On Subspace-ergodic Operators

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Abstract. In this paper, we define subspace-ergodic operators and give examples of these operators. We show that by any given separable infinite dimensional Banach space, subspace-ergodic operators can be constructed. We demonstrate that an invertible operator T is subspace-ergodic if and only if T^{-1} is subspace-ergodic. We prove that the direct sum of two subspace-ergodic operators is subspace-ergodic and if the direct sum of two operators is subspace-ergodic, then each of them is subspace-ergodic. Also, we investigate relations between subspace-ergodic and subspace-mixing operators. For example, we show that if T is subspace-mixing and invertible, then T^n and T^{-n} are subspace-ergodic for any $n \in \mathbb{N}$.

Keywords: *ergodic operators; mixing operators; subspace-ergodic operators; subspace-mixing operators.*

1 Introduction

Let X be a complex and separable Banach space and $B(X)$ be the set of all bounded linear operators on X. Let $\mathbb{N}_0$ be the set of non-negative integers and let $\mathbb{N}$ be the set of natural numbers. We say that T is topologically transitive if for any non-empty open sets $U \subseteq X$ and $V \subseteq X$, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V \neq \emptyset$. One can read more information about these operators in [1-3]. In the statement of [3] and [4], an operator $T \in B(X)$ is called mixing, if for any two non-empty open sets $U \subseteq X$ and $V \subseteq X$, there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V$ non-empty for every $n \geq N$.

Costakis and Sambarino showed in [4] that $T = \lambda B$ is mixing, where λ is a scalar with $|\lambda| > 1$ and B is the backward shift on l^2. It is interesting that one can construct mixing operators on every infinite-dimensional separable Banach space [5].

Theorem 1.1. If X is any infinite-dimensional separable Banach space, then X supports a mixing operator [5].

Let $T: X \to X$ be an operator. Then for any sets $A \subseteq X$ and $B \subseteq X$, the return set from A to B is defined as:

$$N_T(A, B) = \{n \in \mathbb{N}_0; T^n(A) \cap B \neq \emptyset\}.$$

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So, if an operator T is topologically transitive, then $N_T(U,V)$ is non-empty for any open sets $U \subseteq X$ and $V \subseteq X$. As it mentioned in [1], if T is topologically transitive, then $N_T(U,V)$ is infinite. Note that if T is mixing, then $N_T(U,V)$ is cofinite. Remember that we say a set S is cofinite if $\mathbb{N} \setminus S$ is finite.

We call a strictly increasing sequence $(n_k)_k$ of positive integers syndetic if

$$sup_{k\geq 1}(n_{k+1} - n_k) < \infty.$$

We say a subset A of $\mathbb{N}_0$ is syndetic if the strictly increasing sequence of positive integers forming A is syndetic [1]. The complement of syndetic sets does not contain arbitrary long intervals.

We say an operator $T \in B(X)$ is topologically ergodic if for any pair of non-empty open sets $U \subseteq X$ and $V \subseteq X$, $N_T(U,V)$ is syndetic [1]. It is not hard to see that mixing operators are topologically ergodic and topologically ergodic operators are topologically transitive. Grosse-Erdmann and Peris proved in [6] that ergodic operators are weakly-mixing. An operator T is called weakly-mixing if $T \oplus T$ is topologically transitive. Remember that if X and Y are two Banach spaces, then $X \oplus Y = \{(x,y); x \in X, y \in Y\}$ and if $S \in B(X)$ and $T \in B(Y)$, then the operator $S \oplus T: X \oplus Y \to X \oplus Y$ is defined by $(S \oplus T)(x,y) = (Sx, Ty)$.

Madore and Martinez-Avendano introduced subspace-transitive operators in [7]. An operator T is called subspace-transitive with respect to a closed and non-trivial subspace M of X if for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$, $T^{-n}(U) \cap V$ contains a non-empty open subset of M for some $n \in \mathbb{N}_0$. They also defined subspace-hypercyclic operators. For more information, see [8-10]. Also, in [11], one can read interesting properties of subspace-supercyclic operators.

Talebi and Moosapoor defined subspace-mixing operators in [12]. Let M be a closed and non-empty subspace of X. We say an operator $T \in B(X)$ is M-mixing if for any relatively open sets $U \subseteq M$ and $V \subseteq M$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $T^n(U) \cap V \neq \emptyset$.

In this paper, we define subspace-ergodic operators and give examples of these operators. We show that by any given separable infinite-dimensional Banach space, we can construct subspace-ergodic operators. We demonstrate that an invertible operator T is subspace-ergodic if and only if T^{-1} is subspace-ergodic. We prove that the direct sum of two subspace-ergodic operators is subspace-ergodic. Also, we prove that if the direct sum of two operators is subspace-ergodic, then each of them is subspace-ergodic. Moreover, we investigate relations between subspace-ergodic operators and subspace-mixing operators. For example, we show that if T is subspace-mixing and invertible, then T^n and T^{-n} are subspace-ergodic for any $n \in \mathbb{N}$.

2 Definitions and Some Results

First, we define the return set with respect to a subspace. As usual, when we talk about a subspace, it is considered a closed subspace. Also, the idea of this paper is given from subspace-hypercyclic and subspace-mixing operators. So, in the following definitions we will assume that M is a closed subspace.

Definition 2.1. Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. For $A \subseteq M$ and $B \subseteq M$, we define the return set from A to B with respect to M as follows:

$$N_T(A,B)_M = \{n \in \mathbb{N}_0; T^n(A) \cap B \neq \emptyset\}.$$

By Definition 2.1, if T is M-transitive, then $N_T(U,V)_M$ is non-empty for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$. Also, if T is an M-mixing operator, then $N_T(U,V)_M$ is cofinite.

Now we define subspace-ergodic operators as follows:

Definition 2.2. Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. We say that T is M-ergodic or subspace-ergodic with respect to M if for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$, the set $N_T(U,V)_M$ is syndetic.

By definition, it is clear that any ergodic operator is subspace-ergodic since it is sufficient to consider $M := X$.

Example 2.3. Let $T \in B(X)$ be an ergodic operator and let I be the identity operator on X. Then $T^p \oplus \alpha I$ is M-ergodic with respect to $M \coloneqq X \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar α.

Proof. Let α be a scalar. First, we show that $T \oplus \alpha I$ is M-ergodic. Let U_1 and V_1 be non-empty relatively open subsets of M. Thus, there exist non-empty open subsets U and V of X such that $U_1 = U \oplus \{0\}$ and $V_1 = V \oplus \{0\}$. By hypothesis, T is an ergodic operator. Thus, the set $\{n \in \mathbb{N}_0; T^n(U) \cap (V) \neq \emptyset\}$ is syndetic. Note that we have

$$\begin{aligned}(T \oplus \alpha I)^n(U \oplus \{0\}) \cap (V \oplus \{0\}) &= (T^n(U) \oplus \{0\}) \cap (V \oplus \{0\}) \\ &= (T^n(U) \cap V) \oplus (\{0\} \cap \{0\}) \quad (1) \\ &= (T^n(U) \cap V) \oplus \{0\}.\end{aligned}$$

By Eq. (1), we deduce that $\{n \in \mathbb{N}_0 : (T \oplus \alpha I)^n(U_1) \cap (V_1) \neq \emptyset\}$ is syndetic too. Therefore, $T \oplus \alpha I$ is M-ergodic. Now if T is an ergodic operator, then T^p is ergodic for any $p \in \mathbb{N}$ [1, p.62]. Hence, similar to what was shown, $T^p \oplus \alpha I$ is M-ergodic for any $p \in \mathbb{N}$. On the other hand, α is an arbitrary scalar. So, for any scalar α, $T^p \oplus \alpha I$ is M-ergodic.

Similarly, $\alpha I \oplus T^p$ is N-ergodic with respect to $N := \{0\} \oplus X$ for any $p \in \mathbb{N}$ and for any scalar α.

By the above examples, we can gain more subspace-ergodic operators from known ergodic operators. For instance, note to the following example:

Example 2.4. Let T be a weighted shift on l^2 given by

$$T(x_1, x_2, x_3, x_4, \dots) = \left(2x_2, \frac{3}{2}x_3, \frac{4}{3}x_4, \dots\right).$$

As was proved in [1, Example 2.39], for any two non-empty open subsets U and V of l^2, we have $N_T(U, V)$ is syndetic. Hence, T is an ergodic operator and by Example 2.3, $T^p \oplus \alpha I$ is M-ergodic with respect to $M := l^2 \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar α.

As mentioned before, mixing operators are ergodic. Also, it is not hard to see that if T is a mixing operator, then T^p is mixing for any $p \in \mathbb{N}$. So, by Example 2.3, we can create the following example:

Example 2.5. Let $T \in B(X)$ be a mixing operator and let I be the identity operator on X. Then $T^p \oplus \alpha I$ is M-ergodic with respect to $M := X \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar α. For example, let $T = \lambda B$ be the Rolewicz's operator on l^2, where B is the backward shift on l^2 and λ be a scalar with $|\lambda| > 1$. As it said in the introduction, $T = \lambda B$ is mixing. So, for any scalar α, the operator $(\lambda B)^p \oplus \alpha I$ is M-ergodic with respect to $M := l^2 \oplus \{0\}$. Also, it is easy to see that $\alpha I \oplus (\lambda B)^p$ is N-ergodic with respect to $N := \{0\} \oplus l^2$.

Corollary 2.6. Let X be a separable and infinite-dimensional Banach space. Then there exists a subspace-ergodic operator on $X \oplus X$.

Proof. By Theorem 1.1, there exists a mixing operator T on X. Then $T \oplus I$ is the desired operator by Example 2.5.

Now like subspace-hypercyclicity and subspace-supercyclicity, some questions arise for subspace-ergodicity as follows:

Question 1. Let T be an invertible M-ergodic operator. Can we conclude that T^{-1} is also M-ergodic?

Question 2. Let T be an M-ergodic operator. Is T^n is M-ergodic for any $n \in \mathbb{N}$?

Question 3. Let λ be a scalar with $|\lambda| = 1$. Does the M-ergodicity of T imply M-ergodicity of λT ?

Question 4. Let T be an ergodic operator on X. Is there a closed and non-trivial subspace M of X such that T is M-ergodic?

Question 5. Is the direct sum of two subspace-ergodic operators also subspace-ergodic?

In this section, we answer Question 1 affirmatively. Also, we partially answer Question 2 and Question 3. In the next section, we answer Question 5.

In the next theorem, we show that the answer to Question 1 is positive.

Theorem 2.7. Let $T \in B(X)$ be an invertible operator and let M be a closed and non-empty subspace of X. Then T is M-ergodic if and only if T^{-1} is M-ergodic.

Proof. Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. Let $n \in N_T(U,V)_M$. So, $T^n(U) \cap V \neq \emptyset$. T is invertible and

$$T^{-n}(T^n(U) \cap V) \neq \emptyset.$$

Hence, $T^{-n}(V) \cap U \neq \emptyset$. This means that $n \in N_{T^{-1}}(V,U)_M$. So,

$$N_T(U,V)_M \subseteq N_{T^{-1}}(V,U)_M.$$

Similarly, we have

$$N_{T^{-1}}(V,U)_M \subseteq N_T(U,V)_M.$$

Therefore, $N_T(U,V)_M = N_{T^{-1}}(V,U)_M$.

Hence, $N_T(U,V)_M$ is syndetic if and only if $N_{T^{-1}}(V,U)_M$ is syndetic. Therefore, T is M-ergodic if and only if T^{-1} is M-ergodic

Now it is natural to note the powers of a subspace-ergodic operator. In the next theorem, we prove that if T^n is subspace-ergodic for some $n \in \mathbb{N}$, then T is subspace-ergodic too.

Theorem 2.8. Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. If T^n is M-ergodic for some $n \in \mathbb{N}$, then T is also M-ergodic.

Proof. Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. By hypothesis, T^n is M-ergodic. So, $N_{T^n}(U,V)_M$ is syndetic. But if $(T^n)^k(U) \cap V \neq \emptyset$, then $T^{nk}(U) \cap V \neq \emptyset$. Hence, if $k \in N_{T^n}(U,V)_M$, then $k \in N_T(U,V)_M$. So,

$$N_{T^n}(U,V)_M \subseteq N_T(U,V)_M.$$

Let $(m_k)_k$ be the elements of $N_{T^n}(U,V)_M$ and let $(t_k)_k$ be the elements of $N_T(U,V)_M$. Hence,

$$sup_{1\leq k<\infty}(m_{k+1} - m_k) \geq sup_{1\leq k<\infty}(t_{k+1} - t_k).$$

By definition, $sup_{1\leq k<\infty}(m_{k+1} - m_k) < \infty$ and so $sup_{1\leq k<\infty}(t_{k+1} - t_k)$ is less than infinity too. That means $N_T(U,V)_M$ is syndetic and hence, T is an M-ergodic operator.

In the next theorem, we show that if T is an M-mixing operator, then T^n is M-ergodic for any $n \in \mathbb{N}$ and therefore, we have a partial answer to Question 2.

Theorem 2.9. Let $T \in B(X)$ and let M be a closed and non-empty subspace of X. If T is an M-mixing operator, then T^n is an M-ergodic operator for any $n \in \mathbb{N}$.

Proof. It is immediately obtained by Definition 2.2 that T is M-ergodic. Now, let $n > 1$ be an arbitrary natural number. First, we show that T^n is M-mixing.

Suppose that $U \subseteq M$ and $V \subseteq M$ are non-empty relatively open sets. By hypothesis, there exists $N \in \mathbb{N}$ such that $T^k(U) \cap V$ is non-empty for any $k \geq N$. On the other hand, for every $k \in \mathbb{N}$ we have $kn \geq k$. So, $T^{kn}(U) \cap V \neq \emptyset$ for any $k \geq N$. Hence,

$$(T^n)^k(U) \cap V \neq \emptyset, \qquad \text{for any } k \geq N.$$

Therefore, T^n is M-mixing and hence, T^n is M-ergodic. But $n > 1$ is an arbitrary natural number. So, T^n is M-ergodic for any $n \in \mathbb{N}$.

Lemma 2.10. Let $T \in B(X)$ be an invertible operator and let M be a closed and non-empty subspace of X. Then T is M-mixing if and only if T^{-1} is M-mixing.

Proof. Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. By hypothesis, T is an M-mixing operator. So, $N_T(U,V)_M$ is cofinite. But

$$N_T(U,V)_M = N_{T^{-1}}(V,U)_M.$$

Hence, $N_T(U,V)_M$ is cofinite if and only if $N_{T^{-1}}(V,U)_M$ is cofinite. This means that T is M-mixing if and only if T^{-1} is M-mixing.

The next corollary is a direct result of Theorem 2.9 and Lemma 2.10.

Corollary 2.11. Let T be an invertible and M-mixing operator. Then for any $n \in \mathbb{N}$, T^n and T^{-n} are M-ergodic.

Now we mention a theorem from [13] and by this we have a partial answer to Question 3.

Theorem 2.12. Let $T \in B(X)$. Then T is M-mixing with respect to a non-empty and closed subspace M of X if and only if for any non-empty relatively open set $U \subseteq M$ and any 0-neighborhood W in M there exists a positive integer N such that for any $n \geq N$, $T^n(U) \cap W \neq \emptyset$ and $T^n(W) \cap U \neq \emptyset$ [13].

In other words, T is M-mixing if and only if $N_T(U,W)_M$ and $N_T(W,U)_M$ are cofinite, for any non-empty relatively open set $U \subseteq M$ and any 0-neighborhood W in M.

Theorem 2.13. Let $T \in B(X)$ be an M-mixing operator. Let λ be a scalar with $|\lambda| = 1$. Then λT is M-ergodic.

Proof. Let T be a subspace-mixing operator with respect to a closed and non-empty subspace M of X. We show that λT is an M-mixing operator and hence it is M-ergodic. Let $U \subseteq M$ be a relatively open set and let W be a 0-neighborhood in M. By hypothesis, $N_T(U,W)_M$ and $N_T(W,U)_M$ are cofinite. We can find a balanced set W_1, a neighborhood of zero in M such that $W_1 \subseteq W$. Again by hypothesis, $N_T(U,W_1)_M$ is cofinite. So, there exists $N \in \mathbb{N}$ such that

$$T^n(U) \cap W_1 \neq \emptyset \quad (n \geq N). \tag{2}$$

Let $n \geq N$ be an arbitrary natural number. By Eq. (2), $T^n(U) \cap W_1$ is non-empty. So, there exists $x \in U$ such that $T^n x \in W_1$. Since $|\lambda^n| = 1$ and W_1 is a balanced set,

$$\lambda^n T^n x \in \lambda^n W_1 \subseteq W_1.$$

Hence, $\lambda^n T^n x \in W_1$ and therefore, $(\lambda^n T^n)(U) \cap W_1 \neq \emptyset$. Since $W_1 \subseteq W$,

$$(\lambda^n T^n)(U) \cap W \neq \emptyset.$$

Therefore, $N_{\lambda T}(U, W)_M$ is cofinite and similarly, $N_{\lambda T}(W, U)_M$ is cofinite. So, by Theorem 2.12, λT is M-mixing which completes the proof.

3 On the Direct Sum of Two Subspace-ergodic Operators

First, we show that the direct sum of two subspace-ergodic operators is subspace-ergodic. In fact, we show that the answer to Question 5 is positive. In this section, M and N always indicate closed and non-zero subspaces of X and Y respectively.

Theorem 3.1. Let $S \in B(X)$ be an M-ergodic operator and let $T \in B(Y)$ be an N-ergodic operator. Then, $S \oplus T$ is $M \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

Especially, $T \oplus T$ is $N \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

Proof. Let $U \subseteq M$ and $V \subseteq M$ be non-empty relatively open sets. Then,

$$\begin{aligned}(S \oplus T)^n(U \oplus \{0\}) \cap (V \oplus \{0\}) &= (S^n \oplus T^n)(U \oplus \{0\}) \cap (V \oplus \{0\}) \\ &= (S^n(U) \cap V) \oplus (T^n(\{0\}) \cap \{0\}) \\ &= (S^n(U) \cap V) \oplus \{0\}.\end{aligned}$$

So,

$$N_{S\oplus T}\big((U \oplus \{0\}), (V \oplus \{0\})\big)_{M\oplus\{0\}} = N_S(U, V)_M. \tag{3}$$

By hypothesis, $N_S(U, V)_M$ is syndetic. So, by Eq. (3), $N_{S\oplus T}\big((U \oplus \{0\}), (V \oplus \{0\})\big)_{M\oplus\{0\}}$ is syndetic. This means that $S \oplus T$ is $M \oplus \{0\}$-ergodic. Similarly, $S \oplus T$ is $\{0\} \oplus N$-ergodic.

By Theorem 3.1 and Theorem 2.7 we can conclude the following corollary:

Corollary 3.2. Let $S \in B(X)$ be an M-ergodic operator and let $T \in B(Y)$ be an N-ergodic operator. If S and T are invertible operators, then $(S \oplus T)^{-1}$ is $M \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

If $S \in B(X)$ and $T \in B(Y)$ are topologically ergodic operators, then $S^p \oplus T^q$ is topologically ergodic on $X \oplus Y$ for any $p, q \in \mathbb{N}$ [1, p. 173].

Now the question arises if this is also true for subspace-ergodic operators? We partially answer this question in the next theorem.

Theorem 3.3. Let $S \in B(X)$ be an M-ergodic operator and let $T \in B(Y)$ be an N-ergodic operator. Then

(i) if S is an M-mixing operator, then $S^p \oplus T$ is $M \oplus N$-ergodic for any $p \in \mathbb{N}$,

(ii) if T is an N-mixing operator, then $S \oplus T^q$ is $M \oplus N$-ergodic for any $q \in \mathbb{N}$.

Proof. We prove part (i) and the proof of part (ii) is similar. First, we prove that $S \oplus T$ is $M \oplus N$-ergodic. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq N$ be non-empty relatively open sets. By hypothesis, S is an M-mixing operator. So, there exists a natural number p such that for any $n \geq p$,

$$S^n(U_1) \cap (U_2) \neq \emptyset.$$

So,

$$\{n \in \mathbb{N}; n \geq p\} \subseteq N_S(U_1, U_2)_M.$$

By hypothesis, T is an N-ergodic operator and so $N_T(V_1, V_2)_N$ is syndetic. On the other hand,

$$(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) = \big(S^n(U_1) \cap (U_2)\big) \oplus \big(T^n(V_1) \cap (V_2)\big).$$

Hence,

$$\begin{aligned} N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N} &= N_S(U_1, U_2)_M \cap N_T(V_1, V_2)_N \\ &\supseteq \{n \in \mathbb{N}; n \geq p\} \cap N_T(V_1, V_2)_N \\ &= \{n \in \mathbb{N}; n \geq p \text{ and } n \in N_T(V_1, V_2)_N\}. \end{aligned}$$

Since $N_T(V_1, V_2)_N$ is syndetic, we can deduce that $N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N}$ is syndetic. Now note that if S is M-mixing, then S^p is M-mixing for any $p \in \mathbb{N}$ which completes the proof.

Bamerni and Kilicman showed in [14] that the direct sum of two subspace-mixing operators is also subspace-mixing. Now we extend their statement as follows:

Theorem 3.4. Let $S \in B(X)$ be an M-mixing operator and let $T \in B(Y)$ be an N-mixing operator. Then, $(S \oplus T)^n$ is $M \oplus N$-mixing for any $n \in \mathbb{N}$.

Moreover, if S and T are invertible, then $(S \oplus T)^{-n}$ is also $M \oplus N$-mixing for any $n \in \mathbb{N}$. Especially, $(S \oplus T)^n$ and $(S \oplus T)^{-n}$ are $M \oplus N$-ergodic for any $n \in \mathbb{N}$.

Proof. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq N$ be non-empty relatively open sets. S is an M-mixing operator. So, there exists a natural number N_1 such that for any $n \geq N_1$,

$$S^n(U_1) \cap (U_2) \neq \emptyset. \tag{4}$$

On the other hand, T is an N-mixing operator. So, there exists a natural number N_2 such that for any $n \geq N_2$,

$$T^n(V_1) \cap (V_2) \neq \emptyset. \tag{5}$$

Let $p = max\{N_1, N_2\}$. So, by Eq. (4) and Eq. (5), for any $n \geq p$ we have,

$$S^n(U_1) \cap (U_2) \neq \emptyset \quad \text{and} \quad T^n(V_1) \cap (V_2) \neq \emptyset.$$

Hence, for any $n \geq p$,

$$(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) = (S^n(U_1) \cap U_2) \oplus (T^n(V_1) \cap V_2) \neq \emptyset.$$

So, $N_{(S\oplus T)^n}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N}$ is cofinite and hence, $(S \oplus T)^n$ is $M \oplus N$-mixing.

The proof of the rest of the theorem is an easy consequence of Lemma 2.10 and Theorem 2.9.

Finally, we prove that subspace-ergodicity of the direct sum of two operators, indicates subspace-ergodicity of each of them.

Theorem 3.5. Let $S \in B(X)$ and $T \in B(Y)$. If $S \oplus T$ is an $M \oplus N$-ergodic operator, then S is an M-ergodic operator and T is an N-ergodic operator.

Especially, if $T \oplus T$ is $N \oplus N$ ergodic, then T is N-ergodic.

Proof. Let $U_1 \subseteq M$ and $U_2 \subseteq M$ be non-empty relatively open sets. We prove that $N_S(U_1, U_2)_M$ is syndetic.

Suppose that $V_1 \subseteq N$ and $V_2 \subseteq N$ be non-empty relatively open sets. By hypothesis, $S \oplus T$ is $M \oplus N$-ergodic. So, $N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N}$ is syndetic. Let $n \in N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N}$. So,

$$(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \emptyset.$$

And hence,

$$(S^n(U_1) \cap (U_2)) \oplus \left(T^n(V_1) \cap (V_2)\right) \neq \emptyset.$$

Therefore, $S^n(U_1) \cap U_2$ must be non-empty and hence, $n \in N_S(U_1, U_2)_M$. This means

$$N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N} \subseteq N_S(U_1, U_2)_M.$$

Since $N_{S\oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M\oplus N}$ is syndetic, $N_S(U_1, U_2)_M$ is syndetic. Similarly, T is also N-ergodic.

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