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Curvature and the Electromagnetic Field

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Abstract

Maxwell's equations are derived from the curvature tensor and a vector potential. The results are combined with Einstein's equations. Complete solutions to the resulting equation yield simultaneous solutions to both Einstein's and Maxwell's equations. This is a classical theoretical unification of electromagnetism and gravitation.

Keywords: Electromagnetism; Curvature Tensor; Maxwell's Equations; General Relativity; Unified Field Theory

Since the geometrical general theory of relativity was published as a theory of gravitation [1], the notion that electromagnetism was also amenable to a classical, geometrical theory has been pursued in a variety of ways [2], including notable contributions by Weyl [3], Kaluza, and Schrödinger [4]. To date, however, the achievement of this goal is not generally accepted. This is largely due to the fact that the metric tensor is symmetric while the electromagnetic field tensor in antisymmetric. This paper presents a geometrical theory of electromagnetism that is distinct from the various approaches developed by others and introduces the required antisymmetry in a straightforward way. The result is that Maxwell's equations follow, thus unifying the core of classical electromagnetic theory with Einstein's classical gravitational theory. The sole purpose of the present paper is to present this derivation.

The present geometrical theory of electromagnetism begins with the conventional derivation of the fourth-rank curvature tensor $R^{\beta}_{\nu\rho\sigma}$ in four-dimensional spacetime [5]. The lower-case Greek suffixes take on the values $0,1,2,3$; downstairs suffixes denote covariant character; upstairs suffixes denote contravariant character. The summation convention for contraction over pairs of covariant and contravariant components with the same Greek suffix is used.

This derivation begins with a covariant vector A_ν. The first and second covariant derivatives of the vector are $A_{\nu:\rho:\sigma}$ where the colon symbol represents covariant differentiation. Using the symbol $\langle\rho\sigma\rangle$ to denote the tensor preceding it with the suffixes ρ and σ interchanged, the curvature tensor is defined in the following expression:

$$A_{\nu:\rho:\sigma} - \langle\sigma\rho\rangle = A_{\nu:\rho:\sigma} - A_{\nu:\sigma:\rho} = A_\beta R^\beta_{\nu\rho\sigma}.$$

Only the interchange of ρ with σ is used above.

The derivation then employs a sum of the cyclic permutations (c.p.) of the three suffixes in the above equation

$$A_{\nu:\rho:\sigma} - A_{\nu:\sigma:\rho} + c.p. = A_{\nu:\rho:\sigma} - A_{\nu:\sigma:\rho} + A_{\rho:\sigma:\nu} - A_{\sigma:\rho:\nu} + A_{\sigma:\nu:\rho} - A_{\rho:\nu:\sigma}$$

$$= A_\beta \left(R^\beta_{\nu\rho\sigma} + R^\beta_{\rho\sigma\nu} + R^\beta_{\sigma\nu\rho} \right). \tag{0.1}$$

With inherent symmetries in the curvature tensor,

$$R^\beta_{\nu\rho\sigma} + R^\beta_{\rho\sigma\nu} + R^\beta_{\sigma\nu\rho} = 0$$

and the right-hand side of Eq. (0.1) equals zero [6].

The form of the left-hand side of Eq. (0.1) is simplified using the antisymmetric tensor formed from the covariant derivatives of the vector, $B_{\nu\rho} = A_{\nu:\rho} - A_{\rho:\nu}$. With this tensor, the various second derivatives in Eq. (0.1) can be grouped pair-wise based on the suffixes of the second derivatives to become

$$B_{\nu\rho:\sigma} + B_{\rho\sigma:\nu} + B_{\sigma\nu:\rho} = B_{\nu\rho:\sigma} + c.p. = 0. \tag{0.2}$$

This result is a consequence solely of the spacetime geometry. The vector A_ν can be any vector and the result is independent of any physical units which the vector may have. If the metric tensor is dimensionless and the vector A_ν has the units of the electromagnetic vector potential, which are $tesla \cdot meter$ in SI units [7], then the antisymmetric tensor $B_{\nu\sigma}$ has units of the electromagnetic field and Eq. (0.2) gives two of Maxwell's equations [8], [9].

The antisymmetric tensor formed from the covariant derivatives of a vectoris the same as the antisymmetric tensor formed from the ordinary derivatives of the vector,

$$A_{\nu:\rho} - A_{\rho:\nu} = A_{\nu,\rho} - \Gamma^{\alpha}_{\nu\rho} A_{\alpha} - A_{\rho,\nu} + \Gamma^{\alpha}_{\rho\nu} A_{\alpha} = A_{\nu,\rho} - A_{\rho,\nu}$$

where a comma denotes ordinary differentiation and the $\Gamma^{\alpha}_{\nu\rho}$ are Christoffel symbols of the second kind. In similar fashion, it is interesting to write out Eq. (0.2) as follows:

$$B_{\nu\rho:\sigma} + c.p. = \left(A_{\nu,\rho} - \Gamma^{\alpha}_{\nu\rho} A_{\alpha} - A_{\rho,\nu} + \Gamma^{\alpha}_{\rho\nu} A_{\alpha} \right)_{:\sigma} + c.p.$$

$$= B_{\nu\rho,\sigma} - \Gamma^{\beta}_{\nu\sigma} B_{\beta\rho} - \Gamma^{\beta}_{\sigma\rho} B_{\nu\beta} + B_{\rho\sigma,\nu} - \Gamma^{\beta}_{\rho\nu} B_{\beta\sigma} - \Gamma^{\beta}_{\nu\sigma} B_{\rho\beta} + B_{\sigma\nu,\rho} - \Gamma^{\beta}_{\sigma\rho} B_{\beta\nu} - \Gamma^{\beta}_{\rho\nu} B_{\sigma\beta}$$

$$= B_{\nu\rho,\sigma} + B_{\rho\sigma,\nu} + B_{\sigma\nu,\rho} - \Gamma^{\beta}_{\nu\sigma} \left(B_{\beta\rho} + B_{\rho\beta} \right) - \Gamma^{\beta}_{\sigma\rho} \left(B_{\nu\beta} + B_{\beta\nu} \right) - \Gamma^{\beta}_{\rho\nu} \left(B_{\beta\sigma} + B_{\sigma\beta} \right)$$

$$= B_{\nu\rho,\sigma} + c.p. = 0\,.$$

Thus, the gravitational field embodied in the Christoffel symbols does not enter into the Maxwell's equations of Eq. (0.2).

The remaining Maxwell's equations arise following contraction over one suffix in the antisymmetric tensor with the suffix of the covariant differentiation in $B_{\nu\rho:\sigma}$ using the contravariant metric tensor $g^{\rho\sigma}$:

$$g^{\rho\sigma} B_{\nu\rho:\sigma} = \mu_0 J_{\nu}\,, \tag{0.3}$$

where μ_0 is the permeability of a vacuum [9], [10].

The electromagnetic current J_{ν} is then expressible in terms of the vector potential and the Ricci tensor [11]:

$$g^{\rho\sigma} B_{\nu\rho:\sigma} = A_{\nu:\rho}{}^{:\rho} - g^{\rho\sigma} \left(A^{\beta} R_{\beta\rho\nu\sigma} + A_{\rho:\sigma:\nu} \right) = \Box A_{\nu} + A^{\beta} R_{\beta\nu} - \left(A_{\rho}{}^{:\rho} \right)_{:\nu} = \mu_0 J_{\nu}\,,$$

where $\Box$ is the covariant D'Alembertian operator such that $A_{\nu:\rho}{}^{:\rho} = \Box A_{\nu}$. This result is general for the vector potential A_{ν} in any gauge. The result is simplified in the covariant Lorentz gauge for which $A_{\rho}{}^{:\rho} = 0$:

$$\mu_0 J_{\nu} = \Box A_{\nu} + A^{\beta} R_{\beta\nu}\,. \tag{0.4}$$

Unlike the first two Maxwell's equations, the source term equations of Eq. (0.3) and Eq. (0.4) do depend on the curvature. There is curvature at a point if there is any matter at that point, matter being any matter-energy aside from the gravitational field. In the case where sources are present at a point, the curvature obtains contributions due not only from the electromagnetic field but also from the masses of the charged particles of the current. In the case where no sources are present, no other matter is present, and only the free electromagnetic field is present, the curvature obtains contributions solely from the free electromagnetic field and Eq. (0.4) becomes

$$0 = \Box A_\nu + A^\beta R_{\beta\nu} .$$

These results can be applied in Einstein's equations for the gravitational field (neglecting the cosmological constant),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{0.5}$$

Here $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature $g^{\alpha\beta} R_{\alpha\beta}$, $T_{\mu\nu}$ is the matter-energy tensor, and κ is the Einstein gravitational constant that has units of inverse force and equals $8\pi G / c^4 \approx 2.08 \times 10^{-43} N^{-1}$, where G is Newton's constant of gravitation and c is the speed of light in a vacuum [11]. Using Eq. (0.4), the inner product of a vector potential A^μ with Eq. (0.5) yields

$$\mu_0 J_\nu - \Box A_\nu - \frac{1}{2} R A_\nu = \kappa A^\mu T_{\mu\nu} .$$

This equation can be solved for the vector potential,

$$A_\nu = \frac{2\left(\mu_0 J_\nu - \Box A_\nu - \kappa A^\mu T_{\mu\nu} \right)}{R} .$$

This result cannot become singular when the scalar curvature is zero. This is because this equation is applicable only when there is matter at least in the form of an electromagnetic field; the curvature cannot then be zero.

At a point where the only matter-energy is the free electromagnetic field, the matter-energy tensor is the electromagnetic stress-energy tensor [12],

$$T_{\mu\nu} = \frac{1}{\mu_0} \left(B_{\mu\alpha} B_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} B_{\alpha\beta} B^{\alpha\beta} \right).$$

This stress-energy tensor depends only upon the metric, the permeability constant, and derivatives of the vector potential. Given the constants, Einstein's equations for the gravitational field now yield a differential equation for the vector of the free electromagnetic field,

$$\frac{2\left\{\Box A_{\nu} + \frac{\kappa}{\mu_0} A^{\mu}\left[\left(A_{\mu:\alpha} - A_{\alpha:\mu}\right)\left(A_{\nu}^{\ :\alpha} - A^{\alpha}_{\ :\nu}\right) - \frac{1}{4} g_{\mu\nu}\left(A_{\alpha:\beta} - A_{\beta:\alpha}\right)\left(A^{\alpha:\beta} - A^{\beta:\alpha}\right)\right]\right\}}{R} + A_{\nu} = 0 .$$

This equation depends, also, on the metric tensor. Thus, complete solutions to this equation yield simultaneous solutions to both Einstein's and Maxwell's equations for the free electromagnetic field. At points where the current is $J_{\nu} \neq 0$, the differential equation becomes more complicated and may depend upon additional constants, such as mass, and particle velocities associated with the current at such points that may contribute to the matter-energy tensor.

The results presented in this paper employ only the well-established geometric interpretation of general relativity and the conventional use of a vector potential for the electromagnetic field.

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