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Q-GROUPOIDS AND THEIR COHOMOLOGY

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ABSTRACT. We approach Mackenzie's $\mathcal{LA}$-groupoids from a supergeometric point of view by introducing Q-*groupoids*. A Q-groupoid is a groupoid object in the category of Q-manifolds, and there is a faithful functor from the category of $\mathcal{LA}$-groupoids to the category of Q-groupoids. Using this approach, we associate to every $\mathcal{LA}$-groupoid a double complex whose cohomology simultaneously generalizes Lie groupoid cohomology and Lie algebroid cohomology. As examples, we obtain simplicial-type models for equivariant Lie algebroid cohomology and orbifold Lie algebroid cohomology, and we obtain double complexes associated to Poisson groupoids and groupoid-algebroid "matched pairs".

1. INTRODUCTION

$\mathcal{LA}$-groupoids were introduced by Mackenzie [10, 12] as the intermediate objects between double Lie groupoids and double Lie algebroids. An $\mathcal{LA}$-groupoid is a groupoid object in the category of Lie algebroids. A more concrete definition is that an $\mathcal{LA}$-groupoid is a square

$$
(1) \qquad \begin{array}{ccc} \Omega & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ A & \longrightarrow & M \end{array} \ ,
$$

where the horizontal sides carry Lie algebroid structures, and the vertical sides carry Lie groupoid structures whose structure maps (particularly the source, target and multiplication maps) are Lie algebroid morphisms[1]. Examples include the tangent prolongation groupoid

$$
(2) \qquad \begin{array}{ccc} TG & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ TM & \longrightarrow & M \end{array}
$$

of a Lie groupoid $G \rightrightarrows M$ and the cotangent prolongation groupoid

$$
(3) \qquad \begin{array}{ccc} T^*G & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ A^* & \longrightarrow & M \end{array}
$$

of a Poisson groupoid $G \rightrightarrows M$ with Lie algebroid A.

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[1]Mackenzie also requires that the "double-source" map $\Omega \to G \times_M A$ is a surjective submersion. For the purposes of this paper, this condition will not be required.

The notion of $\mathcal{LA}$-groupoids may be considered a simultaneous generalization of those of Lie groupoids and Lie algebroids. Indeed, if $G \rightrightarrows M$ is a Lie groupoid, then the square

$$(4) \qquad\qquad \begin{array}{ccc} G & \longrightarrow & G \\ \downdownarrows & & \downdownarrows \\ M & \longrightarrow & M \end{array} ,$$

where the horizontal sides are trivial Lie algebroids, is an $\mathcal{LA}$-groupoid, and similarly, if $A \to M$ is a Lie algebroid, then the square

$$(5) \qquad\qquad \begin{array}{ccc} A & \longrightarrow & M \\ \downdownarrows & & \downdownarrows \\ A & \longrightarrow & M \end{array} ,$$

where the vertical sides are trivial Lie groupoids, is an $\mathcal{LA}$-groupoid.

In this paper, we approach $\mathcal{LA}$-groupoids from a supergeometric point of view in a way that extends the approach to Lie algebroids, due to Vaintrob [25], in which the sheaf of differential algebras $(\wedge\Gamma(A^*), d_A)$ of a Lie algebroid $A \to M$ is interpreted as the function sheaf of a ($\mathbb{Z}$-graded) supermanifold equipped with a homological vector field. This approach leads us to the notion of a Q-groupoid[2]. A Q-groupoid is a supergroupoid that is equipped with a compatible homological vector field. There is a faithful functor, which we denote as $[-1]$, from the category of $\mathcal{LA}$-groupoids to the category of Q-groupoids.

There is a natural notion of Q-groupoid cohomology (and therefore of $\mathcal{LA}$-groupoid cohomology) that is a generalization of both Lie algebroid cohomology and Lie groupoid cohomology. In general, the cochain complex is a double complex that intertwines the groupoid and algebroid structures of Ω. Of particular interest is the case of the tangent prolongation groupoid of a Lie groupoid G, whose double complex, known as the *de Rham double complex* of G, is a model for the cohomology of the classifying space BG.

The notion of $\mathcal{LA}$-groupoid cohomology is essentially an application of the homotopy-theoretic concept of simplicial structures to the cohomology of Lie algebroids. Recall (see e.g. [1, 4]) that, although the geometric realization of a simplicial manifold does not retain a smooth structure, its (singular) cohomology may be computed in terms of differential forms via the simplicial-de Rham double complex (of which the de Rham double complex of a groupoid is an example). Since an $\mathcal{LA}$-groupoid gives rise to a simplicial object in the category of Lie algebroids, we may view the double complex of an $\mathcal{LA}$-groupoid as a "simplicial-Lie algebroid" double complex. More generally, any Q-groupoid gives rise to a simplicial Q-manifold and, consequently, a "simplicial-Q" double complex. This point of view allows us to intuitively think of $\mathcal{LA}$-groupoid cohomology as the Lie algebroid cohomology of the "geometric realization" of the $\mathcal{LA}$-groupoid. For example, the tangent prolongation groupoid (2) may be viewed as a simplicial model for $T(BG) \to BG$.

There is a close relationship between the supergeometric approach to $\mathcal{LA}$-groupoids and Voronov's supergeometric approach to double Lie algebroids [27]. This relationship will be spelled out in [15] and [17].

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[2]The terminology derives from the term Q-*manifold*, due to Schwarz [22], which refers to a supermanifold equipped with a homological vector field.

The structure of the paper is as follows. In §2, we give a brief introduction to supermanifolds with a $\mathbb{Z}$-grading. In §3, we define supergroupoids and Q-groupoids, and describe the $[-1]$ functor from the category of $\mathcal{LA}$-groupoids to the category of Q-groupoids. In §4, we describe the double complex of a Q-groupoid.

In the remaining sections, we apply the $[-1]$ functor to some examples of $\mathcal{LA}$-groupoids and consider the resulting double complexes. In §5, we reproduce the de Rham double complex, which includes as special cases the simplicial model for equivariant cohomology and a model for orbifold cohomology. In the $\mathcal{LA}$-groupoid point of view, these models naturally generalize to Lie algebroid cohomology. The cotangent prolongation groupoid of a Poisson groupoid is addressed in §6, and in §7 we consider the case of vacant $\mathcal{LA}$-groupoids. Vacant $\mathcal{LA}$-groupoids always arise from a matched pair, consisting of a Lie groupoid and a Lie algebroid that compatibly act on each other.

2. Supermanifolds with a $\mathbb{Z}$-grading

Primarily to fix notation and terminology, we present in this section a brief introduction to $\mathbb{Z}$-graded supermanifolds. A more detailed introduction will come in a future paper [15] (also see [16]). Various notions of supermanifolds with $\mathbb{Z}$-gradings have appeared in the work of Kontsevich [6], Roytenberg [21], Ševera [24], and Voronov [26], among others. All of these notions share the common property of equipping a $\mathbb{Z}_2$-graded supermanifold with an additional $\mathbb{Z}$-grading, whereas we choose to consider supermanifolds that are inherently $\mathbb{Z}$-graded. Our choice requires that functions be polynomial in all coordinates of nonzero degree, so as not to allow, for example, e^ξ for a degree 2 coordinate ξ. We stress, however, that the difference is immaterial to the concerns of the present paper, and readers familiar with another notion of $\mathbb{Z}$-graded supermanifolds will find the material in the sections that follow to be equally valid in their preferred context.

Just as a manifold is locally modelled on $\mathbb{R}^n$, a ($\mathbb{Z}$-graded) supermanifold is locally modelled on a *coordinate superspace* $\mathbb{R}^{\{p_i\}}$.

2.1. **Superdomains and Supermanifolds.** Let $\{p_i\}_{i \in \mathbb{Z}}$ be a nonnegative integer-valued sequence. Denote by $\mathcal{O}^{\{p_i\}}$ the sheaf of graded, graded-commutative algebras on $\mathbb{R}^{p_0}$ defined by

$$(6) \qquad \mathcal{O}^{\{p_i\}}(U) = C^\infty(U) \left[\bigcup_{i \neq 0} \{\xi_i^1, \ldots, \xi_i^{p_i}\} \right]$$

for any open set $U \subseteq \mathbb{R}^{p_0}$, where ξ_i^k is of degree $-i$. In particular, $\xi_i^k \xi_j^\ell = (-1)^{ij} \xi_j^\ell \xi_i^k$.

Definition 2.1. The *coordinate superspace* $\mathbb{R}^{\{p_i\}}$ is the pair $\left(\mathbb{R}^{p_0}, \mathcal{O}^{\{p_i\}} \right)$.

Remark 2.2. The basic premise of supergeometry is that we treat $\mathbb{R}^{\{p_i\}}$ as if it were a space whose sheaf of "smooth functions" is $\mathcal{O}^{\{p_i\}}$. Following this idea, we write $C^\infty\left(\mathbb{R}^{\{p_i\}}\right) \stackrel{\text{def}}{=} \mathcal{O}^{\{p_i\}}$.

There is a natural surjection of sheaves $\text{ev} : C^\infty\left(\mathbb{R}^{\{p_i\}}\right) \to C^\infty\left(\mathbb{R}^{p_0}\right)$, called the *evaluation map*, where the kernel is the ideal generated by all elements of nonzero degree.

Definition 2.3. A *superdomain* $\mathcal{U}$ of dimension $\{p_i\}$ is a pair $(U, C^\infty(\mathcal{U}))$, where U is an open subset of $\mathbb{R}^{p_0}$ and $C^\infty(\mathcal{U}) \stackrel{\text{def}}{=} \mathcal{O}^{\{p_i\}}|_U$. A morphism of superdomains $\mu : \mathcal{U} \to \mathcal{V}$ consists of a smooth map $\mu_0 : U \to V$ and a morphism of sheaves of graded algebras $\mu^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ over μ_0, such that $\text{ev} \circ \mu^* = \mu_0^* \circ \text{ev}$.

Definition 2.4. A *supermanifold* $\mathcal{M}$ of dimension $\{p_i\}$ is a pair $(M, C^\infty(\mathcal{M}))$, where M (the *support*) is a topological space and $C^\infty(\mathcal{M})$ is a sheaf of graded algebras (the *sheaf of functions*) on M that is locally isomorphic to a superdomain of dimension $\{p_i\}$. A morphism of supermanifolds $\mu : \mathcal{M} \to \mathcal{N}$ consists of a map $\mu_0 : M \to N$ and a sheaf morphism $\mu^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$ over μ_0 that is locally a morphism of superdomains.

Remark 2.5. It follows from Definition 2.4 that if $\mathcal{M} = (M, C^\infty(\mathcal{M}))$ is a dimension $\{p_i\}$ supermanifold, then the topological space M automatically has the structure of a p_0-dimensional manifold. The evaluation map describes an embedding of M into $\mathcal{M}$.

In geometric situations, the supermanifolds of interest will often have function sheaves that are nonnegatively-graded. In those situations, the following terminology is useful.

Definition 2.6. A supermanifold $\mathcal{M}$ of dimension $\{p_i\}$ is said to be of *degree* d if p_i is nonzero only when $-d \leq i \leq 0$.

2.2. The functor $[-1]$. Let $E \to M$ be a vector bundle. The sheaf $\wedge\Gamma(E^*)$ may be interpreted as the function sheaf of a supermanifold with support M, which we denote as $[-1]E$. The dimension of $[-1]E$ is $\{p_i\}$, where $p_0 = \dim M$, $p_{-1} = \text{rank}\, E$, and $p_i = 0$ for $i \neq 0, -1$. In particular, $[-1]E$ is a degree 1 supermanifold.

A bundle map

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\downarrow & & \downarrow \\
M & \longrightarrow & M'
\end{array}
$$

induces a sheaf morphism $\phi^* : \wedge\Gamma(E'^*) \to \wedge\Gamma(E^*)$, which may be viewed as a morphism of supermanifolds $[-1]\phi : [-1]E \to [-1]E'$. Thus we have a functor, denoted as $[-1]$, from the category of vector bundles to the category of (degree 1) supermanifolds.

Remark 2.7. The reader should be aware that the notation here differs from that of much of the existing literature, e.g. [6, 21, 24, 26], where the supermanifold with function sheaf $\wedge\Gamma(E^*)$ is denoted as $E[1]$. There are two separate distinctions at work here. The first is that, following a suggestion of Weinstein, we have placed the "degree shift" operator on the left in order to emphasize the fact that it is

a functor [3]. The second distinction is that, in the spirit of supergeometry, we have interpreted the operation $[-1]$ to be a geometric, as opposed to an algebraic, operation. In other words, the *fibres* of $[-1]E$ are of degree -1, whereas in the previous literature $E[1]$ is characterized by the property that the *linear functions* are of degree 1. Because the degree of a homogeneous vector space is opposite in sign to the degree of its dual space, our degree shift operator differs by a sign from the degree shift operator in the existing literature.

2.3. **Homological vector fields.** Let $\mathcal{M}$ be a supermanifold.

Definition 2.8. A *vector field* of degree j on $\mathcal{M}$ is a degree j derivation ϕ of $C^\infty(\mathcal{M})$, i.e. a linear operator such that, for any homogeneous functions $f, g \in C^\infty(\mathcal{M})$,

$$|\phi f| = j + |f| \tag{7}$$

and

$$\phi(fg) = \phi(f)g + (-1)^{j|f|} f\phi(g). \tag{8}$$

The space of degree j vector fields on M is denoted $\mathfrak{X}_j(\mathcal{M})$, and the space of all vector fields is $\mathfrak{X}(\mathcal{M}) \stackrel{\text{def}}{=} \bigoplus_{j \in \mathbb{Z}} \mathfrak{X}_j(\mathcal{M})$.

The bracket $[\phi, \psi] \stackrel{\text{def}}{=} \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$ gives the space of vector fields on $\mathcal{M}$ the structure of a Lie superalgebra. In particular, if ϕ is an odd degree vector field, then $[\phi, \phi] = 2\phi^2$ is not automatically zero.

Definition 2.9. A vector field ϕ on a supermanifold is called *homological* if it is of degree 1 and satisfies the equation $[\phi, \phi] = 0$.

Definition 2.10 ([22])**.** A *Q-manifold* is a supermanifold equipped with a homological vector field.

Let $(\mathcal{M}, \phi)$ and $(\mathcal{N}, \psi)$ be Q-manifolds.

Definition 2.11. A Q-manifold morphism from $\mathcal{M}$ to $\mathcal{N}$ is a morphism of supermanifolds $\mu : \mathcal{M} \to \mathcal{N}$ such that ϕ and ψ are μ-related; that is, if for all $f \in C^\infty(\mathcal{N})$,

$$\mu^*(\psi f) = \phi(\mu^* f). \tag{9}$$

Example 2.12 ([25]). Let $A \to M$ be a Lie algebroid. Then the supermanifold $[-1]A$, equipped with the Lie algebroid differential d_A, is a Q-manifold. Special cases include the odd tangent bundle $[-1]TM$ of a manifold M, equipped with the de Rham differential; the odd cotangent bundle $[-1]T^*M$ of a Poisson manifold M, equipped with the Lichnerowicz-Poisson differential; and $[-1]\mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, equipped with the Chevalley-Eilenberg differential.

Remark 2.13. The Q-manifolds that arise from the construction of Example 2.12 are characterized by the property of being of degree 1. In fact, the $[-1]$ functor gives an equivalence of categories from the category of Lie algebroids to the category of degree 1 Q-manifolds.

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[3]This convention also agrees with the fact that $[-1]E$ is the $\mathbb{Z}$-graded analogue of the $\mathbb{Z}_2$-graded supermanifold ΠE.

3. Supergroupoids and Q-groupoids

A supergroupoid is a groupoid object in the category of supermanifolds. In other words, a supergroupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ is a pair of supermanifolds $(\mathcal{G}, \mathcal{M})$ equipped with surjective submersions $s, t : \mathcal{G} \to \mathcal{M}$ (source and target) and, letting $\mathcal{G}^{(2)}$ be the fibre product $\mathcal{G}\,_s\times_t \mathcal{G}$, maps $m : \mathcal{G}^{(2)} \to \mathcal{G}$ (multiplication), $e : \mathcal{M} \to \mathcal{G}$ (identity), and $i : \mathcal{G} \to \mathcal{G}$ (inverse), satisfying a series of commutative diagrams that describe the various axioms of a groupoid.

Remark 3.1. If $\mathcal{G} \rightrightarrows \mathcal{M}$ is a supergroupoid, then there is an underlying ordinary groupoid $G \rightrightarrows M$, where G and M are the supports of $\mathcal{G}$ and $\mathcal{M}$, respectively.

3.1. **Multiplicative vector fields.** In addition to the multiplication map, there are two natural maps $p_1, p_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ which project onto the first and second component, respectively. We may use these maps to give the following "simplical"[4] characterization of multiplicative vector fields:

Definition 3.2. A vector field ψ on $\mathcal{G}$ is *multiplicative* if there exists a vector field $\psi^{(2)}$ on $\mathcal{G}^{(2)}$ that is p_1-, p_2-, and m-related to ψ.

Remark 3.3. To check whether a vector field ψ is multiplicative, one may first try to obtain a candidate $\psi^{(2)}$ by restricting the product vector field $\psi \times \psi \in \mathfrak{X}(\mathcal{G} \times \mathcal{G})$ to a vector field on the submanifold $\mathcal{G}^{(2)}$. It turns out that $\psi \times \psi$ is tangent to $\mathcal{G}^{(2)}$ if and only if there exists a *base vector field* $\psi^{(0)} \in \mathfrak{X}(\mathcal{M})$ that is s- and t-related to ψ. If this is the case, then it only remains to check whether $\psi \times \psi|_{\mathcal{G}^{(2)}}$ is m-related to ψ.

If ψ is a multiplicative vector field with base vector field $\psi^{(0)}$, then it may be shown (see [16]) that ψ is e-related to $\psi^{(0)}$ and that ψ is i-related to itself. These facts, in addition to Definition 3.2, give us the following proposition, which relates our definition to the one given by Mackenzie and Xu [14]:

Proposition 3.4. *A vector field $\psi \in \mathfrak{X}(\mathcal{G})$ is multiplicative if and only if it is a groupoid morphism[5] $\mathcal{G} \to \bigoplus_{j\in\mathbb{Z}}[j]T\mathcal{G}$.*

3.2. Q-**groupoids.**

Definition 3.5. A Q-*groupoid* is a groupoid object in the category of Q-manifolds or, equivalently, a supergroupoid equipped with a multiplicative, homological vector field.

The following construction extends the $[-1]$ functor to the category of $\mathcal{LA}$-groupoids.

Theorem 3.6. *Let*

$$
\begin{array}{ccc}
\Omega & \longrightarrow & G \\
\downdownarrows & & \downdownarrows \\
A & \longrightarrow & M
\end{array}
$$

---

[4]The three maps p_2, m, and p_1 are face maps for the simplicial (super)manifold associated to $\mathcal{G}$. This is discussed in more detail in §4.

[5]A degree j vector field on a supermanifold $\mathcal{M}$ may be viewed as a section of the bundle $[-j]T\mathcal{M} \to \mathcal{M}$. A general vector field on $\mathcal{M}$ is then a section of the bundle $\bigoplus_{j\in\mathbb{Z}}[j]T\mathcal{M}$. If $\mathcal{G} \rightrightarrows \mathcal{M}$ is a supergroupoid, then $\bigoplus_{j\in\mathbb{Z}}[j]T\mathcal{G}$ inherits a groupoid structure over $T\mathcal{M}$.

be an $\mathcal{LA}$-groupoid. Then $[-1]\Omega$, equipped with the Lie algebroid differential d_Ω, is a Q-groupoid with base $[-1]A$.

Proof. The base vector field is d_A. Since the source and target maps $s, t : \Omega \to A$ are Lie algebroid morphisms, we have that d_Ω and d_A are $[-1]s$- and $[-1]t$-related. We then let $d_\Omega^{(2)} = d_\Omega \times d_\Omega|_{\Omega^{(2)}}$, and observe that this is precisely the Lie algebroid differential for the induced Lie algebroid $\Omega^{(2)} \to G^{(2)}$. Since $m : \Omega^{(2)} \to \Omega$ is a Lie algebroid morphism, we have that $d_\Omega^{(2)}$ is $[-1]m$-related to d_Ω. $\qquad\square$

Remark 3.7. As in Remark 2.13, we may consider the subcategory of degree 1 Q-groupoids. The statement of Remark 2.13 directly generalizes to the following: *The $[-1]$ functor is an equivalence of categories from the category of $\mathcal{LA}$-groupoids to the category of degree 1 Q-groupoids.*

By applying Theorem 3.6 to the tangent prolongation groupoid (2) and the cotangent prolongation groupoid (3), we immediately obtain the examples $[-1]TG \rightrightarrows [-1]TM$, when $G \rightrightarrows M$ is a groupoid, and $[-1]T^*G \rightrightarrows [-1]A^*$, when $G \rightrightarrows M$ is a Poisson groupoid. These examples are described in more detail in §5 and §6, respectively.

Interesting examples of Q-groupoids that are not of degree 1 may arise from Courant algebroids. In particular, the correspondence, due to Roytenberg [21] (also see [24]), between Courant algebroids and degree 2 symplectic Q-manifolds leads us to consider the following example of a degree 2 symplectic Q-groupoid.

Example 3.8. Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. Then $[-1]T^*G \rightrightarrows [-1]A^*$ is a degree 1 symplectic groupoid. We may apply the $[-1]T$ functor to obtain the Q-groupoid $[-1]T([-1]T^*G) \rightrightarrows [-1]T([-1]A^*)$. The degree 2 symplectic structure on $[-1]T([-1]T^*G)$ arises from the canonical symplectic structure on T^*G, and, similarly, the Poisson structure on $[-1]T([-1]A^*)$ arises from the linear Poisson structure on A^*. The relation to Courant algebroids is that, under Roytenberg's correspondence, $[-1]T([-1]T^*G)$ is the degree 2 symplectic Q-manifold associated to the standard Courant algebroid $TG \oplus T^*G$. Although $[-1]T([-1]A^*)$ is only Poisson, we may associate it to a structure on $A \oplus T^*M$ similar to that of a Courant algebroid, except that the bilinear form may be degenerate. We observe that $TG \oplus T^*G$ has a natural Lie groupoid structure over the dual bundle $TM \oplus A^*$, and we suggest that this is a first example of a more general notion of what might be called "Courant groupoids".

4. Cohomology of Q-groupoids

Let $G \rightrightarrows M$ be a Lie groupoid. Then there is an associated simplicial manifold

$$(10) \qquad \cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\Rrightarrow}} G^{(2)} \overset{\longrightarrow}{\underset{\longrightarrow}{\Longrightarrow}} G^{(1)} = G \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} G^{(0)} = M,$$

where $G^{(q)}$ is the manifold of compatible q-tuplets of elements of G. The face maps $\sigma_i^q : G^{(q)} \to G^{(q-1)}$ are defined as follows for $q > 1$:

$$\sigma_0^q(g_1, \ldots, g_q) = (g_2, \ldots, g_q),$$
$$\sigma_i^q(g_1, \ldots, g_q) = (g_1, \ldots, g_i g_{i+1}, \ldots g_q), \qquad 0 < i < q,$$
$$\sigma_q^q(g_1, \ldots, g_q) = (g_1, \ldots, g_{q-1}).$$

Additionally, we have $\sigma_0^1 = s$ and $\sigma_1^1 = t$. The degeneracy maps $\Delta_i^q : G^{(q)} \to G^{(q+1)}$ are the maps that insert unit elements. The simplicial manifold (10) is known as the *nerve* of G.

Remark 4.1. The simplicial point of view introduces duplicate notation for various maps. Particularly, in relation to the notation of §3, we have the following: $\sigma_0^2 = p_2$, $\sigma_1^2 = m$, and $\sigma_2^2 = p_1$.

If $\mathcal{G} \rightrightarrows \mathcal{M}$ is a supergroupoid, then the nerve of $\mathcal{G}$ is a simplicial supermanifold. Suppose that $\mathcal{G}$ is a Q-groupoid with homological vector field ψ. Then, since ψ is multiplicative, there exist natural lifts of ψ to vector fields $\psi^{(q)}$ on $\mathcal{G}^{(q)}$ for all $q \geq 0$, satisfying the property that $\psi^{(q)}$ and $\psi^{(q-1)}$ are σ_i^q-related[6] for all $0 \leq i \leq q$. It immediately follows that the action of the vector fields $\psi^{(q)}$ as derivations commutes with the groupoid coboundary operator $\delta^q : C^\infty(\mathcal{G}^{(q-1)}) \to C^\infty(\mathcal{G}^{(q)})$, defined as

$$\delta^q = \sum_{i=0}^{q} (-1)^i \left(\sigma_i^q\right)^* .$$

Let $C^{p,q}(\mathcal{G}) \overset{\text{def}}{=} C_p^\infty(\mathcal{G}^{(q)})$ denote the space of degree p functions on $\mathcal{G}^{(q)}$. Then $(C^{p,q}(\mathcal{G}), \delta, \psi)$ is a double complex with total differential $D = \psi + (-1)^p \delta$. We call the cohomology ring of the total complex the *Q-groupoid cohomology of $\mathcal{G}$* and denote it as $H_\psi(\mathcal{G})$. In the case where $\mathcal{G}$ arises from an $\mathcal{LA}$-groupoid

$$\begin{array}{ccc} \Omega & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ A & \longrightarrow & M \end{array} ,$$

then we may refer to $H_\psi(\mathcal{G})$ as the *$\mathcal{LA}$-groupoid cohomology* of Ω. Let us first consider the trivial examples.

Example 4.2. If $G \rightrightarrows M$ is a Lie groupoid, then the Q-groupoid that arises from the $\mathcal{LA}$-groupoid (4) is simply $G \rightrightarrows M$ with the zero homological vector field. In this case, $C^{p,q}(G) = 0$ for $p \neq 0$, and the total complex may be directly identified with the smooth Eilenberg-Maclane complex of G.

Example 4.3. If $A \to M$ is a Lie algebroid, then the Q-groupoid arising from the $\mathcal{LA}$-groupoid (5) is $[-1]A \rightrightarrows [-1]A$ with the homological vector field d_A. In this case, $C^{p,q}([-1]A) = \wedge^p \Gamma(A^*)$ for all q. Since $\sigma_i^q = id$ for all i, we have that $\delta^q = id$ for even q and $\delta^q = 0$ for odd q. At the first stage, the spectral sequence for this double complex collapses to the Lie algebroid cohomology complex $(\wedge \Gamma(A^*), d_A)$.

From Examples 4.2 and 4.3, we see that $\mathcal{LA}$-groupoid cohomology generalizes both Lie algebroid cohomology and Lie groupoid cohomology. In the sections that follow, we will consider more interesting examples.

5. The de Rham double complex

Let $G \rightrightarrows M$ be a Lie groupoid. We apply the $[-1]$ functor to the tangent prolongation groupoid (2) and obtain the Q-groupoid $[-1]TG \rightrightarrows [-1]TM$, where the homological vector field is the de Rham differential d.

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[6]It is also the case that $\psi^{(q)}$ and $\psi^{(q+1)}$ are Δ_i^q-related for all $0 \leq i \leq q$. In other words, the nerve of a Q-groupoid is a simplicial Q-manifold.

For each q, there is a natural identification of $([-1]TG)^{(q)}$ with $[-1]T(G^{(q)})$. Thus the space of cochains is

$$(11) \qquad\qquad C^{p,q}([-1]TG) = \Omega^p(G^{(q)}).$$

The double complex $(\Omega^p(G^{(q)}), \delta, d)$ is known as the *de Rham double complex* [7] of G. The de Rham double complex is a special case of the de Rham complex of a simplicial manifold [1], which is a model for the cohomology of the geometric realization. In this case, the simplicial manifold is the nerve of G, whose geometric realization is [23] the classifying space BG. Therefore, the $\mathcal{LA}$-groupoid cohomology of TG is equal to $H^\bullet(BG; \mathbb{R})$.

Since the de Rham double complex is already well-known, $\mathcal{LA}$-groupoid cohomology does not provide any new information about classifying spaces. However, as we illustrate in §5.1 and §5.2, the $\mathcal{LA}$-groupoid point of view may be used to produce interesting generalizations of the de Rham double complex.

Example 5.1. Let Γ be a Lie group that acts (from the right) on a manifold M. Then it may be shown that the classifying space of the action groupoid $M \times \Gamma \rightrightarrows M$ is the homotopy quotient $M \times E\Gamma/\Gamma$. The de Rham double complex of the action groupoid therefore computes the equivariant cohomology $H_\Gamma^\bullet(M)$.

Example 5.2. Let $G \rightrightarrows M$ be an étale groupoid representing an orbifold X. Moerdijk and Pronk [19] have shown that the orbifold cohomology is isomorphic to the cohomology of BG. Therefore, the de Rham double complex of G is a model for the orbifold cohomology of X.

Example 5.3. The double complex of $[-1]T([-1]T^*G) \rightrightarrows [-1]T([-1]A^*)$ (see Example 3.8) is the de Rham double complex of the supergroupoid $[-1]T^*G \rightrightarrows [-1]A^*$. Since this supergroupoid has a linear structure over the ordinary groupoid $G \rightrightarrows M$, the double complex retracts[7] to the de Rham double complex of G, and the Q-groupoid cohomology is just $H^\bullet(BG)$. It may be interesting to see if one can "twist" the homological vector field on $[-1]T([-1]T^*G)$ by introducing a closed 3-form on G.

5.1. **Equivariant Lie algebroid cohomology.** In light of Example 5.1, we may view $\mathcal{LA}$-groupoid cohomology as a generalization of equivariant cohomology. We will now describe how a model for equivariant Lie algebroid cohomology may be obtained as a special case of $\mathcal{LA}$-groupoid cohomology.

Equivariant Lie algebroid cohomology was originally introduced by Ginzburg [5] as a natural generalization of his theory of *equivariant Poisson cohomology*. More recently, Bruzzo, et al. [2] have proven a corresponding localization theorem. The model introduced by Ginzburg is a generalization of the Cartan model, which is in terms of the infinitesimal data of the action. In this section, we introduce a model that is noninfinitesimal and is therefore useful, for example, in the case of a discrete group action. We provide an argument for interpreting this model as giving the cohomology of a "homotopy quotient Lie algebroid". It seems reasonable to expect, as in the case of equivariant de Rham cohomology, that under suitable conditions

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[7]In general for a degree 1 supergroupoid $\mathcal{G} \rightrightarrows \mathcal{M}$, there exists a natural linear structure over the underlying ordinary groupoid $G \rightrightarrows M$, and it follows that $H^\bullet(B\mathcal{G}) = H^\bullet(BG)$. However, it seems plausible that one could construct a supergroupoid (possibly with both positively- and negatively-graded coordinates) for which there does not exist a linear structure over the underlying ordinary groupoid, and for which $H^\bullet(B\mathcal{G})$ does not equal $H^\bullet(BG)$.

(specifically, when the acting group is connected and compact) our model will coincide with Ginzburg's. However, it will be simplest to leave this issue until after the machinery of Q-algebroids has been introduced in [15].

Let $A \to M$ be a Lie algebroid and let G be a Lie group. We equip $A \times TG \to M \times G$ with the product Lie algebroid structure.

Definition 5.4. An A-*action* of G is a (right) action of TG on A such that the action map $\widetilde{s} : A \times TG \to A$ is an algebroid morphism.

Remark 5.5. An A-*action* necessarily possesses an underlying action map $s : M \times G \to M$. In practice, one will often begin with an action of G on M which one will then seek to extend to an A-action.

Proposition 5.6. *Let $\widetilde{s}$ be an A-action of G. Then*

$$(12) \qquad \begin{array}{ccc} A \times TG & \longrightarrow & M \times G \\ \downdownarrows & & \downdownarrows \\ A & \longrightarrow & M \end{array}$$

is an $\mathcal{LA}$-groupoid, where the vertical sides are action groupoids.

Proof. The result is immediate from Definition 5.4 and the fact that $TG \to G$ is an "$\mathcal{LA}$-group". $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 5.7. Let $\widetilde{s} : A \times TG \to A$ be an A-action. The *equivariant Lie algebroid cohomology* of A is the $\mathcal{LA}$-groupoid cohomology of (12).

Example 5.8. Given an action map $s : M \times G \to M$, then $Ts : TM \times TG \to TM$ is the unique TM-action that lifts s. By making the identification $TM \times TG = T(M \times G)$, we recover the de Rham double complex of $G \times M$ and the usual notion of equivariant cohomology.

Example 5.9. If G is a discrete group, then any action $\widetilde{s} : A \times G \to A$ where G acts by Lie algebroid automorphisms is an A-action. The resulting double complex is then of the form $(\wedge^q \Gamma(A^*) \times G^q, d_A, \delta)$, where δ is the coboundary operator for group cohomology with coefficients in $\wedge^q \Gamma(A^*)$. In the case where G is finite, the fact that $H^n(G; \wedge^q \Gamma(A^*))$ vanishes for $n > 0$ implies that the spectral sequence collapses to the complex $\left((\wedge^q \Gamma(A^*))^G, d_A \right)$ of invariant Lie algebroid forms.

We will now describe how the $\mathcal{LA}$-groupoid (12) may be viewed as a "homotopy quotient". Although EG does not naturally have a manifold structure, we may represent EG by the simplicial manifold

$$(13) \qquad \qquad \cdots \; \substack{\longrightarrow\\[-0.5ex]\longrightarrow\\[-0.5ex]\longrightarrow\\[-0.5ex]\longrightarrow} \; G^3 \; \substack{\longrightarrow\\[-0.5ex]\longrightarrow\\[-0.5ex]\longrightarrow} \; G^2 \; \substack{\longrightarrow\\[-0.5ex]\longrightarrow} \; G.$$

The simplicial manifold (13) is just the nerve of the pair groupoid $G \times G \rightrightarrows G$. We may take the tangent bundle of this pair groupoid to obtain the $\mathcal{LA}$-groupoid

$$(14) \qquad \begin{array}{ccc} TG^2 & \longrightarrow & G^2 \\ \downdownarrows & & \downdownarrows \\ TG & \longrightarrow & G \end{array} ,$$

the cohomology of which may be easily shown to be trivial. The $\mathcal{LA}$-groupoid (14) plays the role of $T(EG) \to EG$. We then consider the product $\mathcal{LA}$-groupoid

(15)
$$
\begin{array}{ccc}
A \times TG^2 & \longrightarrow & M \times G^2 \\
\Downarrow\Downarrow & & \Downarrow\Downarrow \\
A \times TG & \longrightarrow & M \times G
\end{array}
\ .
$$

The space of $\mathcal{LA}$-groupoid cochains for (15) is essentially $\wedge\Gamma(A^*) \otimes \Omega(G^\bullet)$, and the total differential splits into $d_A \otimes 1$ and $1 \otimes D_G$, where D_G is the total differential for (14). Since the cohomology of (14) is trivial, we conclude that the cohomology of (15) is equal to the Lie algebroid cohomology of A.

Finally, we observe that (12) is the quotient of (15) by the diagonal action of the $\mathcal{LA}$-group $TG \to G$, where $TG \to G$ acts from the right on (14) by left-multiplication by the inverse on each component. In this sense, we may consider (12) to be the homotopy quotient Lie algebroid

$$
\begin{array}{c}
(A \times T(EG))/TG \\
\downarrow \\
(M \times EG)/G
\end{array}
\ .
$$

If the action of TG on A is free and proper, then the Lie algebroid structure of A passes to the quotient $A/TG \to M/G$. In this case, one can show that the retract from the double complex of (15) to $(\wedge\Gamma(A^*), d_A)$ respects the basic subcomplexes, so the equivariant Lie algebroid cohomology equals the Lie algebroid cohomology $H^*(A/TG)$ of the quotient. This fact can be used, for example, to obtain characteristic classes in $H^*(A/TG)$ associated to the "principal G-bundle of Lie algebroids" $A \to A/TG$.

5.2. **Lie algebroid structures over orbifolds.** One could, motivated by Example 5.2, use the $\mathcal{LA}$-groupoid point of view to represent a Lie algebroid over an orbifold, as follows. Let X be an orbifold represented by an étale Lie groupoid $G \rightrightarrows M$. Then an étale $\mathcal{LA}$-groupoid

$$
\begin{array}{ccc}
\Omega & \longrightarrow & G \\
\Downarrow\Downarrow & & \Downarrow\Downarrow \\
A & \longrightarrow & M
\end{array}
$$

may be viewed as representing a Lie algebroid[8] over X.

In this case, the $\mathcal{LA}$-groupoid cohomology may be interpreted as an "orbifold Lie algebroid cohomology"; for example, if X has a Poisson structure then one could

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[8]Strictly speaking, one should check that the notion of Lie algebroids over orbifolds is well-defined, in the sense of respecting Morita equivalences (see e.g. [18]). We have not made an attempt to show this, and as such, the ideas in this section should be considered tentative.

define orbifold Poisson cohomology by constructing an $\mathcal{LA}$-groupoid of the form[9]

(16)
$$
\begin{array}{ccc}
T^*G & \longrightarrow & G \\
\downdownarrows & & \downdownarrows \\
T^*M & \longrightarrow & M
\end{array}
$$
.

Example 5.10. If $X = M/\Gamma$ is the global quotient of a manifold M by the action of a finite group Γ, then X may be represented by the action groupoid $M \times \Gamma \rightrightarrows M$. A Lie algebroid over X will then be represented by an $\mathcal{LA}$-groupoid of the form

$$
\begin{array}{ccc}
A \times \Gamma & \longrightarrow & M \times \Gamma \\
\downdownarrows & & \downdownarrows \\
A & \longrightarrow & M
\end{array}
$$
,

where Γ acts on A by Lie algebroid automorphisms. In other words, the orbifold Lie algebroid cohomology in this case coincides with the equivariant Lie algebroid cohomology $H_\Gamma(A)$.

6. Poisson groupoids

Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \rightrightarrows M$. There is a naturally induced Lie groupoid structure [3] on $T^*G \rightrightarrows A^*$; this is known as the cotangent prolongation groupoid, or simply the cotangent groupoid. If, furthermore, G is a Poisson manifold with Poisson bivector π, then there is an associated Lie algebroid structure on the bundle $T^*G \rightrightarrows G$. It is then reasonable to ask whether the groupoid structure on T^*G is compatible with the Lie algebroid structure, or more precisely, if the square (3) is an $\mathcal{LA}$-groupoid. It is a theorem, due to Mackenzie [11], that this is the case if and only if (G, π) is a Poisson groupoid.

Now suppose that $(G, \pi) \rightrightarrows M$ is a Poisson groupoid. We may apply the $[-1]$ functor to the cotangent prolongation groupoid to obtain the Q-groupoid $[-1]T^*G \rightrightarrows [-1]A^*$, where the homological vector field is the Lichnerowicz-Poisson differential d_π. The first two rows of the resulting double complex are as follows:

(17)
$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle-\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^\infty(G) & \xrightarrow{d_\pi} & \mathfrak{X}(G) & \xrightarrow{d_\pi} & \mathfrak{X}^2(G) & \xrightarrow{d_\pi} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle-\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^\infty(M) & \xrightarrow{d_{A^*}} & \Gamma(A) & \xrightarrow{d_{A^*}} & \wedge^2\Gamma(A) & \xrightarrow{d_{A^*}} & \cdots
\end{array}
$$

We may explicitly describe the first vertical map $\delta^1 : \wedge^\bullet\Gamma(A) \to \mathfrak{X}^\bullet(G)$. The face maps σ_i^1 are algebra homomorphisms completely determined by the properties

(18)
$$
\left(\sigma_0^1\right)^*(X) = \overleftarrow{X},
$$

(19)
$$
\left(\sigma_1^1\right)^*(X) = \overrightarrow{X},
$$

---

[9]Note that when $G \rightrightarrows M$ is an étale groupoid, there naturally exists a cotangent groupoid $T^*G \rightrightarrows T^*M$. This is clearly different from the cotangent prolongation groupoid (3), which exists even when $G \rightrightarrows M$ is not étale.

for $X \in \Gamma(A)$, where $\overleftarrow{X}$ and $\overrightarrow{X}$ are, respectively, the right- and left-invariant vector fields associated to X. Therefore $\ker \delta^1$ consists of multisections $X \in \wedge^\bullet \Gamma(A)$ such that $\overleftarrow{X} = \overrightarrow{X}$. It is difficult to describe explicitly the second vertical map, since there does not seem to be a simple description of the cochains that would appear in the higher rows; however, it may be seen [20] that $\ker \delta^2$ consists of multiplicative multivector fields on G.

Example 6.1. In the case of a Poisson-Lie group (G, π), the space of (p, q)-cochains is $\wedge^p \mathfrak{g} \otimes C^\infty(G^q)$, and δ is the differential for the smooth group cohomology with coefficients in $\wedge \mathfrak{g}$. If G is compact, then the spectral sequence collapses to $\left((\wedge \mathfrak{g})^G, d_{\mathfrak{g}^*} \right)$, and we obtain the G-invariant Lie algebra cohomology of $\mathfrak{g}^*$.

7. Vacant $\mathcal{LA}$-groupoids

An $\mathcal{LA}$-groupoid (1) is said to be *vacant* if it has trivial core[10] or, equivalently, if the induced map $\Omega \to G \times_M A$ is a diffeomorphism. Since the fibres of this map are vector spaces, it suffices to count dimensions in order to check whether an $\mathcal{LA}$-groupoid is vacant. Examples of vacant $\mathcal{LA}$-groupoids include the $\mathcal{LA}$-groupoids representing Lie algebroid structures over orbifolds (§5.2) and the cotangent groupoid (§6) of a Poisson Lie group.

Mackenzie [10] has shown that every vacant $\mathcal{LA}$-groupoid is isomorphic to a "matched pair" $\mathcal{LA}$-groupoid $G \bowtie A$, constructed out of a compatible pair of actions of G and A on each other. We will review the matched pair construction and then describe the double complexes arising from vacant $\mathcal{LA}$-groupoids.

Let us first recall the notions of groupoid and Lie algebroid actions [9]. If $G \rightrightarrows M$ is a Lie groupoid, then a left action of G on a vector bundle[11] $E \to M$ is a linear map $s^*(E) \to E$, $(g, v) \mapsto g(v)$, such that the diagram

(20)
$$
\begin{array}{ccc}
s^*(E) & \longrightarrow & G \\
\downarrow & & \downarrow t \\
E & \longrightarrow & M
\end{array}
$$

commutes and such that, for all $(g, h, v) \in (s \circ m)^*(E)$, $g(h(v)) = (gh)(v)$.

A left action of G on E induces a Lie groupoid structure on $s^*(E) \rightrightarrows E$, where the source and target maps are given by

(21) $$\widetilde{s}(g, v) = v,$$

(22) $$\widetilde{t}(g, v) = g(v).$$

A pair $((g, v), (h, w))$ is composable if $v = h(w)$. It follows, in particular, that g and h are composable elements of G, and the multiplication is then defined as $(g, v) \cdot (h, w) = (gh, w)$. We leave descriptions of the identity and inverse maps as an exercise for the reader.

Now let $A \to M$ be a Lie algebroid, and let $P \xrightarrow{\pi} M$ be a submersion. Then a (right) action of A on P is a Lie algebra homomorphism $\widetilde{\rho} : \Gamma(A) \to \mathfrak{X}(P)$ that

---

[10]See [10] for a description of the core of an $\mathcal{LA}$-groupoid and its induced Lie algebroid structure.

[11]One can, of course, define Lie groupoid actions on general fibre bundles, but vector bundle actions are sufficient for the present purposes.

lifts the anchor map $\rho : \Gamma(A) \to \mathfrak{X}(M)$, in the sense that for all $X \in \Gamma(A)$, $\widetilde{\rho}(X)$ is π-related to $\rho(X)$.

An action of A on P induces a Lie algebroid structure on $\pi^*(A) \to P$, as follows. The action map $\widetilde{\rho}$ may be extended by $C^\infty(P)$-linearity to obtain the map (which we will also denote $\widetilde{\rho}$)

$$(23) \qquad\qquad \widetilde{\rho} : \Gamma(\pi^*(A)) = C^\infty(P) \otimes \Gamma(A) \to \mathfrak{X}(G),$$

which is the anchor map for the induced Lie algebroid. The Lie bracket of sections is defined by setting

$$(24) \qquad\qquad [1 \otimes X, 1 \otimes Y] = 1 \otimes [X, Y]$$

for $X, Y \in \Gamma(A)$ and extending by the Leibniz rule. The Jacobi identity follows from the Jacobi identity for the bracket on $\Gamma(A)$ and the fact that $\widetilde{\rho}$ is a Lie algebra homomorphism.

Let $G \rightrightarrows M$ be a Lie groupoid and let $A \to M$ be a Lie algebroid (not necessarily the Lie algebroid of G), such that G is equipped with an action on A, and A is equipped with an action on $G \xrightarrow{s} M$. Then, based on the above discussion, we may form a square

$$(25) \qquad\qquad\qquad \begin{array}{ccc} s^*(A) & \longrightarrow & G \\ \big\|\big\downarrow\big\downarrow & & \big\|\big\downarrow\big\downarrow \\ A & \longrightarrow & M \end{array} \quad,$$

where the horizontal sides are Lie algebroids and the vertical sides are Lie groupoids. It is automatically true that $\widetilde{s}$ is a Lie algebroid homomorphism; if $\widetilde{t}$ and $\widetilde{m}$ are also Lie algebroid homomorphisms, then (25) is an $\mathcal{LA}$-groupoid.

Definition 7.1. A *groupoid-algebroid matched pair* is a Lie groupoid $G \rightrightarrows M$ and a Lie algebroid $A \to M$ equipped with mutual actions such that $\widetilde{t}$ and $\widetilde{m}$ are Lie algebroid morphisms.

Remark 7.2. The reader may wish to see Mackenzie [10] for a more concrete set of compatibility conditions for groupoid-algebroid matched pairs.

Suppose that (G, A) is a matched pair. We apply the $[-1]$ functor to (25) to obtain the Q-groupoid $s^*([-1]A) \rightrightarrows [-1]A$, whose homological vector field we denote as $d_{\widetilde{A}}$. The algebra of functions on $s^*([-1]A)$ is $C^\infty(G) \otimes_s \wedge \Gamma(A^*)$. For the higher groupoid cochains, we observe that

$$(s^*([-1]A))^{(q)} = (s \circ p_q^q)^*([-1]A),$$

where $p_q^q : G^{(q)} \to G$ is the projection map onto the last component. So

$$C^\infty\left((s^*([-1]A))^{(q)}\right) = C^\infty(G^{(q)}) \otimes_{s \circ p_q^q} \wedge \Gamma(A^*),$$

and the space of (p, q)-cochains for the double complex is therefore

$$(26) \qquad\qquad C^{p,q}(s^*([-1]A)) = C^\infty(G^{(q)}) \otimes_{s \circ p_q^q} \wedge^p \Gamma(A^*).$$

As is already clear from (26), the double complex intertwines the Lie groupoid cohomology of G and the Lie algebroid cohomology of A. If G has compact t-fibres, then the spectral sequence collapses at the first stage to the G-invariant Lie algebroid complex of A.

Example 7.3 ([13, 16]). Let M be a Poisson manifold, and let H be a Poisson-Lie group with a Poisson action $s : M \times H \to M$. Then there an action of the groupoid $M \times H$ on T^*M, given by the map $\widetilde{t} : s^*(T^*M) \to T^*M$, sending (x, g, η), where $\eta \in T^*_{xg}M$, to $r_g^*\eta$, where r_g is the map given by right-multiplication by g. Additionally, there is an action of T^*M on $M \times H$, given by the map $\widetilde{\rho} : \Omega^1(M) \to \mathfrak{X}(M \times G)$, $\alpha \mapsto \widetilde{\pi}^\sharp(s^*\alpha)$, where $\widetilde{\pi}^\sharp : \Omega^1(M \times H) \to \mathfrak{X}(M \times H)$ arises from the product Poisson structure on $M \times H$. This pair of actions gives $(M \times H, T^*M)$ the structure of a matched pair, so that

(27)
$$
\begin{array}{ccc}
s^*(T^*M) & \longrightarrow & M \times H \\
\downdownarrows & & \downdownarrows \\
T^*M & \longrightarrow & M
\end{array}
$$

is an $\mathcal{LA}$-groupoid.

This $\mathcal{LA}$-groupoid was introduced by Mackenzie [13] in order to describe a general procedure for Poisson reduction. By applying the Lie functor to the vertical sides of (27), one obtains the double Lie algebroid corresponding to the matched pair Lie algebroid structure, due to Lu [8], on $(M \times \mathfrak{h}) \oplus T^*M$.

If H is compact, then the $\mathcal{LA}$-groupoid cohomology of (27) is equal to the H-invariant Poisson cohomology of M.

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