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A New Family of Exponential Type Estimators in the Presence of Non-Response

Ceren Unal* & Cem Kadilar

Department of Statistics, Faculty of Science, Hacettepe University, 06800 Beytepe, Ankara, Turkey
*E-mail: cerenunal@hacettepe.edu.tr

Abstract. We propose families of estimators for the population mean using an exponential function in case of non-response. This situation is examined under two cases, Case I and II. The bias, MSE and minimum MSE are separately obtained for both cases. We compare the proposed estimators theoretically with the main estimators from the literature, such as Hansen and Hurwitz (1946), ratio, regression and exponential estimators. The conditions for which the proposed estimators are most efficient are obtained. Moreover, different empirical studies are conducted to support the theoretical results for both cases.

Keywords: *auxiliary variable; efficiency; exponential type estimators; family of estimators; non-response; population mean.*

1 Introduction

Several authors have introduced different types of estimators to estimate unknown population parameters. When estimating population parameters, the information of an auxiliary variable (x) is generally used for enhancing the efficiency. For instance, Yadav and Mishra [1] and Yadav *et al*. [2] have proposed an estimator for population mean using auxiliary information. For this reason, ratio, product, regression, etc. type estimators have been proposed using auxiliary information to introduce more efficient estimators than others in the literature.

Some of the main estimators to estimate the population mean under the SRSWOR scheme are the following.

The ratio type estimator was proposed by Cochran [3]:

$$t_R = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{1}$$

In Eq. (1), $\bar{X}$ and $\bar{x}$ refer to the population and sample mean of x, respectively, and the sample mean of y is defined as $\bar{y}$. The MSE of the t_R is given by:

$$MSE(t_R) = \lambda \bar{Y}^2\left(C_y^2 + C_x^2 - 2C_{xy}\right), \tag{2}$$

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where $\lambda = \frac{1-f}{n}, C_y^2 = \frac{S_y^2}{\bar{Y}^2}, C_x^2 = \frac{S_x^2}{\bar{X}^2}, C_{xy} = \rho_{xy} C_x C_y$. Besides, $\bar{Y}$ is the population mean of *y*. Here, $f = \frac{n}{N}$ and ρ_{xy} is the population correlation coefficient between *x* and *y*.

Cochran [4] proposed the classical regression estimator and obtained its MSE as follows:

$$t_{reg} = \bar{y} + b(\bar{X} - \bar{x}), \tag{3}$$

$$MSE(t_{reg}) = \bar{Y}^2 \lambda C_y^2 (1 - \rho_{xy}^2), \tag{4}$$

respectively, where *b* is the regression coefficient.

The product, ratio and regression type estimators have equal efficiency when the relation between *x* and *y* is a straight line passing through the origin. However, this situation may not occur most of the time [5]. Recently, estimators have been proposed to take advantage of an exponential function.

Bahl and Tuteja [6] were the first to introduce an exponential type estimator:

$$t_{BT} = \bar{y} exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right). \tag{5}$$

$MSE(t_{BT})$ is obtained as follows:

$$MSE(t_{BT}) = \left(\frac{C_x^2}{4} - C_{xy} + C_y^2\right) \lambda \bar{Y}^2. \tag{6}$$

Following Bahl and Tuteja [6], Yadav and Kadilar [7] proposed as estimator:

$$t_{YK} = k\bar{y}\ exp\ exp\ \left(\frac{(c\bar{X} - c\bar{x})}{(c\bar{X} + c\bar{x}) + 2d}\right), \tag{7}$$

where *c* and *d* are functions of the parameters or real constants. The expression of the minimum MSE for the estimator in Eq. (7) is given by:

$$MSE_{min}(\bar{t}_{YK}) = \bar{Y}^2 \left(1 - \frac{\left(\lambda\left(2\xi^2 C_x^2 - \xi C_{xy}\right) + 1\right)^2}{\lambda\left(C_y^2 + 5\xi^2 C_x^2 - 4\xi C_{xy}\right) + 1}\right) \tag{8}$$

where $\xi = \frac{c\bar{X}}{2(c\bar{X} + d)}$.

After the significant contributions of these studies, Singh *et al*. [8], Kumar and Saini [9], and Singh *et al*. [10] proposed exponential type estimators for the population mean.

In real life, all information on various variables may not be available. Hansen and Hurwitz [11] introduced a new sub-sampling method to deal with non-response situations. In this method a population consist of *N* units, S, and *n* units are drawn from the population under SRSWOR. *N* is composed of N_1 and

N_2 for responding and non-responding units, respectively. Furthermore, n is divided into two units, responding (n_1) and non-responding (n_2). From n_2 a sub-sample of size $r = \frac{n_2}{z}(z > 1)\bar{X}$ is randomly drawn. Here, z means the inverse sampling rate at the second phase sample. In addition to this technique, the methods of Srinath [12] and Bouza [13] can be used as alternatives to Hansen Hurwitz's method for determining the subsample size [14].

Hansen and Hurwitz [11] were the first to introduce an unbiased estimator for non-response situations for the population mean as follows:

$$t_{HH} = w_1\bar{y}_1 + w_2\bar{y}_{2(r)}, \tag{9}$$

where $w_1 = \frac{n_1}{n}$ is the response proportion in the sample. Similarly, $w_2 = \frac{n_2}{n}$ refers to the non-response proportion. In addition, $\bar{y}_{2(r)}$ and $\bar{y}_1$ indicate the sample means of the y contingent on r and n_1 units, respectively. The variance of t_{HH} is given by:

$$V(t_{HH}) = \bar{Y}^2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right), \tag{10}$$

where $C_{y(2)}^2 = \frac{S_{y(2)}^2}{\bar{Y}^2}$ is the coefficient of variation of y for N_2 units. $W_2 = \frac{N_2}{N}$ is the non-response proportion of the population.

The non-response situation will be examined under Case I and Case II. In Case I, non-response is known and exists only in y.

For this case, Rao [15] adapted the ratio estimator and the regression estimator, t_R^* and t_{reg}^*, in Eq. (1) and Eq. (7) respectively, using the Hansen and Hurwitz [11] technique:

$$t_R^* = \frac{\bar{X}}{\bar{x}}\bar{y}^*, \tag{11}$$

$$t_{reg}^* = \bar{y}^* + b^*(\bar{X} - \bar{x}), \tag{12}$$

where $b^* = \frac{S_{xy}^*}{S_x^{*2}}$ and $\bar{y}^*$ is the sample mean of y under the non-response scheme. The $MSE(t_R^*)$ and $MSE(t_{reg}^*)$ are given, respectively, as

$$MSE(t_R^*) = \bar{Y}^2\left(\lambda\left(C_x^2 - 2C_{yx} + C_y^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right), \tag{13}$$

$$MSE(t_{reg}^*) = \bar{Y}^2\left(\lambda C_y^2\left(1 - \rho_{xy}^2\right) + \frac{W_2(z-1)}{n}C_{y(2)}^2\right). \tag{14}$$

Singh *et al*. [16] proposed an exponential type estimator using a similar technique by adapting the t_{BT} estimator as:

$$t_{BT}^* = \bar{y}^* exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right). \tag{15}$$

The MSE for the t_{BT}^{*} estimator is given by:

$$MSE(t_{BT}^{*}) = \bar{Y}^2\left(\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right)\lambda + \frac{W_2(z-1)}{n}C_{y(2)}^2\right). \tag{16}$$

After the significant contributions of these studies, Yunusa and Kumar [17], Dansawad [18], Singh and Usman [19], Pal and Singh [20,21], Yadav *et al*. [22], Sinha and Kumar [23], Pal and Singh [24], Kumar and Kumar [25], Sanaullah *et al*. [26] and Javaid *et al*. [27] have proposed exponential type estimators for the population mean for Case I.

For Case II, non-response exists in both *x* and *y* and $\bar{X}$ is known. Cochran [4] adapted the estimator in Eq. (1) for the Case II as follows:

$$t_R^{**} = \frac{\bar{y}^*}{\bar{x}^*}\bar{X}, \tag{17}$$

where $\bar{x}^*$denotes the sample mean of x in case of non-response. The MSE of the t_R^{**} is:

$$MSE(t_R^{**}) = \lambda\bar{Y}^2\left(C_x^2 - 2C_{yx} + C_y^2\right) + \bar{Y}^2\left(\frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + C_{x(2)}^2 - 2C_{yx(2)}\right)\right), \tag{18}$$

where $C_{x(2)}^2 = \frac{S_{x(2)}^2}{\bar{X}^2}$ and $C_{yx(2)} = C_{x(2)}C_{y(2)}\rho_{yx(2)}$. Here, $\rho_{yx(2)}$ is the population correlation coefficient of the non-response group between *y* and *x*.

Singh *et al*. [16] adapted the exponential type estimator in Eq. (3) using the Hansen and Hurwitz [11] technique for Case II:

$$t_{BT}^{**} = \bar{y}^* exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right). \tag{19}$$

$MSE(t_{BT}^{**})$ is given as:

$$MSE(t_{BT}^{**}) = \bar{Y}^2\left(\lambda\frac{C_x^2}{4} - \lambda C_{yx} + \lambda C_y^2 + + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + \frac{C_{x(2)}^2}{4} - C_{yx(2)}\right)\right) \tag{20}$$

Cochran [4] proposed a classical regression estimator under non-response by adapting the estimator in Eq. (7) as follows:

$$t_{reg}^{**} = \bar{y}^* + b^*(\bar{X} - \bar{x}^*), \tag{21}$$

whose MSE is given as:

$$MSE\left(t_{reg}^{**}\right) = \lambda C_y^2 \bar{Y}^2\left(1-\rho_{xy}^2\right) + \bar{Y}^2\left(\frac{W_2(z-1)}{n}\left(C_{y(2)}^2 - 2\rho_{xy}\frac{C_y}{C_x}C_{yx(2)} + \rho_{xy}^2\frac{C_y^2}{C_x^2}C_{x(2)}^2\right)\right). \quad (22)$$

After Singh *et al*. [16], Kumar and Bhougal [28], Kumar [29], Yadav *et al.* [22], Pal and Singh [20], Singh and Usman [19], Ünal and Kadilar [30, 31], Muneer *et al.* [32], Sanaullah *et al.* [26], Riaz *et al.* [33], Sinha and Kumar [23], Pal and Singh [24], Kumar and Kumar [25] and Sanaullah *et al.* [26] also proposed new estimators, taking advantage of an exponential function for Case II.

In this study, families of estimators taking advantage of an exponential function to estimate the population mean by adapting the estimator in Eq. (7) are proposed for non-response situations. The properties will be examined in Section 2 and comparisons between the proposed estimator and existing estimators from the literature will be made in Section 3 and Section 4, respectively.

2 The Proposed Families of Estimators

Based on Yadav and Kadilar [7], we adapt the exponential type estimators in Eq. (5) to a family of estimators, taking advantage of the exponential function for the population mean for Case I and Case II.

2.1 Case I:

We propose the following family of estimators for the first case:

$$t_{CC1,i} = k\bar{y}^* exp\left[\frac{(a_i\bar{X}+b_i)-(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)+(a_i\bar{x}+b_i)}\right], \qquad i = 1,\ldots,10. \quad (23)$$

Here, k is a chosen constant to make $MSE\left(t_{CC1,i}\right), i = 1,\ldots,10$ min and a_i, b_i either functions of known parameters of x, such as, $\beta_2(x)$, C_x etc. or real numbers.

To obtain expressions for $B\left(t_{CC1,i}\right)$ and $MSE\left(t_{CC1,i}\right), i = 1,\ldots,10$, we consider:

$$\bar{y}^* = \left(\bar{Y}e_y^* + \bar{Y}\right), \bar{x} = (\bar{X}e_x + \bar{X}),$$

Then, $E(e_x) = 0$, $E\left(e_y^* e_x\right) = \lambda C_{xy}$, $E\left(e_y^*\right) = 0$, $E\left({e_y^*}^2\right) = \lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2$. $E(e_x^2) = \lambda C_x^2$,

Now, expressing $t_{CC1,i}, i = 1,\ldots,10$ in terms of e_x and e_y^*, we have:

$$t_{CC1,i} = \bar{Y}\left(k + ke_y^*\right) exp\left(\frac{a_i\bar{X}+b_i-a_i\bar{X}-a_i\bar{X}e_x-b_i}{a_i\bar{X}+b_i+a_i\bar{X}+a_i\bar{X}e_x+b_i}\right), \quad (24)$$

$$= k\bar{Y}\left(1+e_y^*\right) exp\left[\frac{-\theta_i e_x}{2}\left(1+\frac{\theta_i e_x}{2}\right)^{-1}\right] \quad (25)$$

$$= \bar{Y}\left(k + ke_y^* - \frac{k\theta_i}{2}e_x + \frac{3k\theta_i^2}{8}e_x^2 - \frac{k\theta_i}{2}e_y^* e_x\right), i = 1,\ldots,10, \quad (26)$$

where $\theta_i = \frac{a_i\bar{X}}{a_i\bar{X}+b_i}$.

Expanding the right-hand side of Eq. (26), we have:

$$\left(t_{CC1,i} - \bar{Y}\right) = \bar{Y}\left(ke_y^* - \frac{k\theta_i}{2}e_x + \frac{3k\theta_i^2}{8}e_x^2 - \frac{k\theta_i}{2}e_y^* e_x\right) + \bar{Y}(k-1). \quad (27)$$

We take the expectation on both sides of Eq. (27) as the bias and we get:

$$B\left(t_{CC1,i}\right) = \bar{Y}\left((k-1) + \frac{3k\theta_i^2}{8}\lambda C_x^2 - \frac{k\theta_i}{2}\lambda\rho_{xy}C_x C_y\right), i = 1,\ldots,10. \quad (28)$$

We take the square of both sides of $\left(t_{CC1,i} - \bar{Y}\right)$ and then we take the expectation, so we obtain $MSE\left(t_{CC1,i}\right), i = 1,\ldots,10$ as:

$$MSE\left(t_{CC1,i}\right) = \bar{Y}^2(k^2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2\right) + (k-1)^2 + \lambda C_x^2\left(k^2\theta_i^2 - \frac{3k\theta_i^2}{4}\right) - \lambda\rho_{xy}C_y C_x(2k^2\theta_i - k\theta_i)),$$

$$i = 1,\ldots,10. \quad (29)$$

After obtaining the optimal *k* as:

$$k^* = \frac{A_1}{A_2}. \quad (30)$$

here,

$$A_1 = \lambda\left(\frac{3}{4}\theta_i^2 C_x^2 - \theta_i C_{yx}\right) + 2$$

and

$$A_2 = 2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n}C_{y(2)}^2 + \lambda\theta_i^2 C_x^2 - 2\lambda\theta_i C_{yx} + 1\right),$$

we have the min $MSE\left(t_{CC1,i}\right), i = 1,\ldots,10$ estimators as follows:

$$MSE_{min}\left(\bar{t}_{CC1,\iota}\right)^2\left(1 - \frac{A_1^2}{2A_2}\right), \quad i = 1,\ldots,10. \quad (31)$$

Some members of the estimators in Eq. (23) are given in Table 1.

2.2 Case II:

We propose the family of estimators for the second case as follows:

$$t_{CC2,i} = k\bar{y}^* \, exp\, exp\, \left[\frac{(a_i\bar{X}+b_i)-(a_i\bar{x}^*+b_i)}{(a_i\bar{X}+b_i)+(a_i\bar{x}^*+b_i)}\right], i = 1,\dots,10. \quad (32)$$

Table 1 Some members of the family of estimators.

Values			Estimators
θ_i	a_i	b_i	
θ_1	1	1	$t_{CC1,1} = k\bar{y}^* exp \left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2}\right]$
θ_2	1	$\beta_2(x)$	$t_{CC1,2} = k\bar{y}^* exp \left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\beta_2(x)}\right]$
θ_3	1	C_x	$t_{CC1,3} = k\bar{y}^* exp \left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2C_x}\right]$
θ_4	1	ρ	$t_{CC1,4} = k\bar{y}^* exp \left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho}\right]$
θ_5	$\beta_2(x)$	C_x	$t_{CC1,5} = k\bar{y}^* exp \left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right]$
θ_6	C_x	$\beta_2(x)$	$t_{CC1,6} = k\bar{y}^* exp \left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right]$
θ_7	C_x	ρ	$t_{CC1,7} = k\bar{y}^* exp \left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\rho}\right]$
θ_8	ρ	C_x	$t_{CC1,8} = k\bar{y}^* exp \left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2C_x}\right]$
θ_9	$\beta_2(x)$	ρ	$t_{CC1,9} = k\bar{y}^* exp \left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right]$
θ_{10}	ρ	$\beta_2(x)$	$t_{CC1,10} = k\bar{y}^* exp \left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right]$

Using Table 1, we can write some members of $t_{CC2,i}, i = 1,\dots,10$ for Case II.

To obtain $B(t_{CC2,i})$ and $MSE(t_{CC2,i})$ we consider:

$$\bar{y}^* = (\bar{Y} + \bar{Y}e_y^*), \bar{x}^* = (\bar{X} + \bar{X}e_x^*),$$

Then, we have:

$$E(e_x^*) = 0, \qquad E\left(e_x^{*2}\right) = \lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2, \qquad E(e_y^*) = 0,$$

$$E\left(e_y^{*2}\right) = \lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2, \qquad E\left(e_x^* e_y^*\right) = \lambda C_{xy} + \frac{W_2(z-1)}{n} C_{xy(2)}.$$

Now, expressing $t_{CC2,i}, i = 1,\dots,10$ estimators in Eq. (32), we get:

$$t_{CC2,i} = \bar{Y}(k + ke_y^*)\, exp\, exp\, \left(\frac{a\bar{X}+b-a\bar{X}-a\bar{X}e_x^*-b}{a\bar{X}+b+a\bar{X}+a\bar{X}e_x^*+b}\right), \quad (33)$$

$$= k\bar{Y}(1 + e_0^*)\, exp\, exp\, \left[\frac{-\theta_i e_1^*}{2}\left(1 + \frac{\theta_i e_1^*}{2}\right)^{-1}\right] \quad (34)$$

$$= \bar{Y}\left(k + ke_y^* - \frac{k\theta_i e_x^*}{2} + \frac{3k\theta_i^2 e_x^{*2}}{8} - \frac{k\theta_i}{2} e_y^* e_x^*\right), i = 1,\ldots,10. \quad (35)$$

Using the same procedure as was used for the first proposed family, we obtain $B(t_{CC2,i})$ and $MSE(t_{CC2,i})$ as follows:

$$B(t_{CC2,i}) = \bar{Y}(\frac{3k\theta_i^2}{8}\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right) + (k-1) - \frac{k\theta_i}{2}\left(\lambda\rho_{yx} C_y C_x + \frac{W_2(z-1)}{n} C_{yx(2)}\right)), \quad i = 1,\ldots.,10, \quad (36)$$

$$MSE(t_{CC2,i}) = \bar{Y}^2(k^2\left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) + (k-1)^2 + k\theta_i^2\left(k - \frac{3}{4}\right)\left(\lambda C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2\right) - k\theta_i(2k - 1)\left(\lambda C_{yx} + \frac{W_2(z-1)}{n} C_{yx(2)}\right)), \quad i = 1,\ldots,10. \quad (37)$$

Minimization of $MSE(t_{CC2,i})$ with respect to k, the optimal k is obtained as:

$$k^{**} = \frac{B_1}{B_2}, \quad (38)$$

where

$$B_1 = \left(\lambda\left(\frac{3}{4}\theta_i^2 C_x^2 - \theta_i C_{yx}\right) + \frac{W_2(z-1)}{n}\left(\frac{3}{4}\theta_i^2 C_{x(2)}^2 - \theta_i C_{yx(2)}\right) + 2\right)$$

and

$$B_2 = 2\left(\lambda\left(C_y^2 - 2\theta_i C_{yx} + \theta_i^2 C_x^2\right) + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + \theta_i^2 C_{x(2)}^2 - 2\theta_i C_{yx(2)}\right) + 1\right).$$

Using k^{**} in $MSE(t_{CC2,i})$, we have min $MSE(t_{CC2,i})$, $i = 1,\ldots,10$ estimators as follows:

$$MSE_{min}(t_{CC2,i})^2 = \bar{Y}^2\left(1 - \frac{B_1^2}{2B_2}\right), \; i = 1,\ldots,10 \quad (39)$$

3 Efficiency Comparisons

Now we will investigate the efficiencies of $t_{CC1,i}$ and $t_{CC2,i}$, $i = 1,\ldots,10$ given in Eq. (23) and Eq. (32) with various estimators from the literature for Case I and Case II.

3.1 Efficiency Comparisons for Case I

Using Eqs. (10), (13), (14), (16) and (31) we find the efficiency conditions of $t_{CC1,i}, i = 1,\ldots,10$ as follows:

i) $V(t_{HH}) - MSE(t_{CC1,i})_{min} > 0$

$$\left(1-\frac{A_1^2}{2A_2}\right) < \left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) \tag{40}$$

ii) $MSE(t_R^*) - MSE\left(t_{CC1,i}\right)_{min} > 0$

$$\left(1-\frac{A_1^2}{2A_2}\right) < \left(\lambda\left(-2C_{yx} + C_x^2 + C_y^2\right) + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) \tag{41}$$

iii) $MSE(t_{BT}^*) - MSE\left(t_{CC1,i}\right)_{min} > 0$

$$\left(1-\frac{A_1^2}{2A_2}\right) < \left(\lambda\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right) + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) \tag{42}$$

iv) $MSE\left(t_{reg}^*\right) - MSE\left(t_{CC1,i}\right)_{min}$

$$\left(1-\frac{A_1^2}{2A_2}\right) < \left(\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) \tag{43}$$

3.2 Efficiency Comparisons for Case II

Using Eqs. (10), (18), (20), (22) and (39) for $t_{CC2,i}, i = 1,\ldots,10$ we have:

i) $V(t_{HH}) - MSE\left(t_{CC2,i}\right)_{min} > 0$

$$\left(1-\frac{B_1^2}{2B_2}\right) < \left(\lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2\right) \tag{44}$$

ii) $MSE(t_R^{**}) - MSE\left(t_{CC2,i}\right)_{min} > 0$

$$\left(1-\frac{B_1^2}{2B_2}\right) < \left(\lambda\left(C_y^2 + C_x^2 - 2C_{yx}\right) + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + C_{x(2)}^2 - 2C_{yx(2)}\right)\right) \tag{45}$$

iii) $MSE(t_{BT}^{**}) - MSE\left(t_{CC2,i}\right)_{min} > 0$

$$\left(1-\frac{B_1^2}{2B_2}\right) < \left(\lambda\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right) + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + \frac{C_{x(2)}^2}{4} - C_{yx(2)}\right)\right) \tag{46}$$

iv) $MSE\left(t_{reg}^{**}\right) - MSE\left(t_{CC2,i}\right)_{min} > 0$

$$\left(1-\frac{B_1^2}{2B_2}\right) < (\lambda C_y^2\left(1-\rho_{xy}^2\right) + \frac{W_2(z-1)}{n}\left(C_{y(2)}^2 + \rho_{xy}^2 \frac{C_y^2}{C_x^2} C_{x(2)}^2 - 2\rho_{yx}\frac{C_y}{C_x} C_{yx(2)}\right) \tag{47}$$

When the conditions Eq. (40-43) and Eq. (44-47) are satisfied, we infer that $t_{CC1,i}$ and $t_{CC2,i}$ are more efficient than the other estimators for both i values, respectively.

4 Numerical Illustrations

We used different data sets considered by Khare and Sinha [34] and Khare and Srivastava [35] for Case I and II, respectively, to examine the performances of $t_{CC1,i}$ and $t_{CC2,i}, i = 1,\dots,10$ in practice compared to other estimators from the literature.

The percent relative efficiencies (PREs) were also computed using various values of *z* for both cases separately by using the following formula:

$$PRE(*, t_{HH}) = \frac{V(t_{HH})}{MSE(*)} \times 100.$$

4.1 Numerical Illustration of Case I

We used the data set from Khare and Sinha [34] for Case I. The descriptive statistics are given in Table 2. Note that for Population 1:

Table 2 Parameter values for Population I.

$N = 96, n$ =40	$\bar{Y}$ =137.92	C_y =1.32	$\rho_{yx(2)}$ =0.72	$C_{yx(2)}$ =1.408
f =0.4167	$\bar{X}$ =144.87	C_x =0.81	ρ_{yx} =0.77	C_{yx} =0.823
W_2 =0.25	λ =0.0146	$C_{x(2)}$ =0.94	$C_{y(2)}$ =2.08	$\beta_2(x)$ =1.2

The MSE values of $t_{CC1,i}, i = 1,\dots,10$ and $t_{HH}, t_R^*, t_{BT}^*, t_{reg}^*$ for various values of *z* for Case I are given in Table 3.

Table 3 MSE Values of $(t_{CC1,i}, i = 1,\dots,10)$ and other estimators for Population I.

	***z*=4**	***z*=5**	***z*=6**	***z*=7**	***z*=8**
t_{HH}	2026.406	2540.7587	3055.112	3569.4645	4083.817
t_R^*	1751.647	2265.9998	2780.353	3294.7056	3809.058
t_{BT}^*	1843.525	2357.8782	2872.231	3386.584	3900.937
t_{reg}^*	1739.829	2254.1822	2768.535	3282.888	3797.241
$t_{CC1,1}$	1688.787	2107.573	2506.6	2887.233	3250.717
$t_{CC1,2}$	1688.948	2107.728	2506.748	2887.376	3250.855
$t_{CC1,3}$	1688.634	2107.426	2506.458	2887.097	3250.586
$t_{CC1,4}$	1688.601	2107.395	2506.428	2887.068	3250.558
$t_{CC1,5}$	1688.525	2107.321	2506.357	2887.0001	3250.492
$t_{CC1,6}$	1689.175	2107.945	2506.957	2887.577	3251.048
$t_{CC1,7}$	1688.747	2107.535	2506.563	2887.198	3250.683
$t_{CC1,8}$	1688.829	2107.613	2506.638	2887.27	3250.753
$t_{CC1,9}$	1688.498	2107.295	2506.332	2886.976	3250.469
$t_{CC1,10}$	1689.237	2108.004	2507.014	2887.632	3251.101

Based on these results we conclude that all estimators in $(t_{CC1,i}, i = 1,\dots,10)$ are more efficient than the other estimators in Case I.

Table 4 PREs of $(t_{CC1,i}, i = 1,\ldots,10)$ and other estimators for Population I.

	z=4	z=5	z=6	z=7	z=8
t_{HH}	100	100,000	100	100	100
t_R^*	115,6858	112,125	109,8822	108,3394	107,2133
t_{BT}^*	109,9202	107,756	106,3672	105,4001	104,6881
t_{reg}^*	116,4716	112,713	110,3512	108,7294	107,547
$t_{CC1,1}$	119,9918	120,554	121,8827	123,6292	125,6282
$t_{CC1,2}$	119,9804	120,545	121,8755	123,6231	125,6229
$t_{CC1,3}$	120,0027	120,562	121,8896	123,6351	125,6333
$t_{CC1,4}$	120,005	120,564	121,8911	123,6363	125,6343
$t_{CC1,5}$	120,0104	120,568	121,8945	123,6392	125,6369
$t_{CC1,6}$	119,9642	120,532	121,8654	123,6145	125,6154
$t_{CC1,7}$	119,9946	120,556	121,8845	123,6307	125,6295
$t_{CC1,8}$	119,9888	120,551	121,8809	123,6277	125,6268
$t_{CC1,9}$	120,0123	120,570	121,8957	123,6403	125,6378
$t_{CC1,10}$	119,9598	120,529	121,8626	123,6122	125,6134

According to the Table 4, the PRE values of $(t_{CC1,i}, i = 1,\ldots,10)$, especially $t_{CC1,9}$, were better compared to those of the other estimators. We also found that the PRE values of $t_{CC1,i}, i = 1,\ldots,10$ increased with increasing values of *z*.

4.2 Numerical Illustration for Case II

We used the data set from Khare and Srivastava [35] for Case II and the descriptive statistics are given in Table 5. Note that for Population 2:

Table 5 Values of the parameters for Population II.

$N = 70, n = 35$	$\bar{X} = 1755.53$	$\lambda = 0.014$	$\rho_{yx} = 0.778$	$C_{yx} = 0.39$
$f = 0.50$	$\bar{Y} = 981.29$	$C_x = 0.801$	$\rho_{yx(2)} = 0.445$	$C_{yx(2)} = 0.104$
$W_2 = 0.2$	$C_y = 0.625$	$C_{x(2)} = 0.5739$	$C_{y(2)} = 0.409$	$\beta_2(x) = 0.34$

The MSE values of the $t_{CC2,i}, i = 1,\ldots,10$ and $t_{HH}, t_R^{**}, t_{BT}^{**}, t_{reg}^{**}$ estimators for various values of *z* in Case II are given in Table 6.

According to the MSE values, the $t_{CC2,i}, i = 1,\ldots,10$ estimators are more efficient than the other estimators in a non-response situation both in *y* and *x*. The PRE values which are given in Table 7, were also better compared to those of the other estimators. Furthermore, we also found that the PRE values of $(t_{CC2,i}, i = 1,\ldots,10)$ decreased with increasing values of *z*.

Table 6 MSE values of $\left(t_{CC2,i}, i = 1,\ldots,10\right)$ and other estimators for Population II.

	z=4	**z=5**	**z=6**	**z=7**	**z=8**
t_{HH}	8137.694	9056.8011	9975.908	10895.0147	11814.12
t_R^{**}	8231.209	9813.9618	11396.72	12979.4683	14562.22
t_{BT}^{**}	4619.291	5417.1474	6215.003	7012.8594	7810.715
t_{reg}^{**}	4794.201	5684.361	6574.521	7464.6817	8354.842
$t_{CC2,1}$	4576.81	5357.387	6135.116	6910.009	7682.081
$t_{CC2,2}$	4576.791	5357.478	6135.314	6910.313	7682.488
$t_{CC2,3}$	4576.804	5357.415	6135.176	6910.101	7682.204
$t_{CC2,4}$	4576.804	5357.418	6135.183	6910.111	7682.218
$t_{CC2,5}$	4576.852	5357.204	6134.712	6909.39	7681.251
$t_{CC2,6}$	4576.793	5357.467	6135.289	6910.274	7682.436
$t_{CC2,7}$	4576.809	5357.391	6135.125	6910.023	7682.099
$t_{CC2,8}$	4576.811	5357.383	6135.107	6909.996	7682.063
$t_{CC2,9}$	4576.85	5357.213	6134.732	6909.42	7681.292
$t_{CC2,10}$	4576.794	5357.465	6135.285	6910.268	7682.428

Table 7 PREs of $\left(t_{CC2,i}, i = 1,\ldots,10\right)$ and Others for Population II.

	z=4	z=5	z=6	z=7	z=8
t_{HH}	100	100,000	100	100	100
t_R^{**}	98,8639	92,285	87,53315	83,94038	81,12856
t_{BT}^{**}	176,1676	167,188	160,5133	155,3577	151,2553
t_{reg}^{**}	169,7404	159,328	151,7359	145,9542	141,4045
$t_{CC2,1}$	177,8027	169,053	162,6034	157,6701	153,788
$t_{CC2,2}$	177,8035	169,050	162,5982	157,6631	153,7799
$t_{CC2,3}$	177,803	169,052	162,6018	157,668	153,7856
$t_{CC2,4}$	177,803	169,052	162,6016	157,6677	153,7853
$t_{CC2,5}$	177,8011	169,058	162,6141	157,6842	153,8046
$t_{CC2,6}$	177,8034	169,050	162,5988	157,664	153,7809
$t_{CC2,7}$	177,8028	169,052	162,6032	157,6697	153,7877
$t_{CC2,8}$	177,8027	169,053	162,6037	157,6703	153,7884
$t_{CC2,9}$	177,8012	169,058	162,6136	157,6835	153,8038
$t_{CC2,10}$	177,8034	169,050	162,5989	157,6641	153,7811

5 Conclusion

We proposed families of estimators, $t_{CC1,i}, t_{CC2,i}; i = 1,\ldots,10$, taking advantage of an exponential function for estimating the population mean under non-response for two cases. Equations for the bias and minimum MSE of $t_{CC1,i}, t_{CC2,i}; i = 1,\ldots,10$ were also obtained for both cases. In this way, the $t_{CC1,i}, t_{CC2,i}; i = 1,\ldots,10$ estimators were found to be more efficient in theory than the other estimators under the obtained conditions. Using the data sets from Khare and Sinha [34] and Khare and Srivastava [35], we concluded that $t_{CC1,i}$ and $t_{CC2,i}$ are more efficient than the other estimators in Case I and II, respectively. Therefore, the proposed $t_{CC1,i}, t_{CC2,i}; i = 1, \ldots, 10$ estimators are recommended for both cases of non-response situations based on the obtained results.

References

[1] Yadav, S.K. & Mishra, S.S., *Developing Improved Predictive Estimator for Finite Population Mean Using Auxiliary Information*, Statistika-Statistics and Economy Journal, **95**(1), pp. 76-85, 2015.

[2] Yadav, S.K., Dixit, M.K., Dungana, H.N. & Mishra, S.S., *Improved Estimators for Estimating Average Yield Using Auxiliary Variable*, International Journal of Mathematical, Engineering and Management Sciences, **4**(5), pp. 1228-1238, 2019.

[3] Cochran, W.G., *The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce*, The Journal of Agricultural Science, **30**(2), pp. 262-275, 1940.

[4] Cochran, W.G., *Sampling Techniques*, John Wiley and Sons, New York, 1977.

[5] Solanki, R.S., Singh, H.P. & Rathaur, A., *An Alternative Estimator for Estimating the Finite Population Mean Using Auxiliary Information in Sample Surveys*, International Scholarly Research Network Probability and Statistics, pp. 1-14, 2012.

[6] Bahl, S. & Tuteja, R.K., *Ratio and Product Type Exponential Estimators*, Journal of Information and Optimization Sciences, **12**(1), pp. 159-164, 1991.

[7] Yadav, S.K. & Kadilar, C., *Efficient Family of Exponential Estimators for the Population Mean*, Hacettepe Journal of Mathematics and Statistics, **42**(6), pp. 671-677, 2013.

[8] Singh, H.P., Solanki, R.S. & Singh, A.K., *Predictive Estimation of Finite Population Mean Using Exponential Estimators*, Statistika: Statistics and Economy Journal, **94**(1), pp. 41-53, 2014.

[9] Kumar, A. & Saini, M., *Exponential Estimators for Finite Population Mean Using Auxiliary Attribute: A Predıctıve Approach*, **41**(7), pp. 999-1009, 2020.

[10] Singh, R., Chauhan, P., Sawan, N. & Smarandache, F., *Ratio-Product Type Exponential Estimator for Estimating Finite Population Mean Using Information on Auxiliary Attribute, Auxiliary Information and A Priori Values in Construction of Improved Estimators*, **1**, pp. 18-32, 2007.

[11] Hansen, M.H. & Hurwitz, W.N., *The Problem of Non-Response in Sample Surveys*, Journal of the American Statistical Association, **41**(236), pp. 517-529, 1946.

[12] Srinath, K.P., *Multiphase Sampling in Nonresponse Problems*, J. Am. Statist. Assoc. **66**, pp. 583-658, 1971.

[13] Bouza, C.N., *Sobre El Problema De La Fracción De Submuestreo Para El Caso De Las No Respuestas*, Trabajos de Estadística y de Investigación Operativa, **32**, pp. 30-36, 1981.

[14] Bouza-Herrera, C.N., *Subsampling Rules for Item Non-Response of an Estimator Based on the Combination of Regression and Ratio*, Journal of King Saud University-Science, **31**, pp. 171-176, 2019.

[15] Rao, P.S.R.S., *Ratio Estimation with Sub Sampling the Non-Respondents*, Survey Methodology, **12**, pp. 217-230, 1986.

[16] Singh, R., Kumar, M., Chaudhary, M.K. & Smarandache, F., *Estimation of Mean in Presence of Non-Response Using Exponential Estimator*, arXiv preprint arXiv:0906.2462, 2009.

[17] Yunusa, O. & Kumar, S., *Ratio-Cum-Product Estimator Using Exponential Estimator in the Presence of Non-Response*, Journal of Advanced Computing, **3**(1), pp. 1-11, 2014.

[18] Dansawad, N., *A Class of Exponential Estimator to Estimate the Population Mean in the Presence of Non-Response*, Naresuan University Journal: Science and Technology (NUJST), **27**(4), pp. 20-26, 2019.

[19] Singh, G.N. & Usman, M., *Efficient Combination of Various Estimators in the Presence of Non-Response*, Communications in Statistics-Simulation and Computation, pp. 1-35, 2019. DOI:10.1080/03610918.2019.1614618.

[20] Pal, S.K. & Singh, H.P., *A Class of Ratio-Cum-Ratio-Type Exponential Estimators for Population Mean with Sub Sampling the Non-Respondents*, Jordan Journal of Mathematics and Statistics, **10**(1), pp. 73-94, 2017.

[21] Pal, S.K. & Singh, H.P., *Finite Population Mean Estimation Through a Two-Parameter Ratio Estimator Using Auxiliary Information in Presence of Non-Response*, Journal of Applied Mathematics, Statistics, and Informatics, **12**(2), pp. 5-39, 2016.

[22] Yadav, S.K., Subramani, J., Misra, S., Singh, L. & Mishra, S.S., *Improved Estimation of Population Mean in Presence of Non-Response Using Exponential Estimator*, International Journal of Agricultural and Statistical Sciences, **12**(1), pp. 271-276, 2016.

[23] Sinha, R.R. & Kumar, V., *Regression Cum Exponential Estimators for Finite Population Mean Under Incomplete Information*, Journal of Statistics and Management Systems, **20**(3), pp. 355-368, 2017. DOI:10.1080/09720510.2017.1304922.

[24] Pal, S.K. & Singh H.P., *Estimation of Finite Population Mean Using Auxiliary Information in Presence of Non-Response*, Communications in Statistics – Simulation and Computation, **47**(1), pp. 143-165, 2018. DOI:10.1080/03610918.2017.1280161.

[25] Kumar, K. & Kumar, M., *Improved Exponential Ratio and Product Type Estimators for Population Mean in the Presence of Nonresponse*, Bulletin of Mathematics and Statistics Research, **5**(2), pp. 68-76, 2017.

[26] Sanaullah, A., Ehsan, I. & Noor-Ul-Amin, M., *Estimation of Mean for A Finite Population Using Sub-Sampling of Non-Respondents*, Journal of Statistics and Management Systems, **22**(6), pp. 1015-1035, 2019.

[27] Javaid, A., Noor-ul-Amin, M. & Hanif, M., *Modified Ratio Estimator in Systematic Random Sampling Under Non-Response*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **89**(4), pp. 817-825, 2019.

[28] Kumar, S. & Bhougal, S., *Estimation of the Population Mean in Presence of Non-Response*, Communications for Statistical Applications and Methods, **18**(4), pp. 537-548, 2011.

[29] Kumar, S., *Improved Exponential Estimator for Estimating the Population Mean in The Presence of Non-Response*, Communications for Statistical Applications and Methods, **20**(5), pp. 357-366, 2013.

[30] Ünal, C. & Kadilar, C., *Exponential Type Estimator for The Population Mean in the Presence of Non-Response*, Journal of Statistics and Management Systems, **23**(3), pp. 603-615, 2019.

[31] Ünal, C. & Kadilar, C., *Improved Family of Estimators Using Exponential Function for The Population Mean in the Presence of Non-Response*, Communications in Statistics – Theory and Methods, **50**(1), pp. 237-248, 2019.

[32] Muneer, S., Shabbir, J. & Khalil, A., *A Generalized Exponential Type Estimator of Population Mean in the Presence of Non-Response,* Statistics in Transition New Series, **19**(2), pp. 259-276, 2018.

[33] Riaz, S., Nazeer, A., Abbası, J. & Qamar, S., *On the Generalized Class of Estimators for Estimation of Finite Population Mean in the Presence of Non-Response Problem*, Journal of Prime Research in Mathematics, **16**(1), pp. 52-63, 2020.

[34] Khare, B.B. & Sinha, R.R., *On Class of Estimators for Population Mean Using Multi-Auxiliary Characters in the Presence of Non-Response*, Statistics in Transition, **10**(1), pp. 3-14, 2009.

[35] Khare, B.B. & Srivastava, S., *Estimation of Population Mean Using Auxiliary Character in Presence of Non-Response*, The National Academy of Science Letters, **16**, pp. 111-114, 1993.