AERONAUTIKA

LOCALLY LINEARIZED SOLUTION OF LIFTING TRANSONIC FLOW BY METHOD OF PARAMETRIC DIFFERENTIATION

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ABSTRACT

The governing equation of transonic small disturbance flow presents a non-linear partial differential equation which is difficult to solve. The method of parametric differentiation reduces the non-linear partial differential equation of transonic flow into an ordinary differential equation with variable coefficients, which is generally much simpler to solve. Further simplification is introduced, as also done in the method of local linearization, by assuming $(1-M_1^2)$ to vary sufficiently slowly, so that in some part of the analysis its derivatives with respect to x can be disregarded. Based upon these methods, the lifting transonic flow was analyzed. For the subsonic and supersonic parts, closed form solutions were obtained. For the case $M_1 \approx 1$, the method yields an integral equation, which can be solved by an iterative scheme starting from the non-lifting solution.

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LIST OF SYMBOLS

g	— a function defined by equation 4-3
g_x, g_z, g_{xx}, g_{zz}	— first derivatives of g with respect to x, z, second derivatives of g with respect to x, z, respectively
h	— a function defined by equation 3-3
p	— a parameter
u	— perturbation velocity along the x-direction
u_σ	— the derivative of u with respect to σ
u_c	— u velocity due to camber
u_t	— u velocity due to thickness
x	— coordinate variable along longitudinal direction
z	— coordinate variable along lateral direction
B	— a variable defined by equation 4-18
E	— a variable measuring thickness
F	— a variable measuring camber
L	— a function defined by equation 3-1
M	— Mach number
M_1	— local Mach number
M_∞	— free stream Mach number
γ	— specific heat ratio
β	— a variable defined by equation 4-6
φ	— perturbation velocity potential
σ	— dimensionless camber parameter
τ	— dimensionless thickness ratio
ζ	— dummy variable

superscript

'	— differentiation with respect to the argument

subscript

$x, y, \sigma, \tau,$	— differentiation with respect to x, y, σ and τ, respectively

I. INTRODUCTION

Small perturbation potential flow in the transonic regime presents a particularly interesting problem in that the governing differential equation is nonlinear, of mixed type and singular. Physically, in a transonic flow, local particle speeds both greater and less than sonic speed are found mixed-to-gether. Since the body travels at nearly the same speed as the forward going disturbances that it generates, the flow perturbations can be expected to be generally greater near free stream Mach number $M_{\infty} = 1$ than in purely subsonic and supersonic flows. This fact serves to indicate, why linearized theory, as employed in subsonic and supersonic flows, fails to predict the flow behavior in the transonic region. However, in the case of unsteady transonic flow, if the body oscillates rapidly, the nonlinear disturbance accumulation will not have time to develop and hence the linearized equation is applicable. Landahl (1961) has given an extensive account on the problems of unsteady transonic flow.

To solve the governing nonlinear differential equation, some simplification has to be introduced in the small disturbance potential flow, and this has attracted many investigators. In the two dimensional case, the hodograph method has avoided this difficulty by linearizing the differential equation without approximation, namely by interchanging the dependent and independent variables. A differential equation, associated with the name of Tricomi, is then obtained. This differential equation retains the mixed type nature of the original equation. However, complicated boundary conditions are produced as a penalty of the linearization process, confining the variety of cases (amenable to exact treatment) to two dimensional flows past a relatively small class of airfoil shapes. Furthermore, the more important three dimensional cases cannot be treated by this method. Exact solutions for the small perturbation flow can be obtained by the hodograph method, and serve as a reference in the evaluation of other approximate methods applicable for a larger class of airfoil shapes and three dimensional bodies. Guderley (1962) gives an extensive account on the application of the hodograph method to transonic flow problems.

A new method has recently been introduced by Rubbert and Landahl (1965, 1967), namely the method of parametric differentiation. This method reduce the nonlinear partial differential equation of transonic flow into an ordinary differential equation with variable coefficients, which is generally much simpler to solve. A short review on their method will be given, and an extension of their method to the case of lifting flows, but with further simplification of local linearization, will be presented.

II. THE DIFFERENTIAL EQUATION OF TRANSONIC FLOW

The differential equation of transonic small perturbation flow is given by:

$$[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x] \varphi_{xx} + \varphi_{zz} = 0 \qquad (2 - 1)$$

The derivation of this equation is given in the literature (for example, see Ashley and Landahl, 1965), starting from the Kelvin-Bernoulli differential equation for the velocity potential and assuming small disturbances.

III. METHOD OF PARAMETRIC DIFFERENTIATION

The method of parametric differentiation (Rubbert, 1965) reduces the nonlinear differential equation of transonic flow into one in which the nonlinearity is confined in a first order ordinary differential equation, hence it can be solved by existing methods.

Perturbation method has shown to be successful in many problems to minimize the nonlinearity of the problem. The propagation of disturbance in transonic flow is governed by local flow properties, and nonlinear interactions persist if the disturbances are sufficiently large.

A method was formulated, in which the flow about an airfoil is perturbed by a parameter. Let the solution sought be φ and let the governing equation be:

$$L(\varphi) = 0 \qquad (3 - 1)$$

with appropriate boundary condition. Let the solution depend on a parameter p, which can be conveniently chosen, and which may appear either in the differential equation, the boundary condition or both.

Differentiate equation (3 — 1) with respect to p to produce a linear equation in the differentiated variable of the form:

$$\bar{L}(h) = 0 \qquad (3 - 2)$$

where

$$h = \frac{\delta\varphi}{\delta\rho} \qquad (3 - 3)$$

$\bar{L}$ then becomes a linear differential operator for h, whose coefficients involve φ and its derivatives. Let the solution of (3 — 2) be

$$h = h(\varphi, \rho) \qquad (3 - 4)$$

Equation (3 — 3) and (3 — 4) provide a differential equation for φ, which can be solved by integration of

$$d\varphi = h(\varphi,\rho)\, d\rho \qquad (3 - 5)$$

Thus the nonlinearity is retained only in the first order differential equation (3 — 3). The constant arising from the integration of (3 — 5) is provided from a known condition, which is obtained from either the boundary condition or other (exact) solution.

IV. LIFTING FLOW

There exists at the present time no method for the calculation of lifting ransonic flow with satisfactory results for cases in which the measure of camber and thickness are comparable. The essential feature of lifting flow is the presence of a discontinuity in the u velocity component across the x axis. No singularity solution of (2 — 1) in the presence of such discontinuity has been found.

Rubbert (1965) discussed the problem in which the angle of attack and camber are much smaller than the thickness. The change in velocity due to thickness and camber can then be considered as a small perturbation on the non-lifting flow. The boundary condition on the airfoil is given by:

$$\varphi_z = \sigma E'(x) + \tau F'(x) \quad \text{at } z = 0+$$
$$\varphi_z = \sigma E'(x) - \tau F'(x) \quad \text{at } z = 0- \qquad (4-1)$$

where $\tau F(x)$ denotes thickness ratio and $\sigma E(x)$ gives the position of camber line.

An approximate solution may be obtained following the method of parametric differentiation. For the subsonic and supersonic cases, local linearization is employed as well. Following the method of parametric differentiation equation (2 — 1) can be differentiated with respect to σ to give:

$$[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x] \varphi_{\sigma xx} + \varphi_{\sigma zz} - (\gamma + 1) M_\infty^2 \varphi_{xx} \varphi_{\sigma x} = 0 \qquad (4-2)$$

Let

$$g = \frac{\delta\varphi}{\delta\sigma} \qquad (4-3)$$

hence equation (4 — 2) can be rewritten as:

$$\frac{\delta}{\delta x}\left[\{1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x\} g_x \right] + g_{zz} = 0 \qquad (4-4)$$

The boundary conditions accompanying equation 4 — 4 are given by:

$$\frac{\delta\varphi_z}{\delta\sigma} = g_z(x, 0 \pm) = E'(x) \qquad \text{at } z = \pm 0 \qquad (4-5)$$

(on the airfoil)

and $g \longrightarrow 0$ as $\sqrt{x^2 + z^2} \longrightarrow \infty$

As an approximation, write

$$1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x = 1 - M_l^2 = \beta^2 \qquad (4-6)$$

which will then be assumed to vary sufficiently slowly for the subsonic and supersonic cases. This assumption is equivalent to approximating the solution of equation (4 — 4) by expanding the coefficient about the point of interest and retaining the lowest order terms only. Similar assumption was taken in the method of local linearization (Spreiter, 1958). Hence equation (4 — 4) can be recast into

$$(1 - M_1^2)\, g_{xx} + g_{zz} = 0 \qquad 4-7$$

which bears similarity to the differential equation of linearized compressible flow. Working this out for the thickness problem, Rubbert (1965) has shown that such a procedure would lead to the result obtained by the local linearization method. However, the arbitrariness of the local linearization method in the step involved in the substitution of original value of β (see Spreiter, 1959; Rubbert, 1965) is then removed.

Three cases can be distinguished, namely:

(a) $1 - M_1^2 > 0$, for which equation (4 — 7) is elliptic, therefore analogous to the subsonic compressible case,

(b) $1 - M_1^2 < 0$, for which equation (4 — 7) is hyperbolic, hence analogous to the supersonic compressible case,

(c) $1 - M_1^2 = 0$, equation (4 — 7) then degenerates to one similar to the inner equation of the transonic small perturbation flow. A different treatment should be given as will be shown below in point (c).

(a) **Subsonic Flow:**

The governing equation is given by:

$$\beta^2 g_{xx} + g_{zz} = 0 \qquad (4-8)$$

where:

$$\beta^2 = 1 - M_1^2 = 1 - M_\infty^2 - M_\infty^2 (\gamma + 1)\, \varphi_x > 0 \qquad (4-9)$$

The solution satisfying the boundary condition is given by:

$$g_x = U\sigma = \frac{1}{\beta}\, U_{c\sigma} \qquad (4-10)$$

where:

$$U_{c\sigma} = \pm \frac{1}{\pi} \sqrt{\frac{1-x}{x}} \oint_0^1 \frac{E'(x)}{x - x_1} \sqrt{\frac{x_1}{1 - x_1}}\, dx_1 \qquad (4-11)$$

Substitution of the original value of β and integration result in:

$$\frac{2}{3M_\infty^2(\gamma + 1)} \left[1 - M_\infty^2 - M_\infty^2 (\gamma + 1)\, u\right]^{3/2} = U_c + C \qquad (4-12)$$

The lifting flow is here considered as a perturbation by a small parameter measuring camber (and angle of attack) from a known nonlifting (thickness) flow. To evaluate the constant C, then we note that $u = u_t$ if $u_c = 0$, i.e.: if the camber and angle of attack is zero, the solution reduces to the non-lifting case, where u_t is given by

$$u_t = \frac{1}{M_\infty^2\ (\gamma+1)} \{1 - M_\infty^2 - [(1 - M_\infty^2)^{3/2} - \frac{3}{2} M_\infty^2\ (\gamma + 1)\, u_i)\} \qquad (4-13)$$

where:

$$u_i = \frac{1}{\pi} \oint_0^1 \frac{\tau F'(x_1)}{x - x_1} dx_1 \qquad (4-14)$$

Hence

$$C = \frac{2}{3 M_\infty^2 (\gamma + 1)} \left[1 - M_\infty^2 - M_\infty^2 (\gamma + 1) u_t \right]^{3/2} \qquad (4-15)$$

so that

$$u = \frac{1}{M_\infty^2 (\gamma + 1)} \left[(1 - M_\infty^2) - \left\{ \frac{3 M_\infty^2 (\gamma + 1)}{2} u_c + \right. \right.$$

$$\left. \left. [1 - M_\infty^2 - M_\infty^2 (\gamma + 1) u_t]^{3/2} \right\}^{2/3} \right] \qquad (4-16)$$

where

$$u_c = \frac{1}{\pi} \sqrt{\frac{1-x}{x}} \int_0^1 \frac{\sigma E'(x_1)}{x - x_1} \sqrt{\frac{x_1}{1 - x_1}} dx_1 \qquad (4-17)$$

For $\sigma = 0$, $u_c = 0$ and equation (4 — 16) reduces to (4 — 13). We note here that the camber and thickness are related in equation (4 — 16) in a nonlinear fashion, unlike in the subsonic and supersonic linear (compressible) flow cases, where thickness and lifting effects can be superimposed linearly.

(b) **Supersonic Flow:**

Linearized equation gives similar solution for thickness and lifting problems. Hence, applying similar procedure, we obtain:

$$- B^2 g_{xx} + g_{zz} = 0 \qquad (4-17)$$

where

$$B^2 = M_\infty^2 - 1 + M_\infty^2 (\gamma + 1) \varphi_x \qquad (4-18)$$

$$g = \frac{\delta \varphi}{\delta \sigma} \qquad (4-19)$$

subject to the boundary condition:

$$g_z = \frac{\delta \varphi_z}{\delta \sigma} = E'(x) \qquad \text{at } z = \pm 0 \qquad (4-20)$$

Its solution is given by:

$$g_x (x,0) = u_\sigma = - \frac{E'(x)}{B} \qquad (4-21)$$

Substitution of original value of B and integration yields

$$\frac{2}{3 M_\infty^2 (\gamma + 1)} [M_\infty^2 - 1 + M_\infty^2 (\gamma + 1) u]^{3/2} = - E'(x)\sigma + C \qquad (4-22)$$

Now, when $\sigma = 0$, $u = u_t$, hence

$$C = \frac{2}{3\, M_\infty^2 (\gamma + 1)} \left[M_\infty^2 - 1 + M_\infty^2 (\gamma + 1) u_t \right]^{3/2} \qquad (4-23)$$

where

$$u_t = \frac{1}{M_\infty^2 (\gamma + 1)} \left\{ -(M_\infty^2 - 1) + \left[(M_\infty^2 - 1)^{3/2} - \frac{3}{2} M_\infty^2 (\gamma + 1) \tau E'(x) \right]^{2/3} \right\} \qquad (4-24)$$

Hence

$$u = \frac{1}{M_\infty^2 (\gamma + 1)} \left[-(M_\infty^2 - 1) + \left\{ - \frac{3 M_\infty^2 (\gamma + 1) \sigma E'(x)}{2} + [M_\infty^2 - 1 + M_\infty^2 (\gamma + 1) u_t]^{3/2} \right\}^{2/3} \right] \qquad (4-25)$$

which again reduces to (4 — 24) if $\sigma = 0$. Equation (4 — 25) also exhibits the nonlinear relationship between camber and thickness effects.

It is interesting to note the difference between the above formula and the one, in which camber and thickness is combined in one parameter, i.e. by letting

$$z(x,0) = \rho\, R(x) = \tau\, F(x) + \sigma E(x) \qquad (4-26)$$

which then would give

$$u = \frac{1}{M_\infty^2 (\gamma + 1)} \left\{ -(M_\infty^2 - 1) + [-(M_\infty^2 - 1)^{3/2} - \frac{3}{2} M_\infty^2 (\gamma + 1) \rho R'(x)]^{2/3} \right\} \qquad (4-27)$$

In equation (4 — 25), the flow is perturbed by adding small increment of camber or angle of attack, and implicitly this assumes that $\sigma < \tau$. In equation (4 — 27), σ and τ are comparable and combined is one parameter ρ. This result is also based on linearized thin airfoil theory in supersonic flow. For $\sigma \ll \tau$, (4 — 25) and (4 — 27) would give difference of small order.

(c) Transonic Flows at $M_1 \gtrsim 1$

For the transonic case

$$(1 - M_1^2) = (1 - M_\infty^2 - M_\infty^2 (\gamma + 1) \varphi_x] = 0$$

so that equation (4 — 2) reduces to

$$-M_\infty^2 (\gamma + 1) \varphi_{xx} g_x + g_{zz} = 0 \qquad (4-28)$$

which is of the parabolic type, and seems plausible as a transition from elliptic to hyperbolic type. This equation appears in many approximate methods suggested by various investigators. As was commonly done, if φ_{xx} varies sufficiently slowly, the coefficient of g_x can be regarded as a constant, at least as a first approximation in the early part of the analysis.

Equation (4 — 28) was considered by Spreiter (1958, 1964), and Hosokawa (1961), employing the Green's function technique. The solution is given by

$$g(x,z) = -\frac{1}{2\sqrt{\pi k}}\int_0^x \left\{F'(\zeta) + \Delta g(\zeta)\frac{\delta}{\delta t}\right\}\frac{\exp[-Kz^2/_4(x-\zeta)]}{\sqrt{x-\zeta}}\, d\zeta \quad (4-29)$$

where

$$K = M_\infty^2(\gamma+1)\varphi_{xx} \quad (4-30)$$

$$\Delta g(\zeta) = g_{upper}(\zeta) - g_{lower}(\zeta) \quad (4-31)$$

Exact inversion formula can be found, and is given by

$$E'(x) = \lim_{z\to 0}\frac{\delta}{\delta z}\left\{-\frac{1}{2\sqrt{\pi K}}\int_0^x \Delta g(\zeta)\frac{\delta}{\delta z}\frac{\exp[-Kz^2/_4(x-\zeta)]}{\sqrt{x-\zeta}}\, d\zeta\right\} \quad (4-32)$$

and for $0 \leqslant x \leqslant 1$

$$E'(x) = \tfrac{1}{4}\sqrt{\frac{K}{\pi}}\int_0^x \Delta g(\zeta)\frac{d\zeta}{[\sqrt{x-\zeta}]^3} \text{ for } 0 \leqslant x < 1 \quad (4-33)$$

Taking into account the continuity of φ, and hence g, at the leading edge

$$\Delta g(0) = 0 \quad (4-34)$$

so that equation (4 — 33) reduces to

$$E'(x) = \tfrac{1}{2}\sqrt{\frac{K}{\pi}}\int_0^x \{-\Delta g'(\zeta)\}\frac{d\zeta}{\sqrt{x-\zeta}} \quad (4-35)$$

This equation has the form of Abel's integral equation. Therefore, the Kutta condition is not needed here, while it is needed for the singular integral equation for Δg at $M_\infty < 1$, which is difficult to solve, except for $K = 0$.
Inverse transform can be made as

$$\Delta g'(x) = -\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2\sqrt{\pi/K}}{\sqrt{x-\zeta}} E'(\zeta)\, d\zeta \quad (4-36)$$

so that, as $g'(x) = u_\sigma$ and $\Delta g'(x) = g'_{upper}(x) - g'_{lower}(x) = 2u_\sigma$

then

$$u_\sigma = -\frac{1}{\sqrt{\pi K}}\frac{d}{dx}\int_0^x \frac{E'(x_1)}{\sqrt{x-x_1}}\, dx_1 \quad (4-37)$$

Now $K = (\gamma+1)M_\infty^2 U_x$, and we are faced with a differential equation, in which u appears twice as derivatives (firstly with respect to x and secondly with respect to σ), where u is the desired solution. Superficially it seems that not much of a progress has been made. However, in principle, equation (4—37) can be solved by an iterative scheme, in which K is assumed to be constant (i.e. initially equal to its value corresponding to the non-lifting solution, which is used as a starting point). Subsequent value of K is then substituted after each step. If $\frac{\delta}{\delta\sigma}(U_x)$ varies slowly, this process can be expected to converge rapidly.

V. CONCLUDING REMARKS

Based upon the method of parametric differentiation and local linearization technique, the lifting transonic flow was analyzed, in which the change in velocity due to camber was considered as a small perturbation on the non-lifting flow. In the subsonic and supersonic cases, closed form solutions were obtained, which properly reduced to the nonlifting solution, if $\sigma = 0$. In the case of $M_1 \gtrsim 1$, the methods yields an integral equation, which can be solved by an iterative scheme which departs from the nonlifting solution. The accuracy of the analysis remains to be verified by comparing numerical results and experimental data.

The analysis demonstrates the significance of recently introduced approximate methods, specifically the method of parametric differentiation, in solving nonlinear differential equations, in particular, the governing differential equation of transonic flow.

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