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On Size Bipartite and Tripartite Ramsey Numbers for The Star Forest and Path on 3 Vertices

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Abstract. For simple graphs G and H, the size multipartite Ramsey number $m_j(G,H)$ is the smallest natural number t such that any arbitrary red-blue coloring on the edges of $K_{j\times t}$ contains a red G or a blue H as a subgraph. We studied the size tripartite Ramsey numbers $m_3(G,H)$, where $G = mK_{1,n}$ and $H = P_3$. In this paper, we generalize this result. We determine $m_3(G,H)$, where G is a star forest, namely a disjoint union of heterogeneous stars, and $H = P_3$. Moreover, we also determine $m_2(G,H)$ for this pair of graphs G and H.

Keywords: *path; size multipartite Ramsey number; star forest.*

1 Introduction

Given two simple graphs G and H. We use the notation $F \longrightarrow (G,H)$ when for any red-blue coloring of the edges of a graph F we always have a red subgraph G or a blue subgraph H. The *Ramsey number* $r(G,H)$ is defined as the smallest positive integer *n* such that $K_n \longrightarrow (G,H)$, where K_n is the complete graph on *n* vertices. Some values of the Ramsey number for a combination of a star and a path were determined by Parsons [1]. One year before, the *multicolor Ramsey number* for stars was determined by Burr and Roberts [2]. Then, the concept of Ramsey numbers evolved to the *bipartite Ramsey number* $b(G,H)$, which is defined as the smallest positive integer *n* such that $K_{n,n} \longrightarrow (G,H)$. In 1998, the bipartite Ramsey number for a star and a path was completed by Hattingh and Henning [3].

Furthermore, in 2004 Burger and Vuuren [4] generalized the concept of bipartite Ramsey numbers to the size multipartite Ramsey numbers as follows. Let j, l, n, r and s be natural numbers with $n, r \geq 2$. The *size multipartite Ramsey number* $m_j(K_{n\times l}, K_{r\times s})$ is the smallest natural number *t* such that an arbitrary red-blue coloring of the edges of $K_{j\times t}$, where $K_{j\times t}$ is the complete multipartite graph having j partite sets with t vertices per each partite set, necessarily forces a red $K_{n\times l}$ or a blue $K_{r\times s}$ as a subgraph. They also gave some

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properties of the size multipartite Ramsey numbers and determined the exact values of $m_j(K_{2\times2}, K_{3\times1})$, for $j \geq 2$. For the bounds of the size multipartite Ramsey numbers they gave a direct lower bound, a probabilistic lower bound, and a diagonal bipartite upper bound.

Syafrizal, *et al.* [5] generalized this concept by removing the completeness requirement. Thus, the size multipartite Ramsey number, $m_j(G, H)$, is defined as the smallest positive integer *t* such that $K_{j\times t} \longrightarrow (G, H)$. They also determined the size multipartite Ramsey numbers for paths and other graphs [5,6], especially the size multipartite Ramsey numbers for P_3 and stars [7]. Then, Surahmat and Syafrizal [8] gave the size tripartite Ramsey numbers for paths P_n and stars, for $3 \leq n \leq 6$. Meanwhile, the size multipartite Ramsey numbers for stars and cycles have been investigated by Lusiani, *et al.* [9]. They also provided the size tripartite Ramsey numbers for P_3 and a disjoint union of homogeneous stars [10] and the size tripartite Ramsey numbers for stars with paths and cycles [11]. Recently, Jayawardene and Samarasekara [12] determined the size multipartite Ramsey numbers for C_3 and all graphs up to 4 vertices, including the star of order 4. However, the multipartite Ramsey numbers for P_3 and a disjoint union of heterogeneous stars have not been determined.

Here, the generalized concept of the size multipartite Ramsey numbers for a star forest and P_3 is used. The star forest is a disjoint union of heterogeneous stars. In this paper, we determine the size multipartite Ramsey numbers $m_j\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right)$, for j = 2, 3, where $\bigcup_{i=1}^{k} K_{1,n_i}$ is a star forest, for $n_i \geq 1, k \geq 2$ and P_3 is a path on 3 vertices. For $k = 1$, Hattingh and Henning [3] determined $m_2\left(K_{1,r}, P_s\right)$, for $r, s \geq 2$.

For some terms in graph theory used in this paper, we refer to Chartrand [13]. Let G be a finite and simple graph. The vertex and edge sets of graph G are denoted by $V(G)$ and $E(G)$, respectively. A *matching* of a graph G is defined as the set of edges without a common vertex. Let $e = u{\sim}v$ be an edge in G, then u is called *adjacent* to v. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G. The degree $d(v)$ of a vertex v is $|N(v)|$. The *maximum degree* of G is denoted by $\Delta(G)$, where $\Delta(G) = \max\{d(v)|v \in V(G)\}$. The *minimum degree* of G is denoted by $\delta(G)$, where $\delta(G) = \max\{d(v)|v \in V(G)\}$. A star $K_{1,n}$ is the graph on $n + 1$ vertices with one vertex of degree *n*, called the *center* of this star, and *n* vertices of degree 1, called the *leaves*. Any red-blue coloring of graph $K_{j\times t}$ is represented by $K_{j\times t} = F_R \oplus F_B$ or $K_{j\times t} = G_R \oplus G_B$, where F_R and G_R are the red graphs and F_B and G_B are the blue graphs.

2 Bipartite Ramsey Numbers

In this section, we discuss the size bipartite Ramsey number $m_2\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right)$, for $k \geq 2$ and $n_i \geq 1$. We compute the formula of this Ramsey number for any $k \geq 2$ and $n_i \geq 1$. In particular, for $n_i = 1$, for all i, we obtain the value of $m_2\left(kK_{1,1}, P_3\right) = m_2(kP_2, P_3)$, correcting the previous result given by Christou, *et al.* [14]. They showed that $m_2\left(kP_2, K_{1,n}\right) = n + \left\lfloor\frac{k-1}{2}\right\rfloor$, for $k \geq 2$ and $n \geq 1$. For $n = 2$, they had $m_2(kP_2, P_3) = 2 + \left\lfloor\frac{k-1}{2}\right\rfloor$, which is not correct for $k \geq 4$.

Lemma 2.1 $m_2(kP_2, P_3) = \begin{cases} 2, & for\ k = 1 \\ k, & for\ k \geq 2 \end{cases}$

Proof. Let $t = \begin{cases} 2, & for\ k = 1 \\ k, & for\ k \geq 2 \end{cases}$

We consider the coloring of $K_{2\times(t-1)} = F_R \oplus F_B$, such that F_B does not contain P_3. So, $\Delta(F_B) \leq 1$. This is trivial for $k = 1$ since $F_B = K_2$ and F_R is an empty graph. For $k \geq 2$, we choose $F_B = (k-1)P_2$. In this case, we will have no kP_2 in F_R and $F_B \not\supseteq P_3$. So, $m_2(kP_2, P_3) \geq t$.

Now, we show that $m_2(kP_2, P_3) \leq t$. We consider any coloring of $K_{2\times t} = G_R \oplus G_B$, such that G_B does not contain a blue P_3, so $\Delta(G_B) \leq 1$. For $k = 1$, we have $K_{2\times 2} = G_R \oplus G_B$. So, G_B is either a matching graph or an empty graph and G_R is either $2P_2, P_4$ or C_4, which implies $G_R \supseteq 2P_2$. For $k \geq 2$, we have $K_{2\times k} = G_R \oplus G_B$. Let $U = \{u_1, u_2, \dots, u_k\}$ and $V = \{v_1, v_2, \dots, v_k\}$ be two partite sets in $K_{2\times k}$. If G_B is a matching graph, then every vertex in $K_{2\times k}$ is relabeled such that $u_i \sim v_i$ in G_B, for every $i = 1, 2, \dots, k$. We consider a cycle in $K_{2\times k}$, namely $C_k' = u_1v_1u_2v_2u_3v_3 \dots u_kv_ku_1$. So, $E(C_k') - E(G_B)$ contains a red kP_2. Therefore, G_R contains a red kP_2.

In Lemma 2.1 we obtain the size bipartite Ramsey number, $m_2\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right)$, for $n_i = 1$, for all i. So, in Theorems 2.2, 2.4 and 2.5, we determine the size bipartite Ramsey numbers $m_2\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right)$, for all $n_i \geq 1$, for $2 \leq k \leq 4$. For a combination of two stars and P_3, we show this case in Theorem 2.2.

Theorem 2.2 Let n_1 and n_2 be positive integers. Then, $m_2\left(K_{1,n_1} \cup K_{1,n_2}, P_3\right) = \max\{n_1, n_2\} + 1$.

Proof. Let $n_1 \geq n_2 \geq 1$, so we have $\max\{n_1, n_2\} + 1 = n_1 + 1$. To show that $m_2\left(K_{1,n_1} \cup K_{1,n_2}, P_3\right) \geq n_1 + 1$, we consider the coloring of $K_{2\times n_1} = F_R \oplus F_B$, such that F_B does not contain P_3. So, $\Delta(F_B) \leq 1$. We can choose $F_B =$

n_1P_2. Let $U = \{u_1, u_2, \dots, u_{n_1}\}$ and $V = \{v_1, v_2, \dots, v_{n_1}\}$ be two partite sets in $K_{2\times n_1}$. Every vertex in $K_{2\times n_1}$ is relabeled such that $u_i \sim v_i$ in F_B, for every $i = 1, 2, \dots, n_1$. Since $u_i \sim v_j$ in F_R, for $i \neq j$, so F_R does not contain K_{1,n_1}. Therefore, $m_2(K_{1,n_1} \cup K_{1,n_2}, P_3) \geq n_1 + 1$.

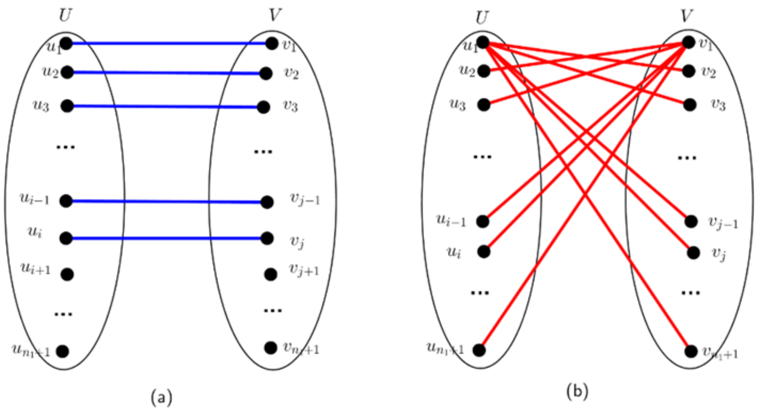

Figure 1 (a). $G_B = pK_2$, for $1 \leq p \leq n+1$ (b). $2K_{1,n_1} \subseteq G_R \subseteq K_{2\times(n_1+1)}$.

Now, we show that $m_2(K_{1,n_1} \cup K_{1,n_2}, P_3) \leq n_1 + 1$. We consider any coloring of $K_{2\times(n_1+1)} = G_R \oplus G_B$, such that G_B does not contain a blue P_3, so $\Delta(G_B) \leq 1$. Let $U = \{u_1, u_2, \dots, u_{n_1+1}\}$ and $V = \{v_1, v_2, \dots, v_{n_1+1}\}$ be two partite sets in $K_{2\times(n_1+1)}$. If G_B is a matching graph, then every vertex in $K_{2\times(n_1+1)}$ is relabeled such that $u_i \sim v_i$ in F_B, for any $i = 1, 2, \dots, n_1 + 1$, see Figure 1(a). Since $u_1 \sim v_j$ and $v_1 \sim u_j$ in G_R, for $2 \leq i, j \leq n_1 + 1$, we find a disjoint union of stars $2K_{1,n_1}$ in G_R, see Figure 1(b).

For the proofs of Theorems 2.4 and 2.5, we use Lemma 2.3, which is stated as follows. Note that we previously defined $\text{sum}(A) = \sum_{i-1}^{k} n_i$, for $A = \{n_1, n_2, \dots, n_k\}$.

Lemma 2.3 *Let* $N = \{n_1, n_2, \dots, n_k\}$, *for* $k \geq 2$ *and* $n_1 \geq n_2, \geq \dots \geq n_k \geq 1$. *For* $1 \leq i \leq 2^k$, *let* $A \in \mathcal{P}(N)$, *where* $A_i \neq A_j$, *for* $i \neq j$. *Let* $B_i = N - A_i$, *and* $L_i = \max\{\text{sum}(A_i) + |B|, \text{sum}(B_i) + |A|\}$. *There exists* $p \in \{1,2,3, \dots, 2^k\}$, *where* A_p or B_p *is not empty (or* A_p *or* B_p *is not* N*), such that* $L_k = \min\{L_i | 1 \leq i \leq 2^k\}$.

Proof. Let $A_r = \emptyset$ and $B_r = N$, for any $r \in \{1,2,3, \dots, 2^k\}$. Then $L_r = \max\{\text{sum}(A_r) + |B_r|, \text{sum}(B_r) + |A_r|\} = \text{sum}(N)$. Let us consider A_s, where $|A_s| \geq 1$ and $B_s = N - A_s$, where $|B_s| \geq 1$. We assume that $|A_s| = t \geq 1$.

Then $L_s = \max\{\text{sum}\,(A_s) + k - t, \text{sum}\,(B_s) + t\}$. Note that $\text{sum}(N) = \text{sum}(A_s) + \text{sum}(B_s) \geq \text{sum}(A_s) + k - t$ and $\text{sum}(N) = \text{sum}(A_s) + \text{sum}(B_s) \geq t + \text{sum}(B_s)$. So, if $A_s = \{1\}$, then $L_s = L_r$. Otherwise, $L_s < L_r$. Then, $L = \min(L_s) < L_r$. Therefore, L is minimum when A and B are not empty.

Theorem 2.4 Let $N = \{n_1, n_2, n_3\}$ be the set of the number of leaves of three stars K_{1,n_1}, K_{1,n_2} and K_{1,n_3}, respectively. If $L = \{\max\{\text{sum}(A) + |B|, \text{sum}(B) + |A|\} | A, B \subseteq N, A \cup B = N, A \cap B = \emptyset\}$, then $m_2\left(\bigcup_{i=1}^{3} K_{1,n_i}, P_3\right) = \min(L)$.

Proof. Let $n_1 \geq n_2 \geq n_3 \geq 1$. We have $L = \{n_1 + n_2 + n_3, \max\{n_1 + 2, n_2 + n_3 + 1\}, n_1 + n_2 + 1, n_1 + n_3 + 1\}$. By Lemma 2.2, $L = \{\max\{n_1 + 2, n_2 + n_3 + 1\}, n_1 + n_2 + 1, n_1 + n_3 + 1\}$. Therefore, $\min(L) = \max\{n_1 + 2, n_2 + n_3 + 1\}$.

Let $t = \max\{n_1 + 2, n_2 + n_3 + 1\}$. To show that $m_2\left(\bigcup_{i=1}^{3} K_{1,n_i}, P_3\right) \geq t$, we consider the coloring of $K_{2\times(t-1)} = F_R \oplus F_B$, such that F_B does not contain a blue P_3, so $\Delta(F_B) \leq 1$. We can choose that $F_B = (t-1)P_2$. Let $U = \{u_1, u_2, \dots, u_{t-1}\}$ and $V = \{v_1, v_2, \dots, v_{t-1}\}$ be two partite sets in $K_{2\times(t-1)}$. Every vertex in $K_{2\times(t-1)}$ is relabeled such that $u_i \sim v_i$ in F_B, for any $i = 1, 2, \dots, t-1$. We have the following two possibilities for the values of $t-1$:

1. For $t - 1 = n_1 + 1$.
   Since $u_1 \sim v_j$, for $2 \leq j \leq n_1 + 1$ in F_R, the star K_{1,n_1} can be constructed by these vertices. Since $v_1 \sim u_i$, for $2 \leq i \leq n_2 + 1 \leq n_1 + 1$ in F_R, the star K_{1,n_2} can be constructed by vertices $v_1, u_2, u_3, \dots, u_{n_2+1}$. However, we cannot construct the star K_{1,n_3}, since we cannot choose any of the remaining vertices as its center.
2. For $t - 1 = n_2 + n_3$.
   Since $u_1 \sim v_j$, for $n_2 + 1 \leq n_2 + n_3$ in F_R, the star K_{1,n_3} can be constructed by these vertices. Since $u_{n_2+1} \sim v_j$, for $1 \leq j \leq n_2$ in F_R, the star K_{1,n_2} can be constructed by these vertices. However, we cannot construct the star K_{1,n_1}, since we cannot choose any of the remaining vertices as its center.

F_R does not contain all stars K_{1,n_1}, K_{1,n_2} and K_{1,n_3}, so F_R does not contain $\bigcup_{i=1}^{3} K_{1,n_i}$.

Now, we show that $m_2\left(\bigcup_{i=1}^{3} K_{1,n_i}, P_3\right) \leq t$. We consider any coloring of $K_{2\times t} = G_R \oplus G_B$, such that G_B does not contain a blue P_3, so $\Delta(G_B) \leq 1$. Then

G_B is a matching graph. Let $U = \{u_1, u_2, \dots, u_t\}$ and $V = \{v_1, v_2, \dots, v_t\}$ be two partite sets in $K_{2\times t}$. Every vertex in $K_{2\times t}$ is relabeled such that $u_i \sim v_j$ in G_B, for some $i = j$. We have the following two possibilities for the values of t:

1. For $t = n_1 + 2$
   Since $u_1 \sim v_j$, for $3 \le j \le n_1 + 2$ in G_R, the star K_{1,n_1} can be constructed by these vertices. Since $v_1 \sim u_2$ and v_1, v_2 are both adjacent to u_i for $3 \le i \le n_1 + 2$ in G_R, the star K_{1,n_2} can be constructed by vertices $v_1, u_2, u_3, \dots, u_{\mathrm{n}_2+1}$, and the star K_{1,n_3} can be constructed by vertices $v_2, u_{n_2+2}, u_{n_2+3}, \cdots, u_{n_3+\mathrm{n}_2+1}$.
2. For $t = n_2 + n_3 + 1$
   Since $u_1 \sim v_j$, for $3 \le j \le n_2 + n_3 + 1$ in G_R, the star K_{1,n_1} can be constructed by vertices $u_1, v_3, v_4, \dots, v_{\mathrm{n}_1+2}$. Since $v_1 \sim u_2$ and v_1, v_2 are both adjacent to u_i for $3 \le i \le n_2 + n_3 + 1$ in G_R, the star K_{1,n_2} can be constructed by vertices $v_1, u_2, u_3, \dots, u_{\mathrm{n}_2+1}$ and the star K_{1,n_3} can be constructed by vertices $v_2, u_{n_2+2}, u_{n_2+3}, \cdots, u_{n_3+\mathrm{n}_2+1}$.

Therefore, G_R contains $\bigcup_{i=1}^{3} K_{1,n_i}$.

Theorem 2.5 Let $N = \{n_1, n_2, n_3, n_4\}$ be the set of the number of leaves of three stars $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}$ and K_{1,n_4}, respectively. If $L = \{\max\{\mathrm{sum}(A) + |B|, \mathrm{sum}(B) + |A|\} | A, B \subseteq N, A \cup B = N, A \cap B = \emptyset\}$, then $m_2\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) = \min(L)$.

Proof. Let $n_1 \ge n_2 \ge n_3 \ge n_4 \ge 1$. We have $L = \{n_1 + n_2 + n_3 + n_4, \max\{n_1 + 3, n_2 + n_3 + n_4 + 1\}, n_1 + n_2 + 2, n_3 + n_4 + 2,\ n_1 + n_2 + n_3 + 1, n_1 + n_2 + n_4 + 1, n_1 + n_3 + n_4 + 1, \max\{n_1 + n_4 + 2, n_2 + n_3 + 2\}\}$.

By Lemma 2.2,

$$L = \{\max\{n_1 + 3, n_2 + n_3 + n_4 + 1\}, n_1 + n_2 + 2, n_3 + n_4 + 2, n_1 + n_2 + n_3 + 1, n_1 + n_2 + n_4 + 1, n_1 + n_3 + n_4 + 1, \max\{n_1 + n_4 + 2,\ n_2 + n_3 + 2\}\}.$$

Therefore,
$\min(L) = \min\{\max\{n_1 + 3, n_2 + n_3 + n_4 + 1\}, \max\{n_1 + n_4 + 2,\ n_2 + n_3 + 2\}\}$.
Then, we have three following possibilities for $\min(L)$:

1. If $n_1 > n_2 + n_3$, then $\min(L) = \min(L') = \max\{n_1 + 3, n_2 + n_3 + n_4 + 1\}$.
2. If $n_1 < n_2 + n_3$, then $\min(L) = \min(L'') = \max\{n_1 + n_4 + 2,\ n_2 + n_3 + 2\}$.
3. If $n_1 = n_2 + n_3$, then $\min(L) = \min(L')$ or $\min(L) = \min(L'')$.

Let $t = \min(L)$. To show that $m_2\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) \ge t$ we consider the coloring of $K_{2\times(t-1)} = F_R \oplus F_B$, such that F_B does not contain a blue P_3, so $\Delta(F_B) \le 1$.

We can choose that $F_B = (t-1)P_2$. Let Let $U = \{u_1, u_2, \dots, u_{t-1}\}$ and $V = \{v_1, v_2, \dots, v_{t-1}\}$ be two partite sets in $K_{2\times(t-1)}$. Every vertex in $K_{2\times(t-1)}$ is relabeled such that $u_i \sim v_i$ in F_B, for every $i = 1, 2, \dots, t-1$. We have four possibilities for the values of $t-1$, as follows:

1. For $t-1 = \min(L') - 1 = n_1 + 2$
   Since $u_1 \sim v_j$, for $2 \le j \le n_1 + 1$ in F_R, the star K_{1,n_1} can be constructed by these vertices. Since $v_1 \sim u_i$, for $n_2 + 2 \le i \le n_2 + n_3 + 1 \le n_1 + 1$ in F_R, the star K_{1,n_2} and K_{1,n_3} can be constructed by vertices $v_1, u_2, u_3, \dots, u_{\mathrm{n}_2+1}$ and $v_{n_1+2}, u_{n_2+2}, u_{n_2+3}, \dots, u_{n_3+\mathrm{n}_2+1}$, respectively. However, we cannot construct the star K_{1,n_4}, since we cannot choose any of the remaining vertices as its center.
2. For $t-1 = \min(L') - 1 = n_2 + n_3 + n_4$
   Since $u_1 \sim v_j$, for $2 \le j \le n_2 + 1$ in F_R, the star K_{1,n_2} can be constructed by these vertices. Since $u_2 \sim v_j$, for $n_2 + 2 \le j \le n_2 + n_3 + 1$ in F_R, the star K_{1,n_3} can be constructed by these vertices. Since $u_3 \sim v_j$, for $j = 1$ and $n_2 + n_3 + 2 \le j \le n_2 + n_3 + n_4$ in F_R, the star K_{1,n_4} can be constructed by these vertices. However, we cannot construct the star K_{1,n_1}, since we cannot choose any of the remaining vertices as its center.
3. For $t-1 = \min(L") - 1 = n_1 + n_4 + 1$
   Since $u_1 \sim v_j$, for $2 \le j \le n_1 + 1$ in F_R, the star K_{1,n_1} can be constructed by these vertices. Since $u_2 \sim v_j$, for $n_1 + 2 \le j \le n_1 + n_4 + 1$ in F_R, the star K_{1,n_4} can be constructed by these vertices. Since $v_1 \sim u_i$, for $3 \le i \le n_2 + 2$ in F_R, the star K_{1,n_2} can be constructed by these vertices. However, we cannot construct the star K_{1,n_3}, since we cannot choose any of the remaining vertices as its center.
4. For $t-1 = \min(L") - 1 = n_2 + n_3 + 1$
   Since $u_1 \sim v_j$, for $2 \le j \le n_2 + 1$ in F_R, the star K_{1,n_2} can be constructed by these vertices. Since $u_2 \sim v_j$, for $n_2 + 2 \le j \le n_2 + n_3 + 1$ in F_R, the star K_{1,n_3} can be constructed by these vertices. Since $v_1 \sim u_i$, for $3 \le i \le n_1 + 2$ in F_R, the star K_{1,n_1} can be constructed by these vertices. However, we cannot construct the star K_{1,n_4}, since we cannot choose any of the remaining vertices as its center.

Since F_R does not contain all stars $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}$ and K_{1,n_4}, therefore F_R does not contain $\bigcup_{i=1}^{4} K_{1,n_i}$.

Now we show that $m_2\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) \le t$. We consider any coloring of $K_{2\times t} = G_R \oplus G_B$, such that G_B does not contain a blue P_3, so $\Delta(G_B) \le 1$. Then G_B is a matching graph. Let $U = \{u_1, u_2, \dots, u_t\}$ and $V = \{v_1, v_2, \dots, v_t\}$ be two partite sets in $K_{2\times t}$. Every vertex in $K_{2\times t}$ is relabeled such that $u_i \sim v_i$ in G_B, for any $i = 1,2,\dots,t$. There are four possibilities for the values of t, as follows.

1. For $t = \min(L') = n_1 + 3$
   Since $u_1 \sim v_j$, for $2 \le j \le n_1 + 1$ in G_R, the star K_{1,n_1} can be constructed by these vertices. Since $v_j \sim u_i$ for $j = 1, n_1 + 2, n_1 + 3$ and $2 \le i \le n_2 + 1$ in G_R, the star K_{1,n_2}, K_{1,n_3} and K_{1,n_4} can be constructed by vertices $\{v_1, u_2, u_3, \dots, u_{n_2+1}\}$, $\{v_{n_1+2}, u_{n_2+2}, u_{n_2+3}, \dots, u_{n_2+n_3+1}\}$ and $\{v_{n_1+3}, u_{n_2+n_3+2}, u_{n_2+n_3+3}, \dots, u_{n_2+n_3+n_4+1}\}$, respectively.
2. For $t = \min(L') = n_2 + n_3 + n_4 + 1$
   Since $v_1 \sim u_i$, for $4 \le i \le n_1 + 3$ in G_R, the star K_{1,n_1} can be constructed by these vertices. Since $v_j \sim u_i$ for $i = 1, 2,3$ and $2 \le j \le n_2 + n_3 + n_4 + 1$ in G_R, the star K_{1,n_2}, K_{1,n_3} and K_{1,n_4} can be constructed by vertices $\{u_1, v_2, v_3, \dots, v_{n_2+1}\}$, $\{u_2, v_{n_2+2}, v_{n_2+3}, \dots, v_{n_2+n_3+1}\}$ and $\{u_3, v_{n_2+n_3+2}, v_{n_2+n_3+3}, \dots, v_{n_2+n_3+n_4+1}\}$, respectively.
3. For $t = \min(L'') = n_1 + n_4 + 2$
   Since $u_i \sim v_j$, for $i = 1,2$ and $2 \le j \le n_1 + n_4 + 1$ in G_R, the star K_{1,n_1} and K_{1,n_4} can be constructed by vertices $u_1, v_2, v_3, \dots, v_{n_1+1}$ and $u_2, v_{n_1+2}, v_{n_1+3}, \dots, v_{n_1+n_4+1}$. Since $v_j \sim u_i$, for $3 \le i \le n_2 + n_3 + 2$ and $j = 1, n_1 + n_4 + 2$ in G_R, the star K_{1,n_2} and K_{1,n_3} can be constructed by vertices $\{v_{n_1+n_4+2}, u_3, u_4, \dots, u_{n_2+2}\}$ and $\{v_1, u_{n_2+3}, u_{n_2+4}, \dots, u_{n_2+n_3+2}\}$, respectively.
4. For $t = \min(L'') = n_2 + n_3 + 2$
   Since $u_i \sim v_j$, for $i = 1,2$ and $2 \le j \le n_2 + n_3 + 1$ in G_R, the star K_{1,n_2} and K_{1,n_3} can be constructed by vertices $u_1, v_2, v_3, \dots, v_{n_2+1}$ and $u_2, v_{n_2+2}, v_{n_2+3}, \dots, v_{n_2+n_3+1}$. Since $v_j \sim u_i$, for $3 \le i \le n_1 + n_4 + 2$ and $j = 1, n_2 + n_3 + 2$ in G_R, the star K_{1,n_1} and K_{1,n_4} can be constructed by vertices $\{v_{n_2+n_3+2}, u_3, u_4, \dots, u_{n_1+2}\}$ and $\{v_1, u_{n_1+3}, u_{n_1+4}, \dots, u_{n_1+n_4+2}\}$, respectively.

Therefore, G_R contains $\bigcup_{i=1}^{4} K_{1,n_i}$.

From Theorems 2.4 and 2.5, we obtain $m_2\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right) = \min(L)$ for $k \in \{2, 3, 4\}$. For $k \ge 5$, it seems that the bipartite Ramsey number for a pair of $\bigcup_{i=1}^{k} K_{1,n_i}$ and P_3 is $\min(L)$. For example, it is easy to see the bipartite Ramsey number for a pair of $2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}$ and P_3. In this case, we calculate that $\min(L) = 19$. Then, $m_2\left(2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}, P_3\right) = 19$, see Figure 2.

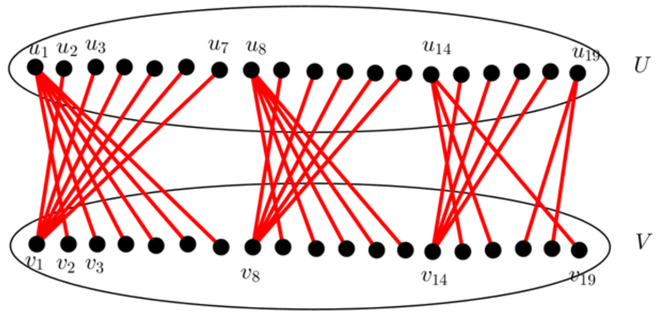

Figure 2 A disjoint union of stars $2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}$ in $K_{2\times 19}$.

Now, we consider another extremal example in Figure 3. Since 100 is too large compared to other numbers of leaves, 100 and the other numbers of leaves are in a different partite set. We calculate that $\min(L) = 109$. Then, $m_2\left(K_{1,100} \cup K_{1,55} \cup 8K_{1,1}, P_3\right) = 109$. Therefore, we present the following conjecture.

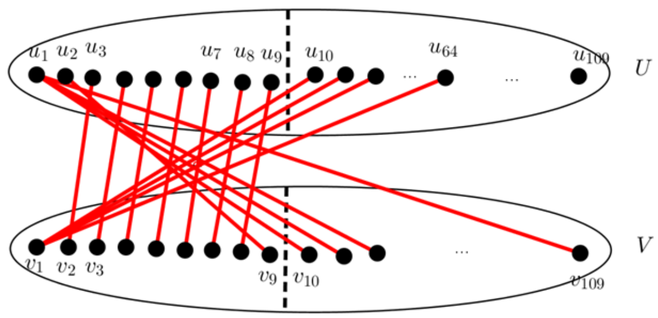

Figure 3 A disjoint union of stars $K_{1,100} \cup K_{1,55} \cup 8K_{1,1}$ in $G_R \subseteq K_{2\times 109}$.

Conjecture 2.1 Let $N = \{n_1, n_2, n_3, \dots, n_k\}$ be the set of the number of leaves of stars K_{1,n_i}, for $n_i \geq 1$, $1 \leq i \leq k$ and $k \geq 2$, respectively. If $L = \{\max\{\mathrm{sum}(A) + |B|, \mathrm{sum}(B) + |A|\} | A, B \subseteq N, A \cup B = N, A \cap B = \emptyset\}$, then $m_2\left(\cup_{i=1}^{k} K_{1,n_i}, P_3\right) = \min(L)$.

3 Tripartite Ramsey Numbers

In this section, the size tripartite Ramsey numbers for a star forest and P_3 is investigated.

Theorem 3.1 *Let $n_1 \geq n_2 \geq 1$ be positive integers. Let $A = \left\lceil \frac{2+n_1+n_2}{3} \right\rceil$ and $B = \left\lceil \frac{n_1+1}{2} \right\rceil$. Then, $m_3\left(K_{1,n_1} \cup K_{1,n_2}, P_3\right) = \max\{A, B\}$.*

Proof. Let $t = \max\{A, B\}$. To show that $m_3(K_{1,n_1} \cup K_{1,n_2}, P_3) \geq t$, we consider the two following possibilities for the value of t:

1. If $A \geq B$, then $t = \left\lceil \frac{2+n_1+n_2}{3} \right\rceil$. We make the edges of graph $K_{3\times(t-1)}$ red. Since $\left|V(K_{3\times(t-1)})\right| = 3t - 3 = 3\left\lceil \frac{2+n_1+n_2}{3} \right\rceil - 3 < 2 + n_1 + n_2 = \left|V(K_{1,n_1} \cup K_{1,n_2})\right|$, $V(K_{3\times(t-1)})$ contains neither a blue P_3 nor the red $K_{1,n_1} \cup K_{1,n_2}$.
2. If $A < B$, then $t = \left\lceil \frac{n_1+1}{2} \right\rceil$. Suppose that $m_3(K_{1,n_1} \cup K_{1,n_2}, P_3) < t$. Then, $n_1 \leq 2(t-1) - 1 = 2\left\lceil \frac{n_1+1}{2} \right\rceil$, which is a contradiction.

Therefore, $m_3(K_{1,n_1} \cup K_{1,n_2}, P_3) \geq t$.

Now, we show that $m_3(K_{1,n_1} \cup K_{1,n_2}, P_3) \leq t$. We consider any coloring of $K_{3\times t} = G_R \oplus G_B$ such that G_B does not contain a blue P_3. Thus, $\Delta(G_B) \leq 1$ and G_B is a matching graph. We consider any two endpoints of a P_2 in G_B, say u and v. We know that $d_{G_R}(u) = d_{G_R}(v) = 2t - 1$. If $n_1 = 2t - 1 - s$, for some nonnegative integers $s \leq \frac{t}{2}$, then $n_2 \leq t - 1 + s$. Then we always have a disjoint union of two stars $K_{1,n_1} \cup K_{1,n_2}$ in G_R with u and v as their centers, respectively, and all vertices that are in the same partite set with v being the leaves of K_{1,n_1}.

Theorem 3.2 Let $n_1 \geq n_2 \geq n_3 \geq 1$ be positive integers. Let $A = \left\lceil \frac{3+n_1+n_2+n_3}{3} \right\rceil$ and $B = \left\lceil \frac{n_1+2}{2} \right\rceil$. Then, $m_3(\bigcup_{i=1}^{3} K_{1,n_i}, P_3) = \max\{A, B\}$.

Proof. Let $t = \max\{A, B\}$. To show that $m_3(\bigcup_{i=1}^{3} K_{1,n_i}, P_3) \geq t$, we consider the following two possibilities for the value of t:

1. If $A \geq B$, then $t = \left\lceil \frac{3+n_1+n_2+n_3}{3} \right\rceil$. We make the edges of graph $K_{3\times(t-1)}$ red. Since $\left|V(K_{3\times(t-1)})\right| = 3t - 3 = 3\left\lceil \frac{3+n_1+n_2+n_3}{3} \right\rceil - 3 < 3 + n_1 + n_2 + n_3 = \left|V(\bigcup_{i=1}^{3} K_{1,n_i})\right|$, $V(K_{3\times(t-1)})$ contains neither a blue P_3 nor the red $\bigcup_{i=1}^{3} K_{1,n_i}$.
2. If $A < B$, then $t = \left\lceil \frac{n_1+2}{2} \right\rceil$. Suppose that $m_3(\bigcup_{i=1}^{3} K_{1,n_i}, P_3) < t$. Then, $n_1 \leq 2(t-1) - 2 = 2\left\lceil \frac{n_1+2}{2} \right\rceil - 4 < n_1$, which is a contradiction.

Therefore, $m_3(\bigcup_{i=1}^{3} K_{1,n_i}, P_3) \geq t$.

Now, we show that $m_3\left(\bigcup_{i=1}^{3} K_{1,n_i}, P_3\right) \le t$. We consider any coloring of $K_{3\times t} = G_R \oplus G_B$, such that G_B does not contain a blue P_3. Thus, $\Delta(G_B) \le 1$ and G_B is a matching graph. Let $U = \{u_1, u_2, \dots, u_t\}, V = \{v_1, v_2, \dots, v_t\}$ and $W = \{w_1, w_2, \dots, w_t\}$ be the three partite sets of graph $K_{3\times t}$. Let $E(G_B) \supseteq E(UV) \cup E(VW) \cup E(UW)$, where $E(UV) = \{u_1v_1, u_2v_2, \dots, u_pv_p\}$, $E(VW) = \{v_{p+1}w_1, v_{p+2}w_2, \dots, v_{p+q}w_q\}$ and $E(UW) = \{u_{p+1}w_{q+1}, u_{p+2}w_{q+2}, \dots, u_{p+r}w_{q+r}\}$. Note that $p+q \le t, q+r \le t$ and $p+r \le t$. For the values of p, q and r, we have four matching possibilities in G_B:

1. $p \ge 1, q = r = 0$, see Figure 4(a).
2. $p \ge 1, q = 0, r \ge 1$, see Figure 4(b) or $p \ge 1, q \ge 1, r = 0$, see Figure 4(c).
3. $p \ge 1, q \ge 1, r \ge 1$, see Figure 5.

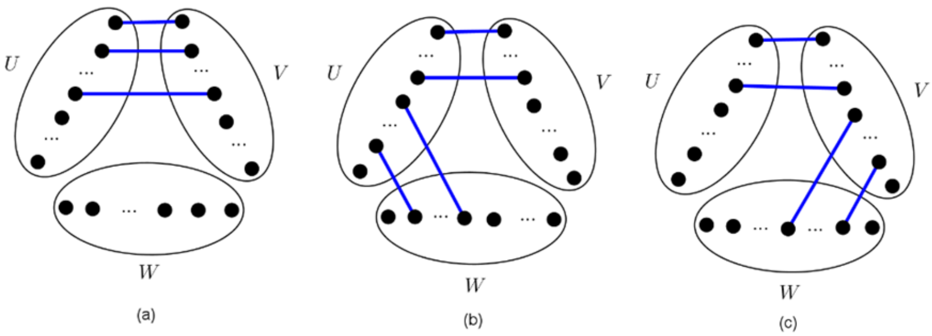

Figure 4 Three matching possibilities in G_B.

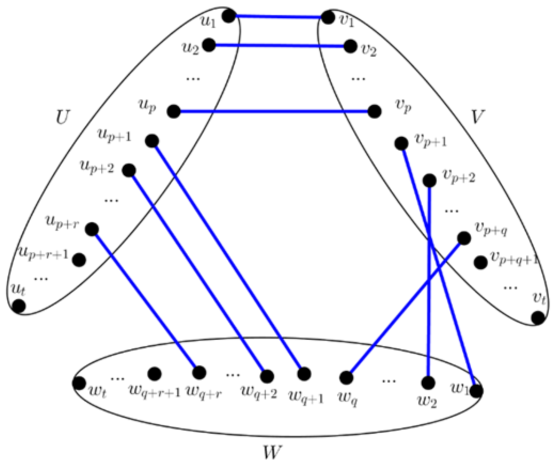

Figure 5 A matching in G_B, if $p \ge 1, q \ge 1, r \ge 1$.

To show the three stars in G_R, we choose the centers of stars K_{1,n_1}, K_{1,n_2} and K_{1,n_3} are u_1, v_1 and either $v_{\text{p}+1}$ (if $p < t$) or w_1 (if $p = t$), respectively. If $t = A$, then $t = \left\lceil \frac{3+n_1+n_2+n_3}{3} \right\rceil \leq \left\lceil \frac{3+3n_1}{3} \right\rceil = 1 + n_1$. If $t = B$, then $t = \left\lceil \frac{n_1+2}{2} \right\rceil \leq n_1 + 1$. Therefore, $t - 1 = n_1$, for all n_1, n_2 and n_3. Then, $t - 1 \leq n_1 \leq 2t - 2$. Let $s_1 = n_1 - (t-2) \geq 1$, so $v_2, v_3, \dots, v_p, v_{p+2}, \dots, v_{p+q}, v_{p+q+1}, \dots, v_t$, $w_1, w_2, w_3, \dots, w_{s_1}$ are the leaves of K_{1,n_1}. We have the following two possibilities to obtain the stars K_{1,n_2} and K_{1,n_3}:

1. If $n_2 \leq t - s_1$, then $w_{s_1+1}, w_{s_1+2}, \dots, w_{s_1+n_2}$ are the leaves of K_{1,n_2}. Since $n_3 \leq n_2 \leq t - s_1 \leq t - 1$, we have $u_2, u_3, \dots, u_{n_3+1}$ are the leaves of K_{1,n_3}, see Figure 6.

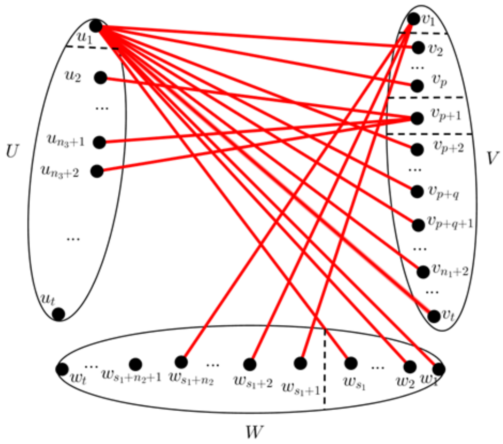

Figure 6 A disjoint union of stars $\bigcup_{i=1}^{3} K_{1,n_i}$ in G_R, if $p < t$.

2. If $t - s_1 \leq n_2 \leq n_1$ and let $s_2 = n_2 - (t - s_1) \geq 1$, then $w_{s_1+1}, w_{s_1+2}, \dots, w_t, u_2, u_3, \dots, u_{s_2+1}$ are the leaves of K_{1,n_2}. Since $n_1 + n_2 = 2t - 2 + s_2$, so $n_3 \leq t - s_2 - 1$. Then, $u_{s_2+2}, u_{s_2+3}, \dots, u_{n_3+s_2+1}$ are the leaves of K_{1,n_3}.

Therefore, we have a disjoint union of stars $\bigcup_{i=1}^{3} K_{1,n_i}$ in G_R, where u_1, v_1 and v_{p+1} are their centers.

Theorem 3.3 *Let $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$ be positive integers. Let $A = \left\lceil \frac{4+n_1+n_2+n_3+n_4}{3} \right\rceil$ and $B = \left\lceil \frac{n_1+3}{2} \right\rceil$. Then, $m_3\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) = \max\{A, B\}$.*

Proof. Let $t = \max\{A, B\}$. To show that $m_3\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) \geq t$, we consider the following two possibilities for the value of t:

1. If $A \geq B$, then $t = \left\lceil \frac{4+n_1+n_2+n_3+n_4}{3} \right\rceil$. We make the edges of graph $K_{3\times(t-1)}$ red. Since $\left|V\left(K_{3\times(t-1)}\right)\right| = 3t - 3 = 3\left\lceil \frac{4+n_1+n_2+n_3+n_4}{3} \right\rceil - 3 < 4 + n_1 + n_2 + n_3 + n_4 = \left|V\left(\bigcup_{i=1}^{4} K_{1,n_i}\right)\right|$, $V\left(K_{3\times(t-1)}\right)$ contains neither a blue P_3 nor the red $\bigcup_{i=1}^{4} K_{1,n_i}$.
2. If $A < B$, then $t = \left\lceil \frac{n_1+3}{2} \right\rceil$. Suppose that $m_3\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) < t$. Then, $n_1 \leq 2(t-1) - 3 = 2\left\lceil \frac{n_1+3}{2} \right\rceil - 5 < n_1$, which is a contradiction.

Therefore, $m_3\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) \geq t$.

Now, we show that $m_3\left(\bigcup_{i=1}^{4} K_{1,n_i}, P_3\right) \leq t$. We consider any coloring of $K_{3\times t} = G_R \oplus G_B$ such that G_B does not contain a blue P_3, so $\Delta(G_B) \leq 1$. Then, G_B is a matching graph, see Figures 4 and 5. The centers of stars $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}$ and K_{1,n_4} are u_1, v_1, v_{p+1} and w_1, respectively. We have the following two possibilities to obtain K_{1,n_1} and K_{1,n_2}:

1. If $n_1 \leq t-2$, then $v_2, v_3, \dots, v_{\mathrm{p}}, v_{\mathrm{p+2}}, \dots, v_{\mathrm{p+q+1}}, \dots, v_{\mathrm{n_1+2}}$ are the leaves of K_{1,n_1}. Since $n_2 \leq n_1 \leq t-2$, we have $w_2, \dots, w_{n_2+1}$ are the leaves of K_{1,n_2}. There are $t - n_2 + 1 \geq 1$ vertices in $W - \{w_1, w_2, \dots, w_{n_2+1}\}$. We have the following two possibilities to obtain K_{1,n_3} and K_{1,n_4}:
   a) If $n_3 \leq t - (n_2+1)$, then $w_{n_2+2}, w_{n_2+3}, \dots, w_{n_2+n_3+1}$ and $u_2, u_3, \dots, u_{\mathrm{n_4+1}}$ are the leaves of K_{1,n_3} and K_{1,n_4}, respectively.
   b) If $t - (n_2+1) < n_3 \leq n_2$, let $s_1 = n_3 - (t - (n_2+1)) \geq 1$, then $w_{n_2+2}, w_{n_2+3}, \dots, w_t, u_2, u_3, \dots, u_{s_1+1}$ are the leaves of K_{1,n_3}. Since $n_1 + n_2 + n_3 \leq 2t - 3 - s_1$, $n_4 \leq t - s_1 - 1$. Then, $u_{s_1+2}, u_{s_1+3}, \dots, u_{s_1+\mathrm{n_4}+1}$ are the leaves of K_{1,n_4}.

2. If $t - 1 \leq n_1 \leq 2t - 3$ and let $s_2 = n_1 - (t-2) \geq 1$, then $v_2, v_3, \dots, v_{\mathrm{p}}, v_{\mathrm{p+2}}, \dots, v_{\mathrm{p+q}}, v_{\mathrm{p+q+1}}, \dots, v_{\mathrm{t}}, w_2, w_3, w_{s_2+1}$ are the leaves of K_{1,n_1}. We have three possibilities:
   a. If $n_2 < t - (s_2+1)$ and let $s_3 = n_3 + n_2 + s_2 + 1 - t \geq 1$, then $w_{s_2+2}, w_{s_2+3}, \dots, w_{s_2+n_2+1}$ are the leaves of K_{1,n_2} and we have two possibilities:
      i. If $n_3 \leq t - (n_2 + s_2 + 1)$, then $w_{s_2+n_2+2}, w_{s_2+n_2+3}, \dots, w_{s_2+n_2+n_3+1}$ are the leaves of K_{1,n_3}. Since $n_4 \leq n_3$, so $u_2, u_3, \dots, u_{\mathrm{n_4+1}}$ are the leaves of K_{1,n_4}.
      ii. If $t - (n_2 + s_2 + 1) < n_3 \leq n_2$, then $w_{s_2+n_2+2}, w_{s_2+n_2+3}, \dots, w_{\mathrm{t}}, u_2, u_3, \dots, u_{s_3+1}$ are the leaves of K_{1,n_3}. Since $n_1 + n_2 + n_3 \leq 2t - 3 +$

s_3, so $n_4 \leq t-1-s_3$. Then, $u_{s_3+2}, u_{s_3+3}, \dots, u_{s_3+n_4+1}$ are the leaves of K_{1,n_4}.

b. If $n_2 = t-(s_2+1)$, then $w_{s_2+2}, w_{s_2+3}, \dots, w_t$ are the leaves of K_{1,n_2}. Since $n_1 + n_2 = 2t - 3 + s_4$, so $n_3 + n_4 \leq t - 1 - s_4$. Then, $u_2, u_3, \dots,$ u_{n_3+1} and $u_{n_3+2}, u_{n_3+3}, \dots, u_{n_3+n_4+1}$ are the leaves of K_{1,n_3} and K_{1,n_4}, respectively.

c. If $t-(s_2+1) < n_2 \leq n_1$ and let $s_4 = n_2 + s_2 + 1 - t \leq 1$, then $w_{s_2+2}, w_{s_2+3}, \dots, w_t, u_2, u_3, \dots, u_{s_4+1}$ are the leaves of K_{1,n_2}. Since $n_1 + n_2 = 2t - 3 + s_4$, so $n_3 + n_4 \leq t - 1 - s_4$. Then, $u_{s_4+2}, u_{s_4+3}, \dots,$ $u_{s_4+n_3+1}$ and $u_{s_4+n_3+2}, u_{s_4+n_3+3}, \dots, u_{s_4+n_3+n_4+1}$ are the leaves of K_{1,n_3} and K_{1,n_4}, respectively.

Therefore, we find a disjoint union of stars $\bigcup_{i=1}^{4} K_{1,n_i}$ in G_R, where u_1, v_1, v_{p+1} and w_1 are their centers.

From Theorems 3.1, 3.2 and 3.3 we obtain that $m_3\left(\bigcup_{i=1}^{k} K_{1,n_i}, P_3\right) = \max\{A, B\}$, where $A = \left\lceil \frac{k+n_1+n_2+\cdots+n_k}{3} \right\rceil$ and $B = \left\lceil \frac{n_1+k-1}{2} \right\rceil$, for $k = 2, 3, 4$. For $k \geq 5$, it seems that the tripartite Ramsey number of $\bigcup_{i=1}^{k} K_{1,n_i}$ and P_3 is also $\max\{A, B\}$. For example, $m_3\left(2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}, P_3\right) = \max\left\{\left\lceil \frac{7+6+6+5+5+4+3+2}{3} \right\rceil, \left\lceil \frac{6+6}{2} \right\rceil\right\} = 13$ and $m_3\left(K_{1,100} \cup K_{1,55} \cup 8K_{1,1}, P_3\right) = \max\left\{\left\lceil \frac{10+100+55+1+1+1+1+1+1+1+1}{3} \right\rceil, \left\lceil \frac{100+9}{2} \right\rceil\right\} = 58$, which can be seen in Figures 7 and 8, respectively.

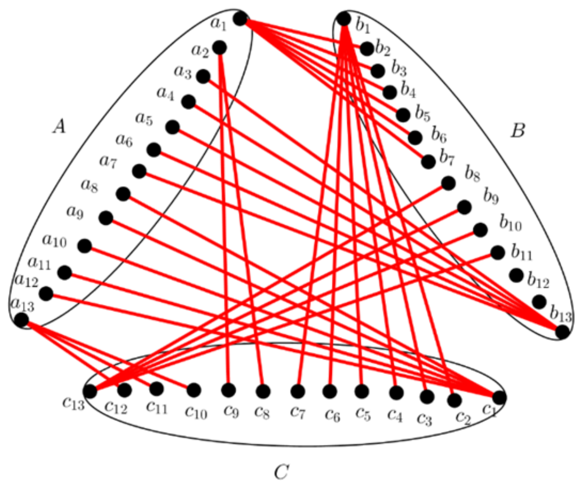

Figure 7 A disjoint union of stars $2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}$ in $K_{3\times13}$.

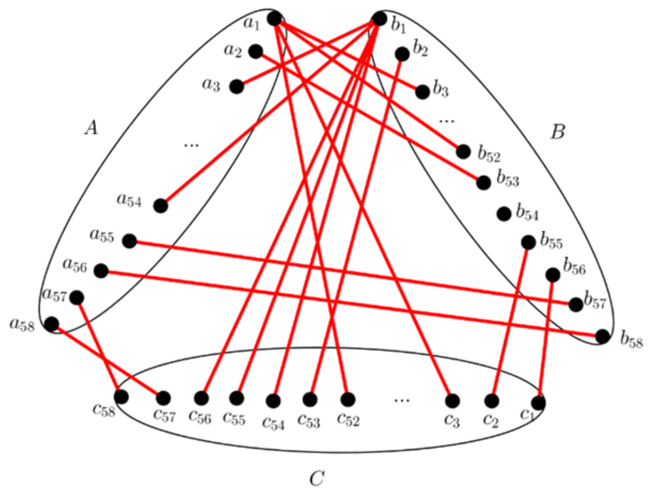

Figure 8 A disjoint union of stars $K_{1,100} \cup K_{1,55} \cup 8K_{1,1}$ in $K_{3\times 58}$.

Note that from these figures, we may have a different way to choose the stars than as mentioned in the proof of Theorem 3.3. Moreover, to obtain $m_3(2K_{1,6} \cup 2K_{1,5} \cup K_{1,4} \cup K_{1,3} \cup K_{1,2}, P_3)$ and $m_3(K_{1,100} \cup K_{1,55} \cup 8K_{1,1}, P_3)$ we cannot use the technique for choosing stars in the proof of Theorem 3.3. So, we would need to develop a new technique to prove the following conjecture.

Conjecture 3.1 Let $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ be positive integers. Let $A = \left\lceil \frac{k+n_1+n_2+\cdots+n_k}{3} \right\rceil$ and $B = \left\lceil \frac{n_1+k-1}{2} \right\rceil$. Then, $m_3(\cup_{i=1}^{k} K_{1,n_i}, P_3) = \max\{A, B\}$.

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