A unifying concept of X chart and X-bar chart when subgroup sizes are equal

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Abstract

X chart and X-bar chart are usually seen as two techniques in SPC with different concepts. In this paper we propose two propositions as a unifying concept for those two charts; one for start-up stage and the other for process control for future observations. Another advantage of this concept lies in the determination of control limits which is not based on approximation method anymore. An exact method will be introduced.

Key words : *X Chart, X-Bar Chart, start-up stage, process control, control limits*

Sari

Konsep untuk bagan kendali X dan bagan kendali $\overline{X}$ bila ukuran subgrupnya sama

Bagan kendali X dan bagan kendali $\overline{X}$ biasanya dipandang sebagai dua teknik Pengendalian Proses Statistis yang memiliki konsep yang berbeda. Dalam tulisan ini kami usulkan dua buah proposisi yang memungkinkan memandang kedua bagan kendali itu dari satu konsep yang sama. Proposisi pertama adalah untuk tahap awal pembuatan bagan kendali (start-up stage) dan yang lain untuk pengendalian proses melalui pengamatan berikutnya. Keuntungan lain dari konsep itu terletak pada batas kendali yang tidak lagi harus ditentukan melalui metode pendekatan seperti yang biasa dikemukakan dalam berbagai pustaka tentang pengendalian proses. Metode eksak akan diperkenalkan.

Kata kunci: *bagan kendali X, bagan kendali $\overline{X}$, tahap awal pembuatan bagan kendali (start-up stage), pengendalian proses, batas kendali.*

1 Introduction

X Chart and X-Bar Chart* are usually seen as two techniques in SPC with different concepts. X Chart or individual chart is usually used in conjunction with moving range and X-Bar Chart is often used together with standard deviation. Furthermore, in those two charts, it is customary to use limiting distribution to determine the control limits either in start-up stage or in process control for future observations.

In this paper we try to develop a unifying concept for constructing:

1. X Chart based on the standard deviation s of individuals.
2. X-Bar Chart in conjunction with standard deviation of subgroup means for special case; when subgroup sizes are equal.

With this concept, the exact distributions for detecting control limits will be obtained. For this purpose, in section 2 we present a unified point of view of individual chart or X Chart and X-Bar Chart. This includes the control limits either in start-up stage or in process control for future observations in X Chart. Section 3 will be devoted to the determination of similar control limits in X-Bar Chart, by using mathematical reason developed in the previous section.

2 X Chart

Individual chart or X Chart is usually used in conjunction with moving range. This can be seen in any literature, even in the most recent ones, such as Badavas (1993), Doty (1991) and Smith (1995). In this case, its control limits are

$$LCL = \overline{X} - k.\overline{MR} \text{ and } UCL = \overline{X} + k.\overline{MR}$$

where k is given by an approximation method.

In this section we try to develop X Chart based on the standard deviation s of individuals which will provide us with exact control limits. Another advantage is that this unify the concept of X Chart and X-Bar Chart. First we introduce the following proposition.

Proposition 1. Let $X_1, X_2, \ldots, X_m$ be random sample from normal distribution $N(\mu, \sigma^2)$. If $\bar{X}$ and s^2 represent respectively sample mean and sample variance, then we have

$$\frac{m}{(m-1)^2}\,\frac{(X_i-\bar{X})^2}{s^2} \sim \textbf{Beta}\left(\frac{1}{2}, \frac{m-2}{2}\right)$$

for all i = 1, 2, ... , m.

Proof.

We know that

$$\frac{(m-1)s^2}{\sigma^2} = \sum_{k=1}^{m} \frac{(X_k-\bar{X})^2}{\sigma^2} \sim \chi^2_{(m-1)}$$

which means that it follows chi-square distribution with (m-1) degrees of freedom.

On the other hand, for all i = 1, 2, ... , m we have

$$(X_i-\bar{X}) \sim N\left(0, \frac{m-1}{m}\sigma^2\right)$$

Consequently,

$$\frac{m}{m-1}\,\frac{(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2_{(1)}$$

Now we write the statistic $\frac{(X_i-\bar{X})^2}{s^2}$ as follows

$$\frac{(X_i-\bar{X})^2}{s^2} = (m-1)\frac{\frac{(X_i-\bar{X})^2}{\sigma^2}}{\sum_{k=1}^{m}\frac{(X_k-\bar{X})^2}{\sigma^2}}$$

or,

$$\frac{(X_i-\bar{X})^2}{s^2} = \frac{(m-1)^2}{m}\,\frac{\frac{m}{(m-1)}\frac{(X_i-\bar{X})^2}{\sigma^2}}{\sum_{k=1}^{m}\frac{(X_k-\bar{X})^2}{\sigma^2}}$$

But the denominator equals

$$\frac{m}{m-1}\,\frac{(X_i-\bar{X})^2}{s^2} + \sum_{k=1}^{m} A_k \frac{(X_k-\bar{X})^2}{\sigma^2}$$

where $A_k = -\frac{1}{m-1}$ for k = i and A_k = 1 otherwise. Now we obtain the following expression

$$\frac{(X_i-\bar{X})^2}{s^2} = \frac{(m-1)^2}{m}\,\frac{\frac{m}{(m-1)}\frac{(X_i-\bar{X})^2}{\sigma^2}}{\frac{m}{m-1}\frac{(X_i-\bar{X})^2}{\sigma^2} + \sum_{k=1}^{m} A_k \frac{(X_k-\bar{X})^2}{\sigma^2}}$$

or,

$$\frac{m}{(m-1)^2}\,\frac{(X_i-\bar{X})^2}{s^2} = \frac{\frac{m}{(m-1)}\frac{(X_i-\bar{X})^2}{\sigma^2}}{\frac{m}{m-1}\frac{(X_i-\bar{X})^2}{\sigma^2} + \sum_{k=1}^{m} \frac{(X_k-\bar{X})^2}{\sigma^2}}$$

We recognize that the numerator is distributed as $\chi^2_{(1)}$. Consequently, the second term of denominator is distributed as $\chi^2_{(m-2)}$. If the numerator and the denominator are divided by 2, then now the numerator is distributed as **Gamma** $(\frac{1}{2}, 1)$ and the second term of denominator is distributed as **Gamma** $(\frac{m-2}{2}, 1)$.

Hence the proof is done.

In practice $X_1, X_2, \ldots, X_m$ represent individual random observations. Its realizations are used as historical data in start-up stage. Hence, in this stage X_i and $\bar{X}$ are not independent. The two statistics X_i and s^2 are also not independent. But if X_f represents future observation, then X_f, $\bar{X}$ and s^2 are independent. This means that control chart in start-up stage and control chart in process control for future observations are different.

2.1 *Start-up Stage*

In this stage, the realization of $X_1, X_2, \ldots, X_m$ are considered as historical data. From the above proposition, we know that

$$\frac{(X_i-\bar{X})^2}{s^2} \sim \frac{(m-1)^2}{m}\,\textbf{Beta}\left(\frac{1}{2}, \frac{m-2}{2}\right)$$

This distribution determines the exact control limits in start-up stage in X Chart. In fact, those exact control limits are

$$LCL = \bar{X} - A.s \text{ and}$$

$$UCL = \bar{X} + A.s$$

where $A^2 = \frac{(m-1)^2}{m}$ **Beta** $\left((1-\alpha), \frac{1}{2}, \frac{m-2}{2}\right)$ and **Beta** $\left((1-\alpha), \frac{1}{2}, \frac{m-2}{2}\right)$ is $(1-\alpha)$-quantile of beta distribution with parameters $\frac{1}{2}$ and $\frac{m-2}{2}$

Multivariate version of Proposition 1 can be seen in Tracy *et al* (1992) and Nomikos *et al* (1995). Its proof can be traced in Gnanadesikan and Kettenring (1972).

2.2 *Process Control for Future Observation*

Let again $X_1, X_2, \ldots, X_m$ be random sample from normal distribution $N(\mu, \sigma^2)$ used in start-up stage and $\bar{X}$ and s^2 represent respectively its sample mean and sample variance. If X_f represents future observation, then X_f and $X_1, X_2, \ldots, X_m$ are independent. It is so between $X_f, \bar{X}$ and s^2. Consequently, we have the following proposition which can be used to determine control limits in process control for future observations.

Proposition 2. Let again $X_1, X_2, \ldots, X_m$ be random sample from normal distribution $N(\mu, \sigma^2)$ used in start-up stage and $\bar{X}$ and s^2 represent respectively its sample mean and sample variance. If X_f represents future observation, then

$$\frac{m}{m+1}\,\frac{(X_f-\bar{X})^2}{s^2} \sim F_{1,(m-1)}$$

Proof.

The fact that X_f and $\bar{X}$ are independent implies that

$$(X_f-\bar{X}) \sim N(0, \frac{m+1}{m}\sigma^2)$$

and hence,

$$\frac{m}{m+1}\,\frac{(X_f-\bar{X})^2}{\sigma^2} \sim \chi^2_{(1)}$$

Now consider the following expression

$$\frac{m}{m+1}\,\frac{(X_f-\bar{X})^2}{\sigma^2} = \frac{\frac{m}{m+1}\,\frac{(X_f-\bar{X})^2}{\sigma^2}}{\frac{s^2}{\sigma^2}}$$

The numerator is distributed as $\chi^2_{(1)}$ and the denominator is distributed as $\chi^2_{(m-1)}$ divided by its degree of freedom. Furthermore X_f, $\bar{X}$ and s^2 are independent which implies that numerator and denominator are independent. Hence we proved the proposition.

Corollary

$$\frac{(X_f-\bar{X})}{s}\sqrt{\frac{m}{m+1}} \sim t_{(m-1)}$$

Based on Proposition 2 and its corollary, control limits in process control for future observations are determined as follows.

$LCL = \bar{X} - B.s$ and

$UCL = \bar{X} + B.s$

where B is the $(1 - \frac{\alpha}{2})$-quantile of student-t distribution with (m-1) degree of freedom.

3 X-bar chart when subgroup sizes are equal n > 1

X-Bar Chart is often used in conjunction with standard deviation. In order to determine its control limits, it is customary to use the limiting distribution (see Badavas (1993), Doty (1991), and Smith (1995)). Here we try to identify the exact distribution which will be applied to calculate those control limits. Now suppose that m represents the number of subgroups and its sizes are equal $n > 1$. If X_{ij} is the j-th item in i-th subgroup; i = 1, 2, . . . , m, j = 1, 2, . . . , n and

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \text{ : sample mean in i-th subgroup}$$

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_i \text{ : grand mean}$$

and $s^2 = \frac{1}{m-1}\sum_{i=1}^{m}(\bar{X}_i - \bar{\bar{X}})^2$

then, according to Proposition 1, we have

$$\frac{m}{(m-1)^2}\,\frac{(\bar{X}_i-\bar{\bar{X}})^2}{s^2} \sim \textbf{Beta}\left(\frac{1}{2}, \frac{m-2}{2}\right)$$

for all i = 1, 2, . . . , m.

Suppose that those m subgroups are used in start-up stage. In this stage $\bar{X}_i$ and $\bar{\bar{X}}$ are not independent. It is so between $\bar{X}_i$ and s^2. If $\bar{X}_f$ represents the sample mean of future subgroup, then $\bar{X}_f$, $\bar{\bar{X}}$ and s^2 are independent. Hence the start-up stage and the process control for future subgroups are as follows.

3.1 *Start-up stage*

In this stage, the exact control limits are

$$LCL = \bar{\bar{X}} - A.s \text{ and}$$

$$UCL = \bar{\bar{X}} + A.s$$

where A^2 is given in section 2.1.

3.2 Process control for future observation

Let again $\bar{X}_i$ be the sample mean of i-th subgroup; i = 1, 2, . . . , m used in start-up stage, $\bar{\bar{X}}$ be the grand mean and s^2 be the variance of subgroup means. Then, according to Proposition 2, we have

$$\frac{m}{m+1} \frac{(\bar{X}_i - \bar{\bar{X}})^2}{s^2} \sim F_{1,(m-1)}$$

Corollary

$$\frac{(\bar{X}_f - \bar{\bar{X}})}{s} \sqrt{\frac{m}{m+1}} \sim t_{(m-1)}$$

This corollary gives us the control limits in process control for future subgroups

$$LCL = \bar{\bar{X}} - B.s \text{ and}$$

$$UCL = \bar{\bar{X}} + B.s$$

where B is the $(1 - \frac{\alpha}{2})$-quantile of student-t distribution with (m − 1) degree of freedom.

4 Concluding remarks

X Chart and X-Bar Chart can be seen as having the same concept based on Proposition 1 for start-up stage and Proposition 2 for process control for future observations. According to these propositions, the control limits in those two charts can be determined through exact distributions and not by an approximation method anymore. This is an advantage of those propositions.

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Reference

1. Badavas P.C. (1993). Real-Time Statistical Process Control. *Prentice Hall Inc.*
2. Doty L.A. (1991). Statistical Process Control; An ASQC Quality Press Book. *Industrial Press.*
3. Gnanadesikan R dan Kettenring J.R. (1972). *Robust Estimates, Residuals, and Outlier Detection with Multiresponse Data.* Biometrics, Vol. 28, No. 1.
4. Nomikos P and MacGregor J.F (1995). *Multivariate SPC Charts for Monitoring Batch Processes.* Technometrics, Vol.37, No.1.
5. Seber G.A.F (1984). Multivariate Observations. *John Wiley and Sons.*
6. Smith G. (1995). Statistical Process Control and Quality Improvement. 2nd Edition. *Prentice Hall.*
7. Tracy N.D. Young J.C dan Mason R.L. (1992). *Multivariate Control Chart for Individual Observations.* Journal of Quality Technology, Vol. 24, No. 2