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Why protective measurement does not establish the reality of the quantum state

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"Protective measurement" refers to two related schemes for finding the expectation value of an observable without disturbing the state of a quantum system, given a single copy of the system that is subject to a "protecting" operation. There have been several claims that these schemes support interpreting the quantum state as an objective property of a single quantum system. Here we provide three counter-arguments, each of which we present in two versions tailored to the two different schemes. Our first argument shows that the same resources used in protective measurement can be used to reconstruct the quantum state in a different way via process tomography. Our second argument is based on exact analyses of special cases of protective measurement, and our final argument is to construct explicit "ψ-epistemic" toy models for protective measurement, which strongly suggest that protective measurement does not imply the reality of the quantum state. The common theme of the three arguments is that almost all of the information comes from the "protection" operation rather than the quantum state of the system, and hence the schemes have no implications for the reality of the quantum state.

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The status of the quantum state is one of the most controversial issues in the foundations of quantum theory. Is it *ontic* (a state of reality) or *epistemic* (a state of knowledge, information, or belief)? The likes of de Broglie and Schrödinger initially conceived of the quantum state as a real physical wave, somewhat akin to a classical field [1], whereas the Copenhagen interpretation views it as a state of knowledge about the outcomes of future experiments [2], more akin to a classical probability measure than a physical field. Einstein also thought that the quantum state represents knowledge [3], but, unlike the Copenhagen school, he thought that this was knowledge about some deeper underlying reality rather than just the outcomes of experiments. In modern parlance, interpretations in which the quantum state is ontic, in the same sense as a classical field, are dubbed ψ-ontic, and those in which it is epistemic, i.e. has the same status as a classical probability measure, are called ψ-epistemic. Most current realist interpretations of quantum theory; such as many-worlds [4–6], de Broglie-Bohm theory [7–10], spontaneous collapse theories [11, 12], and modal interpretations [13]; are ψ-ontic, but the ψ-epistemic view has recently seen something of a revival in the light of quantum information theory [14–19]. In response to this, the question of whether Einstein's view; that the quantum state represents knowledge about a deeper reality; is viable has recently been attacked rigorously, leading to several theorems, collectively known as ψ-ontology theorems, that aim to show that the quantum state must be ontic [20–27] (see [28] for a review).

Protective measurement [29, 30] refers to two distinct, but related, idealized measurement schemes in which a single quantum system can be probed without changing its state. The two schemes are based on the quantum Zeno effect and adiabatic Hamiltonian evolution, and we call them "Zeno protected" and "Hamiltonian protected" measurements respectively. Since protective measurements do not change the state of the system, they can be used to completely determine the unknown quantum state of a single quantum system by performing a tomographically complete sequence of measurements. Thus, as concluded in [30], "this suggests that the wave function up to a phase may be ontological." Such claims have been repeated [31], especially in the context of ψ-ontology theorems [32–35], where it has been claimed that protective measurement provides an equally compelling argument for the reality of the quantum state [32, 35]. In this article, we show that this is not the case. Protective measurements can easily be accounted for on the ψ-epistemic view.

Ever since protective measurements were first proposed, there has been much criticism of the claims surrounding them [36–44]. Much of the early criticism [36–40] was directed towards the claim that protective measurements can be used to determine the unknown state of a quantum system. It is worth emphasizing that this is not exactly the same issue as whether or not the quantum state is ontic. The two issues can only be identified in operationalist approaches to physical theories, in which "what is real" is identified with "what is measurable". However, it is commonplace in physics to argue for the reality of concepts indirectly. For example, the observation of fluctuations in statistical mechanical systems was regarded as good evidence for the reality of atoms, long before we developed methods to manipulate and measure atoms individually. Thus, in broader realist approaches to physics, the quantum state may be ontic, and provably so, even if there is no procedure for measuring it exactly. This is just what the recent ψ-ontology theorems show, at least under certain reasonable assumptions.

On the converse side, it might be thought that the ability to measure something without disturbing the system is at least a sufficient criterion for its reality. Along these lines, the Einstein–Podolsky–Rosen (EPR) criterion for an "element of reality" is [45]:

> If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

However, in the context of protective measurements it is important to be precise about two further subtleties: *where* the information about the quantity comes from and *which* degrees of freedom ought not to be disturbed.

To illustrate the first point, imagine you are handed a quantum system that is prepared in one of two nonorthogonal states. A priori, there is nothing you can do to distinguish them with certainty. On the other hand, if, in addition to the system, you are handed a description of the prepared state written on a piece of paper, then you can easily distinguish the states without even touching the system, just by reading what is written on the paper[1]. In this case, there is no doubt that there is an "element of reality" corresponding to the state of the system, so EPR are not wrong about this, but the "element of reality" corresponds to the configuration of pencil marks on the piece of paper, rather than a property of the system itself. Nobody would claim that such a procedure has any implications for whether the quantum state is ontic (i.e. whether it is an intrinsic property of the system itself), since none of the information about the state actually comes from the system.

In general, if we have access to an additional resource that is correlated with the quantum state of the system, and the quantum state can be determined with the help of this resource, then this does not immediately imply that the quantum state is ontic. Thus, the question of whether protective measurement implies the reality of the quantum state depends on precisely *how much* of the information about the state of the quantum system comes from the protection operation as opposed to the system itself.

There is a wide spectrum of possibilities between having no additional resource and having a complete description of the quantum state, e.g. the piece of paper might give some parameters of the state but not all of them, so we need to determine where the protection operation lies on this spectrum. Our first two arguments show that almost all of the information about the quantity being measured in a protective measurement comes from the protection operation rather than from the system itself, which is something that Rovelli [37] and Uffink [39] have also argued for the Hamiltonian case. In the case of Zeno protected measurements, it is narrowed down to the expectation values of the quantity in a set of orthogonal states, and the only information that comes from this system itself is to pick out one of these orthogonal states as the one that the system is in. For Hamiltonian protected measurements, all of the information comes from the protection operation and the state of the system is entirely superfluous. Thus, we are much closer to the end of the spectrum where we have a complete description of the state of the system written on a piece of paper, from which no conclusion that the quantum state is ontic can be drawn.

Moving on to the second issue, given the role of disturbance in the EPR criterion, it is no surprise that some of the criticism of protective measurement has focussed on whether such procedures can really be implemented without disturbing the system [43, 44]. However, what these critics actually study is the question of whether protective measurement can be implemented in practice without modifying the quantum state of the system, but modifying the quantum state should not be identified with disturbing the system in general. Even if protective measurements do not modify the quantum state, this does not mean that the process does not disturb the underlying degrees of freedom (otherwise known as the *ontic state* of the system), whatever they may be. For example, in classical statistical mechanics, the canonical distribution is invariant under the dynamics of the system, but this does not mean that the microstate of the system is unchanged. In a box of gas, there are a lot of collisions and scattering processes going on, so we can hardly call the dynamics non-disturbing, even in equilibrium. It is only the probability distribution over microstates that is unchanged, and not the microstate itself.

The point is, if you are investigating the question of whether some procedure entails the reality of the quantum state, the possibility that the quantum state is epistemic should be on the table in the first place, in which case the analogy between quantum states and probability distributions is appropriate. If you identify the question of whether the system is disturbed with the question of whether its quantum state is modified then this is tantamount to assuming that the quantum state is identical to the state of reality. It is difficult to argue against the reality of the quantum state based on disturbance if you start with a definition of disturbance that assumes the reality of the quantum state in the first place.

The final argument we present against the idea that protective measurement implies the reality of the quantum state is a pair of ψ-epistemic toy models that reproduce the salient features of Zeno and Hamiltonian protected measurements. In these models the ontic state is disturbed by the protective measurement even though the quantum

[1] A similar argument was made by Rovelli [37].

state, which is represented by a probability measure over the ontic states, is undisturbed in the appropriate limit. Hence, in these models, the situation is analogous to the statistical mechanics example given above, in which the microstate is modified but the equilibrium distribution is unchanged. However, in many ways, the whole issue of disturbance is a bit of a sideshow. A virtue of the recent ψ-ontology theorems is that they work with a precise definition of what it means for a theory to be ψ-ontic, due to Harrigan and Spekkens [3]. This states that a model is ψ-ontic if the probability measures corresponding to nonorthogonal quantum states do not overlap, and is ψ-epistemic otherwise (see [28] for a detailed discussion of this definition). In this sense, we rigorously show that protective measurements exist in ψ-epistemic models.

The remainder of this paper is structured as follows. In §I, we review the basic ideas of protective measurement: the Zeno case in §I A and the Hamiltonian case in §I B. §II presents our first argument, which points out that, given access to the same resources as in protective measurement, you could determine the quantum state in a much more straightforward way using quantum process tomography. In this procedure, it is clear that most of the information is coming from the protection operation, so this sharpens the intuition that the same may be happening in protective measurement. However, to prove this rigorously, we need to analyse protective measurement itself, rather than an alternative procedure, and find a way of determining which information comes from the protection operation and which from the system itself. This we do in our second argument in §III. In §III A, we analyse a Zeno protected measurement of a two-outcome observable and derive the Positive Operator Valued Measure (POVM) corresponding to the whole procedure. The POVM elements contain all the information in the measurement procedure that is independent of the state of the system, and hence they only depend on the protection operation. We find that the POVM elements already contain the expectation values of the measured quantity for an orthogonal set of states, and the only role of the state of the system is to determine which of these orthogonal states the system is in. In §III B, we analyse Hamiltonian protected measurements of quadrature observables made on Gaussian states. These we can solve exactly in the Heisenberg picture, which again cleanly separates the dependence on the system from the dependence on the protection operation, as the Heisenberg operators are independent of the initial state of the system. Here, we find that all of the information about the quadrature expectation value is already contained in the Heisenberg operators and is completely independent of the state of the system. We also find that, for this class of states and measurements, procedures equivalent to protective measurement can be completed in finite time with finite interaction strength, which shows that criticisms of protective measurement based on practical considerations are misdirected. In §IV, we give our final argument, which is to construct explicit ψ-epistemic models of protective measurement. Our Zeno model, discussed in §IV A does not reproduce the quantum predictions exactly, but it does reproduce the salient features that have been thought to imply the reality of the quantum state. For the Hamiltonian case, in §IV B, we exploit the existing ψ-epistemic model for Gaussian quantum theory to exactly reproduce the quadrature measurements considered in §III B. §V concludes and discusses open questions.

Note that throughout we adopt natural units wherein $\hbar = 1$.

I. REVIEW OF PROTECTIVE MEASUREMENTS

Here we review the basic ideas of protective measurement. The paper [30] introduced two methods for protecting the state of a quantum system during the course of a measurement, Zeno protected measurement and Hamiltonian protected measurement. Although the two schemes led the authors to the same conclusions regarding the meaning of the wave function, and the overall forms of our counterarguments apply to both, our detailed analyses of the two schemes are quite different, so it is worth reviewing both of them here.

A. Zeno protected measurements

To describe Zeno protected measurements, it is helpful to introduce the usual colourful characters Alice and Bob. Alice is the person who is trying to determine the quantum state of the system. Bob is the person who initially prepares the system, and it is also his job to act as godfather to the state of the system, protecting it from changing throughout the course of Alice's measurements[2].

The intuition behind Zeno protected measurements is as follows: if Bob makes a *projective* (orthogonal basis) measurement on the system faster than any other process affecting the system, the state will—to first order—remain fixed. This is the quantum Zeno effect [46]. Whilst this is happening, Alice can measure a complete set of observables

[2] We dare not ask how much money Alice has to pay Bob for this protection racket.

and determine the state of the system. Moreover, Alice need not know *which* basis measurement Bob is making, only that such a measurement *is* being made (see Fig. 1 for a schematic of this procedure). In other words, Alice can completely determine the unknown quantum state of a single *protected* system.

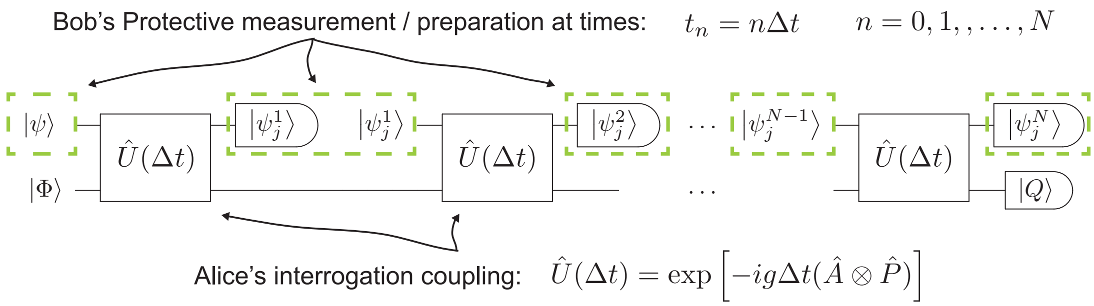

FIG. 1. A quantum circuit schematic of protective measurement. The top wire represents the protected system that Bob prepares and protects. Bob's initial preparation $|\psi\rangle$ is an element of the basis $\{|\psi_j\rangle\}$. Note that the protecting measurement at times t_n will project the protected system onto the state $|\psi_j^n\rangle$. Bob's protection is successful if $|\psi_j^n\rangle = |\psi\rangle$ for all n. Alice prepares the state $|\Phi\rangle$ then continuously couples it to the protected system, but her coupling is puctuated by Bob's interventions. As such her interactions become discretized. Finally she measures in the basis $\{|Q\rangle\}$

In a bit more detail, the procedure works as follows. Bob initially prepares a quantum system in the state $|\psi\rangle$, which is known to him but unknown to Alice, and hands it to Alice at time $t = 0$. For simplicity, we assume that the system has zero internal Hamiltonian. Alice measures observables via the usual von Neumann measurement coupling scheme. At time $t = 0$, she prepares a pointer system in a well-localized state. For example, the pointer could be a particle in one-dimension with canonical operators $(\hat{Q}, \hat{P})$ prepared in a Gaussian state with small uncertainty in Q. Usually, the particle is assumed to have a very large mass, so that we can ignore the kinetic term in its Hamiltonian. If Alice wants to measure the observable $\hat{A}$, she couples $\hat{A}$ to the momentum of the pointer via the Hamiltonian $\hat{H}_I = g\hat{A}\otimes\hat{P}$ from time $t = 0$ to $t = 1/g$, where g is a coupling constant. If the system were not being protected by Bob, then this would be a way of implementing a conventional von Neumann measurement of $\hat{A}$, i.e. if Alice were to measure the position of the pointer at time $t = 1/g$ then she would be very likely to find it close to one of the eigenvalues of $\hat{A}$ with probabilities given approximately by the Born rule, and this becomes exact as the initial uncertainty in Q is decreased to zero. In Fig. 1 Alice's pointer system is the lower wire of the quantum circuit.

However, in actual fact, during the course of Alice's measurement, Bob is doing his best to prevent the state of the system from changing. At times $t_n = n\Delta t$, where $\Delta t = 1/gN$ and $n = 0, 1, 2, \ldots, N$, he sneaks into Alice's lab and instantaneously measures the system in a basis $\{|\psi_j\rangle\}$ that includes $|\psi\rangle$ as one of the basis elements. If any pair of his measurement outcomes differ, then his protection has failed and the whole procedure is aborted. However, in the limit $N \to \infty$, the probability of this happening tends to zero (we give the details of this calculation later). In this limit, it can also be shown that the wavefunction of the pointer simply shifts by $\langle\hat{A}\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle$, so Alice can read off the expectation value of $\hat{A}$ by measuring the pointer position.

If this whole procedure is repeated for a tomographically complete set of observables, then Alice can determine the initial state of the system exactly.

B. Hamiltonian protected measurements

In Hamiltonian protected measurements, Alice's measurement procedure is the same as in the Zeno case. The only thing that is different is the way that Bob protects the system. We assume that Bob has the ability to set the Hamiltonian of the quantum system, e.g. by tuning an external magnetic field if it is a spin system. He initially prepares the system in the state $|\psi\rangle$ and sets the Hamiltonian H_S such that $|\psi\rangle$ is its nondegenerate ground state with finite gap ΔE to the first excited state[3].

In the limit $g \to 0$, we can treat $\hat{H}_I$ as a small perturbation. By the adiabatic theorem, the system and measurement pointer will remain in the ground state of $\hat{H} = \hat{H}_S + \hat{H}_I$ throughout the whole procedure. Since $\hat{H}_I = 0$ for $t < 0$

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[3] Any nondegenerate eigenstate with finite gaps to the neighbouring states would work just as well, but we use the ground state here for simplicity.

and $t > 1/g$, the system will be in the ground state of $\hat{H}_S$ after the measurement is completed[4]. As in the Zeno case, it can again be shown that, in the limit, the pointer shifts by $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$, so the expectation value can be read off from the pointer, and the whole procedure can be repeated for a tomographically complete set of observables in order to determine $|\psi\rangle$.

II. RESOURCE COUNTING ARGUMENT

Our first argument is to examine the resources available to Alice in her quest to determine the state of her quantum system. Firstly, she has a single copy of the system in an unknown state $|\psi\rangle$. If this were all she had then, of course, she would be unable to determine the state. However, there is also Bob's protection operation, which is either a measurement in an orthonormal basis in the Zeno scheme, or the ability to evolve the system according to the unitary $\hat{U}(t) = e^{-i\hat{H}_S t}$ for an arbitrary time t in the Hamiltonian scheme. Both of these operations are correlated to the unknown state of the system and are used an arbitrarily large number of times in the protective measurement procedure, i.e. in the Zeno case, the projective measurement is made an arbitrarily large number of times and in the Hamiltonian case t is arbitrarily large. Given this, it is natural to suspect that most of the information about the unknown state $|\psi\rangle$ is coming from the protection operation rather than from the system itself.

To sharpen this intuition, we argue that if Alice is given the exact same resources that she has available in the protective measurement schemes then she can determine the unknown state of the system in a much more straightforward way. Suppose that instead of Bob applying the protection operation in the way prescribed by protective measurement, Alice has black-box access to it.

In the Zeno case, this means Alice has access to a black box that performs a measurement in a basis $\{|\psi_j\rangle\}$, where $|\psi\rangle$ is one of the basis elements, but Alice does not get to see the outcome of the measurement. Thus, from her point of view, it is a non-selective measurement that implements the quantum channel

$$\mathcal{C}(\hat{\rho}) = \sum_j |\psi_j\rangle \langle \psi_j| \hat{\rho} |\psi_j\rangle \langle \psi_j| \,. \tag{1}$$

Alice can use this black box as many times as she likes and pass whatever systems she likes through it.

Given these resources, Alice can determine $|\psi\rangle$ in a quite straightforward way. First, she puts the system that Bob prepared in state $|\psi\rangle$ to one side for later use. Then, using different systems that she prepares in known states, she performs process tomography [47] on her black-box to determine $\mathcal{C}$. Specifically, if she prepares systems in states $|\phi_j\rangle$, passes them through the black box, and performs measurements in the bases $\{\left|\xi_m^{(k)}\right\rangle\}_m$, then so long as the projectors $|\phi_j\rangle \langle \phi_j|$ span the vector space of linear operators and the projectors $|\xi_m^{(k)}\rangle \langle \xi_m^{(k)}|$ also span this space, then the probabilities

$$p(m|j,k) = \left\langle \xi_m^{(k)} \middle| \mathcal{C}(|\phi_j\rangle \langle \phi_j|) \middle| \xi_m^{(k)} \right\rangle, \tag{2}$$

determine $\mathcal{C}$ uniquely. Thus, by repeating this process a large number of times, she can estimate the probabilities $p(m|j,k)$, and hence $\mathcal{C}$, to arbitrary accuracy. Due to finite sample errors, this will never be exact, but, just as in Zeno protected measurement scheme, $\mathcal{C}$ can be used an arbitrarily large number of times, so both procedures only work exactly in the limit of an infinite number of uses of $\mathcal{C}$.

Having determined $\mathcal{C}$, Alice can calculate the basis $\{|\psi_j\rangle\}$ [5]. Knowing this, she now only needs to know which of the basis states the system was prepared in. However, since she put the system that Bob gave her to one side at the beginning, she can now simply measure it in the basis $\{|\psi_j\rangle\}$. She will get the outcome corresponding to $|\psi\rangle$ with certainty, and thus will have determined the state of the system exactly.

Were Alice to use this procedure in the original scenario, it would look very different to Bob because he will see many different outcomes of his measurement instead of almost always seeing $|\psi\rangle$. For some purposes this difference may be crucial. For example if Bob is a bank unwittingly offering the protection as part of a quantum money scheme [49], then while protective measurement can facilitate successful counterfeiting [50], our alternative would quickly

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[4] One might be concerned that the discontinuous change from $\hat{H} = \hat{H}_S$ to $\hat{H} = \hat{H}_S + g\hat{A} \otimes \hat{P}$ at $t = 0$ and back again at $t = 1/g$ violates the assumptions of the adiabatic theorem. However, we can instead use the measurement interaction $\hat{H}_I = g(t)\hat{A} \otimes \hat{P}$ where $g(t)$ is a smoothly varying function with $\int_{t=0}^{t=T} g(t)\,\mathrm{d}t = 1$ and where $g(t) = 0$ for $t < 0$ and $t > T$.

[5] This is not completely straightforward as Alice only knows $\mathcal{C}$ as a linear map and not the specific decomposition in terms of the projectors $|\psi_j\rangle \langle \psi_j|$ given in eq. (1). However, the fixed point set of $\mathcal{C}$ is the set of operators that are diagonal in the $\{|\psi_j\rangle\}$ basis, and there are several methods for determining the fixed point set of a completely-positive trace-preserving map, e.g. [48].

land Alice in jail. But for understanding what information the protection process provides, the calculations in the following sections lead us to believe this difference is not significant.

For the Hamiltonian case, the situation is similar. Alice's black box now implements the unitary evolution $\hat{U}(t) = e^{-i\hat{H}_S t}$ and has a setting that allows Alice to vary the time duration t. For a fixed value of t, Alice can use process tomography [47, 51] to determine the unitary operator $\hat{U}(t)$. This allows her to determine the eigenvalues $e^{-iE_j t}$ and associated eigenspaces of $\hat{U}(t)$, where the E_j's are the eigenvalues of $\hat{H}_S$. This does not allow her to determine $\hat{H}_S$ uniquely because $e^{-iE_j t}$ is a periodic function of E_j and there may be degeneracies in the spectrum of $\hat{U}(t)$ that are not present in $\hat{H}_S$, e.g. if $t = 2n\pi/(E_j - E_k)$ for some integer n then $e^{-iE_j t} = e^{-iE_k t}$ so there is a degeneracy in $\hat{U}(t)$ even if $E_j \neq E_k$. Nonetheless, by running this procedure for different values of t, Alice can eventually determine $\hat{H}_S$. There are a variety of methods for doing this, e.g. [52], and we omit the details here.

Knowing $\hat{H}_S$, Alice can determine its ground state and then she knows that this is $|\psi\rangle$ without even touching the system[6].

We are not claiming that either of these procedures are models of what is actually going on in protective measurement. They are just designed to show that it is unsurprising that Alice can determine the unknown state given the resources she has available. In our procedures, it is clear that the majority of the information about the unknown state is coming from the protection operation rather than the system itself. In the Zeno case, without even touching the system, the state is narrowed down to one of an orthogonal set, which can then be determined by a conventional projective measurement. Only this information is coming from the system itself. The Hamiltonian case is even simpler as the state is determined without ever touching the system at all. The fact that this can be done with the same resources as protective measurement lends credence to the idea that the same thing may be going on in protective measurements themselves. It is not a rigorous argument that this must be so, nor is it a proof that protective measurement does not imply the reality of the quantum state. For that, we shall have to analyse the details of the protective measurement schemes themselves, which we do in the following sections. Nevertheless, we think that this argument casts doubt on the naive inference from "the quantum state can be measured by protective measurements" to "the quantum state must be real".

III. EXACT ANALYSES

The previous argument made it plausible that most of the information about the unknown state in a protective measurement is coming from the protection operation rather than from the system itself. To demonstrate that this is indeed the case, we need to analyse protective measurement itself rather than alternative procedures that use the same resources.

For the Zeno scheme, we do this for the case of a protective measurement of a two-outcome observable $\hat{A}$ with eigenvalues ± 1. Note that this is sufficient for determining the unknown state of any finite-dimensional system. For example, on a qubit the Pauli operators $\hat{X}$, $\hat{Y}$ and $\hat{Z}$, along with the identity operator, span the vector space of linear operators on the system, and are hence tomographically complete.

Any measurement procedure acting on a quantum system can be represented as a Positive Operator Valued Measure (POVM). In particular, as pointed out by D'Ariano and Yuen [40], this still applies to procedures such as protective measurement that consist of an arbitrarily long sequence of steps. In the case of protective measurement, the POVM depends on the unknown state of the system because the protection operation is correlated with it. However, by writing things in this way we can cleanly separate the information that comes from the protection operation from the information that comes from the quantum system. Parameters that appear in the POVM come from the protection operation because they are independent of what the state of the system is, i.e. if Bob were to give Alice a system prepared in the state $|\phi\rangle$ but protect it with a measurement designed to preserve $|\psi\rangle$ then the POVM would be the same as if Bob gave Alice the correct state $|\psi\rangle$. We find that, as indicated by the previous argument, almost all of the information about $|\psi\rangle$ is already present in the POVM. It contains complete information about the expectation values $\langle\psi_j|\,\hat{A}\,|\psi_j\rangle$ for the basis states $|\psi_j\rangle$ of the protection operation, and the only role of the system is to select which of these applies.

The Hamiltonian case is more tricky to analyse exactly due to its reliance on the adiabatic theorem. However, we can perform an exact analysis for the protective measurement of quadrature observables where the protected state is one of a set of overlapping continuous variable Gaussian states. Again, this is sufficient to determine the unknown state, as the states we consider can be distinguished by their quadrature expectation values.

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[6] If an arbitrary nondegenerate eigenstate is used instead of the ground state, Alice must in addition measure $\hat{H}_S$ on the system to determine $|\psi\rangle$ with certainty.

We describe the dynamics in the Heisenberg picture, which provides another way to separate the dependence on the protection operation from the dependence on the unknown state, and, in this case, we find that the information learned by Alice comes entirely from the protection operation. We also find that there are procedures that yield equivalent information to protective measurement that can be performed with perfect accuracy in finite time, which shows that criticisms of Hamiltonian protected measurements based on practical considerations are misdirected.

A. Zeno protected measurements

In this section, we analyse the effect of a Zeno protected measurement of a two outcome observable $\hat{A}$ with eigenvalues ± 1 that is designed to preserve the state $|\psi\rangle$. We analyse the action of this procedure when it is applied to an arbitrary initial state of the system $\hat{\rho}$, i.e. not necessarily $|\psi\rangle$.

Alice performs her measurement of $\hat{A}$ by coupling the system to a one dimensional pointer with canonical operators $(\hat{Q}, \hat{P})$ via the Hamiltonian

$$\hat{H}_I = g\hat{A} \otimes \hat{P} \tag{3}$$

where g is a coupling parameter. This Hamiltonian acts from $t_0 = 0$ to $t_N = N\Delta t$ where $\Delta t = 1/gN$.

Bob's protection is via $N+1$ projective measurements of the system in the $\{|\psi_j\rangle\}$ basis, where $|\psi\rangle$ is one of the basis elements, evenly spaced at times $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$. If Bob gets different outcomes for any of his measurements then he aborts the procedure.

Let $\Pi_\pm$ be the projectors onto the ± 1 eigenspaces of $\hat{A}$, so that $\hat{A} = \Pi_+ - \Pi_-$.

Between each protection operation the evolution is given by

$$\hat{U}(\Delta t) = \exp[-(i/N)\hat{H}_I] = \exp[-(i/N)\hat{A} \otimes \hat{P}] = \exp[-(i/N)(\hat{\Pi}_+ \otimes \hat{P} - \hat{\Pi}_- \otimes \hat{P})]. \tag{4}$$

Because $[\hat{\Pi}_+ \otimes \hat{P}, \hat{\Pi}_- \otimes \hat{P}] = 0$ we can simplify this to

$$\hat{U}(\Delta t) = \exp[-(i/N)\hat{\Pi}_+ \otimes \hat{P}]\exp[+(i/N)\hat{\Pi}_- \otimes \hat{P}] = \hat{\Pi}_+ \otimes \exp[-(i/N)\hat{P}] + \hat{\Pi}_- \otimes \exp[+(i/N)\hat{P}] \tag{5}$$

Suppose the pointer is initially prepared in the Gaussian state $|\Phi\rangle = \int_{-\infty}^{\infty} \Phi(Q)\,|Q\rangle\, dQ$, where

$$\Phi(Q) = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left(-\frac{Q^2}{2\sigma^2}\right), \tag{6}$$

and is measured in the $\{|Q\rangle\}$ basis at the end of the procedure.

The effect of this procedure can be viewed as a generalized measurement on the system that depends on $|\psi\rangle$ via the protection operation. Specifically, if the outcome of the final measurement is Q then the state of the system is updated via

$$\hat{\rho} \to \frac{\hat{M}_Q \hat{\rho} \hat{M}_Q^\dagger}{\mathrm{Tr}(\hat{E}_Q \hat{\rho})}, \tag{7}$$

where $\hat{M}_Q$ is a Kraus operator, which we shall specify shortly, and $\{\hat{E}_Q\}$ is a POVM given by $\hat{E}_Q = \hat{M}_Q^\dagger \hat{M}_Q$. The probability of getting the Q outcome is given by $\mathrm{Tr}(\hat{E}_Q\hat{\rho})$. Since $\hat{E}_Q$ is independent of the initial state of the system, any information it contains about $|\psi\rangle$ comes from the protection operation rather than the system. The system itself contributes information only to the extent that $\mathrm{Tr}(\hat{E}_Q\hat{\rho})$ depends on $\hat{\rho}$.

The generalized measurement operators are given by

$$M_Q = \sum_j [\langle\psi_j| \otimes \langle Q|]\hat{U}(\Delta t)[|\psi_j\rangle\langle\psi_j| \otimes \hat{\mathbb{I}}]\hat{U}(\Delta t)[|\psi_j\rangle\langle\psi_j| \otimes \hat{\mathbb{I}}] \cdots \hat{U}(\Delta t)[|\psi_j\rangle \otimes |\Phi\rangle] \tag{8}$$

$$= \sum_j |\psi_j\rangle\langle\psi_j|\,\langle Q|\left\{\langle\psi_j|\hat{\Pi}_+|\psi_j\rangle \exp[(-i/N)\hat{P}] + \langle\psi_j|\hat{\Pi}_-|\psi_j\rangle \exp[+(i/N)\hat{P}]\right\}^N |\Phi\rangle \tag{9}$$

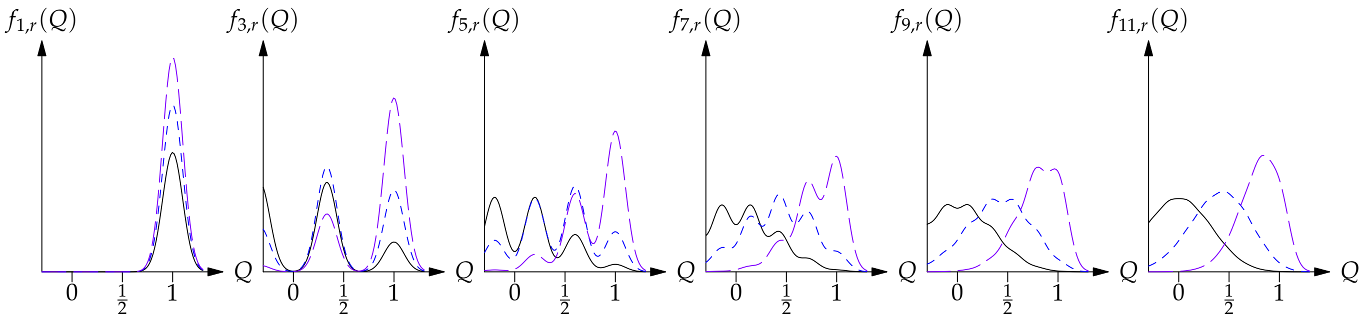

FIG. 2. $f_{N,r}$ for $N = 1,3,5,7,9,11$, $r = 0.5, 0.7, 0.9$ (black, blue short dashes and purple long dashes respectively), and $\sigma = 0.1$. For these parameters the Gaussian approximation of eq. (18) can be visualised at around $N \geq 11$.

note that $\langle\psi_j|\hat{\Pi}_\pm|\psi_j\rangle$ is a real number and the operator exponentials commute. Lets introduce some simplified notation $r_j = \langle\psi_j|\hat{\Pi}_+|\psi_j\rangle$ and $1 - r_j = \langle\psi_j|\hat{\Pi}_-|\psi_j\rangle$. Now we have

$$M_Q = \sum_j |\psi_j\rangle\langle\psi_j| \left\{ \sum_{n=0}^{N} \binom{N}{n} \langle Q| r_j^n (1-r_j)^{N-n} e^{-in\hat{P}/N} e^{+i(N-n)\hat{P}/N} |\Phi\rangle \right\} \tag{10}$$

$$= \sum_j |\psi_j\rangle\langle\psi_j| \left\{ \sum_{n=0}^{N} \binom{N}{n} \langle Q| r_j^n (1-r_j)^{N-n} e^{-i(2n-N)\hat{P}/N} |\Phi\rangle \right\} \tag{11}$$

$$= \sum_j f_{N,r_j}(Q) |\psi_j\rangle\langle\psi_j| \tag{12}$$

where we define $f_{N,r}(Q)$, plotted in Fig. 2, as

$$f_{N,r}(Q) = \sum_{n=0}^{N} \binom{N}{n} r^n (1-r)^{N-n} \langle Q| e^{-i(2n-N)\hat{P}/N} |\Phi\rangle \tag{13}$$

$$= \sum_{n=0}^{N} \binom{N}{n} r^n (1-r)^{N-n} \Phi[Q - (2n-N)/N] \tag{14}$$

The corresponding POVM element is

$$\hat{E}_Q = \hat{M}_Q^\dagger \hat{M}_Q = \sum_j f_{N,r_j}^2(Q) |\psi_j\rangle\langle\psi_j| \,. \tag{15}$$

It is already evident that the POVM contains information about $|\psi\rangle$. All the POVM elements are diagonal in the $\{|\psi_j\rangle\}$ basis and the functions f_{N,r_j} depend on the operator $\hat{A}$ via $r_j = \langle\psi_j|\hat{\Pi}_+|\psi_j\rangle$. If the system happens to be prepared in the state $|\psi\rangle$ then this simply serves to pick out the $f^2_{N,\langle\psi|\hat{\Pi}|\psi\rangle}$ term. It must therefore be the case that all of the information about the expectation value $\langle\psi|\hat{A}|\psi\rangle$ that is observed in the protective measurement is already present in the POVM via the $f^2_{N,\langle\psi|\hat{\Pi}_+|\psi\rangle}$ term, which comes from the protection operation, and the initial state just serves to pick out $|\psi\rangle$ from amongst the orthogonal possibilities $\{|\psi_j\rangle\}$.

To see that this is the case, we examine the large N limit. For large N we can approximate the binomial distribution by the normal distribution:

$$\binom{N}{n} r^n (1-r)^{N-n} \approx \frac{1}{\sqrt{2\pi N r(1-r)}} \exp\left(-\frac{(n-Nr)^2}{2Nr(1-r)} \right). \tag{16}$$

This gives

$$f_{N,r}(Q) \approx \frac{1}{\sqrt{2\pi N r(1-r)}} \int_{-\infty}^{\infty} dn \Phi[Q - (2n-N)/N] \exp\left(-\frac{(n-Nr)^2}{2Nr(1-r)} \right) \tag{17}$$

$$\approx \frac{\sqrt{\sigma}}{\pi^{1/4}\gamma_{N,r}} \exp\left(-\frac{[Q-(2r-1)]^2}{2\gamma_{N,r}^2} \right) \tag{18}$$

(on the second line we performed the integral) where

$$\gamma_{n,p} = \sqrt{\frac{4r(1-r)}{N} + \sigma^2}. \tag{19}$$

For very large N, $\gamma_{N,r} \approx \sigma$ and then $f_{N,r}(Q) \approx \Phi[Q - (2r-1)]$, giving

$$\hat{E}_Q \approx \sum_j |\psi_j\rangle \langle\psi_j| \, \Phi^2[Q - (2r_j - 1)]. \tag{20}$$

So in this limit, the $|\psi_j\rangle \langle\psi_j|$ term of E_Q is a Gaussian centered about the expectation value $\langle\psi_j|\hat{A}|\psi_j\rangle = (2r_j - 1)$.[7] This can be seen by noting that $r_j = \langle\psi_j|\hat{\Pi}_+|\psi_j\rangle$ and $1 - r_j = \langle\psi_j|\hat{\Pi}_-|\psi_j\rangle$ can be related to the expected value of the observable $\hat{A}$ via $\langle\hat{A}\rangle = \langle\hat{\Pi}_+\rangle - \langle\hat{\Pi}_-\rangle = r_j - (1 - r_j) = 2r_j - 1$. From this, we see that the pointer shifts for each of the basis states $|\psi_j\rangle$ are already completely determined by the POVM, and the system simply serves to determine which of these shifts is applied.

Finally, for completeness, the case where Bob aborts the procedure corresponds to the POVM element

$$\hat{E}_{\text{abort}} = \hat{I} - \int_{-\infty}^{\infty} \hat{E}_Q dQ. \tag{21}$$

Using the approximation eq. (18) we have

$$\hat{E}_{\text{abort}} \approx \sum_j |\psi_j\rangle \langle\psi_j| \left(1 - \frac{\sigma}{\sqrt{\pi}\gamma_{N,r}^2} \int_{-\infty}^{\infty} \exp\left(-\frac{[Q-(2r-1)]^2}{\gamma_{N,r}^2}\right) dQ\right) = \sum_j |\psi_j\rangle \langle\psi_j| \left(1 - \frac{\sigma}{\gamma_{N,r_j}}\right), \tag{22}$$

which tends to zero as N tends to infinity.

B. Hamiltonian protected measurements

In the Hamiltonian case, we can again separate the dependence on the protection operation from the dependence on the initial state of the system by working in the Heisenberg picture. The Heisenberg evolved operators depend only on the unitary dynamics of the system, i.e. the protection Hamiltonian and the measurement interaction, and are thus independent of the state of the system.

Consider a system with canonical operators $(\hat{q}', \hat{p}')$ and suppose that Bob wishes to protect the "coherent state"

$$|\psi_{c_q,c_p}\rangle = \frac{1}{\pi^{1/4}} \int_{-\infty}^{\infty} \exp\left(-\frac{(q'-c_q)^2}{2} + iq'c_p\right) |q'\rangle \, dq', \tag{23}$$

where c_q and c_p are constants. Note that these states are nonorthogonal for different values of c_q and c_p, so this class of states is sufficient to see that Hamiltonian protected measurements can distinguish nonorthogonal states.

Bob's protects the system by setting its Hamiltonian to be a (displaced) Harmonic oscillator:

$$\hat{H}_S = \frac{1}{2}\left((\hat{p}' - c_p)^2 + (\hat{q}' - c_q)^2\right). \tag{24}$$

so that $|\psi_{c_q,c_p}\rangle$ is its nondegenerate ground state with eigenvalue $1/2$.

Alice wishes to measure the quadrature observable $\hat{a}_\theta = \cos\theta\hat{q}' + \sin\theta\hat{p}'$, which has expectation value $c_\theta = c_q\cos\theta + c_p\sin\theta$ in the state $|\psi_{c_q,c_p}\rangle$. She does this by coupling the system to a pointer, with canonical operators $(\hat{Q}, \hat{P})$, via the interaction Hamiltonian $H_I = g\hat{a}_\theta \otimes \hat{P}$ for a time duration $1/g$.

Let us change the system co-ordinates to $\hat{q} = (\hat{q}' - c_q)\cos\theta + (\hat{p}' - c_p)\sin\theta$ and $\hat{p} = -(\hat{q}' - c_q)\sin\theta + (\hat{p}' - c_p)\cos\theta$, noting that $[\hat{q}, \hat{p}] = [\hat{q}', \hat{p}'] = i$. In these co-ordinates, the overall Hamiltonian is now

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \hat{q}^2\right) + g(\hat{q} + c_\theta) \otimes \hat{P}. \tag{25}$$

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[7] If we also take the limit $\sigma \to 0$, so that Φ is a Dirac delta, then the whole procedure amounts to a projective measurement of the observable $\sum_j (2r_j - 1) |\psi_j\rangle \langle\psi_j|$ analogous to equation 14 of [39].

Heisenberg's equation gives

$$\frac{d}{dt}\begin{pmatrix}\hat{q}\\ \hat{p}\\ \hat{Q}\\ \hat{P}\end{pmatrix} = i\begin{pmatrix}[\hat{H},\hat{q}]\\ [\hat{H},\hat{p}]\\ [\hat{H},\hat{Q}]\\ [\hat{H},\hat{P}]\end{pmatrix} = \begin{pmatrix}\hat{p}\\ -\hat{q}-g\hat{P}\\ g(\hat{q}+c_\theta)\\ 0\end{pmatrix}, \tag{26}$$

with solution:

$$\begin{pmatrix}\hat{q}(t)\\ \hat{p}(t)\\ \hat{Q}(t)\\ \hat{P}(t)\end{pmatrix} = \begin{pmatrix}\hat{q}_t + g(\cos t - 1)\hat{P}(0)\\ \hat{p}_t - g(\sin t)\hat{P}(0)\\ \hat{Q}(0) + g\left(c_\theta t + \hat{p}(0) - \hat{p}_t\right) + g^2(\sin t - t)\hat{P}(0)\\ \hat{P}(0)\end{pmatrix}, \tag{27}$$

where we have used the notation $\hat{q}_t = \hat{q}(0)\cos t + \hat{p}(0)\sin t$ and $\hat{p}_t = -\hat{q}(0)\sin t + \hat{p}(0)\cos t$ based on the solution without the measurement interaction.

When the measurement is complete at $t = 1/g$, the solution reads:

$$\begin{pmatrix}\hat{q}(1/g)\\ \hat{p}(1/g)\\ \hat{Q}(1/g)\\ \hat{P}(1/g)\end{pmatrix} = \begin{pmatrix}\hat{q}_{1/g} + g(\cos(1/g) - 1)\hat{P}(0)\\ \hat{p}_{1/g} - g(\sin(1/g))\hat{P}(0)\\ \hat{Q}(0) + c_\theta + g\left(\hat{p}(0) - \hat{p}_{1/g} - \hat{P}(0)\right) + g^2(\sin(1/g))\hat{P}(0)\\ \hat{P}(0)\end{pmatrix}, \tag{28}$$

in particular when

1. $g \to \infty$:

$$\begin{pmatrix}\hat{q}(1/g)\\ \hat{p}(1/g)\\ \hat{Q}(1/g)\\ \hat{P}(1/g)\end{pmatrix} = \begin{pmatrix}\hat{q}(0)\\ \hat{p}(0) - \hat{P}(0)\\ \hat{Q}(0) + \hat{q}(0) + c_\theta\\ \hat{P}(0)\end{pmatrix}. \tag{29}$$

2. $g = 1/(2\pi n)$ for $n = 1, 2, \ldots$:

$$\begin{pmatrix}\hat{q}(1/g)\\ \hat{p}(1/g)\\ \hat{Q}(1/g)\\ \hat{P}(1/g)\end{pmatrix} = \begin{pmatrix}\hat{q}(0)\\ \hat{p}(0)\\ \hat{Q}(0) + c_\theta - g\hat{P}(0)\\ \hat{P}(0)\end{pmatrix}, \tag{30}$$

3. $g \to 0$:

$$\begin{pmatrix}\hat{q}(1/g)\\ \hat{p}(1/g)\\ \hat{Q}(1/g)\\ \hat{P}(1/g)\end{pmatrix} = \begin{pmatrix}\hat{q}_{1/g}\\ \hat{p}_{1/g}\\ \hat{Q}(0) + c_\theta\\ \hat{P}(0)\end{pmatrix}. \tag{31}$$

Item 1 is easily recognised as the standard von Neumann scheme for measuring $\hat{a}_\theta = \hat{q} + c_\theta$. A measurement of $\hat{Q}(1/g)$ behaves like a measurement of $\hat{a}_\theta$ with an error given by $\hat{Q}(0)$. If we prepare the pointer in a state sharply peaked at $Q = 0$ then the measurement will be very accurate but, by the uncertainty principle, there will be a large spread in $\hat{P}$, and hence a large disturbance to the system variable $\hat{p}$—this is the usual projective or strong measurement. If we prepare the ancilla in a state sharply peaked at $P = 0$ then disturbance to $\hat{p}$ will be small but by the uncertainty principle there will be a large spread in $\hat{Q}$ and hence the measurement will be inaccurate; this is a "weak" measurement.

Item 3 is the standard Hamiltonian protected measurement scheme. Notice that the system undergoes only its free evolution under $\hat{H}_S$. Hence, if the quantum state is an eigenstate of $\hat{H}_S$, its quantum state is unchanged. A measurement of $\hat{Q}(1/g)$ indeed gives access to c_θ (with error based on the spread in $\hat{Q}(0)$), but this is just a parameter in the Hamiltonian. This happens regardless of the initial state of the system, so we can say quite definitively that the information about c_θ is coming entirely from the protection operation rather than from the system itself [8].

Note that we have only proved that this happens for protective measurements of quadrature observables for the class of states given in eq. (23) and the corresponding protection Hamiltonians given in eq. (24). However, quadrature measurements suffice to completely determine the state of the system, and these states are non-orthogonal, so they would not be perfectly distinguishable by ordinary quantum measurements. If there is an argument that Hamiltonian protected measurements imply the reality of the quantum state in general then it ought to apply to this setup in particular, but in this case it is clear that all the information comes from the protection Hamiltonian.

Finally, from item 2 we see that for a carefully chosen, but nonzero, interaction strength, e.g. $g = 1/(2\pi)$, which corresponds to a finite time duration measurement, the system is undisturbed (for *any* initial state). If we prepare the pointer in a state sharply peaked at $Q - gP = 0$ then we can learn c_θ with arbitrary accuracy. This shows that criticisms of Hamiltonian protected measurement based on analysis of practical considerations are misdirected. The criticism is that Hamiltonian protected measurement only works exactly in the limit $g \to 0$, in which case the measurement would take an infinite amount of time. For finite duration procedures, the state of the system becomes entangled with the measurement device and is hence disturbed [43, 44]. To perform a practical protective measurement, Alice would need to know how small to set g in order to make this disturbance negligible, and it has been argued that this requires knowledge of the Hamiltonian that is tantamount to knowing $|\psi\rangle$ in the first place [43].

However, we have now seen that, for a particular class of nonorthogonal states and protection Hamiltonians, the same effect as an exact protective measurement can be achieved in a fixed finite time duration that is independent of the state and Hamiltonian. Thus, if there is an argument that protective measurements do not imply the reality of the quantum state, it cannot be that finite duration protective measurements necessarily disturb the quantum state.

IV. TOY MODELS

Our arguments so far have shown that most of the information about the unknown state of the system (all of it in the Hamiltonian case) comes from the protection operation rather than the system. This should be enough to convince most people that protective measurement cannot have any implications for the reality of the quantum state. However, we have yet to demonstrate formally that protective measurements can be achieved within models that are ψ-epistemic according to the definitions used in the recent ψ-ontology theorems.

To achieve this, we construct ψ-epistemic toy models of protective measurements in theories that have a well-defined state of reality (ontic state), otherwise known as *ontological* models[9]. In this framework, quantum states are represented by probability measures over the ontic states. The model is called ψ-ontic if the probability measures corresponding to non-orthogonal states do not overlap and is otherwise ψ-epistemic (see [3, 28] for further discussion of this definition).

For Zeno protected measurements, we do not try to exactly reproduce the quantum predictions, but give a toy model for a qubit that reproduces the salient features. Namely, non-orthogonal quantum states are represented by overlapping probability measures, there is a protection operation corresponding to repeated projective measurements, and it is possible to measure the expectation values of enough observables to determine the quantum state without disturbing the system by coupling to a continuous variable pointer system. This model is constructed by modifying Spekkens' well known ψ-epistemic toy theory [16] to allow for continuous coupling to a pointer system.

For Hamiltonian protected measurements, we exhibit a toy theory that reproduces the predictions of the example used in §III B exactly. This just exploits the fact that all the states and Hamiltonians involved are Gaussian, and it is known that Gaussian quantum mechanics can be reproduced by a ψ-epistemic model [53]. Whilst this is not a new theory, it is instructive to track exactly what happens to the ontic state of a system during the course of a protective measurement in this theory. We note that weak values have been analysed by Karanjai *et al* [54] in the Gaussian toy theory in a similar way.

The main lesson of both of these models is that protection can be thought of as an operation that effectively re-prepares the system in its initial state. Thus, determining the quantum state of a system by protective measurements is in closer analogy to performing state tomography on multiple systems that are independently prepared in the same state[10] than it is to having just a single copy of the system.

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[8] Formally, if we specify an initial state of the pointer and then cast a final measurement of $\hat{Q}$ as a POVM on the system, all of the POVM will be proportional to the identity.

[9] You may alternatively call these "classical" models or "hidden variable theories", depending on your personal terminology preferences.

[10] See [55] for an interpretation of quantum state tomography compatible with the ψ-epistemic position.

A. Zeno protected measurements

We will start by constructing a toy-model for state preparations and measurements of a spin-1/2 particle in the x and y directions[11]. This model is based on the Spekkens' toy bit [16], with some modifications to allow for the continuous coupling needed for protective measurements.

Consider a system with an ontic state space consisting of two random variables, X and Y, that each take values ± 1. We denote the state where $X = x$ and $Y = y$ as (x, y) and use $\pm$ as shorthand for ± 1, so the four possible ontic states are $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$.

For concreteness, we can imagine that the system consists of a ball in an opaque box with equal width and breadth, where we place the origin of the x-y coordinate system at the center of the box, as illustrated in Fig. 3. The four possible ontic states then represent which quadrant of the x-y coordinate system the ball is in, e.g. $(+,-)$ represents the state of affairs in which the ball is in the lower right quadrant, with positive x-coordinate and negative y-coordinate.

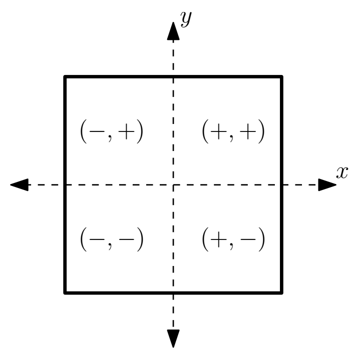

FIG. 3. The ontic state of the system can be thought of as a ball that can be in one of four quadrants of a box. Here, we are looking down on the box along the z-axis and the solid line indicates the border of the box.

Now imagine that we do not have complete control over the position of the ball within the box. In fact, suppose that there are only two things we can do to it. Firstly, we can place a double-partition along the y-axis and separate the box into two pieces according to the sign of the x-coordinate (see Fig. 4). We can then take each of the two pieces and shake them vigorously. By noting which of the pieces rattles when we shake it, we can determine whether the x-coordinate is positive or negative, but doing so causes the y-coordinate to be randomized from the shaking. We call this an X measurement. Alternatively, we can place a double partition along the x-axis, separate the box according to the sign of the y-coordinate, and do the same thing. This allows us to determine the sign of the y-coordinate at the expense of randomizing the x-coordinate. We call this a Y measurement.

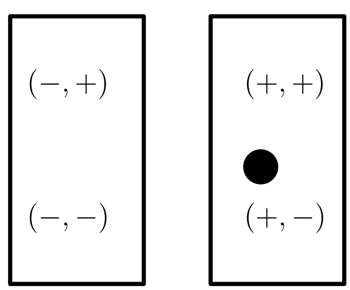

FIG. 4. An X measurement. The box can be split in two by placing a double partition along the y-axis. The two pieces are then shaken vigorously and the sign of the x-coordinate of the ball can be determined by noting which piece of the box rattles. Doing so causes the y-coordinate of the to be randomized, so if, as in the diagram, the ball is initially in $(+,-)$, it will be in either $(+,-)$ or $(+,+)$ with equal probability after the shaking.

In general, we can describe our knowledge about the ontic state of the system at any given time via a probability distribution $\boldsymbol{p} = (p_{++}, p_{+-}, p_{-+}, p_{--})$, where p_{xy} denotes the probability that the ball is in state (x, y). If, at the

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[11] We could easily include z-measurements as well, but having two nonorthogonal states is sufficient for determining whether protective measurement entails the reality of the quantum state.

start, we know nothing about where the ball is then the probability distribution will be $\boldsymbol{p} = (1/4, 1/4, 1/4, 1/4)$. By performing an X measurement and postselecting on the cases where the x-coordinate is found to be positive, we can prepare the system in the state $\boldsymbol{p}^{x+} = (1/2, 1/2, 0, 0)$, and by postselecting on the cases where it is found to be negative we can prepare $\boldsymbol{p}^{x-} = (0, 0, 1/2, 1/2)$. Similarly, with Y measurements we can prepare the states $\boldsymbol{p}^{y+} = (1/2, 0, 1/2, 0)$ and $\boldsymbol{p}^{y-} = (0, 1/2, 0, 1/2)$.

It is easy to see that preparing the system in one of these states followed by performing a sequence of X and Y measurements has the same statistics as preparing a spin-1/2 particle in the states $|x\pm\rangle$ or $|y\pm\rangle$ followed by performing a sequence of measurements of the spin in the x and y directions. Therefore, we can regard this system as an ontological model, or simulation, of such experiments on a spin-1/2 particle.

In this model, the pure states $|x\pm\rangle$, $|y\pm\rangle$ are represented by probability distributions that are spread out over two of the four possible ontic states. Further, the x and y states overlap, e.g. $\boldsymbol{p}^{x+}$ and $\boldsymbol{p}^{y+}$ both assign probability 1/2 to the ontic state $(+,+)$. Therefore, the quantum state is epistemic in this model. Given full knowledge of the ontic state of the system, it is impossible to tell with certainty which quantum state was prepared.

The expectation values of X and Y for the four states we can prepare are shown in Table I. Note that the expectation values completely determine the state so if we can find a method, analogous to protective measurement, of measuring these expectation values without disturbing the state then would be able to determine the state with just a single copy of the system. However, since the quantum state is epistemic in the model, this would show that this feature of Zeno protected measurement does not entail the reality of the quantum state.

State	$\langle X\rangle$	$\langle Y\rangle$
$\boldsymbol{p}^{x+}$	+1	0
$\boldsymbol{p}^{x-}$	-1	0
$\boldsymbol{p}^{y+}$	0	+1
$\boldsymbol{p}^{y-}$	0	-1

TABLE I. Expectation values of the X and Y measurements for each of the four states we can prepare.

In our analogue of Zeno protective measurement, Bob's protection comes from repeated strong measurements; an X measurement to protect $\boldsymbol{p}^{x\pm}$ or a Y measurement to protect $\boldsymbol{p}^{y\pm}$. However, we also need a model for Alice's measurements, i.e. we need to know how to continuously couple a pointer to the observable we want to measure, such that the amount of disturbance is small if the interaction strength is small and the interaction only acts for a short time interval. We turn to this next.

Since our toy model is essentially a classical system, we can in principle perform measurements without any back-action. This type of model is enough to make our point, but we will later describe a model with back-action in order to simulate quantum measurements more precisely.

Our model for a classical measuring device will be a classical point particle in one-dimension, with position Q and momentum P. Suppose that the particle is initially prepared in the state $Q = P = 0$ at time $t = 0$. When we want to measure the x coordinate of the system, we couple the momentum P of the particle to the X variable with the interaction Hamiltonian $H_I = gXP$ from time $t = 0$ to $t = 1/g$, where g is a coupling constant.

With this interaction, the particle will move one unit to the right if $X = +1$ and one unit to the left if $X = -1$, so long as the ball remains undisturbed during the course of the measurement, and the pointer will reach this position at $t = 1/g$. Thus, if we prepare the system in the state $\boldsymbol{p}^{x\pm}$ the pointer will end up at $Q = \pm 1$ with certainty, whereas if we prepare the system in either of the states $\boldsymbol{p}^{y\pm}$ then the pointer will move to $Q = +1$ with probability 1/2 and $Q = -1$ with probability 1/2 because, in those states, the x-coordinate of the system is either positive or negative with 50/50 probability.

Similarly, to measure the y-coordinate of the system, we use the Hamiltonian $H_I = gYP$ in the same way, and now the pointer will move to $Q = \pm 1$ with certainty when the $\boldsymbol{p}^{y\pm}$ states are prepared and will move randomly either to $Q = +1$ or $Q = -1$ when the $\boldsymbol{p}^{x\pm}$ states are prepared.

We can now describe an analogue of Zeno protected measurements in our model. Suppose that the system is initially prepared in one of the states $\boldsymbol{p}^{x\pm}$ or $\boldsymbol{p}^{y\pm}$ and that these are protected by performing either strong X measurements or strong Y measurements respectively at times $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$, where $\Delta t = 1/gN$. At the same time, the system is coupled to the measuring device by one of the continuous processes described above. In the limit $N \to \infty$, we will show that the pointer ends up pointing to the expectation value of the quantity being measured with probability 1, and the system remains in the probability distribution that it started in, i.e. either $\boldsymbol{p}^{x\pm}$ or $\boldsymbol{p}^{y\pm}$. If we repeat this process twice, coupling the pointer to X the first time round and Y the second time round then we can obtain the expectation values of both observables, which serve to identify which probability distribution was prepared with certainty. Thus, if protective measurement were a proof of the reality of the quantum state then this model would analogously be a proof of the reality of the probability distributions $\boldsymbol{p}^{x\pm}$ and $\boldsymbol{p}^{y\pm}$. Since these distributions overlap, i.e. the ontic state is not enough to determine which distribution was prepared, this is clearly preposterous. What is actually happening is that the protective measurement is mostly measuring a property of the measurements that are protecting the system, rather than a property of the initial ontic state of the system. It is the randomization due to this protection that causes the pointer to point to the expectation value.

Let's see how this works in a little more detail. Suppose the system is prepared in $\boldsymbol{p}^{x+}$ and is protected by strongly measuring X every Δt seconds. This will cause the y-coordinate to be randomized every Δt seconds, whilst leaving the x-coordinate as it is. If the continuous measurement interaction is set to measure X then the randomization of the y-coordinate will have no effect on the motion of the pointer because the Hamiltonian $H_I = gXP$ only couples to the x-coordinate. Thus, the measurement will work just as it did without the protection and the pointer will move to $Q = +1$ in time $1/g$. This is the expectation value of X in this case.

If, on the other hand, the continuous measurement interaction is set to measure Y then the y-coordinate is initially random, so the pointer will move $1/N$ units to the right or left with 50/50 probability in the time interval before the first strong X measurement. After each strong X measurement, the y-coordinate is randomized again, so there will be a probability $1/2$ that the pointer continues moving in the same direction for another distance $1/N$ and a probability $1/2$ that it switches direction and moves $1/N$ units in the opposite direction. If we denote the position of the pointer at time t_n as Q_n, the variables Q_n describe a N-step discrete time random walk on the line with step distance $1/N$.

The quantity $\tilde{Q}_n = N(Q_n - Q_{n-1})$ describes the change in position of the pointer in the nth time step, rescaled such that $\tilde{Q}_n = \pm 1$. The variables $\tilde{Q}_n$ are independently and identically distributed with uniform distribution because the y-coordinate is freshly randomized at every step. We can now write the final position of the pointer as

$$Q_N = \frac{1}{N}\sum_{n=1}^{N} \tilde{Q}_n, \tag{32}$$

from which we see that it is the sample mean of N i.i.d. variables with uniform distribution. The variables $\tilde{Q}_n$ have mean 0 and variance 1 so, by the central limit theorem, in the limit $N \to \infty$ the distribution of Q_N approaches a normal distribution with mean 0 and variance $1/N$. Since the variance tends to zero as $N \to \infty$, the limit will be a Dirac δ distribution centred at 0. So, in this limit, the pointer will end up pointing to $Q = 0$ with probability 1, and this is the expectation value of Y.

Note that, in both of these measurements, the system remains in the $\boldsymbol{p}^{x+}$ distribution throughout. Therefore, we can perform a protective measurement of X followed by a protective measurement of Y and obtain both the expectation values. Similarly, if we started with $\boldsymbol{p}^{x-}$ or $\boldsymbol{p}^{y\pm}$ then we could obtain both X and Y expectation values in the same way. This would allow us to tell with certainty which of the four states had been prepared.

Obviously, without the protection, there is no procedure that would allow $\boldsymbol{p}^{x+}$ to be distinguished from $\boldsymbol{p}^{y+}$ with certainty, as both distributions assign probability $1/2$ to the ontic state $(+,+)$. Performing a Y measurement on a system prepared in $\boldsymbol{p}^{x+}$ without protection would yield the values ± 1 with 50/50 probability rather than the expectation value. The reason that we get the expectation value 0 with protection is that the protecting measurement randomizes the y-coordinate, which effectively reprepares the system in an independent copy of $\boldsymbol{p}^{x+}$ each time. Thus, Zeno protective measurement is far more like distinguishing N copies of $\boldsymbol{p}^{x+}$ from N copies of $\boldsymbol{p}^{y+}$ than it is distinguishing a single copy, and this can be done with with arbitrary precision in the limit $N \to \infty$.

Here, we can also uncover an implicit assumption in the argument that Zeno protective measurement implies the reality of the quantum state. Namely, since a measurement of the Pauli observable $\hat{X}$ on the state $|x+\rangle$ does not change the state, it is implicitly assumed that this means that no property of the system has changed at all so, in particular, we still have the same single copy of $|x+\rangle$ that we started with. However, in our model there are two ontic states, $(+,+)$ and $(+,-)$, that can be occupied when we prepare $\boldsymbol{p}^{x+}$ and the protecting measurement randomly switches them. Thus, after a protecting measurement, the y-coordinate is completely uncorrelated from what it was before, so the protective measurement re-prepares the system in a totally independent copy of $\boldsymbol{p}^{x+}$.

This concludes our basic model of Zeno protected measurements. However, in the model constructed so far, our continuous measurements do not simulate the disturbance to the system caused by a quantum measurement. To simulate quantum theory more closely, we would like a model in which the coordinate that is not being measured gets gradually randomized during the course of the measurement, so that it is completely randomized at time $t = 1/g$.

To model this, suppose that, during the course of a measurement of the x-coordinate, the system is subjected to a continuous Markovian evolution, such that the probability of making a transition from $(x,+)$ to $(x,-)$ or vice versa in a time interval $\mathrm{d}t$ is $r\mathrm{d}t$, where r is an arbitrary transition rate parameter that we may set as we please. The probability distribution is then governed by the master equations

$$\frac{\mathrm{d}p_{x+}}{\mathrm{d}t} = -rp_{x+} + rp_{x-} \qquad\qquad \frac{\mathrm{d}p_{x-}}{\mathrm{d}t} = -rp_{x-} + rp_{x+}, \tag{33}$$

which have solution

$$p_{x+}(t) = \frac{1}{2}\left[p_{x+}(0)\left(1+e^{-2rt}\right) + p_{x-}(0)\left(1-e^{-2rt}\right)\right] \tag{34}$$

$$p_{x-}(t) = \frac{1}{2}\left[p_{x-}(0)\left(1+e^{-2rt}\right) + p_{x+}(0)\left(1-e^{-2rt}\right)\right]. \tag{35}$$

In the limit $t \to \infty$ this gives

$$p_{x+} = p_{x-} = \frac{1}{2}[p_{x+}(0) + p_{x-}(0)], \tag{36}$$

so the y-coordinate gets completely randomized, whilst the x-coordinate remains unaffected. If we imagine that this process is going on at the same time as the Hamiltonian coupling $H_I = gXP$ then the pointer will move just as it did before because X does not change, but in the limit $g \to 0$, the back-action will have enough time to completely randomize the y-coordinate. Thus, by combining the Hamiltonian coupling with this back-action, we get the same effect as performing an instantaneous X measurement by shaking the two parts of the box.

Incorporating this disturbance has surprisingly little effect on the analysis of protective measurements. Consider again a system prepared in $\boldsymbol{p}^{x+}$, protected with strong X measurements, and continuously coupled to a pointer measuring X. In this case, the back action causes the system to switch between $(+,+)$ and $(+,-)$. This does not affect the motion of the pointer, which still moves continuously towards $Q = +1$, because the pointer is only coupled to the x-coordinate. It does not affect how the probability distribution of the system evolves because distributions with $p_{++} = p_{+-}$ are stationary states of the back-action evolution given in Eq. (34).

The more interesting case is where we continuously measure Y on a protected system prepared in $\boldsymbol{p}_{x+}$. Again, the back-action does not affect the motion of the pointer, as this is coupled to Y and the back action of a continuous Y measurement only affects the x-coordinate. On the other hand, for a finite N, the back action causes the protecting measurement to sometimes fail, just as it does in the quantum case. This is because it is now possible for a transition of the x-coordinate to occur, and when this happens the protecting measurement yields $X = -1$. However, the failure probability can be made arbitrarily small by taking the limit $N \to \infty$, just as in the quantum case.

In a bit more detail, under the Y-measurement back-action, the probability that the system remains in an ontic state with positive x-coordinate after time Δt is

$$p_{X=+1}(\Delta t) = p_{++}(\Delta t) + p_{+-}(\Delta t) \tag{37}$$

$$= \frac{1}{2}\left[p_{++}(0)(1+e^{-2r\Delta t}) + p_{-+}(0)(1-e^{-2r\Delta t}) + p_{+-}(0)(1+e^{-2r\Delta t}) + p_{--}(0)(1-e^{-2r\Delta t})\right]. \tag{38}$$

For a starting state of $\boldsymbol{p}^{x+}$ we have $p_{++}(0) = p_{+-}(0) = 1/2$ and $p_{-+}(0) = p_{--}(0) = 0$, so

$$p_{X=+1}(\Delta t) = \frac{1}{2}\left(1+e^{-2r\Delta t}\right). \tag{39}$$

If there are a total of N protecting measurements during the course of the measurement then the probability that every protecting measurement gives the $X = +1$ outcome is $p_{X=+1}(\Delta t)^N$. This gives a probability of success of

$$p_{\mathrm{succ}} = \left[\frac{1}{2}\left(1+e^{-2r/gN}\right)\right]^N. \tag{40}$$

We still have the freedom to set r as a function of g and N, and setting $r = g/2N^3$ gives a success probability that converges to 1 as $N \to \infty$.

B. Hamiltonian protected measurements

A natural ψ-epistemic ontological model for the subset of quantum theory used in §III B has already been proposed [53]. In short, for this subset ("Gaussian quantum mechanics") the Wigner representation of states, transformations, and measurements are all non-negative and therefore admit a probabilistic interpretation in terms of a classical phase space, in this case (q', p', Q, P).

The states $|\psi_{c_q,c_p}\rangle$ correspond to Gaussian probability distributions

$$P_{c_q,c_p}(q', p') = \frac{1}{\pi} \exp\left(-(q' - c_q)^2 - (p' - c_p)^2\right). \tag{41}$$

The time evolution follows the classical Hamilton's equations, which matches the evolution of the operators in (26) (with the new phase space variables (q, p, Q, P) defined in an analogous way to the corresponding operators)

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ Q \\ P \end{pmatrix} = \begin{pmatrix} \{q, \mathcal{H}\} \\ \{p, \mathcal{H}\} \\ \{Q, \mathcal{H}\} \\ \{P, \mathcal{H}\} \end{pmatrix} = \begin{pmatrix} p \\ -q - gP \\ g(q + c_\theta) \\ 0 \end{pmatrix}. \tag{42}$$

Solutions to these equations are illustrated in Figs. 5 and 6.

The story is simplest when $P = 0$. We see that the derivative of Q is proportional to a_θ, the variable we are protectively measuring. But meanwhile the system variables (q, p) are evolving according to the free evolution, so that a_θ oscillates around c_θ. The final pointer position Q is the time-average, which will be exactly c_θ in the "protective measuremnt" limit of small g (or equivalently large final time $1/g$). When $P \neq 0$ the system variable p is disturbed by the measuring process, and the "protecting" free evolution smears this disturbance, ironically making the disturbance affect the variable a_θ we are trying to measure, but a_θ still averages to c_θ.

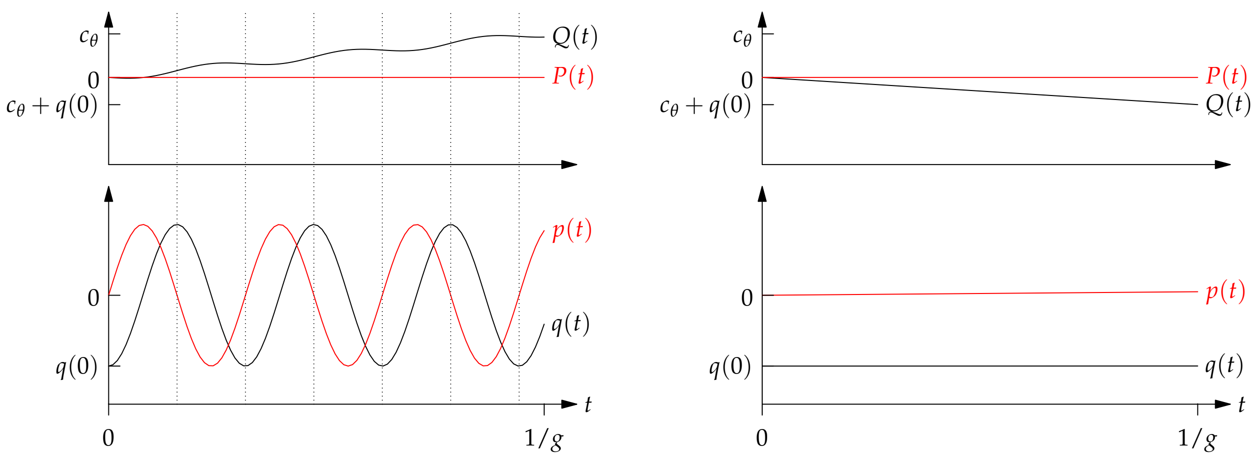

FIG. 5. What happens during measurements of $\hat{a}' = c_\theta + \hat{q}$ according to ψ-epistemic model, in the simple case $P = 0$ where there is no disturbance to the system due to the measurement interaction. On the left is a protective measurement, which due to the time-averaging of the system's free evolution gives the parameter c_θ regardless of the system's initial state. On the right is a traditional von-Neumann type measurement which gives the actual value of $c_\theta + q$.

Notice that if we could violate the uncertainty principle by preparing a pointer system with $Q = 0$ and $P = 0$, we could measure a_θ without causing any disturbance to the system [12]. But without the "protection" from the free evolution, we would simply get a sample from P_{c_q,c_p}—which would not uniquely fix (c_q, c_p). In this model we can see explicitly that the main role of the "protector" is actually to provide a "time ensemble" of system states (q', p') whose time averages are (c_q, c_p).

In fact the basic idea of the above was anticipated in the original paper [30], and the authors offered two responses:

(a) If there are nodes in, say, a 1-dimensional position wavefunction, the particle would have to travel at infinite speed in order to ensure that it is never found there. (This does not arise above because Gaussian wavefunctions do not have nodes.).

[12] That is, without any disturbance to (q, p). Of course our *state of knowledge* about the system would change, but nobody trying to learn about a system should want "protection" from "disturbance" to their knowledge of the system!

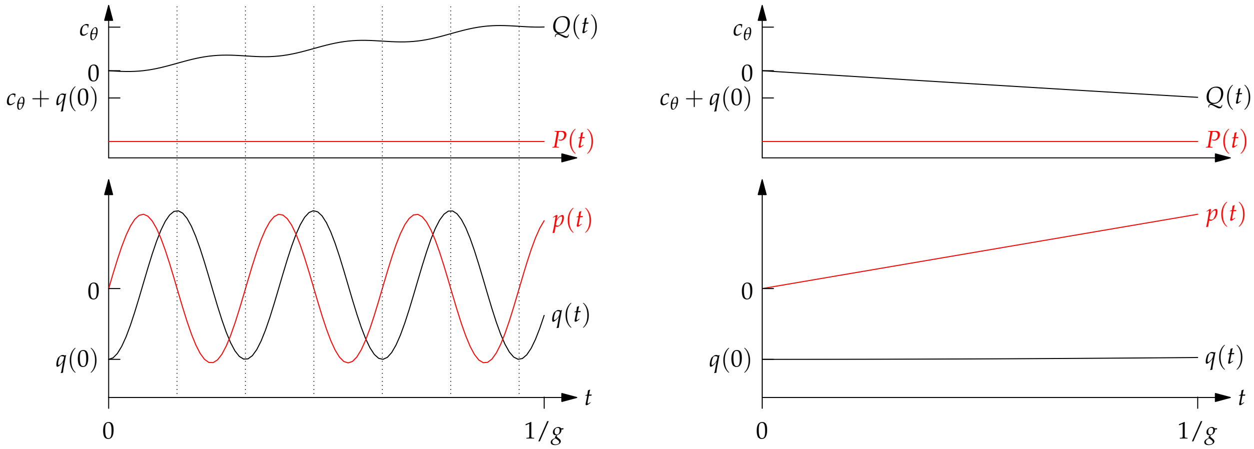

FIG. 6. As in fig. 5, but now with non-zero $P(0)$, so that p is disturbed by the measurement. In the protective case on the left, the free evolution "smears out" the disturbance.

(b) Systems in eigenstates of real Hamiltonains have constant position in Bohmian mechanics. (This does not arise above because the ontological model we use behaves very differently to Bohmian mechanics as applied to the Gaussian quantum mechanics.)

The latter can be easily dismissed in the present context - Bohmian mechanics is a ψ-ontic ontological model and so is clearly useless as a counter-example to claims that protective measurement establishes the reality of the quantum state. The former is a little more compelling, but it is unclear how to actually turn this into an argument for the reality of the wavefunction that is not vunerable to our counter-example. Indeed such an argument would presumably be unable to establish the reality of Gaussian wavefunctions (perhaps it would only establish the reality of wavefunctions with nodes), which would be a rather odd situation. Nodes in a wavefunction might just be a quantum phenomena that is difficult to account for in any ontological model, indepedently of whether it is ψ-ontic or ψ-epistemic.

V. CONCLUSION

In this paper, we have given three arguments that protective measurements do not imply the reality of the quantum state. Firstly, we pointed out that, given the same resources as in protective measurement, Alice could determine the state of the system in a much more straightforward way by doing process tomography on the protection operation. Secondly, we showed that most of the information in a protective measurement comes from the protection operation rather than the system itself — in fact all of it for Hamiltonian protected measurements. In the course of doing this, we found a new procedure equivalent to protective measurement that works for a class of Gaussian states and quadratic Hamiltonians that has a fixed finite interaction strength and time duration. This shows that criticisms of protective measurement based on practical considerations are misdirected. Finally, we constructed explicit ψ-epistemic ontological models of protective measurement, which rigorously establishes that they are compatible with epistemic quantum states.

An implicit underlying assumption of the argument for the reality of the quantum state based on protective measurement is that anything which does not change the quantum state does not affect the system at all. This idea is already incompatible with a ψ-epistemic interpretation of the quantum state, in which we should think of quantum states as more akin to probability measures than to classical fields. Thus, we expect there to be underlying microstates which may be disturbed even if the quantum state stays the same. In fact, this is the mechanism behind protective measurement in our toy models, in which the protection operation effectively prepares the system in an independent copy of the initial state. Indeed, the desire for "protection" is itself suspect in the ψ-epistemic view, since it seems based on the converse assumption that anything which changes the quantum state affects the system. But probability measures can be updated without any change to the system they describe, and indeed in our toy models the "protection" prevents the measurement from actually revealing new information about the initial configuration of the system. If even the nomenclature of protective measurement only makes sense when presupposing the reality of the quantum state, proponents of such reality must be especially careful to avoid begging the question when invoking the protective measurement scheme.

Finally, it would be interesting to extend the analyses of §III and the toy models of §IV beyond the special cases considered to arbitrary protective measurements. In particular, it would be interesting to determine if the finite duration Hamiltonian protected measurements of §III B exist for more general classes of states and observables. Although the present analysis is more than enough to establish that the naive argument from "the quantum state can be measured by protective measurements" to "the quantum state must be real" is incorrect, and that analogous phenomena exist classically, there could be special features of the particular way that protective measurements work in quantum theory that depend on genuinely quantum phenomena. Something similar was recently shown for the phenomenon of "anomalous weak values", which do have a classical analogue [56], but, nonetheless, the specific way they arise in quantum theory is different to the way they arise in classical theories [57]. In particular anomolous weak values provide statistical evidence for (or a "proof" of) contextuality [58]. Thus, we do not wish to claim that there is nothing "quantum" about protective measurements, but rather that there is no convincing argument that they imply the reality of the quantum state, and plenty of compelling evidence that they do not.

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