The uniqueness of almost Moore digraphs with degree 4 and diameter 2

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Abstract

It is well known that *Moore* digraphs of degree $d > 1$ and diameter $k > 1$ do not exist. For degrees 2 and 3, it has been shown that for diameter $k \geq 3$ there are no *almost Moore* digraphs, i.e. the diregular digraphs of order one less than the *Moore* bound. Digraphs with order close to the *Moore* bound arise in the construction of optimal networks. For diameter 2, it is known that *almost Moore* digraphs exist for any degree because the line digraphs of complete digraphs are examples of such digraphs. However, it is not known whether these are the only *almost Moore* digraphs. It is shown that for degree 3, there are no *almost Moore* digraphs of diameter 2 other than the line digraph of K_4. In this paper, we shall consider the *almost Moore* digraphs of diameter 2 and degree 4. We prove that there is exactly one such digraph, namely the line digraph of K_5.

Keywords : ***almost Moore** digraph, complete digraph, line digraph, **Moore** bound, repeat.*

Sari

Ketunggalan graf berarah *Hampir Moore* dengan derajat 4 dan diameter 2

Telah lama diketahui bahwa tidak ada graf berarah *Moore* dengan derajat $d > 1$ dan diameter $k > 1$. Lebih lanjut, untuk derajat 2 dan 3, telah ditunjukkan bahwa untuk diameter $k \geq 3$, tidak ada graf berarah *Hampir Moore*, yakni graf berarah teratur dengan orde satu lebih kecil dari batas *Moore*. Graf berarah dengan orde mendekati batas *Moore* digunakan dalam pengkonstruksian jaringan optimal. Untuk diameter 2, diketahui bahwa graf berarah *Hampir Moore* ada untuk setiap derajat karena graf berarah garis *(line digraph)* dari graf komplit adalah salah satu contoh dari graf berarah tersebut. Akan tetapi, belum dapat dibuktikan apakah graf berarah tersebut merupakan satu-satunya contoh dari graf berarah *Hampir Moore* tadi. Selanjutnya telah ditunjukkan bahwa untuk derajat 3, tidak ada graf berarah Hampir *Moore* diameter 2 selain graf berarah garis dari K_4. Pada makalah ini, kita mengkaji graf berarah Hampir Moore diameter 2 dan derajat 4. Kita buktikan bahwa ada tepat satu graf berarah tersebut, yaitu graf berarah garis dari K_5.

Kata kunci: batas Moore, graf berarah hampir Moore, graf berarah garis, graf berarah komplit, pengulangan.

1 Introduction

By a *digraph* we mean a structure $G = (V, A)$ where $V(G)$ is a nonempty set of distinct elements called *vertices*; and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices u, $v \in V(G)$ called *arcs*. A digraph H is a *subdigraph* of G if $V(H) \subset V(G)$ and $A(H) \backslash \subset A(G)$.

The *order* of a digraph G is the number of vertices in G, i.e., $|V(G)|$. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in G$. Similarly, an *out-neighbour* of a vertex v in a digraph G is a vertex w such that $(v, w) \in G$. For $S \subset V(G)$ denote by $N^-(S)$ (respectively $N^+(S)$) the set of all in-neighbours (respectively out-neighbours) of elements of S, that is $N^-(S) = \{w \in V(G)|(w, v) \in G, v \in S\}$ (respectively, $N^+(S) = \{w \in V(G)|(v, w) \in G, v \in S\}$. The *in-degree (respectively out-degree)* of a vertex $v \in G$ is the number of its in-neighbours (respectively out-neighbours) in G.

If in a digraph G, the in-degree equals the out-degree (= d) for every vertex, then G is called a *diregular* digraph of degree d.

A $v_0 - v_k$ *walk* W of length k in G is an alternating sequence $(v_0a_1v_1a_2 \cdots a_kv_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. A *closed* walk has $v_0 = v_k$. If the arcs $a_1, a_2, \cdots, a_k$ of W are distinct, W is called a *trail*. If, in addition, the vertices $v_0, v_1, \cdots, v_k$ are also distinct, W is called a *path*. A *cycle}* C_k of length k is a closed trail of length $k > 0$ with all vertices distinct (except the first and the last).

The *distance* from vertex u to vertex v in G, denoted by $\delta(u, v)$, is defined as the length of the shortest path from vertex u to vertex v. Note that in general, $\delta(u, v)$ is not necessary equal to $\delta(v, u)$. The *diameter* k of a digraph G is the maximum distance between any two vertices in G.

Let one vertex be distinguished in a diregular digraph of degree d, order n and diameter k. Let n_i, $i = 0, 1, \cdots, k$ be the number of vertices at distance i from the distinguished vertex. Then,

$$n_i \le d^i \qquad \text{for } i = 1, \cdots, k \tag{1}$$

Hence,

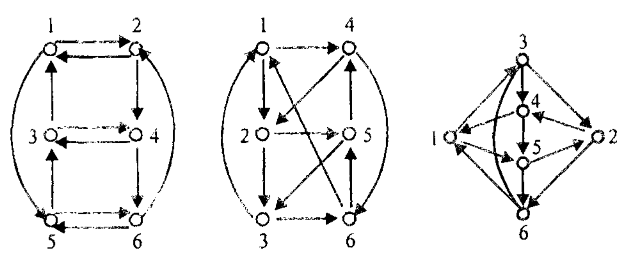
$$n = \sum_{i=0}^{k} n_i \le 1 + d + d^2 + \cdots + d^k \tag{2}$$

If the equality sign holds in (2) then such a digraph is called the *Moore digraph*. The right-hand side of (2) is called the *Moore bound*.

Digraphs with order close to the *Moore* bound arise in the construction of optimal networks [4, 10]. It is well known that except for trivial cases (for $d = 1$ or $k = 1$) *Moore* digraphs do not exist (See [13] or [5] for a simpler proof). The trivial cases are the cycles C_{k+1} of length k+1 and the digraphs K_{d+1} on d+1 vertices.

Since the *Moore* digraphs do not exist for $d \neq 1$ or $k \neq 1$, the problem of the existence of *almost Moore* digraphs, i.e., the diregular digraphs of diameter $k \geq 2$ and degree $d \geq 2$ and order one less than the *Moore* bound, becomes an interesting problem. Such digraphs are denoted by *(d, k)-digraphs*.

Several results have been obtained. The first result in this problem was due to [6] showing that (d, 2)-digraphs do exist, interestingly, one such digraph is the line digraph of K_{d+1}. In particular, there are exactly three non-isomorphic (2,2)-digraphs [12] (see Figure 1), while there is exactly one (3,2)-digraph, i.e., the line digraph of K_4 [3].



r : (1)(2)(3)(4)(5)(6) r : (123)(456) r : (12)(3456)

(a) (b) (c)

Figure 1 The three non-isomorphic (2,2)-digraphs

In [11], Miller and Fris proved that (2, k)-digraphs with $k \geq 3$ do not exist. Subsequently, it was proved that (3, k)-digraphs with $k \geq 3$ do not exist (see [1]).

Every (d, k)-digraph G has the characteristic property that for every vertex $x \in G$ there is a unique vertex $y \in G$ such that there are two walks of lengths not exceeding k from x to y in G [2]. Such a vertex y is called the *repeat* of x, denoted by $r(x)$. If $r(x) = y$ then $r^{-1}(y) = x$. (In general, it may happen that x is on a cycle of length k in digraph G, then $r(x) = x$ and the two walks in question are the trivial walk and the k-cycle itself. Then x is called a *selfrepeat*).

Furthermore, no vertex of a (d, k)-digraph is contained in two cycles of length k.

For $S \subset V(G)$ we define $r(S) = \bigcup_{v \in S} r(v)$ and similarly $r^{-1}(S) \bigcup_{v \in S} r^{-1}(v)$. The function r can be considered as a permutation on the vertex set of G. Figure 1 illustrates the notion of repeat for the three existing (2,2)-digraphs [3]. Each permutation is expressed as a set of permutation cycles.

The following result was proved in [2].

Theorem 1 *For every vertex v of a (d,k)-digraph we have : (a)* $N^+(r(v)) = r(N^+(v))$ *and (b)* $N^-(r(v)) = r(N^-(v))$.

This theorem shows that the mapping $x \mapsto r(x)$ is an *automorphism* of $V(G)$. In what follows we shall therefore refer to r as the *repeat automorphism* of the *almost Moore* digraph G.

In [8], we have proved that if the (4,2)-digraphs contain *one* selfrepeat vertex then the (4,2)-digraphs do not exist, except for the (4,2)-digraphs with *every* vertex is selfrepeat.

In this paper, we shall prove that there is only one such (4,2)-digraph (up to isomorphism), that is the line digraph of K_5. To see that, we have to show that if the (4,2)-digraphs contain no selfrepeat vertices then the (4,2)-digraphs do not exist. By using algebraic techniques,
J. Gimbert [7] shows independently the uniqueness of (4,2)-digraphs.

2 Results

In the following, we assume that the (4,2)-digraph G contains no selfrepeat vertices. Thus, there is no cycle C_2 of length 2 in G.

Lemma 1 *There is no (4,2)-digraph G containing subdigraph of Figure 2 with* $r(c) = a$ *and* $r(s) = c$*, for some* $s \in V(G) \setminus \{b, h\}$.

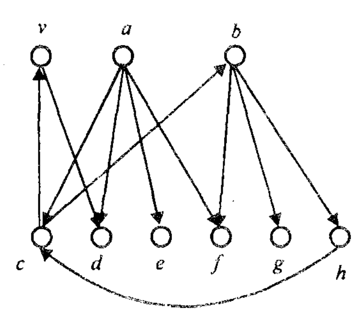


Figure 2

Proof. Since $(c, b) \in G$ then by Theorem 1 we have $(r(c) = a, r(b)) \in G$ and thus $r(b) \in \{d, e, f\}$. To reach a from b, it certainly cannot be done via e or f. It also cannot be done via h since that forces $r(h) = c$, which is a contradiction with $r(s) = c$. Therefore, we have $(g, a) \in G$.

We also have to reach d from b. If we do this via e or f then $r(a) = d$, but $(a, c) \in G$ so by Theorem 1 we have $(r(a) = d, r(c) = a) \in G$. This means that there exists a C_2 : (a, d, a) in G, a contradiction. If we reach d from b via g then $r(g) = d$. This implies $r(b) \in \{e,f\}$. Since $(b, g) \in G$ then by Theorem 1 we have $(e,d) \in G$ or $(f,d) \in G$. Both cases yield $r(a) = d$, a contradiction with $r(g) = d$. Thus, $(h, d) \in G$.

To reach v from b, it cannot be done via h since otherwise there are multiple repeats for h, namely $r(h) = v$ and d. If we do that via g then $r(g) = d$, which is impossible from above. Therefore we have $(e, v) \in G$ or $(f, v) \in G$. Each case implies $r(a) = v$. Since $(a, c) \in G$ by Theorem \ref{iso} we have $(v, a) \in G$. Thus, $r(v) = d$. Applying Theorem \ref{iso} for $(v, a) \in G$, we have $(d, v) \in G$. This creates a C_2 : (d, v, d) in G, which is not possible. Thus we cannot reach v from b in two steps. □

By similar arguments, we can show the two following lemmas.

Lemma 2 *There is no (4, 2)-digraph G containing subdigraph of Figure 3 with* $r(c) = a$ *and* $r(s) = c$, for some $s \in V(G) \setminus \{b, i\}$.

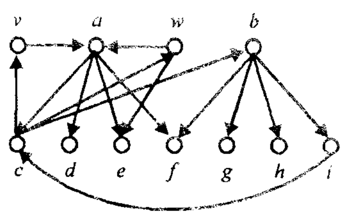


Figure 3

Lemma 3 *Suppose G is the (4,2)-digraph containing subdigraph of Figure 4. If* $r(p) = u$ *and* $u \notin N^+(a) \cup N^+(c) \cup N^+(p)$ then $(t, c) \notin G$.

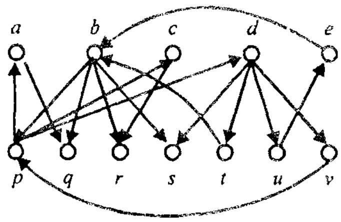


Figure 4

Lemma 4 *Let G a (4,2)-digraph with no selfrepeat vertices. Then there exists a vertex* $v \notin G$ *such that* $d(v, r(v)) = 2$.

Proof. Let x be a vertex of G. If $d(x,r(x)) = 2$ then choose $v = x$ and the proof completes. Otherwise suppose $N^+(x) = \{x_1, x_2, x_3, x_4\}$ and $r(x) = x_1$. Since $(x, x_2) \notin G$, it implies that $(r(x) = x_1, r(x_2)) \notin G$ by Theorem 1. Therefore $d(x_2, r(x_2)) = 2$. The proof completes by choosing $v = x_2$. □

According to Lemma 4 we can label the vertices of G by 0, 1, 2, . . ., 19, such that $d(0, r(0)) = 2$. Without loss of generality assume that $N^+(0) = \{1,2,3,4\}$, $N^+(1) = \{5, 6, 7, 8\}$, $N^+(2) = \{8, 9, 10, 11\}$, $N^+(3) = \{12, 13, 14, 15\}$, and $N^+(4) = \{16, 17, 18, 19\}$. Thus, we have $r(0) = 8$ and due to Theorem 1, $r(\{1, 2, 3, 4\}) = N^+(8)$.

Since G have degree 4, then we know that the vertex 0 has four in-neighbours. But vertices 1, 2, 3, and 4 have to reach 0 in walks of length 2. Thus, we have three essentially different cases as shown in Figure 5

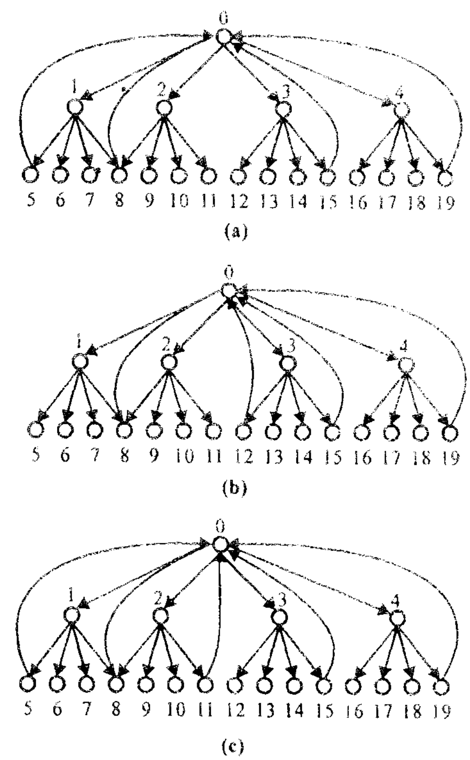


Figure 5 The three different cases of (4, 2)-digraf G

2.1 Case 1

Consider a (4,2)-digraph G containing the subdigraph of Figure 5(a). Thus, we have $r(1) = 0$. From now on, we denote by x, y, and z the remaining out-neighbours of 8 other than 0.

Lemma 5 *Let G be a (4,2)-digraph containing subdigraph of Figure 5(a). Then* $x \in \{9, 10, 11\}$, $y \in \{13, 14, 15\}$, *and* $z \in \{16, 17, 18\}$.

Proof. None of the out-neighbours of 8 is in $\{1, 2\}$ there are no C_2 in G. If one of them, say x, belongs to $\{3,4\}$ then $r(8) = x$. But $(8, 0) \in G$ so we have $(x, 8) \in G$, by Theorem 1. This creates a C_2 : $(x, 8, x)$ in G, which is impossible. None of them is in $\{5, 6, 7, 12, 19\}$, since $r(1) = x$ or $r(8) = 0$, both cases are contradiction with $r(1) = 0$. If two of them, say x and y, are in $\{9, 10, 11\}$ then we have two repeats of 2, namely $r(2) = x$ and y, a contradiction. Thus we have at most one outneighbour of 8 in {9, 10, 11}.

Next we shall show that there is at most one outneighbour of 8 in {13,14,15}. To do this assume there are two, say x and y, are in {13, 14, 15}. Denote by p the remaining vertex such that $(3, p) \in G$ and by v the vertex such that $(v,z) \in G$, then we have the forbidden subdigraph of Lemma 1 in G by letting $a = 8$, $b = 3$, $c = 0$, $d = z$, $e = y$, $f = x$, $g = p$, $h = 12$, and $s = 1$, where $v \in \{2, 3, 4\}$.

Thus we have at most one out-neighbours of 8 in {13, 14, 15}. Similarly, we can show that at most one of them be in {16, 17, 18}. Altogether completes the proof. □

Theorem 2 *There is no (4,2)-digraph containing subdigraph of Figure 5 (a).*

Proof. Suppose that G be a (4,2)-digraph containing subdigraph of Figure 5(a). Due to Lemma 5 we have that $x \in \{9, 10, 11\}$, $y \in \{13, 14, 15\}$, and $z \in \{16, 17, 18\}$. Denote by p and q the two remaining outneighbours of 3 other than 12, then we have the forbidden subdigraph of Lemma 2 in G, by letting $a = 8$, $b = 3$, $c = 0$, $d = z$, $e = x$, $f = y$, $g = p$, $h = q$, $i = 12$, $v = 1$, $w = 2$, and $s = 1$. Thus there is no (4,2)-digraph containing subdigraph of Figure 5a. □

2.2 Case 2

Consider a (4,2)-digraph G containing the subdigraph of Figure 5(b). Thus, $r(4) = 0$. Denote by x, y, and z the remaining out-neighbours of 8 other than 0. Then we can prove the following lemma by applying Lemmas 1 and 2.

Lemma 6 *Let G be a (4,2)-digraph containing a subdigraph of Figure 5(b). Then we have $x \in \{5, 6, 7\}$, $y \in \{9, 10, 11\}$, and $z \in \{13, 14, 15\}$ or we have $x \in \{5, 6, 7\}$, $y \in \{9, 10, 11\}$, and $z \in \{17, 18\}$*

Then the following theorem holds by Lemmas 6 and 2.

Theorem 3 *There is no (4,2)-digraph containing subdigraph of Figure 5(b).*

2.3 Case 3

Consider a (4,2)-digraph G containing the subdigraph of Figure 5(c). Then it is easy to see that the following propositions hold.

Proposition 1 *For each in-neighbour u of 0, we have $r(u) \notin \{0, 3, 4, 5, 6, 7, 9, 10, 11\}$.*

Proposition 2 *If $u, v \in N^+(1) \backslash 8$ or $u, v \in N^+(2) \backslash 8$ then $(u,v) \notin G$.*

By applying Theorem 1, Propositions 1 and 2, we can show the following lemma.

Lemma 7 *Let G be a (4,2)-digraph containing the subdigraph of Figure 5(c). Then (a) if $(5, u) \in G$ then $u \notin \{1, 3, 4, 6, 7, 8, 11, 15, 19\}$ and (b) there is at most one outneighbours of $5 \in \{9, 10\}$.*

Next denote by x, y, and z the outneighbours of 5 other than 0. Then the following lemma holds.

Lemma 8 *Let G a (4,2)-digraph containing subdigraph of Figure 5(c). Then $x \in \{9, 10\}$, $y \in \{12, 13, 14\}$, and $z \in \{16, 17, 18\}$.*

Proof. Due to Lemma 7, we only have to show that there is at most one out-neighbour of 5 in {12, 13, 14} (and {16, 17, 18} respectively). Seeking a contradiction, assume two out-neighbours of 5, say x and y, are in {12, 13, 14}. Denote by p the remaining vertex such that $(3,p) \in G$.

If $z = p$ then we cannot reach 5 from 3. Thus $z \in \{2, 9, 10, 16, 17, 18\}$ and so $(p,5) \in G$. Now, we shall distinguish three cases : (a) $z = 2$, (b) $z \in \{9,10\}$, or (c) $z \in \{16,17,18\}$

In case (a), we have $r(5) = 2$, but then there is no walk of length ≤ 2 from 3 to 2, a contradiction.

In case (b) it can be shown that arcs **(15,** z**)**, $(p, 2)$, **(15, 1)**, **(**x**, 15)**, **(8, 15)**, **(**z**,** p**)**, **(**y**, 8)**, and **(**y**, 11)** must be in G. (See Figure 6). These imply that $r(3) = 15$, $r(15) = 1$, $r(p) = z$, and $r(z) = 5$. But then we cannot reach 19 from 3, a contradiction.

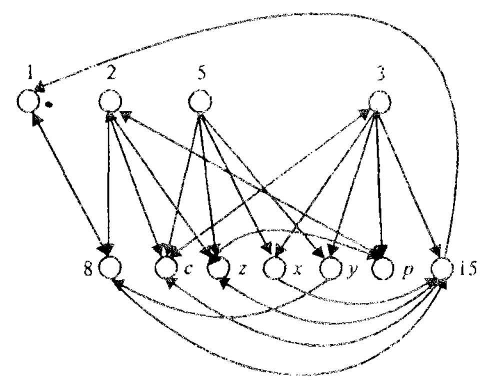


Figure 6.

In case (c) we cannot reach 4 from 3, a contradiction.

Thus, at most one out-neighbour of 5 can be in {12, 13, 14}. With a similar way, we can prove that at most one of them can be in {16, 17, 18}. Therefore, $x \in \{9,10\}$, $y \in \{12, 13, 14\}$, and $y \in \{16, 17, 18\}$. □

Now, we have to prove that the (4,2)-digraph containing subdigraph of Figure 5(c) do not exist. Before proving that, we prove the following propositions. In the following, By applying Lemma 8, we consider a (4,2)-digraph G containing the subdigraph of Figure 5(c) with $x \in \{9,10\}$, $y \in \{12,13,14\}$, and $y \in \{16,17,18\}$. Denote by p the remaining vertex such that $(2,p) \in G$ and $p \notin \{x, 8, 11\}$.

Lemma 9 *There is no (4,2)-digraph containing the subdigraph of Figure 5(c).*

Proof. Seeking a contradiction, assume G a (4,2)-digraph containing the subdigraph of Figure 5(c). We cannot reach 5 from 2 via x, since otherwise there exists a C_2 in

G. Neither can via 11 by Proposition 1. So we shall distinguish two cases : (a) we reach 5 from 2 via 8 or (b) we reach 5 from 2 via p.

In case (a), we can show that arcs **(y,8)**, **(p,3)** **(x,y)**, **(y,11)**, and **(x,19)** must be in G and so $r(1) = 5$, $r(5) = y$, and $r(6) = 0$ (see Figure 7).

But then we cannot reach 15 from 2, a contradiction. Therefore, we cannot reach 5 from 2 via 8.

In case (b), by applying Lemma 3, we have $(8,3) \in G$ and $(8,4) \in G$ (see Figure 8).

Then it forces $(z,19) \notin G$ and $(y,15) \notin G$. Since 5 have to reach 11, 15, and 19, we have the following options :

- $(x,11), (y,19), (z,15) \in G$ or
- $(x,15), (y,19), (z,11) \in G$ or
- $(x,19), (y,11), (z,15) \in G$.

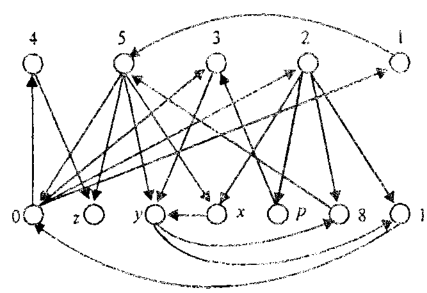


Figure 7

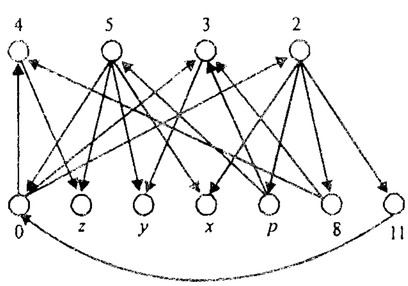


Figure 8

However, all three options are impossible to occur.

This completes the proof. □

From the three cases above then we have the following corollary.

Corollary 1 *There is no (4,2)-digraph without selfrepeat vertices.*

In [8], it showed that the only (4,2)-digraph containing a selfrepeat is the line digraph of K_5. Therefore, together with Corollary 1, we get

Theorem 4 *There is exactly one (4,2)-digraph, namely the line digraph of* K_5.

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