ON THE INFORMATION FLOW REQUIRED FOR THE SCALABILITY OF THE STABILITY OF MOTION OF APPROXIMATELY RIGID FORMATION

A Thesis

by

SAI KRISHNA YADLAPALLI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2005

Major Subject: Mechanical Engineering

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ABSTRACT

On the Information Flow Required for the Scalability of the

Stability of Motion of Approximately Rigid Formation. (May 2005)

Sai Krishna Yadlapalli, B. Tech., Indian Institute of Technology, Madras

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It is known in the literature on Automated Highway Systems that information flow can significantly affect the propagation of errors in spacing in a collection of vehicles. This thesis investigates this issue further for a homogeneous collection of vehicles. Specifically, we consider the effect of information flow on the propagation of errors in spacing and velocity in a collection of vehicles trying to maintain a rigid formation. The motion of each vehicle is modeled using a Linear Time Invariant (LTI) system. We consider *undirected* and *connected* information flow graphs, and assume that that each vehicle can communicate with a maximum of $q(n)$ vehicles, where $q(n)$ may vary with the size n of the collection. The feedback controller of each vehicle takes into account the aggregate errors in position and velocity of the vehicles, with which it is in direct communication. The controller is chosen in such a way that the resulting closed loop system is a Type-2 system. This implies that the loop transfer function must have at least two poles at the origin. We then show that if the loop transfer function has three or more poles at the origin, and if the size of the formation is sufficiently large, then the motion of the collection is unstable. Suppose l is the number of poles of the transfer function relating the position of a vehicle with the control input at the origin of the complex plane, and if the number $\frac{q(n)^{l+1}}{n^l} \to 0$ as $n \to \infty$, then we show that there is a low frequency sinusoidal disturbance with unity maximum amplitude acting on each vehicle such that the maximum errors in spacing response increase at least as much as $O\left(\sqrt{\frac{(n^l)}{q(n)^{l+1}}}\right)$. A consequence of the results presented in this paper is that the maximum of the error in spacing and velocity of any vehicle can be made insensitive to the

size of the collection only if there is at least one vehicle in the collection that communicates with at least $O(\sqrt{n})$ other vehicles in the collection.

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“ None of us is as smart as all of us” – Ken Blanchard

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CHAPTER I

INTRODUCTION

A. Vehicle Formations

Recent advances in a variety of technologies such as communication, computation, sensing and actuation have enabled the development and increased the possibility of deployment of collections of Unmanned Vehicles (UVs) (or simply vehicles) for a wide variety of tasks. Various applications involving unmanned ground and aerial vehicles may be found in [1-19]. For example, UVs are central to automating driving tasks in an Automated Highway System (AHS) [1], the dynamic positioning of mobile offshore bases for creating a runway for large aircrafts and for information gathering in dangerous environments [2]. There seem to be potentially many advantages to deploying UVs in collections for certain tasks: flexibility, ease of reconfiguration and lower cost of deploying collections of smaller UVs as compared to deploying a larger UV being some of them. In order to realize these potential advantages, the problem of coordinating the motion of the collection of vehicles must be addressed and this work is devoted to an analysis of this problem.

It is conceivable that a collection of vehicles will be required to maintain (or remain close to) specified discernible geometric patterns during its motion. We call such a collection of vehicles a formation if every vehicle aids in the maintenance of the specified geometric pattern by coordinating its motion through communication with or sensing other vehicles in the collection. The desired motion of every vehicle in a formation is determined by the desired motion of a few vehicles in the collection so that the specified geometric pattern is maintained. Since vehicles in a formation are coupled dynamically by feedback, errors in spacing and velocity (defined as the deviation in the position and velocity from

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their respective desired values) of a vehicle propagate from one vehicle in the formation to the other.

Of recent interest to the research community is the rigid formation of vehicles, where it is desired that the distance between any two vehicles remain constant throughout the motion. In an AHS, such rigid formations (referred to as a platoons) are desired from the viewpoint of maintaining safety and enhancing the throughput of vehicles on a section of a congested highway [3]. A rigid formation is helpful for localization in partially known environments in the case of mobile robots [4], and in drag reduction via close formation flight [5, 6].

An issue with the design of controllers for vehicles in a collection is that of *collective stability* of the controlled motion of the vehicles [1, 7, 20]. This issue arises because errors in spacing and velocity of a vehicle propagate to others in the collection. Stability of motion of vehicles would require the following: Given *any* $\epsilon > 0$, there must exist a bound $\delta > 0$ on the norm of the *initial* error in the state of *all* vehicles in the collection that will guarantee that the maximum value of the norm in the error of the state of *any* vehicle at any other time from its desired state is bounded by the given ϵ. Collective stability, that is dealt with in this work, requires further that δ be independent of the size of the collection. It is well known in Linear Systems Theory [21], that the stability of solutions of a homogeneous linear constant differential equation can be examined by studying the stability of solution of its *zero* solution. It is also known from Linear Systems theory [21] that the stability of the zero solution can be inferred by examining the *boundedness* of the solution of the linear constant differential equation to bounded forcing functions and is referred to as the Bounded Input, Bounded Output (BIBO) Stability [21]. It is this approach that we take to infer the collective stability of the formation. With a specified controller on each vehicle and with the vehicles starting at their desired positions and velocities, we ask the following question: For any given bound, ϵ, is there a bound, δ, *independent of the size* *of the collection*, on the magnitude of *any* force disturbance that can act on *any* vehicle, so that as the errors propagate with the choice of controllers, they always remain smaller than ϵ? The requirement of the independence of δ from the size of the collection captures the scalability of the stability of motion with the specified controllers. We will say a controller is scalable if the above requirement of collective stability of controlled motion is met. Since no formation can ever be rigid, we will say that an "approximately rigid formation" can be synthesized if one can synthesize a scalable controller.

In this work, we are interested in the synthesis of scalable controllers which take into account an additional consideration - that of spatial shift-invariance (i.e., the controller is independent of the index of the vehicle or the size of the collection). From a practical viewpoint, such a controller will be simple to develop and implement on every vehicle. This is important for applications such as the Adaptive Cruise Control (ACC) System for ground vehicles, because one will not know *a priori* how many vehicles with an ACC System will be placed in succession in traffic. In [8], controllers that used the information about the index of the vehicle in the collection were synthesized; however, for them to achieve an approximately rigid linear formation, the control gains had to increase with the index of the vehicle at least in a linear manner and from a practical viewpoint, this is unrealistic since it will lead to saturation of control effort even with small errors in spacing and velocity. For this reason and for the simplicity of treatment, we only consider the restricted class of controllers for further investigation.

The synthesis of an "approximate rigid formation" is strongly influenced by the communication pattern between the vehicles. If every vehicle in the formation has the information from a reference vehicle in the collection, then errors in the spacing and velocity resulting from a disturbance acting on a vehicle can made to attenuate as it propagates from one vehicle to another [7]. To date, it is believed that the information concerning one vehicle must be available to $O(n)$ vehicles in the formation if one were to construct approximate rigid formations, with n being the size of the collection. The results in [7] and those established in this work point in this direction.

There are other ways of coordinating the motion of vehicles in a collection - some of them stem from practical considerations of the application at hand and some from the availability of information to each vehicle in the formation. For example, methods based on artificial potential are used in mobile robotics for the purpose of collision avoidance with other robots and obstacles [22]. In this case, mobile robots move about in "loose" formations and each robot takes an evasive action *autonomously* if the sensed distance to other robots or obstacles is less than a certain threshold. Another method of coordinating the motion of a vehicle by a driver in the highway is through the use of a two-second rule; in this case, the distance between a vehicle and the vehicle ahead changes linearly with the speed and the formation is not rigid. This method of coordinating the motion of vehicles is used in the design of Adaptive Cruise Control (ACC) Systems for ground vehicles [9]. It has the advantage of guaranteeing that errors in spacing (deviation of the following distance from the actual following distance) and velocity do not amplify as they propagate with just the on-board following distance and relative speed sensors. Clearly, the diverse ways of coordinating the motion of vehicles are representative of the diversity of applications with different requirements.

The following question naturally arises and is the focus of investigation in this thesis: How does a pattern of communication amongst vehicles affect the propagation of errors? Specifically, with a specified pattern of communication amongst them, can an approximately rigid formation be synthesized? If the answer to the latter question is in the affirmative, one can employ the same controller in each of the vehicles irrespective of the size of the collection, i.e., one can design a "scalable" control system with the given information flow.

The main results of this work concerns the necessary conditions on the information structure for the synthesis of approximately rigid formations and are as follows: If the motion of each vehicle can be represented as the motion of a unit mass under the action of a control force and a disturbance force and that the information flow graph is undirected, we show that there is no “scalable” control system if every vehicle can only communicate with at most $q(n)$ vehicles, where n is the size of the collection and $q(n)$ satisfies

$$\lim_{n\to\infty} \frac{q(n)^3}{n^2} = 0.$$

We show this result by constructing a sinusoidal disturbance of atmost unit magnitude acting on each vehicle at an appropriately chosen low frequency that results in a maximum error in spacing of at least $O(\sqrt{\frac{n^2}{q^3(n)}})$. A consequence of this result is that at least one vehicle in the collection must communicate with at least $O(n^{2/3})$ other vehicles in the collection for a “scalable” controller to exist. We also show that if the controller incorporates an integral action, the motion of the collection is necessarily unstable for all sizes of the collection greater than a critical value.

B. Thesis Outline

The following is a brief outline of the chapters that follow.

Chapter II gives an introduction to coordinated vehicle control problem. We precisely define the problem of controlling a string of vehicles in the context of Automated Highway Systems(AHS) and prove the above results.

Chapter III generalizes the one-dimensional formation problem to any general formation in $\Re^3$. In mathematical terms, a string of vehicles is a collection of single-input single-output (SISO) systems, where as for formation flight of UAV’s (Unmanned Air Vehicles), one must consider multiple-input multipe-output systems as well as formations where error propagate in multiple dimensions. We generalize the results of Chapter II for such general formations.

Chapter IV provides a graphical view of the results obtained in the previous chapters. We provide corroborating simulations for a string of vehicles and then for a array of vehicles moving in a straight line (in a non-inertial frame).

Chapter V presents conclusions and gives recommendation for future work.

CHAPTER II

STRING OF VEHICLES

In this chapter, we shall first give a detailed description of the model of the vehicle and other assumptions required for formulation and analysis of the problem considered in this work. We formulate the problem precisely and then present a detailed analysis of the same. We shall then derive the error propagation equations. We conclude the chapter showing the results for a string of vehicles.

We will consider a string of vehicles moving in a straight line in this chapter. The vehicles are indexed in the natural ordering of the string. The first vehicle, which we call reference vehicle, executes maneuvers with bounded velocity and acceleration. This reference vehicle is also referred to as lead vehicle in the AHS literature. For each $i \geq 2$, the i^{th} vehicle desires to maintain a fixed following distance $L_{i,i-1}$ from its predecessor. Initially, all vehicles are assumed to be at their desired position and the the velocities of all the vehicles are identical.

A. Model of the Vehicle

We shall assume that every vehicle can be modeled by transfer function $H(s)$ and is subjected to a controlled force, $u(t)$ and a disturbance $d(t)$. If $x(t)$ is the position of a vehicle measured from the origin of an inertial reference frame, then one may express the Laplace transformation, $X(s)$, of $x(t)$ in terms of the Laplace transformations, $U(s)$ and $D(s)$ of $u(t)$ and $d(t)$ respectively:

$$X(s) = H(s)[U(s) - D(s)] + \frac{(s+a)x(0) + \dot{x}(0)}{s(s+a)}, \tag{2.1}$$

where $H(s) = \frac{1}{s(s+a)}$. There are only two cases to consider: $a = 0$ and $a \neq 0$. The first case corresponds to a point mass model with no damping and the second one corresponds to a point mass model with damping. We represent quantities of interest relevant to the i^{th} vehicle with a subscript i. In particular, the Laplace transformation of the position, $x_i(t)$ of the i^{th} vehicle is related to the inputs, $u_i(t)$ and $d_i(t)$ through the following relation:

$$X_i(s) = H(s)[U_i(s) - D_i(s)] + \frac{(s+a)x_i(0) + \dot{x}_i(0)}{s(s+a)}.$$

The terms $x_i(0)$ and $\dot{x}_i(0)$ represent the initial position and velocity of the i^{th} vehicle.

Even if one assumes that the controlled force, u_i, is the output of some linear time-invariant actuation process, this is a reasonable model for reasons that will be explained later when the structure of the controller considered is discussed.

B. Further Assumptions and Formulation of the Problem

We make the assumption that the information flow graph is undirected; by that we mean that if a vehicle A transmits the information concerning its state directly to a vehicle B, then vehicle B transmits the information concerning its state directly to vehicle A. Therefore, if S_i is the set of vehicles the i^{th} vehicle in the collection can communicate directly with, this assumption implies that $j \in S_i \Rightarrow i \in S_j$. If the i^{th} vehicle, V_i and the j^{th} vehicle, V_j are in direct communication with each other, we refer to the ordered pair (i, j) as a communication link. We also assume that the information available to the i^{th} vehicle in the collection is $x_i(t) - x_j(t) - L_{ij}$, where $j \in S_i$ and L_{ij} is the desired distance to be maintained between the i^{th} and the j^{th} vehicles. We restrict the size of S_i (given by $|S_i|$) to be atmost $q(n)$.

We also assume that the information flow graph representing the communication pattern is *connected*. By connectedness, we mean that every vehicle in the collection *should be able* to communicate with every other vehicle in the collection, even if they are not communicating directly, through a sequence of communication links. We further assume that the structure of the control law used by each vehicle, other than the reference vehicle, is the same. Specifically, we consider the following structure:

$$U_i(s) \quad = -C(s)\sum_{j\in S_i}(X_i(s) - X_j(s) - \tfrac{L_{ij}}{s}), \tag{2.2}$$

where $C(s)$ is a rational scalar transfer function. Let $x_{ref}(t) \in \Re$ be the position of the reference vehicle at time t. The desired position $x_{i,des}(t)$ is related to the position of the reference vehicle x_{ref} through a constant offset L_i, i.e., $x_{i,des}(t) - x_{ref}(t) - L_i \equiv 0$. We define the error in spacing $e_i(t)$, of the i^{th} vehicle to be the deviation of its position from the desired position, i.e., $e_i(t) := x_i(t) - x_{i,des}(t) = x_i(t) - x_{ref}(t) - L_i$.

However, it must be pointed out that such a measurement may not be directly available to the controlled vehicle, since each vehicle may not have directly communicate with or sense the reference vehicle. However, it can be inferred by having the information concerning the reference vehicle passed to each vehicle through appropriate links. While this seems possible, in reality, the information concerning the reference vehicle is delayed as it is passed along the links. In this paper, we do not allow for this possibility of passing the information concerning the reference vehicle along the links.

Since the desired formation corresponds to the vehicles moving as a rigid body in a pure translational maneuver, the desired deviation $L_{ij} := x_{i,des}(t) - x_{j,des}(t)$ is constant throughout the motion and equals $L_i - L_j$. Let $E_i(s)$ be the Laplace transformation of the error in spacing, $e_i(t)$ of the i^{th} vehicle. Let $\bar{x}(t) := x_{ref}(t) - x_{ref}(0)$ be the displacement of the reference vehicle from its initial position at the time t. Then $X_{ref}(s) = \frac{x_{ref}(0)}{s} + \bar{X}(s)$. If all the initial positions of the vehicles were chosen to correspond to the rigid formation, then $x_i(0) - x_{ref}(0) - L_i \equiv 0$. With such a choice of initial conditions and the choice of control law given in equation (2.2) for the vehicle described by equation (2.1) results in the following set of evolution equations for the errors in spacing:

$$
\begin{aligned}
E_i(s) &= X_i(s) - X_{ref}(s) - \frac{L_i}{s} \\
&= H(s)[\underbrace{-C(s)\Big(\sum_{j\in S_i}(X_i(s) - X_j(s) - \frac{L_{ij}}{s})\Big)}_{U_i(s)} - D_i(s)] + \underbrace{\frac{x_i(0)}{s} - X_{ref}(s) - \frac{L_i}{s}}_{-\bar{X}(s)}, \qquad (2.3)\\
&= H(s)[-C(s)\sum_{j\in S_i}(E_i(s) - E_j(s)) - D_i(s)] - \bar{X}(s).
\end{aligned}
$$

There is no loss of generality in choosing the model of the vehicle as considered in Equation (2.1) if the actuation system may be modeled as a Linear Time Invariant (LTI) System. In this case, for some appropriate rational, proper transfer function, $P(s)$, that represents the transfer function from the input, in terms of commanded voltage to the actual force that is applied to the vehicle, we have:

$$
X_i(s) = H(s)[\underbrace{P(s)F_i(s)}_{U_i(s)} - D_i(s)] + \frac{sx_i(0) + \dot{x}_i(0)}{s^2}, \qquad (2.4)
$$

where F_i is the input to the actuation mechanism of the i^{th} vehicle. In this case, the control law

$$
F_i(s) = -\bar{C}(s)\sum_{j\in S_i}(X_i(s) - X_j(s) - \frac{L_{ij}}{s})
$$

results in an error whose Laplace transformation may be expressed as:

$$
\begin{aligned}
E_i(s) &= X_i(s) - X_{ref}(s) - \frac{L_i}{s} \\
&= H(s)[P(s)\underbrace{\Big(-\bar{C}(s)\sum_{j\in S_i}(X_i(s) - X_j(s) - \frac{L_{ij}}{s})\Big)}_{F_i(s)} - D_i(s)] \underbrace{- X_{ref}(s) - \frac{x_i(0)}{s} + \frac{L_i}{s}}_{-\bar{X}(s)}, \qquad (2.5)\\
&= H(s)[-P(s)\bar{C}(s)\sum_{j\in S_i}(E_i(s) - E_j(s)) - D_i(s)] - \bar{X}(s).
\end{aligned}
$$

Equation (2.3) describes the propagation of errors with any possible controller $C(s)$ for the point mass model of a vehicle given by equation (2.1), while equation (2.5) describes the propagation of errors with any possible controller $\bar{C}(s)$ for the more complicated model of a vehicle considered in equation (2.4). It is clear that if $C(s)$ is of the form $P(s)\bar{C}(s)$, equations (2.3) and (2.5) are identical. Since, we allow $C(s)$ to assume such a form, there is no loss of generality in assuming a point mass model of a vehicle given by equation (2.1) and the corresponding controller given by equation (2.2).

Compactly, the error equation in (2.3) may be conveniently expressed as:

$$[I_{n-1} + H(s)C(s)K_1]\boldsymbol{E(s)} = -H(s)\boldsymbol{D(s)} - \boldsymbol{\tilde{X}(s)}, \tag{2.6}$$

where $\boldsymbol{E(s)}$ and $\boldsymbol{D(s)}$ are the respective Laplace transformations of the vector of errors of the following vehicles and the disturbances acting on them; the term $\boldsymbol{\tilde{X}(s)}$ is a vector of dimension $n-1$ and every element of this vector is $\bar{X}(s)$, the term I_{n-1} is an identity matrix of dimension $n-1$ and K_1 is the principal minor obtained by removing the first row and first column of the Laplacian K of the information flow graph defined as follows: For $j \neq i$, $K_{ij} = -1$ if vehicles i and j communicate directly; otherwise $K_{ij} = 0$. The i^{th} diagonal element is then defined as $K_{ii} = -\sum_{j=1, j\neq i}^{n} K_{ij}$. If one uses a mechanical analogy for the collection, the Laplacian K is essentially the stiffness matrix obtained by connecting springs of unit spring constant between vehicles that communicate directly and each vehicle is being viewed as an individual mass.

Fax and Murray [10] have considered a control law for the i^{th} vehicle of the following form (which is different from the control law considered in this paper in equation (2.2)):

$$U_i(s) = -\frac{1}{|S_i|}C(s)\sum_{j\in S_i}(E_i(s) - E_j(s)) \tag{2.7}$$

This kind of control law for a vehicle essentially averages the feedback information from all the vehicles directly communicating with it. With this choice of control law and the model for a vehicle described by Equation (2.1), the equations for errors in spacing can be written as:

$$E_i(s) = H(s)[-\frac{1}{|S_i|}C(s)\sum_{j\in S_i}(E_i(s) - E_j(s)) - D_i(s)] - \bar{X}(s). \tag{2.8}$$

The corresponding error propagation equation may be compactly written as:

$$[I_{n-1} + H(s)C(s)M^{-1}K_1]\boldsymbol{E(s)} = -H(s)\boldsymbol{D(s)} - \boldsymbol{\tilde{X}(s)}, \tag{2.9}$$

where M is the diagonal of K_1.

1. Problem formulation

The following are the objectives of the control law given by equation (2.2):

1. In the absence of any disturbance on every vehicle in the formation, it is desired that for every $i \geq 2$, $\lim_{t\to\infty} e_i(t) = 0$, when the reference vehicle executes a maneuver and its speed asymptotically reaches a constant value.

2. In the presence of disturbances of at most unit in magnitude, it is desired that there exist a constant $M_R > 0$ such that $\max\{|e_i(t)|, |\dot{e}_i(t)|\} \leq M_R$ for every size of the collection and for every $t \geq 0$.

The second objective ensures that the control law given by equation (2.2) is scalable. Since the motion of the collection of vehicles is treated as a LTI system, the motion of each vehicle *modulo* the motion of the reference vehicle is the same as the motion of each vehicle when the reference vehicle is grounded. For this reason, the second objective may be analyzed for the case when the reference vehicle is stationary.

The problem is to determine conditions on the information flow graph (through constraints on K_1) and on the controller (through constraints on $C(s)$) so that these two objectives are met.

C. Analysis

Let us analyze the first requirement: Since the speed of the reference vehicle reaches a constant value, say v_f asymptotically, we have: $\lim_{t\to\infty} \dot{\bar{x}}(t) = v_f = \lim_{s\to 0} s^2\bar{X}(s)$. Therefore, we will have: $\lim_{s\to 0} s^3\bar{X}(s) = 0$.Further, for the analysis of the requirement, we have $\boldsymbol{D(s)} \equiv \mathbf{0}$. Suppose if $\det[I_{n-1} + H(s)C(s)K_1]$ is Hurwitz and $H(s)C(s)$ has at least two poles at origin, we have:

$$
\begin{aligned}
\lim_{s\to 0} s\boldsymbol{E(s)} &= -\lim_{s\to 0}[I_{n-1} + H(s)C(s)K_1]^{-1} s\boldsymbol{\tilde{X}(s)}, \\
&= -\lim_{s\to 0}[s^2 I_{n-1} + s^2 H(s)C(s)K_1]^{-1} \lim_{s\to 0} s^3\boldsymbol{\tilde{X}(s)} = \mathbf{0}.
\end{aligned}
$$

Therefore, the steady state error requirement is readily met if $\det[I_{n-1}+H(s)C(s)K_1]$ is Hurwitz, i.e., if the controlled motion of formations is stable and $H(s)C(s)$ has at least two poles at origin. The second condition, in fact, concerns the stability of the controlled motion of formations.

We will prove the main result concerning the stability of the controlled motion by using a mechanical analogy between the Laplacian of the information flow graph and the stiffness matrix, which essentially provides a way to address the propagation of errors. A route to instability in structural mechanics, for systems that do not have a rigid body mode, is that the smallest eigenvalue of the stiffness matrix goes to zero. In the context of a formation of vehicles, the smallest eigenvalue of the Laplacian K is zero, which corresponds to the rigid body mode, i.e., all vehicles have the same non-trivial displacement. A way to get a system without a rigid body mode is to ground one of the vehicles. As we have noted before, for the purposes of examining the propagation of errors in spacing relative to the reference vehicle in the collection, there is no loss of generality in grounding the reference vehicle. Hence, we set $\bar{X}(s) = 0$ in equation (2.6). It can be seen without much difficulty that the the property of the connectedness of the graph is related to the eigenvalue(s) of the

Laplacian at zero [1].

The mechanical analogy indicates the following line of proof:

1. The smallest eigenvalue, λ, of K_1 goes to zero as $n \to 0$. Let $\boldsymbol{v}$ be the eigen vector of K_1 corresponding to the eigen value λ.

2. Let the inner product of the vector of spacing errors, $\boldsymbol{e(t)}$, with $\boldsymbol{v}$ be the signal $e_v(t)$. Its Laplace transformation, $E_v(s)$, is given by:

$$\begin{aligned} E_v(s) &= < \boldsymbol{v}, \boldsymbol{E(s)} > \\ &= - < \boldsymbol{v}, [I_{n-1} + H(s)C(s)K_1]^{-1} H(s)\boldsymbol{D(s)} > \\ &= -\frac{H(s)}{1 + \lambda H(s)C(s)} D_v(s), \end{aligned}$$

   where $D_v(s) =< \boldsymbol{v}, \boldsymbol{D(s)} >$ and $d_v(t) =< \boldsymbol{v}, \boldsymbol{d(t)} >$, the component of the vector of disturbances acting along the eigen vector $\boldsymbol{v}$. The mechanical analogy indicates the examination of $e_v(t)$ when $d_v(t)$ is a sinusoid at the first natural frequency or close to the first natural frequency.

1. Convergence of the smallest eigenvalue of K_1 to zero

We will start with the following result:

Lemma 1. *Consider information flow graphs where every vehicle in the collection can at most communicate directly with $q(n)$ other vehicles in the collection, $q(n)$ might vary with the size of the collection. Then, for any information flow graph, we have $\lambda \leq \frac{q(n)}{n-1}$.*

---

[1] There is a simple eigenvalue of Laplacian K at zero and hence by grounding one of the vehicles, we get a system without a rigid mode. This follows from the assumption that the underlying information flow graph is connected. Physically, we are eliminating the possibility of two or more seperate collections of vehicles. It is very apparent that grounding one of the vehicles in any one such seperate collection, has no effect on the rigid mode of the others and vice-versa. There are several interesting properties of Laplacian which result from the application of the Perron-Frobenius theorem (refer [23]). One can refer to [24, 25] for a detailed treatment.

Proof. Since K_1 is symmetric, we will use Rayleigh's inequality to get an upper bound for the smallest eigenvalue, λ. For that we construct an assumed mode, $\boldsymbol{v_a}$ in the following way: we keep the reference vehicle grounded and each vehicle to be displaced by one unit. Since, the assumed mode shape indicates the amount by which every mass is displaced, all the elements of v_a are equal. Without any loss of generality, we may set each element to be unity and we represent the corresponding $\boldsymbol{v_a}$ by **1**. One may also identify each edge to be a spring without any loss of generality. This spring-mass analogy makes the rest of the proof easy to follow. By Rayleigh's inequality:

$$\lambda \quad \leq \quad \frac{< \boldsymbol{v_a}, K_1 \boldsymbol{v_a} >}{< \boldsymbol{v_a}, \boldsymbol{v_a} >}, \tag{2.10}$$

where half of the numerator in the above expression is referred to as the "reference" potential energy and half of the denominator is referred to as the "reference" kinetic energy. The reference potential energy is the sum of the potential energy in each spring. It is clear that $< \boldsymbol{v_a}, \boldsymbol{v_a} >$ = $n - 1$, where n is the number of vehicles in the formation. Since **1** is the assumed mode shape, $K_1\mathbf{1}$ is the vector of deflections of the springs - only the springs connected to the reference vehicle will be deflected; the rest will not. Hence, $< \boldsymbol{v_a}, \boldsymbol{v_a} >$ = q_r. Therefore, using equation (2.10) for every information flow graph satisfying the assumptions, we have

$$\lambda \leq \frac{q_r}{n-1} \leq \frac{q(n)}{n-1} \tag{2.11}$$

□

In Chapter IV, various random information flow graphs are considered, which are subject to the constraint that every vehicle can at most communicate directly with a pre-specified number of vehicles. The numerical results obtained for them corroborate Lemma 1.

Remark 1. *The same bound holds even for the (combinatorial) Laplacian ($M^{-1}K_1$) considered by Fax and Murray [10]. We will start by noting that the eigenvalues of $M^{-1}K_1$ are the same as that of $M^{-0.5}K_1M^{-0.5}$. Since M is a diagonal, positive definite matrix, let $M^{0.5}v = w$. The proof is as follows:*

$$\begin{aligned} \lambda &\leq \frac{< \boldsymbol{w}, M^{-0.5}K_1M^{-0.5}\boldsymbol{w} >}{< \boldsymbol{w}, \boldsymbol{w} >} \\ &\leq \frac{< \boldsymbol{v}, K_1\boldsymbol{v} >}{< \boldsymbol{v}, M\boldsymbol{v} >} \\ &\leq \frac{q_r}{n-1} \leq \frac{q(n)}{n-1}. \end{aligned}$$

The second inequality follows from the first because $< \boldsymbol{v}, M\boldsymbol{v} > \geq < \boldsymbol{v}, \boldsymbol{v} >$ by virtue of the information flow graph being connected and therefore, every diagonal entry of M is greater than or equal to 1.

Remark 2. *It is possible that $q(n) \rightarrow \infty$ as $n \rightarrow \infty$ and yet $\lambda \rightarrow 0$. For example, if $q(n)$ increases as $O((n)^\alpha), \alpha < 1$, the quantity $\frac{q(n)}{n} \rightarrow 0$.*

Lemma 1 deals with information flow graphs which are only subject to the constraint that each vehicle may only communicate with a specified number of vehicles. In certain types of regular formations such as a square formation or a cubic formation, where each vehicle can only communicate with vehicles within a certain distance from it, more structure can be imposed on the graphs such as the one dealt in the following proposition:

Proposition 1. *Consider information flow graphs that are connected. Suppose each vehicle in the collection may only communicate directly with m other vehicles in the collection, m being a constant. Further, suppose that the distribution of vehicles is such that the number of vehicles $p(k)$, with k as the length of the communication path to the reference vehicle be $\alpha k^r, \quad k = 1, \ldots, l_0$ for some positive constants α and r. The term l_0 is the diameter of the graph considered. Then, the smallest eigenvalue λ of K_1 goes to zero in the following* *manner: There exists a $N^* > 0$ such that for all $n > N^*$ for any such information flow graph considered,*

$$\lambda \leq \frac{m(r+3)\alpha^{\frac{2}{r+1}}}{(r+1)^{\frac{r+3}{r+1}}} \frac{1}{n^{\frac{2}{r+1}}}. \tag{2.12}$$

Proof. We shall again use Rayleigh's inequality to get an upper bound for the smallest eigenvalue λ, with the assumed mode v_a, constructed in the following way: We find the length of the communication path [2] l_i, of the i^{th} vehicle to the reference vehicle and assign this number to the i^{th} element of the assumed mode. If two vehicles are connected by an edge, the difference between their weights can only be 0, 1 or -1; this is because the weight corresponds to the shortest path between the reference vehicle and the vehicle under consideration. Hence, each spring in the spring-mass system can at most have a deflection of one unit in magnitude. Since there are at most $\frac{mn}{2}$ edges, (because each vehicle is connected to at most m other vehicles and each spring is connected to a pair of vehicles), it follows that the total potential energy is at most $\frac{1}{4}mn$. Let l_0 be the diameter [3] of the information flow graph and let $p(k)$ be the number of vehicles in the collection with k as the length of their communication path to the reference vehicle. Then,

$$< \boldsymbol{v_a}, \boldsymbol{v_a} >= 1^2 p(1) + 2^2 p(2) + \cdots + l_0^2 p(l_0). \tag{2.13}$$

[2]For vehicles A and B that do not communicate directly, the length, l, of the communication path between A and B is the minimum number of intermediate vehicles $V_1, V_2, \ldots, V_l$ such that (1) A and V_1 communicate directly, (2) V_l and B communicate directly and (3) for all $1 \leq i \leq l-1$, V_i and V_{i+1} communicate directly.

[3]The diameter of a graph, l_0, is the maximum value of the length between all possible pairs of vehicles that do not communicate directly.

Therefore using Equation 2.10, we have:

$$\begin{aligned}\lambda &\le \frac{mn}{4}\frac{2}{1^2p(1)+\ldots+(l_0-1)^2p(l_0-1)+l_0^2p(l_0)} \\ &\le \frac{mn}{2\alpha}\frac{1}{1^{2+r}+2^{2+r}+\ldots+(l_0-1)^{2+r}} \\ &\le \frac{mn}{2\alpha}\frac{1}{\int_0^{l_0-1}x^{2+r}dx} \\ &= \frac{mn}{2\alpha}\frac{r+3}{(l_0-1)^{r+3}}.\end{aligned}$$

We now proceed to get a bound for l_0. Since the total number of vehicles, excluding the reference vehicle, in the collection is $n-1$, it follows that $p(1)+\ldots+p(l_0)=n-1$ and hence,

$$\alpha\sum_{k=0}^{l_0-1}k^r \le n-1 \le \alpha\sum_{k=0}^{l_0}k^r.$$

Since

$$\sum_{k=0}^{l_0-1}k^r \le \int_0^{l_0}x^r dx = \frac{l_0^{r+1}}{r+1} \le \sum_{k=0}^{l_0}k^r.$$

it follows that

$$\begin{aligned}n &\le 1+\alpha\frac{(l_0+1)^{r+1}}{r+1} \\ \Rightarrow l_0+1 &\ge \left(\frac{(n-1)(r+1)}{\alpha}\right)^{\frac{1}{r+1}}.\end{aligned}$$

From the above inequality, we are guaranteed that $l_0\to\infty$ as $n\to\infty$ for all information graphs considered. Since

$$\lim_{n\to\infty}\frac{l_0-1}{\left(\frac{n(r+1)}{\alpha}\right)^{\frac{1}{r+1}}} \ge 1,$$

it follows that there exists a $N^*>0$ such that for all $n>N^*$ and for any information flow graph considered in this corollary, we have:

$$l_0 \geq \frac{1}{2^{r+3}}(\frac{n(r+1)}{\alpha})^{\frac{1}{r+1}},$$
$$\Rightarrow \lambda \leq \frac{mn}{2\alpha}\frac{r+3}{(l_0-1)^{r+3}} \leq \frac{m(r+3)\alpha^{\frac{2}{r+1}}}{(r+1)^{\frac{r+3}{r+1}}}\frac{1}{n^{\frac{2}{r+1}}}.$$

□

Remark 3. *If $r < 1$, the bound in the corollary is a tighter one than the one given by Lemma 1.*

Now that we formulated an upper bound on the convergence of λ of K_1 to 0, we shall make use of it, to analyze the propagation of errors due to disturbances acting on the vehicles.

2. Analysis of the propagation of errors

We will focus on showing the following: since $\lambda \rightarrow 0$ as $n \rightarrow \infty$,

1. If $H(s)C(s)$ has exactly two poles at origin, there exists a sinusoidal disturbance acting on each vehicle of at most unit amplitude and of frequency proportional to $\sqrt{\lambda}$ that results in amplitudes of errors in spacing of the order of $O\left(\sqrt{\frac{n}{q(n)^2}}\right)$.
2. If $H(s)C(s)$ has more than two poles at origin, then there is a critical size N^* of the collection such that for all $n > N^*$, at least one root of the equation

$$1 + H(s)C(s)\lambda = 0$$

   has a positive real part; in other words, the controlled motion of the collection is unstable.

We will first show the following:

Lemma 2. *If $H(s)C(s)$ has more than two poles at the origin and if $\lambda \to 0$ as the size of the collection, n, goes to ∞, then there exists a critical size N^* of the formation, such that for any size $n > N^*$ of the formation, the motion of the formation will be unstable.*

Proof. For the problem considered in this section, if $H(s)C(s)$ has more than two poles at zero, it can be factored as $H(s)C(s) = \frac{L(s)}{s^{l+2}}$, $(l > 0)$ for some $L(s)$ that does not have any poles at the origin. We can write the closed loop characteristic equation $\Delta(s)$ as,

$$\Delta(s) := s^{l+2} + \lambda L(s) = 0.$$

We first note that $\Delta(s)$ is Hurwitz only if $L(0) \neq 0$. We further note that $\Delta(s)$ is Hurwitz iff $s^m \Delta(1/s)$ is Hurwitz, where m is the degree of the polynomial $\Delta(s)$. We will now analyze the root locus of $\delta(s) := 1 + \frac{K}{L(1/s)s^{l+2}}, = 1 + \frac{\tilde{L}(s)K}{s^{l+2}}$, where $K := \frac{1}{\lambda}$ and $\tilde{L}(s) = \frac{1}{L(1/s)}$. Since, $\tilde{L}(s)$ is always proper, it is clear that the root locus of $\delta(s)$ has at least $l+2$ asymptotes. Thus, as $K \to \infty$, $(l+2)$ root loci move along lines that make the following angles with the positive real axis.

$$\phi_j = \frac{180^o + 360^o(j-1)}{l+2}, \qquad j = 1, 2, ..., l+2$$

Since $l \geq 1$, it is clear that at least one asymptote, along which one encounters a RHP pole, resulting in the instability of the closed loop as K increases. Hence, if $H(s)C(s)$ has more than two poles at origin, it is evident that there exists a critical size N^* of the formation, such that for any size $n > N^*$ of the formation, the motion of the formation will be unstable.

Hence, we require that $H(s)C(s)$ has not more than two poles at origin to avoid the instability of the formation. But we also derived that $H(s)C(s)$ should have at least two poles at origin to meet the steady state requirement. Hence, to meet both the conditions, $H(s)C(s)$ must have exactly two poles at origin.

Remark 4. *In the above lemma, if the $H(s)C(s)$ has exactly two poles at origin i.e., $l = 0$, and if $L(0)$ is negative, as $|s| \to 0$, there is at least one root of $\Delta(s)$ with positive real part. Hence, the motion of the formation will become unstable. Hence, even for $l = 0$ we require $L(0)$ must be positive so as to avoid instability of motion of the formation.*

□

The following theorem addresses the main result for platoons and it relates the propagation of errors in a platoon due to a disturbance of at most unit magnitude acting on each vehicle:

Theorem 1.

If $H(s)C(s)$ has exactly two poles at the origin and if $L(0)$ is positive, then the errors in spacing grow at least as $O\left(\sqrt{\frac{n^p}{q^{p+1}}}\right)$, where p is the number of poles of the plant transfer function $H(s)$ at the origin. In other words, no control law of the type considered in this paper is scalable to arbitrarily large collections if $\frac{q^{p+1}}{n^p} \to 0$ as $n \to \infty$.

Proof. 1. Consider the transfer function that relates E_v to D_v.

$$\frac{E_v}{D_v}(s) \quad = \quad -\frac{H(s)}{1 + \lambda H(s)C(s)}.$$

Since $L(s)$ does not have a pole at zero, $L(0) \neq 0$. Consider a modal disturbance $\tilde{d}_v(t)$ to be a sinusoid of unit amplitude and of frequency $w = \sqrt{\lambda L(0)} rad/s$, then the amplitude of the modal response $\tilde{e}_v(t)$ is given by the magnitude of the following complex number:

$$\frac{H(jw)}{\underbrace{(1 - \frac{L(jw)}{L(0)})}_{\theta(w)}}.$$

Let p be the number of poles of $H(s)$ at the origin. It is clear from the assumed structure of $H(s)$ that $p = 2$ when $a = 0$ and $p = 1$ when $a \neq 0$. Hence $H(s)$ can be written as $\frac{1}{s^p}\tilde{H}(s)$, such that $\tilde{H}(0) \neq 0$. Since $\theta(w)$ defined above has a root at zero, let $|\theta(w)| = w^\beta |\tilde{\theta}(w)|$, where $\tilde{\theta}(0) \neq 0$ and $\beta \geq 1$. Therefore, the amplitude ratio is

$$\frac{1}{(\sqrt{\lambda L(0)})^{p+\beta}} |\frac{\tilde{H}(jw)}{\tilde{\theta}(w)}|.$$

As $\lambda \to 0$, the amplitude ratio grows to infinity as

$$\frac{\tilde{H}(0)}{|\tilde{\theta}(0)|} \frac{1}{(\sqrt{\lambda L(0)})^{p+\beta}},$$

where $p \geq 1$. Since $\beta \geq 1$ as $\lambda \to 0$, $e_v(t)$ grows at least as

$$\frac{\tilde{H}(0)}{|\tilde{\theta}(0)|} \frac{1}{(\sqrt{\lambda L(0)})^{p+1}}.$$

Since $e_v(t) =< \boldsymbol{v}, \boldsymbol{e(t)} >$, we may express $e_v(t)$ as: $e_v = q_{11}e_1(t) + \ldots + q_{1n}e_n(t)$, for some $q_{11}, \ldots, q_{1n}$. Since v is an eigenvector, we may assume without any loss of generality that $< \boldsymbol{v}, \boldsymbol{v} >= 1$, i.e., $q_{11}^2 + q_{12}^2 + \ldots + q_{1n}^2 = 1$. Each of the errors in spacing is a sinusoid of the frequency, $w = \sqrt{\lambda L(0)}$. Hence, $e_j(t)$ may be expressed as $A_j cos(wt) + B_j sin(wt)$; one may write $e_v = (\sum_{j=1}^n q_{1j}A_j)cos(wt) + (\sum_{j=1}^n q_{1j}B_j)sin(wt)$. It means that either the coefficient of $cos(wt)$ or $sin(wt)$ must increase as $O(\frac{1}{(\sqrt{\lambda})^{p+1}})$. Without any loss of generality, let us say that $(\sum_{j=1}^n q_{1j}A_j)$increases in that fashion. Since

$$(\sum_{i=1}^n q_{1i}A_i) \leq (\sum_{i=1}^n |q_{1i}|) max_{0<i<n+1}|A_i|$$

$$\Rightarrow max_{0<i<n+1}|A_i| \geq O(\frac{1}{(\sqrt{\lambda})^{p+1}}) \frac{1}{||\boldsymbol{v}||_1}$$

.

Since $||\boldsymbol{v}||_2 = 1$, it follows from the equivalence of norms in finite dimensional normed vector spaces [4] that $||\boldsymbol{v}||_1 \leq \sqrt{n}$. Therefore, the maximum amplitude of the errors in spacing over all the vehicles for sufficiently large size of the formation is of $O(\frac{1}{(\sqrt{\lambda})^{p+1}})\frac{1}{\sqrt{n}} = O(\frac{1}{(\sqrt{(n)\lambda^{p+1}})})$. By Lemma 1 we have, $\lambda \leq \frac{q(n)}{n-1}$. Therefore, the errors in the spacing increase as $O\left(\sqrt{\frac{n^p}{q^{p+1}}}\right)$. Hence, a scalable control algorithm requires an information flow graph, where at least one vehicle in the collection communicates directly with at least $O(n^{\frac{p}{p+1}})$.

□

Remark 5. *This theorem may be viewed as a generalization of Theorem* 2.3 *in [7]. Theorem* 2.3 *considers a string of vehicles moving in a straight line, where each vehicle may only communicate with its neighbors.*

Remark 6. *If the errors were governed by equation (8), then the propagation of errors can be analyzed as follows: Since, $M = M^T$ and $K_1 = K_1^T$, we find a matrix of generalized eigenvectors Q such that $Q^T M Q = I; Q^T K_1 Q = \Lambda$. The simultaneous diagonalization of two symmetric positive definite matrices is dealt in vibrations, where M is commonly referred to as the mass matrix and K_1 is referred to as the stiffness matrix. Let $E_Q(s) = QE(s)$ and similarly, $D_Q(s) = QD(s)$. Then:*

$$Q(L + K_1 H(s)C(s))Q^T \boldsymbol{E(s)} = -H(s)QMQ^T \boldsymbol{D(s)}$$

---

[4]Suppose $\boldsymbol{x} \in \Re^n$, the following inequalities hold true for finite dimensional vectors.

$$||\boldsymbol{x}||_\infty \leq ||\boldsymbol{x}||_1 \leq n||\boldsymbol{x}||_\infty$$

$$||\boldsymbol{x}||_\infty \leq ||\boldsymbol{x}||_2 \leq \sqrt{n}||\boldsymbol{x}||_\infty$$

$$\frac{1}{\sqrt{n}}||\boldsymbol{x}||_1 \leq ||\boldsymbol{x}||_2 \leq ||\boldsymbol{x}||_1$$

By the orthogonality relationship, we have:

$$(I + \Lambda H(s)C(s))\boldsymbol{E_Q(s)} = -H(s)\boldsymbol{D_Q(s)}.$$

Let λ *be the smallest generalized eigenvalue, i. e., the smallest of the diagonal elements of* Λ*. Let* $\boldsymbol{v}$ *be the corresponding generalized eigenvalue, i.e.,* $K_1\boldsymbol{v} = \lambda\boldsymbol{v}$*. Define* $\tilde{E}_v =< \boldsymbol{v}, \boldsymbol{E}(s) >$ *and* $\tilde{D}_v =< \boldsymbol{v}, \boldsymbol{D}(s) >$*. We can relate* $\tilde{E}_v(s)$ *to* $\tilde{D}_v(s)$ *as:*

$$\tilde{E}_v(s) = -\frac{H(s)}{1 + \lambda H(s)C(s)}\tilde{D}_v(s),$$

where $\tilde{D}_v$ *is an element of* $\boldsymbol{D_Q}$ *and* $\tilde{E}_v$ *is a corresponding element of* $\boldsymbol{E_Q}$*. Let* $\tilde{e}_v(t) =< \boldsymbol{v}, \boldsymbol{e(t)} >$ *and* $\tilde{d}_v(t) =< \boldsymbol{v}, \boldsymbol{d(t)} >$*. But, we have shown in Theorem 2 that for the equation in above form,* $|\tilde{e}_v(t)|$ *is of* $O(\frac{1}{\sqrt{\lambda}^{p+1}})$*, when* $d(t)$ *is a sinusoid of unit magnitude and of frequency* $\sqrt{\lambda L(0)}$*. Since* $\tilde{e}_v =< \boldsymbol{v}, \boldsymbol{e(t)} >$*, we may write it as*

$$\begin{aligned}
\tilde{e}_v &= < M^{0.5}\boldsymbol{v}, M^{-0.5}\boldsymbol{e(t)} > \\
&\leq ||M^{0.5}\boldsymbol{v}||_2||M^{-0.5}\boldsymbol{e(t)}||_2 \\
&\leq ||M^{-0.5}\boldsymbol{e(t)}||_2 \quad (< \boldsymbol{v}, M\boldsymbol{v} >= 1) \\
&\leq \bar{\sigma}(M^{-0.5})||\boldsymbol{e(t)}||_2 \\
&\leq \sqrt{n-1}||M^{-0.5}\boldsymbol{e(t)}||_\infty < \rho\sqrt{n}||e(t)||_\infty.
\end{aligned}$$

where $\rho = \bar{\sigma}(M^{-0.5}) = \frac{1}{\sqrt{min_i|S_i|}}$*,* $i = \{1, 2...n-1\}$*. Since we are considering information flow graphs which are connected,* ρ *is well-defined and* $\rho \leq 1$*. Therefore,* $||\boldsymbol{e(t)}||_\infty$ *increases at least as* $O(\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\lambda}^{p+1}}) = O(\sqrt{\frac{n^p}{q^{p+1}(n)}})$*, for sufficiently large collections. Hence, it is evident that at least one vehicle in the formation should communicate with at least* $O(n^{\frac{p}{p+1}})$*, for a scalable controller to exist.*

CHAPTER III

VEHICLE FORMATIONS IN HIGHER DIMENSIONS

In this section, we will consider vehicle formations in $\Re^3$. We will consider such maneuvers of the formation, where the desired motion of the reference vehicle automatically specifies the motion of all other vehicles in the formation. A maneuver involving pure translation is an example of one such maneuver.

A. Model of the Vehicle

The index of the reference vehicle is chosen to be 1 without loss of generality as before. The rest of the vehicles may be indexed in any random fashion. Let $(x_i(t), y_i(t), z_i(t))$ denote the position of the i^{th} vehicle in the formation with respect to some fixed inertial frame of reference. We will consider vehicles moving in a three dimensional space and assume that the motion of each vehicle is decoupled in each dimension and hence it can be modelled by a diagonal transfer function matrix $P(s)$. We shall further assume that all its three degrees of freedom are controllable through control forces, $u_{x,i}(t), u_{y,i}(t)$ and $u_{z,i}(t)$. The disturbances acting on the i^{th} vehicle are $d_{x,i}(t), d_{y,i}(t)$ and $d_{z,i}(t)$ in the three directions. Let the Laplace transformations of $x_i(t), y_i(t), z_i(t)$ be respectively $X_i(s), Y_i(s)$ and $Z_i(s)$. Similarly, let $U_{x,i}(s), U_{y,i}(s), U_{z,i}(s)$ and $D_{x,i}(s), D_{y,i}(s), D_{z,i}(s)$ represent the Laplace transformations of $u_{x,i}(t), u_{y,i}(t), u_{z,i}(t)$ and $d_{x,i}(t), d_{y,i}(t), d_{z,i}(t)$ respectively. We will assume the following extension to the vehicle model considered earlier in Chapter II:

$$\begin{pmatrix} X_i(s) \\ Y_i(s) \\ Z_i(s) \end{pmatrix} = P(s) \begin{pmatrix} U_{x,i}(s) - D_{x,i}(s) \\ U_{y,i}(s) - D_{y,i}(s) \\ U_{z,i}(s) - D_{z,i}(s) \end{pmatrix} + \begin{pmatrix} \frac{(s+a)x_i(0)+\dot{x}_i(0)}{s(s+a)} \\ \frac{(s+a)y_i(0)+\dot{y}_i(0)}{s(s+a)} \\ \frac{(s+a)z_i(0)+\dot{z}_i(0)}{s(s+a)} \end{pmatrix}. \tag{3.1}$$

In the above equation, $P(s) = \frac{1}{s(s+a)} I_3$, where I_3 is the identity matrix in R^3. There are only two cases to consider: $a = 0$, and $a \neq 0$. As before, $a = 0$ corresponds to a point mass model for each degree of freedom of the vehicle and there is no damping; the case $a \neq 0$ indicates the presence of linear viscous damping. Let desired trajectory of the reference vehicle be $(x_{ref}(t), y_{ref}(t), z_{ref}(t))$. Let $l_{x,i}, l_{y,i}, l_{z,i}$ be the desired distance between the i^{th} vehicle and the reference vehicle along the x, y and z directions. Let $\delta_x(i,j), \delta_y(i,j)$ and $\delta_z(i,j)$ be the desired distance between vehicles i and j in the x, y and z directions. One may define the error in spacing of the i^{th} vehicle relative to the reference vehicle $(e_{x,i}(t), e_{y,i}(t), e_{z,i}(t))$ as follows:

$$e_{x,i}(t) := x_i - x_{ref}(t) - l_{x,i},$$
$$e_{y,i}(t) := y_i - y_{ref}(t) - l_{y,i},$$
$$e_{z,i}(t) := z_i - z_{ref}(t) - l_{z,i}.$$

We further assume that the structure of the control law used by each vehicle, other than the reference vehicle, is the same. Specifically, we consider the following structure for the other vehicles:

$$\begin{pmatrix} U_{x,i}(s) \\ U_{y,i}(s) \\ U_{z,i}(s) \end{pmatrix} = -C(s) \sum_{j \in S_i} \begin{bmatrix} X_i(s) - X_j(s) - \frac{\delta_x(i,j)}{s} \\ Y_i(s) - Y_j(s) - \frac{\delta_y(i,j)}{s} \\ Z_i(s) - Z_j(s) - \frac{\delta_z(i,j)}{(s)} \end{bmatrix}, \tag{3.2}$$

where $C(s)$ is a 3×3 array of rational transfer functions. Since $C(s)$ is assumed to have cross coupling terms, i.e., $C(s)$ is not a diagonal matrix, one may, without any loss of generality, assume a vehicle model in Equation (3.1). The reasoning follows along the same lines as in the previous chapter. We further assume that initial conditions of the vehicles correspond to the required rigid formation. As a consequence of the choice of the structure of the controller given in Equation (3.2) and the model of vehicle given in Equation (3.1), the dynamics of error propagation can be written in one single equation as follows:

$$\begin{pmatrix} E_{x,i}(s) \\ E_{y,i}(s) \\ E_{z,i}(s) \end{pmatrix} = -P(s)C(s)\sum_{j\in S_i}\begin{pmatrix} E_{x,i}(s)-E_{x,j}(s) \\ E_{y,i}(s)-E_{y,j}(s) \\ E_{z,i}(s)-E_{z,j}(s) \end{pmatrix} - P(s)\begin{pmatrix} D_{x,i}(s) \\ D_{y,i}(s) \\ D_{z,i}(s) \end{pmatrix} - \begin{pmatrix} \bar{X}(s) \\ \bar{Y}(s) \\ \bar{Z}(s) \end{pmatrix},$$

where $\bar{X}_(s) = X_{ref}(s) - X_{ref}(0)$, $\bar{Y}_i(s) = Y_{ref}(s) - Y_{ref}(0)$ and $\bar{Z}_i(s) = Z_{ref}(s) - Z_{ref}(0)$. As in the case of single dimension, the above set of equations can be written as:

$$(I_{3n-3} + P(s)C(s) \otimes K_1)\boldsymbol{E(s)} = -(I_{n-1} \otimes P(s))\boldsymbol{D(s)} - \tilde{\boldsymbol{X}}\boldsymbol{(s)},$$

where I_{3n-3}, I_{n-1} are identity matrices of dimensions $3n-3$ and $n-1$ respectively, K_1 is the principal minor obtained by removing the first row and column of Laplacian K of the information flow graph defined as follows: For $j \neq i$, $K_{ij} = -1$ if vehicles i and j communicate directly; otherwise $K_{ij} = 0$. The i^{th} diagonal element is defined as $K_{ii} = -\sum_{j\neq i} K_{ij}$. As considered earlier in Chapter II, we will assume that the information flow graph is *undirected* and *connected*. Hence, by the virtue of assumptions on information flow graphs, K_1 is symmetric and it cannot have 0 in its spectrum.

The binary operation involving matrices A and B given by $A \otimes B$ indicates the Kronecker product of A and B. We shall refer to $rem(i,j)$ and $mod(i,j)$ as the remainder and quotient obtained respectively when i is divided by j. The term $\boldsymbol{E(s)}$ is the Laplace transformation of the vector of errors in spacing of the vehicles $\boldsymbol{e(t)}$; and if $p := 2 + rem(i,3)$, the i^{th} entry of $\boldsymbol{e(t)}$ is $e_{x,p}$ if $mod(i,3)$ equals 1, is $e_{y,p}$ if $mod(i,3)$ is 2 and is $e_{z,p}$ if $mod(i,3)$ is 0. Likewise, the term $\boldsymbol{D(s)}$ is the Laplace transformation of the vector of disturbances and is constructed in a manner similar to $\boldsymbol{E(s)}$. Similarly, the i^{th} term of the vector $\tilde{\boldsymbol{X}}\boldsymbol{(s)}$ is $\bar{X}(s)$ if $mod(i,3)$ equals 1, is $\bar{Y}(s)$ if $mod(i,3)$ equals 3, and is $\bar{Z}(s)$ otherwise.

Let λ be the smallest eigenvalue of K_1 and let $\boldsymbol{v}$ be the corresponding eigenvector.

Let $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}$ represent the three orthonormal vectors which form the basis of $\Re^3$. One can show that the span of $\{\boldsymbol{p_k} \otimes \boldsymbol{v_1},\ k = 1, 2, 3\}$ is invariant under the action of $(I_{3n-3} + P(s)C(s) \otimes K_1)$. In particular, $(I_{3n-3} + P(s)C(s) \otimes K_1) \begin{bmatrix} \boldsymbol{p_1} \otimes \boldsymbol{v_1} & \boldsymbol{p_2} \otimes \boldsymbol{v_1} & \boldsymbol{p_3} \otimes \boldsymbol{v_1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p_1} \otimes \boldsymbol{v_1} & \boldsymbol{p_2} \otimes \boldsymbol{v_1} & \boldsymbol{p_3} \otimes \boldsymbol{v_1} \end{bmatrix} (I_3 + \lambda P(s)C(s))$.

Define $e_{1,x}(t) =< \boldsymbol{p_1} \otimes \boldsymbol{v_1}, \boldsymbol{e(t)} >, e_{1,y}(t) =< \boldsymbol{p_2} \otimes \boldsymbol{v_1}, \boldsymbol{e(t)} >$ and $e_{1,z}(t) =< \boldsymbol{p_3} \otimes \boldsymbol{v_1}, \boldsymbol{e(t)} >$. Similarly, define $d_{1,x}(t) =< \boldsymbol{p_1} \otimes \boldsymbol{v_1}, \boldsymbol{d(t)} >, d_{1,y}(t) =< \boldsymbol{p_2} \otimes \boldsymbol{v_1}, \boldsymbol{d(t)} >$ and $d_{1,z}(t) =< \boldsymbol{p_3} \otimes \boldsymbol{v_1}, \boldsymbol{d(t)} >$. Then, the Laplace transformations of the signals defined are related by:

$$\gamma(s) \underbrace{\begin{pmatrix} E_{1,x}(s) \\ E_{1,y}(s) \\ E_{1,z}(s) \end{pmatrix}}_{\boldsymbol{E_{1,p}(s)}} = -P(s) \underbrace{\begin{pmatrix} D_{1,x}(s) \\ D_{1,y}(s) \\ D_{1,z}(s) \end{pmatrix}}_{\boldsymbol{D_{1,p}(s)}} - \underbrace{\begin{pmatrix} \bar{X}_1(s) \\ \bar{Y}_1(s) \\ \bar{Z}_1(s) \end{pmatrix}}_{\boldsymbol{\tilde{X}_{1,p}}},$$

where $\gamma(s) := I_3 + \lambda P(s)C(s)$.

B. Analysis

As in the case of platoons considered in Chapter II, the control objectives are as follows:

1. In the absence of any disturbance acting on any vehicle, every vehicle must track its desired position when the speed of the reference vehicle asymptotically reaches a constant value that is different from its initial speed.

2. In the presence of a bounded disturbance of at most unit magnitude, there must exist a $M_R > 0$ such that the errors in spacing and velocity of every vehicle in the collection be bounded by M_R *irrespective* of the size of the collection.

1. Steady-state errors

Let us analyze the first requirement: Since we want the steady state error in spacing to be zero as per the first requirement, it is necessary that $\lim_{s\to 0} s\boldsymbol{E_{1,p}} = 0$ for any possible $\boldsymbol{\Delta v_q} \in \Re^3$ such that $\lim_{s\to 0} s^2\boldsymbol{\tilde{X}_{1,p}} = \boldsymbol{\Delta v_q}$.Therefore,

$$\begin{aligned}\lim_{s\to 0} s\boldsymbol{\tilde{E}_{1,p}} &= -\lim_{s\to 0}(sI_3 + \lambda sP(s)C(s))^{-1}s^2\boldsymbol{\tilde{X}_{1,p}} \\ &= -\lim_{s\to 0}(sI_3 + \lambda sP(s)C(s))^{-1}\boldsymbol{\Delta v_q}.\end{aligned}$$

In the last equation, the Final Value Theorem has been employed with the assumption that the controller is chosen so that the transfer function matrix under consideration is analytic in $Re(s) \geq 0$. The limit on the right hand side of the last equation is zero for all possible $\boldsymbol{\Delta v_q}$ iff $\lim_{s\to 0}(sP(s)C(s))^{-1} = 0$. In other words, $P(s)C(s)$ must be expressible as $\frac{1}{s^2}L(s)$ for some rational $L(s)$ such that $L(0) \neq 0$. We will start with the following lemma that shows the effect of the number of poles of open loop transfer function matrix $P(s)C(s)$ on the over all stability of the closed loop system and the investigation of the propagation of errors in a formation.

Lemma 3. *Consider the following characteristic equation for positive values of* λ*:* $\Delta(s) := det(s^l I + \lambda L(s)) = 0$*, where* $\frac{1}{s^l}L(s)$ *is a square matrix of rational, proper transfer functions with real coefficients.*

1. *If* $l \geq 3$*, there is a* $\lambda_1 > 0$ *such that for all* $\lambda \in (0, \lambda_1)$*, there is a zero of* $\Delta(s)$ *with non-negative real part.*
2. *If* $l = 2$ *and if any of the eigenvalues of* $L(0)$ *is negative or complex, then there is a zero of* $\Delta(s)$ *with non-negative real part.*

Proof. If $det(L(0)) = 0$, then $\Delta(s)$ has a zero at 0 for all λ. Therefore, it is sufficient to consider the case when $det(L(0)) \neq 0$. Let Γ be the Nyquist contour which is indented to the right when poles or zeros of $det(\frac{L(s)}{s^l})$ are encountered. Let $\mu_i(s), i = 1, \ldots, m$ be the characteristic loci (eigen values) of $L(s)$. The multi-variable Nyquist criterion indicates that the sum of the number of encirclements of the Nyquist plots of (the maps of Γ by) $\frac{\lambda\mu_i(s)}{s^l}$ about the point $-1 + j0$ is equal to the excess of the number of zeros of $\Delta(s)$ over the poles of $L(s)$ in the Right Half Plane. Therefore, it is sufficient to show that the Nyquist plot of $\frac{\lambda\mu_i}{s^l}, i = 1, \ldots, m$ has at least one encirclement about the point $-1 + j0$ if $l \geq 3$.

Since the Nyquist plot of $\frac{\mu_i(s)}{s^l}$ intersects the real axis only a finite number of times, we will consider only the maximum absolute value of the finite intersections (not the intersections at infinity). Through an appropriate choice of $\lambda_{1,i}$, all the finite intersections of all Nyquist plots of $\lambda\frac{\mu_i(s)}{s^l}, \lambda \in (0, \lambda_{1,i})$ can be made to occur to the right of the point $-1 + j0$ on the real axis. Define $\lambda_1 := \min_{1 \leq i \leq m} \lambda_{1,i}$. Since the intersections at infinity only correspond to the poles of the transfer function $det(\frac{L(s)}{s^l})$, we consider only the poles on the imaginary axis. Also, the encountering of even number of successive zeros on the Nyquist contour has the same effect of encountering no zeros on the Nyquist contour. Therefore, only the parity of the zeros encountered between successive poles as one traverses the Nyquist contour matters rather than the exact number of zeros. An occurrence of a pole followed by zero followed by a pole either increases by one or does not change the number of encirclements of the Nyquist plot depending on whether the Nyquist plot is starting on the negative real axis prior to encountering the first pole of the pole-zero-pole combination. Since the transfer function $det(\frac{L(s)}{s^l})$ is proper, as $|s| \to \infty$, the eigen values of $\frac{L(s)}{s^l}$ approach constant values. Therefore, as $|s| \to \infty$, the Nyquist plot of $\frac{\lambda\mu_i(s)}{s^l}$ reaches a finite real value along the arc of infinite radius of the Nyquist contour. Therefore, if $l \geq 3$, number of encirclements of the Nyquist plot of $\frac{\lambda\mu_i(s)}{s^l}$ is at least $mod(l, 2) \geq 1$. Hence, for all $\lambda \in (0, \lambda_1)$, the number of encirclements of $\lambda^m\frac{L(s)}{s^l}, \lambda \in (0, \lambda_1)$ about the point $-1 + j0$ is at least $m \geq 1$.

If $l = 2$, it can be seen that if any of eigenvalues of $L(0)$ is negative or complex, the number of encirclements about the point $-1 + j0$ is at least one. Therefore, the total number of encirclements about the point $-1 + j0$ is at least one, implying that at least one root of the characteristic equation $det(s^2 I + \lambda L(s))$ has at least one root with positive real part. □

2. Analysis of propagation of errors

Now that we have obtained the necessary conditions for satisfying the steady state requirement, we shall shift our focus to the analysis of the propagation of errors for bounded disturbances acting on every vehicle.

Theorem 2. *Consider a formation of vehicles with each vehicle following the model described earlier. Further, suppose that the smallest eigenvalue λ of K_1 goes to 0 as the size of the collection, n, increases arbitrarily.*

1. *Let r be the smallest positive integer such that $\lim_{s \to 0} s^r P(s)C(s)$ be bounded. Let $L(s) = s^r P(s)C(s)$. If $r \geq 3$, or if $r = 2$ and any of the eigenvalues of $L(0)$ is not positive, then there is a critical size $N^* > 0$ of the collection such that for all $n > N^*$, the motion of the vehicles in the collection is unstable.*
2. *If $r = 2$, then there is a sinusoidal disturbance acting on each vehicle of the same frequency and at most unit in magnitude such that the error in spacing is of $O\left(\sqrt{\frac{n^l}{q(n)^{l+1}}}\right)$, where l is the smallest positive integer, such that $\lim_{s \to 0} s^l P(s)$ is bounded and not zero.*

Proof. (1) The first part of this theorem is a direct consequence of Lemma 3.

(2) We have shown earlier that $P(s)C(s)$ should have at least two poles at the origin of the complex plane to have zero steady state error even if lead vehicle makes a maneuver such that there is a change in the steady state speed of the collection. Hence, $P(s)C(s) = \frac{1}{s^2}L(s)$

with $L(0)$ having real and positive eigenvalues. Now, consider a sinusoidal disturbance force acting on each vehicle at the following frequency: $\omega = \sqrt{\lambda\mu_i(0)}$ where $\mu_i(0)$ is an eigenvalue of $L(0)$. At that frequency, the amplitude and phase shift are given by: $(I - \lambda\frac{L(j\omega)}{\mu_i(0)})^{-1}P(j\omega)$, which may be expressed as

$$(I - \lambda\frac{L(j\omega)}{\mu_i(0)})^{-1}(j\omega)^l P(j\omega)\frac{1}{(jw)^l},$$

where l is the smallest positive integer such that $\lim_{s\to 0} s^l P(s)$ is bounded. It should be noted that $l = 2$ when $a = 0$ and l = 1 when $a \neq 0$. Since $\lim_{w\to 0}(I - \frac{L(j\omega)}{\mu_i(0)})$ is singular, ω^p for some $p \geq 1$ is a factor of the $det(I - \frac{\lambda L(j\omega)}{\mu_i(0)})$. Hence, we may rewrite, for all sufficiently small w, $(I - \lambda\frac{L(j\omega)}{\mu_i(0)})^{-1}$ as $\frac{1}{\omega^p}\tilde{L}(j\omega)$, where $\tilde{L}(0)$ is bounded. Therefore, the frequency response of the transfer function is given by: $\tilde{L}(j\omega)(-j)^l\frac{1}{\omega^{p+l}}$. Since $\lambda\mu_i(0) = w^2$, the amplitude of errors, $E_{1,p}(t)$ increase as $O(\frac{1}{\sqrt{\lambda}^{p+l}})$ for some $p \geq 1$ for low frequency disturbances.Therefore, the amplitude of at least one entry in $\boldsymbol{E_{1,p}(t)}$ at least increases as $O(\frac{1}{\sqrt{\lambda}^{l+1}})$. Without loss of generality, one can say that $e_{1,x}(t)$ increases in that order. Since, $e_{1,x}(t) =< \underbrace{\boldsymbol{p_1} \otimes \boldsymbol{v_1}}_{\boldsymbol{q'}}, \boldsymbol{e(t)} >$, $||\boldsymbol{e(t)}||_\infty$ is of $O(\frac{1}{||\boldsymbol{q'}||_1\sqrt{\lambda}^{p+l}})$. Since $\boldsymbol{q'} \in \Re^{3n-3}$ is a unit vector, it is true for finite dimensional vectors of 2-norm unity that $||\boldsymbol{q'}||_1 \leq \sqrt{3n-3} < \sqrt{3n}$. Therefore, the maximum amplitude of the errors in spacing over all the vehicles for sufficiently large size of the formation is of $O(\frac{1}{\sqrt{\lambda}^{l+1}\sqrt{3n}})$. By Lemma 1 we have, $\lambda \leq \frac{q(n)}{n-1}$. Therefore, the errors in the spacing increase as $O\left(\sqrt{\frac{n^l}{q(n)^{l+1}}}\right)$ for sufficiently large formations. $\square$

CHAPTER IV

SIMULATIONS

For the purposes of numerical simulation, we consider the motion of collection of vehicles moving in a straight line. Each vehicle is assumed to be a point mass. As mentioned earlier, the control law used is as follows: $U_i(s) = \sum_{j \in S_i} C(s)(x_i - x_j - L_{ij})$, where $j \in S_i$ implies that there exists a communication link between i^{th} vehicle and j^{th} vehicle. We consider a string of vehicles moving in a straight line, where the following vehicle tries to maintain a constant following distance. We describe the corresponding results below:

A. String of Vehicles

We consider a string of vehicles, indexed from 1 to n. The set of vehicles that the first vehicle communicates with directly is the second vehicle, i.e. $S_1 = \{2\}$. For $i = 2, \ldots, n-1$, the set S_i of vehicles the i^{th} vehicle communicates with directly is $\{i-1, i+1\}$ and $S_n = \{n-1\}$. Figure 1 shows the above information topology in a string of 6 vehicles. A lag controller is used for feeding back the error in spacing and is given by $C(s) = \frac{3s+2}{0.01s+1}$. Figure 2 shows the convergence of λ to 0 as the length of the string increases. Figure 3 shows the propagation of errors in spacing in a string of six vehicles. It shows how errors amplify in response to a sinusoidal disturbance acting on the last vehicle along the string, as we move away from the reference vehicle (vehicle indexed 1). The maximum error in spacing increases as n^3 as the size n of the string increases. This result is analogous to a spatially discrete model of a beam, where the first eigenvalue decreases as $\frac{1}{L^2}$, L being the length of a beam. The counterpart for the length of the beam is n, the size of the collection. The decrease in natural frequency is due to a reduction in the "effective stiffness" as the length of the beam is increased. For this reason, the deflection as expected would be larger. The Figure 4 shows an example of the effect stated in Theorem 1. This plot shows the

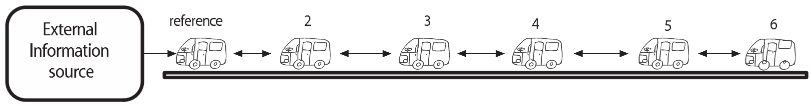

Fig. 1. Predecessor and follower based information flow pattern in the string

disturbance to error gain as a function of frequency. As predicted, the steady state as well as the peak gain increases as N increases. Figure 5 shows the same effect.

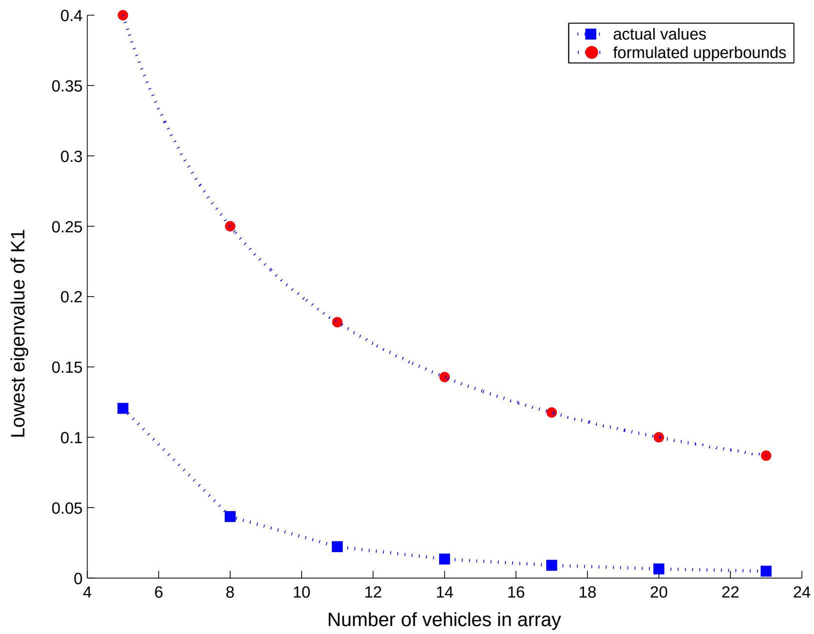

Fig. 2. The variation of λ (lowest eigenvalue of K_1) with n, for a string of n vehicles with each vehicle connected to the vehicles directly behind and ahead of it

The above simulations are repeated with randomly generated information flow graphs. The convergence of λ to 0 for various random graphs with a maximal degree constraint of 4 is shown in Figure 6. It can be observed that though the information flow graphs are random, the upper bound derived in Lemma 1 holds good for all the cases even when the size of the collection is small. The errors in position in response to a sinusoidal disturbance on the last vehicle is shown in Figure 7. One instance of the randomly generated information flow graph is shown in Figure 8. In this case the diameter of the graph is 2 and

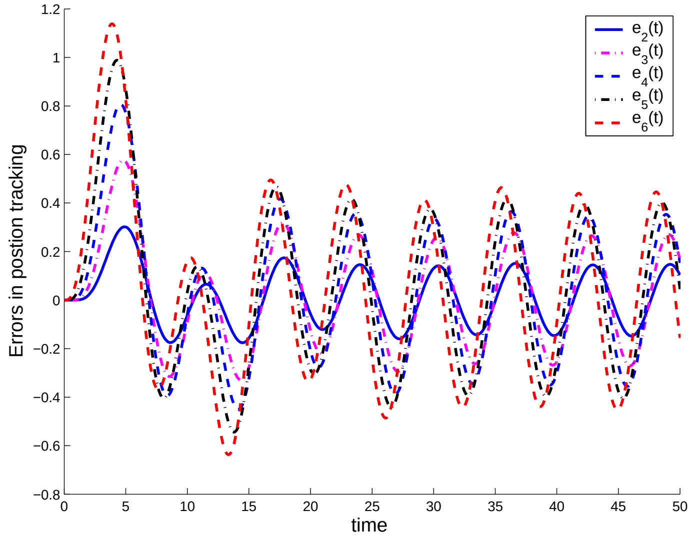

Fig. 3. Propagation of the errors along the string

corresponds to the paths $(1-5-6)$ or $(1-3-4)$ or $(1-5-2)$. It can be seen in Figure 7 that as we move away from the reference vehicle along any of those diameters, the errors amplify. The maximum disturbance to error gain at all frequencies is shown in Figure 9. The variation of the maximal errors of spacing, arising due to sinusoidal disturbance on the last vehicle, with the size of the string is shown in Figure 10. It can be observed that the error to disturbance gain increases with the size of the collection, however in a rather slow manner as opposed to the previous case. This difference may be attributed to the fact that the diameter of the randomly generated information flow graph is typically smaller.

To illustrate the limitations in the sizes of collection that can be considered when an integral action is included in the controller, we consider a controller described by the following transfer function *e.g* $C(s) = \frac{3s^2+2s+1}{s(0.01s+1)}$. However, this strategy will not assure the stability of the motion of the collection of vehicles as shown in Lemma 2. Figure 11 shows the migration of dominant pole to the right half plane as the size of the collection of vehicles increases.

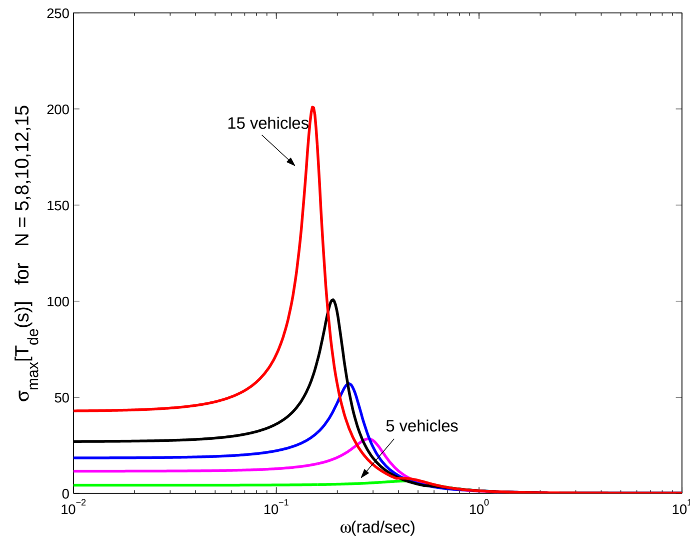

Fig. 4. Variation of the maximum singular value ($\bar{\sigma}$) of the error to disturbance transfer function matrix vs. the size of the string

B. Array of Vehicles

In this section, we consider a square formation of vehicles moving in a straight line. Each vehicle is assumed to be a point mass.We use the same control law in section B of Chapter II. Since information flow graphs considered here have constraints on degree of a graph (the number of vehicles in the collection that a vehicle in the collection can communicate directly with), it is natural to consider Delaunay triangulations as described in [11]. The main advantage of Delaunay triangulation over its counterparts, is that it avoids very long and very short communication links. In a Delaunay triangulation, each vehicle is linked *only* to some of its geographically proximal neighbors. In each of the Delaunay traingulations considered, every vehicle was connected to a maximum of eight other vehicles in the collection. It seems reasonable that there will be such a bound in a Delaunay triangulation, especially when there is a requirement of minimum spacing between vehicles. The convergence of lowest eigenvalue of K_1 to zero with the size of the array for Delaunay triangulated graph is shown in Figure 12. As in the previous case, a lag controller is used for

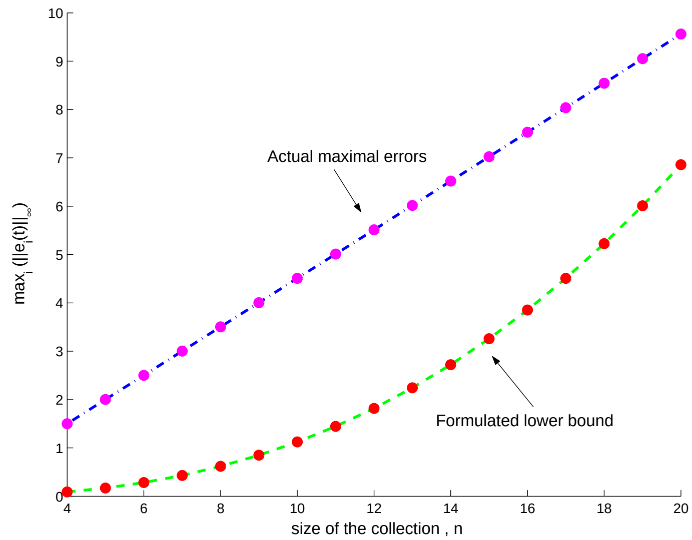

Fig. 5. Variation of maximum spacing error with the size of the string

the feedback of errors, $C(s) = \frac{3s+2}{0.01s+1}$. A Delaunay triangulated square formation is shown in Figure 13. As shown in the figure, the vehicle to the bottom leftmost of the formation is chosen to be the reference vehicle. It is apparent from the figure that the diameter of the graph is along the leading diagonal. Figure 14 shows how the errors in spacing propagate along the diameter of the graph in response to a sinusoidal disturbance of farthest vehicle on the leading diagonal from the reference vehicle. It is not surprising that the errors amplify along the diameter of the graph. Figure 15 shows the maximal disturbance to error gain($\sigma_{max}(T_{de}(s))$, where $T_{de}(s)$ is the closed loop transfer function between disturbance and error as the function of frequency. As expected, since the information of the reference vehicle is only available to a limited number of vehicles, the sensitivity of errors to disturbance increases as the size of the array increases. A similar set of simulations are repeated for randomly generated information flow graphs that obey a pre-specified maximum degree constraint chosen to be 8 for the simulations. It can be seen in Figure 16, that the upper bound derived in Lemma 1 holds good even for randomly generated information flow graphs for square formations. Finally, the plot of disturbance to error gain at various

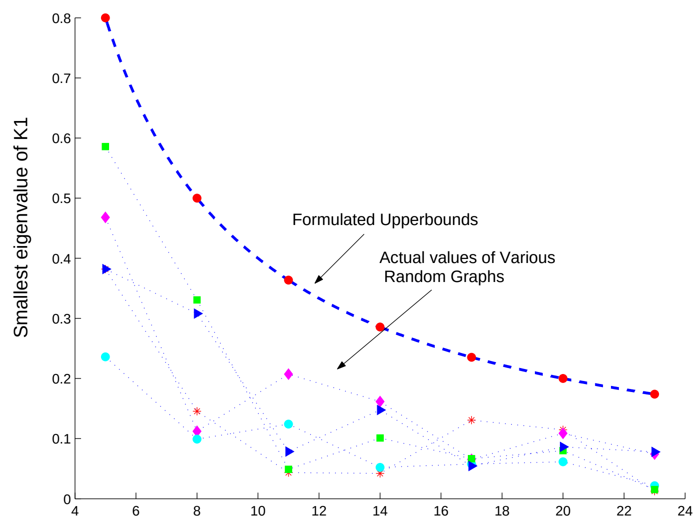

Fig. 6. Variation of λ with n , for a string of n vehicles, connected in a random fashion to a maximum of 4 other vehicles

frequencies is shown in Figure 17.

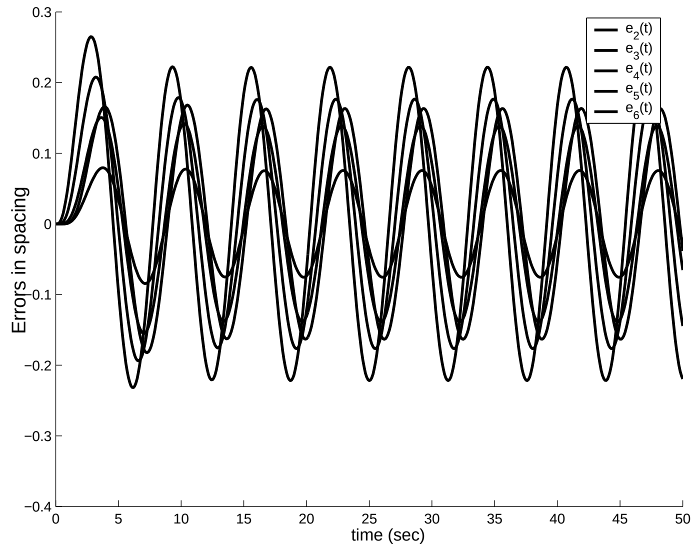

Fig. 7. Propagation of the spacing errors for a randomly connected vehicle string of 6 vehicles

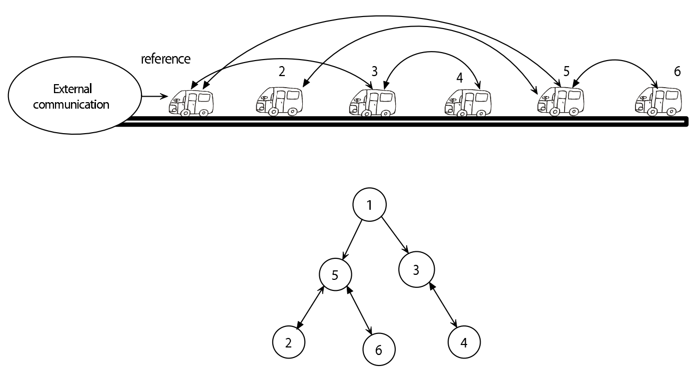

Fig. 8. A random information flow graph

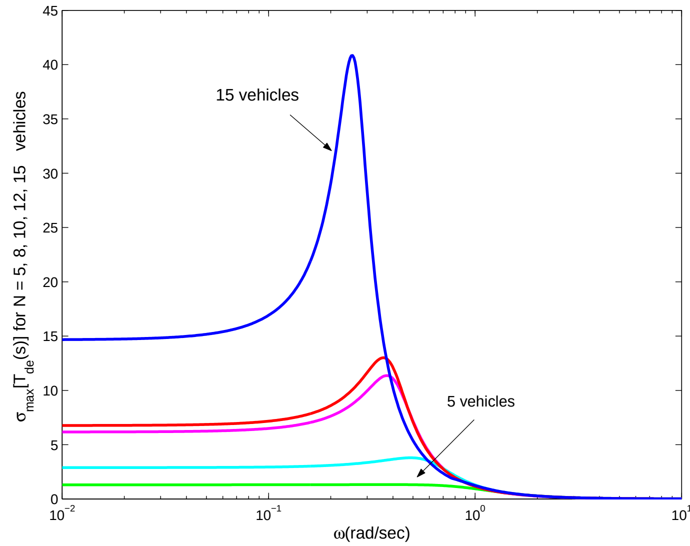

Fig. 9. Variation of the maximum singular value ($\bar{\sigma}$) of the error to disturbance transfer function matrix vs. the size of the string, when vehicles are communicating in random fashion

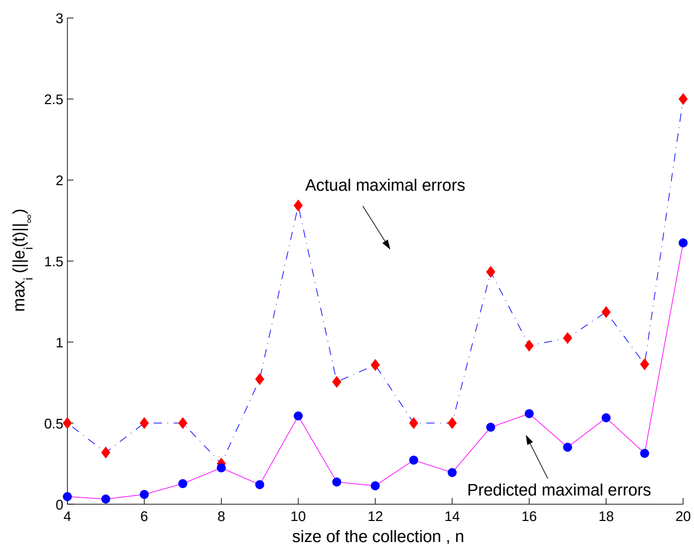

Fig. 10. Variation of maximum spacing error with the size of the string, vehicles connected in a random fashion

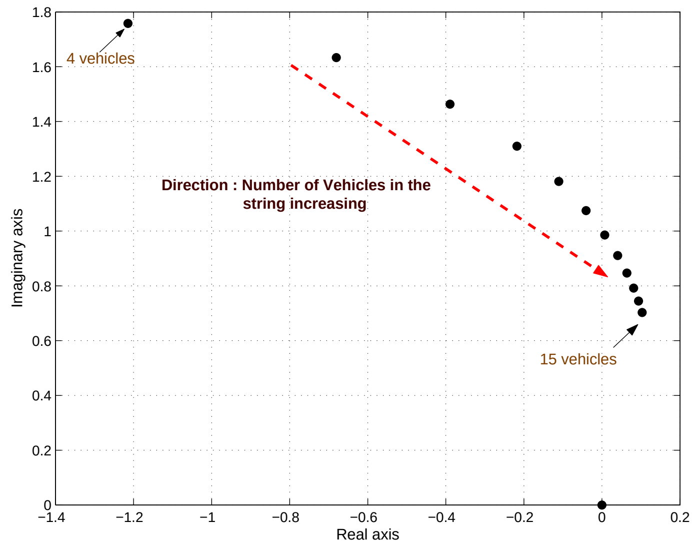

Fig. 11. Plot showing the migration of the dominant pole towards imaginary axis with increase in the number of vehicles

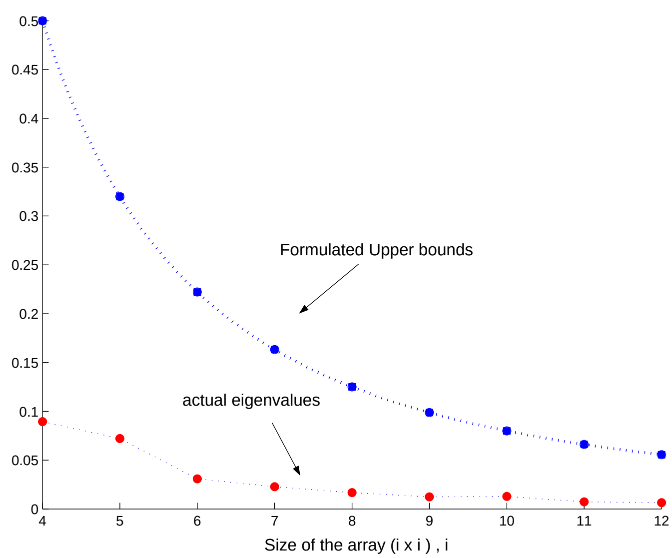

Fig. 12. The convergence of lowest eigenvalue of K_1 of Delaunay triangulated graphs

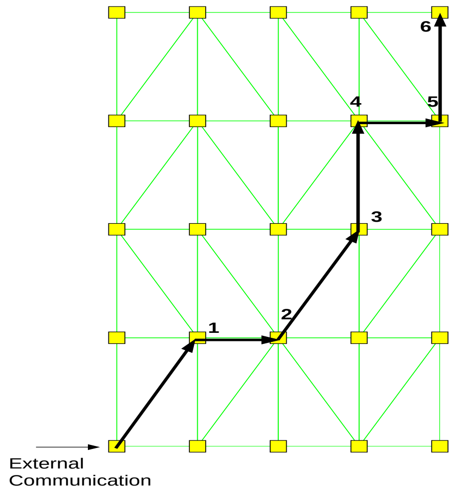

Fig. 13. Delaunay triangulation of a 5 x 5 array of vehicles, arrows showing one of the diameters of the graph

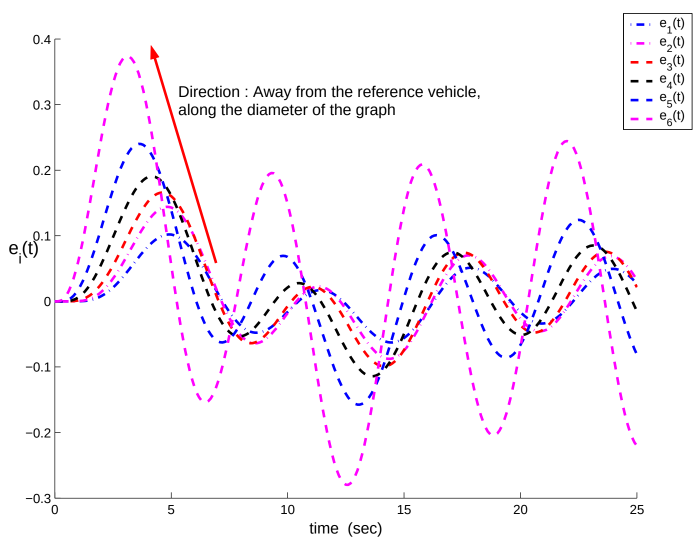

Fig. 14. The propagation of the errors away from the reference vehicle for an array of 5 x 5 vehicles

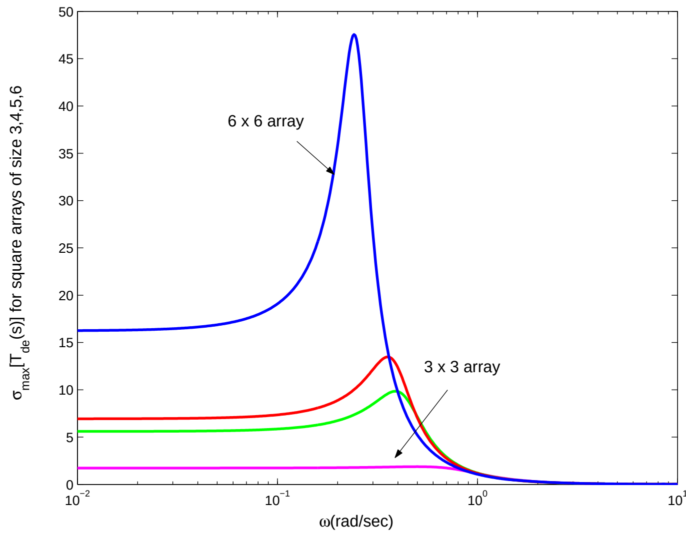

Fig. 15. Variation of the maximum singular value ($\bar{\sigma}$) of the error to disturbance transfer function matrix vs. the size of the array of vehicles communicating through delaunay generated links

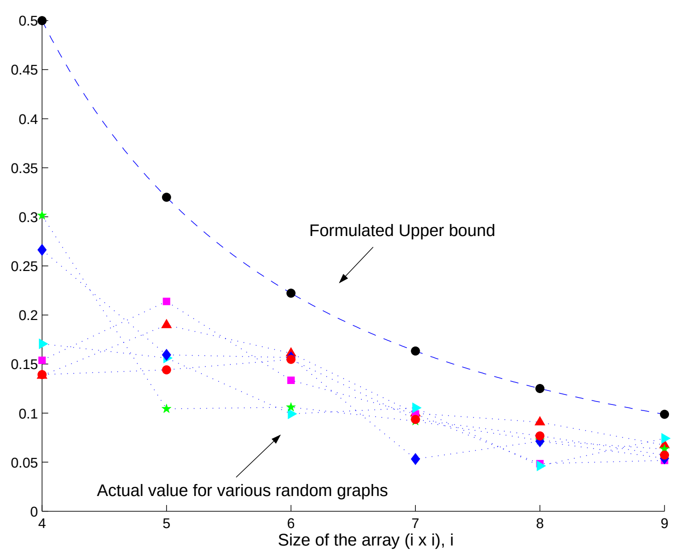

Fig. 16. The convergence of lowest eigenvalue of K_1 for various random generated graphs

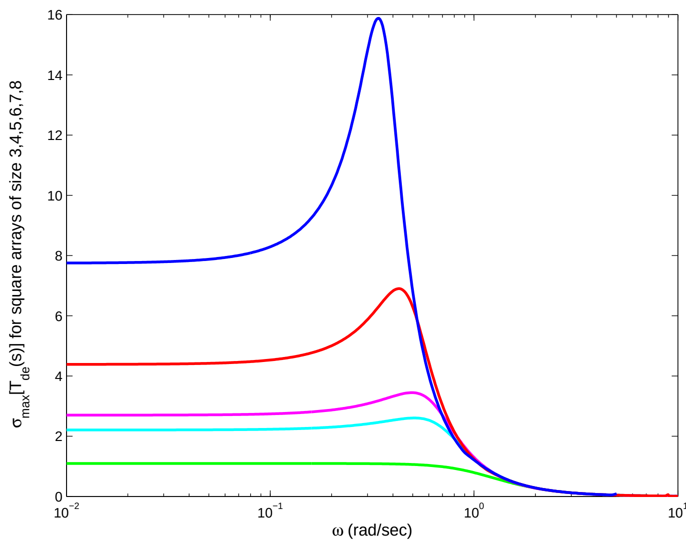

Fig. 17. Variation of the maximum singular value ($\bar{\sigma}$) of the error to disturbance transfer function matrix vs. the size of the array of vehicles communicating in random fashion

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

In this work, we have considered information flow graphs for a collection of vehicles, where there is a constraint on the maximum number of vehicles in the collection every vehicle can communicate with directly. We showed that the motion of collection of vehicles in $\Re^3$ is unstable if the open loop transfer function $P(s)C(s)$ had more than two poles at the origin. We have also shown that $P(s)C(s)$ must have at least two poles at the origin to track ramp inputs resulting from the reference vehicle moving at a constant velocity. We further showed that if $\lambda_1 \rightarrow 0$, there is a disturbance of sufficiently low frequency acting on each vehicle of at most unit magnitude which results in errors in spacing of $O\left(\sqrt{\frac{(n)^l}{q(n)^{l+1}}}\right)$, where l is the number of poles of $P(s)$ at the origin. Hence, to avoid the propagation of errors as the size of the collection increases, one requires at least one vehicle to communicate with $O(n^{1/2})$ other vehicles.

The results presented in the thesis leave several topics for further research.

1. In this thesis, we showed that some simple distributed control architectures result in instability of motion of vehicles. Specifically, we investigated the information flow patterns which can be represented by a undirected graph. An interesting case to study would be the scenario of error propagation, when the information topology corresponds to a directed graph. The case of the directed graph is more general and includes "*one-way*" communication patterns as well, which are researched widely in the literature on AHS.

2. It has been observed that the stability of the motion of the vehicles in a formation and the scalability issues have interesting connections with synchronization of dynamical systems. It is found to have tremendous applications, which include synchronization

of coupled oscillators, modeling populations of interacting biological systems and image processing. It is worthwhile to study the connections between these fields and come up with an unifying framework.

3. So far, in this thesis, we have formulated certain minimum requirements on *how much* information the reference vehicle should communication. One might need to know "how often" the information should be sent for acceptable control.

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