Classification and Dynamics of Class of $\xi^{(as)}$-QSOs

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Abstract. The current study provides a new class of $\xi^{(as)}$-QSO defined on 2D simplex and classifies it into 18 non-conjugate (isomorphic) classes. This classification is based on their conjugacy and the remuneration of coordinates. The current study also examines the limiting points associated with the behavior of trajectories for four classes defined on 2D simplex.

Keywords: *fixed point; limiting point; quadratic stochastic; periodic point; $\xi^{(as)}$-QSO.*

1 Introduction

S. Bernstein developed the concept of the quadratic stochastic operator (QSO) in 1924 [1]. Since then, QSOs have been intensively studied because they emerge in various models in physics [2,3], biology [1,4,5], economics and different branches of mathematics, such as graph theory and probability theory [6-9]. In a biological context, QSOs can be applied in the area of population genetics. QSOs can describe a generation-by-generation distribution when the distribution of the original starting generation is provided. We highlight how these operators can be used to interpret data in population genetics. We can see how these operators function when analogously looking at a biological population closed to reproduction with outside members. For that population, we can assume that each member of this closed biological group has one or more varying species-specific traits: $\{1,\dots,m\}$. Let $x^{(0)}=(x_1^{(0)},\dots,x_m^{(0)})$ be a probability distribution of the species at an initial state and let the heredity coefficient $p_{ij,k}$ be the conditional probability $p(k\backslash i,j)$ that the i^{th} and j^{th} members of the species have reproduced successfully to produce a k^{th} individual. The first generation from this union $x^{(1)}=(x_1^{(1)},\dots,x_m^{(1)})$ can be calculated using the total probability

$$x_k^{(1)}=\sum_{i,j=1}^{m}p(k\backslash i,j)P(i,j),\quad k=\overline{1,m}$$

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Given that no difference exists between the i^{th} and j^{th} members in any generation, the original progenitors i, j are independent, i.e. $P(i,j) = P_i P_j$. This condition suggests that

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1,m}.$$

Consequently, the relation $x^{(0)} \to x^{(1)}$ represents a mapping V, which is known as the evolution operator. Beginning with the selected initial state $x^{(0)}$, the population iteratively develops to the first generation set $x^{(1)} = V(x^{(0)})$ and then to the subsequent set $x^{(2)} = V\left(x^{(1)}\right) = V\left(V\left(x^{(0)}\right)\right) = V^{(2)}(x^{(0)})$ through time. Hence, the discrete dynamical system presents the population system evolution states as follows:

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad \cdots$$

One of the main issues that underlie this theory is finding the limit points of V for an arbitrary starting point $x^{(0)}$. Studying the limit points of QSOs is a complicated task even in 2D simplex and this problem has not yet been solved. Numerous researchers have presented a specific class of QSOs and have examined their behavior, e.g. F-QSO [10], Volterra-QSO [11-14], permutated Volterra-QSO [15,16], ℓ-Volterra QSO [17,18], Quasi-Volterra-QSO [19], non-Volterra-QSO [20,21], strictly non-Volterra QSO [22], non-Volterra operators, and others, produced via measurements [23-25]. Nevertheless, collectively, these classes cannot represent all the QSOs as a set. An attempt was made to study the behavior of nonlinear operators, which is regarded as the main problem in nonlinear operators. However, this problem has not been comprehensively studied because it relies upon a specified cubic matrix, $(P_{ijk})_{i,j,k=1}^{m}$ [26].

Recently, Ganikhodzhaev, *et al*. [27] introduced $\xi^{(as)}$-QSO, which is a new class of QSOs that depend on a partition of the coupled index sets (which have couple traits) $P_m = \{(i,j): i < j\} \subset I \times I$ and $\Delta_m = \{(i,i): i \in N\} \subset I \times I$. In the case of 2D simplex $(m = 3)$, P_3 and Δ_3 have five possible partitions.

In [28,29], the $\xi^{(s)}$-QSO related to $|\xi_1| = 2$ of P_3 with a point partition of Δ_3 was investigated (see Section 2). In [30,31] the $\xi^{(a)}$-QSO related to $|\xi_1| = 2$ of P_3 with a trivial partition of Δ_3 was studied (see Section 2). The $\xi^{(as)}$-QSO related to $|\xi_1| = 3$ of P_3 with a point partition of Δ_3 was examined by

Mukhamedov and Jamal [27,32]. Furthermore, $\xi^{(s)}$-QSO and $\xi^{(a)}$-QSO are related to $|\xi_1|=1$ of P_3 with point and trivial partitions of Δ_3, respectively, as discussed by Alsarayreh, *et al*. [33]. This indicates that all partitions of P_3 were investigated with respect to the point and trivial partitions of Δ_3. Therefore, the main motivation for this study was to introduce new partitions of Δ_3. The current study classifies the operators generated by $\xi^{(as)}$-QSO with cardinality $|\xi_i|=2$ of P_3 and $|\xi_i|=2$ of Δ_3. To demonstrate this in the current report, Section 2 establishes a number of preliminary definitions. Section 3 presents the description and classification of $\xi^{(as)}$-QSOs. Section 4 explains how this study examined the behavior of V_3 and V_{15}, obtained from classes G_3 and G_9, respectively. Section 5 examines the behavior of V_{26} and V_{25}, obtained from classes G_{13} and G_{14}, respectively.

2 Preliminary

Several basic concepts need to be operationalized to ensure clarity of terms.

Definition: QSO is a mapping of the simplex

$$S^{m-1}=\left\{x=(x_1,\cdots,x_m)\in\mathbb{R}^m:\ \sum_{i=1}^{m}x_i=1,\ x_i\geq 0,\ i=\overline{1,m}\right\} \tag{1}$$

into itself with the form

$$x'_k=\sum_{i,j=1}^{m}P_{ij,k}x_ix_j,\quad k=\overline{1,m}, \tag{2}$$

where $V(x)=x'=(x'_1,\cdots,x'_m)$, and $P_{ij,k}$ is a coefficient of heredity that satisfies the following conditions:

$$P_{ij,k}\geq 0,\quad P_{ij,k}=P_{ji,k},\quad \sum_{k=1}^{m}P_{ij,k}=1. \tag{3}$$

Based on the preceding definition, each QSO $V: S^{m-1}\to S^{m-1}$ can be uniquely defined by a cubic matrix $\mathcal{P}=\left(P_{ijk}\right)_{i,j,k=1}^{m}$ with conditions Eq. (1) and Eq. (2). For $V: S^{m-1}\to S^{m-1}$, we specify the set of fixed points as $F_{ix}(V)$. Moreover, for $x^{(0)}\in S^{m-1}$, we indicate the set of limiting points as $\omega_V(x^{(0)})$. It must be remembered that Volterra-QSO is defined by Eq. (2) and condition in Eq. (3), with the additional assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \tag{4}$$

The biological rule of Condition Vol is well-established: an offspring replicates the genotype (trait) of its parent(s). Volterra-QSO exhibits the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \ k \in I, \tag{5}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \ \text{for} \, i \neq k \quad a_{ii} = 0, i \in I \quad a_{ki} = -a_{ik} \text{and} \ |a_{ki}| \leq 1. \tag{6}$$

Remarkably, this kind of operator has received extensive research attention [11-13]. The concept of ℓ-Volterra-QSO was first presented in [17] and formulated as follows.

Let $\ell \in I$ be fixed. Suppose that the genetic heredity coefficient $\{P_{ij,k}\}$ satisfies

$$P_{ij,k} = 0 \ \text{if} \ k \notin \{i, j\} \ \text{for any} \ k \in \{1, \ldots, \ell\}, \ i, j \in I, \tag{7}$$

$$P_{i_0 j_0,k} > 0 \ \text{for some} \, (i_0, j_0), i_0 \neq k, \ j_0 \neq k, \ k \in \{\ell + 1, \ldots, m\}. \tag{8}$$

Therefore, the QSO specified by Eq. (2), additional condition in Eq. (3), Eq. (7) and Eq. (8) is called ℓ -Volterra-QSO.

Remark 1. The following points must be reinforced:

1. An ℓ -Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.
2. No periodic trajectory exists for Volterra-QSO [11]. However, such trajectories exist for ℓ -Volterra-QSO [17].

Following [27], each element $x \in S^{m-1}$ is a probability distribution of set $I = \{1, \ldots, m\}$. Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ be vectors obtained from S^{m-1}. We say that x is equivalent to y if $x_k = 0 \Leftrightarrow y_k = 0$. We denote this relation as $x \sim y$. Let $supp(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that x is singular to y and specify this relation as $x \perp y$ if $supp(x) \cap supp(y) = \varnothing$. Notably, if $x, y \in S^{m-1}$, then $x \perp y$ if and only if $(x, y) = 0$, where $(\cdot, \cdot)$ denotes a standard inner product in $\mathbb{R}^m$. Of further note is that $|\xi_i|$ indicates the cardinality of potential partitions of P_m and Δ_m. If the cardinal of partition of Δ_m is the maximum or minimum, it is named the point partition or trivial partition, respectively. We formulate sets of coupled indexes as follows:

$$P_m = \{(i,j) : i < j\} \subset I \times I, \quad \Delta_m = \{(i,i) : i \in I\} \subset I \times I$$

For any specified pair $(i,j) \in P_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = \left(P_{ij,1}, \cdots, P_{ij,m}\right)$. Evidently, $\mathbb{P}_{ij} \in S^{m-1}$ because of condition in Eq. (3). In this case, let $\xi_1 = \{A_i\}_{i=1}^N$ and $\xi_2 = \{B_i\}_{i=1}^M$ be fixed partitions of P_m and Δ_m, giving: $A_i \bigcap A_j = \varnothing$, $B_i \bigcap B_j = \varnothing$, $\bigcup_{i=1}^N A_i = P_m$, $\bigcup_{i=1}^M B_i = \Delta_m$, where $N, M \le m$.

Definition: [27] QSO $V : S^{m-1} \to S^{m-1}$ given by Eq. (2) and condition in Eq. (3) is considered a $\xi^{(as)}$-QSO with respect to partitions ξ_1 and ξ_2 if the following conditions are satisfied:

1. Let $k \in \{1, \ldots, N\}$ and $(i,j), (u,v) \in A_k$, then $\mathbb{P}_{ij} \sim \mathbb{P}_{uv}$ is considered.
2. Let $k \ne \ell, k, \ell \in \{1, \ldots, N\}$, $(i,j) \in A_k$ and $(u,v) \in A_\ell$, then $\mathbb{P}_{ij} \perp \mathbb{P}_{uv}$ is considered.
3. Let $d \in \{1, \ldots, M\}$ and $(i,i), (j,j) \in B_d$, then $\mathbb{P}_{ii} \sim \mathbb{P}_{jj}$ is considered.
4. Let $s \ne h$, $s, h \in \{1, \ldots, M\}$ and $(u,u) \in B_s$ and $(v,v) \in B_h$, then $\mathbb{P}_{uu} \perp \mathbb{P}_{vv}$ is considered.

Example 1. Let $\xi := \{\{(2,3)\}, \{(1,2),(1,3)\}\}$ and $\xi^* := \{\{(1,1)\}, \{(2,2)\}, \{(3,3)\}\}$ be two possible partitions of P_3 and Δ_3 respectively. It is observable that $P_{12} \sim P_{13}$, $P_{23} \perp (P_{12}, P_{13})$ and $P_{11} \perp P_{22} \perp P_{33}$. Therefore, based on such facts, the following values for $\mathbb{P}_{ij}$ and $\mathbb{P}_{ii}$ can be considered: $\mathbb{P}_{12} = (\alpha, 1-\alpha, 0)$, $\mathbb{P}_{13} = (\alpha, 1-\alpha, 0)$, $\mathbb{P}_{23} = (0,0,1)$ $\mathbb{P}_{11} = (1,0,0)$, $\mathbb{P}_{22} = (0,1,0)$, $\mathbb{P}_{22} = (0,0,1)$. Due to Eq. (2), one can calculate the following:

$$V := \begin{cases} x' = (x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2\alpha x^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + 2(1-\alpha)(x^{(0)})(1-(x^{(0)})) \\ z' = (z^{(0)})^2 + 2x^{(0)}z^{(0)} \end{cases}$$

Hence, the operator V is considered $\xi^{(as)}$-QSO.

3 Classification of $\xi^{(as)}$ - QSO

This section presents the classification of $\xi^{(as)}$-QSOs on 2-dimensional simplex, i.e. $m = 3$ and the cardinality of the potential partitions of P_3 and Δ_3 is equal to 2. The potential partitions of P_3 are listed as follows:

$$\xi_1 := \{\{(1,2)\},\{(1,3)\},\{(2,3)\}\}, |\xi_1| = 3,$$
$$\xi_2 := \{\{(2,3)\},\{(1,2),(1,3)\}\}, |\xi_2| = 2,$$
$$\xi_3 := \{\{(1,3)\},\{(1,2),(2,3)\}\}, |\xi_3| = 2,$$
$$\xi_4 := \{\{(1,2)\},\{(1,3),(2,3)\}\}, |\xi_4| = 2,$$
$$\xi_5 := \{(1,2),(1,3),(2,3)\}, |\xi_5| = 1.$$

The potential partitions of Δ_3 are listed as follows:

$$\xi_1 := \{\{(1,1)\},\{(2,2)\},\{(3,3)\}\}, |\xi_1| = 3,$$
$$\xi_2 := \{(1,1),(2,2),(3,3)\}, |\xi_2| = 1,$$
$$\xi_3 := \{\{(1,1)\},\{(2,2),(3,3)\}\}, |\xi_3| = 2,$$
$$\xi_4 := \{\{(3,3)\},\{(1,1),(2,2)\}\}, |\xi_4| = 2,$$
$$\xi_5 := \{\{(2,2)\},\{(1,1),(3,3)\}\}, |\xi_5| = 2.$$

Proposition 1. For a class of $\xi^{(as)}$-QSO generated from the possible partitions of P_3 and Δ_3 with cardinality equal to 2, we determine the following:

1. The class of all $\xi^{(as)}$-QSOs that correspond to partition ξ_3 of P_3 and partition ξ_5 of Δ_3 is conjugate to the class of all $\xi^{(as)}$-QSOs that corresponds to partition ξ_2 of P_3 and partition ξ_3 of Δ_3.
2. The class of all $\xi^{(as)}$-QSOs that corresponds to the partition ξ_4 of P_3 and partition ξ_4 of Δ_3 is conjugate to the class of all $\xi^{(as)}$-QSOs that corresponds to partition ξ_2 of P_3 and partition ξ_3 of Δ_3.

Proof.

1. Under the general form of QSO given by Eq. (2) and Eq. (3), the coefficients $\left(P_{ij,k}\right)_{i,j,k=1}^{3}$ of operator V of $\xi^{(as)}$-QSOs that correspond to

partition $\xi_5 = \{\{(2,2)\},\{(1,1),(3,3)\}\}$ of Δ_3 and partition $\xi_3 = \{\{(1,3)\},\{(1,2),(2,3)\}\}$ of P_3, satisfying the following conditions:

i. $\mathbb{P}_{11} \sim \mathbb{P}_{33}$ and $\mathbb{P}_{22} \perp \mathbb{P}_{mm}, m = 1,3$; ii. $\mathbb{P}_{12} \sim \mathbb{P}_{23}$ and $\mathbb{P}_{13} \perp (\mathbb{P}_{12}, \mathbb{P}_{23})$, where $\mathbb{P}_{ij} = (p_{ij,1}, p_{ij,2}, p_{ij,3})$. To perform $V_\pi = \pi V \pi^{-1}$ transformation on operator V, where permutation $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and where $V_\pi : x_k' = \sum_{i,j=1}^{3} P_{ij,k}^{\pi} x_i x_j$, $k = \overline{1,3}$, such that $\mathbb{P}_{ij,k}^{\pi} = P_{\pi(i)\pi(j),\pi(k)}$, for any $i,j,k = \overline{1,3}$. Equivalently, $\mathbb{P}_{ij}^{\pi} = \pi \mathbb{P}_{\pi(i)\pi(j)}$ (in vector form) for any $i,j = 1,2,3$. Subsequently, operator V_π corresponding to partitions ξ_3 of Δ_3 and ξ_2 of P_3 is presented by applying the permutation π for the coefficient of V that corresponds to partition ξ_5 of Δ_3 and ξ_3 of P_3. From this, the following relations are derived:

a. $\mathbb{P}_{11} \sim \mathbb{P}_{33}$ and $\mathbb{P}_{22} \perp (\mathbb{P}_{11}, \mathbb{P}_{33})$. By applying the permutation π on $\mathbb{P}_{11}^{\pi} = \mathbb{P}_{33}, \mathbb{P}_{22}^{\pi} = \mathbb{P}_{11}$, and $\mathbb{P}_{33}^{\pi} = \mathbb{P}_{22}$, we obtain $\mathbb{P}_{33} \sim \mathbb{P}_{22}$ and $\mathbb{P}_{11} \perp (\mathbb{P}_{22}, \mathbb{P}_{33})$. Therefore, the properties of ξ_3 of Δ_3 are verified.

b. $\mathbb{P}_{12} \sim \mathbb{P}_{23}$ and $\mathbb{P}_{13} \perp (\mathbb{P}_{12}, \mathbb{P}_{23})$. By applying the permutation π on $\mathbb{P}_{12}^{\pi} = \mathbb{P}_{13}$, $\mathbb{P}_{13}^{\pi} = \mathbb{P}_{23}$, and $\mathbb{P}_{23}^{\pi} = \mathbb{P}_{12}$, we obtain $\mathbb{P}_{12} \sim \mathbb{P}_{13}$ and $\mathbb{P}_{23} \perp (\mathbb{P}_{12}, \mathbb{P}_{13})$. Therefore, the properties of ξ_2 of P_3 are verified.

2. Similarly, the second part of this theorem can be proven by choosing the following permutation: $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. This process completes the proof.

The preceding discussion reveals that any $\xi^{(as)}$-QSO obtained from the class that corresponds with partitions ξ_5 of Δ_3 and ξ_3 of P_3 or ξ_4 of Δ_3 and ξ_4 of P_3 is conjugate to certain $\xi^{(as)}$-QSOs obtained from the class that corresponds to partitions ξ_3 of Δ_3 and ξ_2 of P_3. To investigate the operators of class $\xi^{(as)}$-QSO that correspond to partitions ξ_2 of P_3 and ξ_3 of Δ_3, the coefficient $\left(P_{ij,k}\right)_{i,j,k=1}^{3}$ in special form is selected, as shown in the Tables 1 and 2.

Table 1 The possible values of P_{ii}.

Case	P_{11}	P_{22}	P_{33}
I_1	$(\alpha,\beta,0)$	(0,0,1)	(0,0,1)
I_2	$(\alpha,0,\beta)$	(0,1,0)	(0,1,0)
I_3	$(\beta,\alpha,0)$	(0,0,1)	(0,0,1)
I_4	$(\beta,0,\alpha)$	(0,1,0)	(0,1,0)
I_5	$(0,\alpha,\beta)$	(1,0,0)	(1,0,0)
I_6	$(0,\beta,\alpha)$	(1,0,0)	(1,0,0)

where $\alpha,\beta \in [0,1][0,1]$ such that $\alpha+\beta=1$.

Table 2 The possible values of P_{ij}.

Case	P_{12}	P_{13}	P_{23}
II_1	(1,0,0)	(1,0,0)	(0,0,1)
II_2	(1,0,0)	(1,0,0)	(0,1,0)
II_3	(0,1,0)	(0,1,0)	(1,0,0)
II_4	(0,1,0)	(0,1,0)	(0,0,1)
II_5	(0,0,1)	(0,0,1)	(1,0,0)
II_6	(0,0,1)	(0,0,1)	(0,1,0)

The choices for Cases (I_j, II_i), where $i,j=1,\ldots,6$, provide 36 operators. These operators can be defined accordingly:

$$V_1 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \qquad V_2 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases}$$

$$V_3 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases} \qquad V_4 := \begin{cases} x' = \alpha(x^{(0)})^2 \\ y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_5 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \qquad V_6 := \begin{cases} x' = \alpha(x^{(0)})^2 \\ y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

$$V_7 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 \\ z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \qquad V_8 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 \end{cases}$$

$$V_9 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 \end{cases} \quad V_{10} := \begin{cases} x' = \alpha(x^{(0)})^2 \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_{11} := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 \\ z' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \quad V_{12} := \begin{cases} x' = \alpha(x^{(0)})^2 \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

$$V_{13} := \begin{cases} x' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad V_{14} := \begin{cases} x' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases}$$

$$V_{15} := \begin{cases} x' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 \end{cases} \quad V_{16} := \begin{cases} x' = \beta(x^{(0)})^2 \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_{17} := \begin{cases} x' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 \\ z' = (z^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \quad V_{18} := \begin{cases} x' = \beta(x^{(0)})^2 \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = (z^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

$$V_{19} := \begin{cases} x' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 \\ z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad V_{20} := \begin{cases} x' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 \end{cases}$$

$$V_{21} := \begin{cases} x' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 \end{cases} \quad V_{22} := \begin{cases} x' = \beta(x^{(0)})^2 \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_{23} := \begin{cases} x' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (y^{(0)})^2 + (z^{(0)})^2 \\ z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \quad V_{24} := \begin{cases} x' = \beta(x^{(0)})^2 \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\ z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

$$V_{25} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 \\ z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad V_{26} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 \end{cases}$$

$$V_{27} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 \end{cases} \qquad V_{28} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_{29} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\ y' = \alpha(x^{(0)})^2 \\ z' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \qquad V_{30} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

$$V_{31} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 \\ z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \qquad V_{32} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 \end{cases}$$

$$V_{33} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\ y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 \end{cases} \qquad V_{34} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = \beta(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases}$$

$$V_{35} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2z^{(0)}y^{(0)} \\ y' = \beta(x^{(0)})^2 \\ z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases} \qquad V_{36} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = \beta(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)} \end{cases}$$

As can be seen, this class of $\xi^{(as)}$-QSOs contains 36 operators. These operators are too numerous to explore individually. Thus, two operators V_a and V_b are considered (topologically or linearly) conjugate when a permutation matrix P such that $P^{-1}V_aP = V_b$ exists. Let π be a permutation of set $I = \{1, \dots m\}$. For any vector x, we define $\pi(x) = (x_{\pi(1)}, \dots, x_{\pi(m)})$. Verifying if π is a permutation of set I corresponding to the given permutation matrix P is simple. When true, then $P_x = \pi(x)$. In this case, two operators V_a and V_b are conjugate if and only if $\pi^{-1}V_a\pi = V_b$ for any permutation π. Therefore, one may classify such operators into small classes and examine only the operators within these classes.

Theorem 1. Let $\{V_1, \cdots, V_{36}\}$ be the $\xi^{(as)}$-QSO presented above. Then, these operators are divided into 18 non-isomorphic classes:

$G_1 = \{V_1, V_8\}$	$G_2 = \{V_2, V_7\}$	$G_3 = \{V_2, V_7\}$	$G_4 = \{V_4, V_{12}\}$
$G_5 = \{V_5, V_9\}$	$G_6 = \{V_6, V_{10}\}$	$G_7 = \{V_{13}, V_{20}\}$	$G_8 = \{V_{14}, V_{19}\}$
$G_9 = \{V_{15}, V_{23}\}$	$G_{10} = \{V_{16}, V_{24}\}$	$G_{11} = \{V_{17}, V_{21}\}$	$G_{12} = \{V_{18}, V_{22}\}$
$G_{13} = \{V_{25}, V_{32}\}$	$G_{14} = \{V_{26}, V_{31}\}$	$G_{15} = \{V_{27}, V_{35}\}$	$G_{16} = \{V_{28}, V_{36}\}$
$G_{17} = \{V_{29}, V_{33}\}$	$G_{18} = \{V_{30}, V_{34}\}$.		

Proof. Evidently, the partitions ξ_2 of P_3 and ξ_3 of Δ_3 are invariant only under the permutation $\pi = \begin{pmatrix} x^{(0)} & y^{(0)} & z^{(0)} \\ x^{(0)} & z^{(0)} & y^{(0)} \end{pmatrix}$. Therefore, the given operators should be classified in relation with the remuneration of their coordinates. Consequently, we must perform the $\pi V \pi^{-1}$ transformation on all the operators. Starting with V_1 as the first operator, we obtain the following:

$$V_1\left(\pi^{-1}(x^{(0)}, y^{(0)}, z^{(0)})\right) = V_1\left(x^{(0)}, z^{(0)}, y^{(0)}\right) =$$

$$(\alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}, \beta(x^{(0)})^2,$$

$$(y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)})$$

Thus,

$$\pi V_1 \pi^{-1} = \left(\alpha(x^{(0)})^2 + 2x^{(0)}y^{(0)} + 2x^{(0)}z^{(0)}, (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}, (\alpha-1)\beta(x^{(0)})^2\right)$$
$$= V_8.$$

We can derive the other classes by following the same procedure. The proof is completed using this process.

4 Dynamics of classes G_3 and G_9

This section explores the dynamics of $\xi^{(as)}$-QSO $V_{3,15}: S^2 \to S^2$ selected from G_3 and G_9. Firstly, V_3 is rewritten as follows:

$$V_3 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (1-\alpha)(x^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ z' = (z^{(0)})^2 + (y^{(0)})^2 \end{cases} \tag{9}$$

The operator V_3 can be redrafted as a convex combination $V_3 = \alpha W_1 + (1-\alpha)W_2$, where

$$W_1 := \begin{cases} x' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = 2x^{(0)}\left(1 - x^{(0)}\right) \\ z' = (z^{(0)})^2 + (y^{(0)})^2 \end{cases} \tag{10}$$

and

$$W_2 := \begin{cases} x' = 2y^{(0)}z^{(0)} \\ y' = 2x^{(0)} - (x^{(0)})^2 \\ z' = (z^{(0)})^2 + (y^{(0)})^2 \end{cases} \tag{11}$$

The following regions are introduced and the behavior of $\{W_1^{(n)}\}_{n=1}^{\infty}$ over them is investigated:

$$A_1 := \{x_1^{(0)} \in S^2 : 0 \le x^{(0)}, y^{(0)}, z^{(0)} \le \frac{1}{2}\},$$

$$A_2 := x_1^{(0)} \in S^2 : \frac{1}{2} < x^{(0)} < 1\},$$

$$A_3 := \{x_1^{(0)} \in S^2 : \frac{1}{2} < y^{(0)} < 1\},$$

$$A_4 := \{x_1^{(0)} \in S^2 : \frac{1}{2} < z^{(0)} < 1\}.$$

Proposition 2. The following statements hold for W_1 of V_3:

1. The region A_1 is invariant.
2. Let $x_1^{(0)} \notin Fix(W_1)$, and $x_1 \in A_2$, then, $\{W_1^{(n)}\}_{n=1}^{\infty}$ goes to $A_1 \cup A_3$. Accordingly, if $x_1 \in A_3$, then, $\{W_1^{(n)}\}_{n=1}^{\infty}$ goes to A_1.
3. Let $x_1 \in A_4$, then, $\{W_1^{(n)}\}_{n=1}^{\infty}$ goes to A_1.

Proof.

1. Let $x_1^{(0)} \in A_1$. Then, $0 \le x^{(0)}, y^{(0)}, z^{(0)} \le \frac{1}{2}$. Evidently, $-1 \le 3x^{(0)} - 1 \le \frac{1}{2}$ can be verified. By squaring and adding $-3(y^{(0)} - z^{(0)})^2$, the last inequality becomes $0 \le (3x^{(0)} - 1)^2 - 3(y^{(0)} - z^{(0)})^2 \le 1$, and $9(x^{(0)})^2 - 6x^{(0)} + 1$ $-3(y^{(0)} - z^{(0)})^2 \le 1$ is obtained. By dividing the previous inequality by three after adding two to both parts of the inequality, we derive $3(x^{(0)})^2 - 2x^{(0)} + 1 - (y^{(0)} - z^{(0)})^2 \le 1.$

   Therefore, $2(x^{(0)})^2 + (y^{(0)} + z^{(0)})^2 - (y^{(0)} - z^{(0)})^2 \le 1.$

   Then, $2(x^{(0)})^2 + 4y^{(0)}z^{(0)} \le 1$, which implies that $x' \le \frac{1}{2}$. It is observable that

$y' \le \frac{1}{2} \forall x_1^{(0)}$. Evidently, $0 \le (y^{(0)})^2, (z^{(0)})^2 \le \frac{1}{4}$ is obtained, which implies that $z' \le \frac{1}{2}$. Hence, A_1 is an invariant region, i.e. if an initial point $x_1^{(0)}$ is taken inside this region, then the trajectory will go to a point that is also in this region.

2. The second coordinate of W_1 is less than $\frac{1}{2}$ at any initial point $x_1^{(0)}$, thereby indicating that A_3 is not an invariant region. Subsequently, suppose that A_2 is an invariant region, which indicates that $y' \le x'$ and $z' \le x'$. However,

$$x' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \le (x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2$$

$$\le x^{(0)}(x^{(0)} + y^{(0)} + z^{(0)}) = x^{(0)}.$$

Then $\frac{x'}{x^{(0)}} < 1$, which implies that the first coordinate is a decreasing bounded sequence that converges to zero, thereby contradicting our assumption. Hence, if $x_1^{(0)} \in A_2 \cup A_3$, then $n_{k_1}, n_{k_2} \in \mathbb{N}$, such that the sequences $x^{(n_{k_1})}$ and $y^{(n_{k_2})}$ tend toward invariant region A_1 .

3. Suppose that A_4 is an invariant region; hence, $z' \ge y' + x'$ and $x', y' \le \frac{1}{2}$.. Evidently, $x' \le y'$. By using the last inequality and the first coordinate of W_1, we obtain $y' \ge 2y^{(0)}z^{(0)}$. That is, $z' \le \frac{1}{2}$, which repudiates our assumption. Hence, region A_4 is not invariant.

Theorem 2. Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (10) and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_1)$ belongs to simplex S^2 as initial point. In this case, the two statements below are valid:

1. One has $Fix(W_1) = \left\{ e_1, e_3, (\frac{3-\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}) \right\}$,

2. One has $\omega_{W_1}(x_1^{(0)}) = \left\{ (\frac{3-\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}) \right\}$.

Proof. Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (10), $x_1^{(0)} \notin Fix(W_1)$ be any beginning point that belongs to S^2 and $\{W_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of W_1 that starts at $x_1^{(0)}$.

1. The set of fixed points of W_1 is obtained by finding the solution for the following system of equations:

$$=\begin{cases} x = x^2 + 2yz \\ y = 2x(1-x) \\ z = y^2 + z^2 \end{cases} \quad (12)$$

Depending on the first equation in system Eq. (12), we derive $x - x^2 = 2yz$. Subsequently, the last equation is multiplied by 2 and the new equation is substituted into the second equation in system Eq. (12). We obtain $y(1-4z) = 0$ and find $y = 0$ or $z = \frac{1}{4}$. If $y = 0$, then $x = 0$ or $x = 1$ can be easily found; hence, the fixed points are $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$. If $z = \frac{1}{4}$, then $y = \frac{\sqrt{3}}{4}$ and $x = \frac{3-\sqrt{3}}{4}$ can be found by using the first and third equation in system Eq. (12). Therefore, the fixed point is $\left(\frac{3-\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}\right)$.

2. To investigate the behavior of $\{W_1^{(n)}\}_{n=1}^{\infty}$, the following regions are introduced:

$$B_1 := \{x_1^{(0)} \in S^2 : 0 < z^{(0)} \le x^{(0)} < y^{(0)} < \frac{1}{2}\},$$

$$B_2 := \{x_1^{(0)} \in S^2 : 0 \le y^{(0)} \le z^{(0)} \le x^{(0)} \le \frac{1}{2}\},$$

$$B_3 := \{x_1^{(0)} \in S^2 : 0 < x^{(0)} \le y^{(0)} \le z^{(0)} \le \frac{1}{2}\},$$

$$B_4 := \{x_1^{(0)} \in S^2 : 0 < z^{(0)} \le x^{(0)} \le \frac{1}{3}, \quad \frac{1}{3} < y^{(0)} \le \frac{1}{2}\}.$$

Subsequently, the behavior of $\left\{W_1^{(n)}\right\}_{n=1}^{\infty}$ across all the previously mentioned regions will be explored. After that, the behavior of $\left\{W_1^{(n)}\right\}_{n=1}^{\infty}$ will be simple to describe.

Given that $y', x' \leq \frac{1}{2}$, we can easily conclude $x' \leq y'$, thereby indicating that B_2 cannot possibly be an invariant region. Subsequently, we intend to verify whether B_3 is or is not an invariant region. Let $x_1^{(0)} \in B_3$. Then,

$$z' = (z^{(0)})^2 + (y^{(0)})^2 < (x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2 \leq x^{(0)}z^{(0)} + y^{(0)}z^{(0)} + (z^{(0)})^2$$
$$= z^{(0)}(x^{(0)} + y^{(0)} + z^{(0)}) = z^{(0)}$$

We determine that $\frac{z'}{z^{(0)}} < 1$, which indicates that $z^{(n)}$ is a decreasing bounded sequence. By this we mean that $z^{(n)}$ converges to a fixed point of zero, thereby negating our presumption. Thus, region B_3 is not invariant. Then, we consider a new sequence $x' + z' = 2(x^{(0)})^2 - 2x^{(0)} + 1$. The new sequence has a minimum value of $\frac{1}{2}$, which indicates that all coordinates are greater than zero and smaller than $\frac{1}{2}$. Hence, if $x_1^{(0)} \in B_2 \cup B_3 \cup A_1$, then $n_{k_1}, n_{k_2}, n_{k_3} \in \mathbb{N}$, such that the sequences $x^{(n_{k_1})}$, $y^{(n_{k_2})}$, and $z^{(n_{k_3})}$ return to invariant region B_3. Let $x^{(0)} \leq \frac{1}{3}$. We can easily check whether the maximum value of the first coordinate $x' = (x^{(0)})^2 + 2y^{(0)}(1 - x^{(0)} - y^{(0)})$ occurs when $\left(\frac{1}{3}, \frac{1}{3}\right)$. In this case, $x^{(n)} \leq \frac{1}{3}$ and $z^{(n)} \leq \frac{1}{3}$. Given that all coordinates are equal to one, we conclude that $y^{(n)} \geq \frac{1}{3}$. Therefore, if $x_1^{(0)} \in B_1$, then $n_k \in \mathbb{N}$, such that $\left\{W_1^{(n_k)}\right\}_{n=1}^{\infty}$ returns to B_4. Hence, B_4 is an invariant region.

We have proven that if $x_1^{(0)} \in B_i, i \in \{1, \ldots, 3\}$, then the trajectory $\{W_1^{(n)}\}_{n=1}^{\infty}$ goes to invariant region B_4. Thus, exploring the behavior of $\{W_1^{(n)}\}_{n=1}^{\infty}$ over region B_4 is adequate. Evidently, $y^{(n)} \leq y^{(n+1)}$, i.e. it is a bounded increasing sequence. Given that $y^{(n)} + x^{(n)}$ is a bounded decreasing sequence and $x^{(n)} = y^{(n)} - y^{(n)} + x^{(n)}$, we conclude that $x^{(n)}$ is a decreasing bounded sequence that converges to $\frac{3-\sqrt{3}}{4}$. Thus, we have $y^{(n)}$ converging to $\frac{\sqrt{3}}{4}$. Therefore, $\omega_{W_1}\left(x_1^{(0)}\right) = \{\left(\frac{3-\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}\right)\}$, which is the desired conclusion.

Let the following regions:

$$int\left(S^2\right) = \{x_1^{(0)} \in S^2 : x^{(0)}y^{(0)}z^{(0)} > 0\},$$
$$\overline{int\left(S^2\right)} = \{x_1^{(0)} \in S^2 : x^{(0)}y^{(0)}z^{(0)} = 0\},$$

$$L_1 = \{x_1^{(0)} \in S^2 : x^{(0)} = 0\},$$
$$L_2 = \{x_1^{(0)} \in S^2 : y^{(0)} = 0\},$$
$$L_3 = \{x_1^{(0)} \in S^2 : z^{(0)} = 0\}$$

Theorem 3.

Let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (11), $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2) \cup Per_2(W_2)$ belongs to simplex S^2 as initial point. Then, the three statements below are valid:

1. One has $Fix(W_2) = \left\{e_3, (x^{*}, y^{*}, z^{*})\right\}$, where

$$x^{*} = \frac{-1}{6}\sqrt[3]{t} - \frac{8}{3\sqrt[3]{t} + \frac{5}{3}},$$

$$y^{*} = \frac{-1}{6}\frac{3\sqrt{17}\sqrt[3]{t} + 2\sqrt[3]{t^2} - 24\sqrt{17} - 5\sqrt[3]{t} - 88}{\sqrt[3]{t^2}},$$

$$z^{*} = \frac{-1}{6}\frac{2\sqrt[3]{t^2} - 3\sqrt{17}\sqrt[3]{t} - 11\sqrt[3]{t}, +6\sqrt{17} - 10}{\sqrt[3]{t^2}}, \text{ and } t = \left(98 + 18\sqrt{17}\right).$$

2. One has $Per_2(W_2) = \begin{cases} e_3, (0, y^{\circ}, 1 - y^{\circ}), & if \quad x^{(0)} = 0 \\ e_3, (x^{\circ}, 0, 1 - x^{\circ}), & if \quad y^{(0)} = 0 \end{cases}$

   where $y^{\circ} = \frac{1}{6}(1 + 3\sqrt{57})^{\frac{1}{3}} - \frac{4}{3(1 + 3\sqrt{57})^{\frac{1}{3}}} + \frac{2}{3}$

$$x^{\circ} = \frac{-1}{6}(46 + 6\sqrt{57})^{\frac{1}{3}} - \frac{2}{3(46 + 6\sqrt{57})^{\frac{1}{3}}} + \frac{4}{3}$$

3. One has

$$\omega_{w_2}(x_1^{(0)}) = \begin{cases} (x^{*}, y^{*}, z^{*}) & , if \quad x_1^{(0)} \in int\left(S^2\right) \\ (x^{\circ}, 0, 1 - x^{\circ}), (0, y^{\circ}, 1 - y^{\circ}) & , if \quad x_1^{(0)} \in \overline{int\left(S^2\right)} \\ e_3 & , if \quad x^{(0)}, y^{(0)} = 1 \end{cases}$$

Proof. Let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (11), $x_1^{(0)} \notin Fix(W_2) \cup Per_2(W_2)$ belongs to S^2, and $\{W_2^{(n)}\}_{n=1}^{\infty}$ be a trajectory of W_2 beginning at $x_1^{(0)}$.

1. The set of fixed points of W_2 is obtained by finding the solution for the following system of equations:

$$\begin{cases} x = 2yz \\ y = 2x - x^2 \\ z = z^2 + y^2 \end{cases} \tag{13}$$

Based on the first equation in system Eq. (13), we have $z = \frac{x}{2y}$. By using $z = 1 - y - x$ and the second equation in system Eq. (13), we obtain $3x - 14x^2 + 10x^3 - 2x^4$ $3x - 14x^2 + 10x^3 - 2x^4 = 0$. Thus, the roots of the previous equation are $\{0, x^{\cdot}\}$. By compensating for the values of x, namely, $x = 0$ and $x = x^{\bullet}$ in the second equation in system Eq. (13), we obtain $y = 0$ and $z = 1$ or $y = y^{\bullet}$ and $z = z^{\bullet}$. Therefore, the fixed points of W^2 are e_3 and $(x^{\bullet}, y^{\bullet}, z^{\bullet})$.

2. To find the 2-periodic points of W_2, we should prove that W_2 has no specified order of periodic points in set $S^2 \backslash L_1 \cup L_2$. Evidently, $y^{(n+1)} \geq y^{(n)}$, i.e. the second coordinate of W_2 increases along the iteration of W_2 in set $S^2 \backslash L_2$. Consider a new sequence $x' + y' = 2x^{(0)} - \left(x^{(0)}\right)^2 + 2y^{(0)}(1 - x^{(0)} - y^{(0)})$. Whether $x' + y'$ is a decreasing sequence can easily be checked, thereby indicating that sequence $x^{(n)}$ is decreasing, because $x^{(n)} = y^{(n)} + y^{(n)}$. Thus, the first coordinate of W_2 decreases along the iteration of W_2 in set $S^2 \backslash L_1$, which indicates that W_2 has no specified order 2-periodic points in set $S^2 \backslash L_1 \cup L_2$. Therefore, finding the 2-periodic points of W_2 in $L_1 \cup L_2$ is sufficient. To find the 2-periodic points, the succeeding system of equations should be solved:

$$\begin{cases} x = 2(2x - x^2)(y^2 + z^2) \\ y = 4yz - 4y^2 z^2 \\ z = (2x - x^2)(y^2 + z^2)^2 \end{cases} \tag{14}$$

We start when $x = 0$. Then, we find the solution for $y = 4y - 8y^2 + 8y^3 - 4y^4$. We obtain the following solution: $y = 0$ or $y = y^{\circ}$. If $y = 0$, then $z = 1$. If $y = y^{\circ}$, then $z = 1 - y^{\circ}$. Therefore, e_3 and $(0, y^{\circ}, 1 - y^{\circ})$ are 2-periodic points. On the other hand, if $y = 0$, then the solutions for the equation $x =$

$2(2-x-x^2)(1-x)^2$ are $x=0$ or $x=x^\circ$. If $x=0$, then $z=0$; if $x=x^\circ$, then $z=1-x^\circ$. Therefore, e_3 and $(x^\circ,0,1-x^\circ)$ are 2-periodic points.

3. To investigate the behavior of W_2, the following regions are introduced:

$$\ell_1 := \{x_1^{(0)} \in int\left(S^2\right): 0 < x^{(0)}, y^{(0)}, z^{(0)} \leq \frac{1}{2}\};$$

$$\ell_2 := \{x_1^{(0)} \in int\left(S^2\right): 0 < x^{(0)} \leq z^{(0)} \leq y^{(0)} \leq \frac{1}{2}\}.$$

Evidently, $y' = 2x^{(0)} - (x^{(0)})^2 \geq x^{(0)}$ and $x' = 2y^{(0)}z^{(0)} \leq (y^{(0)})^2 + (z^{(0)})^2 = z'$, which indicates that $x^{(n)} \leq z^{(n)}$ and $x^{(n)} \leq y^{(n)}$. To prove that ℓ_1 is an invariant region, assume that $y' \geq \frac{1}{2}$ by using the first coordinate of W_2. Then we have $x' = 2y^{(0)}z^{(0)}$, which implies that $x' \geq z'$. This relation is a contradiction because $x' \leq z'$. Thus, $y^{(n)} \leq \frac{1}{2}$. Now, suppose that $z' \geq \frac{1}{2}$. By using the first coordinate in W_2, we obtain $x' \geq y'$, which is also a contradiction. Therefore, ℓ_1 is an invariant region. Moreover, if $x_1^{(0)} \in \overline{\ell_1}$, then $n_k \in \mathbb{N}$, such that $W_2^{(n_k)}$ returns to invariant region ℓ_1. Let us complete proving that ℓ_2 is an invariant region. If we assume that $y' \leq z'$, this indicates that $z' = (z^{(0)})^2 + (y^{(0)})^2 \leq 2(z^{(0)})^2$, i.e. $\frac{z'}{z^{(0)}}<1$. Therefore, $z^{(n)}$ is a decreasing bounded sequence. That is, $z^{(n)}$ converges to the fixed point of zero. Moreover, $y^{(n)}$ is an increasing bounded sequence. Thus, $y^{(n)}$ converges to zero. Whether $y^{(n)}$ converges to zero if $x^{(n)}$ converges to zero can be checked. The result implies that the limiting point for W_2 is empty, which is a contradiction. Thus, $n_k \in \mathbb{N}$, such that $z^{(n_k)}$ returns to invariant region $z' \leq y'$, which proves that ℓ_2 is an invariant region. Moreover, if $x_1^{(0)} \in \ell_1$, then $n_k \in \mathbb{N}$, such that $\{W_2^{(n_k)}\}_{n=1}^{\infty}$ returns to invariant region ℓ_2.

Accordingly, the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ can be described. As discussed in the proof of the second part of this theorem, we determine that the first and second coordinates, namely, $x^{(n)}$ and $y^{(n)}$ are decreasing and increasing sequences, respectively. Thus, $x^{(n)}$ and $y^{(n)}$ converge to a certain fixed point. The first and second coordinates of W_2 converge; thus, the third coordinate also converges. Between the two fixed points, the properties mentioned above of W_2 are only satisfied by point $(x^{\bullet}, y^{\bullet}, z^{\bullet})$. Therefore, the limiting point is $\omega_{W_2}(x_1^{(0)}) = (x^{\bullet}, y^{\bullet}, z^{\bullet}),\ \forall x_1^{(0)} \in int\left(S^2\right)$.

To explore the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ when $x_1^{(0)} \in \overline{int\left(S^2\right)}$, consider three cases, i.e. when $x^{(0)} = 0, y^{(0)} = 0$, and $z^{(0)} = 0$. If $x^{(0)} = 0$, then $V^{(1)}((0, y^{(0)}, z^{(0)})) = (x', 0, 1 - x')$ and $V^{(2)}((0, y^{(0)}, z^{(0)})) = (0, y', 1 - y')$. By applying this process to the next iteration, we determine that $V^{(2n+1)}((0, y^{(0)}, z^{(0)})) = (x^{(2n+1)}, 0, 1 - x^{(2n+1)})$ and $V^{(2n)}((0, y^{(0)}, z^{(0)})) = (0, y^{(2n)}, 1 - y^{(2n)})$. That is, the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ in this case will be on the *xz*-plane if *n* is an odd iteration and on the *yz*-plane if *n* is an even iteration. When the preceding process is performed when $y^{(0)} = 0$, we find that $V^{(2n+1)}((x^{(0)}, 0, z^{(0)})) = (0, y^{(2n+1)}, 1 - y^{(2n+1)})$ and $V^{(2n)}((x^{(0)}, 0, z^{(0)})) = (x^{(2n)}, 0, 1 - x^{(2n)})$. That is, the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ in this case will be on the *yz*-plane if *n* is an odd iteration and on the *xz*-plane if *n* is an even iteration. Through the same process, we determine that $V^{(2n+1)}((x^{(0)}, 0, z^{(0)})) = (0, y^{(2n+1)}, 1 - y^{(2n+1)})$ and $V^{(2n)}((x^{(0)}, 0, z^{(0)})) = (x^{(2n)}, 0, 1 - x^{(2n)})$ when $z^{(0)} = 0$. This indicates that $n_k \in \mathbb{N}$, such that the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ in the case of $z^{(0)} = 0$ will be on the *yz*-plane if *n* is an odd iteration and on the *xz*-plane if *n* is an even iteration. Therefore, studying the two cases when $x^{(0)} = 0$ and $y^{(0)} = 0$ is sufficient. Starting with $x^{(0)} = 0$, consider the following function:

$$y^{(2)} = \nu(y^{(0)}) = 4y^{(0)} - 8(y^{(0)})^2 + 8(y^{(0)})^3 - 4(y^{(0)})^4 \tag{15}$$

where $y^{(0)} \in (0,1)$. Now, $Fix(\nu) \cap (0,1) = \{y^{\circ}\}$ can be shown. Through simple calculations, $\nu\left((0, \frac{1}{2}]\right) \subseteq [\frac{1}{2}, 1)$ can be found. Thus, we conclude that $[\frac{1}{2}, 1)$ is sufficient to study the dynamics of ν at interval (0,1).

To investigate the behavior of *v*, the interval $[\frac{1}{2}, 1)$ is divided into three intervals as follows: $I_1 = [\frac{1}{2}, y^{\circ}]$, $I_2 = [y^{\circ}, \frac{1}{2} + \frac{1}{2}\sqrt{\sqrt{2} - 1}]$, and $I_3 = [\frac{1}{2} + \frac{1}{2}\sqrt{\sqrt{2} - 1}, 1)$. Evidently, $\nu(\nu(y^{(0)})) \ge y^{(0)}$ when $y^{(0)} \in I_1$ and

$\nu(\nu(y^{(0)})) \leq y^{(0)}$ when $y^{(0)} \in I_2$. Therefore, two cases should be discussed separately.

a. For any $n \in \mathbb{N}$, $\nu^{(2n+2)}(y^{(0)}) \geq \nu^{(2n)}(y^{(0)}) \forall y^{(0)} \in I_1$ can be easily shown. Thus, $\nu^{(2n)}\left(y^{(0)}\right)$ is an increasing bounded sequence. Furthermore, $\nu^{(2n)}(y^{(0)})$ converges to a fixed point of $\nu^{(2)}$. y° is also a fixed point of $\nu^{(2)}$ and is the only possible point of the convergence trajectory. Hence, sequence $y^{(2n)}$ converges to y°.

b. Similarly, $\left\langle \nu^{(2n+2)}(y^{(0)}) \leq \nu^{(2n)}(y^{(0)}) \right\rangle \forall y^{(0)} \in I_2$. Thus, $\nu^{(2n)}$ is a decreasing bounded sequence. Furthermore, $\nu^{(2n)}(y^{(0)})$ converges to a fixed point of $\nu^{(2)}$. y° is also a fixed point of $\nu^{(2)}$ and it is the only possible point of the convergence trajectory. Hence, sequence $y^{(2n)}$ converges to y°.

To explore the behavior of ν when $y^{(0)} \in I_3$, the following claim is required:

Claim: Let $y^{(0)} \in I_3$. Then, $n_k \in \mathbb{N}$, such that $\nu^{(n_k)} \in I_1 \cup I_2$.

Proof. Let $y^{(0)} \in I_3$. Suppose that the interval I_3 is an invariant interval, which indicates that $y^{(n)} \in I_3$ for any $n \in \mathbb{N}$. Evidently, $\nu^{(n+1)}(y^{(0)}) \leq \nu^{(n)}(y^{(0)})$, which results in $\nu^{(n)}$ being a decreasing bounded sequence and converging to a fixed point ν. However, $Fix(\nu) \cap I_3 = \varnothing$, which is a contradiction. Hence, $n_k \in \mathbb{N}$, such that $\nu^{(n_k)} \in I_1 \cup I_2$.

In accordance with this claim, $y^{(2n)}$ will go to $I_1 \cup I_2$ after several iterations. Thus, sequence $(0, y^{(2n)}, z^{(2n)})$ converges to $(0, y^{\circ}, 1-y^{\circ})$ whenever $x^{(0)} = 0$.

Let $y^{(0)} = 0$ and consider the following function:

$$x^{(2)} = \vartheta(x^{(0)}) = 4x^{(0)} - 10(x^{(0)})^2 + 8(x^{(0)})^3 - 2(x^{(0)})^4, \tag{16}$$

where $x^{(0)} \in (0,1)$. When we do this, $Fix(\vartheta) \cap (0,1) = \{x^{\circ}\}$ can be easily shown. Through simple calculations, we determine $\vartheta\left(\left[0, 1-\frac{1}{2}\sqrt{2}\right] \subseteq [1-\frac{1}{2}\sqrt{2}, 1\right)$ and conclude that $[1-\frac{1}{2}\sqrt{2}, 1)$ is sufficient to study the dynamics of ϑ on (0,1).

To investigate the behavior of ϑ, the invariant interval $[1-\frac{1}{2}\sqrt{2},1)$ is divided into three intervals as follows: $I_1=[1-\frac{1}{2}\sqrt{2},x^{\circ}]$, $I_2=[x^{\circ},\frac{1}{2}]$, and $I_3=[\frac{1}{2},1)$.

Thus, we have two separate cases:

a. Let $x^{(0)}\in I_1$, then $\vartheta(x^{(0)})\in I_2$ and, $\vartheta^{(2)}(x^{(0)})\in I_1$. $\vartheta^{(2n+2)}(x^{(0)})\le\vartheta^{(2n)}(x^{(0)})$ whenever $x^{(0)}\in I_1$ can be easily checked. Therefore, $\vartheta^{(2n)}$ is a decreasing bounded sequence that converges to a fixed point of $\vartheta^{(2)}$. x° is a fixed point of $\vartheta^{(2)}$ and the only possible point of the convergence trajectory. Hence, $\vartheta^{(2)}$ converges to x°.

b. Similarly, let $x^{(0)}\in I_2$, then $\vartheta(x^{(0)})\in I_1$ and $\vartheta^{(2)}(x^{(0)})\in I_2$. $\vartheta^{(2n+2)}(x^{(0)})\ge\vartheta^{(2n)}(x^{(0)})$ whenever $x^{(0)}\in I_2$ can be easily checked. Therefore, $\vartheta^{(2n)}$ is an increasing bounded sequence that converges to a fixed point of $\vartheta^{(2)}$. x° I s a fixed point of $\vartheta^{(2)}$ and the only possible point of the convergence trajectory. Hence, $\vartheta^{(2n)}$ converges to x°.

To explore the behavior of ϑ, when $x^{(0)}\in I_3$, the following claim is required:
Claim: Let $x^{(0)}\in I_3$. Then, $n_k\in\mathbb{N}$, such that $\vartheta^{(n_k)}\in I_1\cup I_2$.

Proof. Let $x^{(0)}\in I_3$. Suppose that interval I_3 is invariant, which indicates that $x^{(n)}\in I_3$ for any $n\in\mathbb{N}$. Evidently, $\vartheta^{(n+1)}(x^{(0)})\le\vartheta^{(n)}(x^{(0)})$, which results in sequence $\vartheta^{(n)}$ being decreasingly bounded and converging to a fixed point of ϑ. However, $Fix(\vartheta)\cap I_3=\varnothing$, which is contradiction. Hence, $n_k\in\mathbb{N}$, such that $\vartheta^{(n_k)}\in I_1\cup I_2$.

In accordance with the claim, $x^{(n)}$ will go to $I_1\cup I_2$ after several iterations. Thus, sequence $(x^{(2n)},0,z^{(2n)})$ converges to $(x^{\circ},0,1-x^{\circ})$ whenever $y^{(0)}=0$. Alternatively, if $x^{(0)}=0$, then

$$V^{(n)}(W_2)=\begin{cases}(0,y^{\circ},1-y^{\circ}) & ,if\quad n=2k\\ \left(x^{\circ},0,1-x^{\circ}\right) & ,if\quad n=2k+1\end{cases}\tag{17}$$

If $y^{(0)}=0$, then

$$V^{(n)}(W_2)=\begin{cases}(0,y^{\circ},1-y^{\circ}) & ,if \quad n=2k+1\\ \left(x^{\circ},0,1-x^{\circ}\right) & ,if \quad n=2k\end{cases} \tag{18}$$

From the preceding demonstrations, we observe that if $x^{(0)}=0$ and n is even, then the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ occurs in $(0,y^{\circ},1-y^{\circ})$, which is equal to the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ when $y^{(0)}=0$ and n is an odd iteration. If $y^{(0)}=0$ and n is an even iteration, then the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ occurs in $(x^{\circ},0,1-x^{\circ})$, which is equal to the behavior of W_2 when $x^{(0)}=0$ and n is an odd iteration. Therefore, the limiting point of W_2 consists of $(x^{\circ},0,1-x^{\circ})$ and $(0,y^{\circ},1-y^{\circ})$ whenever $x^{(0)}\notin int\left(S^2\right)$. If $x^{(0)}=1$, then the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ reaches fixed point e_3 after three iterations; if $y^{(0)}=1$, then the behavior of $\{W_2^{(n)}\}_{n=1}^{\infty}$ reaches fixed point e_3 after one iteration. Therefore, the limiting point in this case includes e_3, which is the desired conclusion.

Subsequently, the behavior of operator $\{V_{15}^{(n)}\}_{n=1}^{\infty}$ selected from class G_9 is explored:

$$V_{15}:=\begin{cases}x'=\left(1-\alpha\right)(x^{(0)})^2+2y^{(0)}z^{(0)}\\ y'=\alpha(x^{(0)})^2+2x^{(0)}\left(1-x^{(0)}\right)\\ z'=(z^{(0)})^2+(y^{(0)})^2\end{cases} \tag{19}$$

The operator V_{15} can be redrafted as a convex combination $V_{15}=\left(1-\alpha\right)W_1+\alpha W_2$, where W_1 and W_2 are equal to the operators given by Eq. (10) and Eq. (11), respectively.

Corollary 1. Let W_1 be a $\xi^{(as)}$-QSO given by Eq. (10) and let $x_1^{(0)}=(x^{(0)},y^{(0)},z^{(0)})\notin Fix(W_1)$ belong to simplex S^2 as initial point. Then, the two statements below are valid:

1. One has $Fix(W_1)=\left\{e_1,e_3,(\frac{3-\sqrt{3}}{4},\frac{\sqrt{3}}{4},\frac{1}{4})\right\}$

2. One has $\omega_{w_1}(x_1^{(0)}) = \left\{ (\frac{3-\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4}) \right\}$.

Let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (11) and let $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2)$ belong to simplex S^2 as initial point. Then, the following statements hold true:

1. One has $Fix(W_2) = \left\{ e_3, (x^{\square}, y^{\square}, z^{\square}) \right\}$.

   where, $x^{\square} = \frac{-1}{6}\sqrt[3]{t} - \frac{8}{3\sqrt[3]{t} + \frac{5}{3}}$,

$$y^{\square} = \frac{-1}{6} \frac{3\sqrt{17}\sqrt[3]{t} + 2\sqrt[3]{t^2} - 24\sqrt{17} - 5\sqrt[3]{t} - 88}{\sqrt[3]{t^2}},$$

$$z^{\square} = \frac{-1}{6} \frac{2\sqrt[3]{t^2} - 3\sqrt{17}\sqrt[3]{t} - 11\sqrt[3]{t} + 6\sqrt{17} - 10}{\sqrt[3]{t^2}}$$

and $t = \left(98 + 18\sqrt{17}\right)$.

2. One has $Per_2(W_2) \begin{cases} e_3, (0, y^{\circ}, 1-y^{\circ}) & , if \quad x^{(0)} = 0 \\ e_3, (x^{\circ}, 0, 1-x^{\circ}) & , if \quad y^{(0)} = 0 \end{cases} \}$

   where, $y^{\circ} = \frac{1}{6}(1+3\sqrt{57})^{\frac{1}{3}} - \frac{4}{3(1+3\sqrt{57})^{\frac{1}{3}}} + \frac{2}{3}$,

$$x^{\circ} = \frac{-1}{6}(46+6\sqrt{57})^{\frac{1}{3}} - \frac{2}{3(46+6\sqrt{57})^{\frac{1}{3}}} + \frac{4}{3}.$$

3. One has

$$\omega_{w_2}(x_1^{(0)}) = \begin{cases} (x^{\square}, y^{\square}, z^{\square}) & , if \quad x_1^{(0)} \in int\left(S^2\right) \\ (x^{\circ}, 0, 1-x^{\circ}), (0, y^{\circ}, 1-y^{\circ}) & , if \quad x_1^{(0)} \notin int\left(S^2\right) \\ e_3 & , if \quad x^{(0)}, y^{(0)} = 1 \end{cases}$$

5 Dynamics of classes G_{13} and G_{14}

In this section, we study the dynamics of $V_{26,25}: S^2 \to S^2$ selected from G_{14} and G_{13}. To start, V_{26} is rewritten as follows:

$$V_{26} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \left(1-\alpha\right)(x^{(0)})^2 \end{cases} \tag{20}$$

The operator V_{26} can be redrafted as a convex combination $V_{26} = \alpha W_1 + (1-\alpha)W_2$, where

$$W_1 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = 0 \end{cases} \tag{21}$$

and

$$W_2 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = 2y^{(0)}z^{(0)} \\ z' = (x^{(0)})^2 \end{cases} \tag{22}$$

Theorem 4. Let W_1 be a $\xi^{(as)}$-QSO given by Eq. (20) and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_1) \cup Per_2(W_1)$ belongs to simplex S^2 as initial point. Then, the three statements below are valid:

1. One has $Fix(W_1) = \left\{\left(\frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}, 0\right)\right\}$
2. One has $Per_2(W_1) = \left\{e_1, e_2, (\frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{(-1+\sqrt{5})^2}{4}, 0)\right\}$,
3. $\omega_{W_1}(x_1^{(0)}) = \{e_1, e_3\}$.

Proof. Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (22), $x_1^{(0)} \notin Fix(W_1) \cup Per_2(W_1)$ belongs to S^2, and $\{W_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of W_1 beginning from point $x_1^{(0)}$.

1. The set of fixed points of W_1 is obtained by finding the solution to the set of equations below:

$$\begin{cases} x = y^2 + z^2 + 2x(1-x) \\ y = x^2 + 2yz \\ z = 0 \end{cases} \tag{23}$$

By substituting the second and third equations Eq. (22) to the first equation, then the first equation in system Eq. (22) becomes $x^4 - 2x^2 + x$, then $x = 0$, $x = 1$, and $x = \frac{\sqrt{5}}{2} - \frac{1}{2}$. $\frac{\sqrt{5}}{2} - \frac{1}{2}$ is verified as the only solution that satisfies system Eq. (22). Hence, the fixed point is only $\left(\frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}, 0\right)$.

2. Let $x_1^{(0)} = (1,0,0)$. Then, $V^{(1)}(x^0, y^0, z^0) = (0,1,0)$ and $V^{(2)}(x^0, y^0, z^0) = (1,0,0)$, which indicates the presence of 2-periodic points. To find all the points, the following system of equations should be solved:

$$\begin{cases} x = 2x^2 - x^4 \\ y = (1-(1-y)^2)^2 \\ z = 0 \end{cases} \tag{24}$$

From the first equation in system Eq. (23), $x \in \left\{0,1, \frac{\sqrt{5}}{2} - \frac{1}{2}\right\}$ then $y \in \left\{1,0, \frac{(-1+\sqrt{5})^2}{4}\right\}$. Therefore, $Per_2(W_1) = \{e_1, e_2, \left(\frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{(-1+\sqrt{5})^2}{4}, 0\right)\}$.

3. Let $x_1^{(0)} \notin Fix(W_1) \cup Per_2(W_1)$. L_3 is an invariant line under W_1. Thus, the behavior of $\{W_1^{(n)}\}_{n=1}^{\infty}$ is explored over this line. Let $x_1^{(0)} \in L_3$. Then, W_1 becomes:

$$\begin{cases} x' = (y^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = (x^{(0)})^2 \\ z' = 0 \end{cases} \tag{25}$$

In this case, the first coordinate of W_1 exhibits the form $x' = \varphi(x^{(0)}) = (1 - x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)})$. Clearly, the function φ decreases on [0,1] and function $\varphi^{(2)}$ is increasing on [0,1]. From the previous two steps, $Fix(\varphi) \cap [0,1] = \left\{\frac{\sqrt{5}}{2} - \frac{1}{2}\right\}$ and $ix(\varphi^2) \cap [0,1] = \left\{0, \frac{\sqrt{5}}{2} - \frac{1}{2}, 1\right\}$, which indicates that intervals $\left[0, \frac{\sqrt{5}}{2} - \frac{1}{2}\right]$ and $\left[\frac{\sqrt{5}}{2} - \frac{1}{2}, 1\right]$ are invariant under the function (φ^2). Evidently, $\varphi^{(2)}(x^{(0)}) \le x^{(0)}$ for any $x^{(0)} \in \left[0, \frac{\sqrt{5}}{2} - \frac{1}{2}\right]$ and $\varphi^{(2)}(x^{(0)}) \ge x^{(0)}$ for any $x^{(0)} \in \left[\frac{\sqrt{5}}{2} - \frac{1}{2}, 1\right]$. If $x^{(0)} \in \left[0, \frac{\sqrt{5}}{2} - \frac{1}{2}\right]$, then $\omega_{\varphi^{(2)}} = \{0\}$; if $x^{(0)} \in \left[\frac{\sqrt{5}}{2} - \frac{1}{2}, 1\right]$, then $\omega_{\varphi^{(2)}} = \{1\}$. In another way,

$$V^{(n)}(W_1) = \begin{cases} \left(\varphi^{(2k)}(x^{(0)}), 1 - \varphi^{(2k)}(x^{(0)}), 0\right) & , if \quad n = 2k \\ \left(\varphi^{(2k)}(\varphi(x^{(0)})), 1 - \varphi^{(2k)}(\varphi(x^{(0)})), 0\right) & , if \quad n = 2k+1 \end{cases} \tag{26}$$

Therefore, the limiting point is $\omega_{W_1}\left(x_1^{(0)}\right) = \{e_1, e_2\}$.

Theorem 5. Let W_2 be a $\xi^{(as)}$-QSO given by Eq. (21) and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2) \cup Per_2(W_2)$ belong to simplex S^2 as initial point. Then, the two statements below are valid:

1. One has $Fix(W_2) = \emptyset$. Moreover ,

   $Per_2(W_2) = \{e_1, e_3, \left(\frac{\sqrt{5}-1}{2}, 0, \frac{1}{16}\left(\sqrt{5}-3\right)^2\left(\sqrt{5}+1\right)^2\right)\}$.

2. One has $\omega_{W_2}\left(x_1^{(0)}\right) = \{e_1, e_3\}$

Proof. Let $W_2: S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by Eq. (21), $x_1^{(0)} \notin Fix(W_2) \cup Per_2(W_2)$ belong to S^2, and $\left\{W_2^{(n)}\right\}_{n=1}^{\infty} = \frac{n!}{r!(n-r)!}$ be a trajectory of W_2 starting at $x_1^{(0)}$.

1. The set of fixed points of W_2 is obtained by finding the solution for the equation set below:

$$\begin{cases} x = y^2 + z^2 + 2x(1-x) \\ y = 2yz \\ z = x^2 \end{cases} \tag{27}$$

   The system provided by Eq. (26) has no solution on [0,1]. Therefore, the set of fixed points is $\emptyset$. The second coordinate of W_2 increases if $z^{(n)} \ge \frac{1}{2}$

and decreases if $z^{(n)} \leq \frac{1}{2}$. In both cases, W_2 has no specified order of periodic points in set $W_2 \backslash L_2$, because the second coordinate of W_2 increases or decreases along the iteration of $W_2 \backslash L_2$. Therefore, finding the 2-periodic points of $W_2 \backslash L_2$ over L_2 is sufficient. To find the 2-periodic points of W_2, the following system of equations should be solved:

$$\begin{cases} x = x^4 + 2x^2(1-x^2) \\ y = 0 \\ z = (1-(1-z)^2)^2 \end{cases} \tag{28}$$

The solution for the first equation in system Eq. (27) is easy to find. Therefore, the periodic points of W_2 are e_1, e_3, and $\left(\frac{\sqrt{5}-1}{2}, 0, \frac{1}{16}\left(\sqrt{5}-3\right)^2\left(\sqrt{5}+1\right)^2\right)$.

2. Let $x_1^{(0)} \notin Fix(W_2) \cup Per_2(W_2)$ and $y^{(0)} = 0$. The first coordinate of W_2 can be rewritten as $x' = (1-x^{(0)})^2 + 2x^{(0)}(1-x^{(0)})$ because the second coordinate is invariant over L_2. The first coordinate is equal to the first coordinate of W_1, which was proven in the previous theorem. Hence, we derive

$$V^{(n)}(W_2) = \begin{cases} \left(\varphi^{(2k)}(x^{(0)}), 0, 1-\varphi^{(2k)}(x^{(0)})\right) & ,if \quad n = 2k \\ \left(\varphi^{(2k)}(\varphi(x^{(0)})), 0, 1-\varphi^{(2k)}(\varphi(x^{(0)}))\right) & ,if \quad n = 2k+1 \end{cases} \tag{29}$$

Therefore, we determine that $\omega_{W_2}\left(x_1^{(0)}\right) = \{e_1, e_3\}$. Let $y^{(0)} \notin L_2$ and $x^{(n)} < \frac{1}{2}$, which indicates that $z^{(n)} < \frac{1}{2}$ and yields $y^{(n+1)} < y^{(n)}$. If $x^{(n)} > \frac{1}{2}$, then the third coordinate $z^{(n)}$ is also smaller than $\frac{1}{2}$, which indicates that $y^{(n+1)} < y^{(n)}$. In the two previous cases, we conclude that $\frac{y^{(n+1)}}{y^{(n)}} < 1$, thereby making $y^{(n+1)}$ is a decreasing bounded sequence that converges to zero, which indicates that studying the dynamics of W_2 over L_2 was sufficient. Therefore, $\omega_{W_2}(x_1^{(0)}) = \{e_1, e_3\}$ for any initial point $x_1^{(0)}$ in S_2.

Subsequently, we explore the behavior of $\left\{V_{25}^{(n)}\right\}_{n=1}^{\infty}$, which is selected from class G_9.

$$V_{25} := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = \alpha(x^{(0)})^2 \\ z' = \left(1-\alpha\right)(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \tag{30}$$

We rewrite V_{25} as a convex combination$V_{25} = \alpha W_1 + (1-\alpha)W_2$, where

$$W_1 := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = (x^{(0)})^2 \\ z' = 2y^{(0)}z^{(0)} \end{cases} \tag{31}$$

and

$$W_2 := \begin{cases} x' = (y^{(0)})^2 + (y^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ y' = 0 \\ z' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \tag{32}$$

Corollary 2 Let $W_1 : S^2 \to S^2$ given by Eq. (30) be a $\xi^{(as)}$-QSO and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2) \cup Per_2(W_2)$ belong to simplex S^2 as initial point. Then, the statements below are valid:

1. One has $Fix(W_1) = \emptyset$. Moreover ,

   $Per_2(W_1) = \{e_1, e_2, \left(\frac{\sqrt{5}-1}{2}, \frac{1}{16}\left(\sqrt{5}-3\right)^2\left(\sqrt{5}+1\right)^2, 0\right)\}$.
2. One has $\omega_{W_2}\left(x_1^{(0)}\right) = \{e_1, e_2\}$

Let $W_2 : S^2 \to S^2$ given by Eq. (31) be a $\xi^{(as)}$-QSO. Then, the three statements below are valid:

1. One has $Fix(W_2) = \left\{\left(\frac{\sqrt{5}}{2} - \frac{1}{2}, 0, \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\right\}$
2. One has $Per_2(W_2) = \{e_1, e_3\}$
3. One has $\omega_{W_2}\left(x_1^{(0)}\right) = \{e_1, e_2\}$

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