Expanding Super Edge-Magic Graphs*

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Abstract. For a graph G, with the vertex set $V(G)$ and the edge set $E(G)$ an edge-magic total labeling is a bijection f from $V(G) \cup E(G)$ to the set of integers $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ with the property that $f(u) + f(v) + f(uv) = k$ for each $uv \in E(G)$ and for a fixed integer k. An edge-magic total labeling f is called super edge-magic total labeling if $f(V(G)) = \{1,2,\cdots,|V(G)|\}$ and $f(E(G)) = \{|V(G)|+1, |V(G)|+2,\cdots,|V(G)|+|E(G)|\}$. In this paper we construct the expanded super edge-magic total graphs from cycles C_n, generalized Petersen graphs and generalized prisms.

Keywords: *Edge-magic; super edge-magic; magic-sum.*

1 Introduction

All graphs considered here are finite, undirected and simple. As usual, the vertex set and edge set will be denoted $V(G)$ and $E(G)$, respectively. The symbol $|A|$ will be denote the *cardinality* of the set A. Other terminologies or notations not defined here can be found in [2,7,15].

Edge-magic total labelings were introduced by Kotzig and Rosa [8] as follow. *An edge-magic total* labeling on G is a bijection f from $V(G) \cup E(G)$ onto $\{1,2,\cdots,|V(G)|+|E(G)|\}$ with the property that, given any edge uv,

$$f(u) + f(v) + f(uv) = k$$

for some constan k. It will be convenient to call $f(u) + f(v) + f(uv)$ the *edge sum* of uv and k the *magic sum* of f. A graph is called *edge-magic total* if it admits any edge-magic total labeling.

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Kotzig and Rosa [9] showed that no complete graph K_n with $n > 6$ is edge-magic total and neither is K_4, and edge-magic total labelings for K_3, K_5 and K_6 for all feasible values of *k*, are described in [14].

In [8] it is proved that every cycle C_n, every *caterpillar* (a graph derived from a path by hanging any number of pendant vertices from vertices of the path) and every complete bipartite graph $K_{m,n}$ (for any m and n) are edge-magic total.

Wallis et.al. [14] showed that all paths P_n and all *n-suns* (a cycle C_n with an additional edge terminating in a vertex of degree 1 attached to each vertex of the cycle) are edge-magic total. It was shown in [16] that the Cartesian product $C_n \times P_m$ admits an edge-magic total labeling for odd *n*.

It is conjectured that all trees are edge-magic total [8] and all wheels W_n are edge-magic total whenever n 3 (mod 4) [4]. Enomoto et.al. [4] showed that the conjectures are true for all trees with less than 16 vertices and wheels W_n for $n \le 30$. Philips et.al. [12] solved the conjecture partially by showing that a wheel W_n, $n \equiv 0$ or 1 (mod 4), is edge-magic total. Slamin et.al [13] showed that for $n \equiv 6$ (mod 8) every wheel W_n has an edge-magic total labeling.

An edge-magic total labeling f is called *super edge-magic total* if $f(V(G)) = \{1, 2, \cdots, |V(G)|\}$ and $f(E(G)) = \{|V(G)|+1, |V(G)|+2, \cdots, |V(G)| + |E(G)|\}$. Enomoto et.al. [4] proved that the complete bipartite graphs $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. They also proved the complete graphs K_n is super edge-magic if and only if $n = 1, 2$ or 3.

In this paper we will construct the super edge-magic total graphs by hanging any number of pendant vertices from vertices of the cycles, generalized prisms and generalized Petersen graphs.

2 Results

For $n \ge 3$ and $p \ge 1$ we denote by $C_n + A_p$ a graph which is obtained by adding *p* vertices and *p* edges to one vertex of cycles C_n (say v_1). The vertex set and the edge set of $C_n + A_p$ are $V(C_n + A_p) = \{v_i : 1 \le i \le n\} \cup \{u_j : 1 \le j \le p\}$ and $E(C_n + A_p) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_n v_1\} \cup \{v_1 u_j : 1 \le j \le p\}$.

Let $(n, p) - sun$ be a graph derived from a cycle C_n, $n \geq 3$, by hanging p pendant vertices from all vertices of the cycle. Let us denote the vertex set of $(n, p) - sun$ by $V((n, p) - sun) = \{v_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}$ and the edge set by $E((n, p) - sun) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i u_{i,j} : 1 \leq i \leq n,$ $1 \leq j \leq p\}$. Observe that $|V((n, p) - sun)| = |E((n, p) - sun)| = n(p+1)$. The cycle $C_n, n \geq 3$, is super edge-magic total if and only if n is odd (see [4]). Now, we shall investigate super edge-magic total labelings for graphs of $C_n + A_p$ and $(n, p) - sun$ which are expanded from a cycle C_n.

Define a vertex labeling f_1 and an edge labeling f_2 of $C_n + A_p$ as follows,

$$f_1(v_i) = \begin{cases} \frac{n+i}{2} & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

$$f_1(u_j) = n + j \quad \text{for } 1 \leq j \leq p,$$

$$f_2(v_i v_{i+1}) = 2(n+p) + 1 - i \quad \text{for } 1 \leq i \leq n-1,$$

$$f_2(v_n v_1) = n + 2p + 1,$$

$$f_2(v_1 u_j) = n + 2p + 1 - j \quad \text{for } 1 \leq j \leq p.$$

Theorem 1. *If n is odd, $n \geq 3$ and $p \geq 1$, then graph $C_n + A_p$ is super edge-magic total.*

Proof. It is easy to verify that the values of f_1 are $1, 2, \cdots, n+p$ and the values of f_2 are $n+p+1, n+p+2, \cdots, 2n+2p$ and furthermore the common edge sum is $k = 2p + \frac{5n+3}{2}$.

Theorem 2. *If n is odd, $n \geq 3$ and $p \geq 1$, then graph $(n, p) - sun$ is super edge-magic total.*

Proof. Label the vertices and the edges of $(n, p) - sun$ in the following way.

$$f_3(v_i) = f_1(v_i) \quad \text{for } 1 \leq i \leq n,$$

$$f_3(u_{1,j}) = nj + 1 \quad \text{for } 1 \leq j \leq p,$$

$$f_3(u_{i,j}) = n(j+1) + 2 - i \quad \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq p,$$

$$f_4(v_i v_{i+1}) = 2n(p+1) + 1 - i \quad \text{for } 1 \leq i \leq n,$$

$$f_4(v_n v_1) = 2np + n + 1,$$

$$f_4(v_i u_{i,j}) = \begin{cases} 2n(p+1) - nj & \text{if } i = 1 \text{ and } 1 \le j \le p, \\ 2np + n(1-j) + \frac{i-1}{2} & \text{if } i \text{ is odd, } 3 \le i \le n \text{ and } 1 \le j \le p, \\ 2n(p+1) - nj + \frac{i-n-1}{2} & \text{if } i \text{ is even,} 2 \le i \le n-1 \text{ and } 1 \le j \le p. \end{cases}$$

We can see that the vertices of $(n,p)-sun$ are labeled by values $1,2,\cdots,$ $n(p+1)$ and the edges are labeled by $n(p+1)+1,\ n(p+1)+2,\ \cdots, 2n(p+1)$. Furthermore, all edges have the same magic number $k = 2n(p+1) + \frac{n+3}{2}$.

A generalized Petersen graph $P(n,m)$, $n \ge 3$ and $1 \le m \le \left\lfloor \frac{n-1}{2} \right\rfloor$, consists of an outer n-cycle $v_1, v_2, \cdots, v_n$ a set of n spokes $v_i z_i$, $1 \le i \le n$, and inner edges $z_i z_{i+m}$, $1 \le i \le n$, with indices taken modulo n.

For $n \ge 5$, $m = 2$ and $p \ge 1$, we denote by $P(n,2) + A_p$ for a graph which is obtained by adding p vertices and p edges to one vertex of $P(n,2)$, say v_1. Hence, $V(P(n,2) + A_p) = V(P(n,2)) \cup \{u_j : 1 \le j \le p\}$ and $E(P(n,2) + A_p) =$ $E(P(n,2)) \cup \{v_1 u_j : 1 \le j \le p\}$.

Let $P(n,2,p)$ be a graph derived from $P(n,2)$, $n \ge 5$, by hanging p pendant vertices from all vertices v_i, $1 \le i \le n$ of $P(n,2)$. Then the vertex set of $P(n,2,p)$ is $V(P(n,2,p)) = V(P(n,2)) \cup \{u_{i,j} : 1 \le i \le n, 1 \le j \le p\}$ and the edge set is $E(P(n,2,p)) = E(P(n,2)) \cup \{v_i u_{i,j} : 1 \le i \le n, 1 \le j \le p\}$.

In [11] it is proved that generalized Petersen graphs $P(n,2)$ are edge-magic total. Fukuchi [6] showed that $P(n,2)$ are super edge-magic total.

Theorem 3. *If n is odd, $n \ge 5$ and $p \ge 1$, then the graph $P(n,2) + A_p$ has a super edge-magic total labeling.*

Proof. Consider a bijection, $f_5 : V(P(n,2) + A_p) \to \{1, 2, \cdots, 2n + p\}$ where,

$$f_5(v_i) = \begin{cases} n + \frac{i}{2} & \text{if } i \text{ is even, } 2 \le i \le n-1, \\ \frac{3n+i}{2} & \text{if } i \text{ is odd, } 1 \le i \le n, \end{cases}$$

$$f_5(z_i)=\begin{cases}\frac{n-i+4}{4} & \text{if } i\equiv 1 \pmod 4,\\ \frac{2n-i+4}{4} & \text{if } i\equiv 2 \pmod 4,\\ \frac{3n-i+4}{4} & \text{if } i\equiv 3 \pmod 4,\\ \frac{4n-i+4}{4} & \text{if } i\equiv 0 \pmod 4,\end{cases}$$

$$f_5(u_j)=2n+j \quad \text{for} \quad 1\le j\le p.$$

We can observe that under the labeling f_5, $\{f_5(v_i)+f_5(v_{i+1}):\ 1\le i\le n\}=\{\frac{5n+1}{2}+i:1\le i\le n\}$ and $\{f_5(z_i)+f_5(z_{i+2}):1\le i\le n\}=\{\frac{n+1}{2}+i:\ 1\le i\le n\}$ with indices taken modulo n. Moreover, $\{f_5(v_i)+f_5(z_i):1\le i\le n\}=\{\frac{3n+1}{2}+i:1\le i\le n\}$ and $\{f_5(v_1)+f_5(u_j):1\le j\le p\}=\{\frac{7n+1}{2}+j:1\le j\le p\}$. The elements of the set $\{f_5(v_i)+f_5(v_{i+1}):1\le i\le n\}\cup\{f_5(z_i)+f_5(z_{i+2}):1\le i\le n\}\cup\{f_5(v_i)+f_5(z_i):1\le i\le n\}\cup\{f_5(v_1)+f_5(u_j):1\le j\le p\}$ form an arithmetic sequence $\frac{n+1}{2}+1$, $\frac{n+1}{2}+2,\cdots,\frac{7n+1}{2}$, $\frac{7n+1}{2}+1,\cdots,\frac{7n+1}{2}+p$. We are able to arrange the values $2n+p+1,\ 2n+p+2,\cdots,\ 5n+2p$ to the edges of $P(n,2)+A_p$ in such way that the resulting labeling is total and every edge $xy\in E(P(n,2)+A_p)$, $f_5(x)+f_5(y)+f_5(xy)=\frac{11n+3}{2}+2p$. Thus we arrive at the desired result.

Theorem 4. *If n is odd, $n\ge 5$ and $p\ge 1$, then the graph $P(n,2,p)$ has a super edge-magic total labeling.*

Proof. Define a bijection, $f_6:V(P(n,2,p))\to\{1,2,\cdots,n(p+2)\}$ as follows,

$$f_6(v_i)=f_5(v_i) \quad \text{and} \quad f_6(z_i)=f_5(z_i) \quad \text{for} \quad 1\le i\le n,$$
$$f_6(u_{1,j})=n(j+1)+1 \quad \text{for} \quad 1\le j\le p,$$
$$f_6(u_{i,j})=n(j+2)+2-i \quad \text{for} \quad 2\le i\le n \text{ and } 1\le j\le p.$$

We can see that under the vertex labeling f_6 the values $f_6(x)+f_6(y)$ of all edges $xy\in E(P(n,2,p))$ constitute an arithmetic sequence $\frac{n+1}{2}+1,\ \frac{n+1}{2}+2,\cdots,\ \frac{7n+1}{2},\ \frac{7n+1}{2}+1,\cdots,\ \frac{7n+1}{2}+np$. If we complete the edge labeling with the consecutive values in the set $\{n(p+2)+1, n(p+2)+2, n(p+2)+3,\cdots, 5n+2np\}$ then we can obtain total labeling where $f_6(x)+f_6(y)+f_6(xy)=\frac{11n+3}{2}+2np$ for every edge $xy\in E(P(n,2,p))$.

In the sequel we shall consider a graph of a generalized prism which can be defined as the Cartesian product $C_n \times P_m$ of a cycle on *n* vertices with a path on *m* vertices.

Let $V(C_n \times P_m) = \{v_{i,k} : 1 \le i \le n \text{ and } 1 \le k \le m\}$ be the vertex set and $E(C_n \times P_m) = \{v_{i,k}v_{i+1,k} : 1 \le i \le n \text{ and } 1 \le k \le m\} \cup \{v_{i,k}v_{i,k+1} : 1 \le i \le n$ and $1 \le k \le m-1\}$ be the edge set, where *i* is taken modulo *n*. For $n \ge 3$, $m \ge 2$ and $p \ge 1$, we will consider a graph $(C_n \times P_m) + A_p$ (respectively a graph $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$) which is obtained by adding *p* vertices and *p* edges to one vertex of $C_n \times P_m$, say $v_{1,m}$ (respectively to all vertices $v_{i,m}$, $1 \le i \le n$ of $C_n \times P_m$). Thus $V((C_n \times P_m) + A_p) = V(C_n \times P_m) \cup \{u_j : 1 \le j \le p\}$,

$$V((C_n \times P_m) + \sum_{i=1}^{n} A_p^i) = V(C_n \times P_m) \cup \{u_{i,j} : 1 \le i \le n, 1 \le j \le p\},$$
$$E((C_n \times P_m) + A_p) = E(C_n \times P_m) \cup \{v_{1,m}u_j : 1 \le j \le p\}, \text{ and}$$
$$E((C_n \times P_m) + \sum_{i=1}^{n} A_p^i) = E(C_n \times P_m) \cup \{v_{i,m}u_{i,j} : 1 \le i \le n, 1 \le j \le p\}.$$

Figueroa-Centeno et.al. [5] showe that the generalized prism $C_n \times P_m$ is super edge-magic if *n* is odd and $m \ge 2$.

The next two theorems show super edge-magic total labelings of graphs $(C_n \times P_m) + A_p$ and $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$.

Theorem 5. *If n is odd,* $n \ge 3$, $m \ge 2$ *and* $p \ge 1$, *then the graph* $(C_n \times P_m) + A_p$ *has a super edge-magic total labeling*.

Proof. If *m* is even, $m \ge 2$, $1 \le k \le m$, $1 \le i \le n$, then we construct a vertex labeling f_7 in the following way,

$$f_7(v_{i,k}) = \begin{cases} n(k-1)+\frac{i+1}{2} & \text{if } i \text{ is odd and } k \text{ is odd,} \\ nk+\frac{i-n+1}{2} & \text{if } i \text{ is even and } k \text{ is odd,} \\ nk+\frac{i-n}{2} & \text{if } i \text{ is odd and } k \text{ is even,} \\ n(k-1)+\frac{i}{2} & \text{if } i \text{ is even and } k \text{ is even,} \end{cases}$$

$$f_7(u_j) = mn + j \quad \text{for} \quad 1 \le j \le p.$$

If m is odd, $m \ge 3$, $1 \le k \le m$, $1 \le i \le n$, then we define a vertex labeling f_8 as follows,

$$f_8(v_{i,k}) = \begin{cases} \frac{n+i}{2}+n(k-1) & \text{if } i \text{ is odd and } k \text{ is odd,} \\ \frac{i}{2}+n(k-1) & \text{if } i \text{ is even and } k \text{ is odd,} \\ nk & \text{if } i=1 \text{ and } k \text{ is even,} \\ n(k-1)+\frac{i-1}{2} & \text{if } i \text{ is odd and } k \text{ is even,} \\ n(k-1)+\frac{n+i-1}{2} & \text{if } i \text{ is even and } k \text{ is even,} \end{cases}$$

$$f_8(u_j) = mn + j \quad \text{for} \quad 1 \le j \le p.$$

It is easy to verify that for each edge $xy \in E((C_n \times P_m) + A_p)$ the values $f_7(x)+f_7(y)$ and $f_8(x)+f_8(y)$ form an arithmetic sequence $\frac{n+1}{2}+1$, $\frac{n+1}{2}+2$, $\cdots$, $2mn-\frac{n-1}{2}, 2mn-\frac{n-3}{2}, \cdots, 2mn-\frac{n-1}{p}+p.$

Let f_9 be a bijection from $E((C_n \times P_m)+A_p)$ onto $\{1, 2, \cdots, 2nm-n+p\}$. We can combine the vertex labeling f_7 (or f_8) and the edge labeling f_9+mn+p such that the resulting labeling is total and the edge sum for each edge $xy \in E((C_n \times P_m)+A_p)$ is equal to $3mn+\frac{3-n}{2}+2p.$

Theorem 6. *If n is odd, $n \ge 3$, $m \ge 2$, and $p \ge 1$, then the graph $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ has a super edge-magic total labeling.*

Proof. Define vertex labeling f_{10} and f_{11} such that :

$$f_{10}(v_{i,k}) = f_7(v_{i,k}) \quad \text{if } m \text{ is even, } 1 \le k \le m, \ 1 \le i \le n,$$
$$f_{11}(v_{i,k}) = f_8(v_{i,k}) \quad \text{if } m \text{ is odd, } 1 \le k \le m, \ 1 \le i \le n,$$
$$f_{10}(u_{1,j}) = f_{11}(u_{1,j}) = n(m+j-1)+1 \quad \text{for} \quad 1 \le j \le p,$$

$$f_{10}(u_{i,j}) = f_{11}(u_{i,j}) = n(m+j) - i + 2 \text{ for } 2 \le i \le n \text{ and } 1 \le j \le p.$$

We can see that vertices of $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ are labeled by values 1, 2, 3, $\cdots, n(m+p)$ and $f_t(x) + f_t(y)$ for all edges $xy \in (C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ and $t \in \{10, 11\}$ constitute an arithmetic sequence $\frac{n+1}{2}+1, \frac{n+1}{2}+2, \cdots, 2mn - \frac{n-1}{2} + np$.

We can complete the edge labeling of $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ with values in the set $\{n(m+p)+1, n(m+p)+2, \cdots, n(3m+2p-1)\}$ consecutively such that the common edge sum is $k = 3mn + 2pn - \frac{n-3}{2}$. Thus the total labeling of $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ is super edge-magic and the theorem is proved.

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