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Existence and Uniqueness Results for Difference ϕ-Laplacian Boundary Value Problems

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Abstract. This paper is devoted to study the existence and uniqueness of solutions to nonlinear difference ϕ-Laplacian boundary value problems with mixed and Dirichlet boundary conditions.

Keywords: *fixed points words; Laplacian problems.*

1 Introduction

The study of nonlinear difference second order boundary value problems has been developed recently in many papers. The first order difference equation with nonlinear functional boundary conditions is considered in [1] where as the n^{th} order difference equations are studied in [2-4]. The existence and uniqueness of solutions for some nonlinear boundary value problems are given [5-6]. Some of these results for second order difference equations have been generalized in [7-8] for ϕ-Laplacian problems. The existence and uniqueness of solutions to difference ϕ-Laplacian problems along with comparison results are given by A. Cabada and V.O. Espinar in [9].

In this paper we study the existence and uniqueness of solution for the equation

$$\Delta(\phi(\Delta u(k))) = f(k,u(k+1)) \tag{1}$$

with different boundary conditions. The results in this paper are generalizations of the results obtained in [9]. The corollaries in this paper are nothing but some of the theorems proved in [9]. The technique of fixed point theory of contraction mapping is utilized in this work to prove existence and uniqueness of solution of solution of Eq. 1.

Throughout this paper we denote $I = \{0,1,2,\ldots,N-1\}$; $P = \{0,1,2,\ldots,N\}$; $J = \{0,1,2,\ldots,N+1\}$. By summation convention we have $\sum_{k=n}^{m}\phi^{-1}(x_k) = 0$ if $m < n$. Suppose B is a set of all real valued functions defined on P and for each

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$x \in B$, define a norm $\|\cdot\|$ on B by $\|x\| = \sum_{k=0}^{N} |x(k)|$ the usual 1-norm in $\Re^{N+1}$. Hence B is a Banach space.

In the sequel we state the following conditions.

(H_1): $\phi : \Re \rightarrow \Re$ is a homeomorphisim and ϕ^{-1} is Lipschitizian function on $\Re$ i.e. there is

$H > 0$ satisfying

$$\left|\phi^{-1}(x) - \phi^{-1}(y)\right| \leq H\left|x - y\right|$$

for all $x,y \in \Re$.

(H_2): A function $f : P \times \Re \rightarrow \Re$ is Lipschitz w.r.t. the second variable i.e. there is $L > 0$ such that

$$\left|f(k,x) - f(k,y)\right| \leq L\left|x - y\right|$$

for all $k \in P$ and $x,y \in \Re$

(H_3): $\phi : \Re \rightarrow \Re$ is nondecreasing.

2 Existence and Uniqueness Results

This section is devoted to the existence and uniqueness of solutions to difference ϕ-Laplacian boundary value problem

$$\Delta(\phi(\Delta u(k))) = f(k,u(k+1))$$

with different boundary conditions. Theorems 2.1, 2.2, and 2.3 in this paper are the generalizations of the Theorems 2.1, 2.2, and 2.3 in [9] respectively.

Theorem 2.1. *Assume (H_1) and (H_2). If* $L \in \left(0, \frac{2}{HN(N+1)}\right)$ *then the problem*

$$(PM_1)\begin{cases} \Delta\left[\phi\left(\Delta u(k)\right)\right] = f\left(k, u(k+1)\right);\ k \in I \\ \Delta u(0) = N_0; u(N+1) = N_1 \end{cases}$$

has a unique solution $u : J \rightarrow R$ for all $N_0, N_1 \in \Re$.

Proof. Define a mapping $T : B \rightarrow B$ by

$$Tv(k) = \phi(N_0) + \sum_{l=0}^{k-1} f\left(l, N_1 - \sum_{s=l+1}^{n} \phi^{-1}(v(s)) \right), \tag{2}$$

for $v \in B$ and $k \in P$. We verify that $u: J \to \Re$ given by

$$u(k) = N_1 - \sum_{s=k}^{N} \phi^{-1}(v(s)) \tag{3}$$

is a solution of problem (PM_1) provided $v \in B$ is a fixed point of T and uniqueness of the solution follows from the existence of unique fixed point of T. From Eq. 3 and by summation convention, it is obvious that $u(N+1) = N_1$. Now,

$$\Delta u(k) = -\sum_{s=k+1}^{N} \phi^{-1}(v(s)) + \sum_{s=k}^{N} \phi^{-1}(v(s))$$

$$= \phi^{-1}(v(k))$$

$$\therefore \phi(\Delta u(k)) = v(k).$$

Since v is a fixed point of T,

$$\phi(\Delta u(k)) = \phi(N_0) + \sum_{l=0}^{k-1} f\left(l, N_1 - \sum_{s=l+1}^{N} \phi^{-1}(v(s)) \right). \tag{4}$$

It follows that

$$\Delta(\phi(\Delta u(k)) = f\left(k, N_1 - \sum_{s=k+1}^{N} \phi^{-1}(v(s)) \right)$$

$$= f(k, u(k+1)).$$

From Eq. 4 it follows that $\phi(\Delta u(0)) = \phi(N_0)$ and injectivity of ϕ implies $\Delta u(0) = N_0$.

Now we prove that T has a unique fixed point in B. For $v_1, v_2 \in B$ and using conditions (H_1), (H_2) we have

$$\left| Tv_1(k) - Tv_2(k) \right| \le \sum_{l=0}^{k-1} \left| f\left(l, N_1 - \sum_{s=l+1}^{N} \phi^{-1}(v_1(s)) \right) - f\left(l, N_1 - \sum_{s=l+1}^{N} \phi^{-1}(v_2(s)) \right) \right|$$

$$\leq L\sum_{l=0}^{k-1}\sum_{s=l+1}^{N}\left|\phi^{-1}(v_1(s))-\phi^{-1}(v_2(s))\right|$$

$$\leq LH\left\|v_1-v_2\right\|k$$

$$\therefore \left\|Tv_1-Tv_2\right\|\leq\sum_{k=0}^{N}LH\left\|v_1-v_2\right\|k$$

$$\leq LH\left\|v_1-v_2\right\|\frac{N(N+1)}{2}.$$

Since $L\in\left(0,\frac{2}{HN(N+1)}\right)$, T is a contraction and hence has a unique fixed point which completes the proof.

Theorem 2.2. *Assume (H_1) and (H_2). If* $L\in\left(0,\frac{2}{HN(N+1)}\right)$, *then the boundary value problem*

$$(PM_2)\begin{cases}\Delta(\phi(\Delta u(k))]=f(k,u(k+1))\\ \mathrm{u}(0)=\mathrm{N}_0;\Delta u(N)=N_1\end{cases}$$

has a unique solution $u : J\rightarrow R$ for all $N_0, N_1\in R$.

Proof. Define a mapping $T : B\rightarrow B$ by

$$Tv(k)=\phi(N_1)-\sum_{l=k}^{N-1}f\left(l,N_0+\sum_{s+0}^{l}\phi^{-1}(v(s))\right), \tag{5}$$

for $k\in P$ and $v\in B$. Similar to Theorem 2.1 it can be verified that $u : J\rightarrow\Re$ given by

$$u(k)=N_0+\sum_{s=0}^{N-1}\phi^{-1}(v(s)) \tag{6}$$

is a desired unique solution of B.V.P.(PM_2) if and only if $v\in B$ is a unique fixed point of T.

Using these theorems following corollaries can be obtained which are the Theorems 2.1 and 2.2 in [9].

Corollary 2.1. *Assume condition (H_1). If* $M\in\left(\frac{-2}{HN(N+1)},0\right)$ *then the problem*

$$-\Delta(\phi(\Delta u(k)))+Mu(k+1)=\sigma(k);k\in I$$

$$\Delta u(0) = N_0; \quad u(N+1) = N_1$$

has a unique solution $u \in B$ for each $\sigma \in B$ and all $N_0, N_1 \in \Re$.

Proof. If we put $f(k,u)=Mu-\sigma(k)$ the above problem reduces to problem (PM_1). Moreover for $u, v \in B$,

$$\left| f(k,u) - f(k,v) \right| = -M \left| u - v \right|.$$

Therefore f is a Lipschitz function with $L = -M$. Since $M \in \left(\frac{-2}{HN(N+1)}, 0\right)$; $L \in \left(0, \frac{2}{HN(N+1)}\right)$ so the conclusion follows from Theorem 2.1.

Corollary 2.2. *Assume condition (H_1) and (H_2). If $M \in \left(\frac{-2}{HN(N+1)}, 0\right)$ then the problem*

$$-\Delta(\phi(\Delta u(k))) + Mu(k+1) = \sigma(k);\ k \in I$$

$$u(0) = N_0;\ \Delta u(N+1) = N_1$$

has a unique solution $u \in B$ for each $\sigma \in B$ and all $N_0, N_1 \in \Re$.

Now we prove the existence and uniqueness of solution for Dirichlet Boundary value problem of ϕ-Laplacian difference equation.

Theorem 2.3. *Assume (H_1), (H_2) and (H_3). If $L \in \left(0, \frac{2}{HN(N+1)}\right)$, then the boundary value problem*

$$(PM_3)\begin{cases} \Delta(\phi(\Delta u(k))] = f(k, u(k+1)) \\ \mathrm{u}(0) = \mathrm{N}_0;\ u(N+1) = N_1 \end{cases}$$

has a unique solution $u: J \to \Re$ for all $N_0, N_1 \in \Re$.

Proof. Define a mapping $T: B \to B$ by

$$Tv(k) = C_v - \sum_{l=k}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v(s)) \right), \tag{7}$$

where C_v is solution of

$$N_1 - N_0 = \sum_{k=0}^{N} \phi^{-1}\left\{ C_v - \sum_{l=k}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v(s)) \right) \right\}. \tag{8}$$

Since ϕ^{-1} is one-one, onto and continuous form $\Re$ to itself, for given $v \in B$ there exists unique $Cv \in \Re$ satisfying Eq. 8. We verify that *$u:J \to \Re$ given by*

$$u(k) = N_0 + \sum_{s=0}^{k-1} \phi^{-1}(v(s)) \tag{9}$$

is a desired unique solution of B.V.P.(PM_3) if and only if $v \in B$ is a unique fixed point of T. From (2.8) it is obvious that $u(0)=N_0$. Now

$$\Delta u(k) = \phi^{-1}(v(k))$$

$$\therefore\ \phi(\Delta u(k)) = C_v - \sum_{l=k}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v(s)) \right)$$

$$\therefore\ \Delta[\phi(\Delta u(k))] = f\left(k, N_0 + \sum_{s=0}^{k} \phi^{-1}(v(s)) \right)$$

$$= f(k; u(k+1)).$$

From Eq. 9 and Eq. 8 it follows that $u(N+1)=N_1$.

Now we prove that T has a unique fixed point. For each $k \in P$ and $v_1, v_2 \in B$, we have

$$Tv_1(k) - Tv_2(k) = C_{v_1} - C_{v_2} - \sum_{l=k}^{N} \left[f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) - f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right) \right]$$

where C_{V_1} *and* C_{V_2} satisfy

$$\sum_{k=0}^{N} \phi^{-1} \left\{ C_{v_1} - \sum_{l=k}^{N} f(l, N_0 + \sum_{s=0}^{1} \phi^{-1}(v_1(s))) \right\} = \sum_{k=0}^{N} \phi^{-1} \left\{ C_{v_2} - \sum_{;=k}^{N} f(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s))) \right\}.$$

There exists $k_0, k_1 \in P$ such that

$$\phi^{-1} \left\{ C_{v_1} - \sum_{l=k_0}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) \right\} \le \phi^{-1} \left\{ C_{v_2} - \sum_{l=k0}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right) \right\}$$

and

$$\phi^{-1} \left\{ C_{v_1} - \sum_{l=k1}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) \right\} \ge \phi^{-1} \left\{ C_{v_2} - \sum_{l=k1}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right) \right\}.$$

Condition *(H₃)* implies that

$$C_{v_1} - \sum_{l=k0}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) \le C_{v_2} - \sum_{l=k0}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right)$$

and

$$C_{v_1} - \sum_{l=k1}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) \ge C_{v_2} - \sum_{l=k1}^{N} f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right)$$

$$\therefore \sum_{l=k1}^{N} \left\{ f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) \right\} - f\left(l, N_0 + \sum_{s=0}^{l} \phi^{1}(v_2(s)) \right) \le C_{v_1} - C_{v_2}$$

$$\le \sum_{l=k0}^{N} \left\{ f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_1(s)) \right) - f\left(l, N_0 + \sum_{s=0}^{l} \phi^{-1}(v_2(s)) \right) \right\}.$$

If we suppose $k < k_0$, then we obtain

$$Tv_1(k) - Tv_2(k) \le LH \left\| v_1 - v_2 \right\| (k_0 - k),$$

and $k > k_0$, then we obtain

$$Tv_1(k) - Tv_2(k) \le LH \left\| v_1 - v_2 \right\| (k - k_0).$$

$$\therefore Tv_1(k) - Tv_2(k) \le LH \left\| v_1 - v_2 \right\| \left\{ \left| k - k_0 \right| \right\}. \tag{10}$$

Similarly we deduce that

$$\therefore Tv_2(k) - Tv_1(k) \le LH \left\| v_1 - v_2 \right\| \left\{ \left| k - k_1 \right| \right\}. \tag{11}$$

Hence from Eq. 10 and Eq. 11 we conclude that, for all $k \in P$

$$\left| Tv_1(k) - Tv_2(k) \right| \le LH \left\| v_1 - v_2 \right\| \max \left\{ \left| k - k_0 \right|, \left| k - k_1 \right| \right\}$$

$$\le LH \left\| v_1 - v_2 \right\| \max \left\{ k, N - k \right\}.$$

As a consequence, we obtain

$$\left\| Tv_1 - Tv_2 \right\| \le LH \frac{N(N+1)}{2} \left\| v_1 - v_2 \right\|.$$

This shows that T is a contraction and hence result holds.

Remark. Similar to Corollaries 2.1 and 2.2, we can obtain a result which is Theorem 2.3 in [9], that the equation

$$-\Delta(\phi(\Delta u(k)) + Mu(k+1) = \sigma(k)\,;\, \sigma \in I$$

has a unique solution for Dirichlet boundary condition $u(0)=N_0$, $u(N+1) = N_1$.

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