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On a Certain Subclass of Meromorphic Functions Defined by a New Linear Differential Operator

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Abstract. In this article, a new linear differential operator $I^k(L_s^a(a_l, b_m)f(z))$ is defined by using the Hadamard product of the q-hypergeometric function and a function related to the Hurwitz-Lerch zeta function. By using this linear differential operator, a new subclass $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$ of meromorphic functions is defined. Some properties and characteristics of this subclass are considered. These include the coefficient inequalities, the growth and distortion properties and the radii of meromorphic starlikeness and meromorphic convexity. Finally, closure theorems and extreme points are introduced.

Keywords: *differential operator; gamma function; Hadamard product; Hurwitz-Lerch zeta function; meromorphic functions; q-hypergeometric function; subordination property.*

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1 Introduction

Let Σ be the class $f(z)$ normalized by:

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk

$$U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\sum S^*(\varphi)$ and $\kappa(\varphi)$ $(\varphi \geq 0)$ the subclasses of Σ consisting of all meromorphic functions, which are, respectively, starlike of order φ and convex of order φ in U^* (see also the recent works [1,2]).

A function f of the form Eq. (1) is in the class of meromorphic starlike of order φ if:

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$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \varphi \qquad (z \in \mathrm{U}^*),$$

and is in the class of meromorphic convex of order φ, if:

$$\Re\left\{-\ (1+\frac{zf''(z)}{f'(z)})\right\} > \varphi \quad (z \in\ \mathrm{U}^*).$$

For functions $f_j(z)$ (j=1,2) defined by:

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{\ n,j} z^n \quad (\ j = 1,2),$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by:

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{\ n,1}\ a_{\ n,2}\ z^n.$$

Let f and F two analytic functions in the unit disk U. We say that f is subordinate to F if there exists an analytic function $\omega(z)$ with $\omega(0) < 1$ $(z \in U)$ such that $f = \mathrm{F}(\omega(z))$.

We denote by $f \prec F$ this subordination.

For real parameters $\alpha_i, \beta_j, i = 1,2, \dots, l, j = 1,2, \dots, m, \alpha_1 \in \mathbb{R}, \beta_j \in \mathbb{R}\{0, -1, -2, \dots\}$) the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by:

$$\begin{aligned} &{}_l\Psi_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;q,z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1,q)_n \dots (\alpha_l,q)_n}{(q,q)_n(\beta_1,q)_n \dots (\beta_m,q)_n} \times \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n \end{aligned} \tag{2}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$ where $q \neq 0$ when $l > m+1$, $(l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U)$.

The q-shifted factorial is defined for $\alpha, q \in \mathbb{R}$ as a product of n factors by:

$$(\alpha; q)_n = \begin{cases} (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}) & (n \in N) \\ 1 & (n = 0) \end{cases} \tag{3}$$

and in terms of basic analogue of the gamma function

$$(q^\alpha, q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad n > 0. \tag{4}$$

It is of interest to note that $\lim_{q\to -1}\left(\frac{(q^{\alpha},q)_n}{(1-q)^n}\right)=(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1)$ is the familiar Pochhammer symbol and that:

$$ {}_l\Psi_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z):=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_l)_n}{(\beta_1)_n\ldots(\beta_m)_n}\frac{z^n}{n!}. \tag{5}$$

Now, for $z\in U$, $0<|q|<1$ and $l>m+1$, the basic hypergeometric function defined in Eq. (2) takes the following form:

$$ {}_l\Psi_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;q,z):=\sum_{n=0}^{\infty}\frac{(\alpha_1,q)_n\ldots(\alpha_l,q)_n}{(q,q)_n(\beta_1,q)_n\ldots(\beta_m,q)_n}z^n, \tag{6}$$

which converges absolutely in the open unit disk U.

Corresponding to the function ${}_l\Psi_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;q,z)$ for meromorphic functions $f\in\Sigma$ consisting functions of the form Eq. (1), Aldweby and Darus [3] and Murugusundaramoorthy and Janani [4] have recently introduced the q analogue of the Liu-Srivastava operator as follows:

$$ {}_l\Omega_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;q,z)\ f(z)={}_lY_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;q,z)*f(z)$$

$$=\frac{1}{z}\,{}_l\Psi_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m\ ;q,z)*f(z)$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{l}(\alpha_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(\beta_i,q)_{n+1}}a_n z^n. \tag{7}$$

Recently, Ghanim [5,6] has introduced the function $G_{s,a}$, defined by:

$$G_{s,a}:=(a+1)^s\left[\Phi\ (z,s,a)-a^s+\frac{1}{z(a+1)^s}\right]$$

$$G_{s,a}=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{a+1}{a+n}\right)^s z^n\ (z\in \mathrm{U}^*). \tag{8}$$

Also, the function $\Phi(z,s,a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, e.g. [[7], p. 121 *et seq.*]; see also [[8-11], p. 194 *et seq.*]):

$$\Phi\ (z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\text{ when }|z|<1;\Re(s)>1\text{ when }|z|=1).$$

Corresponding to the functions $G_{s,a}(z)$ and using the Hadamard product for $f(z) \in \Sigma$ we define a new linear differential operator $L_s^a(\alpha_l, \beta_m)$ on Σ by the following series:

$$L_s^a(\alpha_l, \beta_m) f(z) = {}_l\Omega_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; q, z) f(z) * G_{s,a}(z)$$

$$L_s^a(\alpha_l, \beta_m) f(z)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l} (\alpha_i, q)_{n+1}}{(q,q)_{n+1} \prod_{i=1}^{m} (\beta_i, q)_{n+1}} \left(\frac{a+1}{a+n} \right)^s a_n z^n \quad (z \in \mathrm{U}^*). \tag{9}$$

$$L_s^a(\alpha_l, \beta_m) f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \Gamma_{(n+1,a,s)}(\alpha_l, \beta_m) a_n z^n, \tag{10}$$

where, for convenience:

$$\Gamma_{(n+1,a,s)}(\alpha_l, \beta_m) = \frac{\prod_{i=1}^{l} (\alpha_i, q)_{n+1}}{(q,q)_{n+1} \prod_{i=1}^{m} (\beta_i, q)_{n+1}} \left(\frac{a+1}{a+n} \right)^s \tag{11}$$

The meromorphic functions with the generalized hypergeometric functions have been considered recently by several authors; see, for example see [12-18].

For a function $f \in \Sigma$, we define:

$$I^0(L_s^a(\alpha_l, \beta_m) f(z)) = L_s^a(\alpha_l, \beta_m) f(z),$$

$$I^1(L_s^a(\alpha_l, \beta_m) f(z)) = (1-\gamma)(L_s^a(\alpha_l, \beta_m) f(z)) + \gamma z (L_s^a(\alpha_l, \beta_m) f(z))' + \frac{2}{z},$$

$$I^2(L_s^a(\alpha_l, \beta_m) f(z)) = (1-\gamma)(I^1(L_s^a(\alpha_l, \beta_m) f(z)))$$

$$+ \gamma z (I^1(L_s^a(\alpha_l, \beta_m) f(z)))' + \frac{2}{z},$$

and in general:

$$I^k(L_s^a(\alpha_l, \beta_m) f(z))$$

$$= (1-\gamma)(I^{k-1}(L_s^a(\alpha_l, \beta_m) f(z))) + \gamma z (I^{k-1}(L_s^a(\alpha_l, \beta_m) f(z)))' + \frac{2}{z},$$

$$I^k(L_s^a(\alpha_l,\beta_m)f(z))$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k\frac{\prod_{i=1}^{l}\left(\alpha_i,q\right)_{n+1}}{\left(q,q\right)_{n+1}\prod_{i=1}^{m}\left(\beta_i,q\right)_{n+1}}\left(\frac{a+1}{a+n}\right)^s a_n z^n$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k\Gamma_{(n+1,a,s)}\left(\alpha_l,\beta_m\right)a_n z^n\left(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\right),$$

where: $\Gamma_{(n+1,a,s)}\left(\alpha_l,\beta_m\right)$ is given by Eq. (11).

We note that I^k was studied by Ghanim and Darus [13,19], Challab and Darus [20] and El-Ashwah, Aouf and El-Deeb [21].

Making use of the operator $I^k\left(L_s^a\left(\alpha_l,\beta_m\right)f\left(z\right)\right)$, we say that a function $f(z)\in\Sigma$ is in the class $L_{s,a}^k(\alpha_l,\beta_m;A,B,b)$ if it satisfies the following inequality:

$$1-\frac{1}{b}\left\{\frac{z(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'}{I^k(L_s^a(\alpha_l,\beta_m)f(z))}+1\right\}\prec\frac{1+Az}{1+Bz} \tag{12}$$

or, equivalently, to:

$$\left|\frac{\dfrac{z(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'}{I^k(L_s^a(\alpha_l,\beta_m)f(z))}+1}{B\dfrac{z(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'}{I^k(L_s^a(\alpha_l,\beta_m)f(z))}+[(A-B)b+B]}\right|<1 \tag{13}$$

$(-1\le B<A\le 1;\alpha_i\in\mathbb{R}\ (i=1,2,...,I)$ and $\beta_j\in\mathbb{R}\setminus\mathbb{Z}_0^-\ (j=1,2,...,m);$
$l,m,k\in\mathbb{N}_0;l>m+1;b\in\mathbb{C};z\in\mathrm{U}^*)$

Let Σ^* denote the subclass of Σ consisting of functions of the form:

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}|a_n|z^n. \tag{14}$$

We now write:

$$L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)=L_{s,a}^k(\alpha_l,\beta_m;A,B,b)\cap\Sigma^*.$$

2 Some Basic Properties of the Class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$

We begin this section by proving the necessary and sufficient condition (involving coefficient bounds) for the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$.

Theorem 1. Let the function $f(z)$ defined by Eq. (14) be in the class Σ^*. Then the function $f(z)$ belongs to the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$ if and only if:

$$\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k\left[(n+1)(1-B)-|b|(A-B)\right]\Gamma_{(n+1,a,s)}|a_n|\le|b|(A-B). \quad (15)$$

Proof. Assuming that the inequality Eq. (15) holds true, then aiming to prove Eq. (13), we find that:

$$\left|\frac{z(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'+I^k(L_s^a(\alpha_l,\beta_m)f(z))}{Bz(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'+\left[b(A-B)+B\right](I^k(L_s^a(\alpha_l,\beta_m)f(z))}\right|$$

$$=\left|\frac{\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k(n+1)\Gamma_{(n+1,s,a)}|a_n|z^{n+1}}{b(A-B)+\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k\left[B(n+1)+b(A-B)\right]\Gamma_{(n+1,s,a)}|a_n|z^{n+1}}\right|$$

$$<1 \quad (z\in \mathrm{U}^*) \quad (16)$$

$z\in\partial\boldsymbol{U}=\{z:z\in\mathbb{C}\wedge|z|=1\}$. Hence, by the maximum modulus theorem, we have $f(z)\in L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$.

Conversely, suppose that $f(z)$ is in the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$ with $f(z)$ of the form Eq. (14), then we find from Eq. (13) that:

$$\left|\frac{z(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'+I^k(L_s^a(\alpha_l,\beta_m)f(z))}{Bz(I^k(L_s^a(\alpha_l,\beta_m)f(z)))'+[b(A-B)+B](I^k(L_s^a(\alpha_l,\beta_m)f(z))}\right|$$

$$=\left|\frac{\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k(n+1)\Gamma_{(n+1,s,a)}|a_n|z^{n+1}}{b(A-B)+\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^k\left[B(n+1)+b(A-B)\right]\Gamma_{(n+1,s,a)}|a_n|z^{n+1}}\right|$$

$$<1. \quad (17)$$

If we choose z to be real must be $z\to -1$, we get:

$$\sum_{n=1}^{\infty}\left[1+(n-1)\gamma\right]^{k}\left[(n+1)(1-B)-|b|(A-B)\right]\Gamma_{n+1,a,s}\left|a_n\right|\leq |b|(A-B) \quad (18)$$

which is precisely the assertion Eq. (15) of Theorem 1.

Corollary 1. Let the function $f(z)$ defined by Eq. (14) be in the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$. Then:

$$|a_n|\leq \frac{|b|(A-B)}{[1+(n-1)\gamma]^k[(n+1)(1-B)-|b|(A-B)]\Gamma_{(n+1,a,s)}} \quad (n\geq 1). \quad (19)$$

The result is sharp for the function $f(z)$ given by:

$$f(z)=z^{-1}$$
$$+\frac{|b|(A-B)}{[1+(n-1)\gamma]^k[(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}z^n \quad (n\geq 1). \quad (20)$$

Next, we prove the following growth and distortion properties for the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$.

Theorem 2. If a function $f(z)$ defined by Eq. (14) is in the class $L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$, then for $|z|=r<1$, we have:

$$\left\{1-\frac{|b|(A-B)}{[2(1-B)-|b|(A-B)]\Gamma_{(2,a,s)}}r^2\right\}r^{-2}\leq |f'(z)|\leq$$
$$\left\{1+\frac{|b|(A-B)}{[2(1-B)\text{-}|b|(A-B)]\Gamma_{(2,a,s)}}r^2\right\}r^{-2}. \quad (21)$$

The result is sharp for the function $f(z)$ given by:

$$f(z)=\frac{1}{z}+\frac{|b|(A-B)}{[2(1-B)\text{-}|b|(A-B)]\Gamma_{(2,a,s)}}z. \quad (22)$$

Proof. Let $f(z)\in L_{s,a}^{k,*}(\alpha_l,\beta_m;A,B,b)$. Then we find from Theorem 1 that:

$$[2(1-B)\text{-}|b|(A-B)]\Gamma_{(2,a,s)}\sum_{n=1}^{\infty}\frac{n!}{(n-1)!}|a_n|$$

$$\leq \sum_{n=1}^{\infty} \left[(n+1)(1-B) - |b|(A-B) \right] \left[1+(n-1)\gamma \right]^k \Gamma_{(n+1,a,s)} . |a_n| \leq |b|(A-B),$$

which yields:

$$\sum_{n=1}^{\infty} \frac{n!}{(n-1)!} |a_n| \leq \frac{|b|(A-B)}{[2(1-B)\text{-}|b|(A-B)]\Gamma_{(2,a,s)}}. \tag{23}$$

Now, by differentiating both sides of Eq. (14) with respect to z, we have:

$$f'(z) = \frac{-1}{z^2} + \sum_{n=1}^{\infty} \frac{n!}{(n-1)!} |a_n| z^{n-1} \tag{24}$$

and Theorem 2 follows easily from Eq. (23) and Eq. (24), respectively.

Finally, it is easy to see that the bounds in Eq. (21) are attained for the function $f(z)$ given by Eq. (22).

Next, we determine the radii of meromorphic starlikeness and convexity of order φ for functions in the class $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$.

Theorem 3. Let the function $f(z)$ defined by Eq. (14) be in the class $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$. Then we have:

(i) $f(z)$ is meromorphically starlike of order φ in the disc $|z| < r_1$, that is:

$$\Re\left\{ -\frac{zf'(z)}{f(z)} \right\} > \varphi \qquad (|z| < r_1; 0 \leq \varphi < 1), \tag{25}$$

where:

$$r_1 = \inf_{n \geq 1} \left\{ \frac{[1+(n-1)\gamma]^k [(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)} \frac{(1-\varphi)}{(n+\varphi)} \right\}^{\frac{1}{n+1}} \tag{26}$$

(ii) $f(z)$ is meromorphically convex of order φ in the disc $|z| < r_2$, that is:

$$\Re\left\{ -(1+\frac{zf''(z)}{f'(z)}) \right\} > \varphi \qquad (|z| < r_2; 0 \leq \varphi < 1), \tag{27}$$

where: $|z| < r_2$

$$r_2 = \inf_{n\geq 1} \left\{ \frac{[1+(n-1)\gamma]^{k-1}[(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)} \frac{(1\text{ - }\varphi)}{(n+\varphi)} \right\}^{\frac{1}{n+1}}. \quad (28)$$

Each of these results is sharp for the function $f(z)$ given by Eq. (20).

Proof.

(i) From the definition Eq. (14), we easily get:

$$\left| \frac{\frac{zf'(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}-1+2\varphi} \right| \leq \frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{2(1-\varphi)-\sum_{n=1}^{\infty}(n-1+2\varphi)|a_n||z|^{n+1}}. \quad (29)$$

Thus, we have the desired inequality:

$$\left| \frac{\frac{zf'(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}-1+2\varphi} \right| \leq 1 \quad (0 \leq \varphi < 1), \quad (30)$$

if

$$\sum_{n=1}^{\infty}\left(\frac{n+\varphi}{1-\varphi}\right)|a_n||z|^{n+1} \leq 1. \quad (31)$$

Hence, by Theorem 1, Eq. (31) will be true if:

$$\left(\frac{n+\varphi}{1-\varphi}\right)|z|^{n+1} \leq \frac{[1+(n-1)\gamma]^{k}[(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)}$$

then:

$$|z| \leq \left\{ \frac{[1+(n-1)\gamma]^{k}[(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)} \frac{(1-\varphi)}{(n+\varphi)} \right\}^{\frac{1}{n+1}} \quad (32)$$

The last inequality Eq. (32) leads us immediately to the disc $|z| < r_1$, where r_1 is given by Eq. (26).

(ii) In order to prove the second assertion of Theorem 2, we find from definition Eq. (14) that:

$$\left| \frac{2 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf''(z)}{f'(z)} + 2\varphi} \right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)|a_n||z|^{n+1}}{2(1-\varphi) - \sum_{n=1}^{\infty} n(n-1+2\varphi)|a_n||z|^{n+1}}. \tag{33}$$

Thus, we have the desired inequality:

$$\left| \frac{2 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf''(z)}{f'(z)} + 2\varphi} \right| \leq 1 \quad (0 \leq \varphi < 1), \tag{34}$$

if

$$\sum_{n=1}^{\infty} n\left(\frac{n+\varphi}{1-\varphi}\right)|a_n||z|^{n+1} \leq 1. \tag{35}$$

Hence, by Theorem 1, Eq. (35) will be true if:

$$n\left(\frac{n+\varphi}{1-\varphi}\right)|z|^{n+1} \leq \frac{[1+(n-1)\gamma]^k[(n+1)(1-B)-|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)}. \tag{36}$$

The last inequality Eq. (36) readily yields the disc $|z| < r_2$, where r_2 is defined by Eq. (28). The proof of Theorem 3 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by Eq. (20).

3 Closure Theorems

In this section we first prove:

Theorem 4. The class $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$ is closed under convex linear combinations.

Proof. Let each of the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,j}|z^n \quad (j = 1, 2) \tag{37}$$

be in the class $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$. It is sufficient to show that the function $F(z)$ defined by:

$$F(z) = (1-t)f_1(z) + t\, f_2(z) \quad (0 \leq t \leq 1) \tag{38}$$

is also in the class $L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$. Since

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left[(1-t)\left|a_{n,1}\right| + t\left|a_{n,2}\right| \right] z^n \quad (0 \le t \le 1). \tag{39}$$

with the aid of Theorem 1, we have:

$$\sum_{n=1}^{\infty} [1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)} . \{(1-t)|a_{n,1}| + t|a_{n,2}|\}$$

$$= (1-t)\sum_{n=1}^{\infty} [1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)} |a_{n,1}|$$

$$+ \; t\sum_{n=1}^{\infty} [1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)} |a_{n,2}|$$

$$\le (1-t)|b|(A-B) + t|b|(A-B) = |b|(A-B),$$

which shows that $F(z) \in L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$.

Theorem 5. Let $f_0(z) = \frac{1}{z}$ and $f_n(z) = \frac{1}{z}$

$$+ \frac{|b|(A-B)}{[1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)}} z^n \quad (n \ge 1). \tag{40}$$

Then $f(z) \in L_{s,a}^{k,*}(\alpha_l, \beta_m; A, B, b)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=0}^{\infty} \mu_n \, f_n(z), \tag{41}$$

where: $\mu_n \ge 0$ $(n \ge 0)$ and $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof. Let the function $f(z)$ expressed in the form given by Eq. (41), then

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \mu_n \frac{|b|(A-B)}{[1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)}} z^n \tag{42}$$

and for this function, we have:

$$\sum_{n=1}^{\infty} [1+(n-1)\gamma]^k [(n+1)(1-B) - |b|(A-B)] \Gamma_{(n+1,a,s)} \mu_n$$

$$\times \frac{|b|(A-B)}{[1+(n-1)\gamma]^k[(n+1)(1-B)\text{-}|b|(A-B)]\Gamma_{(n+1,a,s)}}$$

$$= \sum_{n=1}^{\infty} \mu_n |b|(A-B) = |b|(A-B)(1-\mu_0) \leq |b|(A-B), \tag{43}$$

which shows that $f(z) \in L^{k,*}_{s,a}(\alpha_l, \beta_m; A, B, b)$ by Theorem 1.

Conversely, suppose that the function $f(z)$ defined by Eq. (14) belongs to the class $L^{k,*}_{s,a}(\alpha_l, \beta_m; A, B, b)$.

Since

$$|a_n| \leq \frac{|b|(A-B)}{[1+(n-1)\gamma]^k[(n+1)(1-B)-|b|(A-B)]\Gamma_{(n+1,a,s)}} \quad (n \geq 1), \tag{44}$$

by Corollary 1, setting

$$\mu_n = \frac{[1+(n-1)\gamma]^k[(n+1)(1-B)-|b|(A-B)]\Gamma_{(n+1,a,s)}}{|b|(A-B)}|a_n| \ (n \geq 1) \tag{45}$$

and

$$\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n,$$

it follows that

$$f(z) = \sum_{n=0}^{\infty} \mu_n \ f_n(z).$$

This completes the proof of Theorem 5.

4 Conclusion

This research has introduced a new linear differential operator related to the q-hypergeometric function and the Hurwitz Lerch zeta function and some properties were studied. Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

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