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Classification of Cayley Rose Window Graphs

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Abstract

Rose window graphs are a family of tetravalent graphs, introduced by Steve Wilson. Following it, Kovacs, Kutnar and Marusic classified the edge-transitive rose window graphs and Dobson, Kovacs and Miklavic characterized the vertex transitive rose window graphs. In this paper, we classify the Cayley rose window graphs.

1 Introduction

Rose window graphs were introduced in [6] in the following way:

Definition 1.1. *Given natural numbers $n \geq 3$ and $1 \leq a, r \leq n-1$, the Rose Window graph $R_n(a, r)$ is defined to be the graph with vertex set $V = \{A_i, B_i : i \in \mathbb{Z}_n\}$ and four kind of edges: A_iA_{i+1} (**rim** edges), A_iB_i (**inspoke** edges), $A_{i+a}B_i$ (**outspoke** edges) and B_iB_{i+r} (**hub** edges), where the addition of indices are done modulo n.*

In the introductory paper [6], author's initial interest in rose window graphs arose in the context of graph embeddings into surfaces. The author conjectured that rose window graphs are edge-transitive if and only if it belongs to the one of the four families given in Theorem 1.1. The conjecture was proved by Kovacs *et. al.* in [4]. In particular, they proved that

Theorem 1.1. *[4] A rose window graph is edge-transitive if and only if it belongs to one of the four families:*

1. $R_n(2, 1)$.
2. $R_{2m}(m \pm 2, m \pm 1)$
3. $R_{12m}(\pm(3m+2), \pm(3m-1))$ *and* $R_{12m}(\pm(3m-2), \pm(3m+1))$.
4. $R_{2m}(2b, r)$, *where* $b^2 \equiv \pm 1 (mod\ m)$, $2 \leq 2b \leq m$, *and* $r \in \{1, m-1\}$ *is odd.*

A similar characterization for vertex-transitive graphs was proved in [1]:

Theorem 1.2. *[1] A rose window graph $R_n(a, r)$ is vertex-transitive if and only if it belongs to one of the following families:*

1. $R_n(a, r)$, *where* $r^2 \equiv \pm 1 (mod\ n)$ *and* $ra \equiv \pm a (mod\ n)$.
2. $R_{4m}(2m, r)$, *where r is odd and* $(r^2 + 2m) \equiv \pm 1 (mod\ 4m)$.
3. $R_{2m}(m \pm 2, m \pm 1)$
4. $R_{12m}(\pm(3m+2), \pm(3m-1))$ *and* $R_{12m}(\pm(3m-2), \pm(3m+1))$.
5. $R_{2m}(2b, r)$, *where* $b^2 \equiv \pm 1 (mod\ m)$, $2 \leq 2b \leq m$, *and* $r \in \{1, m-1\}$ *is odd.*

As a Cayley graph is always vertex-transitive, a natural question to ask is to characterize the rose-window graphs which are also Cayley graphs. For that, it is sufficient to look for Cayley graphs only in the 5 families mentioned in Theorem 1.2. The main goal of this paper is finding an answer to this question. In particular, we prove the following theorem:

Theorem 1.3. *A rose-window graph $R_n(a,r)$ is Cayley if and only if one of the following holds:*

1. $R_n(a,r)$, *where* $r^2 \equiv \pm 1 (mod\ n)$ *and* $ra \equiv \pm a (mod\ n)$.
2. $R_{4m}(2m,r)$, *where* r *is odd and* $(\mathbf{r^2+2m}) \equiv \mathbf{1}(\mathbf{mod\ 4m})$.
3. $R_{2m}(m \pm 2, m \pm 1)$ *where* **m is a multiple of 2 or 3**.
4. $R_{12m}(\pm(3m+2), \pm(3m-1))$ *and* $R_{12m}(\pm(3m-2), \pm(3m+1))$ *where* $\mathbf{m} \not\equiv \mathbf{0}(\mathbf{mod\ 4})$.
5. $R_{2m}(2b,r)$, *where* $b^2 \equiv \pm 1 (mod\ m)$, $2 \leq 2b \leq m$, *and* $r \in \{1, m-1\}$ *is odd.* □

Before stating the proof, we note a few generic automorphisms and other properties of $R_n(a,r)$. Other automorphisms, specific to any particular family of rose window graphs, will be introduced whenever they are needed.

1. Define $\tau : V \to V$ by $\tau(A_i) = A_{-i}$ and $\tau(B_i) = B_{-i}$. Clearly τ is an automorphism with $\tau^2 = \mathsf{id}$ and hence $R_n(a,r) \cong R_n(-a,r)$.
2. $R_n(a,r) = R_n(a,-r)$.
3. Define $\rho : V \to V$ by $\rho(A_i) = A_{i+1}$ and $\rho(B_i) = B_{i+1}$; and $\mu : V \to V$ by $\mu(A_i) = A_{-i}$ and $\mu(B_i) = B_{-a-i}$. Clearly ρ and μ are automorphisms. As $\rho^n = \mu^2 = \mathsf{id}$ and $\mu\rho\mu = \rho^{-1}$, we have $\langle \rho, \mu \rangle \cong D_n$.
4. If $(n,r) = 1$, then $\zeta : V \to V$ given by $\zeta(A_i) = B_{-ir^{-1}}$ and $\zeta(B_i) = A_{-ir^{-1}}$ is an automorphism and hence $R_n(a,r) \cong R_n(ar^{-1}, r^{-1})$.

Remark 1.1. *In view of the first two observations, it is enough to study $R_n(a,r)$ for $1 \leq a, r \leq \lceil \frac{n}{2} \rceil$.*

The main theorem, which is repeatedly used in the proofs throughout the paper, is the following:

Proposition 1.1. *A vertex-transitive graph G is Cayley if and only if $\mathsf{Aut}(G)$ has a subgroup H which acts regularly on the vertices of G. In particular, non-identity elements of H do not stabilize any vertex.* □

Remark 1.2. *In this context, it is to be noted that if a group of order n acts transitively on a set of order n, then the action is regular.*

2 Family-1 [$R_n(a,r)$: $r^2 \equiv \pm 1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$]

If $r^2 \equiv \pm 1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$, then $\delta : V \to V$ given by $\delta(A_i) = B_{ri}$ and $\delta(B_i) = A_{ri}$ is an automorphism. For proof, see Lemma 2 [6] or Lemma 3.7 [1]. If $r^2 \equiv 1 (mod\ n)$, then $\delta^2 = \mathsf{id}$ and if $r^2 \equiv -1 (mod\ n)$, then $\delta^2 = \tau$, i.e., δ is of order 4.

Theorem 2.1. *If $r^2 \equiv 1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$, then $R_n(a,r)$ is a Cayley graph.*
Proof: Since $R_n(a,r) = R_n(a,-r)$, without loss of generality, we can assume that $ra \equiv -a (mod\ n)$. Consider ρ and δ as defined above. We have $\rho^n = \delta^2 = \mathsf{id}$ and $\delta\rho\delta = \rho^r$. Define

$$\begin{aligned} H = \langle \rho, \delta \rangle &= \langle \rho, \delta : \rho^n = \delta^2 = \mathsf{id}; \delta\rho\delta = \rho^r \rangle \\ &= \{\mathsf{id}, \rho, \rho^2, \ldots, \rho^{n-1}, \delta, \rho\delta, \rho^2\delta, \ldots, \rho^{n-1}\delta\}. \end{aligned}$$

Clearly, H is a subgroup of $\mathsf{Aut}(R_n(a,r))$. It suffices to show that H acts regularly on $R_n(a,r)$. For that we observe that

- $\rho^j(A_i) = A_{i+j}$ and $\rho^j(B_i) = B_{i+j}$, and
- $\rho^j\delta(A_i) = B_{ri+j}$ and $\rho^j\delta(B_i) = A_{ri+j}$.

As $gcd(r,n) = 1$, the map $i \mapsto ri + j$ is a bijection on $\{0, 1, \ldots, n-1\}$. Thus H acts transitively on $R_n(a,r)$. It is also clear from the construction of H, that for any pair of vertices in $R_n(a,r)$, there exists a unique element in H which maps one to the other. Hence, $R_n(a,r)$ is a Cayley graph. □

Lemma 2.2. *If $r^2 \equiv -1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$, then n is even, a is odd and $n = 2a$.*
Proof: Let p be an odd prime factor of n such that $p^i | n$ and $p^{i+1} \nmid n$. Then $r^2 \equiv -1 (mod\ p^i)$ and $r^2 \equiv -1 (mod\ p)$. Again, $p^i | a(r \pm 1)$, i.e., $p | a(r \pm 1)$. If $p | (r \pm 1)$, then $r^2 \equiv 1 (mod\ p)$, a contradiction, as $-1 \not\equiv 1 (mod\ p)$. Thus for all odd prime factors p of n, we have $p^i | a$. Hence, if n is odd, then $n = a$, a contradiction (See Remark 1.1). Thus n is even.

We claim that $2 | n$ but $4 \nmid n$. Because if $4 | n$, then $r^2 \equiv -1 (mod\ 4)$. However, there does not exist any such r. Thus n is 2 times the product of some odd primes. Also, all the odd prime factors of n are also factors of a, as seen above. Thus, if $2 | a$, then $n = a$, a contradiction (See Remark 1.1). Thus $2 \nmid a$ and hence a is odd and $n = 2a$. □

Theorem 2.3. *If $r^2 \equiv -1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$, then $R_n(a,r)$ is a Cayley graph.*
Proof: Let $\alpha = \rho^2; \beta = \rho\delta^2; \gamma = \mu\delta$. Clearly, $\alpha, \beta, \gamma \in \mathsf{Aut}(R_n(a,r))$. It can be easily checked that $\beta\alpha = \alpha^{-1}\beta; \gamma\alpha = \alpha^{-r}\gamma$ and $\gamma^2 = \alpha^{\frac{a-1}{2}}\beta$. Define

$$\begin{aligned} H &= \langle \alpha, \beta, \gamma : \alpha^{n/2} = \beta^2 = \gamma^4 = \mathsf{id}; \beta\alpha = \alpha^{-1}\beta; \gamma\alpha = \alpha^{-r}\gamma; \gamma^2 = \alpha^{\frac{a-1}{2}}\beta \rangle \\ &= \{\alpha^i\beta^j\gamma^k : 0 \le i < n/2, 0 \le j,k \le 1\} \end{aligned}$$

Note that, from the above lemma, $n/2$ and $(a-1)/2$ are positive integers. We claim that the elements in H are distinct. If not, suppose

$$\alpha^{i_1}\beta^{j_1}\gamma^{k_1} = \alpha^{i_2}\beta^{j_2}\gamma^{k_2},\ \text{where } 0 \le i_1, i_2 < n/2, 0 \le j_1, j_2 \le 1, 0 \le k_1, k_2 \le 1,$$

i.e.,

$$\beta^{-j_2}\alpha^{i_1-i_2}\beta^{j_1} = \gamma^{k_2-k_1},\ \text{where } k_2 - k_1 = 0 \text{ or } 1.$$

Now, as $\gamma = \mu\delta$ flips A_i's and B_j's, and α, β maps A_i's to A_j's and B_i's to B_j's, $k_2 - k_1$ must be 0, i.e., $k_1 = k_2$. Thus, we have

$$\alpha^{i_1-i_2} = \beta^{j_2-j_1},\ \text{where } j_2 - j_1 = 0 \text{ or } 1.$$

If $j_2 - j_1 = 1$, then $\alpha^{i_1-i_2} = \beta = \rho\delta^2$. But $\alpha^{i_1-i_2}(A_0) = A_{2(i_1-i_2)}$ (even index) and $\rho\delta^2(A_0) = A_1$ (odd index). Hence, $j_2 - j_1 = 0$, i.e., $j_1 = j_2$. This implies $\alpha^{i_1-i_2} = \mathsf{id}$ and as a result $i_1 = i_2$. Thus the elements of H are distinct and $|H| = n/2 \times 2 \times 2 = 2n$.
We claim that H acts transitively on $R_n(a,r)$. It suffices to show that the stabilizer of A_0 in H, $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$.

Let $\alpha^i\beta^j\gamma^k \in \mathsf{Stab}_H(A_0)$, i.e., $\alpha^i\beta^j\gamma^k(A_0) = A_0$. Since, γ flips A_i's and B_j's, and α, β do not, we have $k = 0$. Thus, $\alpha^i\beta^j(A_0) = A_0$. If $j = 1$, then $\alpha^i\beta(A_0) = \alpha^i\rho\delta^2(A_0) = \rho^{1+2i}\delta^2(A_0) = A_0$, i.e., $A_{1+2i} = A_0$, a contradiction, as the parity of indices on both sides does not match. Thus, $j = 0$ and we have $\alpha^i(A_0) = A_0$. But this implies $A_{2i} = A_0$, i.e., $i = 0$. Hence $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$.

Finally, in view of Remark 1.2, H acts regularly on $R_n(a,r)$ and hence $R_n(a,r)$ is a Cayley graph. □

3 Family-2 [$R_{4m}(2m,r)$: r is odd and $(r^2+2m) \equiv \pm1(mod\,4m)$]

Proposition 3.1. *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm1(mod\ n)$, then*

- *$gcd(r,n) = 1$.*
- *If $\gamma : V \to V$ be defined by $\gamma(A_i) = B_{ri}$ and $\gamma(B_i) = A_{(r+a)i}$, then $\gamma \in \mathsf{Aut}(R_n(a,r))$.*

Proof: Let $n = 4m$ and $a = 2m$, and let if possible, $gcd(r,n) = l > 1$. As r is odd, $l|m$. Thus $r = lt$ and $m = ls$ for some $s,t \in \mathbb{N}$. Thus $n = 4ls, a = 2ls$ and $r = lt$. Now $(r^2 + n/2) \equiv \pm1(mod\ n)$ implies $l^2t^2 + 2ls \equiv \pm1(mod\ 4ls)$, which in turn implies $l|(l^2t^2 + 2ls \pm 1)$, i.e., $l|1$, a contradiction. Thus $gcd(r,n) = 1$.

γ, as defined above, has been shown to be in $\mathsf{Aut}(R_n(a,r))$ in Lemma 3.8 [1]. □

Proposition 3.2. *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv 1(mod\ n)$, then*

- *$r^{-1} = r + a\ (mod\ n)$*
- *$\zeta \in \mathsf{Aut}(R_n(a,r))$ (defined before) takes the following form: $\zeta(A_i) = B_{-(r+a)i}$ and $\zeta(B_i) = A_{-(r+a)i}$, and $\zeta^4 = \mathsf{id}$.*

Proof: $r(r+a) \equiv r^2 + ar \equiv 1 - a + ar \equiv 1 + a(r-1) \equiv 1\ (mod\ n)$. The last equivalence holds as r is odd and $a = n/2$. Thus $r^{-1} = r + a$ (mod n). The form of ζ follows immediately from the fact that $r^{-1} = r + a$ (mod n). □

Theorem 3.1. *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv 1(mod\ n)$, then $R_n(a,r)$ is a Cayley graph.*
Proof: Let $\alpha = \rho^2, \beta = \rho\mu$ and $\sigma = \gamma\zeta^2$, where γ and ζ are as defined in Propositions 3.1 and 3.2. It can be easily checked that $\sigma(A_i) = B_{(r+a)i}$ and $\sigma(B_i) = A_{ri}$; $\alpha^{n/2} = \beta^2 = \sigma^2 = \mathsf{id}$; $\beta\alpha\beta = \alpha^{-1}, \sigma\alpha\sigma = \alpha^r, (\beta\sigma)^2 = \alpha^{\frac{a-r+1}{2}}$. Define

$$H = \langle \alpha, \beta, \sigma : \alpha^{n/2} = \beta^2 = \sigma^2 = \mathsf{id}; \beta\alpha\beta = \alpha^{-1}, \sigma\alpha\sigma = \alpha^r, (\beta\sigma)^2 = \alpha^{\frac{a-r+1}{2}} \rangle$$

$$= \{\alpha^i \beta^j \sigma^k : 0 \le i < n/2, 0 \le j, k \le 1\}$$

We claim that the elements in H are distinct. If not, suppose

$$\alpha^{i_1} \beta^{j_1} \sigma^{k_1} = \alpha^{i_2} \beta^{j_2} \sigma^{k_2}, \text{ where } 0 \le i_1, i_2 < n/2, 0 \le j_1, j_2, k_1, k_2 \le 1,$$

i.e.,

$$\alpha^{i_1 - i_2} \beta^{j_1} \sigma^{k_1 - k_2} = \beta^{j_2}, \text{ where } k_1 - k_2 = 0 \text{ or } 1.$$

Now, as σ flips A_i's and B_j's, and α, β maps A_i's to A_j's and B_i's to B_j's, $k_1 - k_2$ must be 0, i.e., $k_1 = k_2$. Thus, we have

$$\alpha^{i_1 - i_2} = \beta^{j_2 - j_1}, \text{ where } j_2 - j_1 = 0 \text{ or } 1.$$

Since, α maintains the parity of indices and β flips the parity of indices of A_i's and B_i's, $j_2 - j_1$ is even, i.e., $j_1 = j_2$. This implies $\alpha^{i_1 - i_2} = \mathsf{id}$ and as a result $i_1 = i_2$. Thus the elements of H are distinct and $|H| = n/2 \times 2 \times 2 = 2n$.
We claim that H acts transitively on $R_n(a, r)$. In order to prove it, we show that the orbit of A_0, $\mathcal{O}_{A_0}$, under the action of H is the vertex set of $R_n(a, r)$. By orbit-stabilizer theorem, we get

$$|\mathcal{O}_{A_0}| = \frac{|H|}{|\mathsf{Stab}_H(A_0)|}.$$

As the number of vertices in $R_n(a, r)$ is $2n$ and $|H| = 2n$, it is enough to show that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$. Let $\alpha^i \beta^j \sigma^k$ be an arbitrary element of H which stabilizes A_0, i.e., $\alpha^i \beta^j \sigma^k (A_0) = A_0$, with $0 \le i < n/2, 0 \le j, k \le 1$. Now, as σ flips A_i's and B_j's, and α, β maps A_i's to A_j's and B_i's to B_j's, $k = 0$. Thus $\alpha^i \beta^j (A_0) = A_0$, i.e., $\alpha^{-i}(A_0) = \beta^j(A_0)$. Since, α maintains the parity of indices and β flips the parity of indices of A_i's and B_i's, $j = 0$ and hence $i = 0$. Thus $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$.

Finally, in view of Remark 1.2, H acts regularly on $R_n(a, r)$ and hence $R_n(a, r)$ is a Cayley graph. □

In **Family 2**, if $(r^2 + n/2) \equiv -1 (mod\ n)$, we will show that $R_n(a, r)$ is not a Cayley graph. In order to prove it, we recall a few observations and results.

Remark 3.1. *It was noted in [6] and [1], that $R_n(a, r)$ has either one or two or three edge orbits. If it has one edge orbit, then by definition, it is edge transitive, as in Theorem 1.1. If $R_n(a, r)$ has two edge orbits, then one orbit consists of rim and hub edges, and the other consists of spoke edges. If $R_n(a, r)$ has three orbits on edges, then the first one consists of rim edges, the second one consists of hub edges, and the third one consists of spoke edges.*

As **Family 3, 4, 5** in Theorem 1.2 are also edge transitive, they have only one edge orbit. On the other hand, family 1 and 2 in Theorem 1.2, have two edge orbits, as evident from Remark 3.1 and Theorem 3.2.

Theorem 3.2 (Theorem 2.3,[1])**.** *There is an automorphism of $R_n(a, r)$ sending every rim edge to a hub edge and vice-versa if and only if one of the following holds:*

1. *$a \neq n/2$, $r^2 \equiv 1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$;*
2. *$a = n/2$, $r^2 \equiv \pm 1 (mod\ n)$ and $ra \equiv \pm a (mod\ n)$;*

3. n is divisible by 4, $gcd(n, r) = 1$, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1(mod\ n)$.

Corollary 3.3 (Corollary 3.9,[1])**.** *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1(mod\ n)$, then the automorphism group of $R_n(a, r)$ has two edge orbits and the full automorphism group of $R_n(a, r)$, $\mathsf{Aut}(R_n(a, r)) = \langle \rho, \mu, \gamma \rangle$, where γ is as defined in Proposition 3.1.*

Theorem 3.4. *If n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv -1(mod\ n)$, then $R_n(a, r)$ is not a Cayley graph.*
Proof: As evident from Corollary 3.3, the full automorphism group of $R_n(a, r)$ is given by

$$\mathsf{Aut}(R_n(a, r)) = \langle \rho, \mu, \gamma : \rho^n = \mu^2 = \gamma^4 = \mathsf{id}; \mu\rho\mu = \rho^{-1}, \gamma\mu = \rho^a\mu\gamma, \gamma\rho = \rho^{r-a}\mu\gamma^3 \rangle.$$

One can easily check the relations between the generators starting from the definition and conclude that $|\mathsf{Aut}(R_n(a, r))| = n \times 2 \times 4 = 8n$. If possible, let $R_n(a, r)$ be a Cayley graph with a regular subgroup H of $\mathsf{Aut}(R_n(a, r))$ and $|H| = 2n$.

Let $K = \langle \gamma \rangle$. Then $|K| = 4$ and $H \cap K$ is a subgroup of K. As $\gamma^2(A_0) = A_0$, i.e., γ^2 has a fixed point, $\gamma^2 \notin H$. Thus $H \cap K = \{\mathsf{id}\}$ and

$$|HK| = \frac{|H||K|}{|H \cap K|} = 8n.$$

Hence $\mu \in \mathsf{Aut}(R_n(a, r)) = HK$. Thus $\mu = hk$, where $h \in H$ and $k \in K = \{\mathsf{id}, \gamma, \gamma^2, \gamma^3\}$. If $k = \mathsf{id}$, then $\mu = h \in H$. But as $\mu(A_0) = A_0$, i.e., μ has a fixed point, $\mu \notin H$. Thus $k \neq \mathsf{id}$.
If $k = \gamma^2$, then $\mu = h\gamma^2$, i.e., $h = \mu\gamma^2 \in H$. But as $\mu\gamma^2(A_0) = A_0$, $\mu\gamma^2 \notin H$ and hence $k \neq \gamma^2$.
If $k = \gamma$, then $\mu\gamma^{-1} = h$, i.e., $h^{-2} = (\gamma\mu)^2 = \rho^a\gamma^2 \in H$. But, as $\rho^a\gamma^2(A_{a/2}) = A_{a/2}$, by similar argument, $k \neq \gamma$.
If $k = \gamma^3$, then $h^2 = (\mu\gamma)^2 = \rho^a\gamma^2 \in H$. By similar argument as above, $k \neq \gamma^3$.
As all the four possible choices of $k \in K$ leads to contradiction, we conclude that there does not exist any regular subgroup H of $\mathsf{Aut}(R_n(a, r))$ and hence $R_n(a, r)$ is not a Cayley graph. □

4 Family-3 $[R_{2m}(m \pm 2, m \pm 1)]$

As $m + 2 \equiv -(m - 2)\ (mod\ 2m)$ and $m + 1 \equiv -(m - 1)\ (mod\ 2m)$, it suffices to check the family $R_{2m}(m - 2, m - 1)$. It was proved in Section 3.2 of [5], that

$$G := \mathsf{Aut}(R_{2m}(m - 2, m - 1)) = \langle \rho, \mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1} \rangle = K \rtimes \langle \rho\varepsilon_0, \mu\rho^m \rangle \cong \mathbb{Z}_2^m \rtimes D_m,$$

where $K = \langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1} \rangle \cong \mathbb{Z}_2^m$, D_m is the dihedral group and ε_i is the involution given by $(A_i, B_{i-1})(A_{i+m}, B_{i-1+m})(A_{i+1}, B_{i+m})(A_{i+1+m}, B_i)$. Thus $|G| = 2^{m+1}m$. One can easily check that the following relations between the generators hold:

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i;\ \varepsilon_i\rho^m = \rho^m\varepsilon_i;\ \mu\varepsilon_i = \varepsilon_{m-1-i}\mu;$$

$$\rho\varepsilon_i = \varepsilon_{i+1}\rho, \forall i, j \in \{0, 1, \ldots, m - 1\} \text{ and } \varepsilon_0\varepsilon_1 \cdots \varepsilon_{m-1} = \rho^m$$

where the addition of indices of ε_i's are done modulo m. Using this relations, it is easy to see that $\circ(\rho\varepsilon_i) = m$ and $\circ(\mu\rho^i) = 2$.

It follows from definition that $\rho^{2i}\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{i-2}, \varepsilon_{i+1}, \ldots, \varepsilon_{m-1} \in \mathsf{Stab}_G(A_i)$. Again, using the relations between generators, we get $|\langle \rho^{2i}\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{i-2}, \varepsilon_{i+1}, \ldots, \varepsilon_{m-1}\rangle| = 2^{m-1}$. Now, as $R_{2m}(m-2, m-1)$ is a vertex transitive graph, by orbit-stabilizer theorem, it follows that $|G|/|\mathsf{Stab}_G(A_i)| = 2 \times 2m$, i.e., $|\mathsf{Stab}_G(A_i)| = \frac{2^{m+1}m}{4m} = 2^{m-1}$. Thus, we have

$$\mathsf{Stab}_G(A_i) = \langle \rho^{2i}\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{i-2}, \varepsilon_{i+1}, \ldots, \varepsilon_{m-1}\rangle.$$

Similarly, it follows that

$$\mathsf{Stab}_G(B_i) = \langle \rho^{m-2+2i}\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+2}, \ldots, \varepsilon_{m-1}\rangle.$$

Theorem 4.1. *$R_{2m}(m-2, m-1)$ is a Cayley graph, if m is even.*
Proof: In this case, $n = 2m$, $a = m-2$ and $r = m-1$. Now, if m is even, we have

$$r^2 = (m-1)^2 = m^2 - 2m + 1 \equiv 1(mod\ 2m) \equiv 1(mod\ n) \text{ and}$$

$$ra = (m-1)(m-2) = m^2 - 3m + 2 \equiv -m + 2(mod\ 2m) \equiv -a(mod\ n).$$

Thus, if m is even, $R_{2m}(m-2, m-1)$ is a subfamily of **Family-1** and as a result, $R_{2m}(m-2, m-1)$ is a Cayley graph. □

Theorem 4.2. *$R_{2m}(m-2, m-1)$ is a Cayley graph, if m is an odd multiple of* 3.
Proof: Let $m = 3l$. For $i = 0, 1, 2$, denote by Σ_i, the product of all ε_j's such that $j \neq i\ (mod\ 3)$. Note that $\Sigma_i\Sigma_j = \Sigma_k$ for distinct i, j, k's in $\{0, 1, 2\}$ and $\circ(\Sigma_i) = 2$.

Let $\alpha = \rho^2, \beta = \Sigma_0$ and $\gamma = \Sigma_1$. It can be easily checked that $\beta\alpha = \alpha\gamma, \gamma\alpha = \alpha\beta\gamma$ and $\beta\gamma = \gamma\beta$. Define

$$H = \langle \alpha, \beta, \gamma : \circ(\alpha) = m, \circ(\beta) = \circ(\gamma) = 2; \beta\alpha = \alpha\gamma, \gamma\alpha = \alpha\beta\gamma, \beta\gamma = \gamma\beta\rangle.$$

Thus, any element of H can be expressed as $\alpha^i\beta^j\gamma^k$ where $0 \leq i \leq m-1, 0 \leq j, k \leq 1$, i.e., $|H| \leq 4m$.
Claim 1: $|H| = 4m$.
Proof of Claim 1: If not, there exist $0 \leq i_1, i_2 \leq m-1, 0 \leq j_1, j_2, k_1, k_2 \leq 1$ such that $\alpha^{i_1}\beta^{j_1}\gamma^{k_1} = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$, i.e.,

$$\rho^{2(i_1-i_2)} = \alpha^{i_1-i_2} = \beta^{j_2-j_1}\gamma^{k_2-k_1} \text{ (as } \beta\gamma = \gamma\beta).$$

If $j_2 - j_1 = k_2 - k_1 = 0$, then $i_1 = i_2$ (since, $\circ(\rho) = 2m$) and as a result the claim is true. However, if any one or both of $j_2 - j_1$ or $k_2 - k_1$ is 1, then the right hand side is an element of order 2. As a result, the left hand side must be an element of order 2, which implies $2(i_1 - i_2) = m$. However, as m is odd, this can not hold. As a result, the claim is true, i.e., $|H| = 4m$.

As in proof of Theorem 3.1, it is enough to show that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$. Let $\alpha^i\beta^j\gamma^k \in \mathsf{Stab}_H(A_0)$, i.e., $\alpha^i\beta^j\gamma^k(A_0) = A_0$ for some i, j, k with $0 \leq i \leq m-1$, $0 \leq j, k \leq 1$. Therefore,

$$\beta^j\gamma^k(A_0) = A_{2m-2i} \tag{1}$$

Claim 2: $k = 0$.
Proof of Claim 2: If not, let $k = 1$, i.e., $\beta^j\gamma(A_0) = A_{2m-2i}$. Note that

- both ε_0 and ε_{m-1} occurs in the expression of γ, and
- all ε_i's except ε_0 and ε_{m-1} stabilizes A_0.

Thus $A_{2m-2i} = \beta^j\gamma(A_0) = \beta^j\varepsilon_{m-1}\varepsilon_0(A_0) = \beta^j\varepsilon_{m-1}(B_{2m-1}) = \beta^j(A_m)$. If $j = 0$, then we have $A_m = A_{2m-2i}$, which is a contradiction, due to mismatch of parity of indices. If $j = 1$, then we have $\beta(A_m) = A_{2m-2i}$. Note that

- $\mathsf{Stab}_G(A_0) = \mathsf{Stab}_G(A_m) = \langle \mu, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-2}\rangle$.
- ε_0 does not occur in the expression of β, but ε_{m-1} occur in the expression of β.

Thus, we have $A_{2m-2i} = \beta(A_m) = \varepsilon_{m-1}(A_m) = B_{2m-1}$, a contradiction. Hence for $k = 1$, both $j = 0$ or $j = 1$ leads to a contradiction, and as a result $k = 0$.

Thus, from Equation 1, we have $\beta^j(A_0) = A_{2m-2i}$. If $j = 1$, then $A_{2m-2i} = \beta(A_0) = \varepsilon_{m-1}(A_0) = B_{m-1}$, a contradiction. Thus, $j = 0$ and hence we have $A_0 = A_{2m-2i}$ i.e., $2m \equiv 2i\ (mod\ 2m)$, i.e., $i \equiv m \equiv 0\ (mod\ m)$. Thus $i = 0$. This implies that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$ and hence the theorem holds. □

Theorem 4.3. *$R_{2m}(m-2, m-1)$ is not a Cayley graph, if m is odd and $m \not\equiv 0\ (mod\ 3)$.*

Proof: Consider $K = \langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}\rangle$. Then $K \cong \mathbb{Z}_2^m$ and $|K| = 2^m$ as $\circ(\varepsilon_i) = 2$ and $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \forall i, j \in \{0, 1, \ldots, m-1\}$.

If possible, let H be a regular subgroup of G. Then $|H| = 4m$. Thus

$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{2^2 m \cdot 2^m}{|H \cap K|} \leq 2^{m+1}m, \text{ i.e., } |H \cap K| \geq 2.$$

Now, as $|H| = 4m$, where m is odd and $|K| = 2^m$, we have $|H \cap K| = 2$ or 4. We will prove that $|H \cap K| = 4$. In fact, using the next two claims, we prove that $|H \cap K| \neq 2$.

Claim 1: If $|H \cap K| = 2$, then the non-identity element of $H \cap K$ must be $\varepsilon_0\varepsilon_1 \cdots \varepsilon_{m-1} = \rho^m$.

Proof of Claim 1: Let $\alpha = \varepsilon_{l_1}\varepsilon_{l_2} \cdots \varepsilon_{l_p}$ be the non-identity element of $H \cap K$. Let $L = \langle \mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}\rangle$. Then $|L| = 2^{m+1}$ and $K \subsetneq L$ as $\mu \in L \setminus K$. Thus

$$|HL| = \frac{|H||L|}{|H \cap L|} = \frac{4m \cdot 2^{m+1}}{|H \cap L|} \leq |G| = 2^{m+1}m, \text{ i.e., } |H \cap L| \geq 4.$$

As $|H \cap K| = 2$ and $K \subsetneq L$, there exists atleast one element of the form $\beta = \mu\varepsilon_{i_1}\varepsilon_{i_2} \cdots \varepsilon_{i_s}$ in $H \cap L$.

Again, let $L' = \langle \rho\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}\rangle$. By similar arguments, we can deduce that $|H \cap L'| \geq 4$. So there exists an element of the form $\gamma = \rho\mu\varepsilon_{j_1}\varepsilon_{j_2} \cdots \varepsilon_{j_t}$ in $H \cap L'$.

As $\alpha, \beta, \gamma \in H$, it follows that $\beta\alpha\beta^{-1}, \gamma\alpha\gamma^{-1} \in H$. Observe that

$$\beta\alpha\beta^{-1} = (\mu\varepsilon_{i_1}\varepsilon_{i_2} \cdots \varepsilon_{i_s})(\varepsilon_{l_1}\varepsilon_{l_2} \cdots \varepsilon_{l_p})(\mu\varepsilon_{i_1}\varepsilon_{i_2} \cdots \varepsilon_{i_s})^{-1} = \mu(\varepsilon_{l_1}\varepsilon_{l_2} \cdots \varepsilon_{l_p})\mu.$$

As $\mu\varepsilon_i = \varepsilon_{m-1-i}\mu$, $\beta\alpha\beta^{-1}$ is product of some ε_i's and hence $\mathsf{id} \neq \beta\alpha\beta^{-1} \in H \cap K$. Since $|H \cap K| = 2$, then $\alpha = \beta\alpha\beta^{-1}$.

Similarly,

$$\gamma\alpha\gamma^{-1} = (\rho\mu\varepsilon_{j_1}\varepsilon_{j_2} \cdots \varepsilon_{j_t})(\varepsilon_{l_1}\varepsilon_{l_2} \cdots \varepsilon_{l_p})(\rho\mu\varepsilon_{j_1}\varepsilon_{j_2} \cdots \varepsilon_{j_t})^{-1} = \rho(\mu\varepsilon_{l_1}\varepsilon_{l_2} \cdots \varepsilon_{l_p}\mu)\rho^{-1}$$

$$= \rho(\beta\alpha\beta^{-1})\rho^{-1} = \rho\alpha\rho^{-1}.$$

As $\rho\varepsilon_i = \varepsilon_{i+1}\rho$, $\rho\alpha\rho^{-1}$ is product of some ε_i's and hence $\gamma\alpha\gamma^{-1} \in H \cap K$ and by similar arguments, we have $\alpha = \gamma\alpha\gamma^{-1}$.

Thus, using $\rho\varepsilon_i = \varepsilon_{i+1}\rho$, we get

$$\varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p} = \alpha = \rho\alpha\rho^{-1} = \rho(\varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p})\rho^{-1} = \varepsilon_{l_1+1}\varepsilon_{l_2+1}\cdots\varepsilon_{l_p+1} \tag{2}$$

As $K = \langle\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m-1}\rangle \cong \mathbb{Z}_2^m$ and ε_i's corresponds to the standard generators of $\mathbb{Z}_2^m$, i.e., $\varepsilon_i \leftrightarrow (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the only 1 occuring in the $(i+1)$th position, $\varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p}$ corresponds to the vector in $\mathbb{Z}_2^m$ with 1's in $l_1+1, l_2+1, \ldots, l_p+1$ positions and $\varepsilon_{l_1+1}\varepsilon_{l_2+1}\cdots\varepsilon_{l_p+1}$ corresponds to the vector with 1's in $l_1+2, l_2+2, \ldots, l_p+2$ positions. Thus, from Equation 2, we get that all the positions in the vector must be 1, i.e., $\alpha = \varepsilon_0\varepsilon_1\cdots\varepsilon_{m-1} = \rho^m$. Hence the claim is true.

Claim 2: If $|H \cap K| = 2$, then $\rho^m \notin H$

Proof of Claim 2: As $H \cap L$ is a subgroup of H and m is odd, therefore $4 \leq |H \cap L| \mid 4m$ implies $|H \cap L| = 4$. Thus $H \cap L$ is either isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. Note that any non-identity element $\sigma \in H \cap L$ must contain in its expression either ε_0 or ε_{m-1}, as otherwise $\sigma \in \langle\mu, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-2}\rangle = \mathsf{Stab}_G(A_0)$, a contradiction to the fact that σ belongs to a regular subgroup H.

Suppose that $H \cap L$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. As $H \cap K \subsetneq H \cap L$, therefore there exists a non-identity element in $H \cap L$ of the form $\sigma = \mu\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_s}$. As explained earlier, σ must contain in its expression either ε_0 or ε_{m-1}. In fact, in this case, both ε_0 and ε_{m-1} must occur in the expression of σ, as otherwise $\circ(\sigma) = 4$. Note that by Claim 1, $\rho^m \in H \cap L$. Thus, for all the three non-identity elements, ρ^m, σ, σ' (say) in $H \cap L$, both ε_0 and ε_{m-1} must occur. Also as $H \cap L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have $\sigma\sigma' = \rho^m$. But if σ, σ' contains both ε_0 and ε_{m-1}, then ρ^m contains neither ε_0 nor ε_{m-1}, a contradiction. Hence $H \cap L \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose that $H \cap L$ is isomorphic to $\mathbb{Z}_4$. As $\circ(\rho^m) = 2$, there exists a non-identity element $\zeta = \mu\varepsilon_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_s} \in H \cap L$ such that $\langle\zeta\rangle = H \cap L$ and $\zeta^2 = \rho^m$. Note that the number of ε_i's in the expression of ζ^2 is always even but that of ρ^m is m (odd) as $\rho^m = \varepsilon_0\varepsilon_1\cdots\varepsilon_{m-1}$. Hence, $H \cap L \ncong \mathbb{Z}_4$.

Thus, by Claim 1 and 2, we get $|H \cap K| = 4$. As $K \cong \mathbb{Z}_2^m$, we have $H \cap K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that

$$\mathsf{Stab}_G(B_{(m+3)/2}) = \langle\rho\mu, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{(m+1)/2}, \varepsilon_{(m+7)/2}, \ldots, \varepsilon_{m-1}\rangle.$$

Again, as the graph is vertex-transitive, by orbit-stabilizer theorem, we have $G = H \cdot \mathsf{Stab}_G(B_{(m+3)/2})$. Thus, $\rho = hb$, where $h \in H$ and $b \in \mathsf{Stab}_G(B_{(m+3)/2})$.

Claim 3: $\rho\mu$ does not occur in the expression of b.

Proof of Claim 3: If possible, let $b = \rho\mu\varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p}$ and hence $h = \rho b^{-1} = \mu\varepsilon_{t_1}\varepsilon_{t_2}\cdots\varepsilon_{t_p} \in H \cap L$. Again, as $H \cap K \subseteq H \cap L$ and $|H \cap L| = |H \cap K| = 4$, we have $H \cap K = H \cap L$. Thus, $h \in H \cap K \subset K$ and hence h does not contain μ in its expression, a contradiction. Thus Claim 3 is true.

Therefore, by Claim 3, $b = \varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p}$ and $h = \rho b^{-1} = \rho\varepsilon_{l_1}\varepsilon_{l_2}\cdots\varepsilon_{l_p} \in H$.

Let $H \cap K = \{\mathsf{id}, \alpha_1, \alpha_2, \alpha_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $h\alpha_i h^{-1} \in H$. As α_i's, being elements of K, are product of some ε_i's and $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$, $\rho\varepsilon_i = \varepsilon_{i+1}\rho$, we have

$$h\alpha_i h^{-1} = \rho\alpha_i\rho^{-1} = \rho(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_s})\rho^{-1} = \varepsilon_{i_1+1}\varepsilon_{i_2+1}\cdots\varepsilon_{i_s+1} \in K \text{ for } i = 1, 2, 3. \tag{3}$$

Thus $h\alpha_i h^{-1} \in H \cap K = \{\mathsf{id}, \alpha_1, \alpha_2, \alpha_3\}$.
Claim 4: $h\alpha_1 h^{-1} = \alpha_2$ or α_3.
Proof of Claim 4: If $h\alpha_1 h^{-1} = \mathsf{id}$, then $\alpha_1 = \mathsf{id}$, a contradiction.

If $h\alpha_1 h^{-1} = \alpha_1$, then as above, get $\varepsilon_{i_1+1}\varepsilon_{i_2+1}\cdots\varepsilon_{i_s+1} = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_s}$. Now, as in proof of Claim 1, we can argue that this implies $\alpha_1 = \rho^m$. But, in that case, we must have $h\alpha_2 h^{-1} = \alpha_3$ and $h\alpha_3 h^{-1} = \alpha_2$, because otherwise

- $h\alpha_2 h^{-1} = \mathsf{id}$ implies $\alpha_1 = \mathsf{id}$, a contradiction.
- $h\alpha_2 h^{-1} = \alpha_2$ implies $\alpha_2 = \rho^m$, a contradiction, as $\alpha_1 \neq \alpha_2$.
- $h\alpha_2 h^{-1} = \alpha_1$ implies $h\alpha_2 h^{-1} = h\alpha_1 h^{-1}$, i.e., $\alpha_1 = \alpha_2$, a contradiction.

Thus we have $h\alpha_2 h^{-1} = \rho\alpha_2\rho^{-1} = \alpha_3$ and $h\alpha_3 h^{-1} = \rho\alpha_3\rho^{-1} = \alpha_2$. Hence, from Equation 3, we see that both α_2 and α_3 are product of ε_i's and the number of ε_i's occuring in their expressions are same. Thus the number of ε_i's occuring in the expression of $\alpha_2\alpha_3$ is even. However, $\alpha_2\alpha_3 = \alpha = \rho^m = \varepsilon_0\varepsilon_1\cdots\varepsilon_{m-1}$ has odd number of ε_i's occuring in its expression. This is a contradiction and hence $h\alpha_1 h^{-1} \neq \alpha_1$. Thus Claim 4 is true.

Without loss of generality, we can assume that $h\alpha_1 h^{-1} = \alpha_2$. Thus $h\alpha_2 h^{-1}$ is either α_1 or α_3. If $h\alpha_2 h^{-1} = \alpha_1$, we must have $h\alpha_3 h^{-1} = \alpha_3$, a contradiction, as shown in Claim 4. Hence we have $h\alpha_2 h^{-1} = \alpha_3$ and similarly $h\alpha_3 h^{-1} = \alpha_1$. So, by Equation 3, we get $\rho\alpha_1\rho^{-1} = \alpha_2$, $\rho\alpha_2\rho^{-1} = \alpha_3$ and $\rho\alpha_3\rho^{-1} = \alpha_1$. Hence, we have

$$\alpha_1 = \rho\alpha_3\rho^{-1} = \rho(\rho\alpha_2\rho^{-1})\rho^{-1} = \rho^2(\rho\alpha_1\rho^{-1})\rho^{-2} = \rho^3\alpha_1\rho^{-3},\ i.e.,\ \rho^3\alpha_1 = \alpha_1\rho^3.$$

Similarly, we have $\rho^3\alpha_2 = \alpha_2\rho^3$ and $\rho^3\alpha_3 = \alpha_3\rho^3$.

Recall that $H \cap K = \{\mathsf{id}, \alpha_1, \alpha_2, \alpha_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and α_i's are product of some ε_j's. Let

$$\alpha_1 = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_l}; \alpha_2 = \varepsilon_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_p}; \alpha_3 = \varepsilon_{k_1}\varepsilon_{k_2}\cdots\varepsilon_{k_q}.$$

Note that each α_i must contain either ε_0 or ε_{m-1} in its expression, as otherwise it will be an element of $\mathsf{Stab}_G(A_0)$ and hence can not belong to H. As $\alpha_1\alpha_2 = \alpha_3$ and $\alpha_1\alpha_2\alpha_3 = \mathsf{id}$, without loss of generality, we can assume that, among ε_0 or ε_{m-1}, α_1 contains only ε_0, α_2 contains only ε_{m-1} and α_3 contains both ε_0 and ε_{m-1} in their expressions. This happens because if two of the α_i's contain both ε_0 and ε_{m-1} in their expressions, then the their product, i.e., the third α_i, will not have ε_0 or ε_{m-1} in its expression, thereby making it an element of $\mathsf{Stab}_G(A_0)$.

Now, from the relation $\rho^3\alpha_1 = \alpha_1\rho^3$ and using the fact that $\rho\varepsilon_i = \varepsilon_{i+1}\rho$, we get,

$$(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_l})\rho^3 = \rho^3(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_l}) = (\varepsilon_{i_1+3}\varepsilon_{i_2+3}\cdots\varepsilon_{i_l+3})\rho^3,$$

$$\text{i.e., } \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_l} = \varepsilon_{i_1+3}\varepsilon_{i_2+3}\cdots\varepsilon_{i_l+3}.$$

Now, as m is not a multiple of 3, m is of the form $3t+1$ or $3t+2$.

If $m = 3t+1$, then by using the standard generators of $\mathbb{Z}_2^m$, as in the proof of Claim 1, we get that all of $\varepsilon_0, \varepsilon_3, \varepsilon_6, \ldots, \varepsilon_{3t} = \varepsilon_{m-1}$ occurs in the expression of α_1, a contradiction to that fact that among ε_0 or ε_{m-1}, α_1 contains only ε_0.

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Similarly, if $m = 3t + 2$, we get all of

$$\varepsilon_0, \varepsilon_3, \varepsilon_6, \ldots, \varepsilon_{3t} = \varepsilon_{m-2}, \varepsilon_1, \varepsilon_4, \cdots, \varepsilon_{3t+1} = \varepsilon_{m-1}$$

occurs in the expression of α_1, a contradiction.

Thus, we conclude that there does not exist any regular subgroup H of $\mathsf{Aut}(R_{2m}(m-2, m-1))$ and hence $R_{2m}(m-2, m-1)$ is not a Cayley graph, when m is odd and not a multiple of 3. $\square$

5 Family-4 $[R_{12m}(\pm(3m+2), \pm(3m-1))$ and $R_{12m}(\pm(3m-2), \pm(3m+1))]$

As $R_n(a, r) = R_n(a, -r)$ and $R_n(a, r) \cong R_n(-a, r)$, it is enough to check $R_{12m}(3m+2, 3m-1)$ and $R_{12m}(3m-2, 3m+1)$. More precisely, it suffices to work with the family $R_{12m}(3d+2, 9d+1)$ where $d = \pm m$ $(mod\ 12m)$, as mentioned in Section 3.3 of [5]. Define σ as follows:

$$\sigma(A_i) = \begin{cases} A_i & \text{if } i \equiv 0\ (mod\ 3) \\ B_{i-1} & \text{if } i \equiv 1\ (mod\ 3) \\ B_{i-1-3d} & \text{if } i \equiv 2\ (mod\ 3) \end{cases} \quad \text{and} \quad \sigma(B_i) = \begin{cases} A_{i+1} & \text{if } i \equiv 0\ (mod\ 3) \\ A_{i+3d+1} & \text{if } i \equiv 1\ (mod\ 3) \\ B_{i+6d} & \text{if } i \equiv 2\ (mod\ 3) \end{cases}$$

Also, if $m \equiv 2$ $(mod\ 4)$, let $b = d + 1$ and define ω as follows:

$$\omega(A_i) = \begin{cases} A_{bi} & \text{if } i \equiv 0\ (mod\ 3) \\ B_{bi-b} & \text{if } i \equiv 1\ (mod\ 3) \\ B_{b+bi-1} & \text{if } i \equiv 2\ (mod\ 3) \end{cases} \quad \text{and} \quad \omega(B_i) = \begin{cases} A_{bi+1} & \text{if } i \equiv 0\ (mod\ 3) \\ A_{4+bi-4b} & \text{if } i \equiv 1\ (mod\ 3) \\ B_{b+bi-1} & \text{if } i \equiv 2\ (mod\ 3) \end{cases}$$

It was shown in [5], that

$$G := \mathsf{Aut}(R_{12m}(3d+2, 9d+1)) = \begin{cases} \langle \rho, \mu, \sigma, \omega \rangle, & \text{if } m \equiv 2\ (mod\ 4) \\ \langle \rho, \mu, \sigma \rangle, & \text{otherwise} \end{cases}$$

It is to be noted that $m \equiv 2$ $(mod\ 4)$ if and only if $-m \equiv 2$ $(mod\ 4)$. Thus, it is enough to work only with the family $R_{12m}(3m+2, 9m+1)$.

Theorem 5.1. *If m is odd and $m \neq 3$, then $R_{12m}(3m+2, 9m+1)$ is a Cayley graph.*
Proof: As m is odd, $G = \langle \rho, \mu, \sigma \rangle$. It can also be checked that $\sigma\rho^3\sigma = \rho^3; \sigma\mu = \mu\sigma; (\rho\sigma)^3 = \rho^{3(m+1)}; \circ(\sigma) = 2$. Let $\alpha = (\rho\sigma)^2$ and $\beta = \rho^2\mu\sigma$. As m is odd and $m \neq 3$, it can be shown that $\circ(\alpha) = 3m, \circ(\beta) = 8$ and $\beta\alpha = \alpha^{-1}\beta^{-1}$. Define

$$H = \langle \alpha, \beta : \circ(\alpha) = 3m, \circ(\beta) = 8; \beta\alpha = \alpha^{-1}\beta^{-1} \rangle$$

$$= \{\alpha^i\beta^j : 0 \leq i \leq 3m-1; 0 \leq j \leq 7\}$$

Claim 1: The elements in H are distinct.
If not, suppose

$$\alpha^{i_1}\beta^{j_1} = \alpha^{i_2}\beta^{j_2}, \text{ where } 0 \leq i_1, i_2 < 3m, 0 \leq j_1, j_2 \leq 8,$$

i.e.,

$$\alpha^{i_1-i_2} = \beta^{j_2-j_1}. \tag{4}$$

As $\alpha(A_0) = B_1, \alpha^2(A_0) = A_{3m+4}, \alpha^3(A_0) = A_{6m+6}, \alpha^4(A_0) = A_{6m+7}, \ldots, \alpha^{3m}(A_0) = A_0$, any power of α maps A_0 to $A_{0(mod\ 3)}$ or $A_{1(mod\ 3)}$ or $B_{1(mod\ 3)}$. On the other hand, as

$$\beta(A_0) = A_2, \beta^2(A_0) = B_{3m-1}, \beta^3(A_0) = B_{3m+1}, \beta^4(A_0) = A_{6m},$$

$$\beta^5(A_0) = A_{6m+2}, \beta^6(A_0) = B_{6m-1}, \beta^7(A_0) = B_{9m+1}, \beta^8(A_0) = A_0,$$

we see that β, β^2, β^5 and β^6 maps A_0 to $A_{2(mod\ 3)}$. Thus, $j_2 - j_1$ in Equation 4 can take values from $\{0, 3, 4, 7\}$.
If $j_2 - j_1 = 0$, then it is obvious that $i_1 = i_2$ and $j_1 = j_2$.
If $j_2 - j_1 = 4$, squaring Equation 4, we get, $\alpha^{2(i_1-i_2)} = \mathsf{id}$. Therefore, $3m|2(i_1 - i_2)$. Now, as $gcd(2, 3) = 1$ and m is odd, we have $3m|(i_1 - i_2)$, i.e., $i_1 = i_2$ and hence $j_1 = j_2$.
If $j_2 - j_1 = 3$, since $gcd(3, 8) = 1$, then $\circ(\beta^{j_2-j_1}) = 8$. Therefore, $\alpha^{8(i_1-i_2)} = \mathsf{id}$, i.e., $3m|8(i_1 - i_2)$. As m is odd, $3m$ is coprime to 8 and hence, $3m|(i_1 - i_2)$, i.e., $i_1 = i_2$ and $j_1 = j_2$.
The case $j_2 - j_1 = 7$ follows similarly as above. Thus combining all the cases, we see that elements of H are distinct and $H = 3m \times 8 = 24m$.
Claim 2: H acts transitively on $R_{12m}(3m + 2, 9m + 1)$.
In order to prove it, we show that the orbit of A_0, $\mathcal{O}_{A_0}$, under the action of H is the vertex set of $R_{12m}(3m + 2, 9m + 1)$. By orbit-stabilizer theorem, we get

$$|\mathcal{O}_{A_0}| = \frac{|H|}{|\mathsf{Stab}_H(A_0)|}.$$

As the number of vertices in $R_{12m}(3m + 2, 9m + 1)$ is $24m$ and $|H| = 24m$, it is enough to show that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$. Let $\alpha^i\beta^j$ be an arbitrary element of H which stabilizes A_0, i.e., $\alpha^{-i}(A_0) = \beta^j(A_0)$ with $0 \leq i \leq 3m - 1; 0 \leq j \leq 7$. Again, by mimicing the argument used in the proof of *Claim 1*, one can conclude that $j \in \{0, 3, 4, 7\}$.
If $j = 4$, then $\alpha^{-i}(A_0) = \beta^4(A_0) = A_{6m}$. Thus, $-i$ and hence i is a multiple of 3. [since, α^x sends A_0 to $A_{0(mod\ 3)}$, only if x is a multiple of 3] Let $-i = 3k$ and therefore $A_{6m} = \alpha^{3k}(A_0) = A_{k(6m+6)}$, i.e., $12m|k(6m + 6) - 6m$, i.e., $2m|m(k - 1) + k$, i.e., $m|k$ which implies $k = lm$. Again, as $2m|m(k - 1) + lm$, we have $2|k - 1 + l$, i.e., $2|l(m + 1) - 1$. But this is a contradiction, as $m + 1$ is even and hence $l(m + 1) - 1$ is odd. Thus $j \neq 4$.
If $j = 3$, then $\alpha^{-i}(A_0) = \beta^3(A_0) = B_{3m+1}$. As $3m+1 \equiv 1(mod\ 3)$, we have $-i = 3k+1$ [since, α^x sends A_0 to $B_{1(mod\ 3)}$, only if $x \equiv 1(mod\ 3)$] Therefore, $\beta^3(A_0) = B_{3m+1} = \alpha^{3k+1}(A_0) = \alpha^{3k}(B_1)$, i.e., $B_{3m+1} = B_{1+6mk+6k}$. This implies $12m|6mk+6k-3m$, i.e., $4m|2mk+2k-m$,i.e., $m|2k$ and, as m is odd, we have $m|k$. Let $k = lm$. Again, as $4m|2mk + 2lm - m$, we have $4|2k + 2l - 1$. However, this is a contradiction, as $2k + 2l - 1$ is odd and hence $j \neq 3$.
Using similar arguments as above, it can be shown that $j \neq 7$.
Thus, we have $j = 0$ and this, in turn, implies $i = 0$. Hence, $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$.

Finally, in view of Remark 1.2, H acts regularly on $R_{12m}(3m + 2, 9m + 1)$ and hence $R_{12m}(3m + 2, 9m + 1)$ is a Cayley graph, if m is odd and $m \neq 3$. □

Theorem 5.2. *If $m = 3$, then $R_{12m}(3m + 2, 9m + 1)$, i.e, $R_{36}(11, 28)$ is a Cayley graph.*
Proof: This can be checked by a Sage program. □

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Theorem 5.3. *If $m \equiv 0 (mod\ 4)$, then $R_{12m}(3m+2, 9m+1)$ is not a Cayley graph.*
Proof: As $m \not\equiv 2(mod\ 4)$,

$$G = \langle \rho, \mu, \sigma : \rho^n = \mu^2 = \sigma^2 = \mathsf{id}; \mu\rho\mu = \rho^{-1}, \sigma\rho^3\sigma = \rho^3, \sigma\mu = \mu\sigma,$$

$$(\rho\sigma)^3 = \rho^{3(m+1)}, (\rho\sigma\rho)^3 = \rho^{9m+6}\rangle, \text{ where } n = 12m$$

If possible, let $R_{12m}(3m+2, 9m+1)$ be a Cayley graph, H be a regular subgroup of G and $K = \mathsf{Stab}_G(A_0)$. Then $|G| = 96m = 8n$ (See Lemma 7.1 in Appendix), $|H| = 2n = 24m$ and $H \cap K = \{\mathsf{id}\}$.

Let $K' = \langle \rho \rangle$. Then $|K'| = n$ and $|HK'| = \frac{|H||K'|}{|H \cap K'|} = \frac{2n^2}{n/t} \leq |G| = 8n$, where t is a factor of n. Thus, $t \leq 4$, i.e., $t = 1, 2, 3$ or 4. If $t = 1$, then $H \cap K' = K'$, i.e., $\rho \in H$. If $t = 2$, then $H \cap K' = \langle \rho^2 \rangle$, i.e., $\rho^2 \in H$. If $t = 3$, then $H \cap K' = \langle \rho^3 \rangle$, i.e., $\rho^3 \in H$. If $t = 4$, then $H \cap K' = \langle \rho^4 \rangle$, i.e., $\rho^4 \in H$. Combining all the cases, we get that

$$\text{either } \rho^3 \in H \text{ or } \rho^4 \in H. \tag{5}$$

Claim: $\rho^4 \in H$.
Proof of Claim: Suppose that that $\rho^3 \in H$ but $\rho^4 \notin H$. Let $L = \langle \rho, \mu \rangle$. Then $|L| = 2n$. Therefore

$$|HL| = \frac{|H||L|}{|H \cap L|} = \frac{2n \cdot 2n}{2n/t} = 2nt \leq |G| = 8n, \text{ i.e., } t = 1, 2, 3 \text{ or } 4 \text{ and } t \text{ divides } 2n.$$

Therefore, $|H \cap L| = 2n, n, 2n/3$ or $n/2$, i.e., $|H \cap L| \geq n/2$. As $\rho^3 \in H \cap L$, we have $\langle \rho^3 \rangle \subseteq H \cap L$ and $|\langle \rho^3 \rangle| = n/3$. Thus, $(H \cap L) \setminus \langle \rho^3 \rangle \neq \emptyset$.

Now, as $\rho^{2i}\mu(A_i) = A_i$, $\rho^{2i}\mu \notin H$. Similarly, if $\rho^{2i+1}\mu \in H$, then $\rho^3 \cdot \rho^{2i+1}\mu \in H$, i.e., $\rho^{2i+4}\mu \in H$. Note that $2i+4$ is even and hence by previous argument, $\rho^{2i+4}\mu \notin H$, i.e., $\rho^{2i+1}\mu \notin H$. This shows that H does not contain any element of the form $\rho^i\mu$. Moreover, $\mu \notin H$. Now, as $(H \cap L) \setminus \langle \rho^3 \rangle \neq \emptyset$, H must contain an element of the form ρ^i, where i is not a multiple of 3. Again, as $\rho^3 \in H$, either ρ or $\rho^2 \in H$, i.e., $\rho^4 \in H$. This is a contradiction to the assumption that $\rho^4 \notin H$. Thus the claim is true.

Let $K'' = \langle \rho\sigma \rangle$. As $\circ(\rho\sigma) = n$, we have $|K''| = n$ and by similar arguments as above, we get that either $(\rho\sigma)^3 \in H$ or $(\rho\sigma)^4 \in H$.

Case 1: If $\rho^4 \in H$ and $(\rho\sigma)^4 \in H$, then

$$(\rho\sigma)^4 = (\rho\sigma)^3(\rho\sigma) = \rho^{3(m+1)}\rho\sigma = \rho^{3m+4}\sigma = \rho^{12l+4}\sigma = (\rho^4)^{3l+1}\sigma \in H \text{ [letting } m = 4l].$$

As $\rho^4 \in H$, therefore $\sigma \in H$. But as $\sigma(A_0) = A_0$, i.e., σ stabilizes A_0, it can not be in H. This is a contradiction.

Case 2: If $\rho^4 \in H$ and $(\rho\sigma)^3 \in H$, then $(\rho\sigma)^3 = \rho^{3(m+1)} = \rho^{12l+3} = (\rho^4)^{3l}\rho^3 \in H$, where $m = 4l$ i.e., $\rho^3 \in H$. Again, as $\rho^4 \in H$, we have $\rho \in H$. As $\circ(\rho) = n$ and $[H : \langle \rho \rangle] = 2$, $\langle \rho \rangle$ is normal in H.

From definition, it follows that $\mathsf{id}, \mu, \sigma, \mu\sigma \in K$. On the other hand, as $R_{12m}(3m+2, 9m+1)$ is vertex transitive, by orbit-stabilizer theorem, we have

$$|K| = \frac{|G|}{2n} = \frac{8n}{2n} = 4. \text{ Hence, } K = \mathsf{Stab}_G(A_0) = \{\mathsf{id}, \mu, \sigma, \mu\sigma\} \text{ and } |HK| = \frac{2n \cdot 4}{1} = 8n = |G|.$$

Thus, $HK = G$. As $\sigma\rho \in G$, it can be expressed in the form $\alpha\beta$, where $\alpha \in H$ and $\beta \in K = \{\mathsf{id}, \mu, \sigma, \mu\sigma\}$.
If $\beta = \mathsf{id}$, then $\alpha = \sigma\rho \in H$, i.e., $\sigma \in H$ (as $\rho \in H$), which is a contradiction, as H, being a regular subgroup can not contain any non-identity element which stabilizes A_0.
If $\beta = \mu$, then $\sigma\rho = \alpha\mu$, i.e., $\alpha = \sigma\mu\rho^{-1} \in H$, i.e., $\sigma\mu \in H$ (as $\rho \in H$), which is a contradiction.
If $\beta = \sigma$, then $\alpha = \sigma\rho\sigma \in H$. Since $\langle\rho\rangle$ is normal in H, therefore $(\sigma\rho\sigma)\rho(\sigma\rho\sigma)^{-1} \in H$, i.e.,

$$(\sigma\rho\sigma)\rho(\sigma\rho\sigma)^{-1} = (\sigma\rho\sigma)\rho\sigma\rho^{-1}\sigma = (\sigma\rho)^3\rho^{-2}\sigma = \rho^{3m+1}\sigma \in H \Rightarrow \sigma \in H \text{ (as } \rho \in H),$$

a contradiction.
If $\beta = \mu\sigma$, then $\sigma\rho = \alpha\mu\sigma$, i.e., $\alpha = \sigma\rho\mu\sigma \in H$. Since $\langle\rho\rangle$ is normal in H, therefore $(\sigma\rho\mu\sigma)\rho(\sigma\rho\mu\sigma)^{-1} \in H$, i.e.,

$$\begin{aligned}(\sigma\rho\mu\sigma)\rho(\sigma\mu\rho^{-1}\sigma) &= (\sigma\rho\mu\sigma)\rho(\sigma\rho\mu\sigma) = \sigma\rho\mu(\sigma\rho)^2\mu\sigma \\ &= \sigma\rho\mu(\rho^{3m+2}\sigma)\mu\sigma \quad [\text{as } (\sigma\rho)^3 = \rho^{3m+3}, \text{ we have } (\sigma\rho)^2 = \rho^{3m+2}\sigma] \\ &= \sigma\rho\mu\rho^{3m+2}\mu \qquad [\text{as } \sigma\mu = \mu\sigma \text{ and } \sigma^2 = \mathsf{id}] \\ &= \sigma\rho\rho^{-3m-2} = \sigma\rho^{-3m-1} \in H \Rightarrow \sigma \in H(\text{as } \rho \in H), \text{ a contradiction.}\end{aligned}$$

Thus, combining Case 1 and 2, we conclude that there does not exist any regular subgroup H of G, i.e., $R_{12m}(3m+2, 9m+1)$ is not a Cayley graph, if $m \equiv 0(mod\ 4)$. □

5.1 $m \equiv 2(mod\ 4)$

As $m \equiv 2(mod\ 4)$, $G = \langle\rho, \mu, \sigma, \omega\rangle$. It can be checked that $\sigma\rho^3\sigma = \rho^3; \sigma\mu = \mu\sigma; \sigma\omega = \omega\sigma; \omega\rho = \sigma\rho\omega; \omega\mu = \mu\omega\sigma; (\rho\sigma)^3 = \rho^{3(m+1)}; \omega\rho^{3l} = \rho^{3l(m+1)}; (\rho\sigma\rho)^3 = (\rho^2\sigma)^3 = \rho^{9m+6}; \circ(\sigma) = \circ(\omega) = \circ(\sigma\omega) = 2; \circ(\omega\mu) = 4$.

Let $\alpha = \omega\sigma\rho^{4m}\omega\sigma$ and $\beta = \rho^{3m/2}$. Using the above relations, it can be shown that $\circ(\alpha) = 3; \circ(\beta) = 8; \alpha\beta = \beta\alpha$. Define

$$\gamma = \begin{cases} \rho^{8m}\sigma\rho^2\omega, & \text{if } m \text{ is of the form } 12l+2 \text{ or } 12l+6 \\ (\rho^{8m}\sigma\rho^2\omega)^3, & \text{if } m \text{ is of the form } 12l+10. \end{cases}$$

In all the cases, it can be checked that $\circ(\gamma) = 2m$, $\alpha\gamma = \gamma\alpha$ and $\gamma\beta = \beta^{m+1}\gamma$. It is to be noted that $\alpha = \omega\sigma\rho^{4m}\omega\sigma = (\omega\sigma\rho\omega\sigma)^{4m} = [\omega(\sigma\rho\omega)\sigma]^{4m} = (\omega(\omega\rho)\sigma)^{4m} = (\rho\sigma)^{4m}$.

Proposition 5.1. *1.* $\gamma^2 = \begin{cases} \rho^{4m+4}, & \textit{if } m \textit{ is of the form } 12l+2 \textit{ or } 12l+6 \\ \rho^{12}, & \textit{if } m \textit{ is of the form } 12l+10. \end{cases}$

2. $\gamma^m = \begin{cases} \alpha^2\beta^4, & \textit{if } m \textit{ is of the form } 12l+6 \\ \beta^4, & \textit{if } m \textit{ is of the form } 12l+2 \textit{ or } 12l+10. \end{cases}$

Proof: See Appendix.

Theorem 5.4. *If $m \equiv 2(mod\ 12)$, then $R_{12m}(3m+2, 9m+1)$ is a Cayley graph.*
Proof: Let $m = 12l+2$. Therefore $8m = 96l+16$, i.e., $8m-4 = 12(8l+1)$. Then $\gamma^2 = \rho^{4m+4}$.(by Proposotion 5.1) Define

$$H = \langle \alpha, \beta, \gamma : \alpha^3 = \beta^8 = \gamma^{2m} = \mathsf{id}; \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha, \gamma\beta = \beta^{m+1}\gamma, \gamma^m = \beta^4 \rangle.$$

Thus, it is clear that every element of H is of the form $\alpha^i\beta^j\gamma^k$ where $i = 0,1,2$; $j = 0,1,\ldots,7$ and $k = 0,1,\ldots,m-1$.
Claim 1: $H = \{\alpha^i\beta^j\gamma^k : i = 0,1,2; j = 0,1,\ldots,7; k = 0,1,\ldots,m-1\}$.
Proof of Claim 1: If possible, let there exist $i_1, i_2 \in \{0,1,2\}, j_1, j_2 \in \{0,1,\ldots,7\}$ and $k_1, k_2 \in \{0,1,\ldots,m-1\}$, such that $\alpha^{i_1}\beta^{j_1}\gamma^{k_1} = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$. As $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, we have

$$\alpha^{i_2-i_1} = \beta^{j_1-j_2}\gamma^{k_1-k_2}.$$

Case 1: $k_1 - k_2$ is even.
As $\gamma^2 = \rho^{4m+4}$ and $\beta = \rho^{3m/2}$, we have $\alpha^{i_2-i_1} = \rho^x$, i.e., $(\rho\sigma)^{4m(i_2-i_1)} = \rho^x$. This implies that $3 \mid 4m(i_2-i_1)$, i.e., $3|m$ or $3|(i_2-i_1)$. As $3 \nmid m$, we have $3|(i_2-i_1)$, i.e., $i_1 = i_2$. Thus $\beta^{j_1-j_2} = \gamma^{k_2-k_1} = (\gamma^2)^{(k_2-k_1)/2}$, i.e.,

$$(\rho)^{3m(j_1-j_2)/2} = (\rho^{4m+4})^{(k_2-k_1)/2} = \rho^{2(m+1)(k_2-k_1)}. \tag{6}$$

Therefore, $12m \mid [3m(j_1-j_2)/2 - 2(m+1)(k_2-k_1)]$, i.e.,

$$24m \mid 3m(j_1-j_2) - 4(m+1)(k_2-k_1) \tag{7}$$

Thus, $m \mid 4(m+1)(k_2-k_1)$. As $gcd(m,4) = 2$ and $gcd(m, m+1) = 1$, it follows that $\frac{m}{2} \mid (k_2-k_1)$, i.e., $k_2 - k_1 = \frac{m}{2}s$. Since, $0 \le k_2 - k_1 < m$, we have $s = 0$ or 1. Again, as $m+1$ is a multiple of 3, from Equation 7, we get that $12 \mid 3m(j_1-j_2)$, i.e., $2 \mid (j_1-j_2)$. Let $j_1 - j_2 = 2t$. As $0 \le j_1 - j_2 < 8$, we have $t \in \{0,1,2,3\}$. Thus, rewriting Equation 7, we get $24m \mid (6mt - 2m(m+1)s)$, i.e., $12 \mid 3t - (m+1)s$. Thus

$$4 \mid \left(t - \frac{m+1}{3}s\right) = t - (4l+1)s, \text{ where } s \in \{0,1\}, t \in \{0,1,2,3\}. \tag{8}$$

If $s = 1$, then $k_2 - k_1 = m/2 = 6l+1$ is odd, a contradiction. Thus $s = 0$ and hence from Equation 8, we have $4 \mid t$, i.e., $t = 0$. Therefore, we have $j_1 - j_2 = k_1 - k_2 = 0$, and as a result $i_1 = i_2$. Thus Claim 1 is true, if Case 1 holds.
Case 2: $k_1 - k_2$ is odd.
Let $k_1 - k_2 = 2t - 1$. Then we have $\alpha^{i_2-i_1} = \beta^{j_1-j_2}(\gamma^2)^t\gamma^{-1}$. As $\gamma^2 = \rho^{4m+4}$ and $\beta = \rho^{3m/2}$, we have $\gamma\alpha^{i_2-i_1} = \rho^x$. Now $i_2 - i_1 = 0, 1$ or 2. Thus either of $\gamma, \alpha\gamma, \alpha^2\gamma$ is ρ^x. But

$$\gamma(A_0) = \rho^{8m}\sigma\rho^2\omega(A_0) = \rho^{8m}\sigma(A_2) = \rho^{8m}(B_{9m+1}) = B_{5m+1}$$

$$\alpha\gamma(B_0) = (\rho\sigma)^{4m}(A_{8m+3}) = (\rho\sigma)^{48l+8}(A_{8m+3}) = (\rho\sigma)^2((\rho\sigma)^3)^{16l+2}(A_{8m+3}) = A_{9m+3}$$

$$\alpha^2\gamma(A_0) = (\rho\sigma)^{8m}(B_{5m+1}) = (\rho\sigma)((\rho\sigma)^3)^{32l+5}(B_{5m+1}) = (\rho\sigma)(B_{4m}) = B_{10m+1}$$

As each of $\gamma, \alpha\gamma, \alpha^2\gamma$ maps some A_i to some B_j, none of them is equal to ρ^x and hence a contradiction. So $k_1 - k_2$ can not be odd.

Combining Case 1 and 2, we conclude that Claim 1 is true and hence $|H| = 24m = 2n$. So, as in proof of Claim 2 in Theorem 5.1, it suffices to show that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$. Let $\alpha^i\beta^j\gamma^k(A_0) = A_0$.

Claim 2: k is even.

Proof of Claim 2: If possible, let k be odd, say $k = 2t+1$. Then, as α commutes with β and γ, we have $\beta^j\gamma^{2t}\gamma\alpha^i(A_0) = A_0$, i.e., $\gamma\alpha^i(A_0) = \beta^{-j}(\gamma^2)^{-t}(A_0) = \rho^x(A_0) = A_x$, as in Case 2 above. Now, $i = 0, 1$ or 2 and as $\gamma(A_0) = B_{5m+1}$ and $\alpha^2\gamma(A_0) = B_{10m+1}$, we have $i = 1$. This implies $\alpha\beta^j\gamma^{2t+1}(A_0) = A_0$, i.e., $\beta^j(\gamma^2)^t\gamma(A_0) = \alpha^2(A_0) = A_{11m}$, i.e.,

$$A_{11m} = \beta^j(\gamma^2)^t\gamma(A_0) = \beta^j(\gamma^2)^t(B_{5m+1}) = \rho^x(B_{5m+1}) = B_{5m+x+1},\ \text{a contradiction.}$$

Hence the claim is true and let $k = 2t$. Therefore,

$$\beta^j(\gamma^2)^t(A_0) = \alpha^{-i}(A_0).$$

As left side of the above equation is $\rho^x(A_0)$ and $\alpha(A_0) = B_{10m-1}$, we conclude that $i = 0$ or 1. If $i = 1$, then we have $\alpha\beta^j(\gamma^2)^t(A_0) = A_0$. Again as α commutes with β and γ, we have

$$A_0 = \beta^j\gamma^{2t}\alpha(A_0) = \beta^j\gamma^{2t}(B_{10m-1}) = \rho^x(B_{10m-1}) = B_{10m+x-1},\ \text{a contradiction.}$$

Therefore, $i = 0$ and hence we have $\beta^j(\gamma^2)^t(A_0) = A_0$, i.e.,

$$\rho^{4(m+1)t+3j\frac{m}{2}}(A_0) = A_0,\ \text{i.e., } 12m \mid 4(m+1)t + 3j\frac{m}{2} = 12(4l+1)t + 3j(6l+1)$$

Thus $12 \mid 3j(6l+1)$, i.e., $4|j(6l+1)$. However as $6l+1$ is odd and $j \in \{0, 1, \ldots, 7\}$, we have $j = 0$ or 4. If $j = 4$, we have $12m \mid 12(4l+1)t + 12(6l+1)$, i.e., $m = 12l+2 = 2(6l+1)$ divides $(4l+1)t + 12(6l+1)$ and hence $2(6l+1) \mid (4l+1)t$. As $3(4l+1) - 2(6l+1) = 1$, we have $gcd(4l+1, 6l+1) = 1$ and hence $6l+1 \mid t$. However as $0 \le k \le m-1$, we have $0 \le t \le \frac{m-1}{2} < 6l+1$. Thus the only possible value of t is 0 and hence $k = 0$. Therefore, we have $\beta^j(A_0) = A_0$, i.e., $\rho^{3(6l+1)j}(A_0) = A_0$. This implies that $12m = 12(12l+2) \mid 3(6l+1)j$, i.e., $8|j$ and hence $j = 0$.

Thus we have $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$ and the theorem holds. □

Theorem 5.5. *If $m \equiv 6(mod\ 12)$, then $R_{12m}(3m+2, 9m+1)$ is a Cayley graph.*

Proof: Let $m = 12l+6$. Therefore $8m = 96l+48 = 12(8l+4)$. Also note that in this case, $\alpha = (\rho\sigma)^{4m} = ((\rho\sigma)^3)^{4(4l+2)} = \rho^{12(m+1)(4l+2)} = \rho^{12(4l+2)} = \rho^{4m}$. Also $\gamma^2 = \rho^{4m+4}$. (by Proposition 5.1) Define

$$H = \langle \alpha, \beta, \gamma : \alpha^3 = \beta^8 = \gamma^{2m} = \mathsf{id}; \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha, \gamma\beta = \beta^{m+1}\gamma, \gamma^m = \alpha^2\beta^4\rangle.$$

Thus, it is clear that every element of H is of the form $\alpha^i\beta^j\gamma^k$ where $i = 0, 1, 2$; $j = 0, 1, \ldots, 7$ and $k = 0, 1, \ldots, m-1$.

Claim 1: $H = \{\alpha^i\beta^j\gamma^k : i = 0, 1, 2; j = 0, 1, \ldots, 7; k = 0, 1, \ldots, m-1\}$.

Proof of Claim 1: If possible, let there exist $i_1, i_2 \in \{0, 1, 2\}, j_1, j_2 \in \{0, 1, \ldots, 7\}$ and $k_1, k_2 \in \{0, 1, \ldots, m-1\}$, such that $\alpha^{i_1}\beta^{j_1}\gamma^{k_1} = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$. As $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, we have

$$\alpha^{i_2-i_1} = \beta^{j_1-j_2}\gamma^{k_1-k_2}. \tag{9}$$

If k_1-k_2 is odd, say $k_1-k_2=2t-1$, then $\gamma=\alpha^{i_1-i_2}\beta^{j_1-j_2}\gamma^{2t}$. As $\alpha=\rho^{4m}$, the right hand side is of the form ρ^x. On the other hand, $\gamma(A_0)=B_{5m+1}$. Thus $\gamma\neq\alpha^{i_1-i_2}\beta^{j_1-j_2}\gamma^{2t}$. Hence k_1-k_2 is even, say $2t$. Thus, we have $\rho^{4m(i_2-i_1)}=\rho^{(4m+4)t+3\frac{m}{2}(j_1-j_2)}$, i.e.,

$$12m \mid 4(m+1)t+3(6l+3)(j_1-j_2)+4m(i_1-i_2). \tag{10}$$

This implies that $4 \mid 9(2l+1)(j_1-j_2)$, i.e., $4|(j_1-j_2)$. Now as $0\leq j_1-j_2\leq 7$, we have $j_1-j_2=0$ or 4.
Sub-claim 1a: $j_1-j_2=0$.
If possible, let $j_1-j_2=4$. Then, from Equation 10, we have $12m$ divides $4(m+1)t+6m+4m(i_1-i_2)$ and hence $m|4(m+1)t$, i.e., $m|4t$, as $gcd(m.m+1)=1$. Now, as $0\leq 4t=2(k_1-k_2)\leq 2m-2$, we have $4t=0$ or m. However, if $4t=m$, we have $2t=(6l+3)$, an odd number. Thus $4t$ and hence $t=0$. Therefore, from Equation 10, we get $12m \mid 6m+4m(i_1-i_2)$, i.e., $6|4(i_1-i_2)$ which implies $3|(i_1-i_2)$ i.e., $i_1=i_2$. However, this implies that $12m|6m$, a contradiction. Thus Sub-claim 1a is true and $j_1=j_2$. Thus Equation 10 reduces to

$$3m \mid (m+1)t+m(i_1-i_2). \tag{11}$$

Again since $gcd(m,m+1)=1$, this implies that $m|t$. However, as $0\leq t\leq\frac{m-1}{2}$, we have $t=0$ and hence $k_1=k_2$. Thus from Equation 11, we get $3|(i_1-i_2)$, i.e., $i_1=i_2$. Thus Claim 1 is true and $|H|=24m=2n$. So, as in proof of Claim 2 in Theorem 5.1, it suffices to show that $\mathsf{Stab}_H(A_0)=\{\mathsf{id}\}$. Let $\alpha^i\beta^j\gamma^k(A_0)=A_0$. As $\alpha=\rho^{4m},\beta=\rho^{3m/2}$ and $\gamma^2=\rho^{4m+4}$ are powers of ρ and $\gamma(A_0)=B_{5m+1}$, if k is odd, $\alpha^i\beta^j\gamma^k(A_0)=B_x$ for some index x. Thus k is even, say $k=2t$. Thus, we have $\alpha^i\beta^j\gamma^k(A_0)=\rho^{4mi+8(m+1)t+\frac{3mj}{2}}=A_0$, i.e., $12m$ divides $4mi+8(m+1)t+\frac{3mj}{2}$, i.e.,

$$24m \mid 8mi+16(m+1)t+3mj \tag{12}$$

This implies that $m|16(m+1)t$. As $gcd(m,m+1)=1$, we have $m|16t$. Again, as $m=12l+6=2(6l+3)$ and $6l+3$ is odd, we have $m|2t=k$, i.e., $k=t=0$. Thus Equation 12 reduces to $24m \mid 8mi+3mj$, i.e., $24|(8i+3j)$. However, this implies that $8|j$ and $3|i$, i.e., $i=j=0$. Thus $\mathsf{Stab}_H(A_0)=\{\mathsf{id}\}$ and the theorem holds. □

Theorem 5.6. *If $m\equiv 10(mod\ 12)$, then $R_{12m}(3m+2,9m+1)$ is a Cayley graph.*
Proof: Let $m=12l+10$. Therefore $8m=96l+80$, i.e., $8m-8=12(8l+6)$. By Proposition 5.1, we have $\gamma^2=\rho^{12}$. Define

$$H=\langle\alpha,\beta,\gamma:\alpha^3=\beta^8=\gamma^{2m}=\mathsf{id};\alpha\beta=\beta\alpha,\alpha\gamma=\gamma\alpha,\gamma\beta=\beta^{m+1}\gamma,\gamma^m=\alpha^2\beta^4\rangle.$$

Thus, it is clear that every element of H is of the form $\alpha^i\beta^j\gamma^k$ where $i=0,1,2$; $j=0,1,\ldots,7$ and $k=0,1,\ldots,m-1$.
Claim 1: $H=\{\alpha^i\beta^j\gamma^k : i=0,1,2; j=0,1,\ldots,7; k=0,1,\ldots,m-1\}$.
Proof of Claim 1: If possible, let there exist $i_1,i_2\in\{0,1,2\}, j_1,j_2\in\{0,1,\ldots,7\}$ and $k_1,k_2\in\{0,1,\ldots,m-1\}$, such that $\alpha^{i_1}\beta^{j_1}\gamma^{k_1}=\alpha^{i_2}\beta^{j_2}\gamma^{k_2}$. As $\alpha\beta=\beta\alpha$ and $\alpha\gamma=\gamma\alpha$, we have

$$\alpha^{i_2-i_1}=\beta^{j_1-j_2}\gamma^{k_1-k_2}. \tag{13}$$

If $k_1 - k_2$ is odd, say $k_1 - k_2 = 2t - 1$, then $\gamma\alpha^{i_2-i_1} = \beta^{j_1-j_2}\gamma^{2t}$. As $\gamma^2 = \rho^{12}$, the right hand side is of the form ρ^x, i.e., $\gamma\alpha^{i_2-i_1} = \rho^x$. Now $i_2 - i_1 = 0, 1$ or 2. Thus either of $\gamma, \alpha\gamma, \alpha^2\gamma$ is ρ^x. But $\gamma(A_0) = B_{9m+5}, \alpha\gamma(A_0) = B_{10m+5}$, $\alpha^2\gamma(B_0) = A_{5m+7}$. As each of $\gamma, \alpha\gamma, \alpha^2\gamma$ maps some A_i to some B_j, none of them is equal to ρ^x and hence a contradiction. So k_1-k_2 is even, say $k_1 - k_2 = 2t$. As $\gamma^2 = \rho^{12}$ and $\beta = \rho^{3m/2}$, we have $\alpha^{i_2-i_1} = \rho^x$, i.e., $(\rho\sigma)^{4m(i_2-i_1)} = \rho^x$. This implies that $3 \mid 4m(i_2 - i_1)$, i.e., $3|m$ or $3|(i_2 - i_1)$. As 3 does not divide m, we have $3|(i_2 - i_1)$, i.e., $i_1 = i_2$. Thus $\rho^{\frac{3m}{2}(j_1-j_2)} = \beta^{j_1-j_2} = \gamma^{k_2-k_1} = (\gamma^2)^t = \rho^{12t}$, i.e.,

$$24m \mid 3m(j_1 - j_2) - 24t \tag{14}$$

Thus, we have $m|24t$. As $m = 2(6l + 5)$, $(6l + 5)$ is odd and 3 does not divide $(6l + 5)$, we get $\frac{m}{2}|t$. However, as $0 \leq k_2 - k_1 \leq m - 1$, we have $0 \leq t \leq \frac{m-1}{2}$. Hence $t = 0$ and $k_1 = k_2$. Also Equation 14 reduces to $8|(j_1 - j_2)$. Thus $j_1 = j_2$. Hence Claim 1 is true and $|H| = 24m = 2n$.

So, as in proof of Claim 2 in Theorem 5.1, it suffices to show that $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$. Let $\alpha^i\beta^j\gamma^k(A_0) = A_0$.

Claim 2: k is even.

Proof of Claim 2: If possible, let k be odd, say $k = 2t+1$. Then, as α commutes with β and γ, we have $\beta^j\gamma^{2t}\gamma\alpha^i(A_0) = A_0$, i.e., $\gamma\alpha^i(A_0) = \beta^{-j}(\gamma^2)^{-t}(A_0) = \rho^x(A_0) = A_x$, as in the proof of Claim 1 of this theorem. Now, $i = 0, 1$ or 2 and as $\gamma(A_0) = B_{9m+5}$ and $\alpha\gamma(A_0) = B_{10m+5}$, we have $i = 2$. This implies $\alpha^2\beta^j\gamma^{2t+1}(A_0) = A_0$, i.e., $\beta^j(\gamma^2)^t\gamma(A_0) = \alpha(A_0) = A_{7m}$, i.e.,

$$A_{7m} = \beta^j(\gamma^2)^t\gamma(A_0) = \beta^j(\gamma^2)^t(B_{9m+5}) = \rho^x(B_{9m+5}) = B_{9m+x+5}, \text{ a contradiction.}$$

Hence the claim is true and let $k = 2t$. Therefore,

$$\beta^j(\gamma^2)^t(A_0) = \alpha^{-i}(A_0).$$

As left side of the above equation is $\rho^x(A_0)$ and $\alpha^2(A_0) = B_{2m-1}$, we conclude that $i = 0$ or 2. If $i = 2$, then we have $\alpha^2\beta^j(\gamma^2)^t(A_0) = A_0$. Again as α commutes with β and γ, we have

$$A_0 = \beta^j\gamma^{2t}\alpha^2(A_0) = \beta^j\gamma^{2t}(B_{2m-1}) = \rho^x(B_{2m-1}) = B_{2m+x-1}, \text{ a contradiction.}$$

Therefore, $i = 0$ and hence we have $\beta^j(\gamma^2)^t(A_0) = A_0$, i.e.,

$$\rho^{12t+3j\frac{m}{2}}(A_0) = A_0, \text{ i.e., } 12m \mid 12t + 3j\frac{m}{2} = 12t + 3j(6l + 5)$$

Thus $12 \mid 3j(6l+5)$, i.e., $4|j(6l+5)$. However as $6l+5$ is odd and $j \in \{0, 1, \ldots, 7\}$, we have $j = 0$ or 4. If $j = 4$, we have $12m \mid 12t+12(6l+5)$, i.e., $m = 12l+10 = 2(6l+5) \mid t+(6l+5)$ and hence $(6l + 5) \mid t$. However as $0 \leq k \leq m - 1$, we have $0 \leq t \leq \frac{m-1}{2} < 6l + 5$. Thus the only possible value of t is 0 and hence $k = 0$. Therefore, we have $\beta^j(A_0) = A_0$, i.e., $\rho^{3(6l+5)j}(A_0) = A_0$. This implies that $12m = 24(6l+5) \mid 3(6l+5)j$, i.e., $8|j$ and hence $j = 0$.

Thus we have $\mathsf{Stab}_H(A_0) = \{\mathsf{id}\}$ and the theorem holds. □

6 Family-5 [$R_{2m}(2b, r)$: $b^2 \equiv \pm 1$(mod m) and $r \in \{1, m-1\}$ is odd]

Theorem 6.1. *If $b^2 \equiv \pm 1(mod\ m)$ and $r \in \{1, m-1\}$ is odd, then $R_{2m}(2b, r)$ is a Cayley graph.*

Proof: If $r = 1$, then it is clear that the conditions of being in **Family-1** are satisfied, (i.e., $r^2 \equiv 1(mod\ n)$ and $ra \equiv a(mod\ n)$) and hence, by Theorem 2.1, $R_{2m}(2b, r)$ is a Cayley graph. So we are left with the case when $n = 2m$, $a = 2b$, $b^2 \equiv \pm 1(mod\ m)$, $r = m - 1$ and m is even. Observe that, in this case,

$$r^2 = (m-1)^2 = m^2 - 2m + 1 \equiv 1(mod\ 2m) \equiv 1(mod\ n)\ \text{[since, } m \text{ is even].}$$

Also, as $m \mid bm$ i.e., $m|b(r+1)$, we have $br \equiv -b(mod\ m)$, i.e., $2br \equiv -2b(mod\ 2m)$, i.e., $ra \equiv -a(mod\ n)$. Thus, in this case, $r^2 \equiv 1(mod\ n)$ and $ra \equiv -a(mod\ n)$ holds. Hence, by Theorem 2.1, $R_{2m}(2b, r)$ is a Cayley graph. □

Remark 6.1. *The above theorem shows that* **Family-5** *is a subfamily of* **Family-1**. *However, they were shown as different families in Theorem 3.10 in [1].*

Combining the analysis of the rose window graphs in **Families: 1–5**, we have Theorem 1.3.

7 Appendix

Lemma 7.1. *Let $G = \mathsf{Aut}(R_{12m}(3m+2, 9m+1))$, where $m \equiv 0(mod\ 4)$. Then $|G| = 96m$.*

Proof: Since, $R_{12m}(3m+2, 9m+1)$ is vertex-transitive and its order is $24m$ and $\mathsf{Stab}_G(A_0)$ contains $\mathsf{id}, \mu, \sigma, \mu\sigma$, therefore, by orbit-stabilizer theorem, we have $|G| \geq 4 \times 24m = 96m$. Thus, it is enough to show that $|G| \leq 96m$. We also know that

$$G = \langle \rho, \mu, \sigma : \rho^n = \mu^2 = \sigma^2 = \mathsf{id}; \mu\rho\mu = \rho^{-1}, \sigma\rho^3\sigma = \rho^3, \sigma\mu = \mu\sigma,$$
$$(\rho\sigma)^3 = (\sigma\rho)^3 = \rho^{3(m+1)}, (\rho\sigma\rho)^3 = \rho^{9m+6}\rangle, \text{ where } n = 12m.$$

Consider the sets $X = \{\rho^i\sigma\rho^j\mu^k : i \in \{0, 1, 2 \ldots, n-1\}, j \in \{0, 1, 2\}, k \in \{0, 1\}\}$ and $Y = \{\rho^i\mu^k : i \in \{0, 1, 2 \ldots, n-1\}, k \in \{0, 1\}\}$. We claim that all elements are either in X or in Y. It is clear that elements in G which does not involve σ are in Y, due to the relations $\rho^n = \mu^2 = \mathsf{id}$ and $\mu\rho\mu = \rho^{-1}$. Again, as $\sigma\mu = \mu\sigma$ and $\mu\rho = \rho^{-1}\mu$, any element in G can be expressed in the form where μ occurs in the extreme right of the expression. Thus it is enough to show that elements in G which involve only ρ and σ are of the form $\rho^i\sigma\rho^j$ where $i \in \{0, 1, 2 \ldots, n-1\}$ and $j \in \{0, 1, 2\}$. Again, as $\sigma\rho^3 = \rho^3\sigma$, it is clear that the power of ρ lying on the right of σ can be made $0, 1$ or 2. Finally, we deal with elements $\sigma\rho\sigma$ and $\sigma\rho^2\sigma$.

As $(\rho\sigma\rho)^3 = \rho^{9m+6}$, we have $\sigma\rho^2\sigma\rho^2\sigma = \rho^{9m+4}$, i.e.,

$$\sigma\rho^2\sigma = \rho^{9m+4}\sigma\rho^{-2} = \rho^{9m+4}\sigma\rho^{12m-2} = \rho^{9m+4+12m-3}\sigma\rho = \rho^{9m+1}\sigma\rho \in X.$$

As $(\rho\sigma)^3 = \rho^{3(m+1)}$, we have $(\sigma\rho\sigma\rho\sigma) = \rho^{3m+2}$, i.e.,

$$\sigma\rho\sigma = \rho^{3m+2}\sigma\rho^{-1} = \rho^{3m+2}\sigma\rho^{12m-1} = \rho^{3m+2+12m-3}\sigma\rho^2 = \rho^{3m-1}\sigma\rho^2 \in X.$$

Similarly, any other element of G involving ρ and σ can be expressed in the form of elements in X. Thus $G = X \cup Y$ and hence

$$|G| = |X \cup Y| \leq |X| + |Y| \leq (n \times 3 \times 2) + (n \times 2) = 6n + 2n = 8n = 96m.$$

$\square$

Proof of Proposition 5.1 :

1. For $m = 12l + 2$, we have $8m = 96l + 16$, i.e., $8m - 4 = 12(8l + 1)$.

$$\begin{aligned}
\gamma^2 &= (\rho^{8m}\sigma\rho^2\omega)(\rho^{8m}\sigma\rho^2\omega) = \rho^{8m}\rho^{8m-4}\sigma\rho^2\omega\rho^4\sigma\rho^2\omega \quad \text{(as } \rho^{12} \text{ commutes with } \sigma \text{ and } \omega\text{)}\\
&= \rho^{4m-4}\sigma\rho^2(\omega\rho^3)\rho\sigma\rho^2\omega = \rho^{4m-4}\sigma\rho^2(\rho^{3(m+1)}\omega)\rho\sigma\rho^2\omega \quad \text{(as } \omega\rho^{3l} = \rho^{3l(m+1)}\text{)}\\
&= \rho^{7m-1}\sigma\rho^2(\omega\rho)\sigma\rho^2\omega = \rho^{7m-1}\sigma\rho^2(\sigma\rho\omega)\sigma\rho^2\omega = \rho^{7m-1}\sigma\rho^2\sigma\rho\sigma\omega\rho^2\omega\\
&= \rho^{7m-1}\sigma\rho^2\sigma\rho\sigma(\omega\rho\omega)^2 = \rho^{7m-1}\sigma\rho^2\sigma\rho\sigma(\sigma\rho)^2 = \rho^{7m-1}\sigma\rho^2\sigma\rho\sigma(\sigma\rho)(\sigma\rho)\\
&= \rho^{7m-1}\sigma\rho^2\sigma\rho^2\sigma\rho = \rho^{7m-2}\rho\sigma\rho^2\sigma\rho^2\sigma\rho = \rho^{7m-2}(\rho\sigma\rho)(\rho\sigma\rho)(\rho\sigma\rho) = \rho^{7m-2}(\rho\sigma\rho)^3\\
&= \rho^{7m-2}\rho^{9m+6} = \rho^{16m+4} = \rho^{4m+4}
\end{aligned}$$

For $m = 12l + 6$, we have $8m = 96l + 48 = 12(8l + 4)$.

$$\begin{aligned}
\gamma^2 &= (\rho^{8m}\sigma\rho^2\omega)(\rho^{8m}\sigma\rho^2\omega) = \rho^{16m}\sigma\rho^2\omega\sigma\rho^2\omega \quad \text{(as } \rho^{12} \text{ commutes with } \sigma \text{ and } \omega\text{)}\\
&= \rho^{4m}\sigma\rho^2\sigma\omega\rho^2\omega = \rho^{4m}(\sigma\rho\sigma)^2(\omega\rho\omega)^2 = \rho^{4m}(\sigma\rho\sigma)^2(\sigma\rho)^2 \quad \text{(as } \omega\rho = \sigma\rho\omega\text{)}\\
&= \rho^{4m}\sigma\rho^3\sigma\rho = \rho^{4m+4}.
\end{aligned}$$

Similarly, for $m = 12l + 10$, it can be proved that $\gamma^2 = \rho^{12}$.

2. The values of γ^m can be found by raising γ^2 to the power $m/2$, and hence can be checked to have the respective forms.

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