqrt{14}}\phi}\,\Gamma_M\,\epsilon\,, \end{aligned} \tag{9.2}$$

and it is easily verified that (9.1) is supersymmetric.

In the string frame, the metric in the solution (9.1) becomes simply the Minkowski metric $ds^2_{\rm str} = \eta_{MN}dx^M dx^N$, where

$$t = \exp(\tfrac{8}{7}m\,x^0)\,. \tag{9.3}$$

The dilaton is a linear function of the redefined time; $\Phi = -4mx^0 +$ constant.

The solution (9.1) is straightforwardly lifted to ten dimensions, where it gives

$$\begin{aligned} ds_{10}^2 &= e^{2mz}\Big[-(\tfrac{8}{7}m\,t)^{-1/4}dt^2 + (\tfrac{8}{7}m\,t)^{7/4}(dz^2 + dx^i dx^i)\Big]\,, \\ e^{\hat\phi} &= e^{4mz}(\tfrac{8}{7}m\,t)^{7/2}\,. \end{aligned} \tag{9.4}$$

This can again be viewed as a time-dependent supersymmetric cosmological solution, driven purely by the dilaton. In the string frame the metric is again Minkowskian, but now the dilaton is linearly proportional to the light-cone coordinate x^+:

$$ds_{\rm str}^2 = 2dx^+\,dx^- + dx^i\,dx^i\,, \qquad \Phi = x^+\,. \tag{9.5}$$

A metric-dilaton configuration of this kind was also discussed in [59]. It is straightforward to see that the solution preserves half of the supersymmetry, with the Killing spinor given by $\Gamma_+\,\epsilon_0$ where ϵ_0 is a constant spinor.

A further uplift to $D = 11$ using the standard Kaluza-Klein formula

$$ds_{11}^2 = e^{\frac{1}{6}\hat\phi}ds_{10}^2 + e^{-\frac{4}{3}\hat\phi}dy^2 \tag{9.6}$$

yields the Ricci-flat solution

$$ds_{11}^2 = -r^2dt^2 + t^2dr^2 + r^2t^2dx^idx^i + r^{-4}t^{-4}dy^2\,, \tag{9.7}$$

where we have changed from the ten-dimensional coordinate z to a new coordinate r defined by $r = e^{\frac{4}{3}mz}(\frac{8}{7}m\,t)^{1/6}$. The metric (9.7) is a pp-wave. To see this, we introduce new coordinates X_+ and X_- defined by

$$r^2\,t^2 = X_+\,, \qquad \frac{r}{t} = e^{2X_-}\,, \tag{9.8}$$

in terms of which (9.7) becomes

$$ds_{11}^2 = dX_+ dX_- + X_+ dx^i dx^i + X_+^{-2} dy^2 \,. \tag{9.9}$$

Thus, we conclude that in eleven dimensions the solution describes a pp-wave.

The metric (9.9) is a particular example of a more general class of pp-waves, contained within the ansatz

$$ds_D = dX_+ dX_- + X_+^{h_1} dx^{m_1} dx^{m_1} + X_+^{h_2} dy^{m_2} dy^{m_2} + X_+^{h_3} dz^{m_3} dz^{m_3} + \cdots \,. \tag{9.10}$$

Here, we take the index ranges to be

$$1 \le m_1 \le p_1 \,, \qquad p_1 + 1 \le m_2 \le p_1 + p_2 \,, \qquad \text{etc.}\,, \tag{9.11}$$

and so the total dimension is $D = 2 + p_1 + p_2 + \cdots$. The only non-vanishing vielbein components of the Riemann tensor for (9.10) are given by

$$R_{m_i + m_j +} = -\tfrac{1}{2} h_i (h_i - 2) X_+^{-2} \delta_{m_i m_j} \,. \tag{9.12}$$

Thus (9.10) is Ricci-flat if

$$0 = \sum_{i=1} p_i h_i (h_i - 2) \,. \tag{9.13}$$

The pp-wave (9.9) that resulted from lifting our time-dependent cosmological solution to $D = 11$ is the special case with

$$p_1 = 8 \,, \qquad h_1 = 1 \,, \qquad p_2 = 1 \,, \qquad h_2 = -2 \,, \tag{9.14}$$

which clearly satisfies (9.13).

It is possible to consider a generalization of the solution (9.1) by introducing a nonflat metric for the transerse space as

$$ds_9^2 = -dt^2 + (\tfrac{8}{7} m\, t)^2 \gamma_{ij}(x) dx^i dx^i \tag{9.15}$$

with the dilaton still given by (9.1). In order for (9.15) to be a solution, the metric γ_{ij} must be Ricci flat. The Killing spinor in this case is given by

$$\epsilon(t,x) = t^{1/2}\epsilon(x) \tag{9.16}$$

where $\epsilon(x)$ is a Killing spinor which solves the equation

$$\nabla_i \, \epsilon(x) = 0 \tag{9.17}$$

in the background γ_{ij}. The solution with the curved metric γ_{ij} will be supersymmetric as long as we use Ricci flat manifolds that admit Killing spinors. See [60] where the supermembrane in eleven dimensions is treated, and see also [61, 62]. For a discussion of nonflat world volume metrics see [63] where the D8-brane of Romans massive theory is studied. We should emphasize that solutions with the transverse space allowed to be curved always have less supersymmetry than in the flat transverse space case.

CHAPTER X

AdS PP-WAVES I

The pp-waves in M-theory and type-IIB supergravity in general have 16 ”standard” Killing spinors, that is half of the maximum supersymmetry. A large class of these solutions were studied in [64, 65, 66]. For appropriate choices of field strengths and integration constants, supernumerary Killing spinors beyond the 16 standard ones could also arise [64, 65, 66, 67, 68, 69]. These include all of those from the Penrose limits of AdS×sphere arising from non-dilatonic p-branes and/or intersecting p-branes, and of AdS×sphere×sphere, arising from non-standard brane intersections [70].

It is natural to study the pp-waves in AdS background. As mentioned in the introduction the effect of introducing a pp-wave in such a background can be viewed as performing an infinite boost on the boundary conformal field theory [42, 71]. The supersymmetry of the purely gravitational pp-wave in AdS_4 of Kaigorodov [41] and its higher dimensional counterparts were discussed in [42]. These metrics preserve $\frac{1}{4}$ of the supersymmetry, consisting with the fact that in the dual conformal field theory, the original supersymmetry as well as the superconformal symmetry are broken by the boost [42]. Generalizations of the Kaigorodov metric to inhomogeneous solutions were obtained in [72, 73, 74]. For a discussion of the Kaigorodov spacetime see [75].

In this chapter, we show that purely gravitational AdS pp-waves can in fact admit supernumerary supersymmetries [76] for appropriately constrained harmonic functions associated with the pp-waves, extending the result of [71], where only $\frac{1}{4}$ supersymmetric solutions were discussed.

AdS pp-waves can also be supported by a field strength. Their supersymmetry has been studied in [77, 78, 79, 76]. See also [80]. In the case of charged pp-waves of minimum gauged supergravities in $D = 4$ and $D = 5$, it was shown [79, 76] that supernumerary supersymmetry can arise again for appropriately constrained harmonic functions. The new solutions preserve $\frac{1}{2}$ of the supersymmetry, double the number of standard Killing spinors associated with the general pp-wave solutions including the Kaigorodov metric. For pp-waves with $\frac{1}{2}$ supersymmetry in $D = 3$ see [81].

A. Purely gravitational pp-waves

In this section, we consider pure gravitational pp-waves in Einstein gravity with a negative cosmological constant in arbitrary dimensions. The Lagrangian is given by

$$e^{-1}\mathcal{L} = R + (D-1)(D-2)g^2, \tag{10.1}$$

where $e = (-\det(g_{MN}))^{1/2}$. The Killing spinor in this theory satisfies the equation

$$\nabla_M \epsilon = -\tfrac{1}{2} g\, \Gamma_M \epsilon\,. \tag{10.2}$$

We study AdS pp-waves using the metric ansatz

$$ds_D^2 = e^{2g\rho}(-4dx^+ dx^- + H(dx^+)^2 + dz_i\, dz_i) + d\rho^2 \tag{10.3}$$

where the function H depends on x^+, ρ and z_i coordinates. The Einstein equations of motion reduce to

$$\Box H \equiv \Big(\partial_\rho^2 + g(D-1)\partial_\rho + e^{2g\rho}\sum_{i=1}^{D-3}\partial_i^2\Big) H = 0\,, \tag{10.4}$$

where the index i stands here for z_i. To discuss the Killing spinor equations, we make a natural choice for the vielbein basis

$$e^+ = e^{g\rho} dx^+ \,, \quad e^- = e^{g\rho}(-2dx^- + \tfrac{1}{2} H\, dx^+) \,, \quad e^i = e^{g\rho} dz^i \,, \quad e^\rho = d\rho \,, \tag{10.5}$$

such that we have $ds^2 = 2e^+e^- + e^z e^z + e^\rho e^\rho$. In this tangent basis, the spin connections are given by

$$\omega_{-\rho} = g\, e^+ \,, \quad \omega_{+i} = \tfrac{1}{2} e^{-g\rho}\, \partial_i H\, e^+ \,, \quad \omega_{+\rho} = g\, e^- + \tfrac{1}{2} H'\, e^+ \,, \quad \omega_{i\rho} = g\, e^i \,, \tag{10.6}$$

where the prime denotes the derivative ∂_ρ. Note that for the metric in this basis we have $\eta_{+-} = 1$ and $\eta_{++} = \eta_{--} = 0$. In the following we use the notation that all derivatives are with respect to the curved metric and all indices on gamma matrices are vielbein indices. The Killing spinor equations are given by

$$\begin{aligned}
&[\partial_+ + \tfrac{1}{2} g e^{g\rho}\, \Gamma_+(\Gamma_\rho + 1) + \tfrac{1}{4} g e^{g\rho} H\, \Gamma_-(\Gamma_\rho + 1) + \tfrac{1}{4} e^{g\rho} H'\, \Gamma_{-\rho} \\
&\qquad + \tfrac{1}{4} \sum_i^{D-3} \partial_i H\, \Gamma_{-i}]\epsilon = 0 \,, \\
&[\partial_- - g e^{g\rho}\, \Gamma_-(\Gamma_\rho + 1)]\epsilon = 0 \,, \\
&[\partial_i + \tfrac{1}{2} g e^{g\rho}\, \Gamma_i(\Gamma_\rho + 1)]\epsilon = 0 \,, \qquad i = 1, 2, \cdots, D-3, \\
&[\partial_\rho + \tfrac{1}{2} g\, \Gamma_\rho]\epsilon = 0 \,,
\end{aligned} \tag{10.7}$$

where we have $\Gamma_+^2 = \Gamma_-^2 = 0$ and $\{\Gamma_+, \Gamma_-\} = 2$. Thus, we see that a generic pp-wave in a pure Einstein theory with a cosmological constant preserves $\frac{1}{4}$ of the maximally allowed supersymmetry. The projections are given by

$$(\Gamma_\rho + 1)\epsilon = 0 = \Gamma_- \epsilon \,. \tag{10.8}$$

We are interested in finding solutions that preserve more supersymmetry. One might expect that it would be helpful in this case first to analyse the integrability conditions $[\partial_M\,,\partial_N]\epsilon=0$ among the Killing spinor equations. This calculation yields

$$
\begin{aligned}
0 &= [\partial_+\,,\partial_i]\epsilon=-\tfrac{1}{4}[ge^{2g\rho}H'\,\Gamma_i+e^{g\rho}\partial_i H'\,\Gamma_\rho+\sum_j\partial_j\partial_i H\,\Gamma_j]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon=-\tfrac{1}{4}[e^{g\rho}(H''+2gH')\Gamma_\rho+\sum_i\partial_i H'\,\Gamma_i]\Gamma_-\epsilon\,. \qquad (10.9)
\end{aligned}
$$

The integrability conditions are satisfied provided that $\Gamma_-\epsilon=0$. This is an example where integrability conditions are not enough for the existence of the Killing spinors.

To see whether the metrics can admit more supersymmetry than the $\frac{1}{4}$, let us use the less restrictive projection condition

$$
g(\Gamma_\rho+1)\epsilon=\mathrm{i}f\,\Gamma_-\epsilon\,, \qquad (10.10)
$$

where $f=f(x^+,\rho,z_i)$ is to be determined. Substitute this projection into the Killing spinor equations, we have

$$
\begin{aligned}
&\Big[\partial_+ +\tfrac{\mathrm{i}}{2}e^{g\rho}f\,\Gamma_+\Gamma_- -\tfrac{1}{4}\Big(e^{g\rho}H'+\sum_i\Gamma_i\partial_i H\Big)\Gamma_-\Big]\epsilon=0\,, \qquad (10.11)\\
&\partial_-\,\epsilon=0\,,\qquad [\partial_i+\tfrac{\mathrm{i}}{2}e^{g\rho}f\,\Gamma_i\,\Gamma_-]\,\epsilon=0\,,\qquad [\partial_\rho+\tfrac{\mathrm{i}}{2}f\,\Gamma_- -\tfrac{1}{2}g]\epsilon=0\,.
\end{aligned}
$$

The integrability conditions $[\partial_M\,,\partial_N]\epsilon=0$ among these equations are

$$
\begin{aligned}
0 &= [\partial_i\,,\partial_j]\epsilon=-\tfrac{\mathrm{i}}{2}e^{g\rho}(\Gamma_j\,\partial_i f-\Gamma_i\,\partial_j f)\Gamma_-\epsilon\,,\\
0 &= [\partial_i\,,\partial_\rho]\epsilon=\tfrac{\mathrm{i}}{2}[(e^{g\rho}f)'\,\Gamma_i-\partial_i f]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_i]\epsilon=-\tfrac{1}{2}\Big[\mathrm{i}e^{g\rho}(\Gamma_i\,\partial_+ f-\Gamma_+\,\partial_i f)+e^{2g\rho}f^2\,\Gamma_i\\
&\qquad +\tfrac{1}{2}e^{g\rho}\partial_i H'+\tfrac{1}{2}\sum_{j=1}^{D-3}\Gamma_j\partial_j\partial_i H\Big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon=-\tfrac{1}{2}\Big[\mathrm{i}\partial_+ f+e^{g\rho}f^2-\mathrm{i}(e^{g\rho}f)'\,\Gamma_+ +\tfrac{1}{2}\sum_i\Gamma_i\partial_i H'\\
&\qquad +\tfrac{1}{2}e^{g\rho}(H''+gH')\Big]\Gamma_-\epsilon\,. \qquad (10.12)
\end{aligned}
$$

From these integrability conditions we see that if we insist on more supersymmetry than the usual $\frac{1}{4}$ we must set

$$\partial_i f = 0 = \partial_i H' \qquad \text{and} \qquad \partial_i \partial_j H = 0\,, \qquad i \neq j\,. \tag{10.13}$$

We then have

$$\begin{aligned} (e^{g\rho} f)' &= 0\,, & (10.14)\\ \mathrm{i}\partial_+ f + e^{g\rho} f^2 + \tfrac{1}{2} e^{-g\rho} \partial_i \partial_i H &= 0\,, \qquad i = 1, 2, \cdots, D-3\,, & (10.15)\\ \mathrm{i}\partial_+ f + e^{g\rho} f^2 + \tfrac{1}{2} e^{g\rho} (H'' + gH') &= 0\,. & (10.16) \end{aligned}$$

The conditions in (10.13), together with (10.4), implies that H is given by

$$H = \tfrac{1}{2} \sum_{i=1}^{D-3} c_i z_i^2 + \frac{e^{-2g\rho}}{2g^2(D-3)} \sum_{i=1}^{D-3} c_i + b\, e^{-(D-1)g\rho}\,, \tag{10.17}$$

where c_i and b are functions depending on x^+ only. Equation (10.15) implies that all c_i's are equal, and hence we let $c_i = c(x^+)$. From eqs.(10.14) and (10.16) it follows that we must set $b = 0$. It is straightforward to solve for f, given by

$$f = e^{-g\rho} U(x^+)\,, \tag{10.18}$$

where U satisfies the following first-order non-linear equation

$$\mathrm{i} \frac{dU}{dx^+} + U^2 + \tfrac{1}{2} c = 0\,. \tag{10.19}$$

Making use of eq.(10.19) together with the solutions for f and H we can now solve the Killing spinor equations given in (10.12). The Killing spinor solution is

$$\begin{aligned} \epsilon &= e^{\frac{1}{2} g\rho} \Big(1 - \tfrac{\mathrm{i}}{2} U \sum_{i=1}^{D-3} z_i\, \Gamma_i\, \Gamma_-\Big)\Big(1 + \tfrac{\mathrm{i}}{2} g^{-1} f\, \Gamma_-\Big) \times \\ &\quad \times \Big[1 - \tfrac{1}{2}\Big(1 - e^{-\mathrm{i}\int U dx^+}\Big)\Gamma_+\, \Gamma_-\Big]\epsilon_0\,, \qquad (10.20) \end{aligned}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. Thus, the metric preserves $\frac{1}{2}$ of the supersymmetry. It is important that the final result of our Killing spinors (10.20) satisfy the projection condition (10.10), which can be easily verified to be true.

Note that the special case of $c = 0$, $b \neq 0$ is the Kaigorodov metric. The above analysis implies that it preserves $\frac{1}{4}$ of the supersymmetry. In order to have $\frac{1}{2}$ BPS solutions, it is necessary to set the Kaigorodov component to zero.

Note that in general c is any function depending on x^+. The simplest case is that c is a constant. The x^+ dependence of c has no effect on the existence of the Killing spinors, but only modifies the explicit Killing spinor solutions.

B. PP-waves in $D = 4$ gauged supergravity

1. The solution

In this section we continue our investigations of supernumerary supersymmetry by including a U(1) charge. We start with gauged $\mathcal{N} = 2$ Einstein-Maxwell AdS supergravity, whose Lagrangian for the bosonic sector is given by

$$e^{-1}\mathcal{L}_4 = R - \tfrac{1}{4}F_{(2)}^2 + 6g^2, \qquad (10.21)$$

where $F_{(2)} = dA_{(1)}$. The supersymmetry transformation rule for the complex gravitino $\Psi_M = \Psi_M^1 + \mathrm{i}\Psi_M^2$ is [82, 83]

$$\delta\Psi_M = \Big[\nabla_M - \tfrac{\mathrm{i}}{2}gA_M + \tfrac{\mathrm{i}}{8}F_{AB}\,\Gamma^{AB}\,\Gamma_M + \tfrac{1}{2}g\,\Gamma_M\Big]\epsilon\,. \qquad (10.22)$$

We consider the following pp-wave ansatz

$$\begin{aligned} ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H(dx^+)^2 + dz^2) + d\rho^2, \\ A_{(1)} &= g^{-1}S(1 - e^{-g\rho})\,dx^+, \end{aligned} \qquad (10.23)$$

where $H = H(x^+, \rho, z)$ and S is here a function of x^+. The equations of motion imply that H satisfies

$$\Box H \equiv H'' + 3g\, H' + e^{-2g\rho}\partial_z^2 H = -S^2 e^{-4g\rho}\,. \tag{10.24}$$

The solution can be expressed as

$$H = S^2\Big(\tfrac{1}{2}c\, z^2 + g^{-2}\,(\tfrac{1}{2}c\, e^{-2g\rho} - \tfrac{1}{4}e^{-4g\rho} + b\, e^{-3g\rho})\Big) + H_0\,, \tag{10.25}$$

where b and c are functions of x^+ and H_0 satisfies $\Box H_0 = 0$. (Note that the terms associated with b and c actually belong to H_0. We extract them since they are necessary for the solution to reduce under $g \to 0$ to the pp-wave that is the Penrose limit of $\mathrm{AdS}_2 \times S^2$ of the corresponding ungauged theory.)

If we turn off the field strength by setting $S = 0$, and let H_0 depend only on ρ, namely $H_0 = c_0 + b\, e^{-3g\rho}$, then we recover the Kaigorodov metric.

2. Standard supersymmetry

Here we investigate the supersymmetry of the "charged" pp-wave we derived. The Killing spinor equations in this background are given by

$$\begin{aligned}
&[\partial_+ + \tfrac{1}{2}g e^{g\rho}\,\Gamma_+(\Gamma_\rho + 1) + \tfrac{1}{4}g e^{g\rho} H\,\Gamma_-(\Gamma_\rho + 1) + \tfrac{1}{4}\Gamma_{-z}\partial_z H + \tfrac{1}{4}e^{g\rho}H'\,\Gamma_{-\rho}\\
&\qquad + \tfrac{\mathrm{i}}{2}S(e^{-g\rho} - 1) + \tfrac{\mathrm{i}}{4}e^{-g\rho}S\,\Gamma_\rho\,\Gamma_-\,\Gamma_+]\epsilon = 0\,,\\
&[\partial_- - g e^{g\rho}\,\Gamma_-(\Gamma_\rho + 1)]\epsilon = 0\,,\\
&[\partial_z + \tfrac{1}{2}g e^{g\rho}\,\Gamma_z(\Gamma_\rho + 1) + \tfrac{\mathrm{i}}{4}e^{-g\rho}S\,\Gamma_{z\rho}\,\Gamma_-]\epsilon = 0\,,\\
&[\partial_\rho - \tfrac{\mathrm{i}}{4}e^{-2g\rho}S\,\Gamma_- + \tfrac{1}{2}g\,\Gamma_\rho]\epsilon = 0\,.
\end{aligned} \tag{10.26}$$

Imposing the following projections

$$(\Gamma_\rho + 1)\epsilon = 0\,, \qquad \Gamma_-\epsilon = 0\,, \tag{10.27}$$

the Killing spinor equations become

$$[\partial_+ - \tfrac{\mathrm{i}}{2}S]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \qquad \partial_z\,\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{1}{2}g]\epsilon = 0\,. \tag{10.28}$$

Thus the Killing spinor is given by

$$\epsilon = e^{\frac{1}{2}g\rho + \frac{\mathrm{i}}{2}\int S\,dx^+}\epsilon_0\,, \tag{10.29}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$ and $\Gamma_-\epsilon_0 = 0$. The solution therefore preserves $\frac{1}{4}$ of the supersymmetry. We follow the literature [64, 65] and call these spinors the standard Killing spinors, since there is no further requirement on the function H for the existence of the ϵ, as long as H satisfies the equation of motion (10.24).

3. Supernumerary supersymmetry

When the integration constants of H satisfy further conditions, there can arise additional Killing spinors, which are called supernumerary Killing spinors in [64, 65]. In order to obtain these Killing spinors, we consider the integrability conditions $[\partial_M\,,\partial_N]\epsilon = 0$. We find that

$$\begin{aligned}
0 &= [\partial_z\,,\ \partial_\rho]\epsilon = \tfrac{\mathrm{i}}{4}ge^{-g\rho}S\,\Gamma_z\,\Gamma_-(\Gamma_\rho + 1)\epsilon\,,\\
0 &= [\partial_+\,,\partial_-]\epsilon = -\tfrac{\mathrm{i}}{2}gS\,\Gamma_-(\Gamma_\rho + 1)\epsilon\,,\\
0 &= [\partial_+\,,\partial_z]\epsilon = \tfrac{\mathrm{i}}{4}gS(3 - 2\Gamma_+\,\Gamma_-)\Gamma_z(\Gamma_\rho + 1)\epsilon - \tfrac{\mathrm{i}}{4}e^{-g\rho}\partial_+S\,\Gamma_{z\rho}\,\Gamma_-\epsilon\\
&\quad -\tfrac{1}{4}e^{g\rho}\partial_zH'\,\Gamma_\rho\,\Gamma_-\epsilon - \tfrac{1}{4}[ge^{2g\rho}H' + \partial_z^2H + \tfrac{1}{2}e^{-2g\rho}S^2]\Gamma_z\,\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{4}ge^{-g\rho}S(3 - \Gamma_+\,\Gamma_-)(\Gamma_\rho + 1)\epsilon + \tfrac{\mathrm{i}}{4}e^{-2g\rho}\partial_+S\,\Gamma_-\epsilon\\
&\quad -\tfrac{1}{4}\partial_zH'\,\Gamma_z\,\Gamma_-\epsilon + \tfrac{1}{4}e^{g\rho}[gH' + e^{-2g\rho}\partial_z^2H + \tfrac{1}{2}e^{-4g\rho}S^2]\Gamma_\rho\,\Gamma_-\epsilon\,.
\end{aligned} \tag{10.30}$$

To arrive at the last integrability condition we made use of equation (10.24) for H. It is clear that the integrability conditions are satisfied with the projections given in (10.27). However, we now show that it is possible to relax these projections. We find that the integrability conditions can also be satisfied, with the following less restrictive projection

$$g(\Gamma_\rho + 1)\epsilon = \mathrm{i} f\,\Gamma_-\epsilon \tag{10.31}$$

where $f = f(x^+, \rho, z)$. This gives the projected Killing spinor equations

$$\begin{aligned}
&[\partial_+ - \tfrac{\mathrm{i}}{2}S - \tfrac{1}{2}g^{-1}e^{-g\rho}fS\,\Gamma_- + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{2}e^{-g\rho}S)\Gamma_+\Gamma_- \\
&\qquad -\tfrac{1}{4}(e^{g\rho}H' + \Gamma_z\partial_z H)\Gamma_-]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_z + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{2}e^{-g\rho}S)\Gamma_z\,\Gamma_-]\epsilon = 0\,, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{1}{2}e^{-2g\rho}S)\Gamma_- - \tfrac{1}{2}g]\epsilon = 0\,.
\end{aligned} \tag{10.32}$$

The integrability conditions among these equations are

$$\begin{aligned}
0 &= [\partial_z\,, \partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}[\Gamma_z\partial_z f - (e^{g\rho}f)' + \tfrac{1}{2}ge^{-g\rho}S]\Gamma_z\,\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_z]\epsilon = -\tfrac{1}{2}[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{2}e^{-g\rho}\partial_+ S)\Gamma_z - (\mathrm{i}e^{g\rho}\,\Gamma_+ - g^{-1}e^{-g\rho}S)\partial_z f \\
&\quad +(e^{g\rho}f + \tfrac{1}{2}e^{-g\rho}S)^2\,\Gamma_z + \tfrac{1}{2}(e^{g\rho}\partial_z H' + \Gamma_z\partial_z^2 H)]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_\rho]\epsilon = -\tfrac{1}{2}[\mathrm{i}(\partial_+ f - \tfrac{1}{2}e^{-2g\rho}\partial_+ S) + g^{-1}S(e^{-g\rho}f)' \\
&\quad -\mathrm{i}((e^{g\rho}f)' - \tfrac{1}{2}ge^{-g\rho}S)\Gamma_+ + \tfrac{1}{2}e^{g\rho}(H'' + gH') + \tfrac{1}{2}\Gamma_z\partial_z H' \\
&\quad +e^{g\rho}(f^2 - \tfrac{1}{4}e^{-4g\rho}S^2)]\Gamma_-\epsilon\,.
\end{aligned} \tag{10.33}$$

It is clear from these expressions that if we want more supersymmetry than $\frac{1}{4}$ we need again to impose $\partial_z f = 0 = \partial_z H'$. The vanishing of the integrability conditions in this case then yields the equations

$$(e^{g\rho}f)' - \tfrac{1}{2}ge^{-g\rho}S = 0\,,$$

$$
\begin{aligned}
&\mathrm{i}(\partial_+ f + \tfrac{1}{2}e^{-2g\rho}\partial_+ S) + \tfrac{1}{2}e^{-g\rho}\partial_z^2 H + e^{g\rho}(f + \tfrac{1}{2}e^{-2g\rho}S)^2 = 0\,,\\
&\mathrm{i}(\partial_+ f - \tfrac{1}{2}e^{-2g\rho}\partial_+ S) + g^{-1}S(e^{-g\rho}f)' + \tfrac{1}{2}e^{g\rho}(H'' + gH')\\
&\qquad + e^{g\rho}f^2 - \tfrac{1}{4}e^{-3g\rho}S^2 = 0\,. \qquad (10.34)
\end{aligned}
$$

From the first of eqs.(10.34) we obtain

$$
f = -\tfrac{1}{2}e^{-2g\rho}S + e^{-g\rho}U\,, \qquad (10.35)
$$

where $U = U(x^+)$ is in general a complex function. Note that S is a real function. Using the solution for f and the equation for H the remaining two equations in (10.34) gives

$$
\begin{aligned}
&\mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}\partial_z^2 H = 0\,,\\
&\mathrm{i}\frac{dS}{dx^+} - e^{-g\rho}S^2 + 3S\,U + ge^{3g\rho}H' + e^{g\rho}\partial_z^2 H = 0\,. \qquad (10.36)
\end{aligned}
$$

Since the functions S and U depends only on x^+ we need to check that the ρ dependence in the equation for S drops out before we can proceed. For this we need to make use of the solution for H, which is given by (10.25). Setting $H_0 = 0$, and substituting H into eqs.(10.36) we have[1]

$$
\mathrm{i}\frac{dS}{dx^+} - 3S(b\,S - U) = 0\,, \qquad \mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}c\,S^2 = 0\,. \qquad (10.37)
$$

In order to solve these equations we rewrite U into an real and imaginary part $U = u + \mathrm{i}v$. Eqs.(10.37) then yield the following set of equations:

$$
\frac{dS}{dx^+} + 3v\,S = 0\,, \qquad \frac{du}{dx^+} + 2u\,v = 0\,, \qquad S(u - b\,S) = 0\,,
$$

[1]It is straightforward to verify that in general supernumeary supersymmetry requires that H_0 be given by (10.17), which is not the most general solution for $\Box H_0 = 0$.

$$\frac{dv}{dx^+} + v^2 - u^2 - \tfrac{1}{2} c\, S^2 = 0\,. \tag{10.38}$$

We have four equations for the five functions S, u, v, b and c, and so one function will be left arbitrary. We present the solution to eqs.(10.38) in terms of the function b. The solution is given by

$$\begin{aligned} S &= \frac{k}{b^3}\,, \qquad u = \frac{k}{b^2}\,, \qquad v = b^{-1}\frac{db}{dx^+}\,, \\ c &= \frac{2b^5}{k^2}\Big[\frac{d^2 b}{dx^{+2}} - \frac{k^2}{b^3}\Big]\,, \end{aligned} \tag{10.39}$$

where k is an arbitrary constant and we have taken $S \neq 0$. (The case with $S = 0$ was considered in section 2.) Note that the original generic $\frac{1}{4}$ supersymmetric solution depending on the three functions b, c and S now only have one independent function in order for the solution to have the enhanced $\frac{1}{2}$ supersymmetry.

We next turn to presenting the explicit Killing spinors. The Killing spinor equations are

$$\begin{aligned} &[\partial_+ - \tfrac{\mathrm{i}}{2} S - \tfrac{1}{2} g^{-1} e^{-g\rho} f S\, \Gamma_- + \tfrac{\mathrm{i}}{2} U\, \Gamma_+\, \Gamma_- - \tfrac{1}{4}(e^{g\rho} H' + c\, z\, S^2\, \Gamma_z)\Gamma_-]\epsilon = 0\,, \\ &\partial_-\, \epsilon = 0\,, \qquad [\partial_z + \tfrac{\mathrm{i}}{2} U\, \Gamma_z\, \Gamma_-]\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{\mathrm{i}}{2} g^{-1} f'\, \Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,, \end{aligned} \tag{10.40}$$

where f is given by (10.35). The third equation of the above implies $\epsilon = (1 - \frac{i}{2} z\, U\, \Gamma_z\, \Gamma_-) \times \chi(\rho, x^+)$. Substituting this into the fourth equation yields the solution $\chi = e^{\frac{1}{2} g\rho}(1 + \frac{i}{2} g^{-1} f\, \Gamma_-)\eta(x^+)$. The equation for η can be obtained from the first equation of (10.40) after making use of eqs.(10.37). We have

$$\frac{d\eta}{dx^+} - \tfrac{\mathrm{i}}{2}[S - U\, \Gamma_+\, \Gamma_-]\eta = 0\,. \tag{10.41}$$

Note that it requires conspiracy for the z and ρ dependent terms to drop out. Finally, we arrive at the Killing spinor, given by

$$\begin{aligned}\epsilon &= e^{\frac12 g\rho+\frac{\mathrm{i}}{2}\int S\,dx^+}(1-\tfrac{\mathrm{i}}{2}z\,U\,\Gamma_z\,\Gamma_-)(1+\tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\times\\ &\qquad\times\Big[1-\tfrac12(1-e^{-\mathrm{i}\int U dx^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0 \qquad (10.42)\end{aligned}$$

where ϵ_0 is a constant spinor, satisfying the projection

$$(\Gamma_\rho+1)\epsilon_0=0\,. \qquad (10.43)$$

There are two special cases that are worth considering. The first case is that b is set to a constant, implying that $v=0$. It follows then that the functions S and u are constants as well, and $c=-2b^2$. Assuming $S=\mu$ the Killing spinor in this case is given by

$$\begin{aligned}\epsilon &= e^{\frac12 g\rho+\frac{\mathrm{i}}{2}\mu x^+}(1-\tfrac{\mathrm{i}}{2}\mu b\,z\,\Gamma_z\,\Gamma_-)(1+\tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\times\\ &\qquad\times\Big[1-\tfrac12(1-e^{-\mathrm{i}\,\mu b x^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0 \qquad (10.44)\end{aligned}$$

where ϵ_0 is a constant spinor, satisfying the projection $(\Gamma_\rho+1)\epsilon_0=0$. Thus after imposing the condition $c=-2b^2$, the solution has $\frac12$ of the supersymmetry instead of the $\frac14$ for a generic pp-wave solution. The standard Killing spinors are those with an additional projection $\Gamma_-\epsilon_0=0$, in which case, ϵ of (10.44) becomes that in (10.29). The supernumerary Killing spinors are the remaining half with $\Gamma_-\epsilon_0\neq0$.

The function H, for the pp-wave with supernumerary supersymmetry, is given by

$$\begin{aligned}H &= -\mu^2\,b^2z^2-g^{-2}f^2=-\mu^2\Big(b^2z^2+g^{-2}(b^2\,e^{-2g\rho}+\tfrac14e^{-4g\rho}-b\,e^{-3g\rho})\Big)\,,\\ f &= -\tfrac12\mu(e^{-2g\rho}-2b\,e^{-g\rho})\,. \qquad (10.45)\end{aligned}$$

If we set $b=\frac12$, we have $H=-\mu^2\,[\frac14z^2+g^{-2}\sinh^2(\frac12g\rho)\,e^{-3g\rho}]$. We can then take the $g \rightarrow 0$ limit and obtain a pp-wave in ungauged $D = 4$, $\mathcal{N} = 2$ Einstein Maxwell supergravity. The solution is given by

$$\begin{aligned} ds^2 &= -4dx^+ \, dx^- - \tfrac{1}{4}\mu^2(z^2 + \rho^2)\,(dx^+)^2 + dz^2 + d\rho^2, \\ F_{(2)} &= -\mu \, dx^+ \wedge d\rho \, . \end{aligned} \qquad (10.46)$$

This is precisely the pp-wave arising from the Penrose limit of $\mathrm{AdS}_2 \times S^2$, which is known to have supernumerary supersymmetries [64, 65].

Note that in the ansatz (10.23), we could instead have used $A_{(1)} = \mu z \, dx^+$. The metric in this case is identical to that with $A_{(1)}$ given in (10.23). However, we verified that the solution would be non-supersymmetric, because of the explicit $A_{(1)}$ dependence in the supersymmetry transformation rule.

Charged pp-waves with $c = 0$ were also obtained in [80], by performing an infinite boost of the AdS charged black holes. It can be deduced from the above analysis that the solution with $c = 0$ has only the standard supersymmetry. We can also obtain pure gravitional $\frac{1}{2}$-supersymmetric pp-waves by setting $b = \tilde{b}/\mu$ and then sending $\mu \rightarrow 0$.

In [78] a general class of pp-waves that preserve $\frac{1}{4}$ of the supersymmetry were given. PP-waves with $\frac{1}{2}$ of the supersymmetry were also obtained in [79], where the Killing spinors were given in component language, while ours are presented in an elegant form, in terms of constant spinors satisfying a single gamma matrix projection.

The second special case corresponds to the absence of the Kaigorodov component b which can be achieved by taking a degenerate limit of (10.39). It is worth examing on its own. In this case we have the coupled system

$$\frac{dS}{dx^+} + 3v\,S = 0\,, \qquad \frac{dv}{dx^+} + v^2 - \tfrac{1}{2}c\,S^2 = 0\,. \qquad (10.47)$$

This implies a relation between the metric functions c and S, given by

$$c = -\tfrac{2}{3} S^{-3} \frac{d^2 S}{dx^{+2}} + \tfrac{8}{9} S^{-4} \Big(\frac{dS}{dx^+}\Big)^2 . \qquad (10.48)$$

Making use of these equations together with the solutions for H and f the Killing spinor equations (10.40) yield the solution

$$\begin{aligned} \epsilon &= e^{\frac{1}{2}g\rho}\, e^{\frac{\mathrm{i}}{2}\int S\, dx^+} (1 + \tfrac{1}{2} z\, v\, \Gamma_z\, \Gamma_-)(1 + \tfrac{\mathrm{i}}{2} g^{-1} f\, \Gamma_-) \times \\ &\quad \times \Big[1 - \tfrac{1}{2}(1 - e^{\int v\, dx^+})\Gamma_+\, \Gamma_-\Big] \epsilon_0 \end{aligned} \qquad (10.49)$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. For the functions H and f we have

$$\begin{aligned} H &= \tfrac{1}{2} S^2 \Big[c\, z^2 + \tfrac{1}{2} g^{-2} e^{-2g\rho}(2c - e^{-2g\rho})\Big] , \\ f &= -\tfrac{1}{2} e^{-2g\rho} S + \mathrm{i} e^{-g\rho} v\, . \end{aligned} \qquad (10.50)$$

We can consider a special case of eqs.(10.47) by setting $c \equiv$ constant and $v = \tilde{k}S$ where $\tilde{k}$ is a (real) constant. In this case the equations fixes $\tilde{k}$ to $\tilde{k}^2 = -\frac{1}{4}c$ with $c < 0$. The equation for S is

$$\frac{dS}{dx^+} + \tilde{k} S^2 = 0\, , \qquad (10.51)$$

with the solution given by $S(x^+) = 1/(1 + \tilde{k}\, x^+)\, .$

C. PP-waves in $D = 5$ gauged supergravity

1. The solution

For simplicity, we consider simple gauged supergravity in $D = 5$. The Lagrangian for the bosonic sector is given by [84]

$$e^{-1}\mathcal{L}_5 = R - \tfrac{1}{4} F^2_{(2)} + \tfrac{1}{12\sqrt{3}} \epsilon^{MNPQR} F_{MN} F_{PQ} A_R + 12 g^2 . \qquad (10.52)$$

Analogous to the $D = 4$ discussion, we use the following pp-wave ansatz

$$
\begin{aligned}
ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H(dx^+)^2 + dz_1^2 + dz_2^2) + d\rho^2, \\
A_{(1)} &= \tfrac{1}{2}g^{-1}S(1-e^{-2g\rho})\,dx^+, \qquad (10.53)
\end{aligned}
$$

where $S = S(x^+)$. The supergravity equations of motion then reduce to the following

$$
\Box H \equiv H'' + 4gH' + e^{-2g\rho}\sum_{i=1}^{2}\partial_i^2 H = -e^{-6g\rho}S^2. \qquad (10.54)
$$

The solution is given by

$$
H = S^2[\tfrac{1}{2}(c_1 z_1^2 + c_2 z_2^2) + g^{-2}(\tfrac{1}{4}(c_1+c_2)\,e^{-2g\rho} - \tfrac{1}{12}e^{-6g\rho} + b\,e^{-4g\rho})] + H_0\,, \qquad (10.55)
$$

where c_i and b are functions of x^+ and $\Box H_0 = 0$. The generalized Kaigorodov-type metric is obtained by setting $S = 0$ and $H_0 = c_0 + b\,e^{-4g\rho}$ with c_0 and b now being constants.

2. Supersymmetry

The supersymmetry transformation on the gravitino is given by

$$
\delta\Psi_M = [\nabla_M - \tfrac{3\,\mathrm{i}}{2\sqrt{3}}\,gA_M - \tfrac{\mathrm{i}}{16\sqrt{3}}F_{AB}\,(\Gamma_M\,\Gamma^{AB} - 3\,\Gamma^{AB}\,\Gamma_M) + \tfrac{1}{2}g\,\Gamma_M]\epsilon\,, \qquad (10.56)
$$

where ϵ is a complex symplectic spinor. For our pp-wave background, the Killing spinor equations are given by

$$
\begin{aligned}
&[\partial_+ + \tfrac{1}{2}ge^{g\rho}\,\Gamma_+(\Gamma_\rho+1) + \tfrac{1}{4}ge^{g\rho}H\,\Gamma_-(\Gamma_\rho+1) + \tfrac{1}{4}e^{g\rho}H'\,\Gamma_{-\rho} \\
&\qquad + \tfrac{1}{4}\sum_{i=1}^{2}\Gamma_{-i}\partial_i H + \tfrac{3\,\mathrm{i}}{4\sqrt{3}}S(e^{-2g\rho}-1) \\
&\qquad + \tfrac{\mathrm{i}}{8\sqrt{3}}e^{-2g\rho}S\,\Gamma_\rho(\Gamma_+\,\Gamma_- + 3\Gamma_-\,\Gamma_+)]\epsilon = 0\,, \\
&[\partial_- - ge^{g\rho}\,\Gamma_-(\Gamma_\rho+1)]\epsilon = 0\,, \\
&[\partial_i + \tfrac{1}{2}ge^{g\rho}\,\Gamma_i(\Gamma_\rho+1) + \tfrac{\mathrm{i}}{4\sqrt{3}}e^{-2g\rho}S\,\Gamma_{i\rho}\,\Gamma_-]\epsilon = 0\,, \qquad i = 1,2,
\end{aligned}
$$

$$[\partial_\rho - \tfrac{\mathrm{i}}{2\sqrt{3}} e^{-3g\rho} S\,\Gamma_- + \tfrac{1}{2} g\,\Gamma_\rho]\epsilon = 0\,. \qquad (10.57)$$

As in the case of $D = 4$, the standard Killing spinors, which exist for all H satisfying (10.54), arise with the following projections $(\Gamma_\rho + 1)\epsilon = 0$ and $\Gamma_-\epsilon = 0$. The Killing spinor equations become

$$[\partial_+ - \mathrm{i}\tfrac{\sqrt{3}}{4} S]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \qquad \partial_i\,\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{1}{2} g]\epsilon = 0\,. \qquad (10.58)$$

Thus, the generic pp-waves we considered preserve $\frac{1}{4}$ of the standard supersymmetry. In [77], a general class of null solutions with $\frac{1}{4}$ of the supersymmetry were obtained, however the issue of supernumerary supersymmetry was not addressed. We demonstrate below that, as in the case of $D = 4$, supernumerary Killing spinors can also arise.

To obtain the supernumerary Killing spinor and the corresponding conditions on H, we impose the following projection on the spinors

$$g(\Gamma_\rho + 1)\epsilon = \mathrm{i} f\,\Gamma_-\epsilon\,. \qquad (10.59)$$

The Killing spinor equations become

$$\begin{aligned}
&[\partial_+ - \tfrac{3\mathrm{i}}{4\sqrt{3}} S + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{2\sqrt{3}} e^{-2g\rho} S)\Gamma_+\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i\,\Gamma_-\partial_i H \\
&\qquad - \tfrac{1}{4}(e^{g\rho} H' + \sqrt{3} g^{-1} e^{-2g\rho} f\, S)\Gamma_-]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{2\sqrt{3}} e^{-2g\rho} S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{1}{\sqrt{3}} e^{-3g\rho} S)\Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,. \qquad (10.60)
\end{aligned}$$

The integrability conditions among these equations are

$$\begin{aligned}
0 &= [\partial_i\,, \partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}[\partial_i f - (e^{g\rho} f)'\,\Gamma_i + \tfrac{1}{\sqrt{3}} g e^{-2g\rho} S\,\Gamma_i]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_i]\epsilon = -\tfrac{1}{2}[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{2\sqrt{3}} e^{-2g\rho}\partial_+ S)\Gamma_i - (\mathrm{i}e^{g\rho}\,\Gamma_+ - \tfrac{3}{2\sqrt{3}} g^{-1} e^{-2g\rho} S)\partial_i f
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}e^{g\rho}\partial_i H' + \tfrac{1}{2}\sum_j \Gamma_j\partial_j\partial_i H + (e^{g\rho}f + \tfrac{1}{2\sqrt{3}}e^{-2g\rho}S)^2\,\Gamma_i]\Gamma_-\epsilon\,,\\
0 \;=\;& [\partial_+\,,\partial_\rho]\epsilon = -\tfrac{1}{2}[\mathrm{i}(\partial_+ f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}\partial_+ S) + \tfrac{3}{2\sqrt{3}}g^{-1}S(e^{-2g\rho}f)' + \tfrac{1}{2}\sum_i \Gamma_i\partial_i H'\\
&-\mathrm{i}((e^{g\rho}f)' - \tfrac{1}{\sqrt{3}}ge^{-2g\rho}S)\Gamma_+ + \tfrac{1}{2}e^{g\rho}(H'' + gH')\\
&+(f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}S)(e^{g\rho}f + \tfrac{1}{2\sqrt{3}}e^{-2g\rho}S)]\Gamma_-\epsilon\,. \qquad (10.61)
\end{aligned}
$$

To have more supersymmetry than the $\frac{1}{4}$ we need to set

$$
\partial_i f = 0 = \partial_i H' \quad \text{and} \quad \partial_j\partial_i H = 0\,, \quad i \neq j\,. \qquad (10.62)
$$

The integrability conditions then implies

$$
\begin{aligned}
&f = -\tfrac{1}{2\sqrt{3}}e^{-3g\rho}S + e^{-g\rho}U\,,\\
&\mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}\partial_i^2 H = 0\,, \qquad i = 1, 2,\\
&\mathrm{i}\Big(\frac{dS}{dx^+} - \tfrac{2}{\sqrt{3}}e^{2g\rho}\frac{dU}{dx^+}\Big) - g^{-1}e^{3g\rho}S(e^{-2g\rho}f)' - \tfrac{1}{\sqrt{3}}e^{4g\rho}(H'' + gH')\\
&\qquad\qquad -\tfrac{2}{\sqrt{3}}e^{3g\rho}U(f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}S) = 0\,, \qquad (10.63)
\end{aligned}
$$

where $U = U(x^+)$. Substituting in the solution for H, given by (10.55), we find that it is necessary to have that $c_1 = c_2 \equiv c$, and that H_0 is given by (10.17). For simplicity, we set $H_0 = 0$ here since the H_0 represents the pure gravitational component, which was discussed in section 2. The equations for S and U are then given by

$$
\mathrm{i}\frac{dS}{dx^+} - 4S(\sqrt{3}\,b\,S - U) = 0\,, \qquad \mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}c\,S^2 = 0\,. \qquad (10.64)
$$

Substituting $U = u + \mathrm{i}v$ into the above yields the equations

$$
\begin{aligned}
&\frac{dS}{dx^+} + 4v\,S = 0\,, \qquad \frac{du}{dx^+} + 2u\,v = 0\,, \qquad S(u - \sqrt{3}\,b\,S) = 0\,,\\
&\frac{dv}{dx^+} + v^2 - u^2 - \tfrac{1}{2}c\,S^2 = 0\,. \qquad (10.65)
\end{aligned}
$$

The solution to these equations is

$$
\begin{aligned}
S &= \frac{k}{b^2}\,, \qquad u = \frac{\sqrt{3}\,k}{b}\,, \qquad v = \frac{1}{2b}\frac{db}{dx^+}\,,\\
c &= \frac{b^3}{k^2}\Big[\frac{d^2b}{dx^{+2}} - \frac{1}{2b}\Big(\frac{db}{dx^+}\Big)^2\Big] - 6\,b^2,
\end{aligned}
\tag{10.66}
$$

where k is an arbitrary constant and we have taken $S \neq 0$. Note that as in the case of $D = 4$, the original generic $\frac{1}{4}$-supersymmetric metric depending on the four functions b, c_1, c_2 and S now only have one independent function in order for the solution to have the enhanced $\frac{1}{2}$ supersymmetry.

The Killing spinor is calculated from the equations

$$
\begin{aligned}
&[\partial_+ - \tfrac{3\,\mathrm{i}}{4\sqrt{3}}S - \tfrac{3}{4\sqrt{3}}g^{-1}e^{-2g\rho}f S\,\Gamma_- + \tfrac{\mathrm{i}}{2}U\,\Gamma_+\Gamma_-\\
&\qquad -\tfrac{1}{4}(e^{g\rho}H' + c\,S^2(z_1\,\Gamma_1 + z_2\,\Gamma_2))\Gamma_-]\epsilon = 0\,,\\
&\partial_-\,\epsilon = 0\,, \qquad [\partial_i + \tfrac{\mathrm{i}}{2}U\,\Gamma_i\,\Gamma_-]\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{i}{2}g^{-1}f'\,\Gamma_- - \tfrac{1}{2}g]\epsilon = 0\,.
\end{aligned}
\tag{10.67}
$$

The solution is

$$
\begin{aligned}
\epsilon &= e^{\frac{1}{2}g\rho + \mathrm{i}\frac{\sqrt{3}}{4}\int S\,dx^+}(1 - \tfrac{\mathrm{i}}{2}U\,(z_1\,\Gamma_1 + z_2\,\Gamma_2)\Gamma_-)(1 + \tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\times\\
&\qquad \times\Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\int U dx^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,
\end{aligned}
\tag{10.68}
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. As in $D = 4$ we consider two special cases. The first corresponds to $v = 0$, which implies that b, c and S are all constants, with $c = -6b^2$. Letting $S = \mu$ the Killing spinor in this case is given by

$$
\begin{aligned}
\epsilon &= e^{\frac{1}{2}g\rho + \mathrm{i}\frac{\sqrt{3}}{4}\mu x^+}\Big(1 - \mathrm{i}\tfrac{\sqrt{3}}{2}\,\mu b(z_1\,\Gamma_1 + z_2\,\Gamma_2)\,\Gamma_-\Big)(1 + \tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\times\\
&\qquad \times\Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\sqrt{3}\,\mu b x^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,
\end{aligned}
\tag{10.69}
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. Thus the solution preserves half of the supersymmetry. Among all the Killing spinors, the standard ones are those with $\Gamma_- \epsilon_0 = 0$, whilst the remaining half with $\Gamma_- \epsilon_0 \neq 0$ are the supernumerary ones. The function H for the pp-waves with supernumerary supersymmetry is given by

$$\begin{aligned} H &= -3\mu^2 b^2 (z_1^2 + z_2^2) - g^{-2} f^2 \\ &= -\mu^2 [3b^2 (z_1^2 + z_2^2) + g^{-2} (3b^2\, e^{-2g\rho} + \tfrac{1}{12} e^{-6g\rho} - b\, e^{-4g\rho})]\,, \\ f &= -\tfrac{1}{2\sqrt{3}} \mu (e^{-3g\rho} - 6b\, e^{-g\rho})\,. \end{aligned} \tag{10.70}$$

If we further let $b = \frac{1}{6}$, we have $H = -\frac{1}{12}\mu^2(z_1^2 + z_2^2 + 4g^{-2}\sinh^2(g\rho)\, e^{-4g\rho})$. This enables us to take the limit $g \to 0$, giving rise to a pp-wave in the corresponding ungauged $D = 5$ supergravity, given by

$$\begin{aligned} ds^2 &= -4dx^+ dx^- - \tfrac{1}{12}\mu^2\,(z_1^2 + z_2^2 + 4\rho^2)\,(dx^+)^2 + dz_1^2 + dz_2^2 + d\rho^2, \\ F_{(2)} &= -\mu dx^+ \wedge d\rho\,. \end{aligned} \tag{10.71}$$

This pp-wave can also arise from the Penrose limit of $\mathrm{AdS}_3 \times S^2$ or $\mathrm{AdS}_2 \times S^3$, which have supernumerary supersymmetries. Let us work out the Penrose limit of the maximal supersymmetric vacuum solution $\mathrm{AdS}_3 \times S^2$ which is given by

$$\begin{aligned} ds^2 &= 4R^2(-\cosh^2\chi\, dt^2 + d\chi^2 + \sinh^2\chi\, d\varphi^2) + R^2(d\theta^2 + \cos^2\theta\, d\phi^2)\,, \\ F_{(2)} &= \sqrt{3}R\, \cos\theta\, d\theta \wedge d\phi\,. \end{aligned} \tag{10.72}$$

Performing the substitution

$$\begin{aligned} &\frac{\mu}{\sqrt{3}}\, dx^+ = dt + \tfrac{1}{2} d\phi\,, \qquad \frac{\sqrt{3}}{\mu\, R^2}\, dx^- = dt - \tfrac{1}{2} d\phi\,, \\ &\chi = \frac{z}{2R}\,, \qquad \theta = \frac{\rho}{R}\,, \end{aligned} \tag{10.73}$$

and taking the limit $R \to \infty$ we obtain exactly the pp-wave (10.71).

The second case is that of $b=0$, and hence eqs.(10.65) reduces to

$$\frac{dS}{dx^+}+4v\,S=0\,,\qquad \frac{dv}{dx^+}+v^2-\tfrac12 c\,S^2=0\,. \tag{10.74}$$

The Killing spinor is then given by

$$\begin{aligned}\epsilon &= e^{\frac12 g\rho}\,e^{\mathrm{i}\frac{\sqrt3}{4}\int S\,dx^+}(1+\tfrac12 v(z_1\,\Gamma_1+z_2\,\Gamma_2)\Gamma_-)(1+\tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\times\\ &\quad\times\Big[1-\tfrac12(1-e^{\int v\,dx^+})\Gamma_+\Gamma_-\Big]\epsilon_0\,,\end{aligned} \tag{10.75}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho+1)\epsilon_0=0$ and

$$\begin{aligned}H &= \tfrac12 S^2\Big[c\,(z_1^2+z_2^2)+2g^{-2}e^{-2g\rho}(c-\tfrac{1}{12}e^{-4g\rho})\Big]\,,\\ f &= -\tfrac{1}{2\sqrt3}e^{-3g\rho}S+\mathrm{i}e^{-g\rho}v\,.\end{aligned} \tag{10.76}$$

If we specialize to $v=\tilde kS$ and $c=-6\tilde k^2$ where $\tilde k$ is a constant, the system (10.74) simplifies to

$$\frac{dS}{dx^+}+4\tilde kS^2=0\,. \tag{10.77}$$

D. PP-waves in $D=6$ and $D=7$

1. $D=6$

Our next example is in the Romans six-dimensional gauged $\mathcal{N}=(1,1)$ supergravity [31]. The bosonic field content comprises the metric, a dilaton ϕ, a 2-form potential, a $U(1)$ potential and the gauge potentials $A^i_{(1)}$ of $SU(2)$ Yang-Mills. The Lagrangian describing the bosonic sector is [85]

$$\begin{aligned}\mathcal{L} &= R*\mathbb{1}-\tfrac12{*d\phi}\wedge d\phi+(2g_1^2X^2+\tfrac83 g_1g_2X^{-2}-\tfrac29 g_2^2X^{-6})*\mathbb{1}\\ &- \tfrac12 X^4*F_{(3)}\wedge F_{(3)}-\tfrac12 X^{-2}\Big(*G_{(2)}\wedge G_{(2)}+*F^a_{(2)}\wedge F^a_{(2)}\Big)\\ &- A_{(2)}\wedge(\tfrac12 dB_{(1)}\wedge dB_{(1)}+\tfrac13 g_2A_{(2)}\wedge dB_{(1)}+\tfrac{2}{27}g_2^2A_{(2)}\wedge A_{(2)}+\tfrac12 F^a_{(2)}\wedge F^a_{(2)})\,,\end{aligned} \tag{10.78}$$

where $X \equiv e^{-\frac{1}{2\sqrt{2}}\phi}$, $F_{(3)} = dA_{(2)}$, $G_{(2)} = dB_{(1)} + \frac{2}{3}g_2 A_{(2)}$, $F^a_{(2)} = dA^a_{(1)} + \frac{1}{2}g_1\epsilon_{abc}A^b_{(1)} \wedge A^c_{(1)}$. The fermions of this theory comprise symplectic-Majorana gravitinos Ψ_{Mi} and dilatinos λ_i where $i = 1, 2$ is an $SP(1)$ index. The supersymmetry transformations are given by [86]

$$\begin{aligned}
\delta\Psi_{Mi} &= [D_M - \tfrac{1}{48}X^2 F_{ABC}\,\Gamma^{ABC}\,\Gamma_M\Gamma^7 - \tfrac{1}{4\sqrt{2}}(g_1 X + \tfrac{1}{3}g_2 X^{-3})\,\Gamma_M]\epsilon_i \\
&\quad + \tfrac{1}{16\sqrt{2}}(\Gamma_M\,\Gamma^{AB} - 2\Gamma^{AB}\,\Gamma_M)X^{-1}(G_{AB}\,\delta_i{}^j - \mathrm{i}\,\Gamma^7 F_{AB\,i}{}^j)\Gamma^7\epsilon_j\,, \\
\delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}}\Gamma^M\partial_M\phi + \tfrac{1}{24}X^2 F_{ABC}\,\Gamma^{ABC}\,\Gamma^7 + \tfrac{1}{2\sqrt{2}}(g_1 X - g_2 X^{-3})]\epsilon_i \\
&\quad + \tfrac{1}{8\sqrt{2}}X^{-1}(G_{AB}\,\delta_i{}^j - \mathrm{i}\,\Gamma^7 F_{AB\,i}{}^j)\,\Gamma^{AB}\,\Gamma^7\epsilon_j\,.
\end{aligned} \tag{10.79}$$

The gauge covariant derivative is defined as $D_M\epsilon_i = \nabla_M\epsilon_i + \frac{\mathrm{i}}{2}g_1 A_{M\,i}{}^j\epsilon_j$ where $A_{M\,i}{}^j \equiv A^a_M(-\sigma^a)_i{}^j$ with the field strength given by $F_{MNi}{}^j = \partial_M A_{Ni}{}^j + \frac{\mathrm{i}}{2}g_1 A_{Mi}{}^k A_{Nk}{}^j - (M \leftrightarrow N)$ and σ^a are the usual Pauli matrices.

In this chapter, we consider pp-wave solutions supported by only one field strength. Owing to the Chern-Simons modifications to various field strengths, we find that this can only be done with a $U(1)$ vector field coming from the $SU(2)$ Yang-Mills. Thus we consistently set all the remaining form fields to zero, and also without loss of generality (while insisting on AdS background) take $g_1 = g_2 = -3g/\sqrt{2}$. This leads to the pp-wave ansatz

$$\begin{aligned}
ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H(dx^+)^2 + dz_1^2 + dz_2^2 + dz_3^2) + d\rho^2, \\
A_{(1)} &= \tfrac{1}{3}g^{-1}S(1 - e^{-3g\rho})\,dx^+,
\end{aligned} \tag{10.80}$$

where $S = S(x^+)$. The equations of motion reduce to

$$\Box H \equiv H'' + 5gH' + e^{-2g\rho}\sum_{i=1}^{3}\partial_i^2 H = -e^{-8g\rho}S^2, \tag{10.81}$$

and the solution for H is given by

$$H = S^2\Big[\tfrac{1}{2}\sum_{i=1}^{3} c_i z_i + g^{-2}\big(\tfrac{1}{6}(c_1+c_2+c_3)e^{-2g\rho} - \tfrac{1}{24}e^{-8g\rho} + b\, e^{-5g\rho}\big)\Big] + H_0\,, \qquad (10.82)$$

where $\Box H_0 = 0$. The b and c_i are functions of x^+.

We now investigate the supersymmetry of the pp-waves. This is more conveniently done if we rewrite the symplectic Majorana spinors using a Dirac notation. (See [25] for details.) The Killing spinor equations from the gravitino transformation rule are given by

$$\begin{aligned}
&[\partial_+ - \tfrac{\mathrm{i}}{2\sqrt{2}}S + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)\Gamma_+\,\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i\partial_i H\,\Gamma_- \\
&\qquad -\tfrac{1}{2\sqrt{2}}(g^{-1}e^{-3g\rho}fS + \tfrac{1}{\sqrt{2}}e^{g\rho}H')\Gamma_-]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \qquad i = 1,2,3, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{3}{4\sqrt{2}}e^{-4g\rho}S)\Gamma_- - \tfrac{1}{2}g]\epsilon = 0\,, \qquad (10.83)
\end{aligned}$$

where we have made use of the projection condition $g(\Gamma_\rho + 1)\epsilon = \mathrm{i}f\,\Gamma_-\epsilon$ and where $f = f(x^+, \rho, z_i)$. The integrability conditions $[\partial_M\,, \partial_N]\epsilon = 0$ among these projected Killing spinor equations are

$$\begin{aligned}
0 &= [\partial_i\,,\partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}[\partial_i f - (e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)'\,\Gamma_i]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,,\partial_i]\epsilon = -\tfrac{1}{2}[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}\partial_+ S)\Gamma_i + \tfrac{1}{2}\sum_j \Gamma_j\partial_j\partial_i H \\
&\qquad +\tfrac{1}{2}e^{g\rho}\partial_i H' + (e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)^2\,\Gamma_i - (\mathrm{i}e^{g\rho}\,\Gamma_+ - \tfrac{1}{\sqrt{2}}g^{-1}e^{-3g\rho}S)\partial_i f]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,,\partial_\rho]\epsilon = -\tfrac{1}{2}[\mathrm{i}(\partial_+ f - \tfrac{3}{4\sqrt{2}}e^{-4g\rho}\partial_+ S) + \tfrac{1}{2}\sum_i \Gamma_i\partial_i H' \\
&\qquad -\mathrm{i}(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)'\,\Gamma_+ + \tfrac{1}{\sqrt{2}}g^{-1}S(e^{-3g\rho}f)' \\
&\qquad +(f - \tfrac{3}{4\sqrt{2}}e^{-4g\rho}S)(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S) + \tfrac{1}{2}e^{g\rho}(H'' + gH')]\Gamma_-\epsilon\,. \qquad (10.84)
\end{aligned}$$

As before it is required that we set

$$\partial_i f = 0 = \partial_i H' \qquad \text{and} \qquad \partial_j \partial_i H = 0\,, \qquad i \neq j\,, \tag{10.85}$$

and $c_i = c$. The integrability conditions yields after using the solution for H the following results

$$\begin{aligned} & f = -\tfrac{1}{4\sqrt{2}} e^{-4g\rho} S + e^{-g\rho}\, U, \\ & \mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2} c\, S^2 = 0\,, \\ & \mathrm{i}\frac{dS}{dx^+} + \tfrac{1}{12\sqrt{2}} e^{-3g\rho} S[S(7 - 240b\, e^{3g\rho}) + 60\sqrt{2}\, e^{3g\rho}\, U] = 0\,. \end{aligned} \tag{10.86}$$

In the case of $S = 0$, corresponding to purely-gravitational waves, discussed in section 2, the last equation is trivially satisfied. When $S \neq 0$, due to the ρ dependence, we conclude that no supersymmetry enhancement can occur here. This is expected, since in ungauged $D = 6$, $\mathcal{N} = (1,1)$ supergravity, the pp-waves supported by a 2-form field strength also have no supernumerary supersymmetry. The solution does have standard supersymmetry though. The Killing spinor is given by

$$\epsilon = e^{\frac{1}{2} g\rho + \frac{\mathrm{i}}{2\sqrt{2}} \int S\, dx^+} \epsilon_0\,, \tag{10.87}$$

where $(\Gamma_\rho + 1)\epsilon_0 = 0 = \Gamma_- \epsilon_0$. It is easy to verify that the Killing spinor equations associated with both the gravitino and dilatino transformation rules are satisfied. Thus the solution preserves $\frac{1}{4}$ of the supersymmetry.

2. $D = 7$

The Lagrangian for the bosonic sector of half-maximum supergravity in seven dimensions [46] can be written as follows [87]

$$\mathcal{L} \;=\; R * 1\!\!1 - \tfrac{1}{2} {*d\phi} \wedge d\phi - \tfrac{1}{2} X^4 * F_{(4)} \wedge F_{(4)} - \tfrac{1}{2} X^{-2} * F^a_{(2)} \wedge F^a_{(2)}$$

$$
\begin{aligned}
&+\tfrac{1}{2}F^a_{(2)}\wedge F^a_{(2)}\wedge A_{(3)}-\tfrac{1}{2\sqrt{2}}g_2 F_{(4)}\wedge A_{(3)}\\
&+(2g_1^2X^2+2g_1g_2X^{-3}-\tfrac{1}{4}g_2^2X^{-8})*\mathbb{1} \qquad (10.88)
\end{aligned}
$$

where $X=e^{-\frac{1}{\sqrt{10}}\phi}$, $F_{(4)}=dA_{(3)}$ and $F^a_{(2)}=dA^a_{(1)}+\frac{1}{2}g_1\epsilon_{abc}A^b_{(1)}\wedge A^c_{(1)}$. In addition there is a "self-duality" condition that must be imposed, given by

$$
X^4 *F_{(4)}=-\tfrac{1}{\sqrt{2}}g_2A_{(3)}+\tfrac{1}{2}\omega_{(3)}\,, \qquad (10.89)
$$

where $\omega_{(3)}$ is defined as $\omega_{(3)}=A^a_{(1)}\wedge F^a_{(2)}-\frac{1}{6}g_1\,\epsilon_{abc}\,A^a_{(1)}\wedge A^b_{(1)}\wedge A^c_{(1)}$. This theory has a pair of symplectic-Majorana gravitinos ψ_{Mi} and a pair of dilatinos λ_i, where $i=1,2$ is an $SP(1)$ index. The fermionic supersymmetry transformations are given by [86]

$$
\begin{aligned}
\delta\psi_{Mi} &= \nabla_M\epsilon_i+\tfrac{\mathrm{i}}{2}g_1A_{Mi}{}^j\epsilon_j+\tfrac{1}{960}X^2F_{ABCD}(\Gamma_M\,\Gamma^{ABCD}+5\Gamma^{ABCD}\,\Gamma_M)\epsilon_i\\
&\quad -\tfrac{\mathrm{i}}{40\sqrt{2}}X^{-1}(3\Gamma_M\,\Gamma^{AB}-5\Gamma^{AB}\,\Gamma_M)F_{ABi}{}^j\epsilon_j-\tfrac{1}{5\sqrt{2}}(g_1X+\tfrac{1}{4}g_2X^{-4})\Gamma_M\epsilon_i\,,\\
\delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}}\Gamma^M\partial_M\phi+\tfrac{1}{48\sqrt{5}}X^2F_{ABCD}\,\Gamma^{ABCD}]\epsilon_i-\tfrac{\mathrm{i}}{4\sqrt{10}}X^{-1}F_{ABi}{}^j\,\Gamma^{AB}\epsilon_j\\
&\quad +\tfrac{1}{\sqrt{10}}(g_1X-g_2X^{-4})\epsilon_i\,, \qquad (10.90)
\end{aligned}
$$

where $A_{Mi}{}^j\equiv A^a_M(-\sigma^a)_i{}^j$. Owing to the odd-dimensional self-duality condition for the $A_{(3)}$, our standard ansatz for the pp-wave metric does not work for $A_{(3)}$. We thus consider the pp-wave supported only by the $U(1)$ subsector of the $SU(2)$ Yang-Mills. The pp-wave solution is given by

$$
\begin{aligned}
ds^2 &= e^{2g\rho}(-4dx^+\,dx^-+H(dx^+)^2+dz_1^2+dz_2^2+dz_3^2+dz_4^2)+d\rho^2,\\
A_{(1)} &= \tfrac{1}{4}g^{-1}S(1-e^{-4g\rho})\,dx^+, \qquad (10.91)
\end{aligned}
$$

where $S=S(x^+)$ and H satisfies

$$
\Box H\equiv H''+6gH'+e^{-2g\rho}\sum_{i=1}^{4}\partial_i^2H=-e^{-10g\rho}S^2. \qquad (10.92)
$$

Here we have set $g_1 = g_2 = -2\sqrt{2}\,g$. The function H can be solved, given by

$$H = S^2\Big[\tfrac{1}{2}\sum_{i=1}^{4} c_i z_i^2 + g^{-2}(\tfrac{1}{8}\sum_{i=1}^{4} c_i\, e^{-2g\rho} - \tfrac{1}{40} e^{-10g\rho} + b\, e^{-6g\rho})\Big] + H_0\,, \qquad (10.93)$$

with $\Box H_0 = 0$ and b and c_i are functions of x^+.

The projected Killing spinor equations from the gravitino transformation rule are given by

$$\begin{aligned}
&[\partial_+ - \tfrac{\mathrm{i}}{2\sqrt{2}} S + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)\Gamma_+\,\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i \partial_i H\,\Gamma_- \\
&\qquad - \tfrac{1}{2\sqrt{2}}(g^{-1} e^{-4g\rho} f S + \tfrac{1}{\sqrt{2}} e^{g\rho} H')\Gamma_-]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \qquad i = 1,2,3,4, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho} S)\Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,. \qquad (10.94)
\end{aligned}$$

The integrability conditions

$$\begin{aligned}
0 &= [\partial_i\,, \partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}[\partial_i f - (e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)'\,\Gamma_i]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_i]\epsilon = -\tfrac{1}{2}[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho}\partial_+ S)\Gamma_i + \tfrac{1}{2}\sum_j \Gamma_j \partial_j \partial_i H \\
&\quad + \tfrac{1}{2} e^{g\rho}\partial_i H' + (e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)^2\,\Gamma_i - (\mathrm{i} e^{g\rho}\,\Gamma_+ - \tfrac{1}{\sqrt{2}} g^{-1} e^{-4g\rho} S)\partial_i f]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_\rho]\epsilon = -\tfrac{1}{2}[\mathrm{i}(\partial_+ f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho}\partial_+ S) + \tfrac{1}{2}\sum_i \Gamma_i \partial_i H' \\
&\quad - \mathrm{i}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)'\,\Gamma_+ + \tfrac{1}{\sqrt{2}} g^{-1} S (e^{-4g\rho} f)' \\
&\quad + (f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho} S)(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S) + \tfrac{1}{2} e^{g\rho}(H'' + g H')]\Gamma_-\epsilon\,, \qquad (10.95)
\end{aligned}$$

imply that there is no supernumerary Killing spinors in this case. This should be expected since in $D = 7$, even in ungauged supergravities, there is no pp-wave supported by a 2-form field strength that has supernumerary supersymmetry. The solution does have $\frac{1}{4}$ of standard supersymmetry, with the Killing spinor given by

$$\epsilon = e^{\frac{1}{2} g\rho + \frac{\mathrm{i}}{2\sqrt{2}}\int S\, dx^+}\,\epsilon_0\,, \qquad (10.96)$$

where $(\Gamma_\rho + 1)\epsilon_0 = 0 = \Gamma_- \epsilon_0$.

E. Uplifting to M/string theory

In this appendix we uplift the supersymmetric solutions supported by the U(1) charge to ten and eleven dimensions. In the case of the four and five dimensional solutions we uplift those with S being a constant. The four and seven dimensional solutions are embedded in M-theory and the solutions in $D = 5$ and $D = 6$ are uplifted to type-IIB supergravity and to Romans massive theory, respectively.

1. $D = 4$ oxidized to $D = 11$

The embedding formulae to eleven dimensions were obtained in [88] or we can also use the ansatz in [89] after truncating to our case. We obtain

$$\begin{aligned}
d\hat{s}_{11} &= e^{2g\rho}[-4dx^+dx^- - \mu^2(\tfrac{1}{4}z^2 + g^{-2}\sinh^2(\tfrac{1}{2}g\rho)e^{-3g\rho})(dx^+)^2 + dz^2] + d\rho^2 \\
&\quad +4g^{-2}d\xi + g^{-2}[c^2\,(\sigma_1^2 + \sigma_2^2 + h_3^2) + s^2\,(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \tilde{h}_3^2)]\,, \\
\hat{F}_{(4)} &= -6ge^{3g\rho}dx^+ \wedge dx^- \wedge d\rho \wedge dz - \mu g^{-2}[s\,c\,d\xi \wedge \sigma_3 + \tfrac{1}{2}c^2\sigma_1 \wedge \sigma_2 \\
&\quad -s\,c\,d\xi \wedge \tilde{\sigma}_3 + \tfrac{1}{2}s^2\,\tilde{\sigma}_1 \wedge \tilde{\sigma}_2] \wedge dx^+ \wedge dz\,,
\end{aligned} \tag{10.97}$$

where σ_i are the three left-invariant 1-forms on S^3 satisfying $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. They are given by $\sigma_1 + \mathrm{i}\sigma_2 = e^{-\mathrm{i}\psi}(d\theta + \mathrm{i}\sin\theta\, d\varphi)$ and $\sigma_3 = d\psi + \cos\theta\, d\varphi$ in terms of the Euler angles. The $\tilde{\sigma}_i$ are left-invariant 1-forms on a second S^3. We have also defined

$$\begin{aligned}
&c \equiv \cos\xi\,, \qquad s \equiv \sin\xi\,, \\
&h_3 \equiv \sigma_3 - \tfrac{1}{2}\mu(1 - e^{-g\rho})dx^+, \qquad \tilde{h}_3 \equiv \tilde{\sigma}_3 - \tfrac{1}{2}\mu(1 - e^{-g\rho})dx^+, \\
&\epsilon_{(3)} = \sigma_1 \wedge \sigma_2 \wedge h_3\,, \qquad \tilde{\epsilon}_{(3)} = \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{h}_3\,.
\end{aligned} \tag{10.98}$$

In this pp-wave, the internal S^7 is twisted but not flattened. Analogous solution but with untwisted round S^7 can be found in [90].

2. $D = 5$ oxidized to type-IIB

Using the uplifting formulae to type IIB in [88, 91] we obtain for the metric

$$
\begin{aligned}
d\hat{s}_{10}^2 = e^{2g\rho}\,&[-4dx^+dx^- - \tfrac{1}{12}\mu^2(z_1^2 + z_2^2 + 4g^{-2}\sinh^2(g\rho)\,e^{-4g\rho})(dx^+)^2 \\
&+dz_1^2 + dz_2^2] + d\rho^2 + g^{-2}\sum_{i=1}^{3}[d\mu_i^2 + \mu_i^2(d\phi_i \\
&+\tfrac{1}{2\sqrt{3}}\mu(1 - e^{-2g\rho})dx^+)^2\,]\,,
\end{aligned} \tag{10.99}
$$

and for the 5-form field strength $F_{(5)} = G_{(5)} + *G_{(5)}$,

$$
G_{(5)} = -8ge^{4g\rho}dx^+\wedge dx^-\wedge d\rho\wedge d^2z - \tfrac{1}{2\sqrt{3}}\mu g^{-2}\sum_{i=1}^{3} d(\mu_i^2)\wedge d\phi_i\wedge dx^+\wedge d^2z\,. \tag{10.100}
$$

The μ_i are parameterised as

$$
\mu_1 = \sin\theta\,, \qquad \mu_2 = \cos\theta\,\sin\psi\,, \qquad \mu_3 = \cos\theta\,\cos\psi\,, \tag{10.101}
$$

in terms of the angles on a 2-sphere.

3. $D = 6$ oxidized to Romans massive theory

The bosonic sector of Romans massive theory [23] is described by the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{10} = \hat{R}\hat{*}1\!\!1 &- \tfrac12\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \tfrac12 e^{\frac32\hat{\phi}}\hat{*}\hat{F}_{(2)}\wedge\hat{F}_{(2)} - \tfrac12 e^{-\hat{\phi}}\hat{*}\hat{F}_{(3)}\wedge\hat{F}_{(3)} \\
&-\tfrac12 e^{\frac12\hat{\phi}}\hat{*}\hat{F}_{(4)}\wedge\hat{F}_{(4)} - \tfrac12 d\hat{A}_{(3)}\wedge d\hat{A}_{(3)}\wedge\hat{A}_{(2)} - \tfrac16 m\,d\hat{A}_{(3)}\wedge(\hat{A}_{(2)})^3 \\
&-\tfrac{1}{40}m^2(\hat{A}_{(2)})^5 - \tfrac12 m^2 e^{\frac52\hat{\phi}}\hat{*}1\!\!1\,,
\end{aligned} \tag{10.102}
$$

where the field strengths are defined as

$$
\hat{F}_{(2)} = d\hat{A}_{(1)} + m\hat{A}_{(2)}\,, \qquad \hat{F}_{(3)} = d\hat{A}_{(2)}\,,
$$

$$\hat{F}_{(4)} = d\hat{A}_{(3)} + \hat{A}_{(1)} \wedge d\hat{A}_{(2)} + \tfrac{1}{2} m \hat{A}_{(2)} \wedge \hat{A}_{(2)}\,. \qquad (10.103)$$

Note that the Bianchi identities in this theory are given by

$$d\hat{F}_{(4)} = \hat{F}_{(2)} \wedge \hat{F}_{(3)}\,, \qquad d\hat{F}_{(3)} = 0\,, \qquad d\hat{F}_{(2)} = m\hat{F}_{(3)}\,. \qquad (10.104)$$

Using the embedding formulae obtained in [85] we can lift our six dimensional solution to a solution of the above theory. It is given by (with $m = g$)

$$\begin{aligned} d\hat{s}_{10}^2 &= s^{\frac{1}{12}}\,[ds_6^2 + \tfrac{4}{9} g^{-2} d\xi^2 + \tfrac{1}{9} g^{-2} c^2\,(\sigma_1^2 + \sigma_2^2 + h_3^2)]\,, \\ \hat{F}_{(4)} &= \tfrac{10}{81} g^{-3} s^{1/3} c^3\, d\xi \wedge \epsilon_{(3)} - \tfrac{2}{9\sqrt{2}} g^{-2} e^{-3g\rho} S[s^{1/3} c\,\sigma_3 \wedge d\xi \\ &\qquad - \tfrac{1}{2} s^{4/3} c^2\,\sigma_1 \wedge \sigma_2] \wedge dx^+ \wedge d\rho\,, \\ \hat{F}_{(3)} &= 0\,, \qquad \hat{F}_{(2)} = 0\,, \qquad e^{\hat{\phi}} = s^{-5/6}, \end{aligned} \qquad (10.105)$$

where ds_6^2 is given by (10.80) and (10.82), and $s, c, \epsilon_{(3)}$ and σ_i have the same definitions as before and $h_3 = \sigma_3 - \frac{1}{\sqrt{2}} S(1 - e^{-3g\rho})dx^+$.

4. $D = 7$ oxidized to $D = 11$

Using the embedding ansatz in [87] we obtain

$$\begin{aligned} d\hat{s}_{11} &= ds_7^2 + \tfrac{1}{4} g^{-2} d\xi^2 + \tfrac{1}{16} g^{-2} c^2 (\sigma_1^2 + \sigma_2^2 + h_3^2)\,, \qquad (10.106) \\ \hat{A}_{(3)} &= \tfrac{1}{64} g^{-3} (2s + s\,c^2) \epsilon_{(3)} + \tfrac{1}{8\sqrt{2}} g^{-2} S\, e^{-4g\rho} s\, dx^+ \wedge d\rho \wedge \sigma_3\,, \end{aligned}$$

where ds_7^2 is given by (10.91) and (10.93). The field strength $\hat{F}_{(4)} = d\hat{A}_{(3)}$ is

$$\begin{aligned} \hat{F}_{(4)} &= \tfrac{3}{64} g^{-3} c^3\, d\xi \wedge \epsilon_{(3)} + \tfrac{1}{8\sqrt{2}} g^{-2} S\, e^{-4g\rho} c\, dx^+ \wedge d\rho \wedge d\xi \wedge \sigma_3 \\ &\qquad + \tfrac{1}{16\sqrt{2}} g^{-2} S\, e^{-4g\rho}\, s\, c^2\, dx^+ \wedge d\rho \wedge \sigma_1 \wedge \sigma_2\,, \end{aligned} \qquad (10.107)$$

where $s, c, \epsilon_{(3)}$ and σ_i have the same definitions as before and $h_3 = \sigma_3 - \frac{1}{\sqrt{2}} S(1 - e^{-4g\rho})dx^+$.

CHAPTER XI

AdS PP-WAVES II

In this chapter we continue our investigations of AdS pp-waves by studying the pp-waves of $D=5$ and $D=4$ gauged supergravities supported respectively by $U(1)^3$ and $U(1)^4$ gauge fields [92]. We present a detailed analysis of the supersymmetry. In particular, we show that supernumerary supersymmetry can arise beyond the usual $\frac{1}{4}$. We also study the pp-waves of the Freedman-Schwarz model. The supersymmetry enhancement discussed in this chapter forces the solutions to be independent of the light-cone coordinate x^+.

A. PP-waves in five dimensions

Our first example treats $D=5$ gauged supergravity truncated to the $U(1)^3$ subgroup of $SO(6)$. The bosonic sector of this truncated theory is described by the Lagrangian [88]

$$e^{-1}\mathcal{L}_5 = R - \tfrac{1}{2}(\partial\vec{\varphi})^2 + 4g^2\sum_i X_i^{-1} - \tfrac{1}{4}\sum_i X_i^{-2}(F^i_{(2)})^2 + \tfrac{1}{4}\epsilon^{MNPQR}F^1_{MN}F^2_{PQ}A^3_R\,, \quad (11.1)$$

where

$$\begin{aligned} &X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}}\,, \qquad X_1X_2X_3 = 1\,, \\ &\vec{a}_1 = (\tfrac{2}{\sqrt{6}},\sqrt{2})\,, \qquad \vec{a}_2 = (\tfrac{2}{\sqrt{6}},-\sqrt{2})\,, \qquad \vec{a}_3 = (-\tfrac{4}{\sqrt{6}},0)\,, \end{aligned} \quad (11.2)$$

and the field strengths are defined as $F^i_{(2)} = dA^i_{(1)}$. The equations of motion are

$$\begin{aligned} R_{MN} &= \tfrac{1}{2}\partial_M\vec{\varphi}\cdot\partial_N\vec{\varphi} - \tfrac{4}{3}g^2 g_{MN}\sum_i X_i^{-1} \\ &\quad + \tfrac{1}{2}\sum_i X_i^{-2}(F^i_{MP}F^{i\ P}_{N} - \tfrac{1}{6}(F^i_{(2)})^2 g_{MN})\,, \end{aligned}$$

$$\nabla_M(X_i^{-2}F_i^{MN}) = \tfrac{1}{4}\epsilon^{NPQRS}F_{PQ}^j F_{RS}^k\,, \qquad i\neq j\neq k\,,$$
$$\Box\vec{\varphi} = \tfrac{1}{4}\sum_i \vec{a}_i X_i^{-2}(F_{(2)}^i)^2 - 2g^2\sum_i \vec{a}_i X_i^{-1}. \qquad (11.3)$$

The supersymmetry transformations for the fermions are given by

$$\delta\Psi_M = [\nabla_M - \tfrac{\mathrm{i}}{2}g\sum_i A_M^i + \tfrac{1}{6}g\,\Gamma_M\sum_i X_i - \tfrac{\mathrm{i}}{48}(\Gamma_M\Gamma^{AB} - 3\Gamma^{AB}\Gamma_M)\sum_i X_i^{-1}F_{AB}^i]\epsilon\,,$$
$$\delta\vec{\lambda} = [-\tfrac{\mathrm{i}}{4}\Gamma^M\partial_M\vec{\varphi} + \tfrac{1}{16}\Gamma^{AB}\sum_i \vec{a}_i X_i^{-1}F_{AB}^i - \tfrac{\mathrm{i}}{4}g\sum_i \vec{a}_i X_i]\epsilon\,. \qquad (11.4)$$

1. The solution

We use the following pp-wave metric ansatz

$$ds_D = e^{2A}(-4dx^+dx^- + H(dx^+)^2 + dz_a^2) + e^{2B}dr^2\,, \qquad a = 1, 2, \cdots, D-3, \quad (11.5)$$

in arbitrary dimensions. The functions A and B depends on r only while H depends on x^+, z_a and r coordinates. If we set $H = 0$, the pp-waves reduce to AdS-domain wall solutions [93]. It is natural to choose the following vielbein basis

$$e^+ = e^A dx^+\,, \quad e^- = e^A(-2dx^- + \tfrac{1}{2}Hdx^+)\,, \quad e^a = e^A dz^a\,, \quad e^r = e^B dr \quad (11.6)$$

such that we have $ds^2 = 2e^+e^- + e^ae^a + e^re^r$. The vielbein components of the spin connections are

$$\omega_{-r} = A'e^{-B}e^+\,, \quad \omega_{+a} = \tfrac{1}{2}e^{-A}\partial_a H\, e^+\,,$$
$$\omega_{+r} = A'e^{-B}e^- + \tfrac{1}{2}H'e^{-B}e^+\,, \quad \omega_{ar} = A'e^{-B}e^a\,. \qquad (11.7)$$

where the prime denotes the derivative with respect to r. Note that for the metric in this basis we have $\eta_{+-} = 1$ and $\eta_{++} = \eta_{--} = 0$. The derivatives are always with respect to the curved metric. The vielbein components of the Ricci tensor in

D-dimensions are given by

$$
\begin{aligned}
R_{++} &= -\tfrac{1}{2}e^{-2B}\left[H'' + H'((D-1)A' - B')\right] - \tfrac{1}{2}e^{-2A}\sum_a \partial_a\partial_b H] = -\tfrac{1}{2}\Box H\,,\\
R_{+-} &= -e^{-2B}\left[A'' + A'((D-1)A' - B')\right], \qquad R_{ab} = R_{+-}\,\delta_{ab}\,,\\
R_{rr} &= -(D-1)e^{-2B}\left[A'' + A'(A' - B')\right]. \qquad (11.8)
\end{aligned}
$$

It is straightforward to verify that in five dimensions the following

$$
\begin{aligned}
e^{2A} &= (gr)^2\left[H_1H_2H_3\right]^{1/3}, \qquad H_i = 1 + \frac{\ell_i^2}{r^2}\,,\\
e^{2B} &= \frac{1}{(gr)^2\left[H_1H_2H_3\right]^{2/3}}\,, \qquad X_i = H_i^{-1}[H_1H_2H_3]^{1/3}\,,\\
A^i_{(1)} &= g^{-1}S_i(1 - H_i^{-1})\,dx^+ \qquad (11.9)
\end{aligned}
$$

satisfies the equations of motion with $H(x^+, r, z_a)$ obeying the equation

$$
H'' + (4A' - B')H' + e^{-2(A-B)}\sum_a \partial_a\partial_a H + \frac{4g^2}{r^2}e^{-6A}\sum_i S_i^2\ell_i^4 H_i^{-2} = 0\,. \qquad (11.10)
$$

Here the S_i are functions of x^+.

2. Standard supersymmetry

The Killing spinor equations following from the fermionic transformations are given by

$$
\begin{aligned}
&[\partial_+ + \tfrac{1}{2}A'e^{A-B}(\Gamma_+ + \tfrac{1}{2}H\Gamma_-)(\Gamma_r + 1) - \tfrac{1}{4}e^{A-B}H'\,\Gamma_r\,\Gamma_-\\
&\qquad -\tfrac{1}{4}(\partial_1 H\,\Gamma_1 + \partial_2 H\,\Gamma_2)\Gamma_- - \tfrac{\mathrm{i}}{2}\Big(\sum_i S_i(1 - H_i^{-1})\Big)(\Gamma_r + 1)\\
&\qquad + \frac{\mathrm{i}}{6\mathrm{r}^2}\Big(\sum_i S_i\ell_i^2 H_i^{-1}\Big)\Gamma_r\,\Gamma_+\,\Gamma_-]\epsilon = 0\,,\\
&[\partial_- - A'e^{A-B}\,\Gamma_-(\Gamma_r + 1)]\epsilon = 0\,,\\
&[\partial_a + \tfrac{1}{2}A'e^{A-B}\,\Gamma_a(\Gamma_r + 1) - \frac{\mathrm{i}}{6\mathrm{r}^2}\Big(\sum_i S_i\ell_i^2 H_i^{-1}\Big)\Gamma_a\,\Gamma_r\,\Gamma_-]\epsilon = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
&[\partial_r + \frac{1}{6r}\Big(\sum_i H_i^{-1}\Big)\Gamma_r + \frac{\mathrm{i}}{3\mathrm{r}} g e^{-3A}\Big(\sum_i S_i \ell_i^2 H_i^{-1}\Big)\Gamma_-]\epsilon = 0\,, \\
&[\mathrm{i}e^{3A}\Big(\sum_i a_{1i} H_i^{-1}\Big)(\Gamma_r + 1) + g\Big(\sum_i a_{1i} S_i \ell_i^2 H_i^{-1}\Big)\Gamma_r\,\Gamma_-]\epsilon = 0\,, \\
&[\mathrm{i}e^{3A}\Big(\sum_i a_{2i} H_i^{-1}\Big)(\Gamma_r + 1) + g\Big(\sum_i a_{2i} S_i \ell_i^2 H_i^{-1}\Big)\Gamma_r\,\Gamma_-]\epsilon = 0\,, \qquad (11.11)
\end{aligned}
$$

where we have $\Gamma_+^2 = \Gamma_-^2 = 0$ and $\{\Gamma_+\,,\Gamma_-\} = 2$. The last two equations comes from the dilatino transformations. To arrive at these equations we have made use of the solution (11.9). The above Killing spinor equations have the solution

$$
\epsilon = r^{1/2}\,[H_1 H_2 H_3]^{\frac{1}{12}}\epsilon_0 \qquad (11.12)
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_r + 1)\epsilon_0 = 0$ and $\Gamma_-\epsilon_0 = 0$. The solution therefore preserves $\frac{1}{4}$ of the supersymmetry. The Killing spinor for the $\frac{1}{4}$ supersymmetry exist for arbitrary solutions to eq.(11.10).

3. Supernumerary supersymmetry

To investigate the supernumerary supersymmetry we use the less restrictive projection condition

$$
(\Gamma_r + 1)\epsilon = \mathrm{i}f\,\Gamma_-\epsilon \qquad (11.13)
$$

where the function $f = f(x^+, r, z_a)$ is to be determined. Making use of this projection in the Killing spinor equations they become

$$
\begin{aligned}
&[\partial_+ + \tfrac{\mathrm{i}}{2}(A' e^{A-B} f - \frac{1}{3r^2}\mathcal{M})\Gamma_+\,\Gamma_- + \frac{1}{4r^2}(2f\mathcal{M} - r^2 e^{A-B} H')\Gamma_- \\
&\qquad -\tfrac{1}{4}(\partial_1 H\,\Gamma_1 + \partial_2 H\,\Gamma_2)\Gamma_-]\epsilon = 0\,, \\
&[\partial_a + \tfrac{\mathrm{i}}{2}(A' e^{A-B} f - \frac{1}{3r^2}\mathcal{M})\Gamma_a\,\Gamma_-]\epsilon = 0\,, \qquad \partial_-\epsilon = 0\,, \\
&\Big[\partial_r + \frac{\mathrm{i}}{6\mathrm{r}}\Big(f\sum_i H_i^{-1} + 2g e^{-3A}\mathcal{M}\Big)\Gamma_- - \frac{1}{6r}\sum_i H_i^{-1}\Big]\epsilon = 0\,, \\
&\Big[e^{3A} f \sum_i a_{bi} H_i^{-1} - g\sum_i a_{bi} S_i \ell_i^2 H_i^{-1}\Big]\Gamma_-\epsilon = 0\,, \qquad b = 1, 2 \qquad (11.14)
\end{aligned}
$$

where $\mathcal{M} \equiv \sum_i S_i \ell_i^2 H_i^{-1}$. We analyse these equations by calculating the integrability conditions $[\partial_M , \partial_N]\epsilon = 0$ among them. The integrability $[\partial_a , \partial_r]\epsilon = 0$ yields a solution for f with the requirement $\partial_a f = 0$. We have

$$f = \frac{1}{3r^2 A'}\, e^{-(A-B)}(\mathcal{M} + 3r^2 U(x^+))\,, \tag{11.15}$$

where the function U is in general complex. From the integrability $[\partial_+ , \partial_a]\epsilon = 0$ we obtain an equation for U after imposing some restrictions on the pp-wave function $H(x^+, r, z_a)$. The result is

$$\begin{aligned} &\partial_a H' = 0\,, \qquad \partial_a\partial_b H = 0 \qquad \text{for} \qquad a \neq b\,, \\ &\mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}\partial_a\partial_a H = 0\,, \qquad a = 1, 2\,. \end{aligned} \tag{11.16}$$

The equation for U then requires $\partial_1\partial_1 H = \partial_2\partial_2 H$. Investigating the pair of equations given in the last line of eqs.(11.14) we find that they are satisfied provided that the functions S_i and U satisfy two equations among them. Without loss of generality we give the solutions in terms of S_3 and U. They are given by

$$S_1 = \ell_1^{-2}(S_3\ell_3^2 - (\ell_1^2 - \ell_3^2)U)\,, \qquad S_2 = \ell_2^{-2}(S_3\ell_3^2 - (\ell_2^2 - \ell_3^2)U)\,. \tag{11.17}$$

In order to analyse the final integrability $[\partial_+ , \partial_r]\epsilon = 0$ we need to make use of the solution for H. Taking into account the conditions on H given above the solution is given by

$$\begin{aligned} g^4 H(x^+, r, z_a) &= \tfrac{1}{2} c\, g^4 (z_1^2 + z_2^2) + \tfrac{1}{2}|\epsilon_{ijk}|\, K_{ijk}(x^+, r)\,, \\ K_{ijk}(x^+, r) &= -\frac{S_i^2 \ell_i^4}{(\ell_i^2 - \ell_j^2)(\ell_i^2 - \ell_k^2)(r^2 + \ell_i^2)} \\ &\quad + \frac{1}{2(\ell_i^2 - \ell_j^2)^2(\ell_i^2 - \ell_k^2)^2}\Big[(bg^4 + c\,\ell_i^2)(\ell_i^2 - \ell_j^2)(\ell_i^2 - \ell_k^2) \\ &\quad + 2S_i^2\ell_i^4(2\ell_i^2 - \ell_j^2 - \ell_k^2) - 2S_j^2\ell_j^4(\ell_i^2 - \ell_k^2) - 2S_k^2\ell_k^4(\ell_i^2 - \ell_j^2)\Big] \ln(r^2 + \ell_i^2), \end{aligned} \tag{11.18}$$

where $b=b(x^+)$ and $c=c(x^+)$. Then $[\partial_+\,,\partial_r]\epsilon=0$ yields an equation for S_3 given by

$$\mathrm{i}\frac{dS_3}{dx^+}-(2\ell_3)^{-2}[bg^4+c\,\ell_3^2+2U(U(\ell_1^2+\ell_2^2)-2\ell_3^2(2S_3+U))]=0\,. \tag{11.19}$$

We proceed next by making use of the information that S_i, b and c are real functions. Eqs.(11.17) implies that U must also be real. This has the consequence in eq.(11.16) that U and c must be constants with c being given by $c=-2U^2$. Eqs.(11.19) and (11.17) in turn implies that S_i and b must also be constants. Eliminating U from eqs.(11.17) and setting $S_i=\mu_i$ we obtain

$$\epsilon_{ijk}\,\mu_i\ell_i^2(\ell_j^2-\ell_k^2)=0\,. \tag{11.20}$$

Without loss of generality we solve for μ_1 in terms of the other two charges. The function H which gives $\frac{1}{2}$ supersymmetric pp-wave is given by

$$\begin{aligned}
\mu_1 &= \frac{\mu_2\ell_2^2(\ell_1^2-\ell_3^2)-\mu_3\ell_3^2(\ell_1^2-\ell_2^2)}{\ell_1^2(\ell_2^2-\ell_3^2)}\,,\\
b &= -\frac{2(\mu_2\ell_2^2-\mu_3\ell_3^2)(\mu_2\ell_2^4-\mu_3\ell_3^4-3(\mu_2-\mu_3)\ell_2^2\ell_3^2+\ell_1^2(\mu_2\ell_2^2-\mu_3\ell_3^2))}{g^4(\ell_2^2-\ell_3^2)^2}\,,\\
c &= -\frac{2(\mu_2\ell_2^2-\mu_3\ell_3^2)^2}{(\ell_2^2-\ell_3^2)^2}\,,\\
H &= \tfrac{1}{2}c\,(z_1^2+z_2^2)-f^2\,,\\
f &= -\frac{(\mu_2\ell_2^2-\mu_3\ell_3^2)r^2+(\mu_2-\mu_3)\ell_2^2\ell_3^2}{g^2(\ell_2^2-\ell_3^2)r^3[H_1H_2H_3]^{1/2}}\,.
\end{aligned} \tag{11.21}$$

The projected Killing spinor equations become

$$\begin{aligned}
&[\partial_+-\tfrac{1}{2\sqrt{2}}(-c)^{1/2}(\mathrm{i}\,\Gamma_+-f)\Gamma_--\tfrac{1}{4}c\,(z_1\Gamma_1+z_2\Gamma_2)\Gamma_-]\epsilon=0\,,\\
&[\partial_a-\tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2}\,\Gamma_a\,\Gamma_-]\epsilon=0\,,\qquad \partial_-\epsilon=0\,,\\
&[\partial_r-\tfrac{\mathrm{i}}{2}f'\,\Gamma_--\frac{1}{6r}\sum_i H_i^{-1}]\epsilon=0\,.
\end{aligned} \tag{11.22}$$

The Killing spinor is easily obtained given by

$$\begin{aligned}\epsilon &= r^{1/2}[H_1H_2H_3]^{\frac{1}{12}}(1+\tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2}(z_1\Gamma_1+z_2\Gamma_2)\Gamma_-)(1+\tfrac{\mathrm{i}}{2}f\,\Gamma_-)\eta\,,\\ \frac{d\eta}{dx^+} &= \tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2}\,\Gamma_+\,\Gamma_-\,\eta\,.\end{aligned} \tag{11.23}$$

Solving for η we have

$$\begin{aligned}\epsilon &= r^{1/2}[H_1H_2H_3]^{\frac{1}{12}}(1+\tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2}(z_1\Gamma_1+z_2\Gamma_2)\Gamma_-)(1+\tfrac{\mathrm{i}}{2}f\,\Gamma_-)\times\\ &\quad\times[1-\tfrac{1}{2}(1-e^{\frac{\mathrm{i}}{\sqrt{2}}(-c)^{1/2}x^+})\Gamma_+\,\Gamma_-]\epsilon_0\,,\end{aligned} \tag{11.24}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_r+1)\epsilon_0=0$. The solution thus preserve $\frac{1}{2}$ of the supersymmetry. Note that if we set $\mu_i\ell_i^2=\mu$ (which is consistent with eq.(11.20)) we obtain $b=c=0$.

To conclude, demanding supernumerary supersymmetry puts very strong restrictions on the pp-waves with the functions S, b and c (and U) which initially all being functions of x^+ reduces now to constants. This is not the case for minimal gauged supergravity where supernumerary supersymmetry does allow the various functions to have x^+ dependence.

B. PP-waves in four dimensions

In this section we consider a subsector of the $SO(8)$ gauged supergravity where the bosonic fields comprises the metric, four commuting $U(1)$ gauge potentials and three dilatons. The Lagrangian describing this set of fields is [94]

$$e^{-1}\mathcal{L}_4 = R-\tfrac{1}{2}(\partial\vec{\varphi})^2-\tfrac{1}{4}\sum_i X_i^{-2}(F^i_{(2)})^2-V\,, \tag{11.25}$$

where

$$X_i=e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}}\,,\qquad X_1X_2X_3X_4=1\,, \tag{11.26}$$

$$\vec{a}_1 = (1,1,1)\,, \quad \vec{a}_2 = (1,-1,-1)\,, \quad \vec{a}_3 = (-1,1,-1)\,, \quad \vec{a}_3 = (-1,-1,1)\,,$$

and the field strengths are defined as $F^i_{(2)} = dA^i_{(1)}$. The potential is given by

$$V = -4g^2 \sum_{i<j} X_i X_j = -8g^2 \sum_{i=1}^{3} \cosh\varphi_i\,. \tag{11.27}$$

The $\mathcal{N} = 8$ supersymmetry transformations in this bosonic background were also presented in [94]. They are given by

$$\begin{aligned} \delta\Psi^i_{M} &= \nabla_M \epsilon^{(i)} + \sum_j [\,-\mathrm{i}g\,\Omega_{ij} A^j_M + \tfrac{\mathrm{i}}{8}\Omega_{ij} X_j^{-1} F^j_{AB}\Gamma^{AB}\Gamma_M + \tfrac{1}{4} g X_j \Gamma_M]\epsilon^{(i)}, \quad (11.28) \\ \delta\lambda^{ij} &= [\tfrac{\mathrm{i}}{\sqrt{2}}\Gamma^M \partial_M \phi^{ij} - \tfrac{1}{2\sqrt{2}} \sum_k \Omega_{jk} X_k^{-1} F^k_{AB}\Gamma^{AB} + \mathrm{i}\sqrt{2} g \sum_{k,m} f_{ijk}\Omega_{km} X_m]\epsilon^{(i)}, \end{aligned}$$

where we have rewritten them by introducing complex fermions $\Psi^i_M = \Psi^i_{1M} + \mathrm{i}\Psi^i_{2M}$, etc and made the substitutions $g \to \sqrt{2}g$ and $A^i_{(1)} \to -\frac{1}{2\sqrt{2}} A^i_{(1)}$. Note that $i \neq j$ in the spin 1/2 transformations. The three dilatons are given by the following identifications

$$\varphi_1 = \phi^{12} = \phi^{34}\,, \qquad \varphi_2 = \phi^{13} = \phi^{24}\,, \qquad \varphi_3 = \phi^{14} = \phi^{23}\,, \tag{11.29}$$

and note also that $\phi^{ij} = \phi^{ji}$. The function f_{ijk} is defined as

$$f_{ijk} = \begin{cases} |\epsilon_{ijk}| & \text{for} \quad i,j,k \neq 1\,, \\ \delta_{jk} & \text{for} \quad i = 1\,, \\ \delta_{ik} & \text{for} \quad j = 1\,, \end{cases} \tag{11.30}$$

and the matrix Ω is given by

$$\Omega = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{11.31}$$

1. The solution

The four charge pp-wave is given by

$$ds_4^2 = e^{2A}(-4dx^+dx^- + H(dx^+)^2 + dz^2) + e^{2B}dr^2, \tag{11.32}$$

where

$$\begin{aligned} e^{2A} &= (gr)^4\,[H_1H_2H_3H_4]^{1/2}\,, \qquad H_i = 1 + \frac{\ell_i^2}{r^2}\,, \\ e^{2B} &= \frac{1}{(gr)^2\,[H_1H_2H_3H_4]^{1/2}}\,, \qquad X_i = H_i^{-1}[H_1H_2H_3H_4]^{1/4}\,, \\ A^i_{(1)} &= g^{-1}S_i(1-H_i^{-1})\,dx^+ \end{aligned} \tag{11.33}$$

and $S_i = S_i(x^+)$. The function $H(x^+, r, z)$ satisfies the equation

$$H'' + (3A' - B')H' + e^{-2(A-B)}\partial_z\partial_z H + \frac{4g^2}{r^2}e^{-4A}\sum_i S_i^2\ell_i^4H_i^{-2} = 0\,. \tag{11.34}$$

The solution to this equation is similar to the solution in $D = 5$. The four charged pp-wave can be specialized to one, two and three active charges respectively.

2. Supersymmetry

The $\mathcal{N} = 8$ supersymmetry have four different sectors. We begin by analysing the Killing spinor equations for the sector $\epsilon^{(1)}$. The supersymmetry transformations are given by

$$\begin{aligned} \delta\Psi^1_M &= \nabla_M\epsilon^{(1)} + \sum_{i=1}^{4}[-\tfrac{\mathrm{i}}{2}gA^i_M + \tfrac{\mathrm{i}}{16}X_i^{-1}F^i_{AB}\Gamma^{AB}\Gamma_M + \tfrac{1}{4}gX_i\Gamma_M]\epsilon^{(1)}, \\ \delta\lambda^{12} &= [\tfrac{\mathrm{i}}{\sqrt{2}}\Gamma^M\partial_M\varphi_1 - \tfrac{1}{4\sqrt{2}}\Gamma^{AB}(X_1^{-1}F^1_{AB} + X_2^{-1}F^2_{AB} - X_3^{-1}F^3_{AB} - X_4^{-1}F^4_{AB}) \\ &\quad + \tfrac{\mathrm{i}}{\sqrt{2}}g(X_1 + X_2 - X_3 - X_4)]\epsilon^{(1)}, \\ \delta\lambda^{13} &= [\tfrac{\mathrm{i}}{\sqrt{2}}\Gamma^M\partial_M\varphi_2 - \tfrac{1}{4\sqrt{2}}\Gamma^{AB}(X_1^{-1}F^1_{AB} - X_2^{-1}F^2_{AB} + X_3^{-1}F^3_{AB} - X_4^{-1}F^4_{AB} \\ &\quad + \tfrac{\mathrm{i}}{\sqrt{2}}g(X_1 - X_2 + X_3 - X_4)]\epsilon^{(1)}, \end{aligned}$$

$$\delta\lambda^{14} \;=\; [\tfrac{\mathrm{i}}{\sqrt{2}}\Gamma^M\partial_M\varphi_3 - \tfrac{1}{4\sqrt{2}}\Gamma^{AB}(X_1^{-1}F^1_{AB} - X_2^{-1}F^2_{AB} - X_3^{-1}F^3_{AB} + X_4^{-1}F^4_{AB}$$
$$+\tfrac{\mathrm{i}}{\sqrt{2}}g(X_1 - X_2 - X_3 + X_4)]\epsilon^{(1)}. \qquad (11.35)$$

The Killing spinor equations are readily written down and take the form

$$[\partial_+ + \tfrac{1}{2}A'e^{A-B}(\Gamma_+ + \tfrac{1}{2}H\Gamma_-)(\Gamma_r + 1) - \tfrac{1}{4}H'e^{A-B}\,\Gamma_r\,\Gamma_- - \tfrac{1}{4}\partial_z H\,\Gamma_z\,\Gamma_-$$
$$-\frac{\mathrm{i}}{2}\Big(\sum_{i=1}^{4}S_i(1-H_i^{-1})\Big)(\Gamma_r+1) + \frac{\mathrm{i}}{4\mathrm{r}^2}\Big(\sum_i \mathcal{M}_i\Big)\Gamma_r\,\Gamma_+\,\Gamma_-]\epsilon^{(1)} = 0\,,$$
$$[\partial_- - A'e^{A-B}\,\Gamma_-(\Gamma_r+1)]\epsilon^{(1)} = 0\,,$$
$$[\partial_z + \tfrac{1}{2}A'e^{A-B}\,\Gamma_z(\Gamma_r+1) - \frac{\mathrm{i}}{4\mathrm{r}^2}\Big(\sum_i \mathcal{M}_i\Big)\Gamma_z\,\Gamma_r\,\Gamma_-]\epsilon^{(1)} = 0\,,$$
$$[\partial_r + \frac{1}{4r}\Big(\sum_i H_i^{-1}\Big)\Gamma_r + \frac{\mathrm{i}}{4\mathrm{r}}ge^{-2A}\Big(\sum_i \mathcal{M}_i\Big)\Gamma_-]\epsilon^{(1)} = 0\,,$$
$$[\mathrm{i}g(X_1 + X_2 - X_3 - X_4)(\Gamma_r+1) + \frac{1}{r^2}e^{-A}(\mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3$$
$$-\mathcal{M}_4)\Gamma_r\,\Gamma_-]\epsilon^{(1)} = 0\,,$$
$$[\mathrm{i}g(X_1 - X_2 + X_3 - X_4)(\Gamma_r+1) + \frac{1}{r^2}e^{-A}(\mathcal{M}_1 - \mathcal{M}_2 + \mathcal{M}_3$$
$$-\mathcal{M}_4)\Gamma_r\,\Gamma_-]\epsilon^{(1)} = 0\,,$$
$$[\mathrm{i}g(X_1 - X_2 - X_3 + X_4)(\Gamma_r+1) + \frac{1}{r^2}e^{-A}(\mathcal{M}_1 - \mathcal{M}_2 - \mathcal{M}_3$$
$$+\mathcal{M}_4)\Gamma_r\,\Gamma_-]\epsilon^{(1)} = 0\,, \qquad (11.36)$$

where we have defined $\mathcal{M}_i \equiv S_i\ell_i^2 H_i^{-1}$. These equations have the solution

$$\epsilon^{(1)} = r[H_1H_2H_3H_4]^{\frac{1}{8}}\,\epsilon_0^{(1)} \qquad (11.37)$$

where $\epsilon_0^{(1)}$ is a constant spinor satisfying $(\Gamma_r + 1)\epsilon_0^{(1)} = 0 = \Gamma_-\epsilon_0^{(1)}$. Thus $\frac{1}{4}$ of the supersymmetry of the $\epsilon^{(1)}$ sector is preserved (standard supersymmetry). It is easy to see that the same amount of supersymmetry is preserved simultaneously in the other sectors. The pp-wave therefore preserves overall $\frac{1}{4}$ of the $\mathcal{N} = 8$ supersymmetry.

Now let us examine whether the solution admits supernumerary supersymmetry.

We again make use of the ansatz

$$(\Gamma_r + 1)\epsilon^{(1)} = \mathrm{i} f_1\, \Gamma_- \epsilon^{(1)}. \tag{11.38}$$

A similar analysis of the integrability conditions among the projected Killing spinor equations as in five dimensions shows that the functions $S_i(x^+)$, $U(x^+)$, $b(x^+)$ and $c(x^+)$ must again be constants. In $D = 4$ there are now two conditions that must be satisfied among the charges for there to be supernumerary supersymmetry. Hence the pp-wave solution will depend on just two charge parameters. The constraints among the charges are given by

$$\begin{aligned}
\ell_1^2\ell_4^2(\ell_2^2 - \ell_3^2)(\mu_1 - \mu_4) &= \ell_2^2\ell_3^2(\ell_1^2 - \ell_4^2)(\mu_2 - \mu_3)\,, \\
(\ell_2^2 - \ell_3^2)(\mu_1\ell_1^2 - \mu_4\ell_4^2) &= (\ell_1^2 - \ell_4^2)(\mu_2\ell_2^2 - \mu_3\ell_3^2)\,,
\end{aligned} \tag{11.39}$$

where we have set $S_i = \mu_i$. Solving for μ_1 and μ_2 in terms of the other two charges the function H is given by

$$\begin{aligned}
\mu_\alpha &= \frac{\mu_3\ell_3^2(\ell_\alpha^2 - \ell_4^2) - \mu_4\ell_4^2(\ell_\alpha^2 - \ell_3^2)}{\ell_\alpha^2(\ell_3^2 - \ell_4^2)}\,, \qquad \alpha = 1, 2\,, \\
b &= -\frac{2(\mu_3\ell_3^2 - \mu_4\ell_4^2)}{g^6(\ell_3^2 - \ell_4^2)^2}[\mu_3\ell_3^2(\ell_1^2 + \ell_2^2 + \ell_3^2 - 5\ell_4^2) - \mu_4\ell_4^2(\ell_1^2 + \ell_2^2 - 5\ell_3^2 + \ell_4^2)]\,, \\
c &= -\frac{8(\mu_3\ell_3^2 - \mu_4\ell_4^2)^2}{(\ell_3^2 - \ell_4^2)^2}\,, \\
H &= \tfrac{1}{2}c\, z^2 - f_1^2\,, \\
f_1 &= -\frac{(\mu_3\ell_3^2 - \mu_4\ell_4^2)r^2 + (\mu_3 - \mu_4)\ell_3^2\ell_4^2}{g^3(\ell_3^2 - \ell_4^2)r^4[H_1H_2H_3H_4]^{1/2}}\,.
\end{aligned} \tag{11.40}$$

The projected Killing spinor equations are given by

$$[\partial_+ - \tfrac{1}{2\sqrt{2}}(-c)^{1/2}(\mathrm{i}\,\Gamma_+ - f_1)\Gamma_- - \tfrac{1}{4}c\, z\Gamma_z\,\Gamma_-]\epsilon^{(1)} = 0\,,$$

$$[\partial_z - \tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2}\,\Gamma_z\,\Gamma_-]\epsilon^{(1)} = 0\,, \qquad \partial_-\epsilon^{(1)} = 0\,,$$

$$[\partial_r - \tfrac{\mathrm{i}}{2} f_1' \,\Gamma_- - \frac{1}{4r}\sum_i H_i^{-1}]\epsilon^{(1)} = 0\,. \tag{11.41}$$

The solution for the Killing spinor is

$$\begin{aligned}\epsilon^{(1)} \;&=\; r[H_1 H_2 H_3 H_4]^{\frac{1}{8}}(1 + \tfrac{\mathrm{i}}{2\sqrt{2}}(-c)^{1/2} z\,\Gamma_z\,\Gamma_-)(1 + \tfrac{\mathrm{i}}{2} f_1\,\Gamma_-) \times \\ &\quad \times[1 - \tfrac{1}{2}(1 - e^{\frac{i}{\sqrt{2}}(-c)^{1/2} x^+})\Gamma_+\,\Gamma_-]\epsilon_0^{(1)},\end{aligned} \tag{11.42}$$

where $\epsilon_0^{(1)}$ is a constant spinor satisfying $(\Gamma_r + 1)\epsilon_0^{(1)} = 0$. The pp-wave with H given above therefore preserves $\frac{1}{2}$ of the supersymmetry of the $\epsilon^{(1)}$ sector. Consider next the remaining sectors. For this we use the ansatz $A^i_{(1)} = g^{-1}\eta_i\,\mu_i(1 - H_i^{-1})dx^+$. To preserve $\frac{1}{2}$ supersymmetry in the four respective sectors then requires the sign choices:

$$\begin{array}{rlllll} 1: & \eta_1 = & \eta_2 = & \eta_3 = & \eta_4 \\ 2: & \eta_1 = & \eta_2 = & -\eta_3 = & -\eta_4 \\ 3: & \eta_1 = & -\eta_2 = & \eta_3 = & -\eta_4 \\ 4: & \eta_1 = & -\eta_2 = & -\eta_3 = & \eta_4 \end{array} \tag{11.43}$$

Because of the difference in signs the four charge solution will preserve $\frac{1}{2}$ of the supersymmetry of just one sector and $\frac{1}{4}$ of the supersymmetry of each of the remaining sectors.

Although we have focused on solutions with four active charges one can easily also analyse the supersymmetry of solutions with one, two or three active charges. In the following table we present the overall amount of the $\mathcal{N} = 8$ supersymmetry preserved in the various cases.

No. of active charges	Standard supersymmetry	Enhanced supersymmetry
1	$\frac{1}{4}$	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}=\frac{3}{8}$
3	$\frac{1}{4}$	$\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{5}{16}$
4	$\frac{1}{4}$	$\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{5}{16}$

C. PP-waves in the Freedman-Schwarz model

The Lagrangian describing the bosonic sector of the Freedman-Schwarz model is [18]

$$\begin{aligned}\mathcal{L}_4 &= R*\mathbb{1} - \tfrac{1}{2}*d\phi\wedge d\phi - \tfrac{1}{2}e^{2\phi}*d\chi\wedge d\chi + 4(g_1^2+g_2^2)e^{\phi}*\mathbb{1}\\ &\quad -\tfrac{1}{2}e^{-\phi}(*F^a_{(2)}\wedge F^a_{(2)} + *G^a_{(2)}\wedge G^a_{(2)})\\ &\quad -\tfrac{1}{2}\chi(F^a_{(2)}\wedge F^a_{(2)} + G^a_{(2)}\wedge G^a_{(2)})\,, \end{aligned} \tag{11.44}$$

where

$$\begin{aligned} F^a_{(2)} &= dA^a_{(1)} - \tfrac{1}{\sqrt{2}}g_1\epsilon_{abc}A^b_{(1)}\wedge A^c_{(1)}\,, \qquad a=1,2,3,\\ G^a_{(2)} &= dB^a_{(1)} - \tfrac{1}{\sqrt{2}}g_2\epsilon_{abc}B^b_{(1)}\wedge B^c_{(1)}\,. \end{aligned} \tag{11.45}$$

The supersymmetry transformations for the fermions are given by

$$\begin{aligned} \delta\Psi_M &= [\nabla_M - \tfrac{\mathrm{i}}{\sqrt{2}}g_1\alpha_1^a A^a_M - \tfrac{\mathrm{i}}{\sqrt{2}}g_2\alpha_2^a B^a_M - \tfrac{\mathrm{i}}{4}e^{\phi}\,\Gamma_5\partial_M\chi\\ &\quad +\tfrac{\mathrm{i}}{8\sqrt{2}}e^{-\frac{1}{2}\phi}(\alpha_1^a F^a_{AB} - \mathrm{i}\,\Gamma_5\alpha_2^a G^a_{AB})\Gamma^{AB}\Gamma_M + \tfrac{1}{2}e^{\frac{1}{2}\phi}(g_1 - \mathrm{i}g_2\Gamma_5)\Gamma_M]\epsilon\,,\\ \delta\lambda &= [\tfrac{\mathrm{i}}{\sqrt{2}}(\partial_M\phi - \mathrm{i}e^{\phi}\,\Gamma_5\partial_M\chi)\Gamma^M + \tfrac{1}{4}e^{-\frac{1}{2}\phi}(\alpha_1^a F^a_{AB} + \mathrm{i}\,\Gamma_5\alpha_2^a G^a_{AB})\Gamma^{AB}\\ &\quad -\mathrm{i}\sqrt{2}\,e^{\frac{1}{2}\phi}(g_1+\mathrm{i}g_2\Gamma_5)]\epsilon\,, \end{aligned} \tag{11.46}$$

where $\Gamma_5 = \mathrm{i}\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ such that $\Gamma_5^2 = 1$. The α_1^a and α_2^a are two sets Pauli matrices. The gravitino, the dilatino and the (Majorana) spinor ϵ carry a suppressed indice which runs from one to four. In the following we turn off two of the fields F^a_{MN} and G^a_{MN} each. For a vanishing axion ($\chi = 0$) the pp-wave in this theory is given by

$$\begin{aligned}
ds^2 &= (gr)^2(-4dx^+dx^- + H(dx^+)^2 + dz^2) + dr^2\,, \\
H &= \tfrac{1}{2}c\,z^2 - \frac{b}{(gr)^2} - \frac{c\ln(gr)}{2g^2} - \frac{g_2^2S_1^2 + g_1^2S_2^2}{2g_1^2g_2^2(gr)^4}\,, \\
\phi &= -2\ln(gr)\,, \\
A_{(1)} &= g_1^{-1}S_1(x^+)((gr)^{-2} - 1)dx^+, \\
B_{(1)} &= g_2^{-1}S_2(x^+)((gr)^{-2} - 1)dx^+,
\end{aligned} \tag{11.47}$$

where $g = (g_1^2 + g_2^2)^{1/2}$. Now lets look at the supersymmetry of this solution. The Killing spinor equations are given by

$$\begin{aligned}
&[\partial_+ + \tfrac{1}{2}g(\Gamma_+ + \tfrac{1}{2}H\Gamma_-)(\Gamma_r + a) - \tfrac{\mathrm{i}}{\sqrt{2}}(S_1 + S_2)((gr)^{-2} - 1) \\
&\qquad -\tfrac{1}{4}c\,z\Gamma_z\,\Gamma_- - \tfrac{1}{4}grH'\,\Gamma_r\,\Gamma_- - \mathrm{i}\Lambda\,\Gamma_-\,\Gamma_+\,\Gamma_r]\epsilon = 0\,, \\
&[\partial_- - g\Gamma_-(\Gamma_r + a)]\epsilon = 0\,, \\
&[\partial_z + \tfrac{1}{2}g\Gamma_z(\Gamma_r + a) + \mathrm{i}\Lambda\Gamma_z\,\Gamma_-\,\Gamma_r]\epsilon = 0\,, \\
&[\partial_r + \frac{\mathrm{i}\Lambda}{gr}\,\Gamma_- + \frac{1}{2gr}(g_1 - \mathrm{i}g_2\Gamma_5)\Gamma_r]\epsilon = 0\,, \\
&[(\Gamma_r + a) + 2\mathrm{i}g^{-1}\Gamma_-\,\Lambda\,\Gamma_r]\epsilon = 0\,,
\end{aligned} \tag{11.48}$$

where

$$a = g^{-1}(g_1 + \mathrm{i}g_2\Gamma_5) \qquad \text{and} \qquad \Lambda = \frac{g_2S_1 - \mathrm{i}g_1S_2\Gamma_5}{2\sqrt{2}g_1g_2(g_1^2 + g_2^2)^{1/2}r^2}\,. \tag{11.49}$$

It follows from these equations that to obtain the usual $\frac{1}{4}$ supersymmetry for the pp-wave we need surprisingly to impose $g_2^2S_1 = g_1^2S_2$. The Killing spinor can then be obtained and it is given by

$$\epsilon = e^{-\frac{\mathrm{i}}{\sqrt{2}}\int (S_1+S_2)dx^+} r^{1/2}\epsilon_0\,, \tag{11.50}$$

where ϵ_0 is a constant spinor satisfying the projections $(\Gamma_r + a)\epsilon = 0 = \Gamma_-\epsilon$. To investigate the supernumerary supersymmetry we use the projection condition

$$(\Gamma_r + a)\epsilon = \mathrm{i}f\,\Gamma_-\epsilon\,. \tag{11.51}$$

The projected Killing spinor equations are given by

$$\begin{aligned}
&[\partial_+ + \tfrac{\mathrm{i}}{\sqrt{2}}(S_1+S_2) + \mathrm{i}\Gamma_+(\tfrac{1}{2}gf - \bar{a}\bar{\Lambda})\Gamma_- - \tfrac{1}{4}c\,z\,\Gamma_z\,\Gamma_- \\
&\quad -\frac{1}{2\sqrt{2}g_1g_2g^2r^2}(S_1g_2^2 - S_2g_1^2)\Gamma_5 + (2\Lambda f - \tfrac{1}{4}a\,grH')\Gamma_-]\epsilon = 0\,, \\
&[\partial_z + \mathrm{i}\Gamma_z(\tfrac{1}{2}gf - \bar{a}\bar{\Lambda})\Gamma_-]\epsilon = 0\,, \qquad \partial_-\epsilon = 0\,, \\
&[\partial_r + \frac{\mathrm{i}}{\mathrm{r}}(g^{-1}\Lambda + \tfrac{1}{2}\bar{a}f)\Gamma_- - \frac{1}{2r}]\epsilon = 0\,, \\
&[f - 2g^{-1}\bar{a}\bar{\Lambda}]\Gamma_-\epsilon = 0\,.
\end{aligned} \tag{11.52}$$

Here $\bar{a}$ and $\bar{\Lambda}$ are just a and Λ but with Γ_5 replaced by $-\Gamma_5$. We analyse these projected equations by calculating the integrability conditions among them. The condition $[\partial_z\,, \partial_r]\epsilon = 0$ requires $\partial_z f = 0$ and yields a solution for f given by

$$f = 2g^{-1}\bar{a}\bar{\Lambda} + 2g^{-1}U(x^+)\,. \tag{11.53}$$

The integrability $[\partial_+\,, \partial_z]\epsilon = 0$ provides an equation for $U(x^+)$ which is given by

$$\mathrm{i}\frac{dU}{dx^+} - 2U^2 - \tfrac{1}{4}c = 0\,. \tag{11.54}$$

From the last line of eqs.(11.52) we have $f - 2g^{-1}\bar{a}\bar{\Lambda} = 0$. This equation forces U in the solution for f to vanish. From the equation for U we must in turn set $c = 0$.

Considering next the integrability condition $[\partial_+\,,\partial_r]\epsilon = 0$ we first note that

$$2\Lambda f - \tfrac{1}{4} a\, gr H' = \frac{c\, r^2 - 4b}{8(gr)^2}(g_1 - \mathrm{i} g_2 \Gamma_5)\,. \tag{11.55}$$

It follows that the functions S_1 and S_2 must be constants. We need furthermore also to set $b = 0$ (as well as imposing $g_2^2 S_1 = g_1^2 S_2$). Setting $S_i = \mu_i$ the projected Killing spinor equations become

$$\begin{aligned} &[\partial_+ + \tfrac{\mathrm{i}}{\sqrt{2}}(\mu_1 + \mu_2)]\epsilon = 0\,, \qquad \partial_- \epsilon = 0\,, \qquad \partial_z \epsilon = 0\,, \\ &\Big[\partial_r - \frac{\mathrm{i}}{2}\frac{g_1 - \mathrm{i} g_2 \Gamma_5}{(g_1^2 + g_2^2)^{1/2}} f' \,\Gamma_- - \frac{1}{2r}\Big]\epsilon = 0\,. \end{aligned} \tag{11.56}$$

The Killing spinor solution is

$$\epsilon = e^{-\frac{\mathrm{i}}{\sqrt{2}}(\mu_1+\mu_2)x^+} r^{1/2}\Big[1 + \frac{\mathrm{i}}{2}\frac{g_1 - \mathrm{i} g_2 \Gamma_5}{(g_1^2 + g_2^2)^{1/2}} f\,\Gamma_-\Big]\epsilon_0\,, \tag{11.57}$$

where ϵ_0 is a constant spinor. Inserting the Killing spinor in the projection condition (11.51) and using

$$f = \frac{\mu_1}{\sqrt{2} g_1^2 (g_1^2 + g_2^2)^{1/2}\, r^2} \tag{11.58}$$

we obtain $(\Gamma_r + a)\epsilon_0 = 0$. Thus, the pp-wave preserves $\frac{1}{2}$ of the supersymmetry with H given by

$$H = -f^2 = -\frac{\mu_1^2}{2 g_1^4 (g_1^2 + g_2^2) r^4}\,. \tag{11.59}$$

D. PP-waves in six dimensions

In this section we investigate the supersymmetry of pp-waves in Romans theory [31]. We use the conventions of [86]. We consider a subsector of the theory by truncating the 2-form potential and the $U(1)$ potential. The Lagrangian describing the remaining fields is given by

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{4}X^{-2}(F^a_{(2)})^2 + 4g^2(X^2 + \tfrac{4}{3}X^{-2} - \tfrac{1}{9}X^{-6}) \tag{11.60}$$

where $X = e^{-\frac{1}{2\sqrt{2}}\varphi}$ and $F^a_{\scriptscriptstyle(2)} = dA^a_{\scriptscriptstyle(1)} - \frac{1}{\sqrt{2}} g\, \epsilon_{abc} A^b_{\scriptscriptstyle(1)} \wedge A^c_{\scriptscriptstyle(1)}$.

We have here set $g_1 = g_2 = -\sqrt{2} g$ in [86]. The supersymmetry transformations are

$$\begin{aligned} \delta\Psi_{Mi} &= [D_M + \tfrac{1}{4} g (X + \tfrac{1}{3} X^{-3}) \Gamma_M] \epsilon_i - \tfrac{\mathrm{i}}{16\sqrt{2}} (\Gamma_M \Gamma^{AB} - 2\Gamma^{AB}\Gamma_M) X^{-1} F_{ABi}{}^j \epsilon_j \,, \\ \delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}} \Gamma^M \partial_M \varphi - \tfrac{1}{2} g (X - X^{-3})] \epsilon_i - \tfrac{\mathrm{i}}{8\sqrt{2}} \Gamma^{AB} X^{-1} F_{ABi}{}^j \epsilon_j \,. \end{aligned} \tag{11.61}$$

where $D_M \epsilon_i = \nabla_M \epsilon_i - \frac{\mathrm{i}}{\sqrt{2}} g A_{Mi}{}^j \epsilon_j$. To obtain the pp-wave we turn off two of the $SU(2)$ fields. The solution is given by

$$\begin{aligned} e^{2A} &= (gr)^{4/3} H_1^{1/2} \,, \qquad H_1 = 1 + \frac{\ell_1^2}{r^2} \,, \\ e^{2B} &= \frac{1}{(gr)^2 H_1^{3/2}} \,, \qquad e^{\sqrt{2}\varphi} = H_1 \,, \\ A^1_{\scriptscriptstyle(1)} &= g^{-1} S_1(x^+)(1 - H_1^{-1})\, dx^+ \,, \end{aligned} \tag{11.62}$$

and the pp-wave function $H(x^+, r, z_a)$ satisfies the equation

$$H'' + (5A' - B')H' + e^{-2(A-B)} \sum_a \partial_a \partial_a H + \frac{4 S_1^2 \ell_1^4}{g^2 r^6 (gr)^{4/3} H_1^4} = 0 \,. \tag{11.63}$$

The Killing spinor equations are given by

$$\begin{aligned} &[\partial_+ + \tfrac{1}{2} A' e^{A-B} (\Gamma_+ + \tfrac{1}{2} H \Gamma_-)(\Gamma_r + 1) - \tfrac{1}{4} e^{A-B} H' \,\Gamma_r \,\Gamma_- - \tfrac{1}{4} \sum_a \partial_a H \,\Gamma_a \,\Gamma_- \\ &\qquad - \tfrac{\mathrm{i}}{\sqrt{2}} S_1 (1 - H_1^{-1})(\Gamma_r + 1) + \frac{\mathrm{i}}{4\sqrt{2}} \frac{S_1 \ell_1^2}{r^2 H_1} \Gamma_r \,\Gamma_+ \,\Gamma_-] \epsilon = 0 \,, \\ &[\partial_- - A' e^{A-B}\, \Gamma_- (\Gamma_r + 1)] \epsilon = 0 \,, \\ &[\partial_a + \tfrac{1}{2} A' e^{A-B}\, \Gamma_a (\Gamma_r + 1) - \frac{\mathrm{i}}{4\sqrt{2}} \frac{S_1 \ell_1^2}{r^2 H_1} \Gamma_a \,\Gamma_r \,\Gamma_-] \epsilon = 0 \,, \\ &[\partial_r + \frac{\ell_1^2 + 4r^2}{12 r^3 H_1} \Gamma_r + \frac{3\mathrm{i}}{4\sqrt{2}} \frac{S_1 \ell_1^2 (gr)^{1/3}}{g^2 r^4 H_1^2} \Gamma_-] \epsilon = 0 \,, \\ &[g(\Gamma_r + 1) + \frac{\mathrm{i}}{\sqrt{2}} \frac{S_1}{(gr)^{2/3} H_1} \Gamma_r \,\Gamma_-] \epsilon = 0 \,. \end{aligned} \tag{11.64}$$

It is clear from these equations that the pp-waves preserve $\frac{1}{4}$ of the supersymmetry but there is no supernumerary supersymmetry.

E. PP-waves in seven dimensions

In this section we consider gauged $D=7,\mathcal{N}=2$ supergravity where we retain only the metric, two $U(1)$ gauge potentials and two scalars. The other fields are consistently set to zero. This reduced set of fields are described by the Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\vec{\varphi})^2 - \tfrac{1}{4}\sum_{i=1}^{2} X_i^{-2}(F^i_{(2)})^2 - V\,, \tag{11.65}$$

where

$$\begin{aligned} X_i &= e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}}\,, \quad \vec{a}_1 = (\sqrt{2},\sqrt{\tfrac{2}{5}})\,, \quad \vec{a}_2 = (\,-\sqrt{2},\sqrt{\tfrac{2}{5}})\,,\\ V &= \tfrac{1}{2}g^2(X_1^{-4}X_2^{-4} - 8X_1X_2 - 4X_1^{-1}X_2^{-2} - 4X_1^{-2}X_2^{-1})\,. \end{aligned} \tag{11.66}$$

The supersymmetry transformations are given by

$$\begin{aligned} \delta\psi_{M} &= [\nabla_{M} + \tfrac{1}{4}(X_1^{-1}F^1_{MN}\Gamma_{12} + X_2^{-1}F^2_{MN}\Gamma_{34})\Gamma^{N} + \tfrac{1}{4}gX_1^{-2}X_2^{-2}\,\Gamma_{M}\\ &\quad +\tfrac{1}{4}(X_1^{-1}\partial_{N}X_1 + X_2^{-1}\partial_{N}X_2)\Gamma_{M}\Gamma^{N} + \tfrac{1}{2}g(A^1_{M}\Gamma_{12} + A^2_{M}\Gamma_{34})]\epsilon\,,\\ \delta\lambda_1 &= [-\tfrac{1}{8}(3X_1^{-1}\partial_{M}X_1 + 2X_2^{-1}\partial_{M}X_2)\Gamma^{M} - \tfrac{1}{16}X_1^{-1}F^1_{AB}\Gamma^{AB}\,\Gamma_{12}\\ &\quad +\tfrac{1}{4}g(X_1 - X_1^{-2}X_2^{-2}]\epsilon\,,\\ \delta\lambda_2 &= [-\tfrac{1}{8}(2X_1^{-1}\partial_{M}X_1 + 3X_2^{-1}\partial_{M}X_2)\Gamma^{M} - \tfrac{1}{16}X_2^{-1}F^2_{AB}\Gamma^{AB}\,\Gamma_{34}\\ &\quad +\tfrac{1}{4}g(X_2 - X_1^{-2}X_2^{-2}]\epsilon\,. \end{aligned} \tag{11.67}$$

For more details see [95]. The domain wall solution is given by

$$\begin{aligned} e^{2A} &= (gr)[H_0^{1/2}H_1H_2]^{\frac{1}{5}}\,, \qquad H_i = 1 + \frac{\ell_i^2}{r^2}\,,\\ e^{2B} &= \frac{1}{(gr)^2[H_0^{1/2}H_1H_2]^{\frac{4}{5}}}\,, \quad X_i = H_i^{-1}[H_0^{1/2}H_1H_2]^{\frac{2}{5}}\,, \end{aligned} \tag{11.68}$$

where $H_0 = 1 + \ell_0^2/r^2$. The ansatz for the 1-form potential is

$$A^i_{(1)} = g^{-1}S_i(1 - H_i^{-1})\,dx^+ \tag{11.69}$$

and the function $H(x^+, r, z_a)$ satisfies the equation

$$H'' + (6A' - B')H' + e^{-2(A-B)} \sum_a \partial_a \partial_a H + \frac{4g^2}{r^2} e^{-10A} \sum_i S_i^2 \ell_i^4 H_i^{-2} = 0\,. \qquad (11.70)$$

The Killing spinor equations are given by

$$\begin{aligned}
&[\partial_+ + \frac{1}{4rH_0} e^{A-B}(\Gamma_+ + \tfrac{1}{2}H\Gamma_-)(\Gamma_r + 1) - \tfrac{1}{4}e^{A-B}H'\,\Gamma_r\,\Gamma_- - \tfrac{1}{4}\sum_a \partial_a H\,\Gamma_a\,\Gamma_- \\
&\quad + \tfrac{1}{2}(S_1(1 - H_1^{-1})\Gamma_{12} + S_2(1 - H_2^{-1})\Gamma_{34})(\Gamma_r + 1)]\epsilon = 0\,, \\
&[\partial_- - \frac{1}{2rH_0} e^{A-B}\,\Gamma_-(\Gamma_r + 1)]\epsilon = 0\,, \\
&[\partial_a + \frac{1}{4rH_0} e^{A-B}\,\Gamma_a(\Gamma_r + 1)]\epsilon = 0\,, \\
&[\partial_r - \tfrac{1}{2\sqrt{10}}\varphi_2' + \frac{1}{4rH_0}\Gamma_r - \frac{1}{2r} g e^{-5A}(S_1\ell_1^2 H_1^{-1}\Gamma_{12} + S_2\ell_2^2 H_2^{-1}\Gamma_{34})\Gamma_-]\epsilon = 0\,, \\
&[g(\ell_0^2 - \ell_1^2)H_0^{-1}X_1(\Gamma_r + 1) + S_1\ell_1^2 H_1^{-1} e^{-A}\,\Gamma_{12}\,\Gamma_r\,\Gamma_-]\epsilon = 0\,, \\
&[g(\ell_0^2 - \ell_2^2)H_0^{-1}X_2(\Gamma_r + 1) + S_2\ell_2^2 H_2^{-1} e^{-A}\,\Gamma_{34}\,\Gamma_r\,\Gamma_-]\epsilon = 0\,. \qquad (11.71)
\end{aligned}$$

It is clear from these equations that the pp-waves have $\frac{1}{4}$ supersymmetry but no supernumerary supersymmetry.

CHAPTER XII

CONCLUSION

In his dissertation we have discussed new gauged supergravities in diverse dimensions from generalized Kaluza-Klein reductions of the low-energy effective actions of string theory involving the metric, the dilaton, a 3-form field strength and a 2-form field strength. The generalized reduction gauges two global symmetries, namely the homogeneous scaling symmetry (conformal symmetry) of the equations of motion, and also the dilaton shift symmetry of the Lagrangian. The gauged supergravity resulting from this construction has a positive scalar potential, in the form of a single-exponential of the lower-dimensional dilaton. We showed that the reduction is supersymmetric, by explicitly deriving the lower-dimensional supersymmetry transformation rules.

We should emphasize that the generalized reductions of the kind we have considered are in fact related by a U-duality to more conventional reductions considered extensively in the past. As we already mentioned in the first chapter, performing a generalized reduction involving the global shift symmetry of the axion in the type-IIB theory one can establish a T-duality between the type-IIB theory and the massive type-IIA theory [22]. The S-duality of the type-IIB theory implies that one should also consider $SL(2,\mathbb{R})$-related generalized reductions [96], which will involve the global shift symmetry of the dilaton. When one extends the discussion of non-perturbative dualities to lower dimensions, the underlying global Cremmer-Julia type symmetries can only be interpreted as strictly internal symmetries if one also make use of the scaling symmetry of the equations of motion that homogeneously scales the Lagrangian. Thus it is very natural to consider generalized reductions of the kind we have studied in this dissertation.

The new supergravities have the interesting feature that they all admit supersymmetric vacuum solutions of the form (Minkowski)$\times S^3$, and in some cases also (Minkowski)$\times S^2$. These solutions provide novel compactifications of higher dimensional string theories. We have studied in detail the compactifications of the variant $D = 6$, $\mathcal{N} = (1,1)$ theory. In particular we have demonstrated that its S^2 reduction yields $D = 4$, $\mathcal{N} = 2$ supergravity coupled to a vector multiplet which can be further truncated to $\mathcal{N} = 1$ supergravity coupled to a chiral multiplet. Although we cannot obtain a chiral theory from the $M_4 \times S^2$ reduction, chirality might still survive in brane models [15] where chiral families live solely on the branes and not in the bulk. In fact, from a braneworld perspective, the present model provides an alternative framework to the Salam-Sezgin model, where the bulk solution preserves $\mathcal{N} = 2$ supersymmetry, and it is the branes themselves that provide both chirality and an additional halving of supersymmetry to $\mathcal{N} = 1$. It would be of interest to study the resulting braneworld models constructed from the present theory.

We have discussed in detail the embedding of the vacua in brane solutions. In fact, we made us of this connection to the branes to prove the supersymmetry of the vacuum solutions in diverse dimensions. For example, the $(\text{Minkowski})_4 \times S^2$ vacua embeds in the 3-brane in seven dimensions which itself can be viewed as intersecting M5-branes wrapping on a supersymmetric two-cycle of $K3$ in D=11. Note that, the orders of the reductions of the 3-brane can be reversed, by performing the S^2 reduction first, which gives rise to a D=5 domain wall, with a $(\text{Minkowski})_4$ world-volume. We finally arrive at the four-dimensional Minkowski spacetime by performing a braneworld Kaluza-Klein reduction introduced in [97]. (See also, [98, 99, 100, 101, 86].) Instead of reducing on a specific solution as the above we can reduce on the theory itself after truncating out the 2-form field strengths. First, we expect that there should be a consistent reduction of the minimal $D = 7$ supergravity on S^2. To see this, we can study the global symmetry of the theory reduced on T^2. If we for simplicity set two of the three vector fields in $D = 7$ to zero, then the resulting $D = 5$ theory has a global $O(2,3)$ T-duality symmetry, with the scalars parameterising the coset $O(2,3)/(SO(2) \times SO(3))$. Clearly, we can gauge the $SO(3)$ maximal compact subgroup, which is exactly the isometry group of S^2. This is indicative of a consistent S^2 reduction of the $D = 7$ theory [44]. The resulting gauged $D = 5$ supergravity will have a negative exponential scalar potential which can support a domain wall solution. We can then perform the brane-world Kaluza-Klein reduction to $D = 4$.

It is also interesting to note that in our earlier approach, the lower dimensional theory arises first from the generalized Kaluza-Klein reduction on R, and then a standard sphere (S^2 or S^3) reduction, in which case, the reduction makes use of a gauging of the homogeneous scaling symmetry. If we instead perform sphere reduction first, and then the brane-world reduction, it would appear that we do not need to appeal to the homogeneous scaling symmetry. Clearly, the two approaches are equivalent. One feature in common is that in both approaches, the reduction involves warp factors. Thus our first approach is nothing more than giving a symmetry interpretation of the warp factor in the reduction ansatz. In fact, the near-horizon structure of the $(D-5)$-branes (or the $(D-4)$-branes) given by (7.3) (or (7.9)) in D dimensions can be viewed as domain walls written in the conformal frame, with the world-volume being $(\text{Minkowski})_{d-3} \times S^3$ (or $(\text{Minkowski})_{d-2} \times S^2$). Thus the generalized dimensional reduction can be viewed as a special case of the brane-world reduction.

In this dissertation we have also studied in detail the supersymmetry of pp-waves in AdS backgrounds. The introduction of a pp-wave in the AdS background can be viewed as performing an infinite boost in the strong coupled dual conformal field theory with a finite momentum density. The non-vanishing momentum breaks the original supersymmetry and superconformal symmetry, and hence the supersymmetry is now $\frac{1}{4}$ of the unboosted theory. With an appropriate choice for the integration constants, we have shown that purely gravitational pp-waves admit supernumerary supersymmetry in which the solutions double its supersymmetry. We have also studied $U(1)$-charged pp-waves and shown that supernumerary supersymmetry can arise in four and five dimensions. This indicates a novel supersymmetry enhancement associated with the R-charge in the dual three and four dimensional field theories. It is of interest to discover such a phenomenon in the dual quantum field theory in the infinite momentum frame.

There are a number of immediate unsolved issues with pp-waves in an AdS background. First, how does one obtain an AdS pp-wave from a Penrose limit? Second, for a charged solution, supersymmetry enhancement might also occur in six dimensions, since the Penrose limit of $\mathrm{AdS}_3 \times S^3$ is known to have supernumerary supersymmetry. One would in this case set in Romans $F(4)$ theory the dilaton and the $SU(2)$ fields to zero leaving a 2-form and a 1-form potential. We have not been able to obtain the pp-wave solution in this case.

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APPENDIX A

BOSONIC REDUCTION ANSATZ; EINSTEIN FRAME

We begin by reducing the $D = d + 1$ dimensional Ricci tensor to d dimensions by using the metric ansatz in (3.6). We choose the natural vielbein basis

$$\hat{e}^a = e^{m_2 z + \alpha\varphi} e^a , \qquad \hat{e}^z = e^{m_2 z + \beta\varphi}(dz + \mathcal{A}_{(1)}) . \tag{A.1}$$

Thus we have

$$\hat{e}_M{}^A = e^{m_2 z} \begin{pmatrix} e^{\alpha\varphi} e_\mu{}^a & e^{\beta\varphi}\mathcal{A}_\mu \\ 0 & e^{\beta\varphi} \end{pmatrix} , \qquad \hat{e}_A{}^M = e^{-m_2 z} \begin{pmatrix} e^{-\alpha\varphi} e_a{}^\mu & -e^{-\alpha\varphi}\mathcal{A}_a \\ 0 & e^{-\beta\varphi} \end{pmatrix} . \tag{A.2}$$

The determinant of the metric is

$$\sqrt{-\hat{g}} = e^{(d+1)m_2 z + (\beta + d\alpha)\varphi}\sqrt{-g} = e^{(d+1)m_2 z + 2\alpha\varphi}\sqrt{-g} . \tag{A.3}$$

Using the first Cartan structure equation with zero torsion, $d\hat{e}^A = -\hat{\omega}^A{}_B \wedge \hat{e}^B$, we obtain the spin connections

$$\begin{aligned} \hat{\omega}^a{}_b &= \omega^a{}_b + e^{-(m_2 z + \alpha\varphi)}\Big((\alpha\partial_b\varphi - m_2\mathcal{A}_b)\,\hat{e}^a - (\alpha\partial^a\varphi - m_2\mathcal{A}^a)\,\hat{e}_b\Big) \\ &\quad -\tfrac{1}{2}e^{-m_2 z + (\beta - 2\alpha)\varphi}\mathcal{F}^a{}_b\,\hat{e}^z , \\ \hat{\omega}^a{}_z &= e^{-(m_2 z + \alpha\varphi)}(m_2\mathcal{A}^a - \beta\partial^a\varphi)\,\hat{e}^z - \tfrac{1}{2}e^{-m_2 z + (\beta - 2\alpha)\varphi}\mathcal{F}^a{}_b\,\hat{e}^b + m_2 e^{-(m_2 z + \beta\varphi)}\hat{e}^a . \end{aligned} \tag{A.4}$$

From the curvature 2-forms $\hat{\Theta}^A{}_B = d\hat{\omega}^A{}_B + \hat{\omega}^A{}_C \wedge \hat{\omega}^C{}_B = \frac{1}{2}\hat{R}^A{}_{BCD}\hat{e}^C \wedge \hat{e}^D$, we obtain the Ricci tensor with vielbein components

$$\begin{aligned} \hat{R}_{ab} &= e^{-2(m_2 z + \alpha\varphi)}\Big(R_{ab} - \tfrac{1}{2}\partial_a\varphi\,\partial_b\varphi - \alpha\eta_{ab}\Box\varphi \\ &\quad +\alpha m_2(d-1)(\mathcal{A}^c\partial_c\varphi\,\eta_{ab} - \mathcal{A}_a\partial_b\varphi - \mathcal{A}_b\partial_a\varphi) \\ &\quad +\tfrac{1}{2}m_2(d-1)(\nabla_a\mathcal{A}_b + \nabla_b\mathcal{A}_a) + m_2\nabla_c\mathcal{A}^c\eta_{ab} \end{aligned}$$

$$
\begin{aligned}
& +m_2^2(d-1)(\mathcal{A}_a\mathcal{A}_b - \mathcal{A}_{(1)}^2\eta_{ab})\Big) \\
& - m_2^2(d-1)e^{-2(m_2 z+\beta\varphi)}\eta_{ab} - \tfrac{1}{2}e^{-2(m_2 z+d\alpha\varphi)}\mathcal{F}_a{}^c\mathcal{F}_{bc}\,, \\
\hat{R}_{az} &= e^{-2m_2 z+(d-3)\alpha\varphi}\Big(\tfrac{1}{2}\nabla^b(e^{-2(d-1)\alpha\varphi}\mathcal{F}_{ab}) + m_2(d-1)(\beta\partial_a\varphi - m_2\mathcal{A}_a)\Big) \\
& - \tfrac{1}{2}m_2(d-1)e^{-2m_2 z-(d+1)\alpha\varphi}\mathcal{A}^b\mathcal{F}_{ab}\,, \\
\hat{R}_{zz} &= e^{-2(m_2 z+\alpha\varphi)}\Big(-\beta\Box\varphi + m_2\nabla_c\mathcal{A}^c + m_2\beta(d-1)\mathcal{A}^b\partial_b\varphi - m_2^2(d-1)\mathcal{A}_{(1)}^2\Big) \\
& + \tfrac{1}{4}e^{-2(m_2 z+d\alpha\varphi)}\mathcal{F}_{(2)}^2\,. \qquad \text{(A.5)}
\end{aligned}
$$

The Ricci scalar is

$$
\begin{aligned}
\hat{R} &= e^{-2(m_2 z+\alpha\varphi)}\Big(R - 2\alpha\Box\varphi - \tfrac{1}{2}(\partial\varphi)^2 + 2m_2 d\,\nabla_a\mathcal{A}^a - m_2^2 d\,(d-1)\mathcal{A}_{(1)}^2\Big) \\
& - e^{-2m_2 z}\Big(m_2^2\,d\,(d-1)e^{-2\beta\varphi} + \tfrac{1}{4}e^{-2d\alpha\varphi}\mathcal{F}_{(2)}^2\Big)\,. \qquad \text{(A.6)}
\end{aligned}
$$

The reduced Ricci components in (A.5) have been simplified through use of the relations (3.7).

The Laplacian operator acting on the D-dimensional dilaton is given by

$$
e^{2m_2 z+2\alpha\varphi}\,\widehat{\Box}\hat{\phi} = \Box\phi - m_2(d-1)\Big(\mathcal{A}^\mu\partial_\mu\phi - \frac{4}{\hat{a}}\,m_1\,(\mathcal{A}_{(1)}^2 + e^{2(d-1)\alpha\varphi})\Big) - \frac{4}{\hat{a}}\,m_1\nabla_\mu\mathcal{A}^\mu\,, \quad \text{(A.7)}
$$

where $\hat{\phi} = \phi + \frac{4}{\hat{a}}\,m_1 z$, as given by (3.6).

The vielbein components of the various D-dimensional antisymmetric tensors reduce according to

$$
\begin{aligned}
\hat{H}_{a_1\cdots a_n} &= e^{-(m_2+(n-1)m_1)z-n\alpha\varphi}\,H_{a_1\cdots a_n}\,, \\
\hat{H}_{a_1\cdots a_{n-1}z} &= e^{-(m_2+(n-1)m_1)z+(d-n-1)\alpha\varphi}\,H_{a_1\cdots a_{n-1}}\,. \qquad \text{(A.8)}
\end{aligned}
$$

APPENDIX B

FERMIONIC REDUCTION ANSATZ IN $D \leq 10$; EINSTEIN FRAME

In this appendix we provide an arbitrary dimensional generalized ansatz that reduces the fermions in $D = d+1$ to d dimensions. The generalized ansatz we are constructing is such that the standard S^1 reduction ($m_1 = 0 = m_2$) reduces canonical fermionic kinetic terms with a normalization as

$$\hat{e}^{-1}\hat{\mathcal{L}} = \kappa(\hat{\bar{\Psi}}_M \hat{\gamma}^{MNP} \widehat{\nabla}_N \hat{\Psi}_P + \hat{\bar{\lambda}} \hat{\gamma}^M \widehat{\nabla}_M \hat{\lambda}) \tag{B.1}$$

to canonical kinetic terms

$$e^{-1}\mathcal{L} = \kappa(\bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho + \bar{\lambda}\gamma^\mu \nabla_\mu \lambda + \bar{\chi}\gamma^\mu \nabla_\mu \chi) + \text{rest}\,. \tag{B.2}$$

Here κ is an arbitrary coefficient. Performing the split of the gravitino as $\hat{\psi}_A = (\hat{\psi}_a, \hat{\psi}_D)$ an ansatz that accomplishes this is

$$\begin{aligned}
\hat{\epsilon} &= e^{\frac{1}{2}m_2 z} e^{\frac{1}{2}\alpha\varphi}\, \epsilon\,, \\
\hat{\lambda} &= \tfrac{1}{\sqrt{D-2}}\, e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi} (\chi + \sqrt{D-3}\,\lambda)\,, \\
\hat{\psi}_D &= \tfrac{\sqrt{D-3}}{D-2} e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi} \gamma_D (\sqrt{D-3}\,\chi - \lambda)\,, \\
\hat{\psi}_a &= e^{-\frac{1}{2}m_2 z} e^{-\frac{1}{2}\alpha\varphi} (\psi_a - \tfrac{1}{(D-2)\sqrt{D-3}} \gamma_a (\sqrt{D-3}\,\chi - \lambda))\,, \\
\hat{\phi} &= \sqrt{\tfrac{D-3}{D-2}}\,\phi_1 + \tfrac{1}{\sqrt{D-2}}\phi_2 + \sqrt{2(D-2)}\, m_1 z\,, \\
\varphi &= -\tfrac{1}{\sqrt{D-2}}\phi_1 + \sqrt{\tfrac{D-3}{D-2}}\phi_2\,.
\end{aligned} \tag{B.3}$$

Note that, here and elsewhere in this dissertation our convention is always $\alpha > 0$. A consistent truncation of the matter multiplet can be obtained by setting $m_1 = m_2$ and $\phi_2 = 0 = \chi$.

APPENDIX C

EINSTEIN-FRAME TO STRING-FRAME CONVERSION

The D-dimensional Lagrangian in the Einstein frame is given by

$$\begin{aligned} e^{-1}\mathcal{L} &= R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{\hat{a}\phi}H_{(3)}^2 - \tfrac{1}{4}e^{\frac{1}{2}\hat{a}\phi}(F_{(2)}^a)^2 - \tfrac{1}{2}\bar{\Psi}_M\gamma^{MNP}\nabla_N\Psi_P \\ &\quad -\tfrac{1}{2}\bar{\lambda}\,\gamma^M\nabla_M\lambda - \tfrac{1}{2\sqrt{2}}\bar{\lambda}\gamma^N\gamma^M\Psi_N\partial_M\phi + \cdots\,, \end{aligned} \tag{C.1}$$

where $\hat{a} = \sqrt{\frac{8}{D-2}}$, and where we have omitted additional interaction and four-fermi terms. This may be mapped to the string frame Lagrangian

$$\begin{aligned} \tilde{e}^{-1}\widetilde{\mathcal{L}} &= e^{-2\Phi}\Big(\widetilde{R} + 4(\partial\Phi)^2 - \tfrac{1}{12}\widetilde{H}_{(3)}^2 - \tfrac{1}{4}(\widetilde{F}_{(2)}^a)^2 - \tfrac{1}{2}\bar{\tilde{\Psi}}_M\tilde{\gamma}^{MNP}\widetilde{\nabla}_N\tilde{\Psi}_P \\ &\quad -\tfrac{1}{2}\bar{\tilde{\lambda}}\,\tilde{\gamma}^M\widetilde{\nabla}_M\tilde{\lambda} - (\bar{\tilde{\Psi}}_N\tilde{\gamma}^N\tilde{\Psi}^M - \frac{\hat{a}}{2\sqrt{2}}\bar{\tilde{\lambda}}\tilde{\gamma}^N\tilde{\gamma}^M\tilde{\Psi}_N)\partial_M\Phi + \cdots\Big)\,, \end{aligned} \tag{C.2}$$

by the transformations

$$\begin{aligned} g_{MN} &= e^{\frac{1}{2}\hat{a}\phi}\,\tilde{g}_{MN}\,, \qquad H_{MNP} = \widetilde{H}_{MNP}\,, \qquad F_{MN}^a = \widetilde{F}_{MN}^a\,, \qquad \phi = -\hat{a}\,\Phi\,, \\ \epsilon &= e^{\frac{1}{8}\hat{a}\phi}\,\tilde{\epsilon}\,, \qquad \lambda = e^{-\frac{1}{8}\hat{a}\phi}\tilde{\lambda}\,, \qquad \Psi_M = e^{\frac{1}{8}\hat{a}\phi}\tilde{\Psi}_M\,. \end{aligned} \tag{C.3}$$

Note that $\gamma_M = e^{\frac{1}{4}\hat{a}\phi}\,\tilde{\gamma}_M$ i.e. $\gamma_A = \tilde{\gamma}_A$. Furthermore, we have made use of the D-dimensional Majorana flip properties $\bar{\psi}\gamma^M\chi = -\bar{\chi}\gamma^M\psi$ and $\bar{\psi}\gamma^{MNP}\chi = \bar{\chi}\gamma^{MNP}\psi$ for any two anti-commuting spinors ψ and χ.

The bosonic reduction ansätze in the string frame are considerably simpler than their Einstein-frame counterparts. The reduction of the $D = d+1$ dimensional Ricci tensor is given by

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} + \tfrac{1}{\sqrt{2}}\nabla_a\partial_b\varphi - \tfrac{1}{2}\partial_a\varphi\partial_b\varphi - \tfrac{1}{2}e^{-\sqrt{2}\varphi}\mathcal{F}_{ac}\,\mathcal{F}_b{}^c\,, \\ \hat{R}_{az} &= \tfrac{1}{2}e^{\sqrt{2}\varphi}\,\nabla^b(e^{-\frac{3}{\sqrt{2}}\varphi}\mathcal{F}_{ab})\,, \end{aligned}$$

$$\hat{R}_{zz} = \tfrac{1}{\sqrt{2}}\Box\varphi - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{4}e^{-\sqrt{2}\varphi}\mathcal{F}_{(2)}^2\,,$$
$$\hat{R} = R + \sqrt{2}\Box\varphi - (\partial\varphi)^2 - \tfrac{1}{4}e^{-\sqrt{2}\varphi}\mathcal{F}_{(2)}^2\,. \qquad \text{(C.4)}$$

Some useful formulae for the reduction of the scalar fields are:

$$\widehat{\Box}\hat{\Phi} = \widehat{\Box}\Big(\Phi - \frac{\varphi}{\sqrt{8}} - \tfrac{1}{2}(D-2)mz\Big) = \Box\Phi - \tfrac{1}{\sqrt{8}}\Box\varphi - \tfrac{1}{\sqrt{2}}(\partial_\mu\varphi\,\partial^\mu\Phi - \tfrac{1}{\sqrt{8}}(\partial\varphi)^2)$$
$$- \tfrac{1}{2}m(d-1)(\tfrac{1}{\sqrt{2}}\,\mathcal{A}^\mu\partial_\mu\varphi - \nabla_\mu\mathcal{A}^\mu)\,, \qquad \text{(C.5)}$$

$$(\partial\hat{\Phi})^2 = (\partial\Phi)^2 + \tfrac{1}{8}(\partial\varphi)^2 - \tfrac{1}{\sqrt{2}}\,\partial_\mu\varphi\,\partial^\mu\Phi + m(d-1)\mathcal{A}^\mu(\partial_\mu\Phi - \tfrac{1}{\sqrt{8}}\partial_\mu\varphi)$$
$$+ \tfrac{1}{4}m^2(d-1)^2(\mathcal{A}_{(1)}^2 + e^{\sqrt{2}\varphi})\,, \qquad \text{(C.6)}$$

$$\hat{e}_a{}^M\hat{e}_b{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} = \nabla_a\partial_b\Phi - \tfrac{1}{\sqrt{8}}\nabla_a\partial_b\varphi + \tfrac{1}{4}m(d-1)(\nabla_a\mathcal{A}_b + \nabla_b\mathcal{A}_a)\,,$$
$$\hat{e}_a{}^M\hat{e}_z{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} = -\tfrac{1}{2}e^{-\frac{1}{\sqrt{2}}\varphi}\mathcal{F}_a{}^b(\partial_b\Phi - \tfrac{1}{\sqrt{8}}\,\partial_b\varphi) - \tfrac{1}{2\sqrt{2}}m(d-1)e^{\frac{1}{\sqrt{2}}\varphi}\partial_a\varphi$$
$$- \tfrac{1}{4}m(d-1)e^{-\frac{1}{\sqrt{2}}\varphi}\mathcal{A}^b\mathcal{F}_{ab}\,,$$
$$\hat{e}_z{}^M\hat{e}_z{}^N\widehat{\nabla}_M\partial_N\hat{\Phi} = -\tfrac{1}{\sqrt{2}}\partial^\mu\varphi\,(\partial_\mu\Phi - \tfrac{1}{\sqrt{8}}\,\partial_\mu\varphi) - \tfrac{1}{2\sqrt{2}}m(d-1)\mathcal{A}^\mu\partial_\mu\varphi\,. \qquad \text{(C.7)}$$

APPENDIX D

KILLING SPINORS ON THE SPHERE

In this appendix we give expressions for the Killing spinors on S^n derived in [102] and the decomposition of Dirac matrices on product spaces also given there. The Clifford algebra is

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \tag{D.1}$$

where the sign convention for the flat metric is $\eta_{ab} = (-, + \ldots +)$. The metric for an n-sphere with the radius a is

$$ds_n^2 = a^{-2}(d\theta_n^2 + + \sin^2\theta_n ds_{n-1}^2)\,, \tag{D.2}$$

with $ds_1^2 = d\theta_1^2$ and the Ricci tensor for the sphere is given by $R_{ij} = a^2(n-1)g_{ij}$. The solution to the Killing spinor equation

$$\nabla_j \epsilon_\pm = \pm \tfrac{\mathrm{i}}{2} a \Gamma_i \epsilon_\pm \tag{D.3}$$

is

$$\epsilon_\pm = e^{\pm\frac{\mathrm{i}}{2}\theta_n\Gamma_n}\Big(\prod_{j=1}^{n-1} e^{-\frac{1}{2}\theta_j\Gamma_{j,j+1}}\Big)\epsilon_0\,, \tag{D.4}$$

where the Γ matrices satisfy $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$. The exponential factors can be written as

$$e^{\frac{\mathrm{i}}{2}\theta_n\Gamma_n} = \mathbb{1}\cos\tfrac{1}{2}\theta_n + \mathrm{i}\Gamma_n \sin\tfrac{1}{2}\theta_n\,, \qquad e^{-\frac{1}{2}\theta_j\Gamma_{j,j+1}} = \mathbb{1}\cos\tfrac{1}{2}\theta_j - \Gamma_{j,j+1}\sin\tfrac{1}{2}\theta_j\,. \tag{D.5}$$

The above are valid in all dimensions, but in the case of n is even the equation

$$\nabla_j \epsilon_\pm = \tfrac{1}{2} a \gamma_* \Gamma_j \epsilon_\pm \tag{D.6}$$

can also be considered. Here γ_* is the chirality operator on the sphere and satisfies $\gamma_*^2 = 1$. The solution in this case is given by

$$\epsilon_\pm = e^{\pm\frac{i}{2}\theta_n\gamma_*\Gamma_n}\Big(\prod_{j=1}^{n-1} e^{-\frac{1}{2}\theta_j\Gamma_{j,j+1}}\Big)\epsilon_0\,. \tag{D.7}$$

The decomposition of the $D = m + n$ dimensional gamma matrices $\hat{\Gamma}_A$ in terms of the lower dimensional spacetime M_m and the internal space K_n is performed as

$$\begin{aligned}
(m,n) = (\text{even,odd}) \quad &: \quad \hat{\Gamma}_a = \Gamma_a \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_i = \Gamma_* \otimes \Gamma_i\,,\\
(\text{odd,even}) \quad &: \quad \hat{\Gamma}_a = \Gamma_a \otimes \gamma_*\,, \qquad \hat{\Gamma}_i = \mathbb{1} \otimes \Gamma_i\,,\\
(\text{even,even} \quad &: \quad \hat{\Gamma}_a = \Gamma_a \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_i = \Gamma_* \otimes \Gamma_i\,,\\
\text{or} \quad & \quad\;\; \hat{\Gamma}_a = \Gamma_a \otimes \gamma_*\,, \qquad \hat{\Gamma}_i = \mathbb{1} \otimes \Gamma_i\,,\\
(\text{odd,odd}) \quad &: \quad \hat{\Gamma}_a = \sigma_1 \otimes \Gamma_a \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_i = \sigma_2 \otimes \mathbb{1} \otimes \Gamma_i\,,
\end{aligned} \tag{D.8}$$

where Γ_* is the chirality matrix in an even lower-dimensional spacetime, and γ_* is the chirality matrix in an even dimensional internal space. The σ_1 and σ_2 are Pauli matrices and the chirality operator in the total space is $\hat{\Gamma}_* = \sigma_3 \otimes \mathbb{1}$.

APPENDIX E

A GENERAL CLASS OF PP-WAVES

In this appendix we present the AdS pp-waves supported by an arbitrary n-form field strength in any dimensions D. The Lagrangian for such a system is given by

$$e^{-1}\mathcal{L} = R - \frac{1}{2n!}F_{(n)}^2 + (D-1)(D-2)g^2 \tag{E.1}$$

where the field strength is defined as $F_{(n)} = dA_{(n-1)}$. Our pp-wave ansatz is

$$\begin{aligned} ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H\,(dx^+)^2 + dz^2) + d\rho^2\,, \\ A_{(n-1)} &= \Big(zS_1(x^+) - \frac{S_2(x^+)}{g(D-2n+1)}(e^{-(D-2n+1)g\rho} - 1)\Big)\,dx^+ \wedge d^{n-2}z\,. \end{aligned} \tag{E.2}$$

The field strength and its dual are

$$\begin{aligned} F_{(n)} &= -S_1\,dx^+ \wedge dz^{n-1} + S_2 e^{-(D-2n+1)g\rho} d\rho \wedge dx^+ \wedge d^{n-2}z\,, \\ {*F_{(n)}} &= S_1 e^{(D-2n-1)g\rho} d\rho \wedge dx^+ \wedge d^{\,D-n-2}z - S_2 dx^+ \wedge d^{\,D-n-1}z\,. \end{aligned} \tag{E.3}$$

Thus the equation of motion $d{*F_{(n)}} = 0$ is trivially satisfied. The Einstein equation implies

$$\begin{aligned} \Box H &= -S_1^2 e^{-2ng\rho} - S_2^2 e^{-2(D-n)g\rho}\,, \\ \Box &= \partial_\rho^2 + g(D-1)\partial_\rho + e^{-2g\rho}\sum_{i=1}^{D-3}\partial_i^2\,, \end{aligned} \tag{E.4}$$

with the solution given by

$$\begin{aligned} H(x^+,\rho,z_i) &= a + b\,e^{-(D-1)g\rho} + \frac{e^{-2g\rho}}{2g^2(D-3)}\sum_{i=1}^{D-3} c_i + \frac{S_1^2\,e^{-2ng\rho}}{2g^2 n(D-2n-1)} \\ &- \frac{S_2^2\,e^{-2(D-n)g\rho}}{2g^2(D-n)(D-2n+1)} + \tfrac{1}{2}\sum_{i=1}^{D-3} c_i z_i^2\,. \end{aligned} \tag{E.5}$$

The a, b and c_i are functions of x^+. This solution is not valid for $D = 2n-1$ or $D = 2n+1$ which have to be considered separately. We find that

$$H(D = 2n+1, x^+, \rho, z_i) = a + b\, e^{-2ng\rho} + \frac{e^{-2g\rho}}{4g^2(n-1)} \sum_{i=1}^{2(n-1)} c_i$$
$$+ \frac{S_1^2(2ng\rho+1)}{4n^2g^2}\, e^{-2ng\rho} - \frac{S_2^2}{4g^2(n+1)}\, e^{-2(n+1)g\rho} + \tfrac{1}{2} \sum_{i=1}^{2(n-1)} c_i z_i^2\,, \qquad \text{(E.6)}$$

and $H(D = 2n-1)$ can be obtained from $H(D = 2n+1)$ by making the substitution

$$n \to n-1 \qquad \text{and} \qquad S_1 \longleftrightarrow S_2\,. \qquad \text{(E.7)}$$

(This substitution is not performed on the field strength.)

VITA

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