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# A New Type of Coincidence and Common Fixed-Point Theorems for Modified α-Admissible $\mathcal{Z}$-Contraction Via Simulation Function

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**Abstract.** In this manuscript, we introduce the concept of modified α-admissible contraction with the help of a simulation function and use this concept to establish some coincidence and common fixed-point theorems in metric space. An illustrative example that yields the main result is given. Also, several existing results within the frame of metric space are established. The main theorem was applied to derive the coincidence and common fixed-point results for α-admissible $\mathcal{Z}$-contraction.

**Keywords**: *coincidence point; common fixed point; modified $\alpha$-admissible $\mathcal{Z}$-contraction; simulation function; triangular $\alpha$-orbital admissible function.*

## 1 Introduction

Metric fixed-point theory plays an important role in several areas, such as finding solutions for differential equations, integral equations and so forth. In 1906, Maurice Frechet introduced the concept of $(\mathcal{H}, \varpi)$. Standard metric space is a major tool in functional analysis and topology. The Banach contraction principle is one of the foremost outcomes of functional analysis and has led to remarkable speculations. Many authors have used the Banach principle in their research [1,2].

The main idea of this paper is to use the simulation function to associate some fixed-point results as defined by Khojasteh, *et al.* [3]. In 2012, Samet, *et al.* [4] presented fixed-point outcomes for a new category of (α-ψ)-contractive functions. In 2014, Popescu [5] introduced the concept of the triangular α-orbital admissible function and demonstrated several fixed-point results with the aid of generalized α-Geraghty contraction and the triangular α-orbital admissible function. In 2015, Khojasteh, *et al.* [3] introduced $\mathcal{Z}$-contraction, which generalizes the Banach contraction rule by combining various types of nonlinear contractions [6]. In 2016, Karapinar [7] introduced the notion of α-admissible $\mathcal{Z}$-contraction with the aid of a simulation function and established fixed-point results with the assistance of triangular α-orbital admissible

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mapping in the framework of complete metric space. In 2018, Aydi, *et al.* [8] proved fixed-point results for α-admissible $\mathcal{Z}$-contraction by using triangular α-orbital admissible mapping in the context of complete quasi-metric space. Recently, Chandok, *et al.* [9] demonstrated some results via simulation mapping for Geraghty-type contractive functions.

In this paper, ϖ stands for metric and ($\mathcal{H}$, ϖ) denotes metric space. Now, we recollect some elementary results used in sequel. In 2012, Samet, *et al.* [4] presented the concept of the α-admissible function and (α-ψ)-contractive type mappings and established fixed-point results for such mappings.

**Definition 1.1**. [4] Let $\mathcal{Q}$: $\mathcal{H} \to \mathcal{H}$ and α: $\mathcal{H} \times \mathcal{H} \to [0, +\infty)$. Then, $\mathcal{Q}$ is called α-admissible if $\alpha(\mathbf{\Omega}, \mho) \geq 1 \Rightarrow \alpha(\mathcal{Q}\mathbf{\Omega}, \mathcal{Q}\mho) \geq 1$, for each $\mathbf{\Omega}, \mho \in \mathcal{H}$.

**Definition 1.2**. [4] Let Ψ be the class of maps ψ: $[0, +\infty) \to [0, +\infty)$ that fulfills the following conditions:

(i) ψ is upper semi-continuous and strictly increasing;
(ii) $\{\psi^f(\ell)\}$ tends to 0 as $f \to \infty$, for all $\ell > 0$ and $f \in Z_+$;
(iii) $\psi(\ell) < \ell$, for every $\ell > 0$.

These functions are known as comparison functions.

**Definition 1.3**. [4] Let $\mathcal{Q}$:$\mathcal{H} \to \mathcal{H}$ be a given self-mapping in a metric space ($\mathcal{H}$, ϖ). Then, $\mathcal{Q}$ is called aα-ψ mapping of contraction if there are two mappingsψ $\in \Psi$ and α:$\mathcal{H} \times \mathcal{H} \to [0, +\infty)$ with the goal that

$$\alpha(\mathbf{\Omega}, \wp)\, \varpi(\mathcal{Q}\mathbf{\Omega}, \mathcal{Q}\wp) \leq \psi(\varpi(\mathbf{\Omega}, \wp)), \text{ for all } \mathbf{\Omega}, \wp \in \mathcal{H}.$$

**Theorem 1.4.** [4] Let $\mathcal{Q}$:$\mathcal{H} \to \mathcal{H}$ be anα-ψ contractive mapping in ($\mathcal{H}$, ϖ), which is complete, one-one, and onto. Also, $\mathcal{Q}$ fulfils the following conditions:

(i) $\mathcal{Q}$ is continuous;
(ii) $\mathcal{Q}$ is α-admissible;
(iii) there exists $\Omega_0 \in \mathcal{H}$such that $\alpha(\Omega_0, \mathcal{Q}\Omega_0) \geq 1$.

Then, $\mathcal{Q}$ possesses a fixed point in $\mathcal{H}$.

**Theorem 1.5.** [4] Let $\mathcal{Q}$: $\mathcal{H} \to \mathcal{H}$ be an α-ψ contractive mapping in ($\mathcal{H}$, ϖ), which is complete, one-one, and onto. Also, $\mathcal{Q}$ fulfills the following conditions:

(i) If $\{\Omega_n\}$ is a sequence in $\mathcal{H}$ such that $\alpha(\Omega_n, \Omega_{n+1}) \geq 1$ and $\Omega_n = \mathbf{\Omega}$, then $\alpha(\Omega_n \mathbf{\Omega}) \geq 1$;
(ii) $\mathcal{Q}$ is α-admissible;
(iii) there exists $\Omega_0 \in \mathcal{H}$ such that $\alpha(\Omega_0, \mathcal{Q}\Omega_0) \geq 1$.

Then, $\mathcal{Q}$ possesses a fixed point in $\mathcal{H}$.

In 2014, Popescu [5] introduced the concept of triangular α-orbital admissible mappings as follows:

**Definition 1.6.** [5] Let $\mathcal{Q}$: $\mathcal{H} \rightarrow \mathcal{H}$ be a map and α: $\mathcal{H} \times \mathcal{H} \rightarrow [0, +\infty)$ be a function. We say that $\mathcal{H}$ is α-orbital admissible if

$$\alpha(\mathbf{\Omega}, \mathcal{Q}\Omega) \geq 1 \Rightarrow \alpha(\mathcal{Q}\mathbf{\Omega}, \mathcal{Q}^2\Omega) \geq 1.$$

Moreover, $\mathcal{H}$ is called triangular α-orbital admissible if

$$\alpha (\mathbf{\Omega}, \mho) \geq 1 \text{ and } \alpha (\mho, \mathcal{Q}\mho) \geq 1 \Rightarrow \alpha (\mathbf{\Omega}, \mathcal{Q}\mho) \geq 1.$$

And $\mathcal{Q}$ fulfills Definition 1.6.

## 2 Simulation functions

In 2015, Khojasteh, *et al.* [3] introduced the simulation function and used an equivalent to sum up the Banach contraction rule. From there on, Roldan, *et al.* [10] and Argoubi, *et al.* [11] modified the concept of the simulation function and demonstrated some common fixed-point theorems utilizing a new larger class of simulation functions.

**Definition 2.1.** [3] The mapping Λ: $[0, +\infty) \times [0, +\infty) \rightarrow \mathcal{R}$ is known to be a simulation function if the following properties hold:

$(\Lambda_1)$ $\Lambda(0, 0) = 0$;

$(\Lambda_2)$ $\Lambda(e, f) < e-f$, for all $e, f > 0$;

$(\Lambda_3)$ If $\{e_n\}$,$\{f_n\}$ are sequences in $(0, \infty)$ such that

$e_n = f_n = \ell \in (0, \infty)$, then

$sup\Lambda(e_{n,}f_n) < 0$.

Argoubi, *et al.* [11] observed that condition $(\Lambda_1)$ can be relaxed and results can be proved without taking $(\Lambda_1)$ into consideration.

**Definition 2.2** [11] The mapping Λ: $[0, +\infty) \times [0, +\infty) \rightarrow \mathcal{R}$ is known to be a simulation function if it fulfills $(\Lambda_2)$ and $(\Lambda_3)$.

In 2015, Roldan, *et al.* [10] observed that the third condition (namely $(\Lambda_3)$) is symmetric in both arguments of Λ but in proofs this property is not necessary. In fact, in practice the arguments of Λ have different meanings and they play different roles. Then, Roldan, *et al.* slightly modified condition $(\Lambda_3)$ as follows:

($\Lambda_3$) If $\{e_n\}$, $\{f_n\}$ are sequences in (0, ∞) such that
$e_n = f_n = \ell \in (0, \infty)$ and $e_n < f_n$, for each n ∈ $Z_+$, then
$sup\Lambda(e_{n,}f_n)<0$.

The family of all simulation functions Λ: [0, +∞) × [0, +∞) → $\mathcal{R}$ in the Argoubi sense is denoted by $\mathcal{Z}$.

Next, we present some examples of simulation functions.

**Example 2.3** ([3,7,10]) Let $\Lambda_i$: [0, +∞) × [0, +∞) → $\mathcal{R}$, where *i* = 1, 2, 3, 4, 5 is defined as follows:

1. $\Lambda_1$(a, b) = λb−a, ∀ a, b ∈ [0, +∞), where λ ∈ [0, 1).
2. $\Lambda_2$(a, b) = $\frac{b}{b+1}$− a, ∀ a, b ∈ [0, +∞).
3. $\Lambda_3$(a, b) = ψ(b)−$\phi_1$(a)∀ a, b ∈ [0, +∞), where $\phi_1$, ψ: [0, +∞)→[0, +∞) are two continuous functions such that ψ (a) = $\phi_1$(a) = 0 if and only if a = 0 and ψ (a) < a ≤ $\phi_1$(a), ∀ a > 0.
4. $\Lambda_4$(a, b) = b−η(b)−a, ∀ a, b ∈ [0, +∞), where η: [0, +∞) → [0, +∞) is a lower semi continuous function such that η(a) = 0 if and only if a = 0.
5. $\Lambda_5$(a, b) = b−$\int_0^a \varphi(u)du$,∀a, b ∈ [0, +∞), where $\varphi$: [0, +∞) → [0, +∞) is a function such that $\int_0^\varepsilon \varphi(a)da$ exists and $\int_0^\varepsilon \varphi(a)da > \varepsilon$, for each $\varepsilon > 0$.

The authors in [3] utilized the class of auxiliary functions to define $\mathcal{Z}$-contraction as follows:

**Definition 2.4** [3] Let $\mathcal{Q}$ be a self-map in ($\mathcal{H}$, ϖ) and $\Lambda \in \mathcal{Z}$. Then, $\mathcal{Q}$ is a $\mathcal{Z}$-contraction with regard to $\Lambda$, if $\Lambda$(ϖ ($\mathcal{Q}$**Ω**, $\mathcal{Q}$Ʊ), ϖ (**Ω**, Ʊ)) ≥ 0, for every **Ω**, Ʊ ∈ $\mathcal{H}$.

**Theorem 2.5** [3] Let ($\mathcal{H}$, ϖ) be a complete metric space and g: $\mathcal{H} \to \mathcal{H}$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\Lambda$. Then, g has a unique fixed point in $\mathcal{H}$. In 2016, Karapinar [7] introduced α-admissible $\mathcal{Z}$-contraction and established some results as follows:

**Definition 2.6** [7] Let $\mathcal{Q}$: $\mathcal{H} \to \mathcal{H}$ be a given map in ($\mathcal{H}$, ϖ). If there exists $\Lambda \in \mathcal{Z}$, α: $\mathcal{H} \times \mathcal{H} \to$ [0, ∞) such that $\Lambda(\alpha(\Omega,$Ʊ$)$ϖ ($\mathcal{Q}$**Ω**, $\mathcal{Q}$Ʊ), ϖ (**Ω**, Ʊ)) ≥ 0, for every **Ω**, Ʊ ∈ $\mathcal{H}$. Then, $\mathcal{Q}$ is called α-admissible $\mathcal{Z}$-contraction with regard to $\Lambda$.

**Theorem 2.7** [7] Let S be α-admissible $\mathcal{Z}$-contraction with regard to $\Lambda$ in a complete ($\mathcal{H}$, ϖ) satisfying the following conditions:

(i) There exists $x_0 \in \mathcal{H}$ such that $\alpha(x_0, Sx_0) \geq 1$;
(ii) S is triangular α-orbital admissible;
(iii) S is continuous.

Then, there exists $\boldsymbol{\Omega} \in \mathcal{H}$, such that $S\Omega = \Omega$.

**Theorem 2.8** [8] Let S beα-admissible $\mathcal{Z}$-contraction with regard to $\Lambda$ in a complete quasi-metric space and the accompanying conditions are fulfilled:

(i) There occur $x_0 \in \mathcal{H}$ such that $\alpha\,(x_0, Sx_0) \geq 1$ and $\alpha\,(Sx_0, x_0) \geq 1$;
(ii) S is triangular α-orbital admissible;
(iii) S is continuous.

Then, there exist $\boldsymbol{\Omega} \in \mathcal{H}$, such that $S\Omega = \Omega$.

Let $\mathcal{Q}$, $\mathcal{W}$: $\mathcal{H} \rightarrow \mathcal{H}$ be two maps. We identify the set of coincidence and common fixed points of $\mathcal{Q}$ and by ($\mathcal{Q}$, $\mathcal{W}$) and ($\mathcal{Q}$, $\mathcal{W}$), where

($\mathcal{Q}$, $\mathcal{W}$) = {z $\in\mathcal{H}$: $\mathcal{Q}$z = $\mathcal{W}$z} and $\mathcal{C}$ ($\mathcal{Q}$, $\mathcal{W}$) = {z $\in \mathcal{H}$: $\mathcal{Q}$z = $\mathcal{W}$z = z}.

## 3 Main Results

In this section, we present the idea of modified α-admissible $\mathcal{Z}$-contraction with the assistance of a simulation function and utilize this idea to set up outcomes of $\mathcal{C}$ ($\mathcal{Q}$, $\mathcal{W}$) and $\mathcal{CF}$ ($\mathcal{Q}$, $\mathcal{W}$) in ($\mathcal{H}$, ϖ). We likewise give a precedent that yields the principle result. In the displayed work, we broaden the consequences of Karapinar [7]. Additionally, several existing outcomes in the case of metric spaces are constructed. We likewise apply our fundamental theorem to determine coincidence and common fixed-point results for α-admissible $\mathcal{Z}$-contraction. We prove our results by defining modified $\mathcal{Z}$-contraction with respect to $\Lambda$, which is a generalization of the approach of $\mathcal{Z}$-contraction.

**Definition 3.1** Let S, T: $\mathcal{H} \rightarrow \mathcal{H}$ be given maps in ($\mathcal{H}$, ϖ) such that S($\mathcal{H}$)⊆ T($\mathcal{H}$). If there exist$\Lambda \in \mathcal{Z}$, $\psi \in \Psi$, $\alpha$: $\mathcal{H} \times \mathcal{H} \rightarrow [0, \infty)$ such that for each x, y $\in \mathcal{H}$,

$$\Lambda(\alpha\,(Tx, Ty)\,\varpi\,(Sx, Sy), \psi\,(M\,(Tx, Ty)) \geq 0, \tag{1}$$

where $M(Tx,Ty) = max\ \left\{\varpi(Tx,Ty), \frac{\varpi(Tx,Sx)+\varpi(Ty,Sy)}{2}, \ \frac{\varpi(Tx,Sy)\ +\varpi(Ty,Sx)}{2}\right\}$.

Then, S is called a modified α-admissible $\mathcal{Z}$-contraction with regard to T. Further, if T is an identity map, then S converts into α-admissible $\mathcal{Z}$-contraction with regard to $\Lambda$.

**Theorem 3.2** Let S, T: $\mathcal{H} \to \mathcal{H}$ be given functions in a complete metric space $(\mathcal{H}, \varpi)$, such that $S(\mathcal{H}) \subseteq T(\mathcal{H})$ and $T(\mathcal{H})$ is closed. Let S be a modified α-admissible $\mathcal{Z}$-contraction with regard to T that fulfills the following conditions:

(i) S is α-admissible with regard to T;
(ii) there exist $x_0 \in \mathcal{H}$ such that $\alpha(Tx_0, Sx_0) \geq 1$;
(iii) S is triangular α-orbital admissible;
(iv) f $\{Tx_n\}$ is an arrangement in $\mathcal{H}$ such that $\alpha(Tx_n, Tx_{n+1}) \geq 1$ and $Tx_n \to Tu \in T(\mathcal{H})$ as n tends to ∞, then there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\alpha(Tx_{n(k)}, Tu) \geq 1$, for all k.

Then S and T possess a coincidence point.

***Proof***. In view of $S(\mathcal{H}) \subseteq T(\mathcal{H})$, we can select a point $x_1 \in \mathcal{H}$ such that $Sx_0 = Tx_1$. Similarly, we can choose $x_{n+1} \in \mathcal{H}$ such that

$$Sx_n = Tx_{n+1}. \tag{2}$$

Since S is α-admissible with respect to T and using (ii) we get $\alpha(Tx_0, Sx_0) = \alpha(Tx_0, Tx_1) \geq 1$, which implies that $\alpha(Sx_0, Sx_1) = \alpha(Tx_1, Tx_2) \geq 1$.

Using induction, we get

$$\alpha(Tx_n, Tx_{n+1}) \geq 1, \forall n = 0, 1, 2, \ldots \tag{3}$$

If $Sx_{n+1} = Sx_n$ for some n, then by Eq. (2) we obtain $Sx_{n+1} = Tx_{n+1}$.

Thus, $S$ and $T$ have a coincidence point at $x = x_{n+1}$ and we have completed the proof. Further, we assume that $\varpi(Sx_n, Sx_{n+1}) > 0$. Putting $x = x_n$ and $y = x_{n+1}$ in Eq. (1), we get

$$0 \leq \Lambda(\alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1}), \psi(M(Tx_n, Tx_{n+1})),$$

where

$$M(Tx_n, Tx_{n+1}) = max\left\{\varpi(Tx_n, Tx_{n+1}), \frac{\varpi(Tx_n, Sx_n) + \varpi(Tx_{n+1}, Sx_{n+1})}{2}, \frac{\varpi(Tx_n, Sx_{n+1}) + \varpi(Tx_{n+1}, Sx_n)}{2}\right\}.$$

Using Eq. (2), we have

$$M(Tx_n, Tx_{n+1}) = max\left\{\varpi(Sx_{n-1}, Sx_n), \frac{\varpi(Sx_{n-1}, Sx_n) + \varpi(Sx_n, Sx_{n+1})}{2}, \frac{\varpi(Sx_{n-1}, Sx_{n+1}) + \varpi(Sx_n, Sx_n)}{2}\right\}.$$

But,

$$\frac{\varpi(Sx_{n-1}, Sx_{n+1})}{2} \leq \ max\ \{\varpi(Sx_{n-1}, Sx_n), \varpi(Sx_n, Sx_{n+1})\}.$$

Therefore,

$$M(Tx_n, Tx_{n+1}) \leq \ max\ \{\varpi(Sx_{n-1}, Sx_n), \varpi(Sx_n, Sx_{n+1})\}. \tag{4}$$

Using Eq. (1), we get

$$0 \leq \Lambda(\alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1}), \psi(M(Tx_n, Tx_{n+1}))$$
$$< \psi(M(Tx_n, Tx_{n+1})) - \alpha(Tx_n, Tx_{n+1})\overline{\omega}(Sx_n, Sx_{n+1}),$$

which implies that,

$$\alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1}) < \psi(M(Tx_n, Tx_{n+1})).$$

Indeed,

$$\varpi(Sx_n, Sx_{n+1}) \leq \alpha(Tx_n, Tx_{n+1})\overline{\omega}(Sx_n, Sx_{n+1}). \tag{5}$$

By combining the above two inequalities, we get

$$\varpi(Sx_n, Sx_{n+1}) < \psi(M(Tx_n, Tx_{n+1})).$$

Using Eq. (4), we obtain

$$\varpi(Sx_n, Sx_{n+1}) \leq \psi(max\ \varpi Sxn{-}1{,}Sxn,\ \varpi Sxn{,}Sxn{+}1). \tag{6}$$

If

$$max\ \{\varpi(Sx_{n-1}, Sx_n), \varpi(Sx_n, Sx_{n+1})\} = \ \varpi(Sx_n, Sx_{n+1}),\text{ then}$$
$$\varpi(Sx_{n-1}, Sx_n) \leq \varpi(Sx_n, Sx_{n+1}).$$

Using Eq. (6), we get

$$\varpi(Sx_n, Sx_{n+1}) < \psi(\varpi(Sx_n, Sx_{n+1})).$$

Using the properties of comparison functions,

$$\psi(\varpi(Sx_n, Sx_{n+1})) < \varpi(Sx_n, Sx_{n+1}),$$

we get,

$$\varpi(Sx_n, Sx_{n+1}) < \varpi(Sx_n, Sx_{n+1}),$$

which is a counterstatement. Consequently,

$$max\{\varpi(Sx_{n-1}, Sx_n), \varpi(Sx_n, Sx_{n+1})\} = \ \varpi(Sx_{n-1}, Sx_n).$$

Thus,

$$\varpi(Sx_n, Sx_{n+1}) < \psi(\varpi(Sx_{n-1}, Sx_n)).$$

Again, using the properties of comparison functions,

$\psi(\varpi(Sx_{n-1}, Sx_n)) < \varpi(Sx_{n-1}, Sx_n)$,

we have,

$\varpi(Sx_n, Sx_{n+1}) < \varpi(Sx_{n-1}, Sx_n)$.

Hence, we conclude that the sequence $\{\varpi(Sx_{n-1}, Sx_n)\}$ is non-decreasing and bounded below. Accordingly, there exists $r \geq 0$, such that

$\varpi(Sx_{n-1}, Sx_n) = r \geq 0$.

We state that $r = 0$. Let us assume that $r > 0$. Therefore,

$\alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1}) = r$.

Letting $s_n = \alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1})$ and $t_n = \varpi(Sx_{n-1}, Sx_n)$ and taking $\Lambda_3$ into account, we get that

$0 \leq sup\Lambda(\alpha(Tx_n, Tx_{n+1})\varpi(Sx_n, Sx_{n+1}), \varpi(Sx_{n-1}, Sx_n)) < 0$,

which is a counterstatement.

Thus, we have

$$\varpi(Sx_{n-1}, Sx_n) = 0. \tag{7}$$

Now, we assert that $\{Sx_n\}$ is Cauchy. Let us imagine that $\varepsilon > 0$, for each $n \in Z_+$ and $n, m \in Z_+$ with $n > m > Z_+$ such that $\varpi(x_m, x_n) > \varepsilon$.

From Eq. (7), there exists $n_0 \in Z_+$, in order that

$$\varpi(Sx_n, Sx_{n+1}) < \varepsilon, \forall n > n_0. \tag{8}$$

Consider $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$n_0 \leq n_k < m_k < m_{k+1} \text{ and } \varpi(Sx_{m_k}, Sx_{n_k}) > \varepsilon, \forall k. \tag{9}$$

Also,

$$\varpi(Sx_{m_{k-1}}, Sx_{n_k}) \leq \varepsilon, \forall k \tag{10}$$

where $m_k$ is picked as the smallest number $m \in \{n_k, n_{k+1}, n_{k+2},...\}$ such that Eq. (9) is satisfied. Also, $n_k + 1 \leq$ for every k. But, $n_k + 1 \leq m_k$ is infeasible due to Eq. (8) and Eq. (9). Therefore, $n_k + 2 \leq m_k$, for each $k$. This yields that $n_k+1 < m_k < m_k+1$, for all $k$. On account of triangle inequality, Eq. (9) and Eq. (10), we get

$$\begin{aligned}\varepsilon &< \varpi(Sx_{m_k}, x_{n_k}) \\ &\leq \varpi(Sx_{m_k}, Sx_{m_{k-1}}) + \varpi(Sx_{m_{k-1}}, x_{n_k}) \\ &\leq \varpi(Sx_{m_k}, Sx_{m_{k-1}}) + \varepsilon.\end{aligned}$$

Using Eq. (7), we deduce that

$$\varpi(Sx_{m_k}, Sx_{n_k}) = \varepsilon. \tag{11}$$

By using the triangle inequality theorem, we obtain

$$\varpi(Sx_{m_k}, Sx_{n_k}) \leq \varpi(Sx_{m_k}, Sx_{m_{k+1}}) + \varpi(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + \varpi(Sx_{n_{k+1}}, Sx_{n_k}).$$

Also, we have

$$\varpi(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \varpi(Sx_{m_{k+1}}, Sx_{m_k}) + \varpi(Sx_{m_k}, Sx_{n_k}) + \varpi(Sx_{n_k}, Sx_{n_{k+1}}).$$

With the aid of Eq. (7), we find that

$$\varpi(Sx_{m_{k+1}}, Sx_{n_{k+1}}) = \varepsilon. \tag{12}$$

Specifically, there exist $n_1 \in Z_+$ such that for all $k \geq n_1$, we get

$$\varpi(Sx_{m_k}, Sx_{n_k}) > \frac{\varepsilon}{2} > 0 \text{ and } \varpi(Sx_{m_{k+1}}, Sx_{n_{k+1}}) > \frac{\varepsilon}{2} > 0. \tag{13}$$

Moreover, since S fulfills (iii) of Theorem 3.2, we get

$$\alpha(Tx_{m_k}, Tx_{n_k}) \geq 1.$$

Since, T is a modified α-admissible $\mathcal{Z}$-contraction with regard to T, we get

$$\begin{aligned}
0 &\leq \Lambda(\alpha(Tx_{m_k}, Tx_{n_k})\varpi(Sx_{m_k}, Sx_{n_k}), \psi(M(Tx_{m_k}, Tx_{n_k})) \\
&= \Lambda(\alpha(Tx_{m_k}, Tx_{n_k})\varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}), \psi(M(Tx_{m_k}, Tx_{n_k})) \\
&< \psi(M(Tx_{m_k}, Tx_{n_k}) - \alpha(Tx_{m_k}, Tx_{n_k})\varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}) \\
&< \psi(M(Tx_{m_k}, Tx_{n_k})).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
0 &< \varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}) \\
&< \alpha(Tx_{m_k}, Tx_{n_k})\varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}) \\
&< \psi(M(Tx_{m_k}, Tx_{n_k})) \\
&< \varpi(Tx_{m_k}, Tx_{n_k}).
\end{aligned}$$

From the above inequality, together with Eq. (11) and Eq. (12), we conclude that

$$s_n = \alpha(Tx_{m_k}, Tx_{n_k})\varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}) \to \varepsilon \text{ and}$$

$$t_n = \varpi(Tx_{m_k}, Tx_{n_k}) \to \varepsilon.$$

With the aid of $(\Lambda_3)$, we get

$$sup\Lambda(\alpha(Tx_{m_k}, Tx_{n_k})\varpi(Tx_{m_{k+1}}, Tx_{n_{k+1}}), \varpi(Tx_{m_k}, Tx_{n_k})) < 0,$$

which is a counterstatement.

Hence, $\{Tx_n\}$ is a Cauchy sequence. Since T($\mathcal{H}$) is closed, there exist $u \in \mathcal{H}$ such that

$$\lim_{n\to\infty} Tx_n = Tu.$$

Now, we show that S and T possess coincidence point $u \in \mathcal{H}$.

Oppositely, we assume that $\varpi(Su, Tu) > 0$.

Now, we have

$$0 \leq \Lambda(\alpha(Tx_{n_k}, Tu)\varpi(Sx_{n_k}, Su), \psi(M(Tx_{n_k}, Tu))$$

$$< \psi(M(Tx_{n_k}, Tu)) - \alpha(Tx_{n_k}, Tu)\varpi(Sx_{n_k}, Su).$$

Thus,

$$\alpha(Tx_{n_k}, Tu)\varpi(Sx_{n_k}, Su) < \psi(M(Tx_{n_k}, Tu)). \tag{14}$$

Since by condition (iv) of Theorem 3.2 we have

$$\alpha(Tx_{n_k}, Tu) \geq 1.$$

By the use of triangle inequality, we obtain

$$\varpi(Tu, Su) \leq \varpi(Tu, Sx_{n_k}) + \varpi(Sx_{n_k}, Su)$$

$$\leq \varpi(Tu, Sx_{n_k}) + \alpha(Tx_{n_k}, Tu)\varpi(Sx_{n_k}, Su).$$

Using Eq. (14), we get

$$\varpi(Tu, Su) \leq \varpi(Tu, Sx_{n_k}) + \psi(M(Tx_{n_k}, Tu)),$$

where

$$M(Tx_{n_k}, Tu) = max\left\{\varpi(Tx_{n_k}, Tu), \frac{\varpi(Tx_{n_k}, Sx_{n_k}) + \varpi(Tu, Su)}{2}, \frac{\varpi(Tx_{n_k}, Su) + \varpi(Tu, Sx_{n_k})}{2}\right\}.$$

Owing to the above equality, we get

$$\varpi(Tu, Su) \leq \varpi(Tu, Sx_{n_k}) + \psi(M(Tx_{n_k}, Tu))$$

$$\leq \varpi(Tu, Sx_{n_k}) + \psi(max\left\{\varpi(Tx_{n_k}, Tu), \frac{\varpi(Tx_{n_k}, Sx_{n_k}) + \varpi(Tu, Su)}{2}, \frac{\varpi(Tx_{n_k}, Su) + \varpi(Tu, Sx_{n_k})}{2}\right\}).$$

Letting k → ∞ in the above inequality, we get

$$\varpi(Tu, Su) \leq \psi\left(\frac{\varpi(Tu,Su)}{2}\right) \leq \frac{\varpi(Tu,Su)}{2},$$

which is a counterstatement.

Hence, our assumption is faulty and $\varpi(Su, Tu) = 0$, which indicates that $u \in C(\mathrm{T}, \mathrm{S})$.

**Theorem 3.3.** In conjunction with the assumptions of the above theorem, assume that for all $z_1, z_2 \in C(\mathrm{T}, \mathrm{S})$, there exist $z_3 \in \mathcal{H}$ such that $\alpha(\mathrm{T}z_1, \mathrm{T}z_3) \geq 1$ and $\alpha(Tz_2, \mathrm{T}z_3) \geq 1$ and S, T commute at $u \in C(\mathrm{T}, \mathrm{S})$. Then, there exists a unique $u \in \mathcal{H}$ such that $u \in \mathcal{CF}(\mathrm{S}, \mathrm{T})$.

***Proof.*** We give the proof in three steps.

**Step 1**. We want to prove that if $z_1, z_2 \in C(\mathrm{T}, \mathrm{S})$, then $\mathrm{T}z_1 = \mathrm{T}z_2$. By the given assumption, there exist $z \in \mathcal{H}$ such that

$$\alpha(Tz_1, \mathrm{T}z) \geq 1 \text{ and } \alpha(Tz_2, \mathrm{T}z) \geq 1. \tag{15}$$

Also, $\mathrm{S}(\mathcal{H}) \subseteq \mathrm{T}(\mathcal{H})$. Now, we define the sequence $\{z_n\}$ in $\mathcal{H}$ by $\mathrm{T}z_{n+1} = \mathrm{S}z_n$, for all $n \geq 0$ and $z_0 = z$. Since, S is α-admissible with respect to T, we have from Eq. (15) that $\alpha(\mathrm{T}z_1, \mathrm{T}z_n) \geq 1$ and $\alpha(\mathrm{T}z_2, \mathrm{T}z_n) \geq 1$.

Using Eq. (1), we have

$$0 \leq \Lambda(\alpha(\mathrm{T}z_1, \mathrm{T}z_n)\varpi(Sz_1, \mathrm{S}z_n), \psi(\mathrm{M}(Tz_1, \mathrm{T}z_n))$$
$$< \psi(\mathrm{M}(Tz_1, \mathrm{T}z_n)) - \alpha(\mathrm{T}z_1, \mathrm{T}z_n)\varpi(Sz_1, \mathrm{S}z_n), \text{ that is,}$$
$$\alpha(\mathrm{T}z_1, \mathrm{T}z_n)\varpi(Sz_1, \mathrm{S}z_n) < \psi(\mathrm{M}(Tz_1, \mathrm{T}z_n)).$$

Using Eq. (15), we obtain

$$\varpi(Sz_1, \mathrm{S}z_n) \leq \alpha(Tz_1, \mathrm{T}z_n)\varpi(Sz_1, \mathrm{S}z_n). \tag{16}$$

Also,

$$\varpi(Sz_1, \mathrm{S}z_n) = \varpi(\mathrm{T}z_1, \mathrm{T}z_{n+1}).$$

Therefore,

$$\varpi(Tz_1, Tz_{n+1}) \le \psi(M(Tz_1, Tz_n)), \tag{17}$$

where

$$M(Tz_1, Tz_n) = max\left\{\varpi(Tz_1, Tz_n), \frac{\varpi(Tz_1, Sz_1) + \varpi(Tz_n, Sz_n)}{2}, \frac{\varpi(Tz_1, Sz_n) + \varpi(Tz_n, Sz_1)}{2}\right\}.$$

$$\le \max\{\varpi(Tz_1, Tz_n), \varpi(Tz_1, Tz_{n+1})\}.$$

Using Eq. (17), we get

$$\varpi(Tz_1, Tz_{n+1}) \le \psi(\max\{\varpi(Tz_1, Tz_n), \varpi(Tz_1, Tz_{n+1})\}).$$

Let us suppose that $\varpi(Tz_1, Tz_n) > 0$, for each $n \in Z_+$.

Now, if

$$\max\{\varpi(Tz_1, Tz_n), \varpi(Tz_1, Tz_{n+1})\} = \varpi(Tz_1, Tz_{n+1}), \text{ then}$$

$$\varpi(Tz_1, Tz_{n+1}) \le \psi(\varpi(Tz_1, Tz_{n+1})) < \varpi(Tz_1, Tz_{n+1}),$$

which is a counterstatement.

Thus, we have

$$\max\{\varpi(Tz_1, Tz_n), \varpi(Tz_1, Tz_{n+1})\} = \varpi(Tz_1, Tz_n).$$

So, $\varpi(Tz_1, Tz_{n+1}) \le \Psi(\Gamma(Tz_1, Tz_n))$, that is, $\{\varpi(Tz_1, Tz_n)\}$ is a monotonically decreasing sequence in $R_+$. Thus, we can find $\ell \ge 0$ such that

$$\lim_{n\to\infty} \varpi(Tz_1, Tz_n) = \ell.$$

We claim that $\ell = 0$. Suppose that $0 < \ell$.

Using Eq. (15), we get

$$\alpha(Tz_1, Tz_n)\, \varpi(Tz_1, Tz_{n+1}) = \ell.$$

Letting $s_n = \alpha(Tz_1, Tz_n)\varpi(Tz_1, Tz_{n+1})$, $t_n = \varpi(Tz_1, Tz_n)$ and taking $(\Lambda_3)$ into account, we get that

$$\sup\Lambda(\alpha(Tz_1, Tz_n)\varpi(Tz_1, Tz_{n+1}), \varpi(Tz_1, Tz_n)) < 0,$$

which is a counterstatement.

Thus, we have

$$\lim_{n\to\infty} \varpi(Tz_1, Tz_n) = \ell = 0.$$

**Step 2**. With the same approach we can prove that $\lim_{n\to\infty}\varpi(Tz_2, Tz_n) = 0$. Therefore, $Tz_1 = Tz_2$.

**Step 3**. Now, we demonstrate the presence of a common fixed point.

Let $z_1 \in$ **(T, S),** that is, $Tz_1$= $Sz_1$. Due to the commutativity of S and T at their coincidence points, we get

$$T^2 z_1 = TTz_1 = TSz_1 = STz_1. \tag{18}$$

Let us suppose that $Tz_1 = u$. From (18), we get $Tu = Su$. Thus, $u \in$ (T, S). From the given assumption, $Tz_1 = Tu = u = Su$. Then, $u \in (T, S)$. Now, we show that the fixed point is unique. Imagine that S and T possess another common fixed point,$z_3$. Then, $z_3 \in (T, S)$. From the given assumption, we have $z_3 = Tz_3 = Tu = u$.

**Corollary 3.4.** Let S: $\mathcal{H} \to \mathcal{H}$be a given map in a complete ($\mathcal{H}$, ϖ). If there exist $\Lambda \in \mathcal{Z}$ and α: $\mathcal{H} \times \mathcal{H} \to [0, \infty)$, such that for each **Ω**, Ʊ ∈ $\mathcal{H}$,

$$\Lambda(\alpha\,(\boldsymbol{\Omega}, \mho)\,\varpi\,(S\boldsymbol{\Omega}, S\mho), \varpi\,(\boldsymbol{\Omega}, \mho) \geq 0$$

and fulfilling the accompanying conditions:

(i) there exist $\Omega_0 \in \mathcal{H}$such that $\alpha(\Omega_0, S\Omega_0) \geq 1$;
(ii) S is triangular α-orbital admissible;
(iii) if $\{\Omega_n\}$ is an arrangement in $\mathcal{H}$ such that $\alpha(\Omega_n, \Omega_{n+1}) \geq 1$ and $\Omega_n \to \Theta_1 \in \mathcal{H}$ as n tends to ∞, then there exists a subsequence $\{\Omega_{n(k)}\}$ of $\{\Omega_n\}$, such that$\alpha(\Omega_{n(k)}, \Theta_1) \geq 1$, for every $k$.

Then, S possesses a fixed point.

***Proof.*** The result proceeds from main Theorem 3.2.

**Corollary 3.5.** Let S: $\mathcal{H} \to \mathcal{H}$ be a given function in complete ($\mathcal{H}$, ϖ). If there exist $\Lambda \in \mathcal{Z}$ and α: $\mathcal{H} \times \mathcal{H} \to [0, \infty)$ such that

$$\Lambda(\varpi\,(S\boldsymbol{\Omega}, S\mho), M\,(S\boldsymbol{\Omega}, S\mho)) \geq 0, \tag{19}$$

where

$$M(S\Omega, S\mho) = max\left\{\varpi(\Omega, \mho), \frac{\varpi(\Omega,S\Omega)+\varpi(\mho,S\mho)}{2}, \frac{\varpi(\Omega,S\mho)+\varpi(\mho,S\Omega)}{2}\right\},$$

for every **Ω**, Ʊ ∈ $\mathcal{H}$ and fulfilling the accompanying conditions:

(i) there exist $\Omega_0 \in \mathcal{H}$such that $\alpha(\Omega_0, S\Omega_0) \geq 1$;
(ii) S is triangular α-orbital admissible;

(iii) if $\{\Omega_n\}$ is an arrangement in $\mathcal{H}$ such that $\alpha(\Omega_n, \Omega_{n+1}) \geq 1$ and $\Omega_n \to \Theta_1 \in \mathcal{H}$ as $n$ tends to $\infty$, then there exists a subsequence $\{\Omega_{n(k)}\}$ of $\{\Omega_n\}$, such that $\alpha(\Omega_{n(k)}, \Theta_1) \geq 1$ for every k.

Then, S possesses a fixed point.

***Proof***. The result proceeds from main Theorem 3.2 by taking T as the identity function.

**Example 3.6.** Consider $\mathcal{H} = [0, \infty)$ associated with the metric

$$\varpi(\Omega, \mho) = \begin{cases} 0, & if\ \Omega = \mho, \\ \max\{\Omega, \mho\}, & otherwise, \end{cases}$$

for all $\Omega, \mho \in \mathcal{H}$. Define the self-mappings S and T by $S(\ell) = \ell$ and $T(\ell) = 2\ell$ for each $\ell \in \mathcal{H}$, with $\psi(t) = \frac{t}{2}$. Let $\Lambda: \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be defined as

$$\Lambda(\wp, \sigma) = \wp - \frac{\sigma+2}{\sigma+1}\sigma.$$

Now, we formalize the mapping $\alpha: \mathcal{H} \times \mathcal{H} \to [0, \infty)$ as

$$\alpha(\Theta_1, \Theta_2) = \begin{cases} 1, & if (\Theta_1, \Theta_2) \in [0, 1], \\ 0, & otherwise. \end{cases}$$

If $M(T\mathbf{\Omega}, T\mho) = \varpi(T\mathbf{\Omega}, T\mho)$. Thus,

$$\Lambda(\alpha(T\mathbf{\Omega}, T\mho)\varpi(S\Omega, S\mho),$$

$$\begin{aligned} \psi(M(T\Omega, T\mho)) &= \Lambda(\alpha(T\mathbf{\Omega}, T\mho)\varpi(S\Omega, S\mho), \psi(\varpi(T\mathbf{\Omega}, T\mho)) \\ &= \Lambda(\mho, 2\mho) \\ &= 2\mho - \frac{\mho+2}{\mho+1}\frac{\mho}{2} \\ &= \frac{4\mho(\mho+1) - \mho(\mho+2)}{2(\mho+1)} \\ &= \frac{4\mho^2 + 4\mho - \mho^2 - 2\mho}{2(\mho+1)} \\ &= \frac{3\mho^2 + 2\mho}{2(\mho+1)} \geq 0. \end{aligned}$$

If $M(T\mathbf{\Omega}, T\mho) = \frac{\varpi(T\Omega, S\Omega) + \varpi(T\mho, S\mho)}{2}\ or\ \frac{\varpi(T\Omega, S\mho) + \varpi(T\mho, S\Omega)}{2}$.

Thus, we have

$$\Lambda(\alpha(T\mathbf{\Omega}, T\mho)\varpi(S\Omega, S\mho),$$

$$\psi(M(T\Omega, T\mho)) = \Lambda(\mho, \Omega + \mho) = \Omega + \mho - \frac{\mho+2}{\mho+1}\frac{\mho}{2} = \frac{(2\Omega+2\mho)(\mho+1) - \mho(\mho+2)}{2(\mho+1)}$$

$$= \frac{2(\Omega\mho+\Omega)+\mho^2}{2(\mho+1)} \geq 0.$$

Clearly, (S, T) is a modified generalized contractive pair of mappings and $\psi(t) = \frac{t}{2}$. Now, all the assumptions of Theorem 3.2 and Theorem 3.3 are satisfied. Therefore, S and T have a coincidence point. Also, $0 \in \mathcal{C}$ (S, T) and $0 \in \mathcal{CF}$ (S, T). Hence, 0 is a unique common fixed point of S and T.

## 4 Conclusion

In this work, we investigated the existence of a coincidence point of modified α-admissible $\mathcal{Z}$-contraction. The proposed work contributes to the formulation of a unique common fixed point with the help of the commutative property of two maps. The presented theorems enhance various results present in the literature.

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