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L ∞ -Algebra Actions

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L_∞ -ALGEBRA ACTIONS

RAJAN MEHTA AND MARCO ZAMBON

ABSTRACT. We define the notion of action of an L_∞ -algebra \mathfrak{g} on a graded manifold \mathcal{M} , and show that such an action corresponds to a homological vector field on $\mathfrak{g}[1] \times \mathcal{M}$ of a specific form. This generalizes the correspondence between Lie algebra actions on manifolds and transformation Lie algebroids. In particular, we consider actions of \mathfrak{g} on a second L_∞ -algebra, leading to a notion of “semidirect product” of L_∞ -algebras more general than those we found in the literature.

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1. INTRODUCTION

Given a Lie algebra \mathfrak{g} and a manifold M , an infinitesimal action of \mathfrak{g} on M can be encoded by a *transformation Lie algebroid* structure on the vector bundle $\mathfrak{g} \times M$ over M . Whereas the definition of the action involves the infinite-dimensional space of vector fields on M , the transformation Lie algebroid is a finite-dimensional object. Transformation Lie algebroid structures on $\mathfrak{g} \times M$ are characterized by the property that the projection $\mathfrak{g} \times M \rightarrow \mathfrak{g}$ is a Lie algebroid morphism.

In this note we extend this result by passing from ordinary geometry to the setting of graded geometry. That is, given an L_∞ -algebra \mathfrak{g} and a graded manifold \mathcal{M} , we define the notion of L_∞ -action of \mathfrak{g} on \mathcal{M} (Definition 4.3). We show that such an action can be encoded by a homological vector field on $\mathfrak{g}[1] \times \mathcal{M}$, and that the homological vector fields arising in this way are characterized by the property that the projection map $\mathfrak{g}[1] \times \mathcal{M} \rightarrow \mathfrak{g}[1]$ is a Q -manifold morphism (see Theorem 4.4). In other words, $\mathfrak{g} \times \mathcal{M}$ becomes a *transformation L_∞ -algebroid* over \mathcal{M} .

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The zero component of the L_∞ -action amounts to a homological vector field $Q_{\mathcal{M}}$, giving \mathcal{M} the structure of a Q -manifold. On the other hand, if \mathcal{M} is already a Q -manifold, then we may consider L_∞ -actions whose zero component agrees with the fixed homological vector field on \mathcal{M} .

Our main example is the case where \mathcal{M} is the Q -manifold associated to another L_∞ -algebra \mathfrak{h} . In this case, transformation L_∞ -algebroids are just L_∞ -algebra extensions of \mathfrak{g} by \mathfrak{h} . Special cases include L_∞ -modules, representations up to homotopy, and the “adjoint representation” of an L_∞ -algebra (§6).

Another geometrically interesting example arises when \mathcal{M} is a degree 1 N -manifold (§7). In this case, $(\mathcal{M}, Q_{\mathcal{M}})$ is a degree 1 NQ -manifold, which is equivalent to a Lie algebroid.

One possible application of this work is the integration of certain Q -manifold extensions, as follows. Given an L_∞ -algebra \mathfrak{g} and a graded manifold \mathcal{M} , by Theorem 4.4, *any* Q -manifold structure on $\mathfrak{g}[1] \times \mathcal{M}$ projecting to the one on $\mathfrak{g}[1]$ arises from an L_∞ -action of \mathfrak{g} on \mathcal{M} . In some cases this action can be integrated, and the integrated action can be used to construct the integration of the Q -manifold $\mathfrak{g}[1] \times \mathcal{M}$. This integration procedure has been worked out by Sheng and Zhu in [21, 22] when \mathfrak{g} is Lie algebra, \mathcal{M} is a 2-term complex of vector spaces, and the L_∞ -action is linear (a representation up to homotopy). It would be particularly interesting to apply this procedure to integrate extensions of L_∞ -algebras.

We expect our results to hold more generally when $\mathfrak{g}[1]$ is replaced by an arbitrary Q -manifold. Such a result would be relevant to the integration of Courant algebroids [13, 17, 23].

Conventions: Multibrackets of L_∞ -algebras are always denoted by $[\dots]_k$. They are graded skew-symmetric, of degree $2 - k$. Multibrackets of $L_\infty[1]$ -algebras are denoted by $\{\dots\}_k$. They are graded symmetric, of degree one.

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2. REVIEW: L_∞ -ALGEBRAS AS Q -MANIFOLDS

In this section we review L_∞ -algebras and their description in terms of Q -manifolds ([12],[20, §1.1]).

Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a graded vector space. Letting $TV := \bigoplus_{i \geq 0} T^i V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \dots$ be the tensor algebra over V , we define the graded symmetric algebra over V as

$$SV := TV / \langle v \otimes v' - (-1)^{|v||v'|} v' \otimes v \rangle,$$

where $|v|$ denotes the degree of a homogeneous element $v \in V$. Similarly, the graded skew-symmetric algebra over V is defined as

$$\wedge V := TV / \langle v \otimes v' + (-1)^{|v||v'|} v' \otimes v \rangle.$$

We introduce the notation $S^+V := \bigoplus_{i \geq 1} S^i V$ and $\wedge^+V := \bigoplus_{i \geq 1} \wedge^i V$.

For $j \geq 1$, let S_j denote the permutation group in j letters. For any homogeneous elements $v_1, \dots, v_j \in V$ and $\tau \in S_j$, the *Koszul signs* $\epsilon(\tau)$ and $\chi(\tau)$ are defined by the equations

$$v_{\tau_1} \cdots v_{\tau_j} = \epsilon(\tau) v_1 \cdots v_j$$

and

$$v_{\tau_1} \wedge \cdots \wedge v_{\tau_j} = \chi(\tau)v_1 \wedge \cdots \wedge v_j,$$

where the products are taken in SV and $\wedge V$ respectively. The two signs are related by the equation $\chi(\tau) = \text{sgn}(\tau)\epsilon(\tau)$. We stress that $\epsilon(\tau)$ and $\chi(\tau)$ depend on the v_i in addition to τ , although the notation does not make this dependence explicit.

Recall that $\tau \in S_n$ is called an $(i, n-i)$ -unshuffle if it satisfies the inequalities $\tau(1) < \cdots < \tau(i)$ and $\tau(i+1) < \cdots < \tau(n)$. The set of $(i, n-i)$ -unshuffles is denoted by $\text{Sh}(i, n-i)$. Following [12, Def. 2.1]¹ and [10, Def. 5], we make the following definitions.

Definition 2.1. An L_∞ -algebra is a graded vector space V equipped with a collection of linear maps $[\cdots]_k: \wedge^k V \rightarrow V$ of degree $2-k$, for $k \geq 1$, such that

$$\sum_{i=1}^n (-1)^{i(n-i)} \sum_{\tau \in \text{Sh}(i, n-i)} \chi(\tau)[[v_{\tau(1)}, \dots, v_{\tau(i)}]_i, v_{\tau(i+1)}, \dots, v_{\tau(n)}]_{n-i+1} = 0$$

for every collection of homogeneous elements $v_1, \dots, v_n \in V$.

Definition 2.2. An $L_\infty[1]$ -algebra² is a graded vector space V equipped with a collection of linear maps $\{\cdots\}_k: S^k V \rightarrow V$ of degree 1, for $k \geq 1$, such that

$$\sum_{i=1}^n \sum_{\tau \in \text{Sh}(i, n-i)} \epsilon(\tau)\{\{v_{\tau(1)}, \dots, v_{\tau(i)}\}_i, v_{\tau(i+1)}, \dots, v_{\tau(n)}\}_{n-i+1} = 0$$

for every collection of homogeneous elements $v_1, \dots, v_n \in V$.

There is a bijection [25, Rem. 2.1] between L_∞ -algebra structures on a graded vector space V and $L_\infty[1]$ -algebra structures on $V[1]$, the graded vector space defined by $(V[1])_i := V_{i+1}$. The multibrackets are related by the *décalage isomorphism*

$$(1) \quad (\wedge^n V)[n] \cong S^n(V[1]), \quad v_1 \cdots v_n \mapsto v_1 \cdots v_n \cdot (-1)^{(n-1)|v_1| + \cdots + 2|v_{n-2}| + |v_{n-1}|}.$$

We refer to an $L_\infty[1]$ -algebra structure as a *Lie[1] algebra* when only $\{\cdots\}_2$ is non-trivial, and we use the terminology *DGL[1]-algebra* when only $\{\cdots\}_1$ and $\{\cdots\}_2$ are non-trivial.

We provide the notion of $L_\infty[1]$ -algebra morphism [10, Def. 6] [20, Def. 1.7] only in a special case.

Definition 2.3. Let V be an $L_\infty[1]$ -algebra with multibrackets $\{\cdots\}$, and let W be a DGL[1]-algebra with brackets \mathbf{d} and $\{\cdot, \cdot\}$. An $L_\infty[1]$ -algebra morphism from V to W , denoted $V \rightsquigarrow W$, is a degree 0 linear map $\phi: S^+V \rightarrow W$ such that for all $n \geq 1$

$$(2) \quad \sum_{i=1}^n \sum_{\tau \in \text{Sh}(i, n-i)} \epsilon(\tau)\phi_{n-i+1}(\{v_{\tau(1)}, \dots, v_{\tau(i)}\}_i, v_{\tau(i+1)}, \dots, v_{\tau(n)}) \\ = \mathbf{d}\phi_n(v_1, \dots, v_n) + \frac{1}{2} \sum_{j=1}^{n-1} \sum_{\tau \in \text{Sh}(j, n-j)} \epsilon(\tau)\{\phi_j(v_{\tau(1)}, \dots, v_{\tau(j)}), \phi_{n-j}(v_{\tau(j+1)}, \dots, v_{\tau(n)})\}.$$

for every collection of homogeneous elements $v_1, \dots, v_n \in V$.

¹In [12], the multi-brackets have degree $k-2$, instead of $2-k$. They are related to ours by simply inverting the grading of the graded vector space V .

²The name “ $L_\infty[1]$ -algebra” is borrowed from [20]

In the setting of the above definition, a *curved* $L_\infty[1]$ -algebra morphism $V \rightsquigarrow W$ consists of a degree 0 linear map $\phi : SV \rightarrow W$ satisfying, for $n \geq 0$, a variant of (2) where the index j on the right side of the equation runs from 0 to n . A key point is that nontrivially curved morphisms exist even though we are not considering $L_\infty[1]$ -algebras equipped with 0-brackets.

Remark 2.4. If $\phi : SV \rightarrow W$ is a degree 0 linear map, then the zero component $\phi_0 : \mathbb{R} \rightarrow W_0$ gives rise to an element $\phi_0(1) \in W_0$, which by abuse of notation we denote by ϕ_0 . The curved variant of (2) for $n = 0$ then reads $0 = \mathbf{d}\phi_0 + \frac{1}{2}\{\phi_0, \phi_0\}$. In other words, if ϕ is a curved $L_\infty[1]$ -algebra morphism, then ϕ_0 is a Maurer-Cartan element of W .

The notion of (curved) L_∞ -algebra morphism into a DGLA corresponds to Definition 2.3 (and the following paragraph) via the décalage isomorphism (1).

The following fact, whose proof follows easily from the above, is implicit in [11]:

Lemma 2.5. *Let V be an $L_\infty[1]$ -algebra, let $(W, \{\cdot, \cdot\})$ be a Lie[1] algebra, and let $\phi : SV \rightarrow W$ be a degree zero linear map. Then ϕ is a curved $L_\infty[1]$ -algebra morphism from V to W if and only if $\{\phi_0, \phi_0\} = 0$ and $\phi_+ := \sum_{i \geq 1} \phi_i$ is an $L_\infty[1]$ -algebra morphism from V to the DGL[1]-algebra $(W, \{\phi_0, \cdot\}, \{\cdot, \cdot\})$.*

Now we turn to the supergeometric description of L_∞ -algebras.

Definition 2.6. A Q -manifold is a graded manifold (see, for example, [4, 18]) equipped with a *homological* vector field, that is, a degree 1 vector field Q such that $Q^2 = \frac{1}{2}[Q, Q] = 0$.

Given a *finite dimensional* graded vector space V , there is a bijection between $L_\infty[1]$ -algebra structures on V and formal homological vector fields Q on V vanishing at the origin. (The modifier “formal” refers to the fact that Q is a derivation of the formal power series on V .) The correspondence is given by Voronov’s *higher derived bracket* construction [25, Ex. 4.1]. Specifically, the $L_\infty[1]$ -multibrackets induced by a formal homological vector field Q are defined by

$$(3) \quad \{v_1, \dots, v_n\}_n = [[[[Q, \iota_{v_1}], \dots], \iota_{v_n}]]|_0.$$

Here “ $|_0$ ” denotes evaluation at the origin in V , and ι_v is the (formal) vector field that acts on linear functions $\xi \in V^*$ by $\langle v, \xi \rangle$.

3. AN ALTERNATIVE CHARACTERIZATION OF L_∞ -MORPHISMS

The goal of this section is to obtain a useful formula characterizing curved $L_\infty[1]$ -morphisms $U \rightsquigarrow V$ in the case where U is finite-dimensional (but otherwise has an arbitrary $L_\infty[1]$ -algebra structure) and where V is a Lie[1] algebra (but is possibly infinite-dimensional). At first, this may seem like a peculiar set of assumptions, but we will see in §4 that this setting is exactly the one that arises in the context of actions on graded manifolds.

Let U be a finite-dimensional graded vector space, and let V be a (possibly infinite-dimensional) graded vector space. Denote by $\hat{S}(U^*)$ the algebra of formal power series on U . We may consider elements of $\hat{S}(U^*) \otimes V$ to be V -valued formal power series on U .

Let $\text{Hom}(SU, V)$ denote the graded vector space of linear (not necessarily degree-preserving) maps from SU to V . To any $X \in \hat{S}(U^*) \otimes V$, we may associate an element $\phi_X \in \text{Hom}(SU, V)$, given by

$$(4) \quad \phi_X(e_1 \cdots e_n) = [[\cdots [X, \iota_{e_1}], \dots], \iota_{e_n}]]|_0$$

for $e_i \in U$, where 0 denotes the origin in U . By definition, the brackets on the right side of (4) are given by $[X, \iota_e] = -(-1)^{|X||e|} \iota_e X$. (One should think of this bracket as the Lie bracket of vector fields on the graded manifold $U \times V$.)

Lemma 3.1. *The map $\phi : X \mapsto \phi_X$ is a degree-preserving isomorphism from $\hat{S}(U^*) \otimes V$ to $\text{Hom}(SU, V)$.*

Proof. We first observe that ϕ is the direct product of maps $\phi_k : S^k(U^*) \otimes V \rightarrow \text{Hom}(S^k U, V)$, so it suffices to prove that each ϕ_k is an isomorphism. Next, consider the case $V = \mathbb{R}$. In this case, it is fairly easy to see (for example, in terms of a basis) that ϕ induces isomorphisms from $S^k(U^*)$ to $\text{Hom}(S^k U, \mathbb{R}) = (S^k U)^*$.

Returning to the general case, we can identify $\text{Hom}(S^k U, V)$ with $(S^k U)^* \otimes V$ in the obvious way. The result then follows from right-exactness of the tensor product. \square

Now suppose that V is equipped with a Lie[1] algebra structure, with binary bracket $\{\cdot, \cdot\}_V$. Then we may extend the bracket to $\hat{S}(U^*) \otimes V$ by graded linearity:

$$(5) \quad \{f \otimes v, f' \otimes v'\}_V = (-1)^{|f'||v|} f f' \otimes \{v, v'\}_V.$$

The relationship between the operation (5) and the map ϕ is described by the following lemma, which can be proven directly for $X = f \otimes v$.

Lemma 3.2. *The following equation holds for all $X, X' \in \hat{S}(U^*) \otimes V$ and $e_i \in U$:*

$$\begin{aligned} \phi_{\{X, X'\}_V}(e_1 \cdots e_n) = \\ \sum_{j=0}^n \sum_{\tau \in \text{Sh}(j, n-j)} \epsilon(\tau) (-1)^{|X'|(|e_{\tau(1)}| + \cdots + |e_{\tau(j)}|)} \{ \phi_X(e_{\tau(1)} \cdots e_{\tau(j)}), \phi_{X'}(e_{\tau(j+1)} \cdots e_{\tau(n)}) \}_V. \end{aligned}$$

Suppose further that U is equipped with an arbitrary $L_\infty[1]$ -algebra structure, for which the corresponding homological vector field is Q_U . For any degree 0 element $X \in \hat{S}(U^*) \otimes V$, we may ask whether $\phi_X \in \text{Hom}(SU, V)$ is a curved $L_\infty[1]$ -algebra morphism. The following statement expresses this property directly in terms of X .

Proposition 3.3. *Let U be a finite dimensional $L_\infty[1]$ -algebra, V a Lie[1] algebra, and $X \in \hat{S}(U^*) \otimes V$ of degree zero. Then ϕ_X is a curved $L_\infty[1]$ -algebra morphism $U \rightsquigarrow V$ if and only if*

$$Q_U(X) = \frac{1}{2} \{X, X\}_V.$$

Here, the left side is defined by $Q_U(f \otimes v) = Q_U(f) \otimes v$.

Proof. In this case, the curved variant of (2) reduces to

$$(6) \quad \begin{aligned} \sum_{i=0}^n \sum_{\tau \in \text{Sh}(i, n-i)} \epsilon(\tau) \phi_X(\{e_{\tau(1)}, \dots, e_{\tau(i)}\} e_{\tau(i+1)} \cdots e_{\tau(n)}) = \\ \frac{1}{2} \sum_{j=0}^n \sum_{\tau \in \text{Sh}(j, n-j)} \epsilon(\tau) \{ \phi_X(e_{\tau(1)} \cdots e_{\tau(j)}), \phi_X(e_{\tau(j+1)} \cdots e_{\tau(n)}) \}_V \end{aligned}$$

for all $n \geq 0$ and $e_1, \dots, e_n \in U$. Since X is of degree 0, Lemma 3.2 implies that the right side of (6) is equal to $\frac{1}{2} \phi_{\{X, X\}_V}(e_1 \cdots e_n)$.

On the other hand, by repeatedly applying the Jacobi identity and using the fact that X is of degree 0, we have that

$$\begin{aligned} \phi_{Q_U(X)}(e_1 \cdots e_n) &= [[\cdots [[Q_U, X], \iota_{e_1}] \cdots], \iota_{e_n}]|_0 \\ &= \sum_{i=0}^n \sum_{\tau \in \text{Sh}(r, n-i)} \epsilon(\tau) \left[[[\cdots [Q_U, \iota_{e_{\tau(1)}}] \cdots], \iota_{e_{\tau(i)}}], [[\cdots [X, \iota_{e_{\tau(i+1)}}] \cdots], \iota_{e_{\tau(n)}}] \right] |_0. \end{aligned}$$

This equals the left side of (6), as one sees using the identity $[\tilde{Q}, \tilde{X}]|_0 = [\tilde{Q}|_0, \tilde{X}]|_0$ for $\tilde{Q} \in S(U^*) \otimes U$ and $\tilde{X} \in S(U^*) \otimes V$ (recall that $|_0$ denotes evaluation at the origin in U). Therefore, (6) holds for all $e_i \in U$ if and only if $\phi_{Q_U(X)} = \frac{1}{2}\phi_{\{X, X\}_V}$. The result then follows from Lemma 3.1. \square

4. L_∞ -ACTIONS ON GRADED MANIFOLDS

Let \mathcal{M} be a graded manifold, and let U be a finite-dimensional graded vector space. We consider the algebra $\hat{S}(U^*) \otimes C(\mathcal{M})$ of functions on $U \times \mathcal{M}$ that are formal in U and smooth in \mathcal{M} . Slightly abusing notation, we simply refer to this algebra as $C(U \times \mathcal{M})$. Similarly, we use $\mathfrak{X}(U \times \mathcal{M})$ to refer to the space of vector fields that are formal in U and smooth in \mathcal{M} .

Lemma 4.1. *Let \mathcal{M} be a graded manifold, and let U be a finite-dimensional graded vector space. The following are in one-to-one correspondence:*

- (1) degree 1 elements X of $\mathfrak{X}(U \times \mathcal{M})$ that are annihilated by the projection onto U ,
- (2) degree-preserving maps $\phi_X : SU \rightarrow \mathfrak{X}(\mathcal{M})[1]$, and
- (3) (formal in U) graded manifold morphisms $\eta_X : U \times \mathcal{M} \rightarrow T[1]\mathcal{M}$ that cover the identity map on \mathcal{M} .

Proof. (1 \Leftrightarrow 2) The subspace of $\mathfrak{X}(U \times \mathcal{M})$ consisting of vector fields that are annihilated by the projection onto U can be naturally identified with $\hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})$. The isomorphism of Lemma 3.1 (with $V = \mathfrak{X}(\mathcal{M})[1]$) provides a bijection between $\text{Hom}(SU, \mathfrak{X}(\mathcal{M})[1])$ and $\hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})[1]$. In particular, a degree-preserving map from SU to $\mathfrak{X}(\mathcal{M})[1]$ corresponds to a degree 0 element of $\hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})[1]$, which in turn corresponds to a degree 1 element of $\hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})$.

(1 \Leftrightarrow 3) Given a degree 0 element $X \in \hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})[1]$, define a map $\eta_X : U \times \mathcal{M} \rightarrow T[1]\mathcal{M}$ by taking $\eta_X^* : \Omega(\mathcal{M}) \rightarrow C(U \times \mathcal{M})$ to be the unique algebra morphism such that $\eta_X^* \alpha = \langle X, \alpha \rangle$ for all $\alpha \in \Omega^1(\mathcal{M})$ (recall that $\Omega(\mathcal{M}) = C(T[1]\mathcal{M})$). Since $\eta_X^* f = f$ for $f \in \Omega^0(\mathcal{M})$, we see that η_X covers the identity map on \mathcal{M} . Conversely, one can see that any map from $U \times \mathcal{M}$ to $T[1]\mathcal{M}$, formal in U and covering the identity map on \mathcal{M} , is of the form η_X for some $X \in \hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})[1]$ of degree 0. \square

The space $\mathfrak{X}(\mathcal{M})$ of vector fields on \mathcal{M} is a graded Lie algebra, so $\mathfrak{X}(\mathcal{M})[1]$ is a Lie[1] algebra with bracket $\{P, R\} = (-1)^{|P|}[P, R]$, where $|P|$ denotes the degree of P in $\mathfrak{X}(\mathcal{M})$. The sign in this formula arises from the décalage isomorphism.

Now, suppose that U has the structure of an $L_\infty[1]$ -algebra, with homological vector field Q_U . Then we may also view Q_U as a (horizontal) homological vector field on $U \times \mathcal{M}$. The shifted tangent bundle $T[1]\mathcal{M}$ is also a Q -manifold, where the homological vector field is the de Rham operator d .

For any degree 1 element $X \in \hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M}) \subseteq \mathfrak{X}(U \times \mathcal{M})$, let $\phi_X \in \text{Hom}(SU, \mathfrak{X}(\mathcal{M})[1])$ and $\eta_X : U \times \mathcal{M} \rightarrow T[1]\mathcal{M}$ be the corresponding objects as given by Lemma 4.1.

Theorem 4.2. *Let \mathcal{M} be a graded manifold, and let U be a finite-dimensional $L_\infty[1]$ -algebra, with homological vector field Q_U . Let $X \in \hat{S}(U^*) \otimes \mathfrak{X}(\mathcal{M})$ be of degree 1. The following statements are equivalent:*

- (1) $Q_{\text{tot}} := X + Q_U$ is a homological vector field on $U \times \mathcal{M}$.
- (2) ϕ_X is a curved $L_\infty[1]$ -algebra morphism $U \rightsquigarrow \mathfrak{X}(\mathcal{M})[1]$.
- (3) Q_{tot} is η_X -related to d .

Proof. (1 \Leftrightarrow 2) Proposition 3.3 (with $V = \mathfrak{X}(\mathcal{M})[1]$) says that ϕ_X is a curved $L_\infty[1]$ -algebra morphism if and only if $Q_U(X) = \frac{1}{2}\{X, X\} = -\frac{1}{2}[X, X]$, where in the last step we have used the décalage isomorphism to view X as a degree 1 vector field. Using the assumption $Q_U^2 = 0$ and the identity $\frac{1}{2}[X, X] = X^2$, we see that $(X + Q_U)^2 = 0$ if and only if $Q_U(X) = -\frac{1}{2}[X, X]$.

(1 \Leftrightarrow 3) To prove that Q_{tot} is η_X -related to d , it suffices (since $\Omega(\mathcal{M})$ is locally generated by functions and exact 1-forms) to check that

$$Q_{\text{tot}}(\eta_X^* f) = \eta_X^* df \quad \text{and} \quad Q_{\text{tot}}(\eta_X^*(df)) = \eta_X^* d(df)$$

for all $f \in \Omega^0(\mathcal{M})$. The former equation holds automatically by the definition of η_X . In the latter equation, the right side obviously vanishes, and the left side equals $Q_{\text{tot}}^2(\eta_X^* f) = Q_{\text{tot}}^2(f)$. Since Q_{tot}^2 automatically annihilates functions of U , we conclude that $Q_{\text{tot}}^2 = 0$ if and only if Q_{tot} is η_X -related to d . \square

Definition 4.3. Let \mathfrak{g} be a finite-dimensional L_∞ -algebra, and let \mathcal{M} be a graded manifold. An L_∞ -action of \mathfrak{g} on \mathcal{M} is a curved L_∞ -algebra morphism $\mathfrak{g} \rightsquigarrow \mathfrak{X}(\mathcal{M})$.

The following is a direct application of Theorem 4.2.

Theorem 4.4. *Let \mathfrak{g} be a finite-dimensional L_∞ -algebra, and let \mathcal{M} be a graded manifold. There is a one-to-one correspondence between*

- (1) L_∞ -actions of \mathfrak{g} on \mathcal{M}
- (2) homological vector fields on $\mathfrak{g}[1] \times \mathcal{M}$ for which the projection map $\mathfrak{g}[1] \times \mathcal{M} \rightarrow \mathfrak{g}[1]$ is a Q -manifold morphism.

Explicitly, an L_∞ -action is an element of $\text{Hom}(\wedge \mathfrak{g}, \mathfrak{X}(\mathcal{M})) \cong \text{Hom}(S(\mathfrak{g}[1]), \mathfrak{X}(\mathcal{M})[1])$; writing ϕ_X for the latter, where $X \in S(\mathfrak{g}[1]^) \otimes \mathfrak{X}(\mathcal{M})$, the corresponding homological vector field is $Q_{\text{tot}} := X + Q_{\mathfrak{g}[1]}$.*

Remark 4.5. Given an L_∞ -action of \mathfrak{g} on \mathcal{M} , one can geometrically interpret the associated homological vector field on $\mathfrak{g}[1] \times \mathcal{M}$ as giving the structure of a “transformation L_∞ -algebroid.” Indeed, in the case where \mathfrak{g} is an ordinary Lie algebra and M is an ordinary manifold, this construction reduces (up to a degree shift) to that of the usual transformation Lie algebroid. More precisely, if $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ is a Lie algebra morphism, one can form the transformation Lie algebroid $\mathfrak{g} \times M$ over M . The homological vector field on $\mathfrak{g}[1] \times M$ encoding the Lie algebroid structure is exactly Q_{tot} : the summand X encodes the anchor map, while $Q_{\mathfrak{g}[1]}$ encodes the Lie algebroid bracket. The anchor map $\mathfrak{g} \times M \rightarrow TM$ is a Lie algebroid morphism, corresponding to the map $\mathfrak{g}[1] \times M \rightarrow T[1]M$, which by Thm. 4.2 (3) preserves the homological vector fields.

From Lemma 2.5, we see that, given an L_∞ -action of \mathfrak{g} on \mathcal{M} , the zero component of ϕ_X corresponds to a homological vector field $Q_{\mathcal{M}}$ on \mathcal{M} , and the higher components of ϕ_X then give a (noncurved) $L_\infty[1]$ -algebra morphism from \mathfrak{g} to the DGL[1]-algebra $(\mathfrak{X}(\mathcal{M})[1], \{Q_{\mathcal{M}}, \cdot\}, \{\cdot, \cdot\})$. Equivalently, the zero component of the L_∞ -action is $Q_{\mathcal{M}}$ and its higher components give a (noncurved) L_∞ -algebra morphism from \mathfrak{g} to the DGLA

$(\mathfrak{X}(\mathcal{M}), -[Q_{\mathcal{M}}, \cdot], [\cdot, \cdot])$. (The minus sign comes from the décalage isomorphism.) In many situations, \mathcal{M} already comes equipped with a homological vector field, motivating the following definition.

Definition 4.6. Let \mathfrak{g} be a finite-dimensional L_∞ -algebra, and let $(\mathcal{M}, Q_{\mathcal{M}})$ be a Q -manifold. An L_∞ -action of \mathfrak{g} on \mathcal{M} is *compatible with $Q_{\mathcal{M}}$* if the zero component of ϕ_X corresponds to $Q_{\mathcal{M}}$.

The appropriate analogue of Theorem 4.4 in this context is as follows:

Theorem 4.7. *Let \mathfrak{g} be a finite-dimensional L_∞ -algebra, and let $(\mathcal{M}, Q_{\mathcal{M}})$ be a Q -manifold. There is a one-to-one correspondence between*

- (1) L_∞ -actions of \mathfrak{g} on \mathcal{M} compatible with $Q_{\mathcal{M}}$
- (2) homological vector fields on $\mathfrak{g}[1] \times \mathcal{M}$ for which

$$(7) \quad (\mathcal{M}, Q_{\mathcal{M}}) \rightarrow (\mathfrak{g}[1] \times \mathcal{M}, Q_{tot}) \rightarrow (\mathfrak{g}[1], Q_{\mathfrak{g}[1]})$$

is a sequence of Q -manifold morphisms.

5. EQUIVALENCES

Let \mathfrak{g} be a finite-dimensional L_∞ -algebra and let \mathcal{M} be a graded manifold. We use the notation of Theorem 4.4.

Consider the DGLA $(\mathfrak{X}(\mathfrak{g}[1] \times \mathcal{M}), [Q_{\mathfrak{g}[1]}, \cdot], [\cdot, \cdot])$. The Maurer-Cartan equation for a degree 1 element X in this DGLA reads $[Q_{\mathfrak{g}[1]}, X] + \frac{1}{2}[X, X] = 0$, so it is equivalent to $X + Q_{\mathfrak{g}[1]}$ being a homological vector field on $\mathfrak{g}[1] \times \mathcal{M}$. In view of Theorem 4.4, therefore, L_∞ -actions of \mathfrak{g} on \mathcal{M} are in bijection with Maurer-Cartan elements X of the above DGLA which are annihilated by the projection $\mathfrak{g}[1] \times \mathcal{M} \rightarrow \mathfrak{g}[1]$.

Any nilpotent³ element λ of degree 0 in the DGLA acts on the set of Maurer-Cartan elements [7, §1], mapping a Maurer-Cartan element X to

$$(8) \quad X^\lambda := e^{ad_\lambda} X + \frac{1 - e^{ad_\lambda}}{ad_\lambda} [Q_{\mathfrak{g}[1]}, \lambda] = e^{ad_\lambda} (X + Q_{\mathfrak{g}[1]}) - Q_{\mathfrak{g}[1]}.$$

Notice that the action of nilpotent λ 's lying in $(\hat{S}(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(\mathcal{M}))_0$ preserves the set of Maurer-Cartan elements that are annihilated under the projection $\mathfrak{g}[1] \times \mathcal{M} \rightarrow \mathfrak{g}[1]$, generating an equivalence relation there.

Definition 5.1. Two L_∞ -actions of \mathfrak{g} on \mathcal{M} are *equivalent* if the corresponding Maurer-Cartan elements are equivalent by the action of nilpotent elements $\lambda \in (\hat{S}(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(\mathcal{M}))_0$.

Proposition 5.2. *Equivalent L_∞ -actions of \mathfrak{g} on \mathcal{M} induce isomorphic Q -manifold structures on $\mathfrak{g}[1] \times \mathcal{M}$.*

Proof. Consider an L_∞ -action, with corresponding Maurer-Cartan element X (so the associated homological vector field is $X + Q_{\mathfrak{g}[1]}$). For any nilpotent elements $\lambda \in (\hat{S}(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(\mathcal{M}))_0$, we have by (8) that

$$X^\lambda + Q_{\mathfrak{g}[1]} = e^{ad_\lambda} (X + Q_{\mathfrak{g}[1]}).$$

In other words, the two homological vector fields are related by the diffeomorphism of the graded manifold $\mathfrak{g}[1] \times \mathcal{M}$ obtained by exponentiating the degree 0 vector field λ . \square

³This means that for every element of the DGLA there is a power of $ad_\lambda = [\lambda, \cdot]$ annihilating it.

In practice, \mathcal{M} often comes equipped with a fixed homological vector field $Q_{\mathcal{M}}$ (see Definition 4.6). If λ is any degree 0 element of $\hat{S}^+(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(\mathcal{M})$, one can show that e^{ad_λ} always converges⁴ (\hat{S}^+ is defined analogously to the way S^+ is defined in §2). Furthermore, as such a λ vanishes on $\{0\} \times \mathcal{M}$, its action preserves the set of Maurer-Cartan elements whose restriction to $\{0\} \times \mathcal{M}$ agrees with $Q_{\mathcal{M}}$. Hence we define:

Definition 5.3. Two L_∞ -actions of \mathfrak{g} on \mathcal{M} compatible with $Q_{\mathcal{M}}$ are *equivalent* if the corresponding Maurer-Cartan elements are equivalent by the action of elements $\lambda \in (\hat{S}^+(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(\mathcal{M}))_0$.

From Proposition 5.2 we immediately obtain:

Proposition 5.4. *Compatible L_∞ -actions of \mathfrak{g} on \mathcal{M} which are equivalent induce Q -manifold structures on $\mathfrak{g}[1] \times \mathcal{M}$ which are isomorphic by isomorphisms that commute with the maps appearing in (7).*

6. EXTENSIONS OF L_∞ -ALGEBRAS

Let \mathfrak{g} and \mathfrak{h} be finite-dimensional L_∞ -algebras. Following Definition 4.6, we may consider formal L_∞ -actions of \mathfrak{g} on $\mathfrak{h}[1]$ that are compatible with the homological vector field $Q_{\mathfrak{h}[1]}$. By formal L_∞ -actions, we mean curved L_∞ -algebra morphisms from \mathfrak{g} to the graded Lie algebra of formal vector fields on $\mathfrak{h}[1]$.

The following proposition is an application of Theorem 4.7, which also holds in the setting of formal actions and formal vector fields. Notice that Q_{tot} there vanishes at the origin, since the map $\mathfrak{h}[1] \rightarrow \mathfrak{g}[1] \times \mathfrak{h}[1]$ is a Q -manifold morphism.

Proposition 6.1. *There is a one-to-one correspondence between*

- (1) *compatible formal L_∞ -actions of \mathfrak{g} on $\mathfrak{h}[1]$ and*
- (2) *L_∞ -algebra extensions⁵ $\mathfrak{h} \rightarrow \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{g}$.*

Remark 6.2. Given a compatible formal L_∞ -action of \mathfrak{g} on $\mathfrak{h}[1]$ with associated $L_\infty[1]$ -algebra morphism $\phi : \mathfrak{g}[1] \rightsquigarrow \mathfrak{X}(\mathfrak{h}[1])[1]$, we may explicitly describe the corresponding multibrackets on $\mathfrak{g}[1] \times \mathfrak{h}[1]$ as follows. For $e_i \in \mathfrak{g}[1]$, the $\mathfrak{g}[1]$ component of $\{e_1, \dots, e_n\}$ coincides with the $\mathfrak{g}[1]$ -multibracket. The $\mathfrak{h}[1]$ component is $\phi(e_1 \cdots e_n)|_0$.

In the case of mixed entries, we have

$$(9) \quad \{e_1, \dots, e_n, f_1, \dots, f_k\} = [[\dots [\phi(e_1 \cdots e_n), \iota_{f_1}], \dots], \iota_{f_k}]|_0 \in \mathfrak{h}[1]$$

for $e_i \in \mathfrak{g}[1]$, $f_j \in \mathfrak{h}[1]$, and $k > 0$. Because of the compatibility condition $\phi_0 = Q_{\mathfrak{h}[1]}$, equation (9) reduces to the multibrackets for $\mathfrak{h}[1]$ when $n = 0$ (c.f. (3)).

Notice that $\mathfrak{g}[1]$ is an $L_\infty[1]$ -subalgebra of $\mathfrak{g}[1] \times \mathfrak{h}[1]$ if and only if ϕ takes values in vector fields on $\mathfrak{h}[1]$ vanishing at the origin.

Definition 6.3. Let \mathfrak{g} and \mathfrak{h} be finite-dimensional L_∞ -algebras equipped with a compatible formal L_∞ -action ϕ of \mathfrak{g} on $\mathfrak{h}[1]$. The associated L_∞ -algebra structure on $\mathfrak{g} \times \mathfrak{h}$ is the *extension* of \mathfrak{g} by \mathfrak{h} via ϕ , denoted $\mathfrak{g} \times \mathfrak{h}$. We say that this extension is a *semidirect product* when $\mathfrak{g}[1] \times \{0\}$ is an $L_\infty[1]$ -subalgebra.

⁴The idea is to endow $\mathfrak{X}(\mathfrak{g}[1] \times \mathcal{M})$ with a filtration induced by the degree of polynomials in $\hat{S}(\mathfrak{g}[1]^*)$, cf. [6, §1.4].

⁵This means that the two arrows are strict morphisms of L_∞ -algebras and that the sequence is exact. Recall that a strict $L_\infty[1]$ -algebra morphism from V to W is a degree-preserving linear map $V \rightarrow W$ which intertwines the multi-brackets.

Notice that equivalent compatible actions deliver extensions which are L_∞ -isomorphic, by Proposition 5.4.

The notion of extension of L_∞ -algebras was introduced in terms of coderivations by Baez and Frégier [5], who use semidirect products and cocycles in the L_∞ -category to construct such extensions.

Example 6.4 (Lie algebra extensions, see also §7). In the special case where \mathfrak{g} and \mathfrak{h} are ordinary Lie algebras, any L_∞ -algebra extension is necessarily a Lie algebra (since it is concentrated in degree 0). On the other hand, any compatible L_∞ -action of \mathfrak{g} on $\mathfrak{h}[1]$ is given by a pair (σ, ψ) , where $\sigma: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ and $\psi: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{h}$, such that

$$(10) \quad \sigma([x, y]_{\mathfrak{g}}) - [\sigma(x), \sigma(y)] + ad_{\psi(x, y)} = 0 \text{ for all } x, y \in \mathfrak{g},$$

$$(11) \quad \psi(x, [y, z]_{\mathfrak{g}}) + [\sigma(x), \psi(y, z)] + c.p. = 0 \text{ for all } x, y, z \in \mathfrak{g}.$$

The higher components of the L_∞ -action necessarily vanish by degree considerations.

The equations (10)–(11) are exactly those that define a nonabelian 2-cocycle [9] on \mathfrak{g} with values in \mathfrak{h} . Thus, in this special case, Proposition 6.1 reduces to the well-known fact that Lie algebra extensions $\mathfrak{h} \rightarrow \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{g}$ are in correspondence with such nonabelian 2-cocycles. The connection between such extensions and compatible L_∞ -actions was already observed in [24, Prop. 2.7].

Maps $b: \mathfrak{g} \rightarrow \mathfrak{h}$ act on the space of pairs (σ, ψ) as in Def. 5.1, and the action agrees with the one described explicitly in [2, §5].

In the following subsections, we consider some other examples of extensions.

6.1. L_∞ -modules. In this subsection, we observe that the notion of L_∞ -module, due to Lada and Markl [12], can be seen as a special case of L_∞ -actions.

Let $(W[1], \partial)$ be a *differential graded vector space*. In other words, $W[1]$ is a graded vector space, and ∂ is a linear differential on $W[1]$. Then $\text{End}(W[1])$ is a DGLA with the graded commutator bracket $[A, B]_{\text{comm}} = AB - (-1)^{|A||B|}BA$ and differential $[\partial, \cdot]_{\text{comm}}$. According to Lada and Markl [12, Theorem 5.4], a \mathfrak{g} -module structure on $W[1]$ is equivalent to an L_∞ -algebra morphism

$$\Phi: \mathfrak{g} \rightsquigarrow (\text{End}(W[1]), [\cdot, \cdot]_{\text{comm}}, [\partial, \cdot]_{\text{comm}}).$$

We may identify $\text{End}(W[1])$ with the space of linear vector fields on $W[1]$, by mapping A to the unique vector field Y_A such that $[Y_A, \iota_w] = \iota_{Aw}$ for all $w \in W[1]$. This determines an isomorphism of DGLAs

$$(12) \quad (\text{End}(W[1]), [\cdot, \cdot]_{\text{comm}}, [\partial, \cdot]_{\text{comm}}) \cong (\mathfrak{X}_{\text{lin}}(W[1]), [\cdot, \cdot], [Y_\partial, \cdot]).$$

Hence we can view the \mathfrak{g} -module structure as being an example of an L_∞ -action of \mathfrak{g} on $W[1]$ compatible with $-Y_\partial$. (The minus sign arises from the décalage isomorphism; see the paragraph before Definition 4.6.) On the other hand, any L_∞ -action that is compatible with $-Y_\partial$ and *linear* (in the sense that the vector fields on $W[1]$ in the image of the action map are linear) comes from a \mathfrak{g} -module structure. To summarize, we have the following:

Proposition 6.5. *Let \mathfrak{g} be a finite dimensional L_∞ -algebra and $(W[1], \partial)$ a differential graded vector space. There is a one-to-one correspondence between*

- (1) \mathfrak{g} -module structures on $W[1]$
- (2) linear L_∞ -actions of \mathfrak{g} on $W[1]$ that are compatible with $-Y_\partial$.

Now suppose that W is finite-dimensional. Given a linear L_∞ -action of \mathfrak{g} on $W[1]$, the extension $\mathfrak{g} \ltimes W$ only has nonzero multibrackets of the following types:

- (a) multibrackets on \mathfrak{g} , and
- (b) mixed brackets of the form $\wedge \mathfrak{g} \otimes W \rightarrow W$, of which the zero component $W \rightarrow W$ coincides with the differential $-\partial$. More precisely, identifying $\text{End}(W) = \text{End}(W[1])$, we have for all $v_i \in \mathfrak{g}$ and $w \in W$: $[v_1, \dots, v_k, w] = (-1)^{|v_1| + \dots + |v_k|} \Phi(v_1 \wedge \dots \wedge v_k)w$.

In this case, the generalized Jacobi identity in Definition 2.1 reduces to the equation in [12, Definition 5.1] that forms the original definition of \mathfrak{g} -modules in terms of multibrackets. Thus, the linear version of Proposition 6.1 allows us to recover the correspondence of [12, Theorem 5.4].

6.2. The adjoint module of an L_∞ -algebra. In this subsection we make use of Proposition 6.1 to derive the definition of adjoint representation in the L_∞ -category. Let \mathfrak{g} be a finite dimensional L_∞ -algebra, and let $Q_{\mathfrak{g}[1]}$ be the associated homological vector field on $\mathfrak{g}[1]$. Let $Q_{T\mathfrak{g}[1]}$ denote the complete lift (in the sense of Yano and Ishihara [26]) of $Q_{\mathfrak{g}[1]}$ to the tangent bundle $T\mathfrak{g}[1]$. That is, if the linear functions on $T\mathfrak{g}[1]$ are identified with 1-forms on $\mathfrak{g}[1]$, then $Q_{T\mathfrak{g}[1]}$ is the unique vector field on $T\mathfrak{g}[1]$ whose action on linear functions coincides with the Lie derivative $\mathcal{L}_{Q_{\mathfrak{g}[1]}}$.

To make things more explicit, choose linear coordinates ξ^i on $\mathfrak{g}[1]$, and let $\tilde{\xi}^i$ denote the corresponding fiber coordinates on $T\mathfrak{g}[1]$. Write $Q_{\mathfrak{g}[1]} = Q_{\mathfrak{g}[1]}^i \frac{\partial}{\partial \xi^i}$. Then

$$(13) \quad Q_{T\mathfrak{g}[1]} = Q_{\mathfrak{g}[1]}^i \frac{\partial}{\partial \xi^i} + \tilde{\xi}^j \frac{\partial Q_{\mathfrak{g}[1]}^i}{\partial \xi^j} \frac{\partial}{\partial \tilde{\xi}^i}.$$

The complete lift preserves degree and Lie brackets, so $Q_{T\mathfrak{g}[1]}$ is a (formal) homological vector field, giving an L_∞ -algebra structure on $T\mathfrak{g}$, which is an extension of \mathfrak{g} . The kernel, which we denote $\tilde{\mathfrak{g}}$, is isomorphic to \mathfrak{g} as a graded vector space, but has a different L_∞ -algebra structure. Specifically, it can be seen from the coordinate description (13) that the 1-bracket on $\tilde{\mathfrak{g}}$ is the same as that on \mathfrak{g} , but all the higher brackets vanish, so that $\tilde{\mathfrak{g}}$ is a differential graded vector space.

We may canonically identify $T\mathfrak{g}$ with $\mathfrak{g} \times \tilde{\mathfrak{g}}$ as graded vector spaces. By Proposition 6.1, the L_∞ -algebra extension

$$(14) \quad \tilde{\mathfrak{g}} \rightarrow T\mathfrak{g} = \mathfrak{g} \times \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$$

corresponds to an L_∞ -action of \mathfrak{g} on $\tilde{\mathfrak{g}}[1]$. Furthermore, since $Q_{T\mathfrak{g}[1]}$ is a linear vector field, we have that the action is linear. By Proposition 6.5, we obtain:

Proposition 6.6. *Let \mathfrak{g} be a finite-dimensional L_∞ -algebra, and denote its 1-bracket by ∂ . Then the differential graded vector space $(\tilde{\mathfrak{g}}[1], -\partial)$ naturally has the structure of an L_∞ -module (see §6.1) over \mathfrak{g} , whose corresponding extension is the L_∞ -algebra $T\mathfrak{g}$.*

We refer to $\tilde{\mathfrak{g}}[1]$ as the *adjoint module* of \mathfrak{g} .

From the coordinate description (13) and the derived bracket formula (3), we can directly see that the map $\phi_X: \mathfrak{S}(\mathfrak{g}[1]) \rightarrow \mathfrak{X}(\tilde{\mathfrak{g}}[1])[1] \cong \text{End}(\tilde{\mathfrak{g}}[1])[1]$ is given by $\phi_X(e_1, \dots, e_k) = \{e_1, \dots, e_k, \bullet\}$. Applying the décalage isomorphism (1) we obtain the L_∞ -module structure of Proposition 6.6, which (identifying $\text{End}(\tilde{\mathfrak{g}})$ with $\text{End}(\tilde{\mathfrak{g}}[1])$) reads

$$\wedge^+ \mathfrak{g} \rightarrow \text{End}(\tilde{\mathfrak{g}}), \quad v_1 \wedge \dots \wedge v_k \mapsto (-1)^{|v_1| + \dots + |v_k|} [v_1, \dots, v_k, \bullet].$$

Of course, when \mathfrak{g} is an ordinary Lie algebra, we recover the usual adjoint representation $\mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $v \mapsto [v, \bullet]$.

6.3. Representations up to homotopy of Lie algebras. Let \mathfrak{g} be an ordinary finite-dimensional Lie algebra. In this subsection we observe that L_∞ -modules over \mathfrak{g} are in correspondence with representations up to homotopy of \mathfrak{g} , in the sense of [1, 8]. We note that this correspondence is essentially a special case of [19].

Let (E, ∂) be a finite-dimensional differential graded vector space with an L_∞ -module structure over \mathfrak{g} . By Proposition 6.5 the module structure corresponds to a linear L_∞ -action of \mathfrak{g} on E , compatible with the vector field $-Y_\partial$ on E associated to ∂ . By Theorem 4.4, such L_∞ -actions correspond to degree 1 elements

$$X \in S(\mathfrak{g}[1]^*) \otimes \mathfrak{X}_{\text{lin}}(E)$$

whose zero component is $-Y_\partial$, such that $X + Q_{\mathfrak{g}[1]}$ is a homological vector field on $\mathfrak{g}[1] \times E$. Notice that such an X can be regarded as an element

$$\omega \in S(\mathfrak{g}[1]^*) \otimes \text{End}(E)$$

via the identification (12). The condition that $X + Q_{\mathfrak{g}[1]}$ be homological is therefore equivalent to $D^2 = 0$ for the operator $D := Q_{\mathfrak{g}[1]} + \omega$ on $S(\mathfrak{g}[1]^*) \otimes E$. The latter is exactly the definition [1, 8] of a representation up to homotopy of \mathfrak{g} on E .

Thus we have the following.

Proposition 6.7. *There is a one-to-one correspondence between*

- (1) L_∞ -modules (E, ∂) over \mathfrak{g}
- (2) representations up to homotopy of \mathfrak{g} on $(E, -\partial)$.

An L_∞ -module given by maps $f_n: \wedge^n \mathfrak{g} \rightarrow \text{End}_{1-n}(E)$ ($n \geq 1$) corresponds to the representations up to homotopy ω with components $\omega_n \in \wedge^n \mathfrak{g}^* \otimes \text{End}_{1-n}(E)$ given by

$$f_n(v_1, \dots, v_n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \iota_{v_n} \cdots \iota_{v_1} \omega_n.$$

Proof. It remains to prove the formula relating f_n to ω_n . Since the décalage isomorphism (1) does not introduce additional signs, we have

$$\begin{aligned} f_n(v_1, \dots, v_n) &= \phi_X(z_1 \cdots z_n) = [[\dots [X, \iota_{z_1}], \dots], \iota_{z_n}]|_0 \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} [\iota_{z_n}, [\dots, [\iota_{z_1}, X] \dots]]|_0 = (-1)^{\lfloor \frac{n}{2} \rfloor} \iota_{v_n} \cdots \iota_{v_1} \omega_n. \end{aligned}$$

where we write $z_i := v_i[1]$ for $v_i \in \mathfrak{g}$. □

Remark 6.8. It was noted in [1] that a representation up to homotopy of \mathfrak{g} on W^* induces an L_∞ -algebra structure on $\mathfrak{g} \times W$, which “deserves the name semidirect product.” Proposition 6.7 (together with Proposition 6.1) can be interpreted as a justification of their statement, demonstrating that $\mathfrak{g} \times W$ is indeed a semidirect product in the L_∞ category.

7. EXTENSIONS OF LIE ALGEBROIDS

In this section we consider L_∞ -actions of Lie algebras on degree 1 Q -manifolds (see also [27][28]). This allows us to extend Example 6.4 to the case of Lie algebroids.

Let \mathfrak{g} be a Lie algebra. Let $(A, [\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid over M , so that $A[1]$ is a degree 1 Q -manifold, with homological vector field $Q_{A[1]}$ given by the Lie algebroid differential.

Denote by $\text{CDO}(A) \rightarrow M$ the Lie algebroid whose sections are covariant differential operators [16] on the vector bundle $A \rightarrow M$. Let $\Gamma_A(\text{CDO}(A))$ denote the subspace of covariant differential operators Y that respect the Lie algebroid structure, in the sense that

$$Y[a, b]_A = [Ya, b]_A + [a, Yb]_A, \quad \underline{Y}(\rho_A(a)f) = \rho_A(Ya)f + \rho_A(a)(\underline{Y}f)$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$. Here $\underline{Y} \in \mathfrak{X}(M)$ is the symbol of Y .

Then L_∞ -actions of \mathfrak{g} on $A[1]$ compatible with $Q_{A[1]}$ are given by linear maps

$$\begin{aligned}\sigma: \mathfrak{g} &\rightarrow \chi_0(A[1]) = \Gamma_A(\text{CDO}(A)) \\ \psi: \wedge^2 \mathfrak{g} &\rightarrow \chi_{-1}(A[1]) = \Gamma(A),\end{aligned}$$

such that analogues of (10) and (11) are satisfied.

By Theorem 4.4, we obtain a Lie algebroid structure on the vector bundle $(\mathfrak{g} \times M) \oplus A \rightarrow M$, which we denote by $\mathfrak{g} \times A$. Explicitly, the anchor is $(x, a) \mapsto \underline{\sigma}(x) + \rho_A(a)$ for all $x \in \mathfrak{g}$ and $a \in A$, and the bracket is given by

$$[(x_1, a_1), (x_2, a_2)]_{\mathfrak{g} \times A} = ([x_1, x_2]_{\mathfrak{g}}, [a_1, a_2]_A + \sigma(x_1)a_2 - \sigma(x_2)a_1 + \psi(x_1, x_2)).$$

Theorem 4.7 shows that $\mathfrak{g} \times A$ fits into an exact sequence of Lie algebroids $A \rightarrow \mathfrak{g} \times A \rightarrow \mathfrak{g}$, and that all Lie algebroid extensions of \mathfrak{g} by A arise as above. We summarize the above discussion:

Proposition 7.1. *Let \mathfrak{g} be a Lie algebra and A a Lie algebroid over M . There is a one-to-one correspondence between*

- (1) pairs of maps (σ, ψ) satisfying the analogues of eq. (10) and (11)
- (2) Lie algebroid extensions $A \rightarrow (\mathfrak{g} \times M) \oplus A \rightarrow \mathfrak{g}$.

Remark 7.2. See [2, §2] for an interpretation of the above proposition in terms of splittings in the Lie algebra case. See [16, §4.5][3] for a general theory of Lie algebroid extensions (in which \mathfrak{g} is allowed to be any Lie algebroid).

Example 7.3. Let G be a Lie group integrating \mathfrak{g} , and consider a group action of G on A by Lie algebroid automorphisms. Differentiating the action we obtain a map σ as above which moreover preserves Lie brackets. (To see this notice that the Lie algebroid $\text{CDO}(A)$ integrates to the Lie groupoid over M whose arrows are linear isomorphisms between the fibers of A). Hence, setting $\psi = 0$, by Proposition 7.1 we obtain a Lie algebroid extension of A and \mathfrak{g} .

In particular, when A is the trivial Lie algebroid over M , we recover the transformation algebroid $\mathfrak{g} \times M$ as in Remark 4.5.

We now comment on equivalences.

Proposition 7.4. *Two compatible L_∞ -actions of \mathfrak{g} on $A[1]$ are equivalent (in the sense of Definition 5.3) if and only if the associated Lie algebroid structures on $(\mathfrak{g} \times M) \oplus A$ are isomorphic by isomorphisms that commute with the maps appearing in $A \rightarrow (\mathfrak{g} \times M) \oplus A \rightarrow \mathfrak{g}$.*

Proof. One implication is given by Proposition 5.4, so it remains to prove that the existence of an isomorphism of Lie algebroids ψ as above implies that the corresponding compatible L_∞ -actions are equivalent. Consider $\psi: (\mathfrak{g} \times M) \oplus A \rightarrow (\mathfrak{g} \times M) \oplus A$, and denote by ψ^* the induced automorphism on sections of the exterior algebra of $((\mathfrak{g} \times M) \oplus A)^*$. The map ψ must be of the form $\psi(x, a) = (x, a + \phi(x))$ for a linear $\phi: \mathfrak{g} \rightarrow \Gamma(A)$. Let $\lambda \in (S^+(\mathfrak{g}[1]^*) \otimes \mathfrak{X}(A[1]))_0$ be the corresponding vector field on $\mathfrak{g}[1] \times A[1]$, defined by $\iota_{\phi(v)[1]} = [\lambda, \iota_{v[1]}] \in \chi_{-1}(A[1])$. Then one computes

$$\begin{aligned}\psi^*(\eta) &= \eta = e^{-\lambda}(\eta) && \text{for all } \eta \in \mathfrak{g}[1]^* = C_1(\mathfrak{g}[1]) \\ \psi^*(\xi) &= \xi + \phi^*(\xi) = e^{-\lambda}(\xi) && \text{for all } \xi \in \Gamma(A[1]^*) = C_1(A[1]),\end{aligned}$$

implying that the diffeomorphism Ψ of $\mathfrak{g}[1] \times A[1]$ corresponding to ψ satisfies $\Psi^* = e^{-\lambda}$ (where Ψ^* denotes the “pullback of functions”). In other words, Ψ is the time-1 flow of the vector field $-\lambda$.

Consider the Lie algebroid differentials d_1, d_2 encoding the two Lie algebroid structures on $(M \times \mathfrak{g}) \oplus A$, and view them as homological vector fields Q_1, Q_2 on $\mathfrak{g}[1] \times A[1]$. The fact that ψ is a Lie algebroid isomorphism means that $\Psi_*(Q_1) = Q_2$. Here Ψ_* denotes the push-forward of vector fields on $\mathfrak{g}[1] \times A[1]$, which by the above considerations we can write as $e^{ad\lambda}$. Hence $e^{ad\lambda}(Q_1) = Q_2$, which by (8) means that the two compatible L_∞ -actions of \mathfrak{g} on $A[1]$ are equivalent. \square

Example 7.5. Let $\phi: \mathfrak{g} \rightarrow \Gamma(A)$ be any linear map. Then

$$\sigma(x) := [\phi(x), \cdot]_A, \quad \psi(x \wedge y) = [\phi(x), \phi(y)]_A - \phi([x, y]_{\mathfrak{g}})$$

defines an L_∞ -action of \mathfrak{g} on $A[1]$ compatible with $Q_{A[1]}$.

This L_∞ action is obtained twisting the trivial action by means of the linear map ϕ , as in §5. In particular, the Lie algebroid structure we obtain on $(\mathfrak{g} \times M) \oplus A$ is isomorphic to the product Lie algebroid structure. A concrete isomorphism is given by $(x, a_m) \mapsto (x, (a + \phi(x))_m)$ for all $x \in \mathfrak{g}$, $a_m \in A_m$.

Remark 7.6. A geometric situation in which the map $\phi: \mathfrak{g} \rightarrow \Gamma(A)$ emerges canonically is the following. Let G be a Poisson-Lie group, (M, π) a Poisson manifold, and let $A := T^*M$ be the cotangent algebroid. Let $G \times M \rightarrow M$ a Poisson action with (not necessarily equivariant) moment map⁶ $J: M \rightarrow G^*$ [15, §3.3]. This means that J satisfies $\pi(J^*(x^l)) = x_M$ for all $x \in \mathfrak{g}$, where x^l is the left-invariant 1-form on G^* whose value at the identity is x , and x_M the vector field on M given by the action. Then one can simply define $\phi(x) = J^*(x^l)$, and by Example 7.5 obtain a canonical Lie algebroid structure on $(\mathfrak{g} \times M) \oplus A$.

Notice that, for *any* Poisson action, Jiang-Hua Lu [14, §4] defined a Lie algebroid structure on the same vector bundle. Lu’s construction does not require a moment map, whereas the construction that we have described depends on the choice of moment map, so is obviously quite different. In Lu’s Lie algebroid, the mixed brackets take values in both summands, while in ours they take values only in the second summand A .

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⁶When the Poisson structure on G is zero, this recovers the usual notion of moment map into \mathfrak{g}^* .

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