Trees with Certain Locating-Chromatic Number

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Abstract. The locating-chromatic number of a graph G can be defined as the cardinality of a minimum resolving partition of the vertex set $V(G)$ such that all vertices have distinct coordinates with respect to this partition and every two adjacent vertices in G are not contained in the same partition class. In this case, the coordinate of a vertex v in G is expressed in terms of the distances of v to all partition classes. This concept is a special case of the graph partition dimension notion. Previous authors have characterized all graphs of order n with locating-chromatic number either n or $n-1$. They also proved that there exists a tree of order n, $n \geq 5$, having locating-chromatic number k if and only if $k \in \{3,4,\dots,n-2,n\}$. In this paper, we characterize all trees of order n with locating-chromatic number $n-t$, for any integers n and t, where $n > t+3$ and $2 \leq t < \frac{n}{2}$.

Keywords: *color code; leaves; locating-chromatic number; stem; tree.*

1 Introduction

The topic of locating-chromatic number was introduced by Chartrand, *et al.* [1] in 2002. They determined the locating-chromatic numbers of paths, cycles, and double stars. Inspired by Chartrand, *et al.*, other authors have determined the locating-chromatic numbers of some well known classes of graphs, i.e. amalgamation of stars and firecrackers by Asmiati, *et al.* [2,3], Kneser graphs by Behtoei and Omoomi [4], Halin graphs by Purwasih, *et al.* [5], Cartesian product of graphs and joint product graphs by Behtoei and Omoomi [6] and Behtoei [7], and homogeneous lobster graphs by Syofyan, *et al.* [8].

Let $G=(V,E)$ be a connected graph. We define the *distance* as the minimum length of path connecting vertices u and v in G, denoted by $d(u,v)$. A k-*coloring* of G is a function $c: V(G) \to \{1,2,\dots,k\}$ where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Thus, the coloring c induces a partition Π of $V(G)$ into k color classes (independent sets) $C_1, C_2, \dots, C_k$ where C_i is the set of all vertices colored by i for $1 \leq i \leq k$. The *color code* $c_\Pi(v)$ of a vertex v in G is defined as the k-vector $(d(v,C_1), d(v,C_2), \dots, d(v,C_k))$ where $d(v,C_i) = \min\{d(v,x) \mid x \in C_i\}$ for $1 \leq i \leq k$. The k-coloring c of G such that all

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vertices have different color codes is called a *locating coloring* of G. The least integer k is such that there is a locating coloring in G that is called the *locating-chromatic number* of G, denoted by $\chi_L(G)$.

Chartrand, *et al.* in [1] have determined all graphs of order n with locating-chromatic number n, namely a complete multipartite graph of n vertices. Furthermore in Chatrand, *et al.* [9], all graphs of order n with locating-chromatic number $n-1$ were characterized. Chartrand, *et al.* [1] also proved that there exists a tree of order n, $n \geq 5$, having locating-chromatic number k if and only if $k \in \{3,4,\ldots,n-2,n\}$. Recently, Baskoro and Asmiati [10] have characterized all trees with locating-chromatic number 3. In this investigation, we have characterized all trees of order n with locating-chromatic number $n-t$, for any integers n and t, where $n > t+3$ and $2 \leq t < \frac{n}{2}$.

The following results were proved in Chartrand, *et al.* [1].

Lemma 1. *Let G be a simple, connected and non directed graph. Let function c be a locating coloring of G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for every $w \in V(G) \setminus \{u,v\}$, then $c(u) \neq c(v)$.*

Corollary 1. *If G is a connected graph containing a vertex that is adjacent to k leaves of G, then $\chi_L(G) \geq k+1$.*

2 Main Results

In the following theorem, we provide a method to construct a tree T of order n from any tree of smaller order $t+1$ where $n > 5$ and $2 \leq t < \frac{n}{2}$, such that $\chi_L(T) = n-t$. A vertex v of degree ≥ 2 in a tree T is called a *stem* if it is adjacent to a leaf.

Theorem 1. *For any integer n and t, where $n > t+3$ and $2 \leq t < \frac{n}{2}$, let T_{t+1} be any tree of order $t+1$. Let T_n be a tree of order n obtained by joining $n-t-1$ new vertices to a vertex $x \in V(T_{t+1})$, where x is not a stem. Then, $\chi_L(T_n) = n-t$.*

Proof. Let $V(T_n) = \{x, y_i, z_j \mid 1 \leq i \leq n-t-1, 1 \leq j \leq t\}$, where x is adjacent to $n-t-1$ leaves, y_i are the leaves adjacent to x, and z_j are other vertices in T_n. Define a $(n-t)$ −coloring $c: V(T_n) \to \{1,2,\ldots,n-t\}$ as follows:

1. $c(x) = n-t$,

2. $c(y_i) = i$ for $1 \le i \le n-t-1$,
3. $c(z_j) = j$ for $1 \le j \le t$.

Next, we show that the color codes of all vertices under the coloring c are distinct. We only consider pairs of vertices with the same color. The possibilities are for the pairs of vertices y_i and z_j for some i, j. If z_j is adjacent to x, then $c_\Pi(y_i) \ne c_\Pi(z_j)$ because $c_\Pi(y_i)$ contains exactly one entry 1, while $c_\Pi(z_j)$ contains at least two entries 1. If z_j is not adjacent to x then $c_\Pi(y_i) \ne c_\Pi(z_j)$ because $c_\Pi(y_i)$ contains entry 1 in the $(n-t)^{th}$ ordinate, while $c_\Pi(z_j)$ does not contain entry 1 in the $(n-t)^{th}$ ordinate. Since every vertex of T_n has distinct color codes, c is a locating coloring of T_n. So, $\chi_L(T_n) \le n-t$.

Now, since T_n contains a vertex x that is adjacent to $n-t-1$ leaves, then by Corollary 1, $\chi_L(T_n) \ge n-t$. Hence, $\chi_L(T_n) = n-t$. □

In the following theorem, we give a necessary condition of a tree of order n whose locating-chromatic number is $n-t$, where $2 \le t < \frac{n}{2}$.

Theorem 2. *For any integer n and t, where $n > t+3$ and $2 \le t < \frac{n}{2}$, let T_n be a tree of order n. If $\chi_L(T_n) = n-t$, then T_n has exactly one stem with $n-t-1$ leaves.*

Proof. Since $\chi_L(T_n) = n-t$, every stem of T_n is adjacent to at most $n-t-1$ leaves. Suppose that there is no stem of T_n having $n-t-1$ leaves. Then every stem of T_n is adjacent to at most $n-t-2$ leaves. Furthermore, we have a locating coloring for T_n by using $n-t-1$ colors as follows.

Let there be b stems in T_n. First, we denote all stems of T_n by s_i, for $1 \le i \le b$, the leaves of T_n adjacent to s_i by l_{ij}, for $1 \le i \le b$ and $1 \le j \le n-t-2$, and the remaining vertices by v_k, for $0 \le k \le n-4$. Let $N(s_i)$ be the set of neighbors of s_i, for $1 \le i \le b$. For a coloring c of $V(T_n)$, define $c(N(s_i)) = \{c(x) | x \in N(s_i)\}$. Now, define a $(n-t-1)-$coloring c of T_n with the following steps :

1. For all stems s_i, define $c(s_i) = 1$ or 2 such that if there are at least two stems adjacent to the same v_k for some k, then two of these stems receive different colors.
2. For all vertices v_k adjacent to a stem, assign $c(v_k) = a$, for some $a \in \{3,4,5,\ldots,n-t-1\}$ such that $c(v_k) \ne c(v_l)$ for $k \ne l$.

3. For all vertices v_k not adjacent to a stem, define $c(v_k) = a$, for some $a \in \{3,4,5,\ldots,n-t-1\}$ such that $c(v_k) \neq c(v_l)$ if $d(v_k, C_i) = d(v_l, C_i)$ for $i = 1, 2$.
4. For all leaves l_{ij}, define $c(l_{ij}) = a$, for some $a \in \{1,2,\ldots,n-t-1\}$ such that all vertices (including leaves) adjacent to stems s_i and s_p satisfy $c(N(s_i)) \neq c(N(s_p))$ for any $i \neq p$.

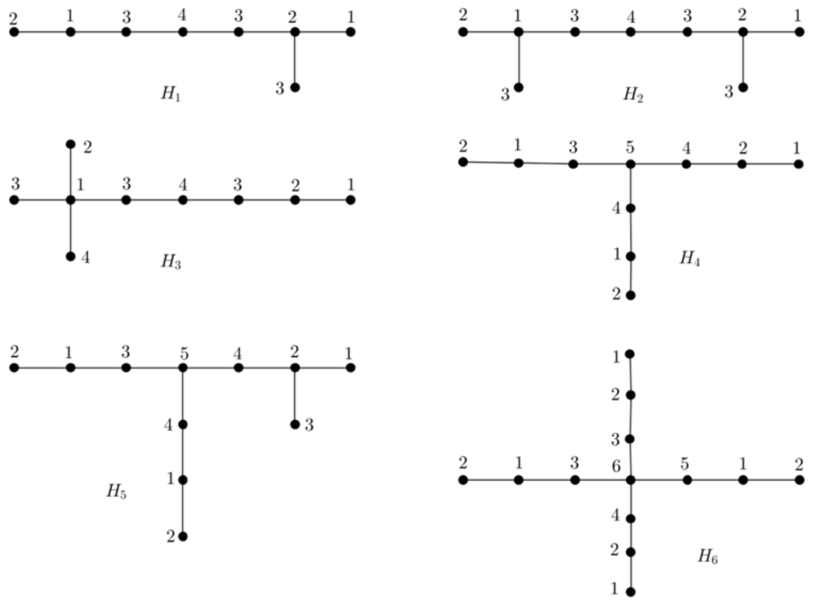

Figure 1 Trees $H_1, H_2, H_3, H_4, H_5, H_6$.

Observe that, with the exception of the six trees depicted in Figure 1, the coloring c can always be done for any tree T_n, $n \geq 6$. Meanwhile, for all trees in Figure 1 we cannot use the coloring c because the number of colors is smaller than the number of vertices v_k adjacent to a stem. However, we can define another coloring for T_n in Figure 1 by $n - t - 1$ colors such that if $n = 8, 9, 10, 11, 13$ then $t = 3, 4, 4, 5, 6$, respectively.

Next, we show that c is a locating coloring of T_n. Let x and y be two vertices of T_n such that $c(x) = c(y)$. We distinguish five cases:

Case 1. $x = s_i$ and $y = s_j$, for $i \neq j$.
Since $c(N(s_i)) \neq c(N(s_j))$ for $i \neq j$ (from Step (4)), we obtain that $c_\Pi(x) \neq c_\Pi(y)$.

Case 2. $x = v_k$ and $y = v_l$, for $k \neq l$.

If v_k is adjacent to a stem s_i and v_l is not adjacent to any stem, then $c_\Pi(v_k) \neq c_\Pi(v_l)$ because $c_\Pi(v_k)$ contains entry 1 in the first or second ordinate, while $c_\Pi(v_l)$ does not contain entry 1 in the first and second ordinate (from Step (1), (2) and (3)).

Case 3. $x = l_{ij}$ and $y = l_{pq}$, for $i \neq p$.
Since $c(N(s_i)) \neq c(N(s_p))$ for $i \neq p$ (from step (4)), we obtain that $c_\Pi(x) \neq c_\Pi(y)$.

Case 4. $x = s_i$ and $y = l_{pq}$.
Since $c(N(s_i)) \neq c(N(s_p))$ for $i \neq p$ (from step (4)), we obtain that $c_\Pi(x) \neq c_\Pi(y)$.

Case 5. $x = v_k$ and $y = l_{ij}$.
Then there are two possibilities for this case:

i) If v_k is adjacent to a stem, then $c_\Pi(v_k) \neq c_\Pi(l_{ij})$ because $c_\Pi(v_k)$ contains at least two entries 1, while $c_\Pi(l_{ij})$ contains exactly one entry 1 (from Step (1),(2),(4)).

ii) If v_k is not adjacent to any stem, then $c_\Pi(v_k) \neq c_\Pi(l_{ij})$ because $c_\Pi(v_k)$ does not contain entry 1 in the first and second ordinate, while $c_\Pi(l_{ij})$ contains entry 1 in the first or second ordinate (from Step (1),(3),(4)).

By the above cases, we prove that c is a locating coloring of T_n. Then $\chi_L(T_n) \leq n - t - 1$, which contradicts $\chi_L(T_n) = n - t$. Hence, there is a stem of T having $n - t - 1$ leaves.

Next, we will show that there is only one stem of T_n having $n - t - 1$ leaves. We suppose that there are two stems of T_n adjacent to $n - t - 1$ leaves. Then, $|V(T_n)| \geq 2(n - t)$. Since $t < \frac{n}{2}$, $|V(T_n)| \geq 2(n - t) > n$, a contradiction with $|V(T_n)| = n$. □

Applying Theorem 1 and Theorem 2, we obtain the following theorem.

Theorem 3. *For any integer n and t, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$, let T_n be a tree of order n. Then $\chi_L(T_n) = n - t$ if and only if T_n has exactly one stem with $n - t - 1$ leaves.*

Based on Theorem 3, we can determine all trees T_n on n vertices with $\chi_L(T_n) = n - t$ for any integers n and t, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$. In particular, if $t = 2, 3$, or 4, all trees T_n with $\chi_L(T_n) = n - t$ are the caterpillars shown in Figures 2, 3, and 4. But for $t \geq 5$, there are trees T_n on n vertices other than caterpillars with $\chi_L(T_n) = n - t$, for example the cases of $t = 5$ and 6, all trees with $\chi_L(T_n) = n - 5$ and $\chi_L(T_n) = n - 6$ are depicted in Figure 5 and Figure 6, respectively. Therefore, as a special case of Theorem 3, we have the following corollary.

First, we give the definition of a caterpillar. Let $P_m = x_1 x_2 \ldots x_m$ be a path with m vertices. A caterpillar $C(m; n_1, n_2, \ldots, n_m)$, is obtained by joining n_i new vertices to every vertex x_i in a path P_m, $n_i \geq 0$, $1 \leq i \leq m$.

Corollary 2. *For any integer n and t, where $n > t + 3$ and $t = 2,3,4$, let T_n be a tree of order n. Then $\chi_L(T_n) = n - t$ if and only if T_n is a caterpillar $C(m; n_1, n_2, \ldots, n_m)$ where $0 \leq n_i \leq n - t - 1$, $2 \leq m \leq t$, $n_1, n_m \neq 0$, and exactly one of n_i is equal to $n - t - 1$.*

All caterpillars in the Corollary 2 are shown in Figure 2, 3, and 4.

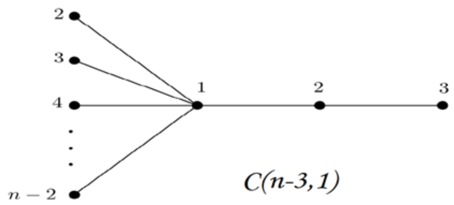

Figure 2 All trees of order $\boldsymbol{n > 5}$ with locating chromatic number $\boldsymbol{n - 2}$.

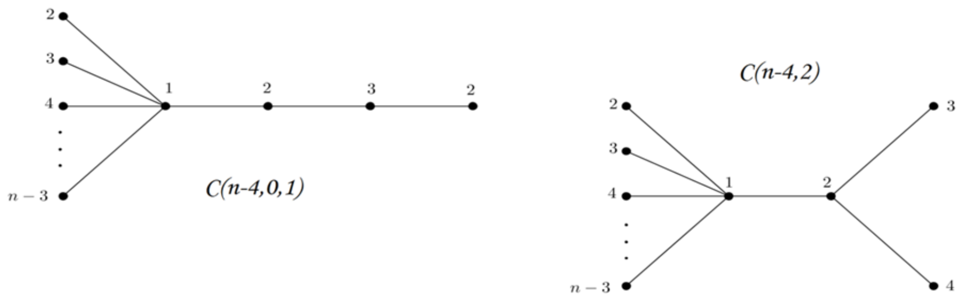

Figure 3 All trees of order $\boldsymbol{n > 6}$ with locating chromatic number $\boldsymbol{n - 3}$.

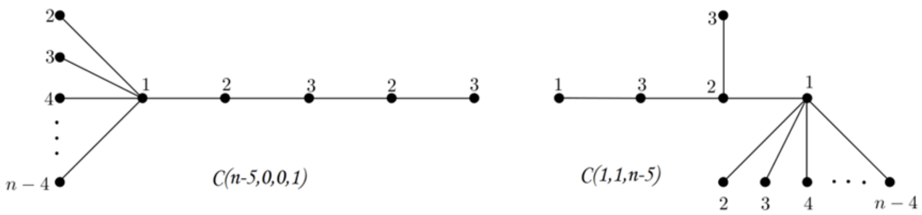

Figure 4 All trees of order $\boldsymbol{n > 7}$ with locating chromatic number $\boldsymbol{n - 4}$.

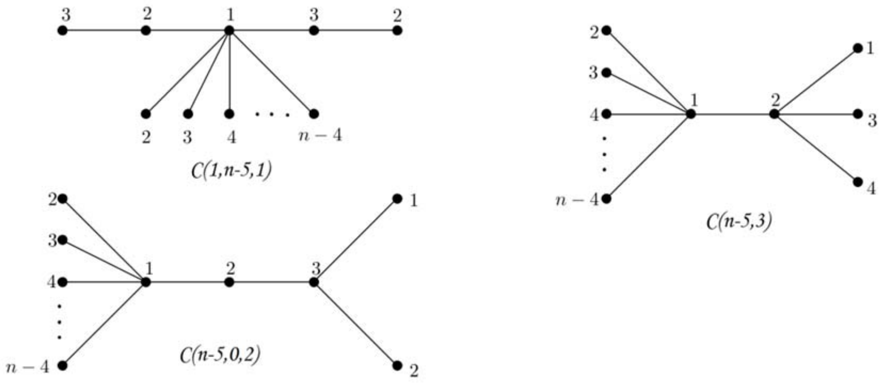

Figure 4 **(*continued*)** All trees of order $\boldsymbol{n > 7}$ with locating chromatic number $\boldsymbol{n-4}$.

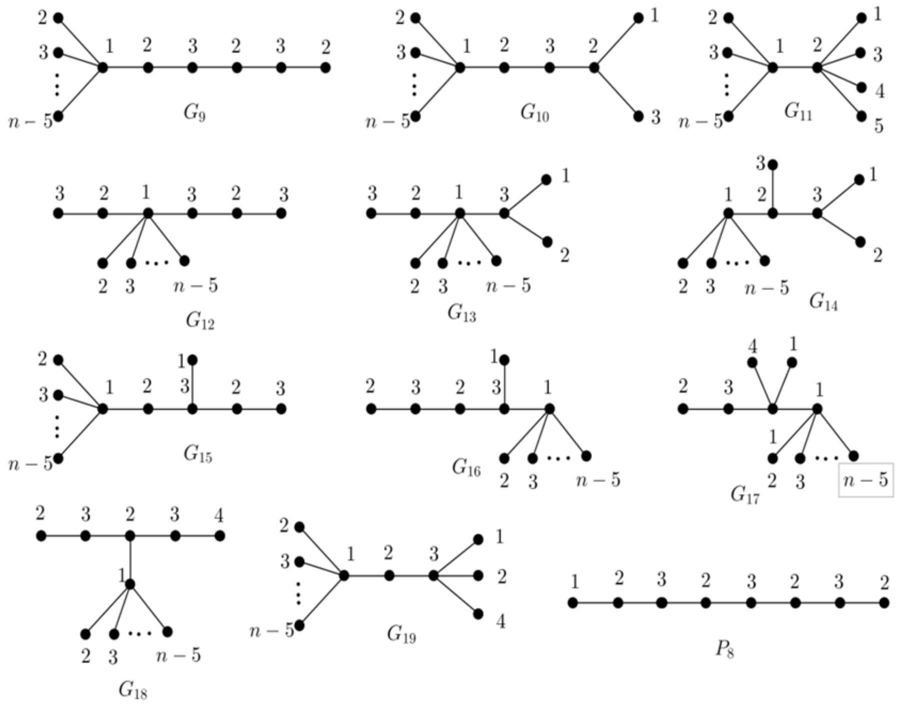

Figure 5 All trees of order $\boldsymbol{n > 8}$ with locating chromatic number $\boldsymbol{n-5}$.

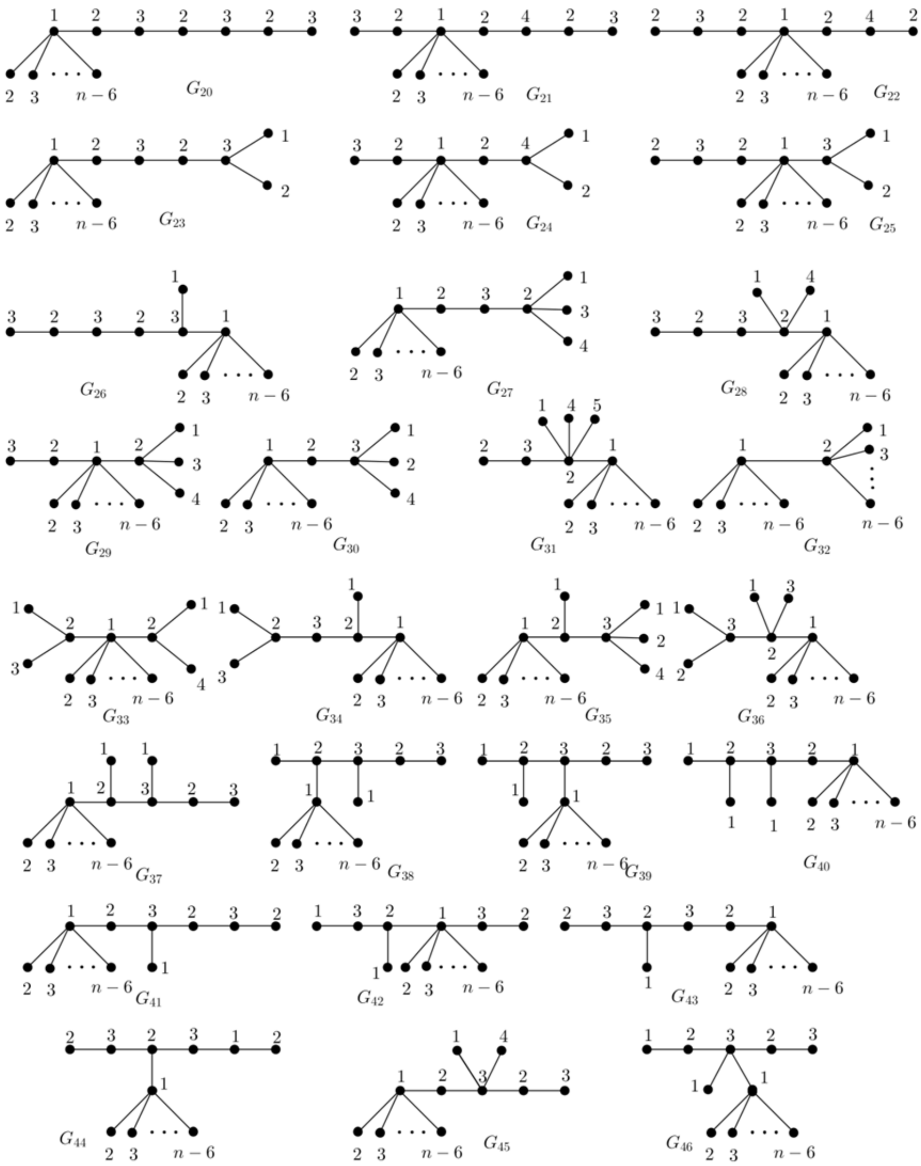

Figure 6 All trees of order $n > 9$ with locating chromatic number $n - 6$.

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